Contents

Abstract

We develop various asymptotic relations between the Riemann zeta function $\zeta(s)$ and the interpolation errors of Lagrange and Hermite interpolation to functions like $|y|^s$ and $y^{2m}\log|y|$. We show that the interpolation nodes of these interpolation processes include zeros of Gegenbauer and Hermite polynomials and polynomials with equidistant zeros. Similar results are valid for the Dirichlet beta function $\beta(s)$ as well. So the results of the monograph serve as the bridge between the theory of zeta functions and polynomial interpolation, one of the most studied areas of analysis.

Several applications of major asymptotics to properties of zeta functions are presented. In particular, we develop new criteria for $\zeta(s) = 0$ and $\beta(s) = 0$ in the critical strip. Other applications include construction of universal exponential sums (in the spirit of Voronin's universality theorem), limit summary formulae for $\zeta(s)$ and $\beta(s)$, and new combinatorial representations for Bernoulli and Euler numbers.

Acknowledgements. We express our gratitude to Yuri Brudnyi for his encouragement. We are also grateful to Vilmos Totik and Viktor Burenkov for their helpful comments.

2010 *Mathematics Subject Classification*: Primary 41A05, 11M06, 26C05, 11M26; Secondary 11B68.

Key words and phrases: Riemann zeta function, Dirichlet zeta function, polynomial interpolation, polynomial approximation, Gegenbauer polynomials, Hermite polynomials, polynomials with equidistant zeros, universal exponential sums, Bernoulli numbers, Euler numbers.

Received 28.1.2013; revised version 15.7.2013.

1. Introduction

In this monograph we combine results from several areas of analysis, such as polynomial interpolation, approximation theory, orthogonal polynomials, and theory of zeta functions, to obtain asymptotic formulae for two zeta functions: the Riemann zeta function $\zeta(s)$ and the Dirichlet beta function $\beta(s)$. Applications include new criteria for $\zeta(s) = 0$ or $\beta(s) = 0$ in the critical strip $\{s \in \mathbb{C} : 0 < \operatorname{Re} s < 1\}$, construction of universal exponential sums, and new combinatorial representations for the Bernoulli and Euler numbers.

The asymptotic behavior of $\zeta(s)$ and $\beta(s)$ (as one of L-functions) is well-studied as $\operatorname{Im} s \to \infty$ (see for example [49, Ch. 1], [43, Ch. 7], and [27, Ch. 6]). However, asymptotic representations of these functions for a fixed s, other than the definitions

$$\zeta(s) := \sum_{n=1}^{N} n^{-s} + o(1), \qquad \beta(s) := \sum_{n=0}^{N} (-1)^n (2n+1)^{-s} + o(1), \tag{1.0.1}$$

as $N \to \infty$, are not known. Here, $\operatorname{Re} s > 1$ for $\zeta(s)$ and $\operatorname{Re} s > 0$ for $\beta(s)$.

We obtain asymptotic formulae of the following types:

1. Pointwise asymptotics.
2. Asymptotics in $L_{p,w}$-spaces, where $p \in (0, \infty)$ and w is a weight.
3. Asymptotic summation formulae.

In asymptotics of the first type, we express the zeta functions through the interpolation errors of Lagrange or Hermite interpolation of even degree N to functions like $|y|^s$ and $y^{2n} \log |y|$ for each fixed $y \in \mathbb{R} \setminus \{0\}$, as $N \to \infty$. Asymptotic summation formulae are corollaries of pointwise asymptotics. In addition, pointwise asymptotics are used to obtain asymptotic formulae in $L_{p,w}$-spaces. So the results of the monograph serve as the bridge between the theory of zeta functions and polynomial interpolation, one of the most studied areas of analysis.

To establish these asymptotics, we introduce and study special classes of polynomials of even degree N, whose zeros along with the origin serve as the interpolation nodes and allow pointwise and $L_{p,w}$-asymptotics for the zeta functions. In particular, Gegenbauer and Hermite polynomials, polynomials with equidistant zeros and Williams–Apostol polynomials belong to these classes.

In Chapter 2 we discuss new integral formulae for the interpolation errors, starting with a fairly general case of polynomial interpolation to functions $f(y) = \int_T \frac{d\mu(t)}{t-z}$, where $T \subseteq \mathbb{C}$ is a set and μ is a complex-valued measure on T (Section 2.1). We provide more details for a subclass of functions of the form $f(y) = \int_{\mathbb{R}} \frac{d\mu(t)}{t-iy}$, $y \in \mathbb{R}$ (Section 2.2). Finally, in Section 2.3 we establish explicit interpolation formulae for special functions $f_{s,l,\nu}(y) :=$

$|y|^s(\operatorname{sgn} y)^l \log^\nu |y|$, where $s \in \mathbb{C}$, $l = 0, 1$ and $\nu = 0,\ 1, \ldots$. Results of Chapter 2 generalize and extend integral interpolation formulae by Hermite [26], Bernstein [5, pp. 92, 98], Lubinsky [38], and the author [19, 20, 21, 23].

The idea of transition from interpolation formulae for $f_{s,l,\nu}$ to asymptotic representations for the zeta functions is explained in Remark 2.3.6. Development of this idea and applications are discussed in Chapters 3–6.

The first step in this direction is made in Chapter 3, where asymptotic properties of special sequences of polynomials are discussed. In Section 3.1 three classes $\mathbb{P}_d, \mathbb{P}_d^*$, and $\mathbb{P}_d^{**}$ of polynomial sequences $\{P_{2N+d}\}_{N=1}^\infty$, $d = 0, 1$, are introduced and some of their properties are studied. Examples of sequences from these classes, including Gegenbauer, Chebyshev, Hermite, Williams–Apostol, Laguerre polynomials and polynomials with equidistant zeros, are provided in Section 3.2. Asymptotic formulae for weighted L_p-quasinorms, $p \in (0, \infty)$, of polynomials from Section 3.2 are established in Section 3.3. Some special cases of these asymptotics for Gegenbauer and Hermite polynomials were obtained by Aptekarev, Buyarov, and Degeza [3].

For $\{P_{2N+d}\}_{N=1}^\infty \in \mathbb{P}_d$, four families of pointwise asymptotic relations between the zeta functions and the interpolation errors with nodes at zeros of $y^{1-kd}P_{2N+d}^k(y)$, where $d = 0, 1$ and $k = 1, 2$, are obtained in Section 4.2. For example, if

$$P_{2N+1}(y) = G_{2N+1}^\lambda(y)/(G_{2N+1}^\lambda)'(0), \qquad \lambda \geq 0,$$

is the normalized Gegenbauer polynomial of degree $2N+1$, then for every $y \in \mathbb{R} \setminus \{0\}$ and $-1 < \operatorname{Re} s < 2N-2$, $s \neq 1, 3, \ldots$,

$$|y|^{s-1} - L_{2N}(y) = -\frac{4\cos(s\pi/2)(1-2^{-s})\Gamma(s)G_{2N+1}^\lambda(y)}{\pi(2N+\lambda)^{s-1}(G_{2N+1}^\lambda)'(0)y}\zeta(s)(1+O((Ny)^{-2})) \qquad (1.0.2)$$

as $N \to \infty$, where L_{2N} interpolates $|y|^{s-1}$ at zeros of $G_{2N+1}^\lambda(y)/y$ and $L_{2N}(0) = 0$. Similarly, for $y \in \mathbb{R} \setminus \{0\}$ and $m = 1, 2, \ldots$,

$$\begin{aligned} &y^{2m}\log|y| - L_{2N}(y) \\ &\qquad = -\frac{(-1)^m(1-2^{-2m-1})(2m)!G_{2N+1}^\lambda(y)}{(2N+\lambda)^{2m}(G_{2N+1}^\lambda)'(0)y}\zeta(2m+1)(1+O((Ny)^{-2})) \qquad (1.0.3)\end{aligned}$$

as $N \to \infty$. Actually, more explicit forms of the remainder terms in (1.0.2) and (1.0.3) are presented in Section 4.2.

More asymptotic relations are given in Sections 4.3 and 4.4. In particular, asymptotic summation formulae for the zeta functions that hold in a broader domain than (1.0.1) are established in Section 4.4. For example, for $\operatorname{Re} s > -1$, $s \neq 1, 3, \ldots$,

$$\begin{aligned}\zeta(s) = &-\frac{\pi^{s+1/2}}{2\cos(s\pi/2)(1-2^{-s})\Gamma(s)}\,\frac{\sqrt{N}}{2^{2N}}\sum_{k=1}^N(-1)^{k+1}k^{s-1}\binom{2N}{N-k} \\ &+ O((\log N)^{\max\{2,\operatorname{Re} s\}}/N),\end{aligned}$$

as $N \to \infty$. Asymptotic representations for $\zeta(2m+1)$ and $\beta(2m)$, $m = 1, 2, \ldots$, are presented in Section 4.4 as well. In particular, asymptotic formulae for Catalan's constant $\beta(2)$ are given.

In Section 5.2, L_p-versions of (1.0.2) and (1.0.3) are obtained. New L_p-criteria for zeros of $\zeta(s)$ and $\beta(s)$ to be in the critical strip are established in Sections 5.3 (for $p = \infty$) and 5.4 (for $0 < p < \infty$). For example, if T_{2N} is the Chebyshev polynomial of the first kind of degree $2N$ and $L_{4N}(y)$ interpolates $|y|^s$ at the zeros of $yT_{2N}^2(y)$ for $0 < \operatorname{Re} s < 1$, then the following statements are valid:

(a) $\zeta(s) = 0$ if and only if for any $p \in (1/2, \infty)$,

$$C_1/N^{\operatorname{Re} s+1+1/p} \le \left(\int_{-1}^{1} \left||y|^{s+1} - L_{4N}(y))\right|^p dy\right)^{1/p} \le C_2/N^{\operatorname{Re} s+1+1/p}.$$

(b) $\zeta(s) \ne 0$ if and only if for any $p \in (1/2, \infty)$,

$$C_3/N^{\operatorname{Re} s+1} \le \left(\int_{-1}^{1} \left||y|^{s+1} - L_{4N}(y)\right|^p dy\right)^{1/p} \le C_4/N^{\operatorname{Re} s+1}.$$

Here, C_i, $1 \le i \le 4$, are positive constants independent of N. We believe that L_p-criteria for $\zeta(s) = 0$ and for $\zeta(s) \ne 0$ in Sections 5.3 and 5.4 are the first ones in terms of the Hermite interpolation error. Special cases of these results for Lagrange interpolation were discussed in [23]. There are numerous other criteria that have been developed for the last 150 years in connection with the celebrated Riemann hypothesis (see the survey [12]).

Three more applications are presented in Chapter 6. The first of them is related to the universality theorem by Voronin [53]. Combining this result with asymptotic representations for the zeta functions, we construct in Section 6.1 universal "exponential" sums, whose shifts along the imaginary axis can approximate continuous functions on the disk $|s - 3/4| < r$, $s \in \mathbb{C}$, $r \in (0, 1/4)$, that are analytic and nonvanishing in the interior of the disk. Real analogues of this result for continuous 2π-periodic functions on $[0, 2\pi)$ and continuous functions on $[-1, 1]$ that do not have sign changes on the corresponding intervals are valid as well. New proofs for the functional equations for $\zeta(s)$ and $\beta(s)$ and new combinatorial formulae for the Bernoulli and the Euler numbers are presented in Sections 6.2 and 6.3, respectively.

Finally, we note that the choice of two zeta functions $\zeta(s)$ and $\beta(s)$ is based on the method developed in this monograph. The question as to whether asymptotic formulae obtained here can be extended to all Dirichlet L-functions remains open.

Notation. Throughout, C, C_1, $C_2, \ldots$ denote positive constants independent of essential parameters. Occasionally we indicate dependence on, or independence of, certain parameters. In particular, $C(a, b, c, \ldots)$, $C_1(a, b, c, \ldots)$, $C_2(a, b, c, \ldots), \ldots$ denote constants that depend on $(a, b, c, \ldots)$. The same symbol does not necessarily denote the same constant at different occurrences.

We also use the generic notation of the set $\mathbb{N}$ of all positive integers, the set $\mathbb{R}$ of all real numbers, and the set $\mathbb{C} = \mathbb{R} + i\mathbb{R}$ of all complex numbers.

Let $D_M(a) := \{z \in \mathbb{C} : |z - a| < M\}$ be the open disk of radius M centered at $a \in \mathbb{C}$, and $\bar{D}_M(a) := \{z \in \mathbb{C} : |z - a| \le M\}$ be the closed disk. Let $\mathcal{P}_n$ be the set of all univariate algebraic polynomials of degree at most n, $n = 0, 1, \ldots$. In addition, $\lfloor x \rfloor$ denotes the floor function of $x \in \mathbb{R}$; and for $\nu < 0$, we set $1/\nu! := 0$ and $f^{(\nu)}(x) := 0$.

2. Integral formulae for the interpolation error term

In this chapter we establish explicit formulae for the interpolation error term for some classes of functions in fairly general settings.

2.1. General formula. In this section we extend the Hermite integral formula to Cauchy-type integrals.

2.1.1. Definitions and historic remarks. Let

$$\omega_{n+1}(z) := (z-z_1)^{k_1}(z-z_2)^{k_2}\dots(z-z_m)^{k_m}, \qquad \sum_{p=1}^{m} k_p = n+1, \tag{2.1.1}$$

be a polynomial from $\mathcal{P}_{n+1}$ with complex distinct zeros z_p of multiplicity k_p, $1 \le p \le m$.

It is well known [54, Sect. 3.1] that for any system of complex numbers

$$\{\gamma_p^{(j-1)}\}_{1\le j\le k_p,\,1\le p\le m}$$

there exists a unique polynomial $P_n(z) \in \mathcal{P}_n$ such that $P_n^{(j-1)}(z_p) = \gamma_p^{(j-1)}$, $1 \le j \le k_p$, $1 \le p \le m$. In particular, the numbers $\gamma_p^{(j-1)}$ can be defined as

$$\gamma_p^{(j-1)} := D_{j-1}f(z_p), \qquad 1 \le j \le k_p,\ 1 \le p \le m,$$

by values of a function $f(z) = D_0 f(z)$ and its "derivatives" $D_{j-1}f(z)$, $j \ge 2$, whose definitions can differ from the mainstream ones. In this case we say that the interpolation polynomial $P_n(z) = P_n(z, f, \omega_{n+1})$ *interpolates* f at zeros z_p of ω_{n+1} of multiplicity k_p, $1 \le p \le m$, $\sum_{p=1}^m k_p = n+1$. If $k_p = 1$, $1 \le p \le m$, that is, $m = n+1$, the polynomial P_n is called the *Lagrange interpolation polynomial* to f at the nodes $\{z_p\}_{p=1}^{n+1}$. Otherwise, P_n is called the *Hermite interpolation polynomial* to f at the nodes $\{z_p\}_{p=1}^m$ with the corresponding multiplicities $\{k_p\}_{p=1}^m$, $\sum_{p=1}^m k_p = n+1$.

The integral formula for the interpolation error term $f(z) - P_n(z)$ is well known for analytic functions f and derivatives $D_{j-1}f(z) = f^{(j-1)}(z)$. Let f be analytic inside of a closed rectifiable Jordan curve $\mathit{\Gamma}$ and let f be continuous on $\mathit{\Gamma}$. In addition, we assume that all zeros z_p, $1 \le p \le m$, of ω_{n+1} are inside of $\mathit{\Gamma}$. Then Hermite [26] (see also [54, Sect. 3.1]) proved the following integral formula:

$$f(z) - P_n(z, f, \omega_{n+1}) = \frac{\omega_{n+1}(z)}{2\pi i}\int_{\mathit{\Gamma}} \frac{f(t)}{(t-z)\omega_{n+1}(t)}\,dt, \tag{2.1.2}$$

which holds for every z inside of $\mathit{\Gamma}$.

Bernstein [5, pp. 92, 98] was the first author who extended this formula for Lagrange interpolation to the nonanalytic function $f(y) = (1-y)^s$, $s > 0$, on $[-1,1]$. Various versions of Bernstein's result were discussed by the author [19, 20, 21, 23]. Lubinsky [38, eq. (18)] found the explicit error formula for $f - P_n$, where

$$f(y) = \int_0^\infty \frac{d\lambda(a)}{1 + (ay)^2} \tag{2.1.3}$$

is a real-valued even function defined on $[-1, 1]\setminus\{0\}$, and $\lambda(a)$ is an increasing function on $(0, \infty)$ such that $0 < \int_0^\infty \frac{d\lambda(a)}{1+a^2} < \infty$. If n is an odd positive integer and $w_{n+1}(y) = T_{n+1}(y)$ is the Chebyshev polynomial, then Lubinsky [38, eq. (18)] showed that $P_n \in \mathcal{P}_{n-1}$ and for $y \in [-1, 1] \setminus \{0\}$,

$$f(y) - P_n(y, f, T_{n+1}) = (-1)^{(n+1)/2} T_{n+1}(y) \int_0^\infty \frac{d\lambda(a)}{(1 + (ay)^2)|T_{n+1}(i/a)|}. \tag{2.1.4}$$

In this chapter we extend the integral representations (2.1.2) and (2.1.4) to more general situations and apply them to the special function $|y|^s (\operatorname{sgn} y)^l \log^\nu |y|$, where $s \in \mathbb{C}$, $l = 0, 1$ and $\nu \in \mathbb{N}$. Here, we discuss an integral representation for $f - P_n$ in a general setting.

Let $T \subset \mathbb{C}$ be a closed set and let μ be a finitely additive complex-valued measure on T with the total variation v_μ. If T is a bounded set, then for every continuous function $F : T \to \mathbb{C}$ the Riemann–Stieltjes integral $\int_T F(t)\, d\mu(t)$ exists. If T is unbounded, then for every continuous function $F : T \to \mathbb{C}$ we define

$$\int_T F(t)\, d\mu(t) := \lim_{R\to\infty} \int_{T\cap \bar{D}_R(0)} F(t)\, d\mu(t), \tag{2.1.5}$$

provided that the limit exists.

Let $Z \subset \mathbb{C} \setminus T$ be an infinite set and let T, Z, and μ satisfy the following condition:

$$\int_T \frac{dv_\mu(t)}{|t - z|} < \infty, \qquad z \in Z. \tag{2.1.6}$$

Remark 2.1.1. Note that condition (2.1.6) implies existence of the integral $\int_T \frac{d\mu(t)}{t-z}$ for all $z \in Z$. In addition, since $(t - z)^{-1}$ is a continuous function on T, condition (2.1.6) is trivially satisfied if T is a bounded closed subset of $\mathbb{C}$.

For given sets T, Z and all measures μ satisfying (2.1.6), we define the class $I(T, Z)$ of all functions $f : Z \to \mathbb{C}$ of the form

$$f(z) = \int_T \frac{d\mu(t)}{t - z}. \tag{2.1.7}$$

Next we define the "derivatives" of f by

$$D_{j-1} f(z) := (-1)^{j-1}(j - 1)! \int_T \frac{d\mu(t)}{(t - z)^j}, \qquad j = 1, 2, \ldots, z \in Z. \tag{2.1.8}$$

Obviously, $D_0 f = f$. Existence of $D_{j-1} f(z)$ for all $z \in Z$ and $j = 1, 2, \ldots$ follows from (2.1.6) and the trivial estimate $|t - z|^{-j} \le (d(z, T))^{1-j} |t - z|^{-1}$, where $d(z, T) := \inf_{\tau\in T} |z - \tau| > 0$ is the distance from z to T. Note that, in most applications, $D_{j-1} f(z)$ coincides with the actual derivative $f^{(j-1)}(z)$.

2.1.2. General theorem

Theorem 2.1.2. *Let $f \in I(T, Z)$, where T, Z, and μ satisfy* (2.1.6)*, and let $z_p \in Z$ be zeros of multiplicity k_p, $1 \le p \le m$, of polynomial* (2.1.1)*. Then there exists a unique polynomial $P_n \in \mathcal{P}_n$, interpolating f at the nodes z_p of multiplicity k_p, $1 \le p \le m$,*

$\sum_{p=1}^{m} k_p = n+1$, *such that*

$$P_n(z) = \int_T \frac{\omega_{n+1}(t) - \omega_{n+1}(z)}{(t-z)\omega_{n+1}(t)}\, d\mu(t), \qquad z \in Z, \tag{2.1.9}$$

and

$$f(z) - P_n(z) = \omega_{n+1}(z) \int_T \frac{d\mu(t)}{(t-z)\omega_{n+1}(t)}, \qquad z \in Z. \tag{2.1.10}$$

Proof. We first consider the special case of $f(z) = (t-z)^{-1}$, $t \in T$, $z \in Z$, that is, the measure μ in (2.1.7) is the Dirac delta function with support in $\{t\}$. Note that $t \neq z_p$, $1 \le p \le m$, since $T \cap Z = \emptyset$. Then the function

$$P_{n,t}(z) := \frac{1}{t-z} - \frac{\omega_{n+1}(z)}{(t-z)\omega_{n+1}(t)} = \frac{\omega_{n+1}(t) - \omega_{n+1}(z)}{(t-z)\omega_{n+1}(t)}, \tag{2.1.11}$$

is a polynomial of degree at most n with complex coefficients, depending on t. Namely, if $\omega_{n+1}(z) = \sum_{q=0}^{n+1} c_q z^q$, then

$$\begin{aligned} P_{n,t}(z) &= \sum_{r=0}^{n} \Big(\sum_{q=0}^{n-r} c_{q+r+1} (t^q/\omega_{n+1}(t)) \Big) z^r \\ &= \sum_{r=0}^{n} \Big(\sum_{q=0}^{n-r} c_{q+r+1} z^q \Big) (t^r/\omega_{n+1}(t)). \end{aligned} \tag{2.1.12}$$

It is easy to verify, by (2.1.11), that $P_{n,t}$ interpolates $(t-z)^{-1}$ at z_p of multiplicity k_p, $1 \le p \le m$ [54, Sect. 3.1], that is,

$$P_{n,t}^{(j-1)}(z_p) = (-1)^{j-1}(j-1)!(t-z_p)^{-j}, \qquad 1 \le j \le k_p,\ 1 \le p \le m. \tag{2.1.13}$$

Indeed, let $P_{n,t}^* \in \mathcal{P}_n$ be the unique polynomial that satisfies (2.1.13) with $P_{n,t}$ replaced by $P_{n,t}^*$. Hence the polynomial $(t-z)P_{n,t}^*(z) - 1$ of degree at most $n+1$ has zeros at z_p of multiplicity k_p, $1 \le p \le m$, $\sum_{p=1}^{m} k_p = n+1$. Therefore, for all $z \in \mathbb{C}$ and $t \neq z_1, z_2, \ldots, z_m$,

$$(t-z)P_{n,t}^*(z) - 1 = C(t)\omega_{n+1}(z),$$

where the constant $C(t) = -1/\omega_{n+1}(t)$ is found by evaluation of this identity at $z = t$. Hence $P_{n,t}^*(z) = P_{n,t}(z)$.

Next, we notice that the rational functions $\psi_r(t) := t^r/\omega_{n+1}(t)$ are continuous on T and

$$|\psi_r(t)| \le C|t - z_1|^{-1}, \qquad 0 \le r \le n,\ t \in T,$$

where C is independent of t and r. Therefore by (2.1.6), ψ_r is μ-integrable on T, $0 \le r \le n$. Hence by (2.1.12), the coefficients of $P_{n,t}$ are μ-integrable on T and $P_n(z) := \int_T P_{n,t}(z)\, d\mu(t)$ is a polynomial from $\mathcal{P}_n$, satisfying the identities

$$P_n^{(j-1)}(z_p) = \int_T P_{n,t}^{(j-1)}(z_p)\, d\mu(t), \qquad 1 \le j \le k_p,\ 1 \le p \le m. \tag{2.1.14}$$

Further, integrating (2.1.13) with respect to $d\mu(t)$ and taking account of (2.1.8), we conclude that P_n interpolates f at z_p of multiplicity k_p, $1 \le p \le m$. Finally, integrating (2.1.11) with respect to $d\mu(t)$, we arrive at (2.1.9) and (2.1.10). ∎

2.1.3. Examples of classes $I(T, Z)$

EXAMPLE 2.1.3. T is a closed rectifiable Jordan curve, Z is a domain with boundary T, and $d\mu(t) = (1/(2\pi i))f(t)dt$, where $f : T \to \mathbb{C}$ is a continuous function. Then the Hermite formula (2.1.2) holds for $z_p \in Z$, $1 \le p \le m$.

EXAMPLE 2.1.4. T is a directed path in $\mathbb{C}$, $Z = \mathbb{C} \setminus T$. Then a function $f \in I(T, Z)$ given by (2.1.7) is called the *Cauchy transform* of the measure μ, and it plays an important role in complex analysis. Some contemporary problems related to this integral are discussed in [52, 40, 51, 29, 30].

EXAMPLE 2.1.5. $T = [a, b]$, $Z = \{z \in \mathbb{C} : \mathrm{Im}\ z > 0\}$, μ is a bounded increasing function on $[a, b]$. Then $f \in I(T, Z)$ if and only if (i) f is analytic on Z; (ii) $\mathrm{Im}\ f(z) \ge 0$ on Z; (iii) f is analytic and positive on $(-\infty, a)$ and f is analytic and negative on (b, ∞). This description can be found in [31, Theorem A6].

EXAMPLE 2.1.6. $T = \mathbb{R}$, $Z = \{z \in \mathbb{C} : \mathrm{Im}\ z > 0\}$, μ is a bounded increasing function on $\mathbb{R}$. Then $f \in I(T, Z)$ if and only if (i) f is analytic on Z; (ii) $\mathrm{Im}\ f(z) \ge 0$ on Z; (iii) $\sup_{y \ge 1} |yf(iy)| < \infty$. This description can be found in [1, Sect. 3.1].

One more special case of Theorem 2.1.2 for $T = \mathbb{R}$ and $Z = i\mathbb{R} \setminus \{0\}$ is studied in Section 2.2.

2.2. Interpolation formulae for $T = \mathbb{R}$ and $Z = i\mathbb{R} \setminus \{0\}$. In this section we discuss an important special case of Theorem 2.1.2.

2.2.1. Special case. Our applications of interpolation formulae are based on a special case of (2.1.10) when $T = \mathbb{R}$ and $Z = i\mathbb{R} \setminus \{0\}$.

Here and in what follows, we slightly change notation from Section 2.1. Let us consider the class $I(\mathbb{R})$ of all functions $f : \mathbb{R} \setminus \{0\} \to \mathbb{C}$ of the form

$$f(y) = \int_{\mathbb{R}} \frac{d\mu(t)}{t - iy}, \tag{2.2.1}$$

where μ is a complex-valued function of bounded variation on every interval of $\mathbb{R}$. To ensure the convergence of the integral in (2.2.1), we assume that μ in (2.2.1) for $f \in I(\mathbb{R})$ satisfies either the condition

$$0 < \int_{\mathbb{R}} \frac{dv_\mu(t)}{(t^2 + y^2)^{1/2}} < \infty, \qquad y \in \mathbb{R} \setminus \{0\}, \tag{2.2.2}$$

which is equivalent to (2.1.6), or the condition

$$0 < \int_{\mathbb{R}} \frac{dv_\mu(t)}{|t|^q} < \infty, \qquad q = 1, \dots, r, \tag{2.2.3}$$

for some $r \in \mathbb{N}$. It is obvious that (2.2.3) implies (2.2.2), that is, (2.2.3) is a more restrictive condition than (2.2.2).

Note that by the definition of the variation of a singular function, the improper Riemann–Stieltjes integral in (2.2.3) can be defined as

$$\int_{\mathbb{R}} \frac{dv_\mu(t)}{|t|^q} := \lim_{\delta \to 0^+} \int_{|t| \ge \delta} \frac{dv_\mu(t)}{|t|^q}, \qquad q = 1, \dots, r.$$

Hence if (2.2.3) holds, then

$$\lim_{\delta \to 0^+} \int_{|t| \le \delta} \frac{d\nu_\mu(t)}{|t|^q} = 0, \qquad q = 1, \ldots, r. \tag{2.2.4}$$

We first discuss differentiability and continuity properties of functions from $I(\mathbb{R})$.

2.2.2. Differentiability and continuity properties

Proposition 2.2.1. *Let $f \in I(\mathbb{R})$.*

(a) *If μ satisfies* (2.2.2)*, then f is infinitely differentiable on $\mathbb{R} \setminus \{0\}$ and, for $y \in \mathbb{R} \setminus \{0\}$,*

$$f^{(j)}(y) = i^j j! \int_{\mathbb{R}} \frac{d\mu(t)}{(t - iy)^{j+1}}, \qquad j = 0, 1, \ldots. \tag{2.2.5}$$

(b) *If μ satisfies* (2.2.3) *for some $r \in \mathbb{N}$, then f is $(r-1)$-differentiable on $\mathbb{R}$ and* (2.2.5) *holds for $y \in \mathbb{R}$ and $j = 0, 1, \ldots, r-1$. Moreover, $f^{(r-1)}$ is continuous on $\mathbb{R}$.*

The proof of Proposition 2.2.1 is based on two lemmas. In the first one we provide three elementary estimates. Let us set for $z \in \mathbb{C} \setminus \{0\}$ and $r \in \mathbb{N}$,

$$\varphi_r(z) := 1 - (1+z)^{-r}, \qquad \psi_r(z) := z^{-1}((1+z)^{-r} - 1) + r.$$

Lemma 2.2.2. (a) *If $z \in \mathbb{C}$ and $0 < |z| \le 1/2$, then*

$$|\psi_r(z)| \le C_1(r)|z|, \tag{2.2.6}$$

where $C_1(r) \le 2^{2r+3}$.

(b) *If $z = ih$, $h \in \mathbb{R} \setminus \{0\}$, then*

$$|\varphi_r(z)| \le C_2(r)|z/(1+z)|, \tag{2.2.7}$$

$$|\psi_r(z)| \le C_3(r)|z/(1+z)|, \tag{2.2.8}$$

where $C_2(r) \le r2^{r-1}$ and $C_3(r) \le r(r+1)2^{r-2}$.

Proof. We first estimate $|\varphi_r(z)|$. For an imaginary $z = ih$, we have

$$|\varphi_r(z)| \le \left|\frac{z}{1+z}\right| \sup_{\tau \in \mathbb{R}} \frac{\sum_{l=0}^{r-1} \binom{r}{l+1} |\tau|^l}{(1+\tau^2)^{(r-1)/2}} \le 2^{r-1} \left|\frac{z}{1+z}\right| \sup_{\tau \in \mathbb{R}} \frac{\sum_{l=0}^{r-1} \binom{r}{l+1} |\tau|^l}{\sum_{l=0}^{r-1} \binom{r-1}{l} |\tau|^l}. \tag{2.2.9}$$

Since $\binom{r}{l+1}/\binom{r-1}{l} = r/(l+1) \le r$ for $0 \le l \le r-1$, (2.2.7) follows from (2.2.9) with $C_2(r) \le r2^{r-1}$.

Next, let $z \in \mathbb{C} \setminus \{0\}$. Then

$$\psi_r(z) = (1+z)^{-r} \sum_{l=0}^{r} \left(r\binom{r}{l} - \binom{r}{l+1} \right) z^l = z(1+z)^{-r} \sum_{l=0}^{r-1} (l+1) \binom{r+1}{l+2} z^l. \tag{2.2.10}$$

If $0 < |z| \le 1/2$, then it follows from (2.2.10) and the estimate $\binom{r+1}{l+2} \le 2^{r+1}$ for $0 \le l \le r-1$ that

$$\begin{aligned}
|\psi_r(z)| &\le |z| \sup_{|z| \le 1/2} |1+z|^{-r} \sum_{l=0}^{r-1} (l+1) \binom{r+1}{l+2} |z|^l \\
&\le 2^{2r+1} |z| \sup_{|z| \le 1/2} \sum_{l=0}^{\infty} (l+1)|z|^l \le 2^{2r+3}|z|.
\end{aligned}$$

Hence (2.2.6) holds. If $z = ih$, $h \in \mathbb{R} \setminus \{0\}$, then from (2.2.10) we obtain

$$|\psi_r(z)| \le \left|\frac{z}{1+z}\right| \sup_{\tau\in\mathbb{R}} \frac{\sum_{l=0}^{r-1}(l+1)\binom{r+1}{l+2}|\tau|^l}{(1+\tau^2)^{(r-1)/2}}$$
$$\le 2^{r-1}\left|\frac{z}{1+z}\right| \sup_{\tau\in\mathbb{R}} \frac{\sum_{l=0}^{r-1}(l+1)\binom{r+1}{l+2}|\tau|^l}{\sum_{l=0}^{r-1}\binom{r-1}{l}|\tau|^l}. \tag{2.2.11}$$

Since $(l+1)\binom{r+1}{l+2}/\binom{r-1}{l} = r(r+1)/(l+2) \le r(r+1)/2$ for $0 \le l \le r-1$, (2.2.8) follows from (2.2.11) with $C_3(r) \le r(r+1)2^{r-2}$. ■

In the next lemma we estimate an integral.

Lemma 2.2.3. *If for some $r \in \mathbb{N}$, μ satisfies the condition*

$$\int_{\mathbb{R}} \frac{dv_\mu(t)}{|t|^r} < \infty, \tag{2.2.12}$$

then for $h \in \mathbb{R} \setminus \{0\}$,

$$I_r(h) := |h| \int_{\mathbb{R}} \frac{dv_\mu(t)}{|t|^r(t^2+h^2)^{1/2}} = o(1) \tag{2.2.13}$$

as $h \to 0$.

Proof. We split the integral into two ones

$$I_r(h) = |h|\left(\int_{|t|\le\sqrt{|h|}} + \int_{|t|>\sqrt{|h|}}\right) \le \int_{|t|\le\sqrt{|h|}} \frac{dv_\mu(t)}{|t|^r} + |h|\int_{|t|>\sqrt{|h|}} \frac{dv_\mu(t)}{|t|^{r+1}}$$
$$= I_1^*(h) + I_2^*(h). \tag{2.2.14}$$

Next by (2.2.4),

$$\lim_{h\to 0} I_1^*(h) = 0, \tag{2.2.15}$$

and by (2.2.12),

$$I_2^*(h) \le \sqrt{|h|}\int_{|t|>\sqrt{h}} \frac{dv_\mu(t)}{|t|^r} \le \sqrt{|h|}\int_{\mathbb{R}} \frac{dv_\mu(t)}{|t|^r} = o(1), \qquad h \to 0. \tag{2.2.16}$$

Finally, (2.2.13) follows from relations (2.2.14)–(2.2.16). ■

Proof of Proposition 2.2.1. (a) We first note that all the integrals on the right-hand side of (2.2.5) exist, by condition (2.2.2). Let us prove (2.2.5) by induction. Assume that f is n-differentiable on $\mathbb{R} \setminus \{0\}$ and (2.2.5) holds for $j = n$. Note that our assumption is valid for $n = 0$, by (2.2.2). Then for any fixed $y \in \mathbb{R} \setminus \{0\}$ and any $h \in \mathbb{R}$ with $0 < |h| < |y|/2$, we obtain, by Lemma 2.2.2(a),

$$\left|\frac{f^{(n)}(y+h) - f^{(n)}(y)}{h} - i^{n+1}(n+1)!\int_{\mathbb{R}} \frac{d\mu(t)}{(t-iy)^{n+2}}\right|$$
$$= n!\left|\int_{\mathbb{R}} \psi_{n+1}\left(-\frac{ih}{t-iy}\right)\frac{d\mu(t)}{(t-iy)^{n+2}}\right|$$
$$\le C_1(n+1)n!|h|\int_{\mathbb{R}} \frac{dv_\mu(t)}{(t^2+y^2)^{(n+3)/2}} = o(1), \qquad h \to 0,$$

where $C_1(n+1)$ is the constant from (2.2.6). Hence (2.2.5) holds for $j = n+1$, $y \neq 0$. Thus f is $(n+1)$-differentiable on $\mathbb{R} \setminus \{0\}$. This proves statement (a).

(b) In view of (a), it suffices to prove (b) for $y = 0$. We prove this by induction. Let f satisfy (2.2.3) for some $r \geq 2$. We assume that f is n-differentiable at zero and

$$f^{(n)}(0) = i^n n! \int_{\mathbb{R}} \frac{d\mu(t)}{t^{n+1}}, \qquad 0 \leq n \leq r-2. \tag{2.2.17}$$

Note that (2.2.17) is valid for $n = 0$, by (2.2.3). Using (2.2.8) for $z = ih$, $h \in \mathbb{R} \setminus \{0\}$, we obtain

$$\begin{aligned}\left|\frac{f^{(n)}(h) - f^{(n)}(0)}{h} - i^{n+1}(n+1)! \int_{\mathbb{R}} \frac{d\mu(t)}{t^{n+2}}\right| &\leq n! \int_{\mathbb{R}} \left|\psi_{n+1}\left(-\frac{ih}{t}\right)\right| \frac{d\nu_\mu(t)}{|t|^{n+2}} \\ \leq C_3(n+1) n! |h| \int_{\mathbb{R}} \frac{d\nu_\mu(t)}{|t|^{n+2}(t^2+h^2)^{1/2}} &= C_3(n+1) n! I_{n+2}(h),\end{aligned} \tag{2.2.18}$$

where $C_3(n+1)$ is the constant from (2.2.8). Since $I_{n+2}(h) = o(1)$ as $h \to 0$, by (2.2.3) and Lemma 2.2.3, we obtain (2.2.5) for $j = n+1$ and $y = 0$ from (2.2.18).

It remains to prove that $f^{(r-1)}$ is continuous on $\mathbb{R}$. Using (2.2.7) for $z = ih$, $h \in \mathbb{R}\setminus\{0\}$, we obtain

$$\begin{aligned}|f^{(r-1)}(h) - f^{(r-1)}(0)| &= (r-1)! \left|\int_{\mathbb{R}} \psi_r\left(-\frac{ih}{t}\right) \frac{d\mu(t)}{t^r}\right| \\ &\leq C(r)|h| \int_{\mathbb{R}} \frac{d\nu_\mu(t)}{|t|^r (t^2+h^2)^{1/2}} = C(r) I_r(h) = o(1),\end{aligned}$$

as $h \to 0$, by (2.2.3) and Lemma 2.2.3. ∎

2.2.3. Interpolation theorems. We first discuss Hermite interpolation at nonzero nodes.

THEOREM 2.2.4. (a) *Let $f \in I(\mathbb{R})$, where μ satisfies condition* (2.2.2), *and let $y_1, \ldots, y_m$ be distinct points from $\mathbb{R} \setminus \{0\}$. Set*

$$H_{n+1}(y) := (y - y_1)^{k_1} \ldots (y - y_m)^{k_m}, \qquad \sum_{p=1}^m k_p = n+1. \tag{2.2.19}$$

Then there exists a unique polynomial $L_n(y) = L_n(y, f(y), H_{n+1}(y)) \in \mathcal{P}_n$ interpolating f at the nodes y_p of multiplicity k_p, $1 \leq p \leq m$, $\sum_{p=1}^m k_p = n+1$, that is,

$$L_n^{(j-1)}(y_p) = f^{(j-1)}(y_p), \qquad 1 \leq j \leq k_p,\ 1 \leq p \leq m, \tag{2.2.20}$$

and such that, for $y \in \mathbb{R} \setminus \{0\}$,

$$L_n(y) = \int_{\mathbb{R}} \frac{H_{n+1}(-it) - H_{n+1}(y)}{(t - iy) H_{n+1}(-it)}\, d\mu(t). \tag{2.2.21}$$

In addition, for $y \in \mathbb{R} \setminus \{0\}$,

$$f(y) - L_n(y) = H_{n+1}(y) \int_{\mathbb{R}} \frac{d\mu(t)}{(t - iy) H_{n+1}(-it)}. \tag{2.2.22}$$

(b) *If μ satisfies condition* (2.2.3) *for $r = 1$, then identities* (2.2.21) *and* (2.2.22) *hold for all $y \in \mathbb{R}$.*

Proof. Proposition 2.2.1(a) shows that in the special case of $T = \mathbb{R}$ and $Z = i\mathbb{R} \setminus \{0\}$, the "derivatives" $D_{j-1}f$, $j = 1, 2, \dots$, defined in (2.1.8), coincide with the corresponding derivatives of f. Thus statement (a) of the theorem follows from Theorem 2.1.2 for $z_p = iy_p$, $1 \le p \le m$, $z = iy$, $P_n(z) = L_n(y)$, $\omega_{n+1}(z) = i^{n+1}H_{n+1}(y)$, $\omega_{n+1}(t) = i^{n+1}H_{n+1}(-it)$.

To prove statement (b), we first note that condition (2.2.3) for $r = 1$ trivially implies (2.2.2). Therefore, (2.2.22) holds for all $y \neq 0$, by statement (a) of Theorem 2.2.4.

Next, f is continuous at zero, by Proposition 2.2.1(b). Further, since

$$|H_{n+1}(-it)|^{-1} \le \prod_{p=1}^{m} |y_p|^{-k_p} < \infty,$$

the function μ_1 defined by $d\mu_1(t) := d\mu(t)/H_{n+1}(-it)$ satisfies the condition

$$\int_{\mathbb{R}} v_{\mu_1}(t)/|t| < \infty.$$

Then the integral on the right-hand side of (2.2.22) can be expressed as $\int_{\mathbb{R}} \frac{d\mu_1(t)}{t-iy}$ and it is continuous at zero, by Proposition 2.2.1(b). Finally letting $y \to 0$ on both sides of (2.2.22), we see that (2.2.22) holds for $y = 0$ as well. Identity (2.2.21) for $y = 0$ follows from (2.2.22) and (2.2.1). ∎

THEOREM 2.2.5. *Let $y_1 = 0$ and $y_2, \dots, y_m$ be distinct points from $\mathbb{R} \setminus \{0\}$. Set*

$$H_{n+1}(y) := y^{k_1}(y - y_2)^{k_2} \dots (y - y_m)^{k_m}, \qquad \sum_{p=1}^{m} k_p = n + 1. \tag{2.2.23}$$

If $f \in I(\mathbb{R})$, where μ satisfies (2.2.3) *for $r = k_1$, then there exists a unique polynomial $L_n(y) = L_n(y, f(y), H_{n+1}(y)) \in \mathcal{P}_n$ interpolating f at the nodes y_p of multiplicity k_p, $1 \le p \le m$, $\sum_{p=1}^{m} k_p = n + 1$, that is,* (2.2.20) *is valid, and such that for all $y \in \mathbb{R}$ identities* (2.2.21) *and* (2.2.22) *hold.*

Proof. Let μ satisfy condition (2.2.3) for $r = k_1$. Assume first that $y_p \neq 0$, $1 \le p \le m$, and $y \neq 0$. Since (2.2.3) implies (2.2.2), we obtain, by Theorem 2.2.4(a),

$$\begin{aligned} &f(y) - L_n(y) \\ &\quad = (y - y_1)^{k_1} \prod_{p=2}^{m} (y - y_p)^{k_p} \int_{\mathbb{R}} \frac{i^{k_1} d\mu(t)}{(t - iy)(t - iy_1)^{k_1} \prod_{p=2}^{m} (-it - y_p)^{k_p}}. \end{aligned} \tag{2.2.24}$$

Then the function μ_2 defined by the relation

$$d\mu_2(t) := \frac{i^{k_1} d\mu(t)}{(t - iy) \prod_{p=2}^{m} (-it - y_p)^{k_p}}$$

satisfies the condition $\int_{\mathbb{R}} |t|^{-j} \, dv_{\mu_2}(t) < \infty$, $1 \le j \le k_1$, by (2.2.3). Therefore, Proposition 2.2.1(b) shows that the function $F(y_1) := \int_{\mathbb{R}} \frac{d\mu_2(t)}{t - iy_1}$ from $I(\mathbb{R})$ is $(k_1 - 1)$-differentiable on $\mathbb{R}$ and $F^{(k_1-1)}(y_1)$ is continuous on $\mathbb{R}$.

Using Proposition 2.2.1(b) again, we see that the integral on the right-hand side of (2.2.24) can be expressed as

$$\int_{\mathbb{R}} \frac{i^{k_1} \, d\mu(t)}{(t - iy)(t - iy_1)^{k_1} \prod_{p=2}^{m} (-it - y_p)^{k_p}} = \frac{i^{-k_1+1}}{(k_1 - 1)!} F^{(k_1-1)}(y_1).$$

Thus for any fixed $y \neq 0$ and fixed $y_p \in \mathbb{R}\setminus\{0\}$, $2 \leq p \leq m$, the right-hand side of (2.2.24) is a continuous function of y_1 on $\mathbb{R}$. Next we show that for fixed nonzero $y, y_2, y_3, \dots, y_m$, the interpolating polynomial $L_n(y) = L_n(y, y_1, y_2, \dots, y_m)$ is a continuous function of y_1 on $\mathbb{R}$ as well. To prove that, we use the classic representation

$$L_n(y, y_1, \dots, y_m) = \sum_{p=1}^{m} \sum_{j=1}^{k_p} f^{(j-1)}(y_p) L_{p,j}(y, y_1, \dots, y_m), \tag{2.2.25}$$

where $L_{p,j}$, $1 \leq j \leq k_p$, are fundamental polynomials of Hermite interpolation at y_p of multiplicity k_p, $1 \leq p \leq m$. Explicit expressions for $L_{p,j}$ were found by Hermite [26]; namely, if $H_{n+1}(y)$ is defined by (2.2.19), then for $1 \leq j \leq k_p$, $1 \leq p \leq m$,

$$L_{p,j}(y, y_1, \dots, y_m) = \frac{H_{n+1}(y)}{(j-1)!(y-y_p)^{k_p-j+1}} \left\{ \frac{(y-y_p)^{k_p}}{H_{n+1}(y)} \right\}_{(y_p)}^{(k_p-j)}, \tag{2.2.26}$$

where $\{(y-y_p)^{k_p}/H_{n+1}(y)\}_{(y_p)}^{(k_p-j)}$ denotes the Taylor polynomial of degree $k_p - j$ for the function $(y-y_p)^{k_p}/H_{n+1}(y)$ at y_p, $1 \leq p \leq m$.

Since $L_{p,j}$, $1 \leq j \leq k_p$, $1 \leq p \leq m$, are continuous functions of y_1 on $\mathbb{R}$ and, by Proposition 2.2.1(b), $f^{(j-1)}(y_1)$, $1 \leq j \leq k_1$, are continuous functions of y_1 on $\mathbb{R}$ as well, we conclude from (2.2.25) that $L_n(y) = L_n(y, y_1, \dots, y_m)$ is a continuous function of y_1 on $\mathbb{R}$. Thus letting $y_1 \to 0$ on both sides of (2.2.24), we arrive at

$$\begin{aligned} f(y) - L_n(y, 0, y_2, \dots, y_m) &= y^{k_1} \prod_{p=2}^{m} (y-y_p)^{k_p} \int_{\mathbb{R}} \frac{i^{k_1} d\mu(t)}{(t-iy)t^{k_1} \prod_{p=2}^{m}(-it-y_p)^{k_p}} \\ &= H_{n+1}(y) \int_{\mathbb{R}} \frac{d\mu(t)}{(t-iy)H_{n+1}(-it)} \end{aligned} \tag{2.2.27}$$

for $y \neq 0$, where H_{n+1} is defined in (2.2.23).

It remains to show that (2.2.27) holds for $y = 0$. Indeed, $f(0) = L_n(0)$ and

$$\begin{aligned} |y|^{k_1} \left| \int_{\mathbb{R}} \frac{d\mu(t)}{(t-iy)t^{k_1} \prod_{p=2}^{m}(-it-y_p)^{k_p}} \right| &\leq C|y|^{k_1} \int_{\mathbb{R}} \frac{dv_\mu(t)}{|t|^{k_1}(t^2+y^2)^{1/2}} \\ &= C|y|^{k_1-1} I_{k_1}(y) = o(1) \end{aligned} \tag{2.2.28}$$

as $y \to 0$, by (2.2.3) and Lemma 2.2.3. Therefore, (2.2.22) holds for all $y \in \mathbb{R}$ and $y_1 = 0$. ■

2.2.4. Even and odd functions. Here, we discuss interpolation formulae for even and odd functions from $I(\mathbb{R})$. Representations for even and odd components of $f \in I(\mathbb{R})$ are given by the formulae ($y \in \mathbb{R} \setminus \{0\}$)

$$f(y) = \int_{\mathbb{R}} \frac{d\mu(t)}{t-iy} = \int_{\mathbb{R}} \frac{t\,d\mu(t)}{t^2+y^2} + iy \int_{\mathbb{R}} \frac{d\mu(t)}{t^2+y^2} = \int_0^\infty \frac{t\,d\mu_e(t)}{t^2+y^2} + iy \int_0^\infty \frac{t\,d\mu_o(t)}{t^2+y^2},$$

where

$$\mu_e(t) = \mu(t) + \mu(-t), \quad \mu_o(t) = \mu(t) - \mu(-t), \quad t \in [0, \infty).$$

On the other hand, if $f : \mathbb{R} \setminus \{0\} \to \mathbb{C}$ is an even function of the form

$$f(y) = \int_0^\infty \frac{t\,d\mu(t)}{t^2+y^2}, \quad y \in \mathbb{R} \setminus \{0\}, \tag{2.2.29}$$

then

$$f(y) = \frac{1}{2} \int_{\mathbb{R}} \frac{d\mu(|t|)}{t - iy}, \qquad y \in \mathbb{R} \setminus \{0\}. \tag{2.2.30}$$

If $f : \mathbb{R} \setminus \{0\} \to \mathbb{C}$ is an odd function of the form

$$f(y) = iy \int_0^\infty \frac{d\mu(t)}{t^2 + y^2}, \qquad y \in \mathbb{R} \setminus \{0\}, \tag{2.2.31}$$

then

$$f(y) = \frac{1}{2} \int_{\mathbb{R}} \frac{\operatorname{sgn} t \, d\mu(|t|)}{t - iy}, \qquad y \in \mathbb{R} \setminus \{0\}. \tag{2.2.32}$$

These two implications come from the fact that, by (2.1.5),

$$\int_{\mathbb{R}} \frac{d\mu(|t|)}{t^2 + y^2} = \lim_{M \to \infty} \int_{-M}^{M} \frac{d\mu(|t|)}{t^2 + y^2} = 0, \qquad \int_{\mathbb{R}} \frac{|t| \, d\mu(|t|)}{t^2 + y^2} = \lim_{M \to \infty} \int_{-M}^{M} \frac{|t| \, d\mu(|t|)}{t^2 + y^2} = 0,$$

respectively. Formulae (2.2.30) and (2.2.32) show that the functions (2.2.29) and (2.2.31) belong to $I(\mathbb{R})$.

Let us denote by $I_e(\mathbb{R})$ or $I_o(\mathbb{R})$ the subclasses of $I(\mathbb{R})$ of all even or odd functions of the form (2.2.29) or (2.2.31), respectively.

Condition (2.2.2), which guarantees existence of the integral in (2.2.1), can be replaced by the following conditions:

$$0 < \int_0^\infty \frac{t \, d\upsilon_\mu(t)}{t^2 + y^2} < \infty, \qquad y \in \mathbb{R} \setminus \{0\}, \tag{2.2.33}$$

if $f \in I_e(\mathbb{R})$, and

$$0 < \int_0^\infty \frac{d\upsilon_\mu(t)}{t^2 + y^2} < \infty, \qquad y \in \mathbb{R} \setminus \{0\}, \tag{2.2.34}$$

if $f \in I_0(\mathbb{R})$. Conditions (2.2.33) and (2.2.34) guarantee the existence of the integrals in (2.2.29) and (2.2.31), respectively. Note that each of these conditions is weaker than (2.2.2). For example, setting $d\mu(t) = t^s \, dt$, $t \in (0, \infty)$, in (2.2.29) and (2.2.31), we see that conditions (2.2.33) and (2.2.34) are equivalent to $-2 < \operatorname{Re} s < 0$ and $-1 < \operatorname{Re} s < 1$, respectively, while for $d\mu(|t|) = |t|^s \operatorname{sgn} t \, dt$, $t \in \mathbb{R} \setminus \{0\}$, in (2.2.30) and (2.2.32), condition (2.2.2) is equivalent to $-1 < \operatorname{Re} s < 0$ (see also (2.3.1) and (2.3.2) below).

THEOREM 2.2.6. *Let $f \in I_e(\mathbb{R})$ or $f \in I_o(\mathbb{R})$, where μ satisfies condition* (2.2.33) *or* (2.2.34), *respectively. Let $n + 1$ be an even positive integer and let $x_1, \dots, x_m$ be distinct points from $(0, \infty)$. Then f is infinitely differentiable on $\mathbb{R} \setminus \{0\}$, and there exists a unique polynomial $L_{n-1} \in \mathcal{P}_{n-1}$ if $f \in I_e(\mathbb{R})$, and there exists a unique polynomial $L_n \in \mathcal{P}_n$ if $f \in I_o(\mathbb{R})$, interpolating f at $\pm x_p$ of multiplicity k_p, $1 \le p \le m$, $\sum_{p=1}^m k_p = (n + 1)/2$, such that*

$$f(y) - L_{n-1}(y) = H_{n+1}(y) \int_0^\infty \frac{t \, d\mu(t)}{(t^2 + y^2) H_{n+1}(-it)}, \qquad y \in \mathbb{R} \setminus \{0\}, \tag{2.2.35}$$

if $f \in I_e(\mathbb{R})$, and

$$f(y) - L_n(y) = iy H_{n+1}(y) \int_0^\infty \frac{d\mu(t)}{(t^2 + y^2) H_{n+1}(-it)}, \qquad y \in \mathbb{R} \setminus \{0\}, \tag{2.2.36}$$

if $f \in I_o(\mathbb{R})$. Here, $H_{n+1}(y) = \prod_{p=1}^m (y^2 - x_p^2)^{k_p}$.

Proof. The proof uses combination of techniques from those of Theorems 2.1.2 and 2.2.4(a). However, we cannot derive (2.2.35) and (2.2.36) directly from (2.2.22) since conditions on μ in Theorem 2.2.6 are weaker than in Theorem 2.2.4(a).

We first show that any function $f \in I_e(\mathbb{R})$ or $f \in I_o(\mathbb{R})$ is infinitely differentiable on $\mathbb{R} \setminus \{0\}$ and for $j = 0, 1, \ldots,$

$$f^{(j)}(y) = \int_0^\infty \frac{d^j}{dy^j}((t^2 + y^2)^{-1})t\, d\mu(t), \qquad y \in \mathbb{R} \setminus \{0\}, \tag{2.2.37}$$

if $f \in I_e(\mathbb{R})$, and

$$f^{(j)}(y) = \int_0^\infty \frac{d^j}{dy^j}(y(t^2 + y^2)^{-1}) d\mu(t), \qquad y \in \mathbb{R} \setminus \{0\}, \tag{2.2.38}$$

if $f \in I_0(\mathbb{R})$. We prove this statement by induction. Formulae (2.2.37) and (2.2.38) are valid for $j = 0$, by the definition of the classes $I_e(\mathbb{R})$ and $I_o(\mathbb{R})$. Assume that these formulae hold for $j = n$. Let $y \in \mathbb{R} \setminus \{0\}$ and $h \in (0, |y|/2)$ and let $f \in I_e(\mathbb{R})$. Then using the Mean Value Theorem, we easily obtain the estimate

$$\begin{aligned}\left|\frac{f^{(n)}(y+h) - f^{(n)}(y)}{h} - D_{n+1,e}(y)\right| \\ \le h \int_0^\infty \max_{z\in[y,y+h]} \left|\frac{d^{n+1}}{dz^{n+1}}((t^2 + z^2)^{-1})\right| t\, dv_\mu(t),\end{aligned} \tag{2.2.39}$$

where $D_{j,e}$, $j = 0, 1, \ldots,$ denotes the right-hand side of (2.2.37). Next (see [24, eq. 0.432]),

$$\begin{aligned}\left|\frac{d^{n+1}}{dz^{n+1}}((t^2 + z^2)^{-1})\right| &= \left|\sum_{q=0}^{\lfloor (n+1)/2\rfloor} c_q(n) z^{n-2q+1}(t^2 + z^2)^{q-n-2}\right| \\ &\le C(n)(t^2 + z^2)^{-(n+3)/2} \le C(n)(t^2 + z^2)^{-1}|z|^{-n-1},\end{aligned} \tag{2.2.40}$$

where $c_q(n)$, $0 \le q \le \lfloor (n + 1)/2\rfloor$, are constants independent of t and z. Combining condition (2.2.33) with estimates (2.2.39) and (2.2.40), we see that the left-hand side of (2.2.39) is $o(1)$ as $h \to 0^+$. A similar estimate holds for $h \in (-|y|/2, 0)$ as well. Therefore, f is $(n + 1)$-differentiable on $\mathbb{R} \setminus \{0\}$ and (2.2.37) holds for $j = n + 1$.

When $f \in I_o(\mathbb{R})$, a similar reasoning shows that f is also $(n + 1)$-differentiable on $\mathbb{R} \setminus \{0\}$ and (2.2.38) holds for $j = n + 1$ if we notice, by (2.2.40), that

$$\begin{aligned}\left|\frac{d^{n+1}}{dz^{n+1}}(z(t^2 + z^2)^{-1})\right| &= \left|z\frac{d^{n+1}}{dz^{n+1}}((t^2 + z^2)^{-1}) + (n + 1)\frac{d^n}{dz^n}((t^2 + z^2)^{-1})\right| \\ &\le C(n)(t^2 + z^2)^{-1}|z|^{-n}.\end{aligned}$$

Thus any function f from $I_e(\mathbb{R})$ or $I_o(\mathbb{R})$ is infinitely differentiable on $\mathbb{R} \setminus \{0\}$ and for $j = 0, 1, \ldots,$ representation (2.2.37) or (2.2.38) holds.

Next we consider the special functions $f(y) = \mathrm{Re}(t - iy)^{-1}$ and $f(y) = i\,\mathrm{Im}(t - iy)^{-1}$, $t \in \mathbb{R}$, $y \in \mathbb{R}\setminus\{0\}$, from $I_e(\mathbb{R})$ and $I_o(\mathbb{R})$, respectively. Note that the measure μ in (2.2.29) and (2.2.31) for these functions is the Dirac delta function with support in $\{t\}$.

Rewriting (2.1.11) for $z_p = ix_p$, $1 \le p \le m$, $z = iy$, $w_{n+1}(z) = i^{n+1}H_{n+1}(y)$, where $H_{n+1}(y) = \prod_{p=1}^{m}(y^2 - x_p^2)^{k_p}$, we obtain

$$L_{n,t}(y) := \frac{1}{t - iy} - \frac{H_{n+1}(y)}{(t - iy)H_{n+1}(-it)} = \frac{H_{n+1}(-it) - H_{n+1}(y)}{(t - iy)H_{n+1}(-it)}. \tag{2.2.41}$$

Since $H_{n+1}(-it)$ is a real-valued polynomial, from (2.2.41) we obtain

$$\begin{aligned} L_{n,t,\mathrm{Re}}(y) &:= \operatorname{Re} L_{n,t}(y) = \frac{t}{t^2 + y^2} - \frac{tH_{n+1}(y)}{(t^2 + y^2)H_{n+1}(-it)} \\ &= \frac{t(H_{n+1}(-it) - H_{n+1}(y))}{(t^2 + y^2)H_{n+1}(-it)}, \end{aligned} \tag{2.2.42}$$

$$\begin{aligned} L_{n,t,\mathrm{Im}}(y) &:= i \operatorname{Im} L_{n,t}(y) = \frac{iy}{t^2 + y^2} - \frac{iyH_{n+1}(y)}{(t^2 + y^2)H_{n+1}(-it)} \\ &= \frac{iy(H_{n+1}(-it) - H_{n+1}(y))}{(t^2 + y^2)H_{n+1}(-it)}. \end{aligned} \tag{2.2.43}$$

Note that $L_{n,t,\mathrm{Re}}(y)$ is an even polynomial of degree at most $n - 1$ and $L_{n,t,\mathrm{Im}}(y)$ is an odd polynomial of degree at most n. Next, if $H_{n+1}(y) = \sum_{q=0}^{(n+1)/2} c_{2q}y^{2q}$, then from (2.2.42) and (2.2.43) we deduce

$$L_{n,t,\mathrm{Re}}(y) = \sum_{r=0}^{(n-1)/2} \Big(\sum_{q=0}^{(n-1)/2-r} c_{2q+2r+2}(-1)^{q+1}(t^{2q+1}/H_{n+1}(-it))\Big) y^{2r}, \tag{2.2.44}$$

$$L_{n,t,\mathrm{Im}}(y) = i \sum_{r=0}^{(n-1)/2} \Big(\sum_{q=0}^{(n-1)/2-r} c_{2q+2r+2}(-1)^{q+1}(t^{2q}/H_{n+1}(-it))\Big) y^{2r+1}. \tag{2.2.45}$$

Further, the rational functions $\psi_q(t) := t^{2q}/H_{n+1}(-it) \ge 0$ are continuous on $\mathbb{R}$ and

$$\sup_{t\in[0,\infty)} \psi_q(t) \le C(t^2 + x_1^2)^{-1}, \qquad 0 \le q \le (n-1)/2,$$

where C is independent of t and q. Therefore, $t\psi_q(t)$ is μ-integrable on $[0,\infty)$ if condition (2.2.33) holds, and $\psi_q(t)$ is μ-integrable on $[0,\infty)$ if condition (2.2.34) holds. Thus by (2.2.44) or (2.2.45), $L_{n,t,\mathrm{Re}}(y)$ or $L_{n,t,\mathrm{Im}}(y)$ is a polynomial of degree at most $n-1$ or n, respectively, with coefficients μ-integrable on $\mathbb{R}$ that depend on t, if conditions (2.2.33) or (2.2.34), respectively, hold. Moreover, $L_{n,t,\mathrm{Re}}(y)$ or $L_{n,t,\mathrm{Im}}(y)$ interpolates $t(t^2+y^2)^{-1}$ or $iy(t^2 + y^2)^{-1}$ at $y = \pm x_p$ of multiplicity k_p, $1 \le p \le m, \sum_{p=1}^{m} k_p = (n + 1)/2$ (see the proof of Theorem 2.1.2).

Then $L_{n-1}(y) := \int_0^\infty L_{n,t,\mathrm{Re}}(y)\, d\mu(t)$ or $L_n(y) := \int_0^\infty L_{n,t,\mathrm{Im}}(y)\, d\mu(t)$ is a polynomial from $\mathcal{P}_{n-1}$ or $\mathcal{P}_n$, and it interpolates $f \in I_e(\mathbb{R})$ or $f \in I_o(\mathbb{R})$, respectively, at $\pm x_p$ of multiplicity k_p, $1 \le p \le m, \sum_{p=1}^{m} k_p = (n + 1)/2$. Integrating (2.2.42) or (2.2.43) with respect to $d\mu(t)$, we arrive at (2.2.35) or (2.2.36), respectively. ∎

Remark 2.2.7. In the special case of $f \in I_e(\mathbb{R})$, where μ is an increasing function satisfying (2.2.33), and for Lagrange interpolation at the Chebyshev nodes (that is, $H_{n+1} = T_{n+1}$) on $[-1, 1] \setminus \{0\}$, Theorem 2.2.6 was proved by Lubinsky [38, eq. (18)]. Actually, Lubinsky established relation (2.1.4) for even functions f of the form $f(y) = \int_0^\infty \frac{d\lambda(a)}{1+(ay)^2}$, $0 < |y| \le 1$, where λ is a nonnegative Borel measure on $[0,\infty)$ satisfying

$0 < \int_0^\infty \frac{d\lambda(a)}{1+a^2} < \infty$. Nevertheless, making the substitution $a = 1/t$, we see that $f \in I_e(\mathbb{R})$ with $d\mu(t) := -td\lambda(1/t)$, where μ satisfies the condition $0 < \int_0^\infty \frac{t\,d\mu(t)}{t^2+1} < \infty$, which is equivalent to (2.2.33) when $0 < |y| \le 1$. Some special cases of Theorem 2.2.6 for Lagrange interpolation were obtained in [23].

2.3. Interpolation formulae for functions $|y|^s(\operatorname{sgn} y)^l \log^\nu |y|$. Asymptotic representations for the zeta functions are based on the integral interpolation formulae obtained in this section.

2.3.1. General nodes. Historically, the function $|y|^s$ plays an important role in Lagrange interpolation and polynomial approximation; see for example [5, 6, 19, 20, 38, 21], where further references are given.

Here, we discuss integral formulae for the Hermite interpolation error term in the case of the function

$$f_{s,l,\nu}(y) := |y|^s(\operatorname{sgn} y)^l \log^\nu |y|,$$

where $s \in \mathbb{C}$, $l = 0$ or $l = 1$, and ν is a nonnegative integer.

It is well known [24, eq. 3.241.2] that for $-1 < \operatorname{Re} s < 0$, the following representations hold:

$$|y|^s = -\frac{2\sin(s\pi/2)}{\pi}\int_0^\infty \frac{t^{s+1}\,dt}{t^2+y^2} = -\frac{\sin(s\pi/2)}{\pi}\int_{\mathbb{R}} \frac{|t|^s \operatorname{sgn} t\,dt}{t-iy}, \qquad y \ne 0, \tag{2.3.1}$$

$$|y|^s \operatorname{sgn} y = \frac{2\cos(s\pi/2)}{\pi} y\int_0^\infty \frac{t^s dt}{t^2+y^2} = -\frac{i\cos(s\pi/2)}{\pi}\int_{\mathbb{R}} \frac{|t|^s\,dt}{t-iy}, \quad y \ne 0. \tag{2.3.2}$$

Hence setting

$$\mu_{s,0}(t) := -\sin(s\pi/2)/(\pi(s+1))|t|^{s+1}, \tag{2.3.3}$$

$$\mu_{s,1}(t) := -i\cos(s\pi/2)/(\pi(s+1))|t|^{s+1} \operatorname{sgn} t, \tag{2.3.4}$$

we see that $f_{s,0,0} \in I(\mathbb{R})$ for $\mu(t) = \mu_{s,0}(t)$, and $f_{s,1,0} \in I(\mathbb{R})$ for $\mu(t) = \mu_{s,1}(t)$, $-1 < \operatorname{Re} s < 0$. Note that $\mu_{s,0}$ and $\mu_{s,1}$ satisfy condition (2.2.2) for $-1 < \operatorname{Re} s < 0$.

Let $y_1, \dots, y_m$ be distinct points from $\mathbb{R}\setminus\{0\}$; let $H_{n+1}(y) = (y-y_1)^{k_1}\cdots(y-y_m)^{k_m}$, $\sum_{p=1}^m k_p = n+1$. Then, by Theorem 2.2.4(a), there exists a unique polynomial $L_n(y, f_{s,l,0}(y), H_{n+1}(y)) \in \mathcal{P}_n$ interpolating $f_{s,l,0}$, where $l = 0$ or $l = 1$, at the zeros y_p of H_{n+1} of multiplicity k_p, $1 \le p \le m$, $\sum_{p=1}^m k_p = n+1$, such that for $-1 < \operatorname{Re} s < 0$ and $y \ne 0$,

$$|y|^s - L_n(y, |y|^s, H_{n+1}(y)) = -\frac{\sin(s\pi/2)}{\pi} H_{n+1}(y)\int_{\mathbb{R}} \frac{|t|^s \operatorname{sgn} t\,dt}{(t-iy)H_{n+1}(-it)}, \tag{2.3.5}$$

and

$$|y|^s \operatorname{sgn} y - L_n(y, |y|^s \operatorname{sgn} y, H_{n+1}(y))$$
$$= -\frac{i\cos(s\pi/2)}{\pi} H_{n+1}(y)\int_{\mathbb{R}} \frac{|t|^s\,dt}{(t-iy)H_{n+1}(-it)}. \tag{2.3.6}$$

If s does not satisfy the condition $-1 < \operatorname{Re} s < 0$, then $f_{s,l,0}$ do not belong to $I(\mathbb{R})$ for $l = 0, 1$. Nevertheless, the following theorem holds:

THEOREM 2.3.1. *Identities* (2.3.5) *and* (2.3.6) *hold in the following cases:*

(a) $-1 < \operatorname{Re} s < n+1$, $y \in \mathbb{R} \setminus \{0\}$, *and* $y_p \in \mathbb{R} \setminus \{0\}$, $1 \le p \le m$;
(b) $0 < \operatorname{Re} s < n+1$, $y \in \mathbb{R}$, *and* $y_p \in \mathbb{R} \setminus \{0\}$, $1 \le p \le m$;
(c) $0 \le k_1 - 1 < \operatorname{Re} s < n+1$, $y \in \mathbb{R}$, *and* $y_1 = 0$, $y_p \in \mathbb{R} \setminus \{0\}$, $2 \le p \le m$.

To prove the theorem, we first discuss some properties of the integral

$$I_l(s,y) := \int_{\mathbb{R}} \frac{|t|^s (\operatorname{sgn} t)^l \, dt}{(t-iy)H_{n+1}(-it)}, \qquad l = 0,1, \tag{2.3.7}$$

where H_{n+1} is the polynomial (2.2.19) and $y_p \in \mathbb{R}$, $1 \le p \le m$.

PROPOSITION 2.3.2. (a) *For each fixed* $y \in \mathbb{R} \setminus \{0\}$ *and fixed* $y_p \in \mathbb{R} \setminus \{0\}$, $1 \le p \le m$, *the integral* $I_l(s,y)$ *is analytic on* $S_n := \{s \in \mathbb{C} : -1 < \operatorname{Re} s < n+1\}$, $l = 0,1$, *and the following formulae hold for* $s \in S_n$*:*

$$\frac{d^j I_l(s,y)}{ds^j} = \int_{\mathbb{R}} \frac{|t|^s (\operatorname{sgn} t)^l \log^j |t| \, dt}{(t-iy)H_{n+1}(-it)}, \qquad j = 1, \dots, \; l = 0,1. \tag{2.3.8}$$

(b) *For each fixed* $y \in \mathbb{R} \setminus \{0\}$, $y_1 = 0$ *and fixed* $y_p \in \mathbb{R} \setminus \{0\}$, $2 \le p \le m$, *the integral* $I_l(s,y)$ *is analytic on* $S_{n,k_1} := \{s \in \mathbb{C} : 0 \le k_1 - 1 < \operatorname{Re} s < n+1\}$, $l = 0,1$, *and formulae* (2.3.8) *hold for* $s \in S_{n,k_1}$.
(c) *For* $y = 0$ *and fixed* $y_p \in \mathbb{R} \setminus \{0\}$, $1 \le p \le m$, *the integral* $I_l(s,y)$ *is analytic on* $S_n^* := \{s \in \mathbb{C} : 0 < \operatorname{Re} s < n+1\}$, $l = 0,1$, *and formulae* (2.3.8) *hold for* $s \in S_n^*$.

Proof. Let us set

$$I_{j,l}(s,y) := \int_{\mathbb{R}} \frac{|t|^s (\operatorname{sgn} t)^l \log^j |t| \, dt}{(t-iy)H_{n+1}(-it)}, \qquad j = 0,1,\dots, \; l = 0,1.$$

If $s_0 \in S_n$, $y \in \mathbb{R} \setminus \{0\}$ and $y_p \in \mathbb{R} \setminus \{0\}$, $1 \le p \le m$, then the following estimates hold:

$$\begin{aligned} |I_{j,l}(s_0,y)| &\le 2 \int_0^\infty \frac{t^{\operatorname{Re} s_0} |\log t|^j \, dt}{(t^2+y^2)^{1/2} \prod_{p=1}^m (t^2+y_p^2)^{k_p/2}} = 2\Big(\int_0^1 + \int_1^\infty\Big) \\ &\le 2\bigg(\Big(|y| \prod_{p=1}^m |y_p|^{k_p}\Big)^{-1} \int_0^1 t^{\operatorname{Re} s_0} \log^j(1/t) \, dt + \int_1^\infty t^{\operatorname{Re} s_0 - n - 2} \log^j t \, dt\bigg) \\ &\le C_1(y,y_1,\dots,y_m) j! \big((\operatorname{Re} s_0 + 1)^{-j-1} + (n+1-\operatorname{Re} s_0)^{-j-1}\big), \end{aligned} \tag{2.3.9}$$

where C_1 is independent of j.

If $s_0 \in S_{n,k_1}$, $y \in \mathbb{R} \setminus \{0\}$ and $y_1 = 0$, $y_p \in \mathbb{R} \setminus \{0\}$, $2 \le p \le m$, then similarly to (2.3.9) we obtain

$$\begin{aligned} |I_{j,l}(s_0,y)| &\le 2 \int_0^\infty \frac{t^{\operatorname{Re} s_0 - k_1} |\log t|^j \, dt}{(t^2+y^2)^{1/2} \prod_{p=2}^m (t^2+y_p^2)^{k_p/2}} \\ &\le 2\bigg(\Big(|y| \prod_{p=2}^m |y_p|^{k_p}\Big)^{-1} \int_0^1 t^{\operatorname{Re} s_0 - k_1} \log^j(1/t) \, dt + \int_1^\infty t^{\operatorname{Re} s_0 - n - 2} \log^j t \, dt\bigg) \\ &\le C_2(y,y_2,\dots,y_m) j! \big((\operatorname{Re} s_0 - k_1 + 1)^{-j-1} + (n+1-\operatorname{Re} s_0)^{-j-1}\big), \end{aligned} \tag{2.3.10}$$

where C_2 is independent of j.

Finally, for $y = 0$ and $y_p \in \mathbb{R} \setminus \{0\}$, $1 \le p \le m$, we obtain

$$|I_{j,l}(s_0, 0)| \le C_3 j!((\operatorname{Re} s_0)^{-j-1} + (n+1-\operatorname{Re} s_0)^{-j-1}), \tag{2.3.11}$$

where C_3 is independent of j.

Next, for any s_0 that belongs to one of the sets S_n, S_{n,k_1}, S_n^*, we replace $|t|^s$ with its Taylor expansion in a small enough neighborhood of s_0 and obtain, by (2.3.9)–(2.3.11),

$$I_l(s,y) = \int_{\mathbb{R}} \sum_{j=0}^{\infty} \frac{|t|^{s_0}(\operatorname{sgn} t)^l \log^j |t|(s-s_0)^j}{j!(t-iy)H_{n+1}(-it)}\, dt = \sum_{j=0}^{\infty} \frac{I_{j,l}(s_0,y)(s-s_0)^j}{j!}. \tag{2.3.12}$$

Therefore, $I_l(s, y)$ is analytic in this neighborhood of s_0. Moreover, it follows from (2.3.12) that $d^j I_l(s,y)/ds^j|_{s=s_0} = I_{j,l}(s_0, y)$. Thus (2.3.8) is established in all cases of the proposition. ∎

Proof of Theorem 2.3.1. (a) Identities (2.3.5) and (2.3.6) hold for $-1 < \operatorname{Re} s < 0$, while their left-hand sides are analytic on $\mathbb{C}$ and their right-hand sides are analytic on $-1 < \operatorname{Re} s < n+1$, by Proposition 2.3.2(a). Thus (2.3.5) and (2.3.6) are valid for $-1 < \operatorname{Re} s < n+1$.

We cannot prove statements (b) and (c) by applying Theorems 2.2.4(b) and 2.2.5, respectively, since the functions $\mu = \mu_{s,0}$ and $\mu = \mu_{s,1}$ from (2.3.3) and (2.3.4) do not satisfy condition (2.2.3) for any $r \ge 1$ and any $s \in \mathbb{C}$. However, the proofs of these statements are similar to those of Theorems 2.2.4(b) and 2.2.5.

(b) If $0 < \operatorname{Re} s < n+1$, then the function μ_1 defined by

$$d\mu_1(t) = |t|^s (\operatorname{sgn} t)^{l+1}/H_{n+1}(-it)\, dt$$

satisfies the condition $\int_{\mathbb{R}} \frac{dv_{\mu_1}(t)}{|t|} < \infty$. Then the function $I_l(s,y) = \int_{\mathbb{R}} \frac{d\mu_1(t)}{t-iy}$ defined in (2.3.7) belongs to $I(\mathbb{R})$, where μ_1 satisfies condition (2.2.3) for $r = 1$. Therefore, $I_l(s, y)$ is continuous at $y = 0$, by Proposition 2.2.1(b). Note that by statement (a) of Theorem 2.3.1, identities (2.3.5) and (2.3.6) hold for $0 < \operatorname{Re} s < n+1$, $y \in \mathbb{R}\setminus\{0\}$, and $y_p \in \mathbb{R}\setminus\{0\}$, $1 \le p \le m$. Letting $y \to 0$ on both sides of (2.3.5) (for $l = 0$) and (2.3.6) (for $l = 1$), we conclude that (2.3.5) and (2.3.6) are valid for all $y \in \mathbb{R}$.

(c) Assume first that $y_p \ne 0$, $1 \le p \le m$, and $y \ne 0$. Then by statement (a),

$$\begin{aligned} |y|^s - L_n(y, |y|^s, H_{n+1}(y)) = -\frac{\sin(s\pi/2)}{\pi}(y-y_1)^{k_1} \prod_{p=2}^{m} (y-y_p)^{k_p} \\ \times \int_{\mathbb{R}} \frac{i^{k_1}|t|^s \operatorname{sgn} t\, dt}{(t-iy)(t-iy_1)^{k_1} \prod_{p=2}^{m}(-it-y_p)^{k_p}}. \end{aligned} \tag{2.3.13}$$

Next, since $1 \le k_1 < \operatorname{Re} s + 1 < n+2$, the function μ_2 defined by

$$d\mu_2(t) = |t|^s \operatorname{sgn} t / \prod_{p=2}^{m} (-it-y_p)^{k_p}\, dt$$

satisfies the condition $\int_{\mathbb{R}} \frac{dv_{\mu_2}(t)}{|t|^j} < \infty$, $1 \le j \le k_1$. Therefore, Proposition 2.2.1(b) shows that the function $F(y_1) := \int_{\mathbb{R}} \frac{d\mu_2(t)}{t-iy_1}$ from $I(\mathbb{R})$ is (k_1-1)-differentiable on $\mathbb{R}$ and $F^{(k_1-1)}(y_1)$ is continuous on $\mathbb{R}$. Using again Proposition 2.2.1(b), we see that the integral on the right-hand side of (2.3.13) is equal to

$$\frac{i^{-k_1+1}}{(k_1-1)!}F^{(k_1-1)}(y_1).$$

Thus for a fixed $y \neq 0$ and fixed $y_p \neq 0$, $2 \le p \le m$, the right-hand side of (2.3.13) is a continuous function of y_1 on $\mathbb{R}$.

Further, it follows from identities (2.2.25) and (2.2.26) that $L_n(y, |y|^s, H_{n+1}(y)) = L_n(y, y_1, y_2, \ldots, y_m)$ is a continuous function of y_1 on $\mathbb{R}$ as well. Hence letting $y_1 \to 0$ on both sides of (2.3.13), we arrive at

$$\begin{aligned} &|y|^s - L_n(y, 0, y_2, \ldots, y_m) \\ &\qquad = -\frac{\sin(s\pi/2)}{\pi} y^{k_1} \prod_{p=2}^{m} (y-y_p)^{k_p} \int_{\mathbb{R}} \frac{i^{k_1}|t|^s \operatorname{sgn} t\, dt}{(t-iy)t^{k_1}\prod_{p=2}^{m}(-it-y_p)^{k_p}} \end{aligned} \tag{2.3.14}$$

for $y \neq 0$. It remains to show that (2.3.14) holds for $y = 0$. Indeed,

$$|0|^s - L_n(0, 0, y_2, \ldots, y_m) = 0,$$

and

$$\begin{aligned} |y|^{k_1}\left|\int_{\mathbb{R}} \frac{|t|^s \operatorname{sgn} t\, dt}{(t-iy)t^{k_1}\prod_{p=2}^{m}(-it-y_p)^{k_p}}\right| &\le 2|y|^{k_1}\int_0^\infty \frac{t^{\operatorname{Re} s-k_1}\, dt}{(t^2+y^2)^{1/2}\prod_{p=2}^{m}(t^2+y_p^2)^{k_p/2}} \\ &\le 2|y|^{k_1}\left(\left(\prod_{p=2}^{m} y_p\right)^{-1}\int_0^1 \frac{t^{\operatorname{Re} s-k_1}\, dt}{(t^2+y^2)^{1/2}} + \int_1^\infty t^{\operatorname{Re} s-n-2}\, dt\right) \\ &\le |y|^{k_1}\left(C_1\int_0^1 \frac{t^{\operatorname{Re} s-k_1}\, dt}{(t^2+y^2)^{1/2}} + C_2\right) = o(1), \end{aligned}$$

as $y \to 0$, since

$$|y|^{k_1}\int_0^1 \frac{t^{\operatorname{Re} s-k_1}\, dt}{(t^2+y^2)^{1/2}} \le |y|^{k_1-1}\int_0^{\sqrt{|y|}} t^{\operatorname{Re} s-k_1}\, dt + |y|^{k_1-1/2}\int_{\sqrt{|y|}}^1 t^{\operatorname{Re} s-k_1}\, dt = o(1),$$

as $y \to 0$. Thus (2.3.14) is valid for $y = 0$. Therefore, statement (c) is established for $|y|^s$, while for $|y|^s \operatorname{sgn} y$ it can be proved similarly. ■

As a corollary of Theorem 2.3.1, we obtain interpolation formulae for $f_{s,l,\nu}$.

COROLLARY 2.3.3. *Let ν be a positive integer. Then the formulae*

$$\begin{aligned} &|y|^s \log^\nu |y| - L_n(y, |y|^s \log^\nu |y|, H_{n+1}(y)) \\ &\qquad = -\frac{H_{n+1}(y)}{\pi}\sum_{q=0}^{\nu}\binom{\nu}{q}(\pi/2)^q \sin((s+q)\pi/2)\int_{\mathbb{R}} \frac{|t|^s \log^{\nu-q}|t|\, \operatorname{sgn} t\, dt}{(t-iy)H_{n+1}(-it)}, \end{aligned} \tag{2.3.15}$$

$$\begin{aligned} &|y|^s \log^\nu |y| \operatorname{sgn} y - L_n(y, |y|^s \log^\nu |y|\, \operatorname{sgn} y, H_{n+1}(y)) \\ &\qquad = -\frac{iH_{n+1}(y)}{\pi}\sum_{q=0}^{\nu}\binom{\nu}{q}(\pi/2)^q \cos((s+q)\pi/2)\int_{\mathbb{R}} \frac{|t|^s \log^{\nu-q}|t| dt}{(t-iy)H_{n+1}(-it)}, \end{aligned} \tag{2.3.16}$$

hold in cases (a)–(c) *of Theorem* 2.3.1.

Proof. Proposition 2.3.2 shows that both sides of identities (2.3.5) and (2.3.6) are analytic functions of s in cases (a), (b), and (c) of Theorem 2.3.1 (for $y = 0$ in case (c), both sides of (2.3.5) and (2.3.6) are identically zero). Then using relations (2.3.8) of Proposition 2.3.2, we differentiate ν times both sides of (2.3.5) and (2.3.6) with respect to s and arrive at (2.3.15) and (2.3.16) in cases (a)–(c) of Theorem 2.3.1. ■

2.3.2. Symmetric nodes. Since $f_{s,l,\nu}$ are either even or odd functions, formulae (2.3.5), (2.3.6), (2.3.15), and (2.3.16) can be simplified for symmetric nodes.

In the following corollary we consider interpolation formulae only for functions $f_{s,0,0}$, $f_{s,1,0}$, $f_{2m,0,1}$, and $f_{2m+1,1,1}$, where $m = 0, 1, \dots$, and for even (statement (a)) and odd (statement (b)) polynomials H_{n+1}. Some of these identities will be used in Chapter 4 to obtain asymptotic representations for the zeta functions.

COROLLARY 2.3.4. *Let $x_1, \dots, x_M$ be distinct points from $(0, \infty)$ and let N and k_p, $1 \le p \le M$, be positive integers. Then the following statements hold:*

(a) *Let*

$$n = 2N - 1, \qquad H_{n+1}(y) = G_{2N}(y) := \prod_{p=1}^{M} (y^2 - x_p^2)^{k_p}, \qquad G_{2N}(0) \neq 0,$$

$$H_{n+1}(-it) = G_{2N}(it) = (-1)^N \prod_{p=1}^{M} (t^2 + x_p^2)^{k_p}, \qquad \sum_{p=1}^{M} k_p = N.$$

Then for each of the functions $f_{s,0,0}$, $f_{s,1,0}$, $f_{2m,0,1}$, and $f_{2m+1,1,1}$ there exists a unique polynomial $L_n \in \mathcal{P}_n$ ($L_n = L_{2N-2} \in \mathcal{P}_{2N-2}$ for even functions and $L_n = L_{2N-1} \in \mathcal{P}_{2N-1}$ for odd ones) interpolating the corresponding function at $\pm x_p$ of multiplicity k_p, $1 \le p \le M$, such that for $y \in \mathbb{R} \setminus \{0\}$,

$$\begin{aligned} |y|^s - L_{2N-2}(y, |y|^s, G_{2N}(y)) &= -\frac{2\sin(s\pi/2)}{\pi} G_{2N}(y) \\ &\times \int_0^\infty \frac{t^{s+1}\,dt}{(t^2+y^2)G_{2N}(it)}, \qquad -2 < \operatorname{Re} s < 2N-1,\ s \neq 0, 2, \dots; \end{aligned} \tag{2.3.17}$$

$$\begin{aligned} |y|^s \operatorname{sgn} y - L_{2N-1}(y, |y|^s \operatorname{sgn} y, G_{2N}(y)) &= \frac{2\cos(s\pi/2)}{\pi} y G_{2N}(y) \\ &\times \int_0^\infty \frac{t^{s}\,dt}{(t^2+y^2)G_{2N}(it)}, \qquad -1 < \operatorname{Re} s < 2N,\ s \neq 1, 3, \dots; \end{aligned} \tag{2.3.18}$$

$$\begin{aligned} &y^{2m} \log|y| - L_{2N-2}(y, y^{2m}\log|y|, G_{2N}(y)) \\ &\quad = (-1)^{m+1} G_{2N}(y) \int_0^\infty \frac{t^{2m+1}\,dt}{(t^2+y^2)G_{2N}(it)}, \qquad m = 0, 1, \dots, N-1; \end{aligned} \tag{2.3.19}$$

$$\begin{aligned} &y^{2m+1} \log|y| - L_{2N-1}(y, y^{2m+1}\log|y|, G_{2N}(y)) \\ &\quad = (-1)^{m+1} y G_{2N}(y) \int_0^\infty \frac{t^{2m+1}\,dt}{(t^2+y^2)G_{2N}(it)}, \qquad m = 0, 1, \dots, N-1. \end{aligned} \tag{2.3.20}$$

(b) *Let*

$$n = 2N, \qquad H_{n+1}(y) = yG_{2N}(y) = y \prod_{p=1}^{M} (y^2 - x_p^2)^{k_p},$$

$$H_{n+1}(-it) = G_{2N}(it) = (-1)^N \prod_{p=1}^{M} (t^2 + x_p^2)^{k_p}, \qquad \sum_{p=1}^{M} k_p = N.$$

Then for each of the functions $f_{s,0,0}$, $f_{s,1,0}$, $f_{2m,0,1}$, and $f_{2m+1,1,1}$ there exists a unique polynomial $L_n \in \mathcal{P}_n$ ($L_n = L_{2N} \in \mathcal{P}_{2N}$ for even functions and $L_n = L_{2N-1} \in \mathcal{P}_{2N-1}$ for odd ones) interpolating the corresponding function at the origin of multiplicity 1 *and at $\pm x_p$ of multiplicity k_p, $1 \le p \le M$, such that for $y \in \mathbb{R} \setminus \{0\}$,*

$$\begin{aligned}|y|^s - L_{2N}(y, |y|^s, yG_{2N}(y)) &= \frac{2\sin(s\pi/2)}{\pi} y^2 G_{2N}(y)\\ &\times \int_0^\infty \frac{t^{s-1}\,dt}{(t^2+y^2)G_{2N}(it)}, \qquad 0 < \operatorname{Re} s < 2N+1,\ s \ne 2,4,\dots; \end{aligned} \tag{2.3.21}$$

$$\begin{aligned}|y|^s \operatorname{sgn} y - L_{2N-1}(y, |y|^s \operatorname{sgn} y, yG_{2N}(y)) &= \frac{2\cos(s\pi/2)}{\pi} y G_{2N}(y)\\ &\times \int_0^\infty \frac{t^{s}\,dt}{(t^2+y^2)G_{2N}(it)}, \qquad 0 < \operatorname{Re} s < 2N,\ s \ne 1,3,\dots; \end{aligned} \tag{2.3.22}$$

$$\begin{aligned}&y^{2m}\log|y| - L_{2N}(y, y^{2m}\log|y|, yG_{2N}(y))\\ &\quad= (-1)^m y^2 G_{2N}(y) \int_0^\infty \frac{t^{2m-1}\,dt}{(t^2+y^2)G_{2N}(it)}, \qquad m = 1,2,\dots,N; \end{aligned} \tag{2.3.23}$$

$$\begin{aligned}&y^{2m+1}\log|y| - L_{2N-1}(y, y^{2m+1}\log|y|, yG_{2N}(y))\\ &\quad= (-1)^{m+1} y G_{2N}(y) \int_0^\infty \frac{t^{2m+1}\,dt}{(t^2+y^2)G_{2N}(it)}, \qquad m = 0,1,\dots,N-1. \end{aligned} \tag{2.3.24}$$

Proof. We first prove relations (2.3.18) and (2.3.20). Using (2.3.6) from Theorem 2.3.1(a) for $H_{n+1} = G_{2N}$, $y \in \mathbb{R} \setminus \{0\}$, and $-1 < \operatorname{Re} s < 2N$, we obtain

$$\begin{aligned}|y|^s \operatorname{sgn} y - L_{2N-1}(y, |y|^s \operatorname{sgn} y, G_{2N}(y)) &= \frac{-i\cos(s\pi/2)}{\pi} G_{2N}(y) \int_{\mathbb{R}} \frac{|t|^s\,dt}{(t-iy)G_{2N}(-it)}\\ = \frac{-i\cos(s\pi/2)}{\pi} G_{2N}(y) \lim_{R\to\infty} &\left(\int_{-R}^{R} \frac{t|t|^s\,dt}{(t^2+y^2)G_{2N}(it)} + iy \int_{-R}^{R} \frac{|t|^s\,dt}{(t^2+y^2)G_{2N}(it)} \right)\\ = \frac{2\cos(s\pi/2)}{\pi} y G_{2N}(y) &\int_0^\infty \frac{t^s\,dt}{(t^2+y^2)G_{2N}(it)}.\end{aligned}$$

Thus (2.3.18) is valid. Next similarly using (2.3.16) from Corollary 2.3.3 for $s = 2m+1$, $m = 0,1,\dots,N-1$, and $\nu = 1$, we obtain

$$\begin{aligned}y^{2m+1}\log|y| - L_{2N-1}(y, y^{2m+1}\log|y|, G_{2N}(y)) &= -\frac{i(-1)^{m+1}}{2} G_{2N}(y) \int_{\mathbb{R}} \frac{|t|^{2m+1}\,dt}{(t-iy)G_{2N}(it)}\\ &= \frac{(-1)^{m+1}}{2} y G_{2N}(y) \int_{\mathbb{R}} \frac{|t|^{2m+1}\,dt}{(t^2+y^2)G_{2N}(it)}.\end{aligned}$$

Hence (2.3.20) follows.

All other identities are corollaries of (2.3.18) and (2.3.20). Indeed, since L_{2N-1} in (2.3.18) and (2.3.20) are odd polynomials, the following implications are valid: (2.3.18) $\Rightarrow$(2.3.22) and (2.3.20)$\Rightarrow$(2.3.24). Next, multiplying both sides of (2.3.18) by y, using the identity

$$L_{2N}(y, yf(y), yG_{2N}(y)) = yL_{2N-1}(y, f(y), G_{2N}(y)) \tag{2.3.25}$$

for suitable odd functions f satisfying the condition $yf(y)|_{y=0} = 0$, and replacing s with $s-1$, we arrive at (2.3.21). We can obtain similarly (2.3.23) from (2.3.20) and (2.3.25).

Finally, replacing s with $s+2$ in (2.3.21), dividing both sides by y^2, and using the identity

$$L_{2N}(y, y^2 f(y), yG_{2N}(y)) = y^2 L_{2N-2}(y, f(y), G_{2N}(y)) \tag{2.3.26}$$

for suitable even functions f satisfying the condition $y^2 f(y)|_{y=0} = 0$, we arrive at (2.3.17). We can obtain similarly (2.3.19) from (2.3.23) and (2.3.26).

It remains to prove identities (2.3.25) and (2.3.26). Indeed, let f be an even function which is $(k_p - 1)$-differentiable at x_p, $1 \le p \le M$, $\sum_{p=1}^M k_p = N$, and $y^2 f(y)|_{y=0} = 0$. Then the polynomial

$$L_{2N-2}(y, f(y), G_{2N}(y)) = L_{2N-1}(y, f(y), G_{2N}(y))$$

interpolates f at $\pm x_p$ of multiplicity k_p, $1 \le p \le M$, $\sum_{p=1}^M k_p = N$. Next, for each $\pm x_p$ and $0 \le j \le k_p - 1$, $1 \le p \le M$,

$$\begin{aligned}(d^j/dy^j)(y^2 f(y))_{y=\pm x_p} &= x_p^2 f^{(j)}(\pm x_p) \pm 2jx_p f^{(j-1)}(\pm x_p) + j(j-1)f^{(j-2)}(\pm x_p)\\ &= x_p^2 L_{2N-2}^{(j)}(\pm x_p) \pm 2jx_p L_{2N-2}^{(j-1)}(\pm x_p) + j(j-1)L_{2N-2}^{(j-2)}(\pm x_p)\\ &= (d^j/dy^j)(y^2 L_{2N-2}(y))_{y=\pm x_p}.\end{aligned}$$

Therefore, the polynomial $y^2 L_{2N-2}(y, f(y), G_{2N}(y)) \in \mathcal{P}_n$ interpolates $y^2 f(y)$ at $\pm x_p$ of multiplicity k_p, $1 \le p \le M$, $\sum_{p=1}^M k_p = N$, and, in addition, it interpolates $y^2 f(y)$ at the origin of multiplicity 1. Hence (2.3.26) follows. Identity (2.3.25) can be proved similarly. ∎

Remark 2.3.5. Note that an identity like (2.3.17) for $s > 0$ and Lagrange interpolation was established by Bernstein [5, p. 98] who used (2.1.2) and the Cauchy theorem to prove his result. Various versions of Bernstein's identity were found by the author [19, 20, 21, 23]. In particular, the case of $0 < \operatorname{Re} s < 2N$ and the Chebyshev nodes was discussed in [21]. A similar identity for $s > 0$ and the Chebyshev nodes was found by Lubinsky [38] as a corollary of (2.1.4).

Remark 2.3.6. In Section 4.2 we shall use the following integral representations for $\zeta(s)$ and $\beta(s)$ (see Proposition 4.2.1):

$$2\Gamma(s)\beta(s) = \int_0^\infty \frac{t^{s-1}}{\cosh t}\,dt, \qquad \operatorname{Re} s > 0; \tag{2.3.27}$$

$$2(1-2^{-s})\Gamma(s)\zeta(s) = \int_0^\infty \frac{t^{s-1}}{\sinh t}\,dt, \qquad \operatorname{Re} s > 1; \tag{2.3.28}$$

$$2^{1-s}(1-2^{1-s})\Gamma(s+1)\zeta(s) = \int_0^\infty \frac{t^s}{\cosh^2 t}\,dt, \qquad \operatorname{Re} s > -1; \tag{2.3.29}$$

$$2^{1-s}\Gamma(s+1)\zeta(s) = \int_0^\infty \frac{t^s}{\sinh^2 t}\,dt, \qquad \operatorname{Re} s > 1. \tag{2.3.30}$$

On the other hand, for any increasing sequence $\{\beta_N\}_{N=1}^\infty$ of positive numbers with $\lim_{N\to\infty} \beta_N = \infty$, formulae (2.3.21) and (2.3.23) for large enough N can be written in the form ($\operatorname{Re} s > 0$, $s \ne 2, 4, \ldots,$ $m = 1, 2, \ldots$):

$$\begin{aligned}|y|^s - L_{2N}(y, |y|^s, yG_{2N}(y)) &= \frac{2\sin(s\pi/2)}{\pi\beta_N^s} G_{2N}(y)\\ &\times\left(\int_0^\infty \frac{t^{s-1}\,dt}{G_{2N}(it/\beta_N)} - \frac{1}{(\beta_N y)^2}\int_0^\infty \frac{t^{s+1}\,dt}{(1+t^2/(\beta_N y)^2)G_{2N}(it/\beta_N)}\right),\end{aligned} \tag{2.3.31}$$

$$|y|^{2m}\log|y| - L_{2N}(y,|y|^{2m}\log|y|, yG_{2N}(y)) = \frac{(-1)^m}{\beta_N^{2m}} G_{2N}(y)$$
$$\times\left(\int_0^\infty \frac{t^{2m-1}\,dt}{G_{2N}(it/\beta_N)} - \frac{1}{(\beta_N y)^2}\int_0^\infty \frac{t^{2m+1}\,dt}{(1+t^2/(\beta_N y)^2)G_{2N}(it/\beta_N)}\right). \quad (2.3.32)$$

The idea of transition from formulae (2.3.31) and (2.3.32) to pointwise asymptotic representations for $\zeta(s)$ and $\beta(s)$ is based on the following simple observation. Let $\{P_{2N+d}\}_{N=1}^\infty$ be a sequence of even or odd polynomials $P_{2N+d} \in \mathcal{P}_{2N+d}$ having all real and distinct zeros and normalized by $P_{2N+d}^{(d)}(0) = 1$, $N = 1, 2, \ldots$, $d = 0, 1$. In addition, we assume that $\beta_N^d P_{2N+d}(z/\beta_N) = \cos(z - d\pi/2)(1 + o(1))$ as $N \to \infty$, uniformly in some complex γ_N-neighborhood of zero, where $\lim_{N\to\infty}\gamma_N = \infty$. Then setting $G_{2kN}(z) = y^{-kd+1}P_{2N+d}^k(z)$, $k = 1, 2$, $d = 0, 1$, in (2.3.31) and (2.3.32), we can show that under some conditions the first integral on the right-hand sides of (2.3.31) and (2.3.32) approaches the corresponding integral on the right-hand sides of (2.3.27)–(2.3.30) as $N \to \infty$, while the second integral will serve as a part of the remainder.

Properties of polynomial sequences $\{P_{2N+d}\}_{N=1}^\infty$ are discussed in Chapter 3. Four families of asymptotic formulae for $\zeta(s)$ and $\beta(s)$ are presented in Chapter 4. Moreover, it is possible to extend these formulae to a broader domain compared with original representations (2.3.27)–(2.3.30).

3. Asymptotic properties of special sequences of polynomials

In this chapter we introduce special sequences of polynomials and study their asymptotic properties. Here and in what follows, d denotes an index that takes the values of 0 or 1.

3.1. Three classes of special sequences of polynomials. In this section we discuss properties of sequences of normalized polynomials with real zeros that satisfy certain asymptotic conditions.

Let $\beta = \beta_{(d)} = \{\beta_N\}_{N=1}^{\infty}$ and $\gamma = \gamma_{(d)} = \{\gamma_N\}_{N=1}^{\infty}$ be increasing sequences of positive numbers and $\delta = \delta_{(d)} = \{\delta_N\}_{N=1}^{\infty}$ a decreasing sequence of positive numbers, satisfying the conditions

$$\lim_{N\to\infty} \beta_N = \lim_{N\to\infty} \gamma_N = \lim_{N\to\infty} \delta_N^{-1} = \infty. \tag{3.1.1}$$

3.1.1. Class $\mathbb{P}_d(\beta, \gamma, \delta)$. We first define our major class of special sequences of polynomials. This class will be used in pointwise asymptotics for the zeta functions.

Definition 3.1.1. Let $\Pi_d = \{P_{2N+d}\}_{N=1}^{\infty}$ be a sequence of even (if $d = 0$) or odd (if $d = 1$) polynomials $P_{2N+d} \in \mathcal{P}_{2N+d}$ satisfying the following properties for $N = 1, \dots$:

(C1) *Real zeros*: P_{2N+d} has only real zeros of multiplicity 1.

(C2) *Normalization*:

$$P_{2N+d}^{(d)}(0) = 1. \tag{3.1.2}$$

(C3) *Strong asymptotics*: There exists a positive constant C_1 independent of N and z such that, for any $z \in \mathbb{C}$ with $|z| \le \gamma_N$,

$$|\beta_N^d\, P_{2N+d}(z/\beta_N) - \cos(z - d\pi/2)| \le C_1 \delta_N \min\{|z|^2, 1\} \begin{cases} \cosh|z|, & d = 0, \\ \sinh|z|, & d = 1. \end{cases} \tag{3.1.3}$$

Then we write $\Pi_d \in \mathbb{P}_d = \mathbb{P}_d(\beta, \gamma, \delta)$.

The following asymptotic property of zeros of $P_{2N+d} \in \Pi_d$ holds:

Proposition 3.1.2. *Let $0 < z_{1,N,d} < z_{2,N,d} < \cdots < z_{N,N,d}$ be positive zeros of a polynomial $P_{2N+d} \in \Pi_d$, $N = 1, 2, \dots$, $\Pi_d \in \mathbb{P}_d(\beta, \gamma, \delta)$. Then*

$$\lim_{N\to\infty} \beta_N z_{p,N,d} = (2p + d - 1)\pi/2, \qquad p = 1, 2, \dots, \tag{3.1.4}$$

and

$$\sup_{N\in\mathbb{N}} \beta_N z_{p,N,d} \le C(p, d), \qquad p = 1, 2, \dots. \tag{3.1.5}$$

Proof. Property (C3) of Definition 3.1.1 shows that for any $M > 0$,

$$\lim_{N\to\infty} \beta_N P_{2N+d}(z/\beta_N) = \cos(z - d\pi/2) \tag{3.1.6}$$

uniformly in the disk $D_M(0)$. Next note that the functions on both sides of (3.1.6) are analytic in $D_M(0)$ and, in addition, all positive zeros of $\cos(z-d\pi/2)$ are $\lambda_{d,p} := (2p+d-1)\pi/2$, $p=1,2,\ldots$. Then by the Hurwitz theorem [50, Sect. 3.4.5], for any $\rho\in(0,\pi/2)$ and a fixed $p=1,2,\ldots$, there exists a number $\nu=\nu(\rho,p)$ such that for all $N>\nu$, a polynomial $P_{2N+d}(z/\beta_N)$ has the only zero in each disk $D_\rho(\lambda_{d,m}), 1\le m\le p$, and moreover, $P_{2N+d}(z/\beta_N)$ has no zeros in $(0,\lambda_{d,p})\setminus\bigcap_{m=1}^p D_\rho(\lambda_{d,m})$. Thus $\beta_N z_{m,N,d}\in D_\rho(\lambda_{d,m})$, $1\le m\le p$, and this proves (3.1.4). Finally, (3.1.5) follows immediately from (3.1.4). ∎

3.1.2. Class $\mathbb{P}_d^*(\beta,\gamma,\delta)$. The following class contains special sequences of polynomials with explicit properties of their coefficients. This class will be used in Section 3.2 to show that some special sequences of polynomials belong to $\mathbb{P}_d(\beta,\gamma,\delta)$.

Definition 3.1.3. Let $\Pi_d=\{P_{2N+d}\}_{N=1}^\infty$ be a sequence of even (if $d=0$) or odd (if $d=1$) polynomials $P_{2N+d}\in\mathcal{P}_{2N+d}$ with real distinct zeros of the form

$$\beta_N^d P_{2N+d}(z/\beta_N)=\sum_{m=0}^{N}\frac{(-1)^m}{(2m+d)!}\mu_{2m+d,N}z^{2m+d},\qquad N=1,2,\ldots,\tag{3.1.7}$$

where the coefficients $\mu_{2m+d,N}$, $0\le m\le N$ satisfy the following properties:

(D1) *Positivity*: $\mu_{2m+d,N}>0$, $1\le m\le N$.
(D2) *Normalization*: $\mu_{d,N}=1$.
(D3) *Strong asymptotics*: There exists a positive constant C_2 independent of N with

$$\sup_{0\le m\le\gamma_N}|\mu_{2m+d,N}-1|\le C_2\delta_N.$$

(D4) *Boundedness*:

$$\sup_N\sup_{0\le m\le\gamma_N}\mu_{2m+d,N}=C_0<\infty.$$

Then we write $\Pi_d\in\mathbb{P}_d^*=\mathbb{P}_d^*(\beta,\gamma,\delta)$.

Remark 3.1.4. Note that property (D1) follows from (D2) by Descartes' rule of signs.

Some properties of polynomial sequences from $\mathbb{P}_d^*$ and a relationship between the classes $\mathbb{P}_d^*$ and $\mathbb{P}_d$ are discussed in the next two propositions. We use the following additional notation in these propositions: For an integer $k\ge0$, $a\in\mathbb{R}$, and $\alpha=\{\alpha_N\}_{N=1}^\infty$, let

$$\alpha_k:=\{\alpha_{N+k}\}_{N=1}^\infty,\qquad \alpha-a:=\{\alpha_N-a\}_{N=1}^\infty,\qquad a\alpha:=\{a\alpha_N\}_{N=1}^\infty.$$

In addition, we note that in the next proposition a sequence $\gamma=\{\gamma_N\}_{N=1}^\infty$ may include negative numbers. In order to define $\mathbb{P}_d^*$ for such a sequence, we assume by the convention that property (D3) is always true for negative γ_N.

The following proposition is helpful in constructing new classes $\mathbb{P}_d^*$ from existing ones.

Proposition 3.1.5. *The following two statements are equivalent:*

(A) $\{P_{2N+d}\}_{N=1}^\infty\in\mathbb{P}_d^*(\beta,\gamma,\delta)$.
(B) *For every integer* $s\ge0$,

$$\left\{\frac{P^{(s)}_{2N+s+|d-s^*|}(z)}{P^{(s+|d-s^*|)}_{2N+s+|d-s^*|}(0)}\right\}_{N=1}^\infty\in\begin{cases}\mathbb{P}^*_{|d-s^*|}(\beta_{\lfloor s/2\rfloor},\gamma_{\lfloor s/2\rfloor}-\lfloor s/2\rfloor,\delta_{\lfloor s/2\rfloor}), & d\ge s^*,\\ \mathbb{P}^*_{|d-s^*|}(\beta_{\lfloor s/2\rfloor+1},\gamma_{\lfloor s/2\rfloor+1}-\lfloor s/2\rfloor-1,\delta_{\lfloor s/2\rfloor+1}), & d<s^*,\end{cases}$$

where $s^*:=s-2\lfloor s/2\rfloor$.

Proof. Since statement (A) is a special case of (B) for $s = 0$, we prove the implication (A)⇒(B). Assume that property (A) is valid. We first consider the case $d \ge s^*$. Then using expansion (3.1.7), we obtain

$$\begin{aligned}
&\beta^{d-s}_{N+\lfloor s/2\rfloor} P^{(s)}_{2N+s+d-s^*}(z/\beta_{N+\lfloor s/2\rfloor}) \\
&\qquad = (\beta^{d}_{N+\lfloor s/2\rfloor} P_{2N+2\lfloor s/2\rfloor+d}(z/\beta_{N+\lfloor s/2\rfloor}))^{(s)} \\
&\qquad = \Big(\sum_{m=0}^{N+\lfloor s/2\rfloor} \frac{(-1)^m}{(2m+d)!}\mu_{2m+d,N+\lfloor s/2\rfloor} z^{2m+d}\Big)^{(s)} \\
&\qquad = (-1)^{\lfloor s/2\rfloor} \sum_{m=0}^{N} \frac{(-1)^m}{(2m+d-s^*)!}\mu_{2m+2\lfloor s/2\rfloor+d,N+\lfloor s/2\rfloor} z^{2m+d-s^*}. \qquad (3.1.8)
\end{aligned}$$

In addition, differentiating (3.1.8) $d - s^*$ times, we obtain

$$\beta^{s^*-s}_{N+\lfloor s/2\rfloor} P^{(s+d-s^*)}_{2N+s+d-s^*}(0) = (-1)^{\lfloor s/2\rfloor}\mu_{2\lfloor s/2\rfloor+d,N+\lfloor s/2\rfloor}, \qquad (3.1.9)$$

since $(d - s^*)! = 1$. Then (3.1.8) and (3.1.9) imply the identity

$$\begin{aligned}
&\frac{\beta^{d-s^*}_{N+\lfloor s/2\rfloor} P^{(s)}_{2N+s+d-s^*}(z/\beta_{N+\lfloor s/2\rfloor})}{P^{(s+d-s^*)}_{2N+s+d-s^*}(0)} \\
&\qquad\qquad = \sum_{m=0}^{N} \frac{(-1)^m}{(2m+d-s^*)!}\frac{\mu_{2m+2\lfloor s/2\rfloor+d,N+\lfloor s/2\rfloor}}{\mu_{2\lfloor s/2\rfloor+d,N+\lfloor s/2\rfloor}} z^{2m+d-s^*}. \qquad (3.1.10)
\end{aligned}$$

Setting

$$\mu^*_{2m+d-s^*,N} := \mu_{2m+2\lfloor s/2\rfloor+d,N+\lfloor s/2\rfloor}/\mu_{2\lfloor s/2\rfloor+d,N+\lfloor s/2\rfloor}, \qquad 0 \le m \le N,$$

we see that $\mu^*_{2m+d-s^*,N} > 0$, $0 \le m \le N$, and $\mu^*_{d-s^*,N} = 1$. Moreover, it follows from (D4) and the relation $\mu_{2\lfloor s/2\rfloor+d,N+\lfloor s/2\rfloor} = 1 + O(\delta_{N+\lfloor s/2\rfloor})$ as $N \to \infty$ that

$$\begin{aligned}
C^*_0 &:= \sup_N \sup_{0\le m\le N} \mu^*_{2m+d-s^*,N} \\
&\le C \sup_N \sup_{0\le m\le N+\lfloor s/2\rfloor} \mu_{2m+d,N+\lfloor s/2\rfloor} \le CC_0 < \infty.
\end{aligned}$$

Finally, using property (D3), we obtain, as $N \to \infty$,

$$\begin{aligned}
&\sup_{0\le m\le \gamma_{N+\lfloor s/2\rfloor}-\lfloor s/2\rfloor} |\mu^*_{2m+d-s^*,N} - 1| \\
&\quad\le \sup_{0\le m\le \gamma_{N+\lfloor s/2\rfloor}-\lfloor s/2\rfloor} (|\mu_{2m+2\lfloor s/2\rfloor+d,N+\lfloor s/2\rfloor} - 1| + |\mu_{2\lfloor s/2\rfloor+d,N+\lfloor s/2\rfloor} - 1|) \\
&\quad\quad \div \mu_{2\lfloor s/2\rfloor+d,N+\lfloor s/2\rfloor} \\
&\quad\le 2C_2\delta_{N+\lfloor s/2\rfloor}/(1 + O(\delta_{N+\lfloor s/2\rfloor})) \le C\delta_{N+\lfloor s/2\rfloor}.
\end{aligned}$$

Thus the coefficients $\mu^*_{2m+d-s^*,N}$, $0 \le m \le N$, satisfy properties (D1)–(D4) for the class $\mathbb{P}^*_{|d-s^*|}(\beta_{\lfloor s/2\rfloor}, \gamma_{\lfloor s/2\rfloor} - \lfloor s/2\rfloor, \delta_{\lfloor s/2\rfloor})$. Therefore, statement (B) holds for $d \ge s^*$.

The case $d < s^*$, that is, $d = 0$ and $s^* = 1$, can be proved similarly. Indeed, by (3.1.7),

$$\begin{aligned}\beta_{N+\lfloor s/2\rfloor}^{-s} P_{2N+s+1}^{(s)}(z/\beta_{N+\lfloor s/2\rfloor}) &= (P_{2N+2\lfloor s/2\rfloor+2}(z/\beta_{N+\lfloor s/2\rfloor}))^{(s)} \\ &= \Bigg(\sum_{m=0}^{N+\lfloor s/2\rfloor+1} \frac{(-1)^m}{(2m)!}\mu_{2m,N+\lfloor s/2\rfloor+1} z^{2m}\Bigg)^{(s)} \\ &= (-1)^{\lfloor s/2\rfloor+1}\sum_{m=0}^{N} \frac{(-1)^m}{(2m+1)!}\mu_{2m+2\lfloor s/2\rfloor+2,N+\lfloor s/2\rfloor+1} z^{2m+1}. \qquad (3.1.11)\end{aligned}$$

Hence

$$\beta_{N+\lfloor s/2\rfloor}^{-s-1} P_{2N+s+1}^{(s+1)}(0) = (-1)^{\lfloor s/2\rfloor+1}\mu_{2\lfloor s/2\rfloor+2,N+\lfloor s/2\rfloor+1}. \qquad (3.1.12)$$

Next, it follows from (3.1.11) and (3.1.12) that

$$\frac{\beta_{N+\lfloor s/2\rfloor} P_{2N+s+1}^{(s)}(z/\beta_{N+\lfloor s/2\rfloor})}{P_{2N+s+1}^{(s+1)}(0)} = \sum_{m=0}^{N} \frac{(-1)^m}{(2m+1)!}\,\frac{\mu_{2m+2\lfloor s/2\rfloor+2,N+\lfloor s/2\rfloor+1}}{\mu_{2\lfloor s/2\rfloor+2,N+\lfloor s/2\rfloor+1}} z^{2m+1}.$$

The proof of the fact that the coefficients

$$\mu_{2m,N}^* := \mu_{2m+2\lfloor s/2\rfloor+2,N+\lfloor s/2\rfloor+1}/\mu_{2\lfloor s/2\rfloor+2,N+\lfloor s/2\rfloor+1}, \qquad 0 \le m \le N,$$

satisfy properties (D1)–(D4) for the class $\mathbb{P}_1^*(\beta_{\lfloor s/2\rfloor+1}, \gamma_{\lfloor s/2\rfloor+1} - \lfloor s/2\rfloor - 1, \delta_{\lfloor s/2\rfloor+1})$ is similar to that in the case $d \ge s^*$. Therefore, statement (B) holds for $d < s^*$ as well. ∎

Proposition 3.1.6. *If there exist constants $h \in (0,1)$ and $C_3 > 0$ such that*

$$h^{\gamma_N} \le C_3\delta_N, \qquad N = 1, 2, \dots, \qquad (3.1.13)$$

then

$$\mathbb{P}_d^*(\beta, b\gamma, \delta) \subseteq \mathbb{P}_d(\beta, \gamma, \delta), \qquad (3.1.14)$$

where $b = b(h) \in (e/2, \infty)$ is the only solution to the equation $(e/(2b))^{2b} = h$.

Proof. We first note that (D2) implies (C2). Next, let $\{P_{2N+d}\}_{N=1}^{\infty} \in \mathbb{P}_d^*(\beta, b\gamma, \delta)$. Then by (3.1.7),

$$\begin{aligned}&\beta_N^d P_{2N+d}(z/\beta_N) - \cos(z - d\pi/2) \\ &\quad= \sum_{m=0}^{\lfloor b\gamma_N\rfloor} \frac{(-1)^m}{(2m+d)!}(\mu_{2m+d,N} - 1)z^{2m+d} + \sum_{m=\lfloor b\gamma_N\rfloor+1}^{N} \frac{(-1)^m}{(2m+d)!}\mu_{2m+d,N} z^{2m+d} \\ &\qquad+ \sum_{m=\lfloor b\gamma_N\rfloor+1}^{\infty} \frac{(-1)^{m+1}}{(2m+d)!} z^{2m+d} = I_1(z) + I_2(z) + I_3(z). \qquad (3.1.15)\end{aligned}$$

Further, taking account of the elementary estimate ($y > 0$)

$$H_d(y) := \sum_{m=1}^{\infty} \frac{y^{2m+d}}{(2m+d)!} \le \min\{y^2, 1\}\begin{cases}\cosh y, & d = 0,\\ \sinh y, & d = 1,\end{cases} \qquad (3.1.16)$$

we deduce from properties (D2) and (D3) that

$$|I_1(z)| \le \sup_{1\le m\le b\gamma_N} |\mu_{2m+d,N} - 1| H_d(|z|) \le C_2\delta_N H_d(|z|). \qquad (3.1.17)$$

In addition, it follows from property (D4) that

$$|I_2(z)| + |I_3(z)| \le (1 + C_0) \sum_{m=\lfloor b\gamma_N\rfloor+1}^{\infty} \frac{|z|^{2m+d}}{(2m+d)!}$$
$$\le (1 + C_0)(2 + d)! \frac{|z|^{2\lfloor b\gamma_N\rfloor}}{(2\lfloor b\gamma_N\rfloor + 2)!} \sum_{m=1}^{\infty} \frac{|z|^{2m+d}}{(2m+d)!}. \tag{3.1.18}$$

If $1 \le |z| \le \gamma_N$, then from (3.1.16), (3.1.18) and (3.1.13) we have

$$|I_2(z)| + |I_3(z)| \le (1 + C_0)(2 + d)! \left(\frac{e|z|}{2\lfloor b\gamma_N\rfloor + 2}\right)^{2\lfloor b\gamma_N\rfloor+2} H_d(|z|)$$
$$\le (1 + C_0)(2 + d)! \left(\frac{e|z|}{2b\gamma_N}\right)^{2b\gamma_N} H_d(|z|) \le (1 + C_0)(2 + d)! \left(\frac{e}{2b}\right)^{2b\gamma_N} H_d(|z|)$$
$$\le (1 + C_0)(2 + d)! C_3 \delta_N H_d(|z|). \tag{3.1.19}$$

If $0 \le |z| < 1$, then from (3.1.16), (3.1.18) and (3.1.13) we obtain

$$|I_2(z)| + |I_3(z)| \le (1 + C_0)(2 + d)! C_4 (e/(2b\gamma_N))^{2b\gamma_N} H_d(|z|)$$
$$\le (1 + C_0)(2 + d)! C_5 (e/(2b))^{2b\gamma_N} H_d(|z|)$$
$$\le (1 + C_0)(2 + d)! C_5 C_3 \delta_N H_d(|z|). \tag{3.1.20}$$

Therefore, relations (3.1.15), (3.1.16), (3.1.17), (3.1.19), and (3.1.20) imply (3.1.3) with $|z| \le \gamma_N$ and $C_1 = C_2 + (1 + C_0)(2 + d)! C_3 (1 + C_5)$. ■

Remark 3.1.7. We note that if condition (3.1.13) holds and

$$\{P_{2N+d}\}_{N=1}^{\infty} \in \mathbb{P}_d^*(\beta, b\gamma, \delta),$$

then for any integer $s \ge 0$, large enough N, and for

$$|z| \le \begin{cases} \gamma_{N+\lfloor s/2\rfloor} - \lfloor s/2\rfloor/b, & d \ge s^*, \\ \gamma_{N+\lfloor s/2\rfloor} - (\lfloor s/2\rfloor + 1)/b, & d < s^*, \end{cases}$$

where $s^* = s - 2\lfloor s/2\rfloor$ and b is defined in Proposition 3.1.6, the following strong asymptotic is valid:

$$\left| \frac{\beta_{N+\lfloor s/2\rfloor}^{|d-s^*|} P_{2N+s+|d-s^*|}^{(s)}(z/\beta_{N+\lfloor s/2\rfloor})}{P_{2N+s+|d-s^*|}^{(s+|d-s^*|)}(0)} - \frac{\cos^{(s)}(z - d\pi/2)}{\cos^{(s+|d-s^*|)}(d\pi/2)} \right|$$
$$\le C_1^* \delta_{N+\lfloor s/2\rfloor} \min\{|z|^2, 1\} \begin{cases} \cosh|z|, & |d - s^*| = 0, \\ \sinh|z|, & |d - s^*| = 1. \end{cases} \tag{3.1.21}$$

Inequality (3.1.21) follows from Propositions 3.1.5, 3.1.6, and the identity

$$\cos^{(s)}(z - d\pi/2)/\cos^{(s+|d-s^*|)}(-d\pi/2) = \cos(z - |d - s^*|\pi/2).$$

3.1.3. Class $\mathbb{P}_d^{**}(\beta, \gamma, \delta, W, k)$. Finally, we define a class of special sequences of polynomials associated with some weights w and an index k that takes the values of 1 or 2. This class will be used for asymptotic formulae for the zeta functions in $L_{p,w}$-spaces.

Definition 3.1.8. Let $k = 1$ or $k = 2$ and let $W := \{w(y, N)\}_{N=1}^{\infty}$ be a sequence of even integrable weights on $(-a, a)$, where $a = 1$ or $a = \infty$. Let $\Pi_d = \{P_{2N+d}\}_{N=1}^{\infty}$ be a sequence of even (if $d = 0$) or odd (if $d = 1$) polynomials $P_{2N+d} \in \mathcal{P}_{2N+d}$, $N = 1, 2, \dots$, satisfying properties (C1)–(C3) (that is, $\Pi_d \in \mathbb{P}_d(\beta, \gamma, \delta)$). In addition, we assume that the weights and the polynomials have the following properties:

(C4) $w(0, N) = 1$, $N = 1, 2, \dots$, and the sequence $\{w(y/\beta_N, N)\}_{N=1}^{\infty}$ converges uniformly to 1 on any finite interval $[-B, B] \subset (-\infty, \infty)$ as $N \to \infty$.

(C5) There exists an even measurable function $\psi(y) \geq 0$, $y \in (-a, a)$, independent of N, such that the following properties hold:

(C5.1) ψ is continuous at zero and $\psi(0) \geq 1$.

(C5.2) There exists a nondecreasing sequence of positive numbers $a_N \geq 1$ such that

$$|P_{2N+d}(y)|w(y, N) \leq \beta_N^{-d}\psi(y/a_N), \quad N = 1, 2, \dots, \quad y \in (-a, a).$$

For $a = 1$ we assume $a_n = 1$, $N = 1, 2 \dots.$

(C5.3) There exist $p_0 = p_0(k)$ and $p_1 = p_1(k)$, $0 \leq p_0 < p_1 \leq \infty$, such that for all $p \in (p_0, p_1)$ and any $c \in (0, a)$, $\int_c^a y^{-(2+kd)p}\psi^{kp}(y/a_N)dy < \infty$.

(C5.4) For $N = 1, 2 \dots$ and each $p \in (p_0, p_1)$, where $p_0(k)$ and $p_1(k)$ satisfy (C5.3),

$$\int_0^a |y^{-dk}P_{2N+d}^k(y)w^k(y, N)|^p dy < \infty.$$

Then we write $\Pi_d \in \mathbb{P}_d^{**} = \mathbb{P}_d^{**}(\beta, \gamma, \delta, W, k)$.

REMARK 3.1.9. For example the function $\psi(y) = 1$ satisfies properties (C5.1) and (C5.3) with $p_0 = 0$, $p_1 = \infty$ if $a = 1$ and $p_0 = 1/(2 + kd)$, $p_1 = \infty$ if $a = \infty$. In addition, (C5.2)$\Rightarrow$(C5.4) for this function if $a = 1$.

REMARK 3.1.10. If a weight $w(y) = w(y, N)$ does not depend on N and $w(y)$ is continuous at zero with $w(0) = 1$, then property (C4) is trivially satisfied.

REMARK 3.1.11. In the next section we consider examples of nonempty classes $\mathbb{P}_d$, $\mathbb{P}_d^*$, and $\mathbb{P}_d^{**}$ with certain β, γ, δ and W.

REMARK 3.1.12. We remark that the polynomial sequence

$$\Pi_d = \{P_{2N+d}\}_{N=1}^{\infty} = \left\{ z^d \left(1 + \sum_{m=1}^{N} \frac{(-1)^m(1 + \delta_N)}{(2m + d)!}(\beta_N z)^{2m} \right) \right\}_{N=1}^{\infty}$$

satisfies properties (D1)–(D4) of Definition 3.1.3 for any β, γ, and δ. However, $\Pi_d \notin \mathbb{P}_d^*(\beta, \gamma, \delta)$, since for $N \geq 3$ not all zeros of P_{2N+d} are real. Indeed, it suffices to show that the polynomial

$$Q_N(y) := \frac{1}{1 + \delta_N} + \sum_{m=1}^{N} \frac{(-1)^m}{(2m + d)!} y^m$$

has less than N real zeros. Since the quadratic polynomial

$$Q_N^{(N-2)}(y) = \frac{(-1)^{N-2}(N - 2)!}{(2N + d - 4)!} \left(1 - \frac{y}{2(2N + 2d - 3)} + \frac{y^2}{8(2N + 2d - 1)(2N + 2d - 3)} \right)$$

has no real zeros for $N \geq 3$, we conclude by Rolle's Theorem that not all zeros of Q_N are real.

3.2. Examples of sequences from the classes $\mathbb{P}_d$, $\mathbb{P}_d^*$, and $\mathbb{P}_d^{**}$. Examples of polynomial sequences presented here include normalized Gegenbauer, Chebyshev, Hermite, Williams–Apostol, and Lommel polynomials and, in addition, normalized polynomials with equidistant zeros.

We recall that $d = 0$ or $d = 1$ and $k = 1$ or $k = 2$. Here, we find some classical representatives of the classes $\mathbb{P}_d$, $\mathbb{P}_d^*$, and $\mathbb{P}_d^{**}$. In addition, if positive zeros $z_{p,N,d}$ of P_{2N+d} can be found explicitly, we compute $P'_{2N+d}(z_{p,N,d})$ and $P''_{2N+d}(z_{p,N,d})/P'_{2N+d}(z_{p,N,d})$, $1 \le p \le N$, $N = 1, 2, \ldots$. These computations are used in Section 4.4 for asymptotic summation formulae.

3.2.1. Normalized Gegenbauer polynomials on $[-1, 1]$. For $\lambda \ge 0$ and $N = 1, 2, \ldots$, we define

$$\beta_N := 2N + \lambda + d, \tag{3.2.1}$$

$$\begin{aligned} P_{2N+d}(y) &:= C_{2N+d}^{\lambda}(y)/(C_{2N+d}^{\lambda})^{(d)}(0) \\ &= \frac{N!}{2^d \Gamma(N+\lambda+d)} \sum_{m=0}^{N} \frac{(-1)^m}{(2m+d)!} \frac{\Gamma(N+m+\lambda+d) 2^{2m+d}}{(N-m)!} y^{2m+d} \end{aligned} \tag{3.2.2}$$

(see [17, Sect. 10.9]). Then by (3.1.7), (3.2.1), and (3.2.2),

$$\mu_{2m+d,N} = \frac{N! \Gamma(N+m+\lambda+d)}{\Gamma(N+\lambda+d)(N-m)!(N+\lambda/2+d/2)^{2m}}, \qquad 0 \le m \le N,\ N = 1, 2, \ldots. \tag{3.2.3}$$

In addition, for $N = 1, 2, \ldots$, we define $a = a_N := 1$ and

$$w(y, N) := (1-y^2)^{\lambda/2 - 1/4}, \quad \psi(y) := (1-y^2)^{-1/4}, \quad p_0(k) := 0, p_1(k) := 4/k. \tag{3.2.4}$$

Then the following property holds for the *normalized Gegenbauer polynomials* P_{2N+d} on $[-1, 1]$:

Proposition 3.2.1. *Let $\gamma = \{\gamma_N\}_{N=1}^{\infty}$ be an increasing sequence satisfying the conditions*

$$\gamma_N = o(N^{2/3}), \quad \gamma_N \ge C \log(N+1), \qquad N = 1, 2, \ldots, \tag{3.2.5}$$

and let $\delta = \{\delta_N\}_{N=1}^{\infty}$ be a decreasing sequence satisfying the conditions

$$\delta_N \ge \gamma_N^3/N^2, \quad N = 1, 2, \ldots, \qquad \lim_{N\to\infty} \delta_N = 0. \tag{3.2.6}$$

Then there exists a constant $b > e/2$ such that for the sequences $\beta = \{\beta_N\}_{N=1}^{\infty}$ and $W = \{w(y, N)\}_{N=1}^{\infty}$ defined by (3.2.1) *and* (3.2.4), *respectively, the sequence of polynomials $\{P_{2N+d}\}_{N=1}^{\infty}$ defined by* (3.2.2) *belongs to $\mathbb{P}_d(\beta, b^{-1}\gamma, \delta)$, $\mathbb{P}_d^*(\beta, \gamma, \delta)$, and $\mathbb{P}_d^{**}(\beta, b^{-1}\gamma, \delta, W, k)$.*

Proof. We first verify properties (D1)–(D4) in Definition 3.1.3. Property (D4) is satisfied by (3.2.3) and the relations

$$\begin{aligned} \mu_{2m+d,N} &= (N+\lambda/2+d/2)^{-2m} \prod_{\nu=0}^{m-1} (N-\nu)(N+\nu+\lambda+d) \\ &\le \left(\frac{N^2 + N(\lambda+d)}{N^2 + N(\lambda+d) + (\lambda/2+d/2)^2} \right)^m \le 1, \qquad 1 \le m \le N, \end{aligned}$$

while properties (D1) and (D2) are trivially satisfied. Further, we note that there exist constants $C_1 > 0$ and $C_2 > 0$, independent of m and N, such that for $0 \le m \le N/\sqrt{2C_1}$,

$$\begin{aligned}\mu_{2m+d,N} &\ge \left(\frac{N^2 - m^2 + (N-m)(\lambda+d)}{N^2 + N(\lambda+d) + (\lambda/2+d/2)^2}\right)^m \\ &= \left(1 - \frac{m^2 + m(\lambda+d) + (\lambda/2+d/2)^2}{N^2 + N(\lambda+d) + (\lambda/2+d/2)^2}\right)^m \\ &\ge (1 - C_1 m^2/N^2)^m \ge e^{-C_2 m^3/N^2}.\end{aligned}$$

Then using (3.2.6) for $0 \le m \le \gamma_N = o(N^{2/3})$, we obtain

$$\mu_{2m+d,N} \ge e^{-C_2\gamma_N^3/N^2} \ge e^{-C_2\delta_N} = 1 - O(\delta_N)$$

as $N \to \infty$. Hence $\sup_{0\le m\le\gamma_N}(1-\mu_{2m+d,N}) = O(\delta_N)$ as $N \to \infty$, and this yields (D3). Thus $\{P_{2N+d}\}_{N=1}^\infty \in \mathbb{P}_d^*(\beta,\gamma,\delta)$ since all zeros of the Gegenbauer polynomials are real and distinct.

Next, the second condition in (3.2.5) along with the first condition in (3.2.6) yields the estimate $\delta_N \ge C(\log^3(N+1))/N^2$, which implies the inequalities $\log 1/\delta_N \le C_1 \log(N+1) \le C_2\gamma_N$. Thus setting $h := e^{-C_2}$, we obtain $\delta_N \ge h^{\gamma_N}$, $N = 1, 2, \ldots$. Therefore by Proposition 3.1.6, there exists $b \in (e/2,\infty)$ such that $\{P_{2N+d}\}_{N=1}^\infty \in \mathbb{P}_d(\beta, b^{-1}\gamma, \delta)$.

Further, the weight $w(y) = w(y,N)$ and the function $\psi(y)$ from (3.2.4) satisfy properties (C4) (see Remark 3.1.10) and (C5.1). In addition, $\int_c^1 y^{-(2+kd)p}(1-y^2)^{-kp/4}\,dy < \infty$ for any $c \in (0,1)$, where $p \in (0,4/k)$, $k = 1,2$. Thus properties (C5.3) and (C5.4) are satisfied as well. Finally, we prove the inequality (with $\lambda \ge 0$)

$$|P_{2N+d}(y)|(1-y^2)^{\lambda/2-1/4} \le CN^{-d}(1-y^2)^{-1/4}, \quad y \in (-1,1),\ N = 1,2,\ldots, \tag{3.2.7}$$

which establishes property (C5.2). We first note that for $\lambda = 0$, P_{2N+d} coincides with the Chebyshev polynomial of the first kind $(2N+d)^{-d}T_{2N+d}$. Thus (3.2.7) is trivial for $\lambda = 0$. To prove (3.2.7) for $\lambda > 0$, we use the following Bernstein inequality [48, Th. 7.32.3]:

$$\max_{y\in[-1,1]}(1-y^2)^{\lambda/2}|C_{2N+d}^\lambda(y)| \le N^{\lambda-1}/\Gamma(\lambda), \quad \lambda > 0. \tag{3.2.8}$$

Next, we note that for $\lambda > 0$,

$$(C_{2N+d}^\lambda)^{(d)}(0) = \frac{(-1)^N 2^d \Gamma(N+\lambda+d)}{\Gamma(\lambda)N!} = \frac{(-1)^N N^{\lambda+d-1}}{\Gamma(\lambda)}(1+o(1)) \tag{3.2.9}$$

as $N \to \infty$. Then (3.2.7) for $\lambda > 0$ follows from (3.2.8) and (3.2.9). Thus property (C5.2) is satisfied. Therefore $\{P_{2N+d}\}_{N=1}^\infty \in \mathbb{P}_d^{**}(\beta, b^{-1}\gamma, \delta, W, k)$. ∎

3.2.2. Normalized Chebyshev polynomials on $[-1,1]$**.** The Chebyshev polynomials of the first and second kind, normalized by (C2), are special cases of the normalized Gegenbauer polynomials for $\lambda = 0$ and $\lambda = 1$, respectively, so Proposition 3.2.1 is valid for them. Here, we include some additional formulae for Chebyshev polynomials, their derivatives, and zeros.

Normalized Chebyshev polynomials of the first kind on $[-1,1]$. For $N = 1, 2, \ldots$, we define $a_N = a = 1$ and

$$\beta_N := 2N + d, \tag{3.2.10}$$

$$\begin{aligned} P_{2N+d}(y) &:= C^0_{2N+d}(y)/(C^0_{2N+d})^{(d)}(0) = (-1)^N(2N+d)^{-d}T_{2N+d}(y) \\ &= (-1)^N(2N+d)^{-d}\cos((2N+d)\arccos y), \end{aligned} \tag{3.2.11}$$

$$\begin{aligned} w(y,N) &:= (1-y^2)^{\lambda/2-1/4}, \quad \psi(y) := (1-y^2)^{-1/4}, \\ p_0(k) &:= 0, \quad p_1(k) := 4/k. \end{aligned} \tag{3.2.12}$$

Positive zeros ($p = 1,\dots,N$, $N = 1,2,\dots$):

$$z_{p,N,d} := \cos\frac{(2p-1)\pi}{4N+2d}. \tag{3.2.13}$$

Derivatives ($p = 1,\dots,N$, $N = 1,2,\dots$):

$$P'_{2N+d}(z_{p,N,d}) = \frac{(-1)^{N+p+1}(2N+d)^{1-d}}{\sin\frac{(2p-1)\pi}{4N+2d}}, \tag{3.2.14}$$

$$P''_{2N+d}(z_{p,N,d}) = \frac{(-1)^{N+p+1}(2N+d)^{1-d}\cos\frac{(2p-1)\pi}{4N+2d}}{\sin^3\frac{(2p-1)\pi}{4N+2d}}, \tag{3.2.15}$$

$$\frac{P''_{2N+d}(z_{p,N,d})}{P'_{2N+d}(z_{p,N,d})} = \frac{\cos\frac{(2p-1)\pi}{4N+2d}}{\sin^2\frac{(2p-1)\pi}{4N+2d}}. \tag{3.2.16}$$

REMARK 3.2.2. For $d = 0$ and $P_{2N}(y) = T_{2N}(y)$ the weight $w(y,N) = 1$ on $[-1,1]$ satisfies properties (C4) and (C5) of Definition 3.1.8 with $\psi(y) = 1$ and $p_0(k) = 0$, $p_1(k) = \infty$. Therefore $\{T_{2N}\}_{N=1}^\infty \in \mathbb{P}_d^{**}(\beta, b^{-1}\gamma, \delta, W, k)$ with this weight, $\beta_N = 2N$, and $\gamma_N, \delta_N, N = 1,2,\dots$, satisfying conditions (3.2.5) and (3.2.6).

Normalized Chebyshev polynomials of the second kind on $[-1,1]$. For $N = 1,2,\dots$, we define $a_N = a = 1$ and

$$\beta_N := 2N+d+1, \tag{3.2.17}$$

$$\begin{aligned} P_{2N+d}(y) &:= C^1_{2N+d}(y)/(C^1_{2N+d})^{(d)}(0) = (-1)^N(2N+d+1)^{-d}U_{2N+d}(y) \\ &= (-1)^N(2N+d+1)^{-d}\sin((2N+d)\arccos y)/\sqrt{1-y^2}, \end{aligned} \tag{3.2.18}$$

$$\begin{aligned} w(y,N) &:= (1-y^2)^{1/4}, \quad \psi(y) := (1-y^2)^{-1/4}, \\ p_0(k) &:= 0, \quad p_1(1) := 4, \quad p_1(2) := 2. \end{aligned} \tag{3.2.19}$$

Positive zeros ($p = 1,\dots,N$, $N = 1,2,\dots$):

$$z_{p,N,d} := \cos\frac{p\pi}{2N+d+1}. \tag{3.2.20}$$

Derivatives ($p = 1,\dots,N$, $N = 1,2,\dots$):

$$P'_{2N+d}(z_{p,N,d}) = \frac{(-1)^{N+p+1}(2N+d+1)^{1-d}}{\sin^2\frac{p\pi}{2N+d+1}}, \tag{3.2.21}$$

$$P''_{2N+d}(z_{p,N,d}) = \frac{3(-1)^{N+p+1}(2N+d+1)^{1-d}\cos\frac{p\pi}{2N+d+1}}{\sin^4\frac{p\pi}{2N+d+1}}, \tag{3.2.22}$$

$$\frac{P''_{2N+d}(z_{p,N,d})}{P'_{2N+d}(z_{p,N,d})} = \frac{3\cos\frac{p\pi}{2N+d+1}}{\sin^2\frac{p\pi}{2N+d+1}}. \tag{3.2.23}$$

Note that formulae (3.2.14)–(3.2.16) and (3.2.21)–(3.2.23) can be easily verified by straightforward calculation.

3.2.3. Normalized polynomials with equidistant zeros on $[-1,1]$**.** For $N=1,2,\dots,$ we define $a_N = a = 1$ and

$$\beta_N := (2N+d-1)\pi/2, \tag{3.2.24}$$

$$P_{2N+d}(y) := y^d \prod_{p=1}^{N}\left(1-\left(\frac{2N+d-1}{2p+d-1}\right)^2 y^2\right), \tag{3.2.25}$$

$$\begin{aligned} w(y,N) &:= \left(\sqrt{(1-y^2)}((1+y)^{1+y}(1-y)^{1-y})^{(2N+d-1)/2}\right)^{-1};\\ \psi(y) &:= 1; \quad p_0(k) := 0, \quad p_1(k) := \infty. \end{aligned} \tag{3.2.26}$$

Positive zeros ($p=1,\dots,N,\ N=1,2,\dots$):

$$z_{p,N,d} := \frac{2p+d-1}{2N+d-1}. \tag{3.2.27}$$

Derivatives ($p=1,\dots,N,\ N=1,2,\dots$):

$$P'_{2N+d}(z_{p,N,d}) = \left(\frac{2p+d-1}{2N+d-1}\right)^d \frac{(-1)^p 2^{2N-1}(2N+d-1)!(2N+d-1)}{(2N+d-1)!!^2 p^d \binom{2N+d-1}{N-p}}, \tag{3.2.28}$$

$$\frac{P''_{2N+d}(z_{p,N,d})}{P'_{2N+d}(z_{p,N,d})} = (2N+d-1)\sum_{n=N-p+1}^{N+p+d-1}\frac{1}{n}, \tag{3.2.29}$$

where

$$(2N+d-1)!! := \begin{Bmatrix} \prod_{l=1}^{N}(2l-1), & d=0,\\ \prod_{l=1}^{N}(2l), & d=1, \end{Bmatrix} = \begin{cases} \frac{(2N-1)!}{2^{N-1}(N-1)!}, & d=0,\\ 2^N N!, & d=1. \end{cases} \tag{3.2.30}$$

Note that the following asymptotic relation for the constant in (3.2.28) follows from Stirling's formula:

$$\begin{aligned} \frac{2^{2N-1}(2N+d-1)!}{(2N+d-1)!!^2} &= \begin{cases} \frac{2^{4N-3}(N-1)!^2}{(2N-1)!}, & d=0,\\ \frac{(2N)!}{2N!^2}, & d=1, \end{cases}\\ &= N^{-1/2}2^{d-2}\pi^{1/2-d}2^{2N}(1+o(1)), \quad N\to\infty. \end{aligned} \tag{3.2.31}$$

We first prove the formulae for the derivatives.

Lemma 3.2.3. *For* $p=1,\dots,N,\ N=1,2,\dots,$ *relations* (3.2.28) *and* (3.2.29) *hold.*

Proof. By straightforward calculation,

$$\begin{aligned} \frac{d}{dy}\left(\frac{P_{2N+d}(y)}{y^d}\right)\Bigg|_{y=z_{p,N,d}} &= -\frac{2(2N+d-1)}{2p+d-1}\prod_{i=1,\,i\neq p}^{N}\left(1-\left(\frac{2p+d-1}{2i+d-1}\right)^2\right)\\ &= -\frac{2^{2N-1}(2N+d-1)(2p+d-1)}{(2N+d-1)!!^2}\prod_{i=1,\,i\neq p}^{N}(i-p)(i+p+d-1) \end{aligned}$$

$$= \frac{(-1)^p 2^{2N-1}(2N+d-1)(p-1)!(N-p)!(N+p+d-1)!}{(2N+d-1)!!^2(p+d-1)!}$$
$$= \frac{(-1)^p 2^{2N-1}(2N+d-1)(2N+d-1)!}{(2N+d-1)!!^2 p^d \binom{2N+d-1}{N-p}}. \tag{3.2.32}$$

Then (3.2.28) follows from (3.2.32) and (3.2.27). Next we obtain

$$\frac{d^2}{dy^2}\left(\frac{P_{2N+d}(y)}{y^d}\right)\bigg|_{y=z_{p,N,d}}$$
$$= \frac{d}{dy}\left(-2y\sum_{j=1}^{N}\left(\frac{2N+d-1)}{2j+d-1}\right)^2\prod_{i=1,\, i\neq j}^{N}\left(1-\left(\frac{2N+d-1}{2i+d-1}\right)^2 y^2\right)\right)\bigg|_{y=z_{p,N,d}}$$
$$= -2\left(\frac{2N+d-1}{2p+d-1}\right)^2\prod_{i=1,\, i\neq p}^{N}\left(1-\left(\frac{2p+d-1}{2i+d-1}\right)^2\right)$$
$$+\,8\sum_{j=1,\, j\neq p}^{N}\left(\frac{2N+d-1}{2j+d-1}\right)^2\prod_{i=1,\, i\neq j,\, i\neq p}^{N}\left(1-\left(\frac{2p+d-1}{2i+d-1}\right)^2\right)$$
$$= \prod_{i=1,\, i\neq p}^{N}\left(1-\left(\frac{2p+d-1}{2i+d-1}\right)^2\right)$$
$$\times\left(-2\left(\frac{2N+d-1}{2p+d-1}\right)^2+8(2N+d-1)^2\sum_{j=1,\, j\neq p}^{N}\frac{1}{(2j+d-1)^2-(2p+d-1)^2}\right)$$
$$= \frac{2(2N+d-1)^2}{2p+d-1}\prod_{i=1,\, i\neq p}^{N}\left(1-\left(\frac{2p+d-1}{2i+d-1}\right)^2\right)$$
$$\times\left(\sum_{j=1,\, j\neq p}^{N}\frac{1}{j-p}-\sum_{j=1}^{N}\frac{1}{j+p+d-1}\right)$$
$$= (2N+d-1)\frac{d}{dy}\left(\frac{P_{2N+d}(y)}{y^d}\right)\bigg|_{y=z_{p,N,d}}\left(\sum_{n=N-p+1}^{N+p+d-1}\frac{1}{n}-\frac{d}{p}\right).$$

Hence

$$(2N+d-1)\left(\sum_{n=N-p+1}^{N+p+d-1}\frac{1}{n}-\frac{d}{p}\right) = \left(\frac{d^2}{dy^2}\left(\frac{P_{2N+d}(y)}{y^d}\right)\Big/\frac{d}{dy}\left(\frac{P_{2N+d}(y)}{y^d}\right)\right)\bigg|_{y=z_{p,N,d}}$$
$$= \frac{P''_{2N+d}(z_{p,N,d})}{P'_{2N+d}(z_{p,N,d})}-\frac{2d}{z_{p,N,d}}.$$

Therefore, (3.2.29) is established. ∎

Next, we prove the following property of the normalized polynomials P_{2N+d} with equidistant zeros on $[-1,1]$.

Proposition 3.2.4. *Let* $\gamma = \{\gamma_N\}_{N=1}^{\infty}$ *be an increasing sequence of positive numbers satisfying the conditions*

$$\gamma_N = o(N^{1/2}), \qquad \lim_{N\to\infty} \gamma_N = \infty, \tag{3.2.33}$$

and let $\delta = \{\delta_N\}_{N=1}^{\infty}$ *be a decreasing sequence satisfying the conditions*

$$\delta_N \geq \gamma_N^2/N, \quad N = 1, 2, \ldots, \qquad \lim_{N\to\infty} \delta_N = 0. \tag{3.2.34}$$

Then for the sequences $\beta = \{\beta_N\}_{N=1}^{\infty}$ *and* $W = \{w(y,N)\}_{N=1}^{\infty}$ *defined by* (3.2.24) *and* (3.2.26), *respectively, the sequence of polynomials* $\{P_{2N+d}\}_{N=1}^{\infty}$ *defined by* (3.2.25) *belongs to* $\mathbb{P}_d(\beta,\gamma,\delta)$, *and* $\mathbb{P}_d^{**}(\beta,\gamma,\delta,W,k)$.

The proof is based on the following properties of P_{2N+d}:

Lemma 3.2.5. (a) *The following representations for* P_{2N+d} *hold* ($z \in \mathbb{C}$):

$$\begin{aligned} P_{2N+d}(z) &= ((2N+d-1)\pi/2)^{-d} \cos((2N+d-1)\pi z/2 - d\pi/2) \\ &\quad\times \frac{\Gamma(N(1+z) + (d-1)z/2 + 2^{d-1})\Gamma(N(1-z) - (d-1)z/2 + 2^{d-1})}{\Gamma^2(N+2^{d-1})}, \end{aligned} \tag{3.2.35}$$

$$\begin{aligned} P_{2N+d}(y)w(y,N) &= ((2N+d-1)\pi/2)^{-d} \cos((2N+d-1)\pi y/2 - d\pi/2) \\ &\quad\times \left(1 + O\left(\frac{1}{N(1-y^2)}\right)\right), \qquad y \in (-1,1). \end{aligned} \tag{3.2.36}$$

(b) *If* $\delta_N \searrow 0$ *and* $\gamma_N = \sqrt{N\delta_N} \nearrow \infty$, *then uniformly on the closed disk* $\bar{D}_{\gamma_N}(0)$,

$$\begin{aligned} &((2N+d-1)\pi/2)^d P_{2N+d}(z/((2N+d-1)\pi/2)) \\ &\qquad = \cos(z - d\pi/2)(1 + O(\delta_N \min\{|z|^2, 1\})). \end{aligned} \tag{3.2.37}$$

The constants O *in* (3.2.36) *and* (3.2.37) *are independent of* N, y *and* N, z, *respectively.*

Proof. Identity (3.2.35) is a combination of two known results in the theory of the gamma function [16, eqs. (1.2.8) and (1.2.9)], while the asymptotic relation (3.2.36) was proved by Stirling's formula in [20, Lemma 2(b)].

To prove (3.2.37), we first note that (3.2.35) implies the identity

$$\begin{aligned} &((2N+d-1)\pi/2)^d P_{2N+d}(z/((2N+d-1)\pi/2))/\cos(z - d\pi/2) \\ &\qquad = \Gamma^{-2}(N+2^{d-1})\Gamma(N+2^{d-1}+z/\pi)\Gamma(N+2^{d-1}-z/\pi). \end{aligned} \tag{3.2.38}$$

To find the asymptotic behavior of the right-hand side of (3.2.38) as $N \to \infty$, we need asymptotic properties of the gamma function. Stirling's formula [16, eq. (1.18.1)] shows that for $M > 0$ and $w \in \mathbb{C}$ with $M + w \to \infty$,

$$\begin{aligned} \log \Gamma(M+w) &= (M+w-1/2)\log(M+w) \\ &\qquad - M - w + (1/2)\log(2\pi) + 1/(12(M+w)) + O(|M+w|^{-2}), \end{aligned}$$

which yields the following relations as $M \to \infty$ and $|w| = o(M^{2/3})$:

$$\begin{aligned} \log \frac{\Gamma(M+w)}{\Gamma(M)} &= (M+w-1/2)\log(1+w/M) - w + w\log M - \frac{w}{12M(M+w)} + O(|M|^{-2}) \\ &= (M+w-1/2)\left(\frac{w}{M} - \frac{w^2}{2M^2} + O\left(\frac{|w|^3}{M^3}\right)\right) - w + w\log M + O\left(\frac{|w|}{M^2}\right) \\ &= \frac{w(w-1)}{2M} + w\log M + O\left(\frac{|w|^3}{M^3}\right). \end{aligned}$$

Hence if $|w| = o(\sqrt{M})$ as $M \to \infty$, then

$$\frac{\Gamma(M+w)}{\Gamma(M)} = M^w\left(1 + \frac{w(w-1)}{2M} + O\left(\frac{|w|^3}{M^3}\right)\right). \tag{3.2.39}$$

Note that for a fixed w this asymptotic is well known [16, eq. (1.18.4)]. Then it follows from (3.2.39) that for $|w| = o(\sqrt{M})$ as $M \to \infty$,

$$\Gamma^{-2}(M)\Gamma(M+w)\Gamma(M-w) = 1 + O(|w|^2/M). \tag{3.2.40}$$

Next, we set $M = N + 2^{d-1} \to \infty$ and $w = z/\pi$ and take account of the relations

$$|w| = O(\gamma_N) = o(\sqrt{M}), \qquad |w|^2/M \le C(\gamma_N^2/N)\min\{|z|^2, 1\} \le C\delta_N \min\{|z|^2, 1\}$$

that are valid for $z \in \bar{D}_{\gamma_N}(0)$, by (3.2.33) and (3.2.34). Then we obtain (3.2.37) from (3.2.38) and (3.2.40). ∎

Proof of Proposition 3.2.4. Properties (C1) and (C2) of Definition 3.1.1 are satisfied trivially. Next, property (C3) is satisfied because of Lemma 3.2.5(b). Therefore, $\{P_{2N+d}\}_{N=1}^\infty \in \mathbb{P}_d(\beta, \gamma, \delta)$. Since the function $\psi(y) = 1$ and numbers $p_0(k) = 0$, $p_1(k) = \infty$ satisfy properties (C5.1) and (C5.3) and, in addition, (C5.2) implies (C5.4) (see Remark 3.1.9), it suffices to verify properties (C4) and (C5.2) of Definition 3.1.8. We first note that $w(0, N) = 1$, and for β_N given by (3.2.24),

$$w(y/\beta_N, N) = (1 - y^2/\beta_N^2)^{-\beta_N/\pi}(1 + y/\beta_N)^{-y/\pi - 1/2}(1 - y/\beta_N)^{y/\pi - 1/2}.$$

Then using the elementary estimates

$$e^{-2t} \le 1 - t \le e^{-t}, \qquad e^{t/2} \le 1 + t \le e^t, \tag{3.2.41}$$

for $t \in [0, 1/2]$, we obtain

$$\begin{aligned}
&(1 - y^2/\beta_N^2)^{-\beta_N/\pi} = 1 + O(y^2/\beta_N), \qquad (1 + y/\beta_N)^{-y/\pi - 1/2} = 1 + O(y/\beta_N),\\
&(1 - y/\beta_N)^{y/\pi - 1/2} = 1 + O(y/\beta_N),
\end{aligned}$$

where the constants O are independent of y and N. Therefore, for any $[-B, B] \subset (-1, 1)$, $\sup_{y \in [-B,B]} |W(y/\beta_N, N) - 1| \le C/\beta_N$. Thus (C4) is satisfied. Next, from (3.2.36) we obtain

$$\begin{aligned}
&\sup_{y \in (-1,1)} |P_{2N+d}(y)| w(y, N)\\
&\quad \le ((2N + d - 1)\pi/2)^{-d}\left(1 + C \sup_{y \in (-1,1)} \frac{|\cos((2N + d - 1)\pi y/2 - d\pi/2)|}{N(1 - y^2)}\right)\\
&\quad \le C((2N + d - 1)\pi/2)^{-d},
\end{aligned} \tag{3.2.42}$$

that is, property (C5.2) is satisfied as well. Thus $\{P_{2N+d}\}_{N=1}^\infty \in \mathbb{P}_d^{**}(\beta, \gamma, \delta, W, k)$. ∎

3.2.4. Normalized Hermite polynomials on $(-\infty, \infty)$. For $N = 1, 2, \ldots$, we define

$$\beta_N := \sqrt{4N + 2d + 1}, \tag{3.2.43}$$

$$P_{2N+d}(y) := H_{2N+d}(y)/H_{2N+d}^{(d)}(0) = N! \sum_{m=0}^{N} \frac{(-1)^m}{(2m + d)!} \frac{2^{2m}}{(N - m)!} y^{2m+d} \tag{3.2.44}$$

(see [17, Sect. 10.13]). Then by (3.1.7), (3.2.43), and (3.2.44),

$$\mu_{2m+d,N} = \frac{N!}{(N-m)!(N+d/2+1/4)^m}, \qquad 0 \le m \le N,\ N = 1, 2, \dots. \tag{3.2.45}$$

In addition, we define $a = \infty$ and for $N = 1, 2, \dots,$

$$\begin{aligned} w(y, N) &:= e^{-y^2/2}, \quad \psi(y) := |1-y^2|^{-1/4}, \quad a_N := \sqrt{4N+2d}, \\ p_0(k) &:= (2+kd+k/2)^{-1}, \quad p_1(k) := 4/k. \end{aligned} \tag{3.2.46}$$

Note that a_N in (3.2.46) is the $(2N+d)$th Mhaskar–Rakhmanov–Saff number for the exponential weight $e^{-y^2/2}$ (see [37, p. 124]).

Then the following property holds for the normalized Hermite polynomials P_{2N+d} on $(-\infty, \infty)$:

Proposition 3.2.6. *Let $\gamma = \{\gamma_N\}_{N=1}^\infty$ be an increasing sequence satisfying the conditions*

$$\gamma_N = o(N^{1/2}), \quad \gamma_N \ge C\log(N+1), \qquad N = 1, 2, \dots, \tag{3.2.47}$$

and let $\delta = \{\delta_N\}_{N=1}^\infty$ be a decreasing sequence satisfying the conditions

$$\delta_N \ge \gamma_N^2/N, \quad N = 1, 2, \dots, \qquad \lim_{N\to\infty} \delta_N = 0. \tag{3.2.48}$$

Then there exists a constant $b > e/2$ such that for the sequences $\beta = \{\beta_N\}_{N=1}^\infty$ and $W = \{w(y, N)\}_{N=1}^\infty$ defined by (3.2.43) *and* (3.2.46)*, respectively, the sequence of polynomials $\{P_{2N+d}\}_{N=1}^\infty$ defined by* (3.2.44) *belongs to $\mathbb{P}_d(\beta, b^{-1}\gamma, \delta)$, $\mathbb{P}_d^*(\beta, \gamma, \delta)$, and $\mathbb{P}_d^{**}(\beta, b^{-1}\gamma, \delta, W, k)$.*

Proof. Properties (D1) and (D2) of Definition 3.1.3 are trivially satisfied, while property (D4) follows from (3.2.45) and the relations

$$\mu_{2m+d,N} = \prod_{\nu=0}^{m-1}\left(\frac{N-\nu}{N+d/2+1/4}\right) < 1, \qquad 1 \le m \le N. \tag{3.2.49}$$

Next, we have the inequality

$$1 - \mu_{2m+d,N} \le Cm^2/N, \quad 0 \le m \le C_1\sqrt{N}, \quad N = 1, 2, \dots, \tag{3.2.50}$$

where C and C_1 are independent of m and N. Indeed, by (3.2.49), for $0 \le m \le C_1\sqrt{N}$, $N = 1, 2, \dots,$

$$\mu_{2m+d,N} \ge \left(1 - \frac{m}{N+1}\right)^m \ge \exp\left(-\frac{2m^2}{N+1}\right) \ge 1 - Cm^2/N. \tag{3.2.51}$$

Thus (3.2.50) follows from (3.2.51). Since $\gamma_N = o(N^{1/2})$, we obtain from (3.2.50) and (3.2.48) the estimate

$$\sup_{0\le m\le\gamma_N} (1 - \mu_{2m+d,N}) \le C\gamma_N^2/N \le C\delta_N.$$

Therefore, property (D3) is satisfied, so $\{P_{2N+d}\}_{N=1}^\infty \in \mathbb{P}_d^*(\beta, \gamma, \delta)$ since all zeros of the Hermite polynomials are real and distinct.

Similarly to the proof of Proposition 3.2.1, we conclude that (3.2.47) and (3.2.48) imply (3.1.13) for some $h \in (0, 1)$. Therefore by Proposition 3.1.6, there exists $b \in (e/2, \infty)$ such that $\{P_{2N+d}\}_{N=1}^\infty \in \mathbb{P}_d(\beta, b^{-1}\gamma, \delta)$.

Next we note that properties (C4) (see Remark 3.1.10), (C5.1), and (C5.4) of Definition 3.1.8 are satisfied trivially. Further, it is easy to see that for any $c > 0$ and $p \in (p_0(k), p_1(k))$,

$$\int_c^\infty \frac{dy}{y^{(2+kd)p}|1-y^2/a_N^2|^{kp/4}} = a_N^{1-2p} \int_{c/a_N}^\infty \frac{dy}{y^{(2+kd)p}|1-y^2|^{kp/4}} < \infty.$$

Thus property (C5.3) is satisfied for $p_0(k) = (2 + kd + k/2)^{-1}$, $p_1(k) = 4/k$, $k = 1, 2$. Finally, we shall show that property (C5.2) is satisfied. Indeed,

$$\sup_{y\in\mathbb{R}} |H_{2N+d}(y)|e^{-y^2/2}|1-y^2/a_N^2|^{1/4} \le C a_N^{-1/2} h_{2N+d}^{1/2}, \tag{3.2.52}$$

where a_N is defined in (3.2.46) and $h_{2N+d} := \int_{\mathbb{R}} H_{2N+d}^2(y)e^{-y^2/2}dy$. This estimate was established by Levin and Lubinsky (see [34], [37, p. 147], and [35, p. 325]) for more general exponential weights. To show that (3.2.52) implies (C5.2), we note that

$$h_{2N+d}^{1/2} = (\sqrt{\pi}2^{2N+d}(2N+d)!)^{1/2},$$
$$H_{2N+d}^{(d)}(0) = (-1)^N(2N+d)!2^d/N!$$

(see [17, Sect. 10.13]). Hence by Stirling's formula,

$$\frac{|H_{2N+d}^{(d)}(0)|}{h_{2N+d}^{1/2}} = \frac{2^{d/2}(2N+d)!^{1/2}}{\pi^{1/4}2^N N!} = 2^d\pi^{-1/2}N^{d/2-1/4}(1+o(1)) \tag{3.2.53}$$

as $N \to \infty$. Then it follows from (3.2.52) and (3.2.53) that

$$\sup_{y\in\mathbb{R}} |P_{2N+d}(y)|e^{-y^2/2}|1-y^2/a_N^2|^{1/4} \le C_1 a_N^{-1/2} N^{1/4-d/2} \le C_2 N^{-d/2}. \tag{3.2.54}$$

Thus property (C5.2) is satisfied. Therefore, $\{P_{2N+d}\}_{N=1}^\infty \in \mathbb{P}_d^{**}(\beta, b^{-1}\gamma, \delta, W, k)$. ∎

3.2.5. Normalized Williams–Apostol polynomials on $(-\infty,\infty)$. The polynomials of degree n ($n = 1, 2, \ldots$) defined by the formulae

$$\begin{aligned} A_n(y) &:= (1/2)((y+i)^n + (y-i)^n) = (1+y^2)^{n/2}T_n(y/\sqrt{1+y^2}) \\ &= (1+y^2)^{n/2}\cos(n \operatorname{arccot} y), \end{aligned} \tag{3.2.55}$$

$$\begin{aligned} W_n(y) &:= (-i/2)((y+i)^{n+1} - (y-i)^{n+1}) = (1+y^2)^{n/2}U_n(y/\sqrt{1+y^2}) \\ &= (1+y^2)^{(n+1)/2}\sin((n+1)\operatorname{arccot} y), \end{aligned} \tag{3.2.56}$$

are called the *Williams–Apostol polynomials of the first* and *second kind*, respectively. The polynomials W_n of odd degree were introduced by Williams [56] in 1971 in connection with a problem of asymptotic representations for $\zeta(2m)$ (see Remark 4.4.6 below). Two years later, Apostol [2] independently introduced W_n for odd n and studied their properties. Polynomials A_n are special cases of two polynomial sequences introduced and studied by Cvijovič and Klinowski in the 1990s–2000s [13, 14, 15]. Note that all even or odd polynomials of these sequences are reduced to A_n and W_n.

Normalized Williams–Apostol polynomials of the first kind on $(-\infty,\infty)$. For $N = 1, 2, \ldots$, we define

$$\beta_N := 2N + d, \tag{3.2.57}$$

$$\begin{aligned} P_{2N+d}(y) &:= A_{2N+d}(y)/A^{(d)}_{2N+d}(0) = (-1)^N(2N+d)^{-d}A_{2N+d}(y) \\ &= (2N+d)^{-d}\sum_{m=0}^{N}(-1)^m\binom{2N+d}{2m+d}y^{2m+d}. \end{aligned} \tag{3.2.58}$$

Then by (3.1.7), (3.2.57), and (3.2.58),

$$\mu_{2m+d,N} = \frac{(2N+d)!}{(2N-2m)!(2N+d)^{2m+d}}, \qquad 0 \le m \le N,\ N = 1,2,\dots. \tag{3.2.59}$$

In addition, we define $a = \infty$ and, for $\alpha \ge 0$ and $N = 1,2,\dots$,

$$\begin{aligned} w(y,N) &:= (1+y^2)^{-N-d/2-\alpha}, \qquad \psi(y) := 1, \qquad a_N := 1, \\ p_0(k) &:= \max\{(2\alpha k+kd)^{-1},(2+kd)^{-1}\}, \qquad p_1(k) = \infty. \end{aligned} \tag{3.2.60}$$

Positive zeros ($p = 1,\dots,N$, $N = 1,2,\dots$):

$$z_{p,N,d} := \cot\frac{(2p-1)\pi}{4N+2d}. \tag{3.2.61}$$

Derivatives ($p = 1,\dots,N$, $N = 1,2,\dots$):

$$P'_{2N+d}(z_{p,N,d}) = \frac{(-1)^{N+p+1}(2N+d)^{1-d}}{\left(\sin\frac{(2p-1)\pi}{4N+2d}\right)^{2N+d-2}}, \tag{3.2.62}$$

$$P''_{2N+d}(z_{p,N,d}) = \frac{2(-1)^{N+p+1}(2N+d-1)(2N+d)^{1-d}\cos\frac{(2p-1)\pi}{4N+2d}}{\left(\sin\frac{(2p-1)\pi}{4N+2d}\right)^{2N+d-3}}, \tag{3.2.63}$$

$$\frac{P''_{2N+d}(z_{p,N,d})}{P'_{2N+d}(z_{p,N,d})} = (2N+d-1)\sin\frac{(2p-1)\pi}{2N+d}. \tag{3.2.64}$$

Note that formulae (3.2.62)–(3.2.64) can be easily verified by straightforward calculation.

Normalized Williams–Apostol polynomials of the second kind on $(-\infty,\infty)$. For $N = 1,2,\dots$, we define

$$\beta_N := 2N+d+1, \tag{3.2.65}$$

$$\begin{aligned} P_{2N+d}(y) &:= W_{2N+d}(y)/W^{(d)}_{2N+d}(0) = (-1)^N(2N+d+1)^{-d}W_{2N+d}(y) \\ &= (2N+d+1)^{-d}\sum_{m=0}^{N}(-1)^m\binom{2N+d+1}{2m+d}y^{2m+d}. \end{aligned} \tag{3.2.66}$$

Then by (3.1.7), (3.2.65), and (3.2.66), for $N = 1,2,\dots$,

$$\mu_{2m+d,N} = \frac{(2N+d+1)!}{(2N-2m+1)!(2N+d+1)^{2m+d}}, \qquad 0 \le m \le N. \tag{3.2.67}$$

In addition, we define $a = \infty$ and, for $\alpha \ge 0$ and $N = 1,2,\dots$,

$$\begin{aligned} w(y,N) &:= (1+y^2)^{-N-d/2-\alpha-1/2}, \qquad \psi(y) := 1, \qquad a_N := 1, \\ p_0(k) &:= \max((2\alpha k+kd)^{-1},(2+kd)^{-1}), \qquad p_1(k) = \infty. \end{aligned} \tag{3.2.68}$$

Positive zeros ($p = 1,\dots,N$, $N = 1,2,\dots$):

$$z_{p,N,d} := \cot\frac{p\pi}{2N+d+1}. \tag{3.2.69}$$

Derivatives ($p = 1, \ldots, N$, $N = 1, 2, \ldots$):

$$P'_{2N+d}(z_{p,N,d}) = \frac{(-1)^{N+p+1}(2N+d+1)^{1-d}}{\left(\sin \frac{p\pi}{2N+d+1}\right)^{2N+d-1}}, \tag{3.2.70}$$

$$P''_{2N+d}(z_{p,N,d}) = \frac{2(-1)^{N+p+1}(2N+d)(2N+d+1)^{1-d}\cos \frac{p\pi}{2N+d+1}}{\left(\sin \frac{p\pi}{2N+d+1}\right)^{2N+d-2}}, \tag{3.2.71}$$

$$\frac{P''_{2N+d}(z_{p,N,d})}{P'_{2N+d}(z_{p,N,d})} = (2N+d)\sin \frac{2p\pi}{2N+d+1}. \tag{3.2.72}$$

Again, formulae (3.2.70)–(3.2.72) can be verified by straightforward calculation.

The following property holds for the normalized Williams–Apostol polynomials P_{2N+d} of the first and second kind on $(-\infty, \infty)$:

Proposition 3.2.7. *Let $\gamma = \{\gamma_N\}_{N=1}^{\infty}$ be an increasing sequence satisfying the conditions*

$$\gamma_N = o(N^{1/2}), \quad \gamma_N \geq C \log(N+1), \qquad N = 1, 2, \ldots,$$

and let $\delta = \{\delta_N\}_{N=1}^{\infty}$ be a decreasing sequence satisfying the conditions

$$\delta_N \geq \gamma_N^2/N, \quad N = 1, 2, \ldots, \qquad \lim_{N\to\infty} \delta_N = 0.$$

Then there exists a constant $b > e/2$ such that for the sequences $\beta = \{\beta_N\}_{N=1}^{\infty}$ and $W = \{w(y, N)\}_{N=1}^{\infty}$ defined by (3.2.57) *or* (3.2.65) *and* (3.2.60) *or* (3.2.68), *respectively, the sequence of polynomials $\{P_{2N+d}\}_{N=1}^{\infty}$ defined by* (3.2.58) *or* (3.2.66) *belongs to $\mathbb{P}_d(\beta, b^{-1}\gamma, \delta)$, $\mathbb{P}_d^*(\beta, \gamma, \delta)$, and $\mathbb{P}_d^{**}(\beta, b^{-1}\gamma, \delta, W, k)$.*

Proof. Let us consider the polynomials of the first kind, that is, β_N and P_{2N+d}, $N = 1, 2, \ldots$, defined by (3.2.57) and (3.2.58), respectively. Properties (D1) and (D2) of Definition 3.1.3 are trivially satisfied, while property (D4) follows from (3.2.59) and the relations

$$\mu_{2m+d,N} = \prod_{\nu=0}^{2m+d-1} \left(1 - \frac{\nu}{2N+d}\right) < 1, \qquad 1 \leq m \leq N. \tag{3.2.73}$$

Next, property (D3) is satisfied since

$$1 - \mu_{2m+d,N} \leq Cm^2/N, \quad 0 \leq m \leq C_1\sqrt{N}, \qquad N = 1, 2, \ldots,$$

which follows from (3.2.73) and the estimates ($0 \leq m \leq C_1\sqrt{N}$, $N = 1, 2, \ldots$)

$$\mu_{2m+d,N} \geq \left(1 - \frac{2m+d-1}{2N+d}\right)^{2m+d} \geq e^{-C_2m^2/N} \geq 1 - Cm^2/N.$$

Therefore, $\{P_{2N+d}\}_{N=1}^{\infty} \in \mathbb{P}_d^*(\beta, \gamma, \delta)$ since all zeros of the Williams–Apostol polynomials are real and distinct.

Similarly to the proof of Propositions 3.2.1 and 3.2.6, we conclude that $\{P_{2N+d}\}_{N=1}^{\infty} \in \mathbb{P}_d(\beta, b^{-1}\gamma, \delta)$ for some $b \in (e/2, \infty)$.

Next we note that by Remark 3.1.9, properties (C5.1) and (C5.3) are satisfied for $p \in ((2+kd)^{-1}, \infty)$. In addition, it is easy to see that the integral in property (C5.4) is finite for $p \in ((2\alpha k+kd)^{-1}, \infty)$ (see also Theorem 3.3.13 below). So properties (C5.3) and (C5.4) are satisfied for $p_0(k) := \max\{(2\alpha k+kd)^{-1}, (2+kd)^{-1}\}$, $p_1(k) = \infty$. In addition, since $\alpha \ge 0$, property (C5.2) follows from (3.2.55), (3.2.57), and (3.2.60). It remains to note that by (3.2.41),

$$w(y/\beta_N, N) = (1 + y^2/(2N+d)^2)^{-N-d/2-\alpha} = 1 + O(y^2/N).$$

Thus property (C4) is satisfied. Therefore, $\{P_{2N+d}\}_{N=1}^{\infty} \in \mathbb{P}_d^{**}(\beta, b^{-1}\gamma, \delta, W, k)$, and the proposition is established for the Williams–Apostol polynomials of the first kind. For the polynomials of the second kind the proof is similar. ∎

3.2.6. Normalized Lommel polynomials on $(-\infty, \infty)$**.** Let us set

$$\begin{aligned} Q_{2N+d}(y) &:= (y/2)^{2N+d} R_{N,d+1/2}(y) \\ &= \sum_{m=0}^{N} \frac{(-1)^m (2N-m)!\Gamma(2N-m+d+1/2)}{m!(2N-2m)!\Gamma(m+d+1/2)2^{2m+d}} y^{2m+d} \\ &= \sum_{m=0}^{N} \frac{(-1)^m}{(2m+d)!} \frac{(4N-2m+d)!}{(2N-2m)!2^{4N-2m+d}} y^{2m+d}, \end{aligned} \tag{3.2.74}$$

where $R_{N,\nu}$ is a Lommel "polynomial" (it is a polynomial in $1/y$; see [55, Sect. 9.61]). Note that the last identity in (3.2.74) is established by the Legendre duplication formula for the gamma function.

For any $\beta = \{\beta_N\}_{N=1}^{\infty}$ and $N = 1, 2, \ldots$, we define

$$P_{2N+d}(y) := \beta_N^{-d} Q_{2N+d}(\beta_N y)/Q_{2N+d}^{(d)}(0). \tag{3.2.75}$$

Then the coefficients

$$\mu_{2m+d,N} = \frac{(4N-2m+d)!(2N)!2^{2m}}{(4N+d)!(2N-2m)!}, \qquad 1 \le m \le N,$$

are found by (3.1.7), (3.2.74), and (3.2.75). The following property holds for the normalized Lommel polynomials P_{2N+d} on $(-\infty, \infty)$.

Proposition 3.2.8. *Let $\gamma = \{\gamma_N\}_{N=1}^{\infty}$ be an increasing sequence satisfying the conditions*

$$\gamma_N = o(N^{1/2}), \quad \gamma_N \ge C \log(N+1), \qquad N = 1, 2, \ldots,$$

and let $\delta = \{\delta_N\}_{N=1}^{\infty}$ be a decreasing sequence satisfying the conditions

$$\delta_N \ge \gamma_N^2/N, \quad N = 1, 2, \ldots, \qquad \lim_{N\to\infty} \delta_N = 0.$$

Then there exists a constant $b > e/2$ such that for any sequence $\beta = \{\beta_N\}_{N=1}^{\infty}$, the sequence of polynomials $\{P_{2N+d}\}_{N=1}^{\infty}$ defined by (3.2.75) belongs to $\mathbb{P}_d^*(\beta, \gamma, \delta)$.

Proof. It suffices to show that properties (D3) and (D4) are satisfied. Property (D4) follows from the relations

$$\mu_{2m+d,N} = \prod_{\nu=0}^{2m-1} \frac{4N-2\nu}{4N-\nu+d} < 1, \qquad 1 \le m \le N, \tag{3.2.76}$$

while property (D3) is a consequence of (3.2.76) and the estimates

$$\mu_{2m+d,N} \ge \left(\frac{4N-4m+2}{4N+1}\right)^{2m} \ge \left(1 - \frac{m}{N}\right)^{2m} \ge e^{-2m^2/N} \ge 1 - Cm^2/N.$$

Since all zeros of P_{2N+d} are real and distinct [55, Sect. 9.71], the proposition is established. ■

REMARK 3.2.9. This example shows that for any β there exists a sequence $\{P_{2N+d}\}_{N=1}^{\infty}$ that belongs to $\mathbb{P}_d^*(\beta,\gamma,\delta)$, where δ and γ are defined in Proposition 3.2.8.

3.2.7. Normalized Laguerre polynomials on $(-\infty,\infty)$**.** In this counterexample we show that a sequence of classical even polynomials on $(-\infty,\infty)$ does not belong to $\mathbb{P}_0(\beta,\gamma,\delta)$ for any β, γ and δ. For $\alpha>-1$, $\alpha\neq -1/2$, and $N=1,2,\dots$, we define

$$P_{2N}(y):=L_N^{\alpha}(y^2)/L_N^{\alpha}(0)=\Gamma(1+\alpha)\sum_{m=0}^{N}\frac{(-1)^m\prod_{\nu=0}^{m-1}(N-\nu)}{m!\Gamma(\alpha+m+1)}y^{2m}.$$

If there exist β, γ and δ such that $\{P_{2N}\}_{N=1}^{\infty}\in\mathbb{P}_0(\beta,\gamma,\delta)$, then by property (C3), for any $M>0$, $\lim_{N\to\infty}P_{2N}(z/\beta_N)=\cos z$ uniformly on the disk $\bar{D}_M(0)$. Then by a classical theorem of complex analysis [50, Sect. 2.8], the mth coefficient of $P_{2N}(z/\beta_N)$ converges to the mth Taylor coefficient of $\cos z$, $m=1,2,\dots$. Therefore,

$$\lim_{N\to\infty}N/\beta_N^2=\left(\frac{\Gamma(m+\alpha+1)m!}{\Gamma(\alpha+1)(2m)!}\right)^{1/m},\qquad m=1,2,\dots,$$

which is valid if and only if $\alpha=-1/2$ (Hermite polynomials) and $\{\beta_N\}_{N=1}^{\infty}$ satisfies the condition $\lim_{N\to\infty}N/\beta_N^2=1/4$. Thus $\{P_{2N}\}_{N=1}^{\infty}\notin\mathbb{P}_0(\beta,\gamma,\delta)$. Note that

$$\lim_{N\to\infty}P_{2N}(y/(2\sqrt{N}))=\Gamma(\alpha+1)(y/2)^{-\alpha}J_\alpha(y)$$

(see [17, Sect. 10.12]).

3.3. Asymptotic formulae for L_p-quasinorms of special polynomials. Here, we establish asymptotics for weighted L_p-quasinorms

$$\left(\int_{-a}^{a}|y^{-d}P_{2N+d}(y)w(y,N)|^p\,dy\right)^{1/p},\qquad 0<p<\infty,$$

of the normalized Gegenbauer, Chebyshev, Hermite, and Williams–Apostol polynomials and polynomials with equidistant zeros introduced in Section 3.2. These formulae are used in Sections 5.2 and 5.4 for L_p-asymptotics and L_p-error criteria for $\zeta(s)=0$ or $\beta(s)$=0.

3.3.1. Technical lemma. The proofs of most of the asymptotic formulae in this section are based on the following technical lemma:

LEMMA 3.3.1. *For $N=1,2,\dots$, $d=0,1$, $p\in(0,\infty)$, $q\in(-1,\infty)$, $\lambda\in\mathbb{R}$, the integral*

$$F_{N,d,p,q,\lambda}:=\left(\int_0^{\pi/2}|\cos((2N+d+\lambda)\theta-\lambda\pi/2)|^p\cos^{-dp}\theta\sin^q\theta\,d\theta\right)^{1/p}$$

satisfies the asymptotic property

$$\lim_{N\to\infty}\lambda_{N,d,p}F_{N,d,p,q,\lambda}=A_{d,p,q},\tag{3.3.1}$$

where

$$\lambda_{N,d,p} := \begin{cases} 1, & d=0,\ 0<p<\infty \text{ or } d=1,\ 0<p<1, \\ (\log N)^{-1}, & d=1,\ p=1, \\ N^{1/p-1}, & d=1,\ 1<p<\infty, \end{cases} \tag{3.3.2}$$

$$A_{d,p,q} := \begin{cases} \left[\dfrac{\Gamma(p/2+1/2)\Gamma(1/2-dp/2)\Gamma(q/2+1/2)}{2\sqrt{\pi}\Gamma(p/2+1)\Gamma(q/2-dp/2+1)}\right]^{1/p}, & d=0,\ 0<p<\infty, \\ & \text{or } d=1,\ 0<p<1, \\ 2/\pi, & d=1,\ p=1, \\ 2^{1-1/p}\left(\int_0^\infty |(\sin t)/t|^p\,dt\right)^{1/p}, & d=1,\ 1<p<\infty. \end{cases} \tag{3.3.3}$$

REMARK 3.3.2. It is known that for even positive $p=2m$,

$$\int_0^\infty \left(\frac{\sin t}{t}\right)^{2m} dt = \frac{\pi}{2(2m-1)!}\sum_{\nu=1}^{m}(-1)^{\nu+m}\nu^{2m-1}\binom{2m}{m-\nu}.$$

This result can be found in [9, Excercise 22, p. 518], where it is attributed to Wolstenholme. The exact value of $I_p := \int_0^\infty |(\sin t)/t|^p\,dt$, $p>1$, is unknown for $p\neq 2m$. Various properties of I_p and $I_p^{1/p}$ are discussed in [8].

To prove the asymptotic (3.3.1), we use the following generalization of Fejér's Lemma [18]:

LEMMA 3.3.3 ([3, Lemma 2.1]). *Let g be continuous and π-periodic on $\mathbb{R}$ and let $f\in L_1[0,\pi]$. If γ is a measurable and a.e. finite function on $[0,\pi]$, then*

$$\lim_{n\to\infty}\int_0^\pi g(n\theta+\gamma(\theta))f(\theta)\,d\theta = (1/\pi)\int_0^\pi g(\theta)\,d\theta\int_0^\pi f(\theta)\,d\theta.$$

Proof of Lemma 3.3.1. We consider three cases.

CASE 1. Let $d=0$, $0<p<\infty$ or $d=1$, $0<p<1$. Then

$$F^p_{N,d,p,q,\lambda} = \frac{1}{2}\int_0^\pi |\cos(N\theta+d\theta/2+\lambda\theta/2-\lambda\pi/2)|^p\cos^{-dp}(\theta/2)\sin^q(\theta/2)\,d\theta.$$

Using Lemma 3.3.3 for $g(\theta)=|\cos\theta|^p$, $f(\theta)=\cos^{-dp}(\theta/2)\sin^q(\theta/2)$, and $\gamma(\theta)=d\theta/2+\lambda\theta/2-\lambda\pi/2$, we obtain

$$\lim_{N\to\infty}\lambda^p_{N,d,p}F^p_{N,d,p,q,\lambda} = \frac{1}{2\pi}\int_0^\pi|\cos\theta|^p\,d\theta\int_0^\pi\cos^{-dp}(\theta/2)\sin^q(\theta/2)\,d\theta = A^p_{d,p,q}. \tag{3.3.4}$$

CASE 2. Let $d=1$, $p=1$. Using the estimate

$$\sup_{\theta\in(0,\pi/2]}|1/\sin\theta-1/\theta|<\infty, \tag{3.3.5}$$

we obtain for any $\varepsilon\in(0,\pi)$,

$$\begin{aligned} F_{N,1,1,q,\lambda} &= \int_0^{\pi/2}|\sin((2N+1+\lambda)\theta)|(\sin\theta)^{-1}\cos^q\theta\,d\theta \\ &= \int_0^{\pi/2}|\sin(2N\theta)|(\sin\theta)^{-1}\cos^q\theta\,d\theta + O(1) \end{aligned}$$

$$
\begin{aligned}
&= \int_0^{\pi N} |\sin\theta|\theta^{-1}\cos^q(\theta/(2N))\,d\theta + O(1)\\
&= \int_0^{\varepsilon N} |\sin\theta|\theta^{-1}\cos^q(\theta/(2N))\,d\theta + O(1).
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\liminf_{N\to\infty}(\log N)^{-1}\left(\min\{1,\cos^q(\varepsilon/2)\}\int_0^{\varepsilon N}|\sin\theta|\theta^{-1}\,d\theta + O(1)\right)\\
&\quad\le \liminf_{N\to\infty}(\log N)^{-1}F_{N,1,1,q,\lambda} \le \limsup_{N\to\infty}(\log N)^{-1}F_{N,1,1,q,\lambda}\\
&\quad\le \limsup_{N\to\infty}(\log N)^{-1}\left(\max\{1,\cos^q(\varepsilon/2)\}\int_0^{\varepsilon N}|\sin\theta|\theta^{-1}\,d\theta + O(1)\right).
\end{aligned} \tag{3.3.6}
$$

Taking account of the asymptotic for the Lebesgue constant [57, Sect. 2.12]

$$
\begin{aligned}
L_N &:= \frac{2}{\pi}\int_0^{\pi N}|\sin\theta|\theta^{-1}\,d\theta + O(1) = \frac{2}{\pi}\int_0^{\varepsilon N}|\sin\theta|\theta^{-1}\,d\theta + O(1)\\
&= (4/\pi^2)\log N + O(1),
\end{aligned} \tag{3.3.7}
$$

we deduce from (3.3.6) the estimates

$$
\begin{aligned}
(2/\pi)\min\{1,\cos^q(\varepsilon/2)\} &\le \liminf_{N\to\infty}(\log N)^{-1}F_{N,1,1,q,\lambda} \le \limsup_{N\to\infty}(\log N)^{-1}F_{N,1,1,q,\lambda}\\
&\le (2/\pi)\max\{1,\cos^q(\varepsilon/2)\}.
\end{aligned} \tag{3.3.8}
$$

Letting $\varepsilon\to 0$ in (3.3.8), we arrive at

$$
\lim_{N\to\infty}(\log N)^{-1}F_{N,1,1,q,\lambda} = 2/\pi = A_{1,1,q}. \tag{3.3.9}
$$

CASE 3. Let $d=1$, $1<p<\infty$. Using Hölder's inequality and estimate (3.3.5), we obtain

$$
\begin{aligned}
F_{N,1,p,q,\lambda} &= \left(\int_0^{\pi/2}|\sin((2N+1+\lambda)\theta)|^p\sin^{-p}\theta\cos^q\theta\,d\theta\right)^{1/p}\\
&= \left(\int_0^{\pi/2}|\sin(2N\theta)|^p\sin^{-p}\theta\cos^q\theta\,d\theta\right)^{1/p} + O(1)\\
&= \left(\int_0^{\pi/2}|\sin(2N\theta)|^p\theta^{-p}\cos^q\theta\,d\theta\right)^{1/p} + O(1)\\
&= \left((2N)^{p-1}\int_0^{\pi N}|\sin\theta|^p\theta^{-p}\cos^q(\theta/(2N))\,d\theta\right)^{1/p} + O(1).
\end{aligned} \tag{3.3.10}
$$

Further, for any $\alpha\in(1/p,1)$ we obtain

$$
J_N := \int_0^{\pi N}|\sin\theta|^p\theta^{-p}\cos^q(\theta/(2N))\,d\theta = \int_0^{N^\alpha} + \int_{N^\alpha}^{\pi N} = J_{N,1}+J_{N,2}. \tag{3.3.11}
$$

Then as $N\to\infty$,

$$
J_{N,2} \le N^{-\alpha p}\int_0^{\pi N}\cos^q(\theta/(2N))\,d\theta \le CN^{1-\alpha p} = o(1), \tag{3.3.12}
$$

and

$$J_{N,1} = \int_0^\infty \left|\frac{\sin\theta}{\theta}\right|^p d\theta + \int_0^{N^\alpha} \left|\frac{\sin\theta}{\theta}\right|^p F_{N,q}(\theta)\,d\theta + o(1) = \int_0^\infty \left|\frac{\sin\theta}{\theta}\right|^p d\theta + o(1), \qquad (3.3.13)$$

where $F_{N,q}(\theta) := \cos^q(\theta/(2N)) - 1$ and the last equality in (3.3.13) follows from the elementary estimates

$$\begin{aligned}\max_{\theta\in[0,N^\alpha]} |F_{N,q}(\theta)| &\le C \max_{\theta\in[0,N^\alpha]} |F_{N,|q|}(\theta)| \\ &\le C|q| \max_{\theta\in[0,N^\alpha]} (1-\cos(\theta/(2N))) \max_{\theta\in[0,N^\alpha]} \max\{1, \cos^{|q|-1}(\theta/(2N))\} \\ &\le CN^{2\alpha-2} = o(1).\end{aligned}$$

Collecting (3.3.11)–(3.3.13), we see that

$$\begin{aligned}(2N)^{p-1} &\int_0^{\pi N} |\sin\theta|^p \theta^{-p} \cos^q(\theta/(2N))\,d\theta \\ &= (2N)^{p-1} \int_0^\infty |(\sin\theta)/\theta|^p\,d\theta + o(N^{p-1}). \qquad (3.3.14)\end{aligned}$$

Finally, the limit relation (3.3.1) for $d = 1$, $1 < p < \infty$ follows from (3.3.10) and (3.3.14), while (3.3.1) for the other two cases is a consequence of (3.3.4) and (3.3.9). ■

3.3.2. Normalized Gegenbauer polynomials on $[-1, 1]$**.** Let us set

$$\tau_{N,d,p} := \begin{cases} 1, & d = 0,\ 0 < p < 4, \\ N, & d = 1,\ 0 < p < 1, \\ N/\log N, & d = 1,\ p = 1, \\ N^{1/p}, & d = 1,\ 1 < p < 4, \end{cases} \qquad (3.3.15)$$

$$\mathcal{G}_{d,p} := \begin{cases} \left[\dfrac{\Gamma(p/2+1/2)\Gamma(1-p/4)}{\Gamma(p/2+1)\Gamma(3/2-p/4)}\right]^{1/p}, & d = 0,\ 0 < p < 4, \\ \left[\dfrac{\Gamma(p/2+1/2)\Gamma(1/2-p/2)\Gamma(1-p/4)}{2^p\sqrt{\pi}\,\Gamma(p/2+1)\Gamma(3/2-3p/4)}\right]^{1/p}, & d = 1,\ 0 < p < 1, \\ 2/\pi, & d = 1,\ p = 1, \\ \left(\int_0^\infty |(\sin t)/t|^p\,dt\right)^{1/p}, & d = 1,\ 1 < p < 4. \end{cases} \qquad (3.3.16)$$

Then the following theorem holds:

THEOREM 3.3.4. *For* $\lambda \ge 0$ *and* $p \in (0, 4)$,

$$\lim_{N\to\infty} \tau_{N,d,p} \left(\int_{-1}^1 \left| \frac{(1-y^2)^{\lambda/2-1/4} C_{2N+d}^\lambda(y)}{y^d (C_{2N+d}^\lambda)^{(d)}(0)} \right|^p dy \right)^{1/p} = \mathcal{G}_{d,p}. \qquad (3.3.17)$$

Proof. The proof is based on the following asymptotic formula as $N \to \infty$ for the Gegenbauer polynomials with $\lambda \ge 0$ and $y = \cos\theta$, where $C/N \le \theta \le \pi - C/N$:

$$\begin{aligned}(\sin\theta)^{\lambda-1/2} &\frac{C_{2N+d}^\lambda(\cos\theta)}{(C_{2N+d}^\lambda)^{(d)}(0)} \\ &= (-1)^N (2N)^{-d} \left(\frac{(1+o(1))\cos((2N+d+\lambda)\theta - \pi\lambda/2)}{\sin^{1/2}\theta} + \frac{O(1)}{N\sin^{3/2}\theta} \right). \qquad (3.3.18)\end{aligned}$$

Here, $o(1)$ and $O(1)$ are constants independent of θ. Relation (3.3.18) follows from the classical asymptotic formula for the Jacobi polynomials [48, eq. 8.21.18]

$$\begin{aligned}(\sin\theta)^{\lambda-1/2}&P_{2N+d}^{(\lambda-1/2,\lambda-1/2)}(\cos\theta)\\ &=(2N+d)^{-1/2}2^{\lambda}\pi^{-1/2}\left(\frac{\cos((2N+d+\lambda)\theta-\pi\lambda/2)}{\sin^{1/2}\theta}+\frac{O(1)}{N\sin^{3/2}\theta}\right),\end{aligned}\tag{3.3.19}$$

where $C/N\le\theta\le\pi-C/N$ and $N\to\infty$. Indeed, if we take account of the relations

$$P_{2N+d}^{(\lambda-1/2,\lambda-1/2)}(y)=\frac{\Gamma(2N+d+\lambda+1/2)\Gamma(2\lambda)}{\Gamma(\lambda+1/2)\Gamma(2N+d+2\lambda)}C_{2N+d}^{\lambda}(y),\tag{3.3.20}$$

$$(C_{2N+d}^{\lambda})^{(d)}(0)=\frac{(-1)^N2^d\Gamma(N+d+\lambda)}{\Gamma(\lambda)\Gamma(N+1)},\tag{3.3.21}$$

then it follows that

$$\begin{aligned}(\sin\theta)^{\lambda-1/2}&\frac{C_{2N+d}^{\lambda}(\cos\theta)}{(C_{2N+d}^{\lambda})^{(d)}(0)}\\ &=\frac{(-1)^N(2N+d)^{-1/2}2^{\lambda}\pi^{-1/2}\Gamma(\lambda+1/2)\Gamma(\lambda)\Gamma(2N+d+2\lambda)\Gamma(N+1)}{\Gamma(2\lambda)\Gamma(2N+d+\lambda+1/2)\Gamma(N+d+\lambda)}\\ &\quad\times\left(\frac{\cos((2N+d+\lambda)\theta-\pi\lambda/2)}{\sin^{1/2}\theta}+\frac{O(1)}{N\sin^{3/2}\theta}\right).\end{aligned}\tag{3.3.22}$$

Then (3.3.22) implies (3.3.18), by Stirling's formula and the Legendre duplication formula.

Let now $p\in(0,4)$. Then

$$\begin{aligned}(2I_{N,d,p})^{1/p}&:=\left(\int_{-1}^{1}\left|\frac{(1-y^2)^{\lambda/2-1/4}C_{2N+d}^{\lambda}(y)}{y^d(C_{2N+d}^{\lambda})^{(d)}(0)}\right|^p dy\right)^{1/p}\\ &=\left(2\int_0^{\pi/2}\left((\sin\theta)^{\lambda-1/2}\left|\frac{C_{2N+d}^{\lambda}(\cos\theta)}{\cos^d\theta\,(C_{2N+d}^{\lambda})^{(d)}(0)}\right|\right)^p\sin\theta\,d\theta\right)^{1/p}.\end{aligned}\tag{3.3.23}$$

Next, using the Bernstein-type inequality (3.2.7), we obtain

$$\begin{aligned}J_{N,d,p,1}&:=\int_0^{1/N}\left((\sin\theta)^{\lambda-1/2}\left|\frac{C_{2N+d}^{\lambda}(\cos\theta)}{\cos^d\theta\,(C_{2N+d}^{\lambda})^{(d)}(0)}\right|\right)^p\sin\theta\,d\theta\\ &\le CN^{-dp}\int_0^{1/N}(\sin\theta)^{1-p/2}\,d\theta\le CN^{p/2-2-dp}=o(N^{-dp}).\end{aligned}\tag{3.3.24}$$

In addition,

$$\begin{aligned}(2N)^{-dp}&\int_0^{1/N}\left|\frac{\cos((2N+d+\lambda)\theta-\pi\lambda/2)}{\cos^d\theta\sin^{1/2}\theta}\right|^p\sin\theta\,d\theta\\ &\le CN^{-dp}\int_0^{1/N}(\sin\theta)^{1-p/2}\,d\theta=o(N^{-dp}).\end{aligned}\tag{3.3.25}$$

Further, we consider two cases.

Case 1. Let $d = 0$, $0 < p < 4$ or $d = 1$, $0 < p < 1$. We first split the integral $I_{N,d,p}$ from (3.3.23) into two ones and obtain the relations

$$J_{N,d,p,2} := \int_{1/N}^{\pi/2} \le I_{N,d,p} = \int_0^{1/N} + \int_{1/N}^{\pi/2} = J_{N,d,p,1} + J_{N,d,p,2}, \tag{3.3.26}$$

where $J_{N,d,p,1}$ is estimated in (3.3.24). Next, it follows from (3.3.18) that

$$\begin{aligned} J_{N,d,p,2} = (2N)^{-dp} \int_{1/N}^{\pi/2} \bigg| & \frac{(1+o(1))\cos((2N+d+\lambda)\theta - \pi\lambda/2)}{\cos^d\theta \sin^{1/2}\theta} \\ & + \frac{O(1)}{N\cos^d\theta\sin^{3/2}\theta}\bigg|^p \sin\theta\, d\theta, \end{aligned} \tag{3.3.27}$$

where the second term under the integral is estimated as follows

$$\begin{aligned} (2N)^{-dp} & \int_{1/N}^{\pi/2} \frac{|O(1)|^p}{N^p(\cos\theta)^{dp}(\sin\theta)^{3p/2-1}}\, d\theta \\ & \le CN^{-p(d+1)}\left(\int_{1/N}^{\pi/3} \frac{d\theta}{(\sin\theta)^{3p/2-1}} + \int_{\pi/3}^{\pi/2} \frac{d\theta}{(\cos\theta)^{dp}}\right) \\ & \le CN^{-p(d+1)}(N^{3p/2-2}\log N + C_1) = o(N^{-dp}), \end{aligned} \tag{3.3.28}$$

since $\min\{2-3p/2+p(d+1), p(d+1)\} > dp$. Collecting now (3.3.24)–(3.3.28), we conclude that for $0 < p < 1$,

$$I_{N,d,p} = (2N)^{-dp}(1+o(1)) \int_0^{\pi/2} \frac{|\cos((2N+d+\lambda)\theta - \pi\lambda/2)|^p}{(\cos\theta)^{dp}(\sin\theta)^{p/2-1}}\, d\theta + o(N^{-dp}), \tag{3.3.29}$$

while for $1 \le p < 4$,

$$\begin{aligned} I_{N,d,p}^{1/p} &= (2N)^{-d}(1+o(1)) \\ &\times \left(\int_0^{\pi/2} \frac{|\cos((2N+d+\lambda)\theta - \pi\lambda/2)|^p}{(\cos\theta)^{dp}(\sin\theta)^{p/2-1}}\, d\theta\right)^{1/p} + o(N^{-d}). \end{aligned} \tag{3.3.30}$$

Applying Lemma 3.3.1 to the integrals on the right-hand sides of (3.3.29) and (3.3.30) for $q = 1 - p/2$, we find that for $d = 0$, $0 < p < 4$ or $d = 1$, $0 < p < 1$,

$$\lim_{N\to\infty} (2N)^d I_{N,d,p}^{1/p} = A_{d,p,1-p/2}. \tag{3.3.31}$$

Thus (3.3.17) for these cases follows from (3.3.31) and (3.3.23).

Case 2. Let $d = 1$, $1 \le p < 4$. We split $I_{N,1,p}$ from (3.3.23) differently

$$\begin{aligned} J_{N,1,p,2} := \int_{1/N}^{\pi/2-1/N^2} & \le I_{N,1,p} = \int_0^{1/N} + \int_{1/N}^{\pi/2-1/N^2} + \int_{\pi/2-1/N^2}^{\pi/2} \\ & = J_{N,1,p,1} + J_{N,1,p,2} + J_{N,1,p,3}, \end{aligned} \tag{3.3.32}$$

where $J_{N,1,p,1}$ is estimated in (3.3.24). Next, (3.3.18) yields

$$\begin{aligned} J_{N,1,p,2} = (2N)^{-p} \int_{1/N}^{\pi/2-1/N^2} \bigg| & \frac{(1+o(1))\cos((2N+1+\lambda)\theta - \pi\lambda/2)}{\cos\theta\sin^{1/2}\theta} \\ & + \frac{O(1)}{N\cos\theta\sin^{3/2}\theta}\bigg|^p \sin\theta\, d\theta, \end{aligned} \tag{3.3.33}$$

and, in addition,

$$
\begin{aligned}
&(2N)^{-p}\int_{1/N}^{\pi/2-1/N^2}\frac{|O(1)|^p}{N^p(\cos\theta)^p(\sin\theta)^{3p/2-1}}\,d\theta\\
&\qquad\le CN^{-2p}\biggl(\int_{1/N}^{\pi/3}\frac{d\theta}{(\sin\theta)^{3p/2-1}}+\int_{\pi/3}^{\pi/2-1/N^2}\frac{d\theta}{(\cos\theta)^p}\biggr)\\
&\qquad\le CN^{-1}(N^{-(p/2+1)}\log N+N^{-1}\log N+N^{-(2p-1)})=o(N^{-1}). \qquad (3.3.34)
\end{aligned}
$$

To estimate $J_{N,1,p,3}$, we use property (C3) of Definition 3.1.1, which holds for the Gegenbauer polynomials with

$$
\beta_N=2N+\lambda+1,\qquad \delta_N=N^{-2}\log^3 N,\qquad \gamma_N=\log N,\qquad \lambda\ge 0,
$$

due to Proposition 3.2.1. Then it follows from (3.1.3) that

$$
\sup_{|y|\le 1/N^2}\left|\frac{C_{2N+1}^\lambda(y)}{y(C_{2N+1}^\lambda)'(0)}\right|\le C,
$$

which immediately implies the inequality

$$
\sup_{\pi/2-1/N^2\le\theta\le\pi/2}\left|\frac{C_{2N+1}^\lambda(\cos\theta)}{\cos\theta\,(C_{2N+1}^\lambda)'(0)}\right|\le C,
$$

Hence

$$
J_{N,1,p,3}\le C\int_{\pi/2-1/N^2}^{\pi/2}(\sin\theta)^{(\lambda-1/2)p+1}\,d\theta\le CN^{-2}=o(N^{-1}). \qquad (3.3.35)
$$

We also need the following estimate:

$$
\begin{aligned}
&(2N)^{-p}\int_{\pi/2-1/N^2}^{\pi/2}\frac{|\cos((2N+1+\lambda)\theta-\pi\lambda/2)|^p}{(\cos\theta)^p(\sin\theta)^{p/2-1}}\,d\theta\\
&\qquad\le CN^{-p}\int_0^{1/N^2}\frac{|\sin((2N+1+\lambda)\theta)|^p}{(\sin\theta)^p}\,d\theta\le CN^{-2}=o(N^{-1}). \qquad (3.3.36)
\end{aligned}
$$

Collecting now (3.3.24), (3.3.32)–(3.3.36), we conclude that

$$
\begin{aligned}
I_{N,1,p}^{1/p}&=(2N)^{-1}(1+o(1))\\
&\qquad\times\biggl(\int_0^{\pi/2}\frac{|\cos((2N+d+\lambda)\theta-\pi\lambda/2)|^p}{(\cos\theta)^p(\sin\theta)^{p/2-1}}\,d\theta\biggr)^{1/p}+o(N^{-1/p}). \qquad (3.3.37)
\end{aligned}
$$

Applying Lemma 3.3.1 to the integral on the right-hand side of (3.3.37) for $q=1-p/2$, we deduce that for $d=1$, $1\le p<4$,

$$
\lim_{N\to\infty}(2N)\lambda_{N,d,p}I_{N,d,p}^{1/p}=A_{d,p,1-p/2}. \qquad (3.3.38)
$$

Therefore (3.3.17) for this case follows from (3.3.38) and (3.3.23). This completes the proof of Theorem 3.3.4. ■

Remark 3.3.5. Aptekarev, Buyarov, and Dehesa [3, Theorem 2] found the asymptotic behavior of $(\int_{-1}^1|C_n^\lambda(y)|^p(1-y^2)^{\lambda-1/2}\,dy)^{1/p}$ as $n\to\infty$. Note that Theorem 3.3.4 for $d=0$ and $p=2$ follows from this result.

3.3.3. Normalized even Chebyshev polynomials of the first kind on $[-1,1]$**.** The corresponding asymptotic formula for the weight $w(y,N) = (1-y^2)^{-1/4}$ follows from (3.3.17). Here, we consider the case of $w(y,N) = 1$ and $d = 0$.

THEOREM 3.3.6. *For any* $p \in (0,\infty)$*,*

$$\lim_{N\to\infty}\left(\int_{-1}^{1} |T_{2N}(y)|^p\,dy\right)^{1/p} = \left(\frac{2\Gamma(p/2+1/2)}{\sqrt{\pi}\,\Gamma(p/2+1)}\right)^{1/p}. \tag{3.3.39}$$

Proof. Since

$$\left(\int_{-1}^{1} |T_{2N}(y)|^p\,dy\right)^{1/p} = \left(2\int_0^{\pi/2} |\cos(2N\theta)|^p \sin\theta\,d\theta\right)^{1/p},$$

formula (3.3.39) follows from Lemma 3.3.1 for $\lambda = d = 0$ and $q = 1$. ■

3.3.4. Normalized polynomials with equidistant zeros on $[-1,1]$**.** Let $\{P_{2N+d}\}_{N=1}^{\infty}$ be the sequence of polynomials (3.2.25) with the weight $w(y,N)$ defined by (3.2.26). Let us set

$$\tau^*_{N,d,p} := \begin{cases} 1, & d=0,\ 0<p<\infty, \\ N, & d=1,\ 0<p<1, \\ N/\log N, & d=1,\ p=1, \\ N^{1/p}, & d=1,\ 1<p<\infty, \end{cases} \tag{3.3.40}$$

$$\mathcal{E}_{d,p} := \begin{cases} \left[\dfrac{2\Gamma(p/2+1/2)}{\sqrt{\pi}\Gamma(p/2+1)}\right]^{1/p}, & d=0,\ 0<p<\infty, \\ \left[\dfrac{2\Gamma(p/2+1/2)}{\pi^{p+1/2}\Gamma(p/2+1)(1-p)}\right]^{1/p}, & d=1,\ 0<p<1, \\ 4/\pi^2, & d=1,\ p=1, \\ (2/\pi)^{1/p}\left(\int_0^\infty |(\sin t)/t|^p\,dt\right)^{1/p}, & d=1,\ 1<p<\infty. \end{cases} \tag{3.3.41}$$

Then the following theorem holds:

THEOREM 3.3.7. *For* $p \in (0,\infty)$*,*

$$\lim_{N\to\infty} \tau^*_{N,d,p}\left(\int_{-1}^{1} |y^{-d}P_{2N+d}(y)w(y,N)|^p\,dy\right)^{1/p} = \mathcal{E}_{d,p}. \tag{3.3.42}$$

Proof. First we split the integral

$$\begin{aligned}\int_{-1}^{1} |y^{-d}P_{2N+d}(y)w(y,N)|^p\,dy &= 2\int_0^1 |y^{-d}P_{2N+d}(y)w(y,N)|^p\,dy \\ &= 2I_{N,d,p} = 2\left(\int_0^{1-1/\sqrt{N}} + \int_{1-1/\sqrt{N}}^1\right) = 2(J_{N,d,p,1} + J_{N,d,p,2}).\end{aligned} \tag{3.3.43}$$

Next, it follows from the asymptotic formula (3.2.36) that

$$\begin{aligned} J_{N,d,p,1} &= ((2N+d-1)\pi/2)^{-dp} \\ &\quad\times \int_0^{1-1/\sqrt{N}} \left| y^{-d}\cos((2N+d-1)\pi y/2 - d\pi/2)\left(1+\frac{O(1)}{N(1-y^2)}\right)\right|^p dy \\ &= (1+O(N^{-1/2}))^p((2N+d-1)\pi/2)^{-dp} \end{aligned}$$

$$\begin{aligned}
&\times \int_0^{1-1/\sqrt{N}} |y^{-d}\cos((2N+d-1)\pi y/2 - d\pi/2)|^p\,dy \\
&= (1+O(N^{-1/2}))^p((2N+d-1)\pi/2)^{-dp} \\
&\times\left(\int_0^1 |y^{-d}\cos((2N+d-1)\pi y/2 - d\pi/2)|^p\,dy + O(N^{-1/2})\right). \qquad (3.3.44)
\end{aligned}$$

In addition, by inequality (3.2.42),

$$J_{N,d,p,2} = O(N^{-1/2})((2N+d-1)\pi/2)^{-dp}. \qquad (3.3.45)$$

Therefore, combining (3.3.43)–(3.3.45) we obtain

$$\begin{aligned}
I_{N,d,p} &= (1+O(N^{-1/2}))^p((2N+d-1)\pi/2)^{-dp} \\
&\times\left(\int_0^1 |y^{-d}\cos((2N+d-1)\pi y/2 - d\pi/2)|^p\,dy + O(N^{-1/2})\right). \qquad (3.3.46)
\end{aligned}$$

It remains to note that as $N\to\infty$,

$$\begin{aligned}
&\int_0^1 |y^{-d}\cos((2N+d-1)\pi y/2 - d\pi/2)|^p\,dy \\
&= (1+o(1))\begin{cases}
\dfrac{\Gamma(p/2+1/2)}{\sqrt{\pi}\,\Gamma(p/2+1)}, & d=0,\ 0<p<\infty, \\
\dfrac{\Gamma(p/2+1/2)}{\sqrt{\pi}\,\Gamma(p/2+1)(1-p)}, & d=1,\ 0<p<1, \\
(2/\pi)\log N, & d=1,\ p=1, \\
(\pi N)^{p-1}\int_0^\infty |(\sin t)/t|^p\,dt, & d=1,\ 1<p<\infty.
\end{cases} \qquad (3.3.47)
\end{aligned}$$

Indeed, in the case $d=0$, $0<p<\infty$ or $d=1$, $0<p<1$, we use Lemma 3.3.3; in the case of $d=1$, $p=1$, we use the asymptotic value of the Lebesgue constant; and the case of $d=1$, $1<p<\infty$ can be easily established by straightforward calculation. Thus (3.3.42) follows from (3.3.43), (3.3.46), and (3.3.47). ∎

3.3.5. Normalized Hermite polynomials on $(-\infty,\infty)$. Let us set

$$\tau^{**}_{N,d,p} := \begin{cases}
N^{-1/(2p)}, & d=0,\ 0<p<8/3, \\
N^{1-1/(2p)}, & d=1,\ 0<p<1, \\
N^{1/2}/\log N, & d=1,\ p=1, \\
N^{1/(2p)}, & d=1,\ 1<p<8/3,
\end{cases} \qquad (3.3.48)$$

$$\mathcal{H}_{d,p} := \begin{cases}
\left[\dfrac{2\Gamma(p/2+1/2)\Gamma(1-p/4)}{\Gamma(p/2+1)\Gamma(3/2-p/4)}\right]^{1/p}, & d=0,\ 0<p<8/3, \\
\left[\dfrac{\Gamma(p/2+1/2)\Gamma(1/2-p/2)\Gamma(1-p/4)}{2^{2p-1}\sqrt{\pi}\,\Gamma(p/2+1)\Gamma(3/2-3p/4)}\right]^{1/p}, & d=1,\ 0<p<1, \\
2/\pi, & d=1,\ p=1, \\
\left(\int_0^\infty |(\sin t)/t|^p\,dt\right)^{1/p}, & d=1,\ 1<p<8/3.
\end{cases} \qquad (3.3.49)$$

Then the following theorem holds:

THEOREM 3.3.8. *For $p \in (0, 8/3)$,*

$$\lim_{N\to\infty} \tau^{**}_{N,d,p} \left(\int_{\mathbb{R}} \left| \frac{e^{-y^2/2} H_{2N+d}}{y^d H^{(d)}_{2N+d}(0)} \right|^p dy \right)^{1/p} = \mathcal{H}_{d,p}. \tag{3.3.50}$$

REMARK 3.3.9. For $d = 0$ this result is proved in [3, Theorem 3]. We shall use this fact in the reasoning for $d = 1$.

The proof of Theorem 3.3.8 is based on three technical lemmas. In the first of them we discuss an estimate for the inverse of the function $\theta(\varphi) = 2\varphi + \sin(2\varphi)$.

LEMMA 3.3.10. *Let $\theta(\varphi) := 2\varphi + \sin(2\varphi) : [0, \pi/2] \to [0, \pi]$ and let $\varphi(\theta) : [0, \pi] \to [0, \pi/2]$ be the inverse of θ. Then for $p \in [1, \infty)$,*

$$\varphi'(\theta)\varphi^{-p}(\theta) - 4^{p-1}\theta^{-p} = O(\theta^{2-p}), \qquad \theta \in [0, \pi].$$

Proof. Note first that since $\theta'(\varphi) > 0$ on $[0, \pi/2)$, $\varphi(\theta)$ is a continuous and increasing function on $[0, \pi]$ and $\varphi(0) = 0$. Next, since $\theta(\varphi) = 4\varphi + O(\varphi^3)$, $\varphi \in [0, \pi/2]$, it is easy to conclude that

$$\varphi(\theta) = \theta/4 + O(\theta^3), \qquad \theta \in [0, \pi]. \tag{3.3.51}$$

Hence for $\theta \in [0, \pi]$,

$$\begin{aligned}
\frac{\varphi'(\theta)}{\varphi^p(\theta)} - \frac{4^{p-1}}{\theta^p} &= \frac{1}{\big(2 + 2\cos(2\varphi(\theta))\big)\varphi^p(\theta)} - \frac{4^{p-1}}{\theta^p} \\
&= \frac{1}{4\cos^2(\theta/4 + O(\theta^3))(\theta/4 + O(\theta^3))^p} - \frac{4^{p-1}}{\theta^p} \\
&= \frac{4^{p-1}}{\theta^p(1 + O(\theta^2))(1 + O(\theta^2))^p} - \frac{4^{p-1}}{\theta^p} = O(\theta^{2-p}). \ \blacksquare
\end{aligned}$$

Next, we find asymptotics of two integrals.

LEMMA 3.3.11. *Let $A_{d,p,q}$ be defined in* (3.3.3)*:*

(a) *For $N = 1, 2, \dots$, $d = 0, 1$, $p \in (0, 4)$, and $\delta \in [0, \pi/2)$, the integral*

$$I_{N,d,p}(\delta) := \left(\int_\delta^{\pi/2} |\sin((N + d/2 + 1/4)(\sin(2\varphi) - 2\varphi) + 3\pi/4)|^p \times \cos^{-dp}\varphi \sin^{1-p/2}\varphi \, d\varphi \right)^{1/p},$$

satisfies the limit property

$$\lim_{\delta\to 0+} \lim_{N\to\infty} \lambda_{N,d,p} I_{N,d,p}(\delta) = A^*_{d,p}, \tag{3.3.52}$$

where $\lambda_{N,d,p}$ is defined by (3.3.2) *and*

$$A^*_{d,p} := \begin{cases} A_{d,p,1-p/2}, & d = 0,\ 0 < p < 4 \text{ or } d = 1,\ 0 < p \le 1, \\ 2^{1-1/p} A_{d,p,1-p/2}, & d = 1,\ 1 < p < 4. \end{cases}$$

(b) *The following relation holds for $0 < p < 4$:*

$$\lim_{\delta\to 0+} \lim_{N\to\infty} \left(\int_\delta^{\pi/2} |\sin((N + d/2 + 1/4)(\sin(2\varphi) - 2\varphi) + 3\pi/4)|^p \sin^{1-p/2}\varphi \, d\varphi \right)^{1/p} = A_{0,p,1-p/2}.$$

Proof. To prove statement (a), we consider three cases.

Case 1. Let $d = 0$, $0 < p < 4$ or $d = 1$, $0 < p < 1$. Then

$$\begin{aligned}
I^p_{N,d,p}(\delta) &= \int_\delta^{\pi/2} |\cos((N + d/2 + 1/4)(2\varphi - \sin(2\varphi)) - \pi/4)|^p \cos^{-dp}\varphi \sin^{1-p/2}\varphi \, d\varphi \\
&= \int_0^\pi |\cos((N + d/2 + 1/4)\theta - \pi/4)|^p \\
&\quad \times (\cos\varphi(\theta))^{-dp}(\sin\varphi(\theta))^{1-p/2}\varphi'(\theta)\chi_{[0,\delta_1]}(\theta)\, d\theta, \qquad (3.3.53)
\end{aligned}$$

where $\varphi(\theta)$ is the inverse of $\theta(\varphi) = 2\varphi - \sin(2\varphi)$ and $\chi_{[0,\delta_1]}$ is the characteristic function of $[0, \delta_1]$, $\delta_1 := 2\delta - \sin(2\delta)$. Then using Lemma 3.3.3 for $g(y) = |\cos y|^p$, and

$$\gamma(\theta) = d\theta/2 + \theta/4 - \pi/4, \qquad f(\theta) = (\cos\varphi(\theta))^{-dp}(\sin\varphi(\theta))^{1-p/2}\varphi'(\theta)\chi_{[0,\delta_1]}(\theta),$$

we obtain from (3.3.53)

$$\begin{aligned}
\lim_{N\to\infty} I_{N,d,p}(\delta) &= (1/\pi)\int_0^\pi |\cos\theta|^p \, d\theta \int_{\delta_1}^\pi (\cos\varphi(\theta))^{-dp}(\sin\varphi(\theta))^{1-p/2}\varphi'(\theta)\, d\theta \\
&= (1/\pi)\int_0^\pi |\cos\theta|^p \, d\theta \int_\delta^{\pi/2} \cos^{-dp}\varphi \sin^{1-p/2}\varphi \, d\varphi.
\end{aligned}$$

This establishes (3.3.52).

Case 2. Let $d = 1$, $p = 1$. Using estimate (3.3.5), we obtain

$$\begin{aligned}
I_{N,1,1}(\delta) &= \int_0^{\pi/2-\delta} |\sin(N(2\varphi + \sin(2\varphi)) + (3/4)(2\varphi + \sin(2\varphi)))| \sin^{-1}\varphi \cos^{1/2}\varphi \, d\varphi \\
&= \int_0^{\pi/2-\delta} |\sin(N(2\varphi + \sin(2\varphi)))| \sin^{-1}\varphi \cos^{1/2}\varphi \, d\varphi + O(1) \\
&= \int_0^{\pi/2-\delta} |\sin(N(2\varphi + \sin(2\varphi)))| \varphi^{-1} \cos^{1/2}\varphi \, d\varphi + O(1). \qquad (3.3.54)
\end{aligned}$$

Next making the substitution $\theta = 2\varphi + \sin(2\varphi)$ in the last integral in (3.3.54) and applying Lemma 3.3.10 for $p = 1$, we obtain

$$\begin{aligned}
I_{N,1,1}(\delta) &= \int_0^{\pi-\delta_1} |\sin N\theta| \varphi'(\theta)(\cos\varphi(\theta))^{1/2}/\varphi(\theta)\, d\theta + O(1) \\
&= \int_0^{\pi-\delta_1} |\sin N\theta| \theta^{-1}(\cos\varphi(\theta))^{1/2}\, d\theta + O(1) \\
&= \int_0^{\mu N} |\sin\theta| \theta^{-1}(\cos\varphi(\theta/(2N)))^{1/2}\, d\theta + O(1), \qquad (3.3.55)
\end{aligned}$$

where $\delta_1 := 2\delta - \sin(2\delta)$ and μ is any number from $(0, \pi - \delta_1]$. Since the inverse function $\varphi(\theta)$ is increasing on $[0, \pi]$, from (3.3.55) we obtain

$$\begin{aligned}
&\liminf_{N\to\infty} (\log N)^{-1}\left(\cos^{1/2}(\varphi(\mu/2))\int_0^{\mu N} |\sin\theta|\theta^{-1}\, d\theta + O(1)\right) \\
&\qquad \le \liminf_{N\to\infty} (\log N)^{-1} I_{N,1,1}(\delta) \le \limsup_{N\to\infty} (\log N)^{-1} I_{N,1,1}(\delta) \\
&\qquad \le \limsup_{N\to\infty} (\log N)^{-1}\left(\int_0^{\mu N} |\sin\theta|\theta^{-1}\, d\theta + O(1)\right). \qquad (3.3.56)
\end{aligned}$$

Then the relations

$$(2/\pi)\cos^{1/2}(\varphi(\mu/2)) \le \liminf_{N\to\infty}(\log N)^{-1} I_{N,1,1}(\delta) \le \limsup_{N\to\infty}(\log N)^{-1} I_{N,1,1}(\delta) \le 2/\pi \tag{3.3.57}$$

follow from (3.3.7) and (3.3.56). Finally, letting $\mu \to 0$ in (3.3.57), we arrive at

$$\lim_{N\to\infty}(\log N)^{-1} I_{N,1,1}(\delta) = 2/\pi,$$

since $\varphi(0) = 0$. This proves (3.3.52).

Case 3. Let $d = 1$, $1 < p < 4$. We first note that by Hölder's inequality and estimate (3.3.5), we obtain

$$\begin{aligned} I_{N,1,p}(\delta) &= \bigg(\int_0^{\pi/2-\delta} |\sin(N(2\varphi+\sin(2\varphi)) + (3/4)(2\varphi+\sin(2\varphi)))|^p \\ &\qquad\qquad \times \sin^{-p}\varphi\cos^{1-p/2}\varphi\,d\varphi\bigg)^{1/p} \\ &= \bigg(\int_0^{\pi/2-\delta} |\sin(N(2\varphi+\sin(2\varphi)))|^p \sin^{-p}\varphi\cos^{1-p/2}\varphi\,d\varphi\bigg)^{1/p} + O(1) \\ &= \bigg(\int_0^{\pi/2-\delta} |\sin(N(2\varphi+\sin(2\varphi)))|^p \varphi^{-p}\cos^{1-p/2}\varphi\,d\varphi\bigg)^{1/p} + O(1). \end{aligned} \tag{3.3.58}$$

Making the substitution $\theta = 2\varphi + \sin(2\varphi)$ and applying Lemma 3.3.10, from (3.3.58) we have

$$\begin{aligned} I_{N,1,p}(\delta) &= \bigg(4^{p-1}\int_0^{\pi-\delta_1} |\sin N\theta|^p \theta^{-p}(\cos\varphi(\theta))^{1-p/2}\,d\theta\bigg)^{1/p} \\ &\quad + O\bigg(\bigg(\int_0^{\pi-\delta_1} |\sin N\theta|^p\theta^{2-p}(\cos\varphi(\theta))^{1-p/2}\,d\theta\bigg)^{1/p}\bigg) + O(1), \end{aligned} \tag{3.3.59}$$

where $\delta_1 := 2\delta - \sin(2\delta)$.

Next we show that

$$R_N := \int_0^{\pi-\delta_1} |\sin N\theta|^p\theta^{2-p}(\cos\varphi(\theta))^{1-p/2}\,d\theta = o(N^{p-1}) \tag{3.3.60}$$

as $N \to \infty$. Indeed, for $1 < p < 3$,

$$R_N \le C\int_0^{\pi-\delta_1} |\sin N\theta|^p\theta^{2-p}\,d\theta \le C\int_0^{\pi-\delta_1}\theta^{2-p}\,d\theta = O(1), \tag{3.3.61}$$

since $\varphi[0,\pi-\delta_1] = [0,\pi/2-\delta]$. If $p = 3$, then

$$R_N \le C\int_0^{\pi-\delta_1} |\sin N\theta|^3\theta^{-1}\,d\theta \le C\int_0^{\pi-\delta_1} |\sin N\theta|/\theta\,d\theta \le C\log N. \tag{3.3.62}$$

Finally, if $p > 3$, then

$$\begin{aligned} R_N &\le C\int_0^{\pi-\delta_1} |\sin N\theta|^p\theta^{2-p}\,d\theta = CN^{p-3}\bigg(\int_0^1 |\sin\theta|^p\theta^{2-p}\,d\theta + \int_1^{N(\pi-\delta_1)} |\sin\theta|^p\theta^{2-p}\,d\theta\bigg) \\ &\le CN^{p-3}\bigg(\int_0^1 |\sin\theta|^p\theta^{-p}\,d\theta + \int_1^\infty \theta^{2-p}\,d\theta\bigg) \le CN^{p-3}. \end{aligned} \tag{3.3.63}$$

Therefore (3.3.60) follows from (3.3.61)–(3.3.63).

It remains to compute the principal term in (3.3.59). For any $\alpha \in (1/p, 1)$ we obtain

$$\begin{aligned} J_{N,p}(\delta) &:= 4^{p-1}\int_0^{\pi-\delta_1} |\sin N\theta|^p\theta^{-p}(\cos\varphi(\theta))^{1-p/2}\,d\theta \\ &= 4^{p-1}N^{p-1}\left(\int_0^{N^\alpha} + \int_{N^\alpha}^{N(\pi-\delta_1)}\right)|\sin\theta|^p\theta^{-p}(\cos\varphi(\theta/N))^{1-p/2}\,d\theta \\ &= 4^{p-1}N^{p-1}(J_{N,p,1}+J_{N,p,2}), \end{aligned} \tag{3.3.64}$$

where

$$J_{N,p,2} \le N^{1-\alpha p}\int_0^{\pi-\delta_1}(\cos\varphi(\theta))^{1-p/2}\,d\theta \le CN^{1-\alpha p} = o(1) \tag{3.3.65}$$

as $N \to \infty$. Further,

$$J_{N,p,1} = \int_0^\infty |\sin\theta|^p\theta^{-p}\,d\theta + \int_0^{N^\alpha}|\sin\theta|^p\theta^{-p}F^*_{N,p}(\theta)\,d\theta + o(1), \tag{3.3.66}$$

as $N \to \infty$, where $F^*_{N,p} := (\cos\varphi(\theta/N))^{1-p/2} - 1$, and by (3.3.51),

$$\begin{aligned} \max_{\theta\in[0,N^\alpha]}|F^*_{N,p}(\theta)| &\le C\max_{\theta\in[0,N^\alpha]}|(\cos\varphi(\theta/N))^{|1-p/2|}-1| \\ &\le C\varphi^2(N^{\alpha-1}) \le CN^{2\alpha-2} = o(1) \end{aligned} \tag{3.3.67}$$

as $N \to \infty$. Then (3.3.66) and (3.3.67) imply

$$J_{N,p,1} = \int_0^\infty |\sin\theta|^p\theta^{-p}\,d\theta + o(1), \tag{3.3.68}$$

as $N \to \infty$. It follows from (3.3.58)–(3.3.60), (3.3.64), (3.3.65), and (3.3.68) that

$$\lim_{N\to\infty} N^{1/p-1}I_{N,1,p}(\delta) = 4^{1-1/p}\left(\int_0^\infty |\sin\theta|^p\theta^{-p}\,d\theta\right)^{1/p}.$$

Hence (3.3.52) follows for $p \in (1,4)$.

This completes the proof of statement (a) of Lemma 3.3.11. The proof of statement (b) follows that of Case 1 above. ∎

We also need an asymptotic formula for Hermite polynomials.

Lemma 3.3.12. *Let $\varepsilon_1 \in (0,\pi/2)$ be a fixed number. For $N = 1,2,\dots$ and $\varepsilon_1 \le \varphi \le \pi/2$,*

$$\begin{aligned} &e^{-(4N+2d+1)(\cos^2\varphi)/2}H_{2N+d}((4N+2d+1)^{1/2}\cos\varphi)/H^{(d)}_{2N+d}(0) \\ &\quad= (-1)^N(1+o(1))(4N)^{-d/2} \\ &\qquad\times\big(\sin((N+d/2+1/4)(\sin(2\varphi)-2\varphi)+3\pi/4)(\sin\varphi)^{-1/2}+O(N^{-1})\big), \end{aligned} \tag{3.3.69}$$

as $N \to \infty$, where $o(1)$ and $O(N^{-1})$ in (3.3.69) are independent of φ.

Proof. Using the Plancherel–Rotach asymptotic representation for Hermite polynomials [48, Theorem 8.22.9(a)] and taking account of $H^{(d)}_{2N+d}(0) = (-1)^N 2^d(2N+d)!/N!$, we obtain

$$\begin{aligned} &e^{-(4N+2d+1)(\cos^2\varphi)/2}H_{2N+d}((4N+2d+1)^{1/2}\cos\varphi)/H^{(d)}_{2N+d}(0) \\ &\quad= (-1)^N 2^{N+1/4-d/2}N!/(\pi^{1/4}(2N+d)!^{3/2}) \\ &\qquad\times\big(\sin((N+d/2+1/4)(\sin(2\varphi)-2\varphi)+3\pi/4)(\sin\varphi)^{-1/2}+O(N^{-1})\big), \end{aligned} \tag{3.3.70}$$

where $\varphi \in (\varepsilon_1, \pi - \varepsilon_1)$, $\varepsilon_1 \in (0, \pi/2)$. Then (3.3.69) follows from (3.3.70), by Stirling's formula. ■

Proof of Theorem 3.3.8. Throughout the proof we set $\varepsilon_1 := \arccos(1-\varepsilon)$. The following relation was proved in [3, Theorem 3] in a more general setting ($0 < p < 8/3$):

$$\lim_{N\to\infty} (4N)^{d/2-1/(2p)} \left(\int_0^{\infty} \left| \frac{e^{-y^2/2} H_{2N+d}(y)}{H_{2N+d}^{(d)}(0)} \right|^p dy \right)^{1/p} = 2^{-1/p} \mathcal{H}_{0,p}, \tag{3.3.71}$$

where $\mathcal{H}_{d,p}$ is defined by (3.3.49). In particular, relations (3.3.50) and (3.3.71) are equivalent for $d = 0$. So it remains to prove (3.3.50) for $d = 1$.

For any fixed $\varepsilon \in (0,1)$, $d = 0, 1$, and $p \in (0, 4)$, by Lemma 3.3.12 we obtain

$$\begin{aligned}
&(4N)^{d/2-1/(2p)} \left(\int_0^{\sqrt{4N+2d+1}\,(1-\varepsilon)} \left| \frac{e^{-y^2/2} H_{2N+d}(y)}{H_{2N+d}^{(d)}(0)} \right|^p dy \right)^{1/p} \\
&\quad = (1+o(1))(4N)^{d/2} \\
&\qquad \times \left(\int_{\varepsilon_1}^{\pi/2} \left| \frac{e^{-(4N+2d+1)(\cos^2\varphi)/2} H_{2N+d}((4N+2d+1)^{1/2}\cos\varphi)}{H_{2N+d}^{(d)}(0)} \right|^p \sin\varphi \, d\varphi \right)^{1/p} \\
&\quad = (1+o(1)) \Bigg(\int_{\varepsilon_1}^{\pi/2} \Bigg| \frac{\sin\big((N+d/2+1/4)(\sin(2\varphi)-2\varphi)+3\pi/4\big)}{(\sin\varphi)^{1/2}} \\
&\qquad\qquad + O(N^{-1}) \Bigg|^p \sin\varphi \, d\varphi \Bigg)^{1/p}
\end{aligned} \tag{3.3.72}$$

as $N \to \infty$. Then (3.3.72) shows that

$$\begin{aligned}
&(4N)^{d/2-1/(2p)} \left(\int_0^{\sqrt{4N+2d+1}\,(1-\varepsilon)} \left| \frac{e^{-y^2/2} H_{2N+d}(y)}{H_{2N+d}^{(d)}(0)} \right|^p dy \right)^{1/p} \\
&\quad = (1+o(1)) \left(\int_{\varepsilon_1}^{\pi/2} |\sin((N+d/2+1/4)(\sin(2\varphi)-2\varphi)+3\pi/4)|^p (\sin\varphi)^{1-p/2} \, d\varphi \right)^{1/p} \\
&\qquad + O(N^{-1})
\end{aligned}$$

if $p \in [1, 4)$, and

$$\begin{aligned}
&(4N)^{d/2-1/(2p)} \left(\int_0^{\sqrt{4N+2d+1}\,(1-\varepsilon)} \left| \frac{e^{-y^2/2} H_{2N+d}(y)}{H_{2N+d}^{(d)}(0)} \right|^p dy \right)^{1/p} \\
&\qquad = (1+o(1)) \Bigg(\int_{\varepsilon_1}^{\pi/2} |\sin((N+d/2+1/4)(\sin(2\varphi)-2\varphi)+3\pi/4)|^p (\sin\varphi)^{1-p/2} \, d\varphi \\
&\qquad\quad + O(N^{-p}) \Bigg)^{1/p}
\end{aligned}$$

if $p \in (0, 1)$. Hence by statement (b) of Lemma 3.3.11,

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \lim_{N\to\infty} (4N)^{d/2-1/(2p)} \left(\int_0^{\sqrt{4N+2d+1}\,(1-\varepsilon)} \left| \frac{e^{-y^2/2} H_{2N+d}(y)}{H_{2N+d}^{(d)}(0)} \right|^p dy \right)^{1/p} \\
&\qquad\qquad = A_{0,p,1-p/2} = 2^{-1/p} \mathcal{H}_{0,p}, \qquad 0 < p < 4.
\end{aligned} \tag{3.3.73}$$

Next we deduce from (3.3.71) and (3.3.73) that

$$\lim_{\varepsilon\to 0}\lim_{N\to\infty}(4N)^{d/2-1/(2p)}\bigg(\int_{\sqrt{4N+2d+1}\,(1-\varepsilon)}^{\infty}\bigg|\frac{e^{-y^2/2}H_{2N+d}(y)}{H^{(d)}_{2N+d}(0)}\bigg|^p\,dy\bigg)^{1/p}=0,\quad 0<p<8/3. \tag{3.3.74}$$

Since $\sup_{2\le N\le\infty}\lambda_{N,d,p}<\infty$, where $\lambda_{N,d,p}$ is defined by (3.3.2), from (3.3.74) we obtain

$$\lim_{\varepsilon\to 0}\lim_{N\to\infty}(4N)^{d-1/(2p)}\lambda_{N,d,p}\bigg(\int_{\sqrt{4N+2d+1}\,(1-\varepsilon)}^{\infty}\bigg|\frac{e^{-y^2/2}H_{2N+d}(y)}{y^dH^{(d)}_{2N+d}(0)}\bigg|^p\,dy\bigg)^{1/p}=0,\quad 0<p<8/3.$$

Hence for $0<p<8/3$,

$$\begin{aligned}
&2\lim_{N\to\infty}(4N)^{dp-1/2}\lambda^p_{N,d,p}\int_0^{\infty}\bigg|\frac{e^{-y^2/2}H_{2N+d}(y)}{y^dH^{(d)}_{2N+d}(0)}\bigg|^p\,dy\\
&\quad=2\lim_{\varepsilon\to 0}\lim_{N\to\infty}(4N)^{dp-1/2}\lambda^p_{N,d,p}\int_0^{\sqrt{4N+2d+1}\,(1-\varepsilon)}\bigg|\frac{e^{-y^2/2}H_{2N+d}(y)}{y^dH^{(d)}_{2N+d}(0)}\bigg|^p\,dy\\
&\qquad+2\lim_{\varepsilon\to 0}\lim_{N\to\infty}(4N)^{dp-1/2}\lambda^p_{N,d,p}\int_{\sqrt{4N+2d+1}\,(1-\varepsilon)}^{\infty}\bigg|\frac{e^{-y^2/2}H_{2N+d}(y)}{y^dH^{(d)}_{2N+d}(0)}\bigg|^p\,dy\\
&\quad=2\lim_{\varepsilon\to 0}\lim_{N\to\infty}(4N)^{dp-1/2}\lambda^p_{N,d,p}\int_0^{\sqrt{4N+2d+1}\,(1-\varepsilon)}\bigg|\frac{e^{-y^2/2}H_{2N+d}(y)}{y^dH^{(d)}_{2N+d}(0)}\bigg|^p\,dy,
\end{aligned}\tag{3.3.75}$$

if the last double limit exists. We shall show below that it exists by direct computation. Note that

$$N^{d-1/(2p)}\lambda_{N,d,p}=\tau^{**}_{N,d,p}. \tag{3.3.76}$$

To finish the proof of Theorem 3.3.8 we consider two cases.

Case 1. Let $d=0$, $0<p<8/3$ or $d=1$, $0<p<1$. Similarly to (3.3.72), we deduce by Lemma 3.3.12 that

$$\begin{aligned}
&(4N)^{d-1/(2p)}\bigg(\int_0^{\sqrt{4N+2d+1}\,(1-\varepsilon)}\bigg|\frac{e^{-y^2/2}H_{2N+d}(y)}{y^dH^{(d)}_{2N+d}(0)}\bigg|^p\,dy\bigg)^{1/p}\\
&=(1+o(1))\bigg(\int_{\varepsilon_1}^{\pi/2}\bigg|\frac{\sin((N+d/2+1/4)(\sin(2\varphi)-2\varphi)+3\pi/4)}{(\sin\varphi)^{1/2}(\cos\varphi)^d}+\frac{O(N^{-1})}{(\cos\varphi)^d}\bigg|^p\sin\varphi\,d\varphi\bigg)^{1/p}
\end{aligned}$$

as $N\to\infty$. This shows that

$$\begin{aligned}
&(4N)^{d-1/(2p)}\bigg(\int_0^{\sqrt{4N+2d+1}\,(1-\varepsilon)}\bigg|\frac{e^{-y^2/2}H_{2N+d}(y)}{y^dH^{(d)}_{2N+d}(0)}\bigg|^p\,dy\bigg)^{1/p}\\
&\qquad=(1+o(1))\bigg(\int_{\varepsilon_1}^{\pi/2}|\sin((N+d/2+1/4)(\sin(2\varphi)-2\varphi)+3\pi/4)|^p\\
&\qquad\qquad\times(\cos\varphi)^{-dp}(\sin\varphi)^{1-p/2}\,d\varphi\bigg)^{1/p}+O(N^{-1})
\end{aligned}$$

if $p \in [1, 8/3)$, and

$$\begin{aligned}
(4N)^{d-1/(2p)} & \Biggl(\int_0^{\sqrt{4N+2d+1}\,(1-\varepsilon)} \left| \frac{e^{-y^2/2} H_{2N+d}(y)}{y^d H^{(d)}_{2N+d}(0)} \right|^p dy \Biggr)^{1/p} \\
&= (1+o(1)) \Biggl(\int_{\varepsilon_1}^{\pi/2} |\sin((N+d/2+1/4)(\sin(2\varphi)-2\varphi)+3\pi/4)|^p \\
&\qquad \times (\cos\varphi)^{-dp} (\sin\varphi)^{1-p/2}\, d\varphi + O(N^{-p}) \Biggr)^{1/p}
\end{aligned}$$

if $p \in (0,1)$. Hence by Lemma 3.3.11,

$$\begin{aligned}
2 \lim_{\varepsilon\to 0} \lim_{N\to\infty} (4N)^{d-1/(2p)} & \Biggl(\int_0^{\sqrt{4N+2d+1}\,(1-\varepsilon)} \left| \frac{e^{-y^2/2} H_{2N+d}(y)}{y^d H^{(d)}_{2N+d}(0)} \right|^p dy \Biggr)^{1/p} \\
&= \left(\frac{\Gamma(p/2+1/2)\Gamma(1/2-dp/2)\Gamma(1-p/4)}{\sqrt{\pi}\,\Gamma(p/2+1)\Gamma(3/2-p/4-dp/2)} \right)^{1/p}. \qquad (3.3.77)
\end{aligned}$$

Thus (3.3.50) follows from (3.3.75) and (3.3.77).

Note that within the argument for Case 1, we actually provide a new proof of (3.3.71) when $d = 0$, by using (3.3.71) for $d = 1$ and Lemmas 3.3.11 and 3.3.12.

Case 2. Let $d = 1$, $1 \le p < 8/3$. In this case we set $v_N := (4N+3)^{-1-\alpha}$, where $\alpha \in (1/2, 3/5)$, and then split the integral over $[0, \sqrt{4N+3}\,(1-\varepsilon)]$ into two integrals:

$$\begin{aligned}
(4N)^{p-1/2} & \lambda^p_{N,1,p} \int_0^{\sqrt{4N+3}\,(1-\varepsilon)} \left| \frac{e^{-y^2/2} H_{2N+1}(y)}{y H'_{2N+1}(0)} \right|^p dy \\
&= (4N)^{p-1/2} \lambda^p_{N,1,p} \Biggl(\int_0^{v_N\sqrt{4N+3}} + \int_{v_N\sqrt{4N+3}}^{\sqrt{4N+3}\,(1-\varepsilon)} \Biggr) \left| \frac{e^{-y^2/2} H_{2N+1}(y)}{y H'_{2N+1}(0)} \right|^p dy \\
&= I_{N,1} + I_{N,2}. \qquad (3.3.78)
\end{aligned}$$

Then choosing $\gamma_N = (4N+3)^{-1/2+\alpha}$, $\delta_N = \gamma_N^2/N \le CN^{-2+2\alpha}$, and recalling that $\beta_N = \sqrt{4N+3}$ (see (3.2.43)), $N = 1, 2, \dots$, we see that $\{H_{2N+1}(y)/H'_{2N+1}(0)\}_{N=1}^\infty \in \mathbb{P}_1(\beta, b^{-1}\gamma, \delta)$, by Proposition 3.2.6. Therefore using inequality (3.1.3), we obtain

$$\sup_{|z|\le(4N+3)^{-\alpha}} \left| \frac{H_{2N+1}(z/\sqrt{4N+3})}{(z/\sqrt{4N+3}) H'_{2N+1}(0)} \right| \le \sup_{|z|\le(4N+3)^{-1/2+\alpha}} \left| \frac{H_{2N+1}(z/\sqrt{4N+3})}{(z/\sqrt{4N+3}) H'_{2N+1}(0)} \right| \le C.$$

Hence

$$\begin{aligned}
I_{N,1} &\le \frac{(4N)^{p-1/2}\lambda^p_{N,1,p}}{\sqrt{4N+3}} \int_0^{(4N+3)^{-\alpha}} \left| \frac{H_{2N+1}(z/\sqrt{4N+3})}{(z/\sqrt{4N+3}) H'_{2N+1}(0)} \right|^p dz \\
&\le C^p N^{p-1-\alpha} \lambda^p_{N,1,p} = O(N^{-\alpha}) = o(1), \qquad 1 \le p < 8/3, \qquad (3.3.79)
\end{aligned}$$

as $N \to \infty$. Further by Lemma 3.3.12,

$$\begin{aligned}
I_{N,2}^{1/p} &= (1+o(1))(4N)^{1/2}\lambda_{N,1,p} \\
&\quad\times\bigg(\int_{\varepsilon_1}^{\arccos v_N}\bigg|\frac{e^{-(4N+3)(\cos^2\varphi)/2}H_{2N+1}((4N+3)^{1/2}\cos\varphi)}{\cos\varphi H'_{2N+1}(0)}\bigg|^p\sin\varphi\,d\varphi\bigg)^{1/p} \\
&= (1+o(1))\lambda_{N,1,p} \\
&\quad\times\bigg(\int_{\varepsilon_1}^{\arccos v_N}\bigg|\frac{\sin\big((N+3/4)(\sin(2\varphi)-2\varphi)+3\pi/4\big)}{(\sin\varphi)^{1/2}\cos\varphi}+\frac{O(N^{-1})}{\cos\varphi}\bigg|^p\sin\varphi\,d\varphi\bigg)^{1/p} \\
&= (1+o(1))\lambda_{N,1,p}\bigg(\int_{\varepsilon_1}^{\arccos v_N}\big|\sin\big((N+3/4)(\sin(2\varphi)-2\varphi)+3\pi/4\big)\big|^p \\
&\qquad\qquad\times(\cos\varphi)^{-p}(\sin\varphi)^{1-p/2}\,d\varphi\bigg)^{1/p}+O(M_N),
\end{aligned}\tag{3.3.80}$$

where for $1\le p<8/3$,

$$\begin{aligned}
M_N &:= \lambda_{N,1,p}N^{-1}\bigg(\int_{\varepsilon_1}^{\arccos v_N}\cos^{-p}\varphi\sin\varphi\,d\varphi\bigg)^{1/p} \\
&\le C\lambda_{N,1,p}N^{-1}\begin{Bmatrix} v_N^{1/p-1}, & p>1, \\ \log N, & p=1, \end{Bmatrix}\le CN^{(1+\alpha)(1-1/p)-1}\log N=o(1)
\end{aligned}\tag{3.3.81}$$

as $N\to\infty$. Next,

$$\begin{aligned}
&\lambda_{N,1,p}\bigg(\int_{\varepsilon_1}^{\arccos v_N}\big|\sin\big((N+3/4)(\sin(2\varphi)-2\varphi)+3\pi/4\big)\big|^p(\cos\varphi)^{-p}(\sin\varphi)^{1-p/2}\,d\varphi\bigg)^{1/p} \\
&= \lambda_{N,1,p}\bigg(\int_{\varepsilon_1}^{\pi/2}\big|\sin\big((N+3/4)(\sin(2\varphi)-2\varphi)+3\pi/4\big)\big|^p(\cos\varphi)^{-p}(\sin\varphi)^{1-p/2}\,d\varphi\bigg)^{1/p} \\
&\quad+O(Q_N),
\end{aligned}\tag{3.3.82}$$

where by Lemma 3.3.10,

$$\begin{aligned}
Q_N &:= \lambda_{N,1,p}\bigg(\int_{\arccos v_N}^{\pi/2}\big|\sin\big((N+3/4)(\sin(2\varphi)-2\varphi)+3\pi/4\big)\big|^p(\cos\varphi)^{-p} \\
&\qquad\qquad\times(\sin\varphi)^{1-p/2}\,d\varphi\bigg)^{1/p} \\
&= \lambda_{N,1,p}\bigg(\int_0^{\arcsin v_N}\big|\sin\big((N+3/4)(\sin(2\varphi)+2\varphi)\big)\big|^p(\sin\varphi)^{-p}(\cos\varphi)^{1-p/2}\,d\varphi\bigg)^{1/p} \\
&\le C_1\lambda_{N,1,p}\bigg(\int_0^{CN^{-1-\alpha}}|\sin(4N+3)\theta|^p\theta^{-p}\,d\theta\bigg)^{1/p} \\
&\le C_2\lambda_{N,1,p}N^{1-1/p}\bigg(\int_0^{CN^{-\alpha}}|\sin\theta|^p\theta^{-p}\,d\theta\bigg)^{1/p}\le C_3N^{-\alpha}=o(1)
\end{aligned}\tag{3.3.83}$$

as $N\to\infty$. Therefore collecting (3.3.78)–(3.3.83), we arrive at

$$(4N)^{1-1/(2p)}\lambda_{N,1,p}\bigg(2\int_0^{\sqrt{4N+3}\,(1-\varepsilon)}\bigg|\frac{e^{-y^2/2}H_{2N+1}(y)}{yH'_{2N+1}(0)}\bigg|^p\,dy\bigg)^{1/p}$$

$$
\begin{aligned}
&= (1+o(1))\lambda_{N,1,p}\bigg(2\int_{\varepsilon_1}^{\pi/2}\big|\sin\big((N+3/4)(\sin(2\varphi)-2\varphi)+3\pi/4\big)\big|^p \\
&\qquad\qquad\qquad\qquad \times(\cos\varphi)^{-p}(\sin\varphi)^{1-p/2}\,d\varphi\bigg)^{1/p}+o(1) \qquad (3.3.84)
\end{aligned}
$$

as $N\to\infty$. Finally, (3.3.50) follows from (3.3.75), (3.3.76) (3.3.84), and Lemma 3.3.11.

This completes the proof of Theorem 3.3.8. ∎

3.3.6. Normalized Williams–Apostol polynomials on $(-\infty,\infty)$**.** The following theorem holds:

THEOREM 3.3.13. *For* $\alpha\ge 0$ *and* $p\in((2\alpha+d)^{-1},\infty)$,

$$
\begin{aligned}
&\lim_{N\to\infty}\tau^*_{N,d,p}\bigg(\int_{\mathbb{R}}\bigg|\frac{(1+y^2)^{-(N+d/2+\alpha)}A_{2N+d}(y)}{y^dA^{(d)}_{2N+d}(0)}\bigg|^p\,dy\bigg)^{1/p}\\
&=\lim_{N\to\infty}\tau^*_{N,d,p}\bigg(\int_{\mathbb{R}}\bigg|\frac{(1+y^2)^{-(N+d/2+\alpha+1/2)}W_{2N+d}(y)}{y^dW^{(d)}_{2N+d}(0)}\bigg|^p\,dy\bigg)^{1/p}=\mathcal{D}_{d,p,\alpha}, \qquad (3.3.85)
\end{aligned}
$$

where $\tau^*_{N,d,p}$ *is defined in* (3.3.40) *and*

$$
\mathcal{D}_{d,p,\alpha}:=\begin{cases}
\left[\dfrac{\Gamma(p/2+1/2)\Gamma(\alpha p-1/2)}{\Gamma(p/2+1)\Gamma(\alpha p)}\right]^{1/p}, & d=0,\ 0<p<\infty,\\
\left[\dfrac{\Gamma(p/2+1/2)\Gamma(1/2-p/2)\Gamma(\alpha p+dp/2-1/2)}{2^p\sqrt{\pi}\,\Gamma(p/2+1)\Gamma(\alpha p)}\right]^{1/p}, & \\
 & d=1,\ 0<p<1,\\
2/\pi, & d=1,\ p=1,\\
\left(\int_0^\infty|(\sin t)/t|^p\,dt\right)^{1/p}, & d=1,\ 1<p<\infty.
\end{cases} \qquad (3.3.86)
$$

Proof. By definitions (3.2.55) and (3.2.58) for the Williams–Apostol polynomials of the first kind,

$$
\begin{aligned}
\int_{\mathbb{R}}\bigg|\frac{(1+y^2)^{-(N+d/2+\alpha)}A_{2N+d}(y)}{y^dA^{(d)}_{2N+d}(0)}\bigg|^p\,dy&=(2N+d)^{-dp}\int_{\mathbb{R}}\bigg|\frac{\cos((2N+d)\operatorname{arccot}y)}{y^d(1+y^2)^\alpha}\bigg|^p\,dy\\
&=2(2N+d)^{-dp}\int_0^{\pi/2}\frac{|\cos(2N+d)\theta|^p(\sin\theta)^{2\alpha p+dp-2}}{\cos^{dp}\theta}\,d\theta. \qquad (3.3.87)
\end{aligned}
$$

Applying Lemma 3.3.1 for $\lambda=0$, $q=2\alpha p+dp-2$ to the integral $F^p_{N,d,p,2\alpha p+dp-2,0}$ on the last right-hand side of (3.3.87), we establish the theorem in this case.

Similarly, by (3.2.56) and (3.2.66), we obtain

$$
\begin{aligned}
&\int_{\mathbb{R}}\bigg|\frac{(1+y^2)^{-(N+d/2+\alpha+1/2)}W_{2N+d}(y)}{y^dW^{(d)}_{2N+d}(0)}\bigg|^p\,dy\\
&\qquad=2(2N+d+1)^{-dp}\int_0^{\pi/2}\frac{|\sin(2N+d+1)\theta|^p(\sin\theta)^{2\alpha p+dp-2}}{\cos^{dp}\theta}\,d\theta\\
&\qquad=2(2N+d+1)^{-dp}F^p_{N,d,p,2\alpha p+dp-2,1}.
\end{aligned}
$$

Then by Lemma 3.3.1 for $\lambda=1$, $q=2\alpha p+dp-2$, we arrive at (3.3.85). ∎

4. Pointwise asymptotic relations between the interpolation error and zeta functions

In this chapter we present pointwise asymptotic formulae for the zeta functions. Here and subsequently, k denotes an index that takes the values of 1 or 2. We also recall that d takes the values of 0 or 1. In addition, throughout the section we use the following notation:

$$h_d(t) := \begin{cases} \cosh t, & d = 0, \\ \sinh t, & d = 1, \end{cases} \tag{4.0.1}$$

$$D_{d,k,N} := \{s \in \mathbb{C} : 2N - 2 > \operatorname{Re} s > -2 + kd\},$$

$$\mathcal{E}^*_{N,\tau} := \delta_N + \gamma_N^\tau e^{-\gamma_N}. \tag{4.0.2}$$

Note that if $\lim_{N\to\infty} \delta_N = 0$ and $\lim_{N\to\infty} \gamma_N = \infty$, then for any $\tau \in \mathbb{R}$,

$$\lim_{N\to\infty} \mathcal{E}^*_{N,\tau} = 0. \tag{4.0.3}$$

4.1. Estimates for integrals. Here, we establish estimates for some integrals that will serve as remainder terms in asymptotic relations. In addition, we show that these integrals are analytic functions of a parameter s.

THEOREM 4.1.1. *Let $\{P_{2N+d}\}_{N=1}^\infty \in \mathbb{P}_d(\beta, \gamma, \delta)$ for some sequences $\beta = \{\beta_N\}_{N=1}^\infty$, $\gamma = \{\gamma_N\}_{N=1}^\infty$ and $\delta = \{\delta_N\}_{N=1}^\infty$, satisfying* (3.1.1) (*see Definition* 3.1.1). *Then there exists $N_0 \in \mathbb{N}$ such that:*

(a) *For $s \in D_{d,k,N}$, $N \ge N_0$, and $y \in \mathbb{R}$,*

$$|\Delta_{2N+d,k,s}(y)| := \left| \int_0^\infty \frac{t^{s-1}}{1 + t^2/(\beta_N y)^2} \left(\frac{i^{kd}}{\beta_N^{kd} P_{2N+d}^k(it/\beta_N)} - \frac{1}{h_d^k(t)} \right) dt \right|$$
$$\le C(s)\mathcal{E}^*_{N,\operatorname{Re} s-(k-1)d}, \tag{4.1.1}$$

where $C(s)$ is independent of y and N, and for any fixed $r > 0$ and $\varepsilon > 0$,

$$\sup_{-2+kd+\varepsilon \le \operatorname{Re} s \le r} C(s) < \infty. \tag{4.1.2}$$

(b) *For every $y \in \mathbb{R} \setminus \{0\}$ and a fixed $N \ge N_0$, the function $\Delta_{2N+d,k,s}(y)$ is analytic in s on $D_{d,k,N}$.*

As an immediate corollary of Theorem 4.1.1(a) and relation (4.0.3), we obtain the following result:

COROLLARY 4.1.2. *If* $\{P_{2N+d}\}_{N=1}^{\infty} \in \mathbb{P}_d(\beta, \gamma, \delta)$, *then*

$$\sup_{y\in\mathbb{R}} |\Delta_{2N+d,k,s}(y)| \le C\mathcal{E}^*_{N,\mathrm{Re}\; s-(k-1)d}, \qquad N \in \mathbb{N}, \tag{4.1.3}$$

$$\lim_{N\to\infty} \sup_{y\in\mathbb{R}} |\Delta_{2N+d,k,s}(y)| = 0. \tag{4.1.4}$$

To prove Theorem 4.1.1, we need the following simple estimate:

LEMMA 4.1.3. *Let* $\{P_{2N+d}\}_{N=1}^{\infty} \in \mathbb{P}_d(\beta, \gamma, \delta)$. *Then there exists* $N_0 \in \mathbb{N}$ *such that for all* $N \ge N_0$ *and any* $t \in [0, \gamma_N]$,

$$\left| \frac{i^d}{\beta_N^d P_{2N+d}(it/\beta_N)} - \frac{1}{h_d(t)} \right| \le \frac{C\delta_N \min\{t^2, 1\}}{h_d(t)}, \tag{4.1.5}$$

where C *is independent of* t *and* N.

Proof. Using (3.1.3) for $z = it$, $t \in [0, \gamma_N]$, we obtain

$$|\beta_N^d P_{2N+d}(it/\beta_N) - i^d h_d(t)| \le C_1\delta_N \min\{t^2, 1\} h_d(t), \tag{4.1.6}$$

whence, for large enough N,

$$\beta_N^d |P_{2N+d}(it/\beta_N)| \ge (1/2) h_d(t). \tag{4.1.7}$$

Then (4.1.5) follows from (4.1.6) and (4.1.7). ∎

Proof of Theorem 4.1.1. (a) We first prove (4.1.1) for $k = 1$. Let us assume that $2N-2 > \mathrm{Re}\, s > -2+d$. Then

$$\begin{aligned} |\Delta_{2N+d,1,s}(y)| &\le \int_0^\infty \frac{t^{\mathrm{Re}\, s-1}}{1+t^2/(\beta_N y)^2} \left| \frac{i^d}{\beta_N^d P_{2N+d}(it/\beta_N)} - \frac{1}{h_d(t)} \right| dt \\ &= \int_0^{\gamma_N} + \int_{\gamma_N}^\infty = I_1(y) + I_2(y). \end{aligned} \tag{4.1.8}$$

Next, for $-2 < \mathrm{Re}\, s < 2N-2$,

$$I_2(y) \le \int_{\gamma_N}^\infty \frac{t^{\mathrm{Re}\, s-1}\, dt}{\beta_N^d |P_{2N+d}(it/\beta_N)|} + \int_{\gamma_N}^\infty \frac{t^{\mathrm{Re}\, s-1}\, dt}{h_d(t)} = I_{2,1}(y) + I_{2,2}(y), \tag{4.1.9}$$

where by the asymptotic formula for the incomplete gamma function $\Gamma(\mu, z)$ (see [17, Sect. 9.2]),

$$I_{2,2}(y) \le C_d \int_{\gamma_N}^\infty t^{\mathrm{Re}\, s-1} e^{-t}\, dt = C_d \Gamma(\mathrm{Re}\, s, \gamma_N) \le C_d C_2(s) \gamma_N^{\mathrm{Re}\, s-1} e^{-\gamma_N}. \tag{4.1.10}$$

Here, $C_0 = 2$ and $C_1 = 2/(1-e^{-2\gamma_1})$. However, to prove (4.1.2), we need an estimate of $C_2(s)$ in (4.1.10). It is possible to show that

$$C_2(s) \le \begin{cases} \max\{2, \Gamma(\mathrm{Re}\, s)\gamma_1^{1-\mathrm{Re}\, s} e^{\gamma_1}\}, & \mathrm{Re}\, s > 1, \\ 1, & \mathrm{Re}\, s \le 1. \end{cases} \tag{4.1.11}$$

Indeed, this inequality is trivial for $\mathrm{Re}\, s \le 1$. For $\mathrm{Re}\, s > 1$ and $\gamma_N \ge 2(\mathrm{Re}\, s - 1)$, we obtain

$$\int_{\gamma_N}^\infty t^{\mathrm{Re}\, s-1} e^{-t}\, dt = \int_{\gamma_N}^\infty (t^{\mathrm{Re}\, s-1} e^{-t/2}) e^{-t/2}\, dt \le 2\gamma_N^{\mathrm{Re}\, s-1} e^{-\gamma_N}, \tag{4.1.12}$$

while for $\operatorname{Re} s > 1$ and $\gamma_1 \le \gamma_N < 2(\operatorname{Re} s - 1)$,

$$\int_{\gamma_N}^{\infty} t^{\operatorname{Re} s - 1} e^{-t}\, dt \le \Gamma(\operatorname{Re} s) \le \Gamma(\operatorname{Re} s)(\gamma_N/\gamma_1)^{\operatorname{Re} s - 1} e^{\gamma_1 - \gamma_N}. \tag{4.1.13}$$

Then (4.1.11) follows from (4.1.12) and (4.1.13).

Further, if $\{z_{p,N,d}\}_{p=1}^{N}$ are positive zeros of P_{2N+d}, then by the normalization property (3.1.2),

$$\begin{aligned} I_{2,1}(y) &= \int_{\gamma_N}^{\infty} \frac{t^{\operatorname{Re} s - 1}\, dt}{t^d \prod_{p=1}^{N} (t^2/(\beta_N z_{p,N,d})^2 + 1)} \\ &\le \frac{\prod_{p=1}^{\lfloor \operatorname{Re} s/2 \rfloor + 1} (\gamma_N^2/(\beta_N z_{p,N,d})^2 + 1)}{\beta_N^d |P_{2N+d}(i\gamma_N/\beta_N)|} \int_{\gamma_N}^{\infty} \frac{t^{\operatorname{Re} s - 1}\, dt}{\prod_{p=1}^{\lfloor \operatorname{Re} s/2 \rfloor + 1} (t^2/(\beta_N z_{p,N,d})^2 + 1)} \\ &\le \gamma_N^{\operatorname{Re} s - 2\lfloor \operatorname{Re} s/2 \rfloor - 2} \prod_{p=1}^{\lfloor \operatorname{Re} s/2 \rfloor + 1} (\gamma_N^2 + \beta_N^2 z_{p,N,d}^2)/(\beta_N^d |P_{2N+d}(i\gamma_N/\beta_N)|). \end{aligned} \tag{4.1.14}$$

Here, $\prod_{p=1}^{0} := 1$ (for $-2 < \operatorname{Re} s < 0$). Next, it follows from inequality (3.1.5) of Proposition 3.1.2 that

$$\sup_{1 \le p \le \lfloor \operatorname{Re} s/2 \rfloor + 1} \sup_{n \in \mathbb{N}} \beta_N z_{p,N,d} \le C_3(\lfloor \operatorname{Re} s/2 \rfloor). \tag{4.1.15}$$

In addition, by (4.1.7), for $N \ge N_0$ we have

$$\beta_N^d |P_{2N+d}(i\gamma_N/\beta_N)| \ge (1/2) h_d(\gamma_N) \le (C_d^{-1}/2) e^{\gamma_N}, \tag{4.1.16}$$

where $C_0 = 2$ and $C_1 = 2/(1 - e^{-2\gamma_1})$. Therefore, combining (4.1.14) with (4.1.15) and (4.1.16), for $N \ge N_0$ we obtain

$$I_{2,1}(y) \le C_4 \gamma_N^{\operatorname{Re} s} (1 + C_3^2(\lfloor \operatorname{Re} s/2 \rfloor)/\gamma_1^2)^{\lfloor \operatorname{Re} s/2 \rfloor + 1} e^{-\gamma_N}.$$

Thus

$$I_{2,1}(y) \le C_5(\lfloor \operatorname{Re} s/2 \rfloor) \gamma_N^{\operatorname{Re} s} e^{-\gamma_N}. \tag{4.1.17}$$

Note that for any fixed $r > 0$,

$$\sup_{-2 < \operatorname{Re} s \le r} C_5(\lfloor \operatorname{Re} s/2 \rfloor) < \infty. \tag{4.1.18}$$

Then (4.1.9), (4.1.10), and (4.1.17) imply an estimate

$$I_2(y) \le C_6(s) \gamma_N^{\operatorname{Re} s} e^{-\gamma_N}, \qquad -2 < \operatorname{Re} s < 2N - 2, \tag{4.1.19}$$

where, by (4.1.11) and (4.1.18), the constant $C_6(s) \le C_2(s) + C_5(\lfloor \operatorname{Re} s/2 \rfloor)$ satisfies

$$\sup_{-2 < \operatorname{Re} s \le r} C_6(s) < \infty \tag{4.1.20}$$

for any fixed $r > 0$.

It remains to estimate $I_1(y)$. If $\operatorname{Re} s > -2 + d$, then for $N \ge N_0$, by Lemma 4.1.3,

$$\begin{aligned} I_1(y) &\le \int_0^{\gamma_N} t^{\operatorname{Re} s - 1} \left| \frac{i^d}{\beta_N^d P_{2N+d}(it/\beta_N)} - \frac{1}{h_d(t)} \right| dt \\ &\le C_7 \delta_N \int_0^{\infty} \frac{t^{\operatorname{Re} s + 1}}{h_d(t)}\, dt \le C_8 \Gamma(\operatorname{Re} s - d + 2) \delta_N, \end{aligned} \tag{4.1.21}$$

where C_7 and C_8 are independent of s and N. Thus (4.1.1) for $k=1$ follows from (4.1.8), (4.1.19), and (4.1.21) for $s \in D_{d,1,N}$, while (4.1.2) is a consequence of (4.1.20) and (4.1.21).

To prove (4.1.1) for $k=2$, we note that

$$|\Delta_{2N+d,2,s}(y)| \le A_2 \int_0^\infty \frac{t^{\operatorname{Re} s-d-1}}{1+t^2/(\beta_N y)^2} \left| \frac{i^d}{\beta_N^d P_{2N+d}(it/\beta_N)} - \frac{1}{h_d(t)} \right| dt, \tag{4.1.22}$$

where

$$\begin{aligned} A_2 &:= \sup_{t>0} t^d \left| \frac{i^d}{\beta_N^d P_{2N+d}(it/\beta_N)} + \frac{1}{h_d(t)} \right| \\ &\le \sup_{t>0} \left(\frac{1}{\prod_{p=1}^N (t^2/(\beta_N z_{p,N})^2+1)} + \frac{t^d}{h_d(t)} \right) \le 2. \end{aligned} \tag{4.1.23}$$

Finally, the integral on the right-hand side of (4.1.22) can be estimated by (4.1.8), (4.1.19), and (4.1.21) with s replaced by $s-d$. Therefore for $\operatorname{Re} s > -2+2d$,

$$|\Delta_{2N+d,2,s}(y)| \le C(s)\mathcal{E}^*_{N,\operatorname{Re} s-d},$$

where $C(s)$ satisfies (4.1.2). This proves statement (a) for $k=2$.
(b) Let $-2+kd < \operatorname{Re} s < 2N-2$, where $N \ge N_0$. We first estimate the integrals

$$\begin{aligned} A_{p,s}(y) &:= \int_0^\infty \frac{t^{s-1}\log^p t}{1+t^2/(\beta_N y)^2} \left(\frac{i^{kd}}{\beta_N^{kd} P^k_{2N+d}(it/\beta_N)} - \frac{1}{h^k_d(t)} \right) dt \\ &= \int_0^1 + \int_1^\infty = I_1 + I_2, \qquad p=0,1,\dots, \end{aligned} \tag{4.1.24}$$

similarly to the proof of statement (a). Indeed, using Lemma 4.1.3, we have

$$\begin{aligned} |I_1| &\le A_k C\delta_N \int_0^1 \frac{t^{\operatorname{Re} s-d(k-1)+1}\log^p(1/t)}{h_d(t)}\, dt \\ &\le C_1 \int_0^1 t^{\operatorname{Re} s-kd+1}\log^p(1/t)dt = C_1 p!(\operatorname{Re} s-kd+2)^{-p-1}, \end{aligned} \tag{4.1.25}$$

where $A_1=1$, $A_2 \le 2$ (see (4.1.5) and (4.1.23)), and C_1 is independent of p and s.

Next taking account of (4.1.15), we obtain

$$\begin{aligned} |I_2| &\le \int_1^\infty \frac{t^{\operatorname{Re} s-1}\log^p t\, dt}{\prod_{\nu=1}^N (t^2/(\beta_N z_{\nu,N,d})^2+1)} + \int_1^\infty \frac{t^{\operatorname{Re} s-1}\log^p t\, dt}{h^k_d(t)} \\ &\le \prod_{\nu=1}^N (\beta_N z_{\nu,N,d})^2 \int_1^\infty t^{\operatorname{Re} s-2N-1}\log^p t\, dt + C_2 \int_1^\infty t^{\operatorname{Re} s-2N-1}\log^p t\, dt \\ &\le C_3 p!(2N-\operatorname{Re} s)^{-p-1}, \end{aligned} \tag{4.1.26}$$

where $C_2 = C_2(N)$ and $C_3 = C_3(N)$ are independent of p and s. Thus relations (4.1.24)–(4.1.26) imply the estimate

$$|A_{p,s}(y)| \le Cp!((\operatorname{Re} s-dk+2)^{-p-1} + (2N-\operatorname{Re} s)^{-p-1}), \qquad p=0,1,\dots, \tag{4.1.27}$$

where C is independent of p and s.

Let $s_0 \in D_{d,k,N}$. Then we replace t^{s-1} in the definition of $\Delta_{2N+d,k,s}$ with its Taylor expansion in a neighborhood of s_0 and obtain, by (4.1.27),

$$\begin{aligned}\Delta_{2N+d,k,s} &= \int_0^\infty \sum_{p=0}^\infty \frac{t^{s_0-1}\log^p t}{p!(1+t^2/(\beta_N y)^2)}(s-s_0)^p\left(\frac{i^{kd}}{\beta_N^{kd}P_{2N+d}^k(it/\beta_N)} - \frac{1}{h_d^k(t)}\right)dt\\ &= \sum_{p=0}^\infty \frac{A_{p,s}(y)}{p!}(s-s_0)^p.\end{aligned}$$

Therefore, $\Delta_{2N+d,k,s}$ is analytic in s in a neighborhood of $s_0 \in D_{d,k,N}$. This proves statement (b). ∎

4.2. Asymptotic relations for $\zeta(s)$ and $\beta(s)$. In this section we present our major pointwise asymptotic relations between the zeta functions and the interpolation errors of Lagrange and Hermite interpolation to functions like $|y|^s$ and $y^{2m}\log|y|$.

The Riemann zeta function is defined by the series $\zeta(s) := \sum_{n=1}^\infty n^{-s}$ for $\operatorname{Re} s > 1$. It is well known that $\zeta(s)$ can be extended to an analytic function on $\mathbb{C}\setminus\{1\}$.

The Dirichlet beta function $\beta(s) := \sum_{n=0}^\infty (-1)^n(2n+1)^{-s}$ is defined for $\operatorname{Re} s > 0$ but can be extended analytically to a function on $\mathbb{C}$. It is a Dirichlet special L-function with the character χ modulo 4 defined by

$$\chi(n) = \begin{cases}(-1)^{(n-1)/2}, & n \text{ odd},\\ 0, & n \text{ even},\end{cases} \qquad n = 0, 1, \ldots.$$

The functions $\zeta(s)$ and $\beta(s)$ are closely related, and they have similar properties. We define the "generalized" zeta function by

$$\zeta_{d,k}(s) := \begin{cases}\beta(s), & d = 0,\ k = 1,\\ \zeta(s), & d = 0,\ k = 2 \text{ or } d = 1,\ k = 1, 2.\end{cases} \tag{4.2.1}$$

An integral representation for $\zeta_{d,k}(s)$ is given below (see also Remark 2.3.6).

Proposition 4.2.1 (see [16, Sect. 1.12]). *For* $\operatorname{Re} s > 1 - k(1-d)$,

$$C_{d,k}(s)\zeta_{d,k}(s) = \int_0^\infty \frac{t^{s+k-2}}{h_d^k(t)}\,dt, \tag{4.2.2}$$

where h_d *is defined by* (4.0.1) *and*

$$C_{d,k}(s) := 2^{1-(k-1)s}(1-2^{k-s-1})^{d(3-2k)+k-1}\Gamma(s+k-1).$$

In particular,

$$\begin{aligned} C_{0,1}(s) &= 2\Gamma(s), & C_{1,1}(s) &= 2(1-2^{-s})\Gamma(s),\\ C_{0,2}(s) &= 2^{1-s}(1-2^{1-s})\Gamma(s+1), & C_{1,2}(s) &= 2^{1-s}\Gamma(s+1).\end{aligned} \tag{4.2.3}$$

Remark 4.2.2. It is easy to see that $C_{d,k}(s)$ defined by (4.2.3) are analytic functions in s for $\operatorname{Re} s > 1 - k(1-d)$. Moreover, the function $C_{d,k}(s)\zeta_{d,k}(s)$ from Proposition 4.2.1 can be extended to an analytic function on the domain

$$D_{d,k} := \{s \in \mathbb{C} : \operatorname{Re} s > -1 - k(1-d),\ s \neq 1 - k(1-d)\}. \tag{4.2.4}$$

Indeed, we first note that $C_{d,k}(s)\zeta_{d,k}(s)$ is undefined at several exceptional points $s = s_{d,k}$ from this domain, where $s_{0,1} = -1$, $s_{0,2} = -2, 1$, and $s_{1,1} = 0$. Then $C_{d,k}(s)\zeta_{d,k}(s)$ is analytic at each $s \in D_{d,k}$, $s \neq s_{d,k}$ since $\zeta_{d,k}(s)$ and $C_{d,k}(s)$ are analytic on this set. It remains to define this function at the exceptional points as the finite limits

$$C_{d,k}(s_{d,k})\zeta_{d,k}(s_{d,k}) := \lim_{s \to s_{d,k}} C_{d,k}(s)\zeta_{d,k}(s), \qquad d = 0, 1,\ k = 1, 2,$$

which exist due the well-known properties of $\Gamma(s)$, $\beta(s)$, and $\zeta(s)$.

We now recall two interpolation formulae (2.3.21) and (2.3.23) that hold for every $y \in \mathbb{R} \setminus \{0\}$,

$$\begin{aligned} |y|^s - L_{2N}(y, |y|^s, yG_{2N}(y)) &= \frac{2\sin(s\pi/2)}{\pi} G_{2N}(y) \\ &\times \int_0^\infty \frac{t^{s-1}\,dt}{(1+(t/y)^2)G_{2N}(it)}, \qquad 0 < \operatorname{Re} s < 2N+1,\ s \neq 2, 4, \ldots, \end{aligned} \tag{4.2.5}$$

$$\begin{aligned} &y^{2m} \log|y| - L_{2N}(y, y^{2m} \log|y|, yG_{2N}(y)) \\ &\quad = (-1)^m G_{2N}(y) \int_0^\infty \frac{t^{2m-1}\,dt}{(1+(t/y)^2)G_{2N}(it)}, \qquad m = 1, \ldots, N, \end{aligned} \tag{4.2.6}$$

where G_{2N} is an even polynomial of degree $2N$ with all real zeros, $G_{2N}(0) \neq 0$, and L_{2N} interpolates a function at the origin of multiplicity 1 and at the zeros of G_{2N} of the corresponding multiplicities.

Let us assume that $\{P_{2N+d}\}_{N=1}^\infty \in \mathbb{P}_d(\beta, \gamma, \delta)$ and choose $G_{2kN}(y) = y^{-kd} P_{2N+d}^k(y)$ with multiplicities k of all nonzero interpolation nodes. Then replacing N with kN and s with $s+k(1-d)-1$ in (4.2.5), we deduce that for every $y \in \mathbb{R}\setminus\{0\}$ and $2kN - k(1-d) + 2 > \operatorname{Re} s > 1 - k(1-d)$, $s + k(1-d) - 1 \neq 2, 4, \ldots$,

$$\begin{aligned} &|y|^{s+k(1-d)-1} - L_{2kN}(y, |y|^{s+k(1-d)-1}, y^{-kd+1} P_{2N+d}^k(y)) \\ &\quad = \frac{2\sin\big((s+k(1-d)-1)\pi/2\big) i^{kd}}{\pi} \frac{P_{2N+d}^k(y)}{y^{kd}} \int_0^\infty \frac{t^{s+k-2}\,dt}{(1+(t/y)^2)P_{2N+d}^k(it)}. \end{aligned} \tag{4.2.7}$$

In addition, from (4.2.6) we obtain

$$\begin{aligned} &y^{2m} \log|y| - L_{2kN}(y, y^{2m} \log|y|, y^{-kd+1} P_{2N+d}^k(y)) \\ &\quad = (-1)^m i^{kd} \frac{P_{2N+d}^k(y)}{y^{kd}} \int_0^\infty \frac{t^{2m+kd-1}\,dt}{(1+(t/y)^2)P_{2N+d}^k(it)}, \qquad m = 1, \ldots, kN. \end{aligned} \tag{4.2.8}$$

Combining now (4.2.7) and (4.2.8) with Proposition 4.2.1 and Theorem 4.1.1, we obtain our major asymptotic.

However, before stating this result, we need to define

$$L_{2kN}(y) := L_{2kN}(y, |y|^{s+k(1-d)-1}, y^{-kd+1} P_{2N+d}^k(y))$$

for $\operatorname{Re} s + k(1-d) - 1 \leq 0$ since the function $|y|^{s+k(1-d)-1}$ is undefined at the origin. We recall that for $\operatorname{Re} s + k(1-d) - 1 > 0$,

$$\begin{aligned} L_{2kN}(y) &= \sum_{p=1}^{2N} |z_{p,N,d}|^{s+k(1-d)-1} l_p(y) \\ &\qquad + (k-1)(s+k(1-d)-1) \sum_{p=1}^{2N} |z_{p,N,d}|^{s+k(1-d)-2} l_p^*(y) \end{aligned} \tag{4.2.9}$$

is the Lagrange (for $k=1$) or Hermite (for $k=2$) interpolation polynomial of degree $2kN$ that interpolates the function $|y|^{s+k(1-d)-1}$ at the origin of multiplicity 1 and at the zeros $\{z_{p,N,d}\}_{p=1}^{2N}$ of $y^{-d}P_{2N+d}(y)$ of multiplicity 1 (if $k=1$) or 2 if ($k=2$). In the above formula, l_p and l_p^*, $1 \le p \le 2N$, in (4.2.9) are the corresponding fundamental polynomials of Lagrange or Hermite interpolation.

Here and in what follows, we define L_{2kN} for $\operatorname{Re} s + k(1-d) - 1 \le 0$ by (4.2.9) as well, that is, $L_{2kN}(0) = 0$. Certainly in this case, L_{2kN} is not the interpolation polynomial for $|y|^{s+k(1-d)-1}$ anymore, though we use the same notation. Actually, L_{2kN} is the corresponding interpolation polynomial for the function

$$\begin{cases} |y|^{s+k(1-d)-1}, & y \ne 0, \\ 0, & y = 0. \end{cases}$$

Theorem 4.2.3. *Let $\{P_{2N+d}\}_{N=1}^{\infty} \in \mathbb{P}_d(\beta,\gamma,\delta)$. Then there exists $N_1 \in \mathbb{N}$ such that the following statements are valid:*

(a) *For any $y \in \mathbb{R}\backslash\{0\}$ and for $-1-k(1-d) < \operatorname{Re} s < 2N-k-1$, $s+k(1-d)-1 \ne 0, 2, \dots,$ $N \ge N_1$,*

$$\begin{aligned} &|y|^{s+k(1-d)-1} - L_{2kN}(y, |y|^{s+k(1-d)-1}, y^{-kd+1}P_{2N+d}^k(y)) \\ &\quad = \frac{2\sin((s+k(1-d)-1)\pi/2)}{\pi\beta_N^{s+k(1-d)-1}} C_{d,k}(s)\zeta_{d,k}(s)\frac{P_{2N+d}^k(y)}{y^{kd}} \\ &\qquad - \frac{2\sin((s+k(1-d)-1)\pi/2)}{\pi(\beta_N y)^2\beta_N^{s+k(1-d)-1}}\,\frac{P_{2N+d}^k(y)}{y^{kd}}\int_0^\infty \frac{t^{s+k}\,dt}{(1+t^2/(\beta_N y)^2)h_d^k(t)} \\ &\qquad + \frac{2\sin((s+k(1-d)-1)\pi/2)P_{2N+d}^k(y)}{\pi y^{kd}\beta_N^{s+k(1-d)-1}}\Delta_{2N+d,k,s+k-1}(y), \end{aligned} \tag{4.2.10}$$

where

$$\sup_{y\in\mathbb{R}} |\Delta_{2N+d,k,s+k-1}(y)| \le C(s)\mathcal{E}^*_{N,\operatorname{Re} s+(k-1)(1-d)}, \tag{4.2.11}$$

and for any fixed $r > 0$ and $\varepsilon > 0$,

$$\sup_{-1-k(1-d)+\varepsilon \le \operatorname{Re} s \le r} C(s) < \infty. \tag{4.2.12}$$

(b) *For any $y \in \mathbb{R}\setminus\{0\}$ and $m = 1, \dots, N$,*

$$\begin{aligned} &y^{2m}\log|y| - L_{2kN}(y, y^{2m}\log|y|, y^{-kd+1}P_{2N+d}^k(y)) \\ &\quad = \frac{(-1)^m C_{d,k}(2m+1-k(1-d))\zeta_{d,k}(2m+1-k(1-d))}{\beta_N^{2m}}\frac{P_{2N+d}^k(y)}{y^{kd}} \\ &\qquad - \frac{(-1)^m}{(\beta_N y)^2\beta_N^{2m}}\frac{P_{2N+d}^k(y)}{y^{kd}}\int_0^\infty \frac{t^{2m+kd+1}\,dt}{(1+t^2/(\beta_N y)^2)h_d^k(t)} \\ &\qquad + C_1\frac{P_{2N+d}^k(y)}{y^{kd}\beta_N^{2m}}\Delta_{2N+d,k,2m+kd}(y), \end{aligned} \tag{4.2.13}$$

where

$$\sup_{y\in\mathbb{R}} |\Delta_{2N+d,k,2m+kd}(y)| \le C\mathcal{E}^*_{N,2m+d}. \tag{4.2.14}$$

REMARK 4.2.4. We recall that $\mathcal{E}^*_{N,\tau}$ is defined in (4.0.2) and for any $\tau \in \mathbb{R}$, $\lim_{N\to\infty} \mathcal{E}^*_{N,\tau} = 0$ by (4.0.3). In addition, we note that on the right-hand side of (4.2.13),

$$\zeta_{d,k}(2m+1-k(1-d)) = \begin{cases} \beta(2m), & d=0,\ k=1, \\ \zeta(2m-1), & d=0,\ k=2, \\ \zeta(2m+1), & d=1,\ k=1,2. \end{cases} \tag{4.2.15}$$

In particular, (4.2.15) shows that in the case of $m=1$, $d=0$, $k=2$ in Theorem 4.2.3(b) we can use the agreement of Remark 4.2.2 to define

$$\begin{aligned} C_{d,k}(2m+1-k(1-d))\zeta_{d,k}(2m+1-k(1-d)) &= C_{0,2}(1)\zeta(1) \\ &:= \lim_{s\to 1} C_{0,2}(s)\zeta(s) = \log 2. \end{aligned}$$

Finally, we remark that each of the asymptotic formulae (4.2.10) and (4.2.13) contains the principal term with $C_{d,k}(s)\zeta_{d,k}(s)$, the integral term, and the remainder term. The remainder term is estimated in (4.2.11) and (4.2.14). Estimates of the integral term and various representations for the zeta functions based on the asymptotics (4.2.10) and (4.2.13) are presented in Sections 4.3, 4.4, and Chapter 5.

Proof of Theorem 4.2.3. We first note that for $1-k(1-d) < \operatorname{Re} s < 2N-k-1$,

$$\begin{aligned} i^{kd}\int_0^\infty \frac{t^{s+k-2}\,dt}{(1+(t/y)^2)P^k_{2N+d}(it)} &= \frac{i^{kd}}{\beta_N^{s+k-1}}\int_0^\infty \frac{t^{s+k-2}\,dt}{(1+t^2/(\beta_N y)^2)P^k_{2N+d}(it/\beta_N)} \\ &= \frac{1}{\beta_N^{s+k(1-d)-1}}\left(\int_0^\infty \frac{t^{s+k-2}\,dt}{(1+t^2/(\beta_N y)^2)h^k_d(t)} + \varDelta_{2N+d,k,s+k-1}(y)\right), \end{aligned} \tag{4.2.16}$$

where, by Theorem 4.1.1(a), for $-1-k(1-d) < \operatorname{Re} s < 2N-k-1$, $N \ge N_0$,

$$\sup_{y\in\mathbb{R}} |\varDelta_{2N+d,k,s+k-1}(y)| \le C(s)\mathcal{E}^*_{N,\operatorname{Re} s+(k-1)(1-d)}, \tag{4.2.17}$$

and for any fixed $r>0$ and $\varepsilon>0$,

$$\sup_{-1-k(1-d)+\varepsilon\le \operatorname{Re} s\le r} C(s) < \infty. \tag{4.2.18}$$

Next by Proposition 4.2.1, for $1-k(1-d) < \operatorname{Re} s < 2N-k-1$, $N \ge N_0$, we obtain

$$\begin{aligned} \int_0^\infty \frac{t^{s+k-2}\,dt}{(1+t^2/(\beta_N y)^2)h^k_d(t)} &= \int_0^\infty \frac{t^{s+k-2}\,dt}{h^k_d(t)} - \frac{1}{(\beta_N y)^2}\int_0^\infty \frac{t^{s+k}\,dt}{(1+t^2/(\beta_N y)^2)h^k_d(t)} \\ &= C_{d,k}(s)\zeta_{d,k}(s) - \frac{1}{(\beta_N y)^2}\int_0^\infty \frac{t^{s+k}\,dt}{(1+t^2/(\beta_N y)^2)h^k_d(t)}. \end{aligned} \tag{4.2.19}$$

If $s+k(1-d)-1 = 2m$, $m=1,\dots,N$, then combining (4.2.16) and (4.2.19) with (4.2.8), we arrive at (4.2.13). Since (4.2.14) follows from (4.2.17), statement (b) of Theorem 4.2.3 is established.

Next, combining (4.2.16) and (4.2.19) with (4.2.7), we arrive at (4.2.10) for $1-k(1-d) < \operatorname{Re} s < 2N-k-1$, $s+k(1-d) \ne 0,2,\dots$, $N \ge N_0$.

To show that (4.2.10) holds in a larger domain

$$D^*_{d,k,N} := \{s\in\mathbb{C} : -1-k(1-d) < \operatorname{Re} s < 2N-k-1,\ s+k(1-d)\ne 0,2,\dots\}$$

for $N \geq N_1$, we notice that for a fixed $y \in \mathbb{R} \setminus \{0\}$, both sides of equality (4.2.10) are analytic functions in s on $D^*_{d,k,N}$. Indeed, it follows from (4.2.9) that the left-hand side of (4.2.10) is an entire function in s. Next, by Remark 4.2.2, the function $C_{d,k}(s)\zeta_{d,k}(s)$ is analytic on the domain $D_{d,k}$ defined by (4.2.4). Therefore it is analytic on $D^*_{d,k,N} \subseteq D_{d,k}$, while for large enough N, $\Delta_{2N+d,k,s+k-1}$ is analytic in s on $D^*_{d,k,N}$, by Theorem 4.1.1(b). It remains to note that the integral $\int_0^\infty \frac{t^{s+k}\,dt}{(1+t^2/(\beta_N y)^2)h_d^k(t)}$ is an analytic function in s for $-1-k(1-d) < \operatorname{Re} s$.

Therefore, equality (4.2.10) can be extended to all $s \in D^*_{d,k,N}$, $N \geq N_1$. In addition, estimates (4.2.11) and (4.2.12) follow from (4.2.17) and (4.2.18), respectively. This completes the proof of Theorem 4.2.3. ∎

Remark 4.2.5. Note that Theorem 4.2.3 can be used to establish asymptotics for the interpolation errors

$$\Delta(y) := f(y) - L_{2kN-2}(y, f(y), y^{-kd}P^k_{2N+d}(y)),$$

since by (2.3.26),

$$\Delta(y) = y^{-2}(f(y) - L_{2kN}(y, f(y), y^{-kd+1}P^k_{2N+d}(y))). \tag{4.2.20}$$

Here, $f(y) = |y|^{s+k(1-d)-1}$, $\operatorname{Re} s > -1-k(1-d)$, or $f(y) = y^{2m}\log|y|$, $m = 1, 2, \ldots$.

4.3. Some corollaries. Simplified pointwise asymptotics are presented in this section. The following corollary is a direct consequence of Theorem 4.2.3.

Corollary 4.3.1. *Let $\{P_{2N+d}\}_{N=1}^\infty \in \mathbb{P}_d(\beta,\gamma,\delta)$ and $\operatorname{Re} s > -1-k(1-d)$. Let $\{y_N\}_{N=1}^\infty$ be a sequence of positive numbers such that $\lim_{N\to\infty}\beta_N y_N = \tau$ for some $\tau \in [0,\infty]$. Then for large enough N the following statements hold:*

(a) *Let $\tau = \infty$. Then for $s + k(1-d) - 1 \neq 0, 2, \ldots$,*

$$\begin{aligned} y_N^{s+k(1-d)-1} &- L_{2kN}(y_N, |y|^{s+k(1-d)-1}, y^{-kd+1}P^k_{2N+d}(y)) \\ &= \frac{2\sin((s+k(1-d)-1)\pi/2)}{\pi\beta_N^{s+k(1-d)-1}}\frac{P^k_{2N+d}(y_N)}{y_N^{kd}}(C_{d,k}(s)\zeta_{d,k}(s) \\ &\qquad + O((\beta_N y_N)^{-2}) + O(\mathcal{E}^*_{N,\operatorname{Re} s+(k-1)(1-d)})), \qquad N \to \infty, \end{aligned} \tag{4.3.1}$$

where the constants $C_1(s)$ and $C_2(s)$ in $O((\beta_N y_N)^{-2}$ and $O(\mathcal{E}^*_{N,\operatorname{Re} s+(k-1)(1-d)})$, respectively, satisfy the property

$$\sup_{-1-k(1-d)+\varepsilon\leq \operatorname{Re} s\leq r}\max\{C_1(s), C_2(s)\} \leq \infty \tag{4.3.2}$$

for any fixed $r > 0$ and $\varepsilon > 0$. In addition, for a fixed $m \in \mathbb{N}$, $1 \leq m \leq N$,

$$\begin{aligned} y_N^{2m}\log y_N &- L_{2kN}(y_N, y^{2m}\log|y|, y^{-kd+1}P^k_{2N+d}(y)) \\ &= \frac{(-1)^m}{\beta_N^{2m}}\frac{P^k_{2N+d}(y_N)}{y_N^{kd}}(C_{d,k}(2m+1-k(1-d))\zeta_{d,k}(2m+1-k(1-d)) \\ &\qquad + O((\beta_N y_N)^{-2}) + O(\mathcal{E}^*_{N,2m+d})), \qquad N \to \infty. \end{aligned} \tag{4.3.3}$$

(b) *Let* $0<\tau<\infty$. *Then for* $s+k(1-d)-1\neq 0,2,\ldots,$

$$
\begin{aligned}
&y_N^{s+k(1-d)-1}-L_{2kN}(y_N,|y|^{s+k(1-d)-1},y^{-kd+1}P^k_{2N+d}(y))\\
&\qquad=\frac{2\sin((s+k(1-d)-1)\pi/2)}{\pi\beta_N^{s+k(1-d)-1}}\frac{P^k_{2N+d}(y_N)}{y_N^{kd}}\Bigg(C_{d,k}(s)\zeta_{d,k}(s)\\
&-\int_0^\infty\frac{t^{s+k}\,dt}{(\tau^2+t^2)h^k_d(t)}+O(|\tau-\beta_Ny_N|)+O(\mathcal{E}^*_{N,\operatorname{Re}s+(k-1)(1-d)})\Bigg),\qquad N\to\infty,
\end{aligned}\tag{4.3.4}
$$

and for a fixed $m\in\mathbb{N}$, $1\le m\le N$,

$$
\begin{aligned}
&y_N^{2m}\log y_N-L_{2kN}(y_N,y^{2m}\log|y|,y^{-kd+1}P^k_{2N+d}(y))\\
&\quad=\frac{(-1)^m}{\beta_N^{2m}}\frac{P^k_{2N+d}(y_N)}{y_N^{kd}}\Bigg(C_{d,k}(2m+1-k(1-d))\zeta_{d,k}(2m+1-k(1-d))\\
&\quad-\int_0^\infty\frac{t^{2m+1+kd}\,dt}{(\tau^2+t^2)h^k_d(t)}+O(|\tau-\beta_Ny_N|)+O(\mathcal{E}^*_{N,2m+d})\Bigg),\qquad N\to\infty.
\end{aligned}\tag{4.3.5}
$$

(c) *Let* $\tau=0$. *Then for* $\operatorname{Re}s>1-k(1-d)$ *and* $s+k(1-d)-1\neq 2,4,\ldots,$ *as* $N\to\infty$,

$$
\begin{aligned}
&y_N^{s+k(1-d)-1}-L_{2kN}(y_N,|y|^{s+k(1-d)-1},y^{-kd+1}P^k_{2N+d}(y))\\
&\quad=\frac{P^k_{2N+d}(y_N)}{\beta_N^{s+k(1-d)-1}y_N^{kd}}\Bigg(O(\mathcal{E}^*_{N,\operatorname{Re}s+(k-1)(1-d)})\\
&\quad+\begin{cases}O((\beta_Ny_N)^{\operatorname{Re}s+k(1-d)-1}), & 1-k(1-d)<\operatorname{Re}s<3-k(1-d),\\ O((\beta_Ny_N)^2\log(1/(\beta_Ny_N))), & \operatorname{Re}s=3-k(1-d),\\ O((\beta_Ny_N)^2), & \operatorname{Re}s>3-k(1-d)\end{cases}\Bigg),
\end{aligned}\tag{4.3.6}
$$

and for a fixed $m\in\mathbb{N}$, $1\le m\le N$, *as* $N\to\infty$,

$$
\begin{aligned}
&y_N^{2m}\log y_N-L_{2kN}(y_N,y^{2m}\log|y|,y^{-kd+1}P^k_{2N+d}(y))\\
&\quad=\frac{P^k_{2N+d}(y_N)}{\beta_N^{2m}y_N^{kd}}\Bigg(O(\mathcal{E}^*_{N,2m+d})+\begin{cases}O((\beta_Ny_N)^2\log(1/(\beta_Ny_N))), & m=1,\\ O((\beta_Ny_N)^2), & m=2,3,\ldots\end{cases}\Bigg).
\end{aligned}\tag{4.3.7}
$$

Proof. Statements (a) and (b) follow immediately from Theorem 4.2.3. To prove statement (c), we first note that by Proposition 4.2.1,

$$
\begin{aligned}
&C_{d,k}(s)\zeta_{d,k}(s)-\frac{1}{(\beta_Ny_N)^2}\int_0^\infty\frac{t^{s+k}\,dt}{(1+t^2/(\beta_Ny_N)^2)h^k_d(t)}\\
&\qquad=\int_0^\infty\frac{t^{s+k-2}\,dt}{(1+t^2/(\beta_Ny_N)^2)h^k_d(t)}.
\end{aligned}\tag{4.3.8}
$$

Denoting the right-hand side integral of (4.3.8) by I_N, we obtain for $1-k(1-d)<\operatorname{Re}s<3-k(1-d)$,

$$
|I_N|\le(\beta_Ny_N)^{\operatorname{Re}s+k(1-d)-1}\int_0^\infty\frac{t^{\operatorname{Re}s+k(1-d)-2}\,dt}{1+t^2}=C(\beta_Ny_N)^{\operatorname{Re}s+k(1-d)-1}.\tag{4.3.9}
$$

Next, for $\operatorname{Re}s>3-k(1-d)$,

$$
|I_N|\le(\beta_Ny_N)^2\int_0^\infty\frac{t^{\operatorname{Re}s+k-4}\,dt}{h^k_d(t)}=C(\beta_Ny_N)^2.\tag{4.3.10}
$$

Further, for $\operatorname{Re} s = 3 - k(1-d)$, and large enough N,

$$\begin{aligned}
|I_N| &\le (\beta_N y_N)^2 \int_0^\infty \frac{t^{1+kd}\,dt}{((\beta_N y_N)^2 + t^2) h_d^k(t)} \\
&= (\beta_N y_N)^2 \Big(\int_0^{\beta_N y_N} + \int_{\beta_N y_N}^1 + \int_1^\infty \Big) \\
&\le (\beta_N y_N)^2 \Big(\int_0^{\beta_N y_N} \frac{t\,dt}{(\beta_N y_N)^2 + t^2} + \int_{\beta_N y_N}^1 \frac{dt}{t} + \int_1^\infty \frac{dt}{t(t^{-d} h_d(t))^k} \Big) \\
&= (\beta_N y_N)^2 ((\log 2)/2 + \log(1/\beta_N y_N) + C). \qquad (4.3.11)
\end{aligned}$$

Then (4.3.6) and (4.3.7) follow from statements (a) and (b), respectively, of Theorem 4.2.3 and from relations (4.3.8)–(4.3.11). ∎

In particular, for a fixed $y > 0$, we can set $y_N = y$, $N = 1, 2, \ldots$. Since $\lim_{N\to\infty} \beta_N y = \infty$ and $\lim_{N\to\infty} \mathcal{E}^*_{N, \operatorname{Re} s + (k-1)(1-d)} = 0$, by (4.0.3), from Corollary 4.3.1(a) we obtain

Corollary 4.3.2. *Let* $\{P_{2N+d}\}_{N=1}^\infty \in \mathbb{P}_d(\beta, \gamma, \delta)$ *and let* $\operatorname{Re} s > -1 - k(1-d)$ *and* $s + k(1-d) - 1 \ne 0, 2, \ldots$. *Then for* $y > 0$,

$$\begin{aligned}
&\lim_{N\to\infty} \beta_N^{s+k(1-d)-1} (y^{s+k(1-d)-1} - L_{2kN}(y, |y|^{s+k(1-d)-1}, y^{-kd+1} P^k_{2N+d}(y))) \\
&\quad = (2/\pi) \sin((s + k(1-d) - 1)\pi/2) y^{-kd} P^k_{2N+d}(y) C_{d,k}(s) \zeta_{d,k}(s), \qquad N \to \infty. \qquad (4.3.12)
\end{aligned}$$

Remark 4.3.3. Special cases of Corollary 4.3.2 for a real $s > 0$, $s+k(1-d)-1 \ne 0, 2, \ldots$, have been discussed in several publications. Bernstein [5, p. 99] in a weaker form and more recently the author [19] in the present form established (4.3.12) for $d = 0$, $k = 1$, and $P_{2N} = (-1)^N T_{2N}$. In the case of $d = 1$, $k = 1$, and $P_{2N+1} = (-1)^N (2N+2)^{-1} U_{2N+1}$, relation (4.3.12) was proved in [19], while for $d = 1$, $k = 1$, and $P_{2N+1} = (-1)^N (2N+1)^{-1} T_{2N+1}$, it was recently established by Revers in [46]. For polynomials P_{2N+d} with equidistant zeros and $d = 0, 1$, $k = 1$, Theorem 2 in [20] can be derived from Corollary 4.3.2 by using asymptotic (3.3.26) and identity (4.2.20). Special cases of this result for $s - d = 1$ and $s - d = 3$ were established earlier by Byrne et al. [10] and Revers [45], respectively.

In the case of a complex s with $\operatorname{Re} s > 0$, $s \ne 0, 2, \ldots$, a weaker version of relation (4.2.12) for $d = 0$, $k = 1$, $P_{2N} = (-1)^N T_{2N}$, was obtained in [21]. A different asymptotic for $\zeta(s)$ with $y^{s+k(1-d)-1}$ replaced by a Lommel's function was found in [23].

For some P_{2N+d} and $y_N = y$, $N = 1, 2, \ldots$, we can simplify (4.3.1) and (4.3.3) by finding the asymptotic value of $P_{2N+d}(y)$. This is discussed in the next corollary.

Corollary 4.3.4. (a) *Let* $P_{2N+d}(y) := C^\lambda_{2N+d}(y)/(C^\lambda_{2N+d})^{(d)}(0)$ *be the normalized Gegenbauer polynomial* (*see* (3.2.2)) *and let* $y \in (0,1)$ *be a number such that* $(\arccos y)/\pi$ *is irrational. Then there exists a subsequence* $\{N_p\}_{p=1}^\infty$ *such that*

$$\begin{aligned}
&\lim_{p\to\infty} (-1)^{kN_p} (2N_p)^{s+k-1} (y^{s+k(1-d)-1} - L_{2kN_p}(y, |y|^{s+k(1-d)-1}, y^{-kd+1} P^k_{2N_p+d}(y))) \\
&\quad = \frac{2\sin((s+k(1-d)-1)\pi/2)}{\pi y^{kd}(1-y^2)^{k\lambda/2}} C_{d,k}(s) \zeta_{d,k}(s), \qquad \operatorname{Re} s > -1 - k(1-d), \qquad (4.3.13)
\end{aligned}$$

and for $m = 1, 2, \ldots,$

$$\lim_{p\to\infty} (-1)^{kN_p}(2N_p)^{2m+kd}(y^{2m}\log y - L_{2kN_p}(y, y^{2m}\log|y|, y^{-kd+1}P^k_{2N+d}(y)))$$
$$= \frac{(-1)^m}{y^{kd}(1-y^2)^{k\lambda/2}} C_{d,k}(2m+1-k(1-d))\zeta_{d,k}(2m+1-k(1-d)). \qquad (4.3.14)$$

(b) *Let* $P_{2N}(y) = T_{2N}(y)$ *be the Chebyshev polynomials of the first kind. Then for every* $y \in (0,1)$, *there exists a subsequence* $\{N_p\}_{p=1}^{\infty}$ *such that* (4.3.13) *and* (4.3.14) *are valid for* $d = \lambda = 0$.

Proof. Let $y \in (0,1)$ be a fixed number and let $y = \cos\theta$. Then by the asymptotic formula (3.3.18) for $\lambda \geq 0$,

$$\frac{C^{\lambda}_{2N+d}(y)}{(C^{\lambda}_{2N+d})^{(d)}(0)}$$
$$= (-1)^N(2N)^{-d}\left(\frac{(1+o(1))\cos((2N+d+\lambda)\theta - \pi\lambda/2)}{\sin^{\lambda}\theta} + \frac{O(1)}{N\sin^{\lambda+1}\theta}\right)$$

as $N \to \infty$. If θ/π is irrational, then the sequence $\{2\pi(N\theta/\pi - \lfloor N\theta/\pi\rfloor)\}_{N=1}^{\infty}$ is dense on $[0, 2\pi)$, by the generalized Dirichlet theorem [11, Sect. 8.1]. Therefore there exists a subsequence $\{N_p\}_{p=1}^{\infty}$ such that $\lim_{p\to\infty}\cos((2N_p + d + \lambda)\theta - \pi\lambda/2) = 1$. Thus (4.3.13) and (4.3.14) follow from (4.3.1) and (4.3.3), respectively, if we set $y_N = y$ and recall that $\beta_N = 2N + \lambda + d$, $N = 1, 2, \ldots$. This establishes statement (a).

If $d = 0$ and $P_{2N}(y) = T_{2N}(y) = \cos(2N\theta)$, where $\theta/\pi = r/l$ is a rational number, then choosing $N_p = pl$, we obtain $T_{2N_p}(y) = 1$, $p = 1, 2, \ldots$. Taking into account statement (a), we see that (4.3.13) and (4.3.14) are valid for $d = \lambda = 0$ and any $y \in (0,1)$. ∎

4.4. Asymptotic summation formulae for zeta functions. Here, we discuss more asymptotic relations, which follow from Corollary 4.3.1(a). In particular, a number of explicit summation formulae for zeta functions are given in Corollary 4.4.4. We first prove the following general result.

THEOREM 4.4.1. *Let* $\{P_{2N+d}\}_{N=1}^{\infty} \in \mathbb{P}_d(\beta,\gamma,\delta)$ *and let* $\{z_{p,N,d}\}_{p=1}^{N}$ *be the set of all positive zeros of* P_{2N+d}, $N = 1, 2, \ldots$. *Then the following statements hold:*

(a) *For* $k = 1$, $-2 + d < \operatorname{Re} s < 2N - 2$, $s - d \neq 0, 2, \ldots,$ *and* $C_{d,1}(s) \neq 0$,

$$\zeta_{d,1}(s) = -\frac{\pi}{\sin((s-d)\pi/2)C_{d,1}(s)}\beta_N^{s-d}\sum_{p=1}^{N}\frac{z^{s-1}_{p,N,d}}{P'_{2N+d}(z_{p,N,d})} + O(\mathcal{E}^*_{N,\operatorname{Re}s}), \qquad (4.4.1)$$

where the constant $C_3(s)$ *in* $O(\mathcal{E}^*_{N,\operatorname{Re}s})$ *satisfies the property*

$$\sup_{\varepsilon\leq\operatorname{Re}s\leq 1-\varepsilon} C_3(s) < \infty \qquad (4.4.2)$$

for any $\varepsilon \in (0, 1/2)$.

(b) *For* $k = 1$ *and* $m = 1, 2, \ldots,$

$$\zeta_{d,1}(2m+d) = \frac{2(-1)^{m+1}}{C_{d,1}(2m+d)}\beta_N^{2m}\sum_{p=1}^{N}\frac{z^{2m+d-1}_{p,N,d}\log z_{p,N,d}}{P'_{2N+d}(z_{p,N,d})} + O(\mathcal{E}^*_{N,2m+d}). \qquad (4.4.3)$$

(c) *For* $k=2$, $-3+2d<\operatorname{Re}s<2N-3$, $s-2d+1\neq 0,2,\dots$, *and* $C_{d,2}(s)\neq 0$,

$$\zeta_{d,2}(s)=-\frac{\pi}{\sin((s-2d+1)\pi/2)C_{d,2}(s)}\beta_N^{s-2d+1}$$
$$\times\sum_{p=1}^{N} z_{p,N,d}^{s-1}\frac{s-z_{p,N,d}P''_{2N+d}(z_{p,N,d})/P'_{2N+d}(z_{p,N,d})}{P'^2_{2N+d}(z_{p,N,d})}+O(\mathcal{E}^*_{N,\operatorname{Re}s-d+1}),\qquad(4.4.4)$$

where the constant $C_4(s)$ *in* $O(\mathcal{E}^*_{N,\operatorname{Re}s-d+1})$ *satisfies the property*

$$\sup_{\varepsilon\le\operatorname{Re}s\le 1-\varepsilon} C_4(s)<\infty\qquad(4.4.5)$$

for any $\varepsilon\in(0,1/2)$.
(d) *For* $k=2$ *and* $m=1,2,\dots$, $m>1-d$,

$$\zeta_{d,2}(2m+2d-1)=\frac{2(-1)^{m+1}}{C_{d,2}(2m+2d-1)}\beta_N^{2m}\sum_{p=1}^{N} z_{p,N,d}^{2m+2d-2}$$
$$\times\frac{(2m+2d-1-z_{p,N,d}P''_{2N+d}(z_{p,N,d})/P'_{2N+d}(z_{p,N,d}))\log z_{p,N,d}+1}{P'^2_{2N+d}(z_{p,N,d})}$$
$$+O(\mathcal{E}^*_{N,2m+d}).\qquad(4.4.6)$$

Proof. It follows from Corollary 4.3.1(a) that for $-1-k(1-d)<\operatorname{Re}s<2N-k-1$, $s+k(1-d)-1\neq 0,2,\dots$, and for any sequence $\{y_N\}_{N=1}^{\infty}$ of positive numbers such that $\lim_{N\to\infty}\beta_N y_N=\infty$, the following asymptotic relation holds:

$$\frac{\beta_N^{s+k(1-d)-1}(y_N^{s+k(1-d)-1}-L_{2kN}(y_N,|y|^{s+k(1-d)-1},y^{-kd+1}P^k_{2N+d}(y)))}{y_N^{-kd}P^k_{2N+d}(y_N)}$$
$$=\frac{2\sin((s+k(1-d)-1)\pi/2)}{\pi}(C_{d,k}(s)\zeta_{d,k}(s)+O((\beta_N y_N)^{-2})$$
$$+O(\mathcal{E}^*_{N,\operatorname{Re}s+(k-1)(1-d)})),\qquad(4.4.7)$$

where $O(\mathcal{E}^*_{N,\operatorname{Re}s+(k-1)(1-d)})$ is independent of y_N and the constants $C_1(s)$ and $C_2(s)$ in $O((\beta_N y_N)^{-2})$ and $O(\mathcal{E}^*_{N,\operatorname{Re}s+(k-1)(1-d)})$, respectively, satisfy property (4.3.2). Next, for each N we can find $y_{N,1}>0$ such that

$$(\beta_N y)^{s+k(1-d)-1}/(y^{-kd}|P^k_{2N+d}(y)|)<\mathcal{E}^*_{N,\operatorname{Re}s+(k-1)(1-d)},\qquad y\ge y_{N,1}.\qquad(4.4.8)$$

Further, let

$$\beta_N^{s+k(1-d)-1}L_{2kN}(y)=\sum_{p=0}^{2kN}A_{p,k}y^p,\qquad y^{-kd}P^k_{2N+d}(y)=\sum_{p=0}^{2kN}B_{p,k}y^p,$$

where $B_{2kN,k}\neq 0$ because $y^{-d}P_{2N+d}(y)$ has $2N$ zeros. Then for each N we can find $y_{N,2}>0$ such that

$$\left|\frac{\beta_N^{s+k(1-d)-1}L_{2kN}(y)}{y^{-kd}P^k_{2N+d}(y)}-\frac{A_{2kN,k}}{B_{2kN,k}}\right|=\left|\frac{A_{2kN,k}+\sum_{p=1}^{2kN}A_{2kN-p,k}y^{-p}}{B_{2kN,k}+\sum_{p=1}^{2kN}B_{2kN-p,k}y^{-p}}-\frac{A_{2kN,k}}{B_{2kN,k}}\right|$$
$$<\mathcal{E}^*_{N,\operatorname{Re}s+(k-1)(1-d)},\qquad y\ge y_{N,2}.\qquad(4.4.9)$$

In addition, we choose $y_{N,3} > 0$ such that

$$(\beta_N y)^{-2} < \mathcal{E}^*_{N,\mathrm{Re}\, s+(k-1)(1-d)}, \qquad y \geq y_{N,3}. \tag{4.4.10}$$

Setting now $y_N := \max\{y_{N,1}, y_{N,2}, y_{N,3}\}$, we see that (4.0.3) and (4.4.10) yield

$$\lim_{N\to\infty} \beta_N y_N = \infty.$$

Therefore collecting relations (4.4.7) - (4.4.10), we obtain

$$\begin{aligned}\zeta_{d,k}(s) = &-\frac{\pi}{2\sin((s+k(1-d)-1)\pi/2)C_{d,k}(s)}\frac{A_{2kN,k}}{B_{2kN,k}}\\ &+ O(\mathcal{E}^*_{N,\mathrm{Re}\, s+(k-1)(1-d)}),\end{aligned} \tag{4.4.11}$$

where the constant $C(s)$ in $O(\mathcal{E}^*_{N,\mathrm{Re}\, s+(k-1)(1-d)})$ satisfies the property

$$\sup_{\varepsilon\leq \mathrm{Re}\, s\leq 1-\varepsilon} C(s) \leq C \sup_{\varepsilon\leq \mathrm{Re}\, s\leq 1-\varepsilon} |\sin((s+k(1-d)-1)\pi/2)C_{d,k}(s)|^{-1} < \infty \tag{4.4.12}$$

for any $\varepsilon \in (0, 1/2)$. Then properties (4.4.2) and (4.4.5) follow from (4.4.12). To prove statements (a) and (b) of Theorem 4.4.1, it remains to find the ratio $A_{2kN,k}/B_{2kN,k}$.

If $k = 1$, then we use the Lagrange interpolation formula

$$L_{2N}(y, f(y), yG_{2N}(y)) = \sum_{p=-N,\, p\neq 0}^{N} \frac{f(z_{p,N,d})yG_{2N}(y)}{(y-z_{p,N,d})z_{p,N,d}G'_{2N}(z_{p,N,d})}, \tag{4.4.13}$$

where $G_{2N}(y) := y^{-d}P_{2N+d}(y)$ and $z_{-p,N,d} = -z_{p,N,d}$, $1 \leq p \leq N$, are negative zeros of P_{2N+d}. Note that (4.4.13) holds for $f(y) = |y|^{s-d}$, $-2+d < \mathrm{Re}\, s < 2N-2$, $s-d \neq 0, 2, \ldots$, even in the case of $-2+d < \mathrm{Re}\, s \leq d$ when f is discontinuous at the origin (see the corresponding agreement after formula (4.2.9)). It follows from (4.4.13) that the leading coefficient of $\beta_N^{s-d}L_{2N}(y, f(y), yG_{2N}(y))$ is

$$A_{2N,1}(f) = B_{2N,1}\beta_N^{s-d} \sum_{p=-N,\, p\neq 0}^{N} \frac{f(z_{p,N,d})}{z_{p,N,d}^{-d+1}P'_{2N+d}(z_{p,N,d})}, \tag{4.4.14}$$

where $B_{2N,1}$ is the leading coefficient of $y^{-d}P_{2N+d}(y)$. Hence

$$\begin{aligned}\frac{A_{2N,1}}{B_{2N,1}} &= \beta_N^{s-d} \sum_{p=-N,\, p\neq 0}^{N} \frac{|z_{p,N,d}|^{s-1}(\mathrm{sgn}\, z_{p,N,d})^{1-d}}{P'_{2N+d}(z_{p,N,d})}\\ &= 2\beta_N^{s-d}\sum_{p=1}^{N} \frac{z_{p,N,d}^{s-1}}{P'_{2N+d}(z_{p,N,d})}.\end{aligned} \tag{4.4.15}$$

Thus (4.4.1) follows from (4.4.11) and (4.4.15), and statement (a) is established.

If $k = 2$, then we use the Hermite interpolation formula (see [26])

$$\begin{aligned}L_{4N}(y, f(y), yG_{2N}^2(y)) = &\sum_{p=-N,\, p\neq 0}^{N} \frac{yG_{2N}^2(y)}{(y-z_{p,N,d})^2}\\ &\times \Bigg(f(z_{p,N,d})\bigg(\frac{1}{z_{p,N,d}G'^2_{2N}(z_{p,N,d})} - \frac{y-z_{p,N,d}}{z_{p,N,d}^2G'^2_{2N}(z_{p,N,d})} - \frac{G''_{2N}(z_{p,N,d})(y-z_{p,N,d})}{z_{p,N,d}G'^3_{2N}(z_{p,N,d})}\bigg)\\ &+ \frac{f'(z_{p,N,d})(y-z_{p,N,d})}{z_{p,N,d}G'^2_{2N}(z_{p,N,d})}\Bigg),\end{aligned} \tag{4.4.16}$$

where $G_{2N}(y) = y^{-d}P_{2N+d}(y)$. Note that (4.4.16) holds for $f(y) = |y|^{s-2d+1}$, $-3+2d < \operatorname{Re} s < 2N-3$, $s-2d+1 \neq 0, 2, \dots$ (see the corresponding agreement after (4.2.9)). It follows from (4.4.16) that the leading coefficient of $\beta_N^{s-2d+1} L_{4N}(y, f(y), yG_{2N}^2(y))$ is

$$A_{4N,2}(f) = B_{4N,2}\beta_N^{s-2d+1} \sum_{p=-N,\, p\neq 0}^{N} \biggl(-\frac{f(z_{p,N,d})}{z_{p,N,d}^2 G_{2N}'^2(z_{p,N,d})} - \frac{f(z_{p,N,d})G_{2N}''(z_{p,N,d})}{z_{p,N,d}G_{2N}'^3(z_{p,N,d})} + \frac{f'(z_{p,N,d})}{z_{p,N,d}G_{2N}'^2(z_{p,N,d})} \biggr), \qquad (4.4.17)$$

where $B_{4N,2} = B_{2N,1}^2$ is the leading coefficient of $y^{-2d}P_{2N+d}^2(y)$. Then (4.4.17) yields

$$\frac{A_{4N,2}}{B_{4N,2}} = 2\beta_N^{s-2d+1} \sum_{p=1}^{N} z_{p,N,d}^{s-1} \frac{sP_{2N+d}'(z_{p,N,d}) - z_{p,N,d}P_{2N+d}''(z_{p,N,d})}{P_{2N+d}'^3(z_{p,N,d})}. \qquad (4.4.18)$$

Thus (4.4.4) follows from (4.4.11) and (4.4.18), and statement (c) is established.

The proof of statements (b) and (d) follows that of statements (a) and (c). Similarly to the proof of asymptotic (4.4.11), we can prove the relation ($m = 1, 2, \dots$, $m > k(1-d)/2$)

$$\zeta_{d,k}(2m+1-k(1-d)) = \frac{(-1)^{m+1}}{C_{d,k}(2m+1-k(1-d))} \frac{A_{2kN,k}^*}{B_{2kN,k}} + O(\mathcal{E}_{N,2m+d}^*), \qquad (4.4.19)$$

where $A_{2kN,k}^*$ is the leading coefficient of $\beta_N^{2m} L_{2kN}(y, f(y), yG_{2N}^k(y))$. Next we note that formulae (4.4.13) and (4.4.16) hold for $f(y) = y^{2m}\log|y|$, $m = 1, 2, \dots$. Therefore, using (4.4.14) for $k=1$ and (4.4.17) for $k=2$, we obtain

$$\frac{A_{2N,1}^*}{B_{2N,1}} = 2\beta_N^{2m} \sum_{p=1}^{N} \frac{z_{p,N,d}^{2m+d-1}\log z_{p,N,d}}{P_{2N+d}'(z_{p,N,d})}, \qquad (4.4.20)$$

$$\frac{A_{4N,2}^*}{B_{4N,2}} = 2\beta_N^{2m} \sum_{p=1}^{N} z_{p,N,d}^{2m+2d-2} \times \frac{((2m+2d-1)\log z_{p,N,d}+1)P_{2N+d}'(z_{p,N,d}) - z_{p,N,d}\log z_{p,N,d}P_{2N+d}''(z_{p,N,d})}{P_{2N+d}'^3(z_{p,N,d})}. \qquad (4.4.21)$$

Thus statements (b) and (d) follow from (4.4.19)–(4.4.21). This completes the proof of Theorem 4.4.1. ■

Remark 4.4.2. One of the conditions on s in statements (a) and (c) of Theorem 4.4.1 is $C_{d,k}(s) \neq 0$. Note that if $\operatorname{Re} s > -1-k(1-d)$ and $s+k(1-d)-1 \neq 0, 2, \dots$, then $C_{0,1}(s) \neq 0$ and $C_{1,2}(s) \neq 0$, while the solution set for $C_{1,1}(s) = 0$ is $\{(2\pi n/\log 2)i\}_{n=-\infty}^{\infty}$ and the solution set for $C_{0,2}(s) = 0$ is $\{1 + (2\pi n/\log 2)i\}_{n=-\infty}^{\infty}$.

Remark 4.4.3. Note that for $N = 1, 2, \dots$, the following identities hold:

$$\sum_{p=1}^{N} \frac{z_{p,N,d}^{2m+d-1}}{P_{2N+d}'(z_{p,N,d})} = 0, \qquad 1 \le m < N, \qquad (4.4.22)$$

$$\sum_{p=1}^{N} z_{p,N,d}^{2m+2d-2} \frac{2m+2d-1-z_{p,N,d}P_{2N+d}''(z_{p,N,d})/P_{2N+d}'(z_{p,N,d})}{P_{2N+d}'^2(z_{p,N,d})} = 0 \qquad (4.4.23)$$

if $1 \le m < 2N$. Indeed, $y^{2m} - L_{2kN}(y, y^{2m}, y^{-kd+1}P_{2N+d}^k(y)) = 0$ for all $y \in \mathbb{R}$. Then the leading coefficient of L_{2kN} is 0, and combining this fact with (4.4.15) and (4.4.18), we arrive at (4.4.22) and (4.4.23).

In particular, if P_{2N+d} is the normalized polynomial with equidistant zeros, we deduce from (3.2.27), (3.2.28) (3.2.29), (4.4.22), and (4.4.23) the following combinatorial identities:

$$\sum_{p=1}^{N} (-1)^{p+1} \binom{2N+d-1}{N-p} (2p+d-1)^{2m+d-1} = 0, \qquad 1 \le m < N, \tag{4.4.24}$$

$$\sum_{p=1}^{N} \binom{2N+d-1}{N-p}^2 (2p+d-1)^{2m+2d-2} \times \left((2p+d-1) \sum_{n=N-p+1}^{N+p+d-1} 1/n - (2m+2d-1) \right) = 0, \qquad 1-d \le m < 2N. \tag{4.4.25}$$

Note that for $d = 1$ identity (4.4.24) is known [44, Sect. 4.22, #34].

Below we obtain special asymptotic summation formulae for the zeta functions by applying Theorem 4.4.1 to Chebyshev polynomials, polynomials with equidistant zeros, and Williams–Apostol polynomials with explicit formulae for their zeros.

To find the convergence rate in these asymptotics, we need to estimate $\mathcal{E}_{n,\tau}^*$ by choosing γ_N and δ_N, $N = 1, 2, \ldots$. We begin with the Chebyshev polynomials. For any $\varepsilon \in (0, 2/3)$ we choose $\gamma_N = N^\varepsilon$ and $\delta_N = N^{-2+3\varepsilon}$, $N = 1, 2, \ldots$. Since the Chebyshev polynomials $\{P_{2N+d}\}_{N=1}^\infty$ of the first and second kinds are special cases of Gegenbauer polynomials with $\lambda = 0$ and $\lambda = 1$, respectively, Proposition 3.2.1 shows that $\{P_{2N+d}\}_{N=1}^\infty \in \mathbb{P}_d(\beta, C(\varepsilon)\gamma, \delta)$. Then

$$\mathcal{E}_{N,\tau}^* = \delta_N + \gamma_N^\tau e^{-\gamma_N} \le C_1(\varepsilon, \tau) N^{-2+3\varepsilon}. \tag{4.4.26}$$

For the polynomials with equidistant zeros $\{P_{2N+d}\}_{N=1}^\infty$, we set $\gamma_N = \log(N+1)$, $\delta_N = N^{-1}\log^2(N+1)$. Then $\{P_{2N+d}\}_{N=1}^\infty \in \mathbb{P}_d(\beta, \gamma, \delta)$, by Proposition 3.2.4, so

$$\mathcal{E}_{N,\tau}^* = \delta_N + \gamma_N^\tau e^{-\gamma_N} \le C_3(\tau) N^{-1} (\log N)^{\max\{2,\tau\}}. \tag{4.4.27}$$

Finally, setting $\gamma_N = N^\varepsilon$ and $\delta_N = N^{-1+2\varepsilon}$, $N = 1, 2, \ldots$, $\varepsilon \in (0, 1/2)$, for Williams–Apostol polynomials $\{P_{2N+d}\}_{N=1}^\infty$ of the first and second kinds, we see that

$$\{P_{2N+d}\}_{N=1}^\infty \in \mathbb{P}_d(\beta, C(\varepsilon)\gamma, \delta),$$

by Proposition 3.2.7. Then

$$\mathcal{E}_{N,\tau}^* \le C_2(\varepsilon, \tau) N^{-1+2\varepsilon}. \tag{4.4.28}$$

Therefore, using Theorem 4.4.1, estimates (4.4.26)–(4.4.28), identities (4.4.24), (4.4.25), and the corresponding formulae for $z_{p,N,d}$, $P'_{2N+d}(z_{p,N,d})$, and

$$P''_{2N+d}(z_{p,N,d}) / P'_{2N+d}(z_{p,N,d})$$

from Section 3.2, we obtain by straightforward calculation the following explicit asymptotic formulae for $\zeta_{d,k}(s)$. We recall that the constants $C_{d,k}(s)$ are defined in (4.2.3).

COROLLARY 4.4.4. *Let* $\operatorname{Re} s > -1 - k(1-d)$, $s + k(1-d) - 1 \neq 0, 2, \ldots$, $C_{d,k}(s) \neq 0$ *and let* $m = 1, 2, \ldots$, $m > (k-1)(1-d)$, *where* $d = 0, 1$ *and* $k = 1, 2$ *are indices. Then the following limit formulae for* $\zeta_{d,k}(s)$ *hold with the convergence rate of* $O(N^{-2+3\varepsilon})$, $\varepsilon \in (0, 2/3)$, *for Chebyshev polynomials,* $O(N^{-1}(\log N)^{s_N})$ *for polynomials with equidistant zeros, and* $O(N^{-1+2\varepsilon})$, $\varepsilon \in (0, 1/2)$, *for Williams–Apostol polynomials. Here,* $s_N := \max\{2, \operatorname{Re} s + (k-1)(1-d)\}$. *In addition, the constants* $C(s)$ *in the remainder term satisfy the property*

$$\sup_{\varepsilon_1 \le \operatorname{Re} s \le 1 - \varepsilon_1} C(s) < \infty \tag{4.4.29}$$

for any $\varepsilon_1 \in (0, 1/2)$.

(a) ***Normalized Chebyshev polynomials of the first kind:***

$$\zeta_{d,1}(s) = \frac{\pi}{\sin((s-d)\pi/2) C_{d,1}(s)} \lim_{N\to\infty} (-1)^{N+1} (2N+d)^{s-1} \times \sum_{p=1}^{N} (-1)^{p+1} \left(\cos \frac{(2p-1)\pi}{4N+2d} \right)^{s-1} \sin \frac{(2p-1)\pi}{4N+2d}, \tag{4.4.30}$$

$$\zeta_{d,1}(2m+d) = \frac{2(-1)^m}{C_{d,1}(2m+d)} \lim_{N\to\infty} (-1)^{N+1} (2N+d)^{2m+d-1} \times \sum_{p=1}^{N} (-1)^{p+1} \left(\cos \frac{(2p-1)\pi}{4N+2d} \right)^{2m+d-1} \sin \frac{(2p-1)\pi}{4N+2d} \log\cos \frac{(2p-1)\pi}{4N+2d}, \tag{4.4.31}$$

$$\zeta_{d,2}(s) = \frac{\pi}{\sin((s-2d+1)\pi/2) C_{d,2}(s)} \lim_{N\to\infty} (2N+d)^{s-1} \times \sum_{p=1}^{N} \left(\cos \frac{(2p-1)\pi}{4N+2d} \right)^{s-1} \left((s+1)\cos^2 \frac{(2p-1)\pi}{4N+2d} - s \right), \tag{4.4.32}$$

$$\zeta_{d,2}(2m+2d-1) = \frac{2(-1)^m}{C_{d,2}(2m+2d-1)} \lim_{N\to\infty} (2N+d)^{2m+2d-2} \times \sum_{p=1}^{N} \left(\cos \frac{(2p-1)\pi}{4N+2d} \right)^{2m+2d-2} \left(\left((2m+2d)\cos^2 \frac{(2p-1)\pi}{4N+2d} - (2m+2d-1) \right) \times \log\cos \frac{(2p-1)\pi}{4N+2d} - \sin^2 \frac{(2p-1)\pi}{4N+2d} \right). \tag{4.4.33}$$

(b) ***Normalized Chebyshev polynomials of the second kind:***

$$\zeta_{d,1}(s) = \frac{\pi}{\sin((s-d)\pi/2) C_{d,1}(s)} \lim_{N\to\infty} (-1)^{N+1} (2N+d+1)^{s-1} \times \sum_{p=1}^{N} (-1)^{p+1} \left(\cos \frac{p\pi}{2N+d+1} \right)^{s-1} \sin^2 \frac{p\pi}{2N+d+1}, \tag{4.4.34}$$

$$\zeta_{d,1}(2m+d) = \frac{2(-1)^m}{C_{d,1}(2m+d)} \lim_{N\to\infty} (-1)^{N+1} (2N+d+1)^{2m+d-1} \times \sum_{p=1}^{N} (-1)^{p+1} \left(\cos \frac{p\pi}{2N+d+1} \right)^{2m+d-1} \sin^2 \frac{p\pi}{2N+d+1} \log\cos \frac{p\pi}{2N+d+1}, \tag{4.4.35}$$

$$\zeta_{d,2}(s) = \frac{\pi}{\sin((s-2d+1)\pi/2)C_{d,2}(s)} \lim_{N\to\infty} (2N+d+1)^{s-1}$$
$$\times \sum_{p=1}^{N} \left(\cos\frac{p\pi}{2N+d+1}\right)^{s-1} \sin^2\frac{p\pi}{2N+d+1}\left((s+3)\cos^2\frac{p\pi}{2N+d+1} - s\right), \qquad (4.4.36)$$

$$\zeta_{d,2}(2m+2d-1) = \frac{2(-1)^m}{C_{d,2}(2m+2d-1)} \lim_{N\to\infty} (2N+d+1)^{2m+2d-2}$$
$$\times \sum_{p=1}^{N} \left(\cos\frac{p\pi}{2N+d+1}\right)^{2m+2d-2}$$
$$\times \sin^2\frac{p\pi}{2N+d+1}\left(\left((2m+2d+2)\cos^2\frac{p\pi}{2N+d+1} - (2m+2d-1)\right)\right.$$
$$\left.\times \log\cos\frac{p\pi}{2N+d+1} - \sin^2\frac{p\pi}{2N+d+1}\right). \qquad (4.4.37)$$

(c) ***Normalized polynomials with equidistant zeros:***

$$\zeta_{d,1}(s) = \frac{\pi^{s+1/2}}{\sin((s-d)\pi/2)C_{d,1}(s)2^{s+d-2}}$$
$$\times \lim_{N\to\infty} \frac{\sqrt{N}}{2^{2N}} \sum_{p=1}^{N} (-1)^{p+1} \binom{2N+d-1}{N-p} (2p+d-1)^{s-1}, \qquad (4.4.38)$$

$$\zeta_{d,1}(2m+d) = \frac{(-1)^m \pi^{2m+d-1/2}}{C_{d,1}(2m+d)2^{2m+2d-3}} \lim_{N\to\infty} \frac{\sqrt{N}}{2^{2N}}$$
$$\times \sum_{p=1}^{N} (-1)^{p+1} \binom{2N+d-1}{N-p} (2p+d-1)^{2m+d-1} \log(2p+d-1), \qquad (4.4.39)$$

$$\zeta_{d,2}(s) = \frac{\pi^{s+1}}{\sin((s-2d+1)\pi/2)C_{d,2}(s)2^{s+2d-3}} \lim_{N\to\infty} \frac{N}{2^{4N}}$$
$$\times \sum_{p=1}^{N} \binom{2N+d-1}{N-p}^2 (2p+d-1)^{s-1} \left((2p+d-1)\sum_{n=N-p+1}^{N+p+d-1} \frac{1}{n} - s\right), \qquad (4.4.40)$$

$$\zeta_{d,2}(2m+2d-1) = \frac{(-1)^m \pi^{2m+2d-1}}{C_{d,2}(2m+2d-1)2^{2m+4d-5}}$$
$$\times \lim_{N\to\infty} \frac{N}{2^{4N}} \sum_{p=1}^{N} \binom{2N+d-1}{N-p}^2 (2p+d-1)^{2m+2d-2}$$
$$\times \left(\left((2p+d-1)\sum_{n=N-p+1}^{N+p+d-1} \frac{1}{n} - (2m+2d-1)\right) \log(2p+d-1) + 1\right). \qquad (4.4.41)$$

Note that in these formulae we use the expression for the constant in (3.2.28) *asymptotically simplified by* (3.2.31).

(d) ***Normalized Williams–Apostol polynomials of the first kind:***

$$\zeta_{d,1}(s) = \frac{\pi}{\sin((s-d)\pi/2)C_{d,1}(s)} \lim_{N\to\infty} (-1)^{N+1}(2N+d)^{s-1} \times \sum_{p=1}^{N} (-1)^{p+1} \left(\cos\frac{(2p-1)\pi}{4N+2d}\right)^{s-1} \left(\sin\frac{(2p-1)\pi}{4N+2d}\right)^{2N+d-s-1}, \qquad (4.4.42)$$

$$\zeta_{d,1}(2m+d) = \frac{2(-1)^m}{C_{d,1}(2m+d)} \lim_{N\to\infty} (-1)^{N+1}(2N+d)^{2m+d-1} \times \sum_{p=1}^{N} (-1)^{p+1} \left(\cos\frac{(2p-1)\pi}{4N+2d}\right)^{2m+d-1} \left(\sin\frac{(2p-1)\pi}{4N+2d}\right)^{2N-2m-1} \times \log\cot\frac{(2p-1)\pi}{4N+2d}, \qquad (4.4.43)$$

$$\zeta_{d,2}(s) = \frac{\pi}{\sin((s-2d+1)\pi/2)C_{d,2}(s)} \lim_{N\to\infty} (2N+d)^{s-1} \sum_{p=1}^{N} \left(\cos\frac{(2p-1)\pi}{4N+2d}\right)^{s-1} \times \left(\sin\frac{(2p-1)\pi}{4N+2d}\right)^{4N+2d-s-3} \left(2(2N+d-1)\cos^2\frac{(2p-1)\pi}{4N+2d} - s\right), \qquad (4.4.44)$$

$$\zeta_{d,2}(2m+2d-1) = \frac{2(-1)^m}{C_{d,2}(2m+2d-1)} \lim_{N\to\infty} (2N+d)^{2m+2d-2} \times \sum_{p=1}^{N} \left(\cos\frac{(2p-1)\pi}{4N+2d}\right)^{2m+2d-2} \left(\sin\frac{(2p-1)\pi}{4N+2d}\right)^{4N-2m-2} \times \left(\left(2(2N+d-1)\cos^2\frac{(2p-1)\pi}{4N+2d} - (2m+2d-1)\right)\log\cot\frac{(2p-1)\pi}{4N+2d} + 1\right). \qquad (4.4.45)$$

(e) ***Normalized Williams–Apostol polynomials of the second kind:***

$$\zeta_{d,1}(s) = \frac{\pi}{\sin((s-d)\pi/2)C_{d,1}(s)} \lim_{N\to\infty} (-1)^{N+1}(2N+d+1)^{s-1} \times \sum_{p=1}^{N} (-1)^{p+1} \left(\cos\frac{p\pi}{2N+d+1}\right)^{s-1} \left(\sin\frac{p\pi}{2N+d+1}\right)^{2N+d-s}, \qquad (4.4.46)$$

$$\zeta_{d,1}(2m+d) = \frac{2(-1)^m}{C_{d,1}(2m+d)} \lim_{N\to\infty} (-1)^{N+1}(2N+d+1)^{2m+d-1} \sum_{p=1}^{N} (-1)^{p+1} \times \left(\cos\frac{p\pi}{2N+d+1}\right)^{2m+d-1} \left(\sin\frac{p\pi}{2N+d+1}\right)^{2N-2m} \log\cot\frac{p\pi}{2N+d+1}, \qquad (4.4.47)$$

$$\zeta_{d,2}(s) = \frac{\pi}{\sin((s-2d+1)\pi/2)C_{d,2}(s)} \lim_{N\to\infty} (2N+d+1)^{s-1} \sum_{p=1}^{N} \left(\cos\frac{p\pi}{2N+d+1}\right)^{s-1} \times \left(\sin\frac{p\pi}{2N+d+1}\right)^{4N+2d-s-1} \left(2(2N+d)\cos^2\frac{p\pi}{2N+d+1} - s\right), \qquad (4.4.48)$$

$$\zeta_{d,2}(2m+2d-1) = \frac{2(-1)^m}{C_{d,2}(2m+2d-1)} \lim_{N\to\infty} (2N+d+1)^{2m+2d-2}$$
$$\times \sum_{p=1}^{N} \left(\cos \frac{p\pi}{2N+d+1}\right)^{2m+2d-2} \left(\sin \frac{p\pi}{2N+d+1}\right)^{4N-2m}$$
$$\times \left(\left(2(2N+d)\cos^2 \frac{p\pi}{2N+d+1} - (2m+2d-1)\right) \log\cot \frac{p\pi}{2N+d+1} + 1\right). \qquad (4.4.49)$$

Remark 4.4.5. Using the identity [44, Sect. 4.2.2, #25]

$$\sum_{p=1}^{N} (-1)^{p+1}(2p+d-1)^{s-1}\binom{2N+d-1}{N-p}$$
$$= \frac{2^{2N+d-1}\Gamma(s)\sin((s-d)\pi/2)}{\pi} \int_0^\infty \frac{\sin^{2N+d-1} t}{t^s}\, dt,$$

where the integral on the right-hand side is convergent for $N \geq 1$ and $\operatorname{Re} s > d$, we obtain from (4.4.38) the following limit integral representation:

$$\zeta_{d,1}(s) = \frac{\pi^{s-1/2}}{2^{s(1-d)}(2^s-1)} \lim_{N\to\infty} \sqrt{N} \int_0^\infty \frac{\sin^{2N+d-1} t}{t^s}\, dt, \qquad \operatorname{Re} s > d.$$

Remark 4.4.6. Another limit formula:

$$\zeta(n) = (\pi/2)^n \lim_{N\to\infty} N^{-n} \sum_{p=1}^{N} \cot^n(p\pi/(2N+1)), \qquad (4.4.50)$$

was established independently by Williams [56] and Apostol [2] for even positive integers n. Apostol [2] states that (4.4.50) is valid for odd $n > 1$ as well.

Remark 4.4.7. Setting $d = 0$ and $m = 1$ in formulae (4.4.31), (4.4.35), (4.4.39), (4.4.43), and (4.4.47), we can obtain new limit representations for Catalan's constant $\zeta_{0,1}(2) = \beta(2)$. In particular, it follows from (4.4.39) that

$$\beta(2) = -\pi^{3/2} \lim_{N\to\infty} \frac{\sqrt{N}}{2^{2N}} \sum_{p=1}^{N} (-1)^{p+1}\binom{2N-1}{N-p}(2p-1)\log(2p-1).$$

In general, formulae (4.4.31), (4.4.33), (4.4.35), (4.4.37), (4.4.39), (4.4.41), (4.4.43), (4.4.45), (4.4.47), and (4.4.49) give various limit representations for $\zeta(n)$ and $\beta(n)$, where $n > 1$ is an integer. In particular, these results imply new limit representations for the Bernoulli numbers B_{2n} and the Euler numbers E_{2n} (see Section 6.3 for definitions).

5. Asymptotic relations between the L_p-interpolation error and zeta functions

In this chapter we obtain asymptotic representations for the zeta functions in the L_p-metric. We recall that d and k are indices that take the values of $0, 1$ and $1, 2$, respectively.

5.1. L_p-asymptotics for the integral term. The right-hand side of each asymptotic in (4.2.10) and (4.2.13) contains the principal term with $\zeta_{d,k}(s)$, the integral term, and the remainder term. In this section we find the L_p-asymptotic behavior of the integral term. Note that if $\zeta(s) \neq 0$, then this term is the second remainder term, while for $\zeta(s) = 0$ it becomes the principal term.

Let

$$I^*_{N,k}(y) := \frac{P^k_{2N+d}(y)}{\beta_N^{s+k(1-d)-1}(\beta_N y)^2 y^{kd}} \int_0^\infty \frac{t^{s+k}}{(1+t^2/(\beta_N y)^2)h^k_d(t)}\, dt,$$

where h_d is defined in (4.0.1), be the integral term from (4.2.10) or (4.2.13) without the constant. Then:

THEOREM 5.1.1. *Let* $\{P_{2N+d}\}_{N=1}^\infty \in \mathbb{P}^{**}_d(\beta,\gamma,\delta,W,k)$ *and let* $\operatorname{Re} s > -1-k(1-d)$, *and*

$$p \in \big((kd+2)^{-1}, (\max\{0, -\operatorname{Re} s - k(1-d)+1\})^{-1}\big) \cap (p_0(k), p_1(k)),$$

where $p_0 = p_0(k)$ *and* $p_1 = p_1(k)$ *are numbers from properties* (C5.3) *and* (C5.4) *of Definition* 3.1.8. *Then*

$$\lim_{N\to\infty} \beta_N^{(\operatorname{Re} s+k(1-d)-1)p+1} \int_{-a}^{a} |I^*_{N,k}(y) w^k(y,N)|^p\, dy = I^*, \tag{5.1.1}$$

where

$$I^* = I^*_{d,k,s,p} := \int_{-\infty}^{\infty} \left|\frac{\cos(y-d\pi/2)}{y^d}\right|^{kp} \left|\int_0^\infty \frac{t^{s+k}\, dt}{(y^2+t^2)h^k_d(t)}\right|^p dy < \infty. \tag{5.1.2}$$

Here, $a = 1$ *or* $a = \infty$ (*cf. Definition* 3.1.8).

Note that if $p \in (0,\infty)$, then $(\operatorname{Re} s + k(1-d)-1)p+1 > 0$ if and only if $p < (\max\{0, -\operatorname{Re} s - k(1-d)+1\})^{-1}$. To prove the theorem, we need some technical estimates.

LEMMA 5.1.2. *For* $\operatorname{Re} s > -1-k(1-d)$ *and* $F(y) := \int_0^\infty \frac{t^{s+k}\, dt}{(y^2+t^2)h^k_d(t)}$, *the following statements hold:*

(a) *For* $y \in (0,1]$,

$$|F(y)| \le C + C_1 \begin{cases} y^{\operatorname{Re} s+k(1-d)-1}, & \operatorname{Re} s + k(1-d) - 1 \neq 0,\\ \log(1/y), & \operatorname{Re} s + k(1-d) - 1 = 0. \end{cases} \tag{5.1.3}$$

(b) *Let $H(y)$ be continuous on $(0,\infty)$ and $K \in (1,\infty]$ be a fixed number or $K=\infty$. Let H and K satisfy the conditions*

$$\sup_{y\in(0,1]} y^{-kd}|H(y)| \le A, \qquad \sup_{y\in(1,K)} |H(y)| \le A \tag{5.1.4}$$

for some $A>0$. Then for $(kd+2)^{-1} < p < (\max\{0, -\operatorname{Re} s - k(1-d)+1\})^{-1}$,

$$\int_0^K y^{-kdp}|H(y)F(y)|^p\,dy \le CA^p. \tag{5.1.5}$$

Proof. (a) For $y\in(0,1]$ we obtain

$$\begin{aligned}|F(y)| &\le \int_0^\infty \frac{t^{\operatorname{Re} s+k}\,dt}{(y^2+t^2)h_d^k(t)} \le \frac{1}{y^2}\int_0^y \frac{t^{\operatorname{Re} s+k}\,dt}{h_d^k(t)} + \int_y^1 \frac{t^{\operatorname{Re} s+k-2}\,dt}{h_d^k(t)} + \int_1^\infty \frac{t^{\operatorname{Re} s+k-2}\,dt}{h_d^k(t)}\\ &\le C_2 y^{\operatorname{Re} s+k(1-d)-1} + C_3\begin{Bmatrix} y^{\operatorname{Re} s+k(1-d)-1}, & \operatorname{Re} s+k(1-d)-1\ne 0,\\ \log 1/y, & \operatorname{Re} s+k(1-d)-1=0,\end{Bmatrix} + C_4. \end{aligned}\tag{5.1.6}$$

Then (5.1.6) yields (5.1.3).

(b) Using (5.1.3) and (5.1.4), we obtain

$$\begin{aligned}\int_0^K y^{-kdp}|H(y)F(y)|^p\,dy &= \int_0^1 + \int_1^K\\ &\le A^p\int_0^1 |F(y)|^p\,dy + A^p\int_1^K y^{-kdp-2p}\left|\int_0^\infty \frac{t^{\operatorname{Re} s+k}\,dt}{(1+(t/y)^2)h_d^k(t)}\right|^p dy \le CA^p.\end{aligned}$$

Hence (5.1.5) follows. ∎

Proof of Theorem 5.1.1. We first note that the double integral I^* defined by (5.1.2) is convergent. This fact follows from Lemma 5.1.2(b) for $H(y)=\cos^k(y-d\pi/2)$, $A=1$, and $K=\infty$. Next, we consider two cases.

Case 1. Let $p\in(0,1]$. Then for any fixed $B\in[1,\min\{\gamma_N, a\beta_N/2\}]$, where N is large enough, we obtain

$$\begin{aligned} I(N) &:= \beta_N^{(\operatorname{Re} s+k(1-d)-1)p+1}\int_0^a |I^*_{N,k}(y)w^k(y,N)|^p\,dy\\ &= \beta_N^{(\operatorname{Re} s+k(1-d)-1)p}\int_0^{a\beta_N} |I^*_{N,k}(y/\beta_N)w^k(y/\beta_N,N)|^p\,dy\\ &\le \int_0^B \left|\frac{\cos(y-d\pi/2)}{y^d}\right|^{kp}\left|\int_0^\infty \frac{t^{s+k}\,dt}{(y^2+t^2)h_d^k(t)}\right|^p w^{kp}(y/\beta_N,N)dy\\ &\quad+\int_B^{a\beta_N} |\beta_N^{kd}P^k_{2N+d}(y/\beta_N)w^k(y/\beta_N,N)|^p\frac{1}{y^{(2+kd)p}}\left(\int_0^\infty \frac{t^{\operatorname{Re} s+k}\,dt}{(1+(t/y)^2)h_d^k(t)}\right)^p dy\\ &\quad+\int_0^B |\beta_N^{kd}P^k_{2N+d}(y/\beta_N)-\cos^k(y-d\pi/2)|^p\frac{w^{kp}(y/\beta_N,N)}{y^{(2+kd)p}}\\ &\quad\times\left|\int_0^\infty \frac{t^{s+k}\,dt}{(1+(t/y)^2)h_d^k(t)}\right|^p dy\\ &= I_{N,1}(B)+I_{N,2}(B)+I_{N,3}(B). \end{aligned}\tag{5.1.7}$$

To estimate the integrals in (5.1.7), we need, firstly, the relations ($p > 0$)

$$w^{kp}(y/\beta_N, N) = 1 + o(1), \qquad \sup_{y\in[0,B]} w^{kp}(y/\beta_N, N) = 1 + o(1), \qquad N \to \infty, \tag{5.1.8}$$

where $o(1)$ in the first relation is independent of $y \in [0, B]$. These follow from property (C4) of Definition 3.1.8. Secondly, we need the inequality

$$\beta_N^d |P_{2N+d}(y/\beta_N)| w(y/\beta_N, N) \le \psi(y/(a_N\beta_N)), \qquad y \in (-a\beta_N, a\beta_N), \tag{5.1.9}$$

from property (C5.2) of Definition 3.1.8, where $\psi \ge 0$ is an even function on $(-a, a)$ independent of N that satisfies properties (C5.1)–(C5.3), $N = 1, 2, \ldots$.

Then it follows from the first relation in (5.1.8) that, as $N \to \infty$,

$$\begin{aligned} I_{N,1}(B) &= (1 + o(1)) \int_0^B \left|\frac{\cos(y - d\pi/2)}{y^d}\right|^{kp} \left|\int_0^\infty \frac{t^{s+k}\, dt}{(y^2 + t^2)h_d^k(t)}\right|^p dy \\ &= (1 + o(1))I^* - (1 + o(1)) \int_B^\infty |\cos(y - d\pi/2)|^{kp} \frac{1}{y^{(2+kd)p}} \left|\int_0^\infty \frac{t^{s+k}\, dt}{(1 + (t/y)^2)h_d^k(t)}\right|^p dy, \end{aligned} \tag{5.1.10}$$

where the last integral term does not exceed $C_1(1 + o(1))/B^{(2+kd)p-1}$. Therefore, as $N \to \infty$,

$$(1 + o(1))I^* - C_1(1 + o(1))/B^{(2+kd)p-1} \le I_{N,1}(B) \le (1 + o(1))I^*, \tag{5.1.11}$$

where C_1 is independent of B and N but $o(1)$ depends on B.

Next, by property (C5.1) there exists $\mu \in (0, a/2)$ such that ψ is bounded on $[0, 2\mu]$. Since $a_N \ge 1$, we conclude that $\psi(y/(a_N\beta_N))$ is bounded on $[0, \mu\beta_N]$ by a constant independent of N. Since $B \in [1, a\beta_N]$, inequality (5.1.9) and the second relation in (5.1.8) yield

$$\begin{aligned} I_{N,2}(B) &\le C_2 \int_B^{a\beta_N} \frac{\psi^{kp}(y/(a_N\beta_N))}{y^{(2+kd)p}}\, dy \\ &= C_2 \left(\int_B^{\mu\beta_N} + \int_{\mu\beta_N}^{a\beta_N}\right) = C_2(I_{N,2,1} + I_{N,2,2}), \end{aligned} \tag{5.1.12}$$

where C_2 is independent of N and B. Then

$$I_{N,2,1} \le \sup_{y\in[0,\mu\beta_N]} \psi^{kp}(y/(a_N\beta_N)) \int_B^{a\beta_N} y^{-(2+kd)p}\, dy \le C_3/B^{(2+kd)p-1}, \tag{5.1.13}$$

where C_3 is independent of N and B, and for $p_0(k) < p < p_1(k)$,

$$I_{N,2,2} \le (1/\beta_N^{(2+kd)p-1}) \sup_{N\in\mathbb{N}} \frac{1}{a_N^{(2+kd)p-1}} \int_{\mu/a_N}^{a/a_N} \frac{\psi^{kp}(y)\, dy}{y^{(2+kd)p}} \le C_4/\beta_N^{(2+kd)p-1}, \tag{5.1.14}$$

where C_4 is independent of N and B. Note that in the case $a = a_N = 1$ the last inequality in (5.1.14) follows from property (C5.3) of Definition 3.1.8, and in the case $a = \infty$ it follows from properties (C5.1), (C5.3), and L'Hospital's rule. Indeed, if $a = \infty$, then (5.1.14) is trivial for $\sup_{N\in\mathbb{N}} a_N < \infty$, since by property (C5.2), $\inf_{N\in\mathbb{N}} a_N \ge 1$. If

$\lim_{N\to\infty} a_N = \infty$, then

$$\sup_{N\in\mathbb{N}} \frac{1}{a_N^{(2+kd)p-1}} \int_{\mu/a_N}^{a/a_N} \frac{\psi^{kp}(y)\,dy}{y^{(2+kd)p}} \le \sup_{z\ge 1} \frac{1}{z^{(2+kd)p-1}} \int_{\mu/z}^{a/z} \frac{\psi^{kp}(y)\,dy}{y^{(2+kd)p}} \le C_4,$$

since for $\lambda := (2+kd)p-1 > 0$,

$$\lim_{z\to\infty} \frac{1}{z^\lambda} \int_{\mu/z}^{a/z} \frac{\psi^{kp}(y)\,dy}{y^\lambda} = \lambda^{-1}\psi^{kp}(0)(\mu^{-\lambda} - a^{-\lambda}).$$

Collecting relations (5.1.12)–(5.1.14), we obtain

$$I_{N,2}(B) \le C_5(1/B^{(2+kd)p-1} + 1/\beta_N^{(2+kd)p-1}), \tag{5.1.15}$$

where C_5 is independent of N and B.

Further, Property (C3) of Definition 3.1.1 yields the estimates

$$\max_{y\in[0,B]} |\beta_N^d P_{2N+d}(y/\beta_N) - \cos(y - d\pi/2)| \le C_6(B)\min\{y^2, 1\}\delta_N, \tag{5.1.16}$$

$$\max_{y\in[0,B]} |\beta_N^d P_{2N+d}(y/\beta_N) + \cos(y - d\pi/2)| \le C_7(B), \tag{5.1.17}$$

since $B \in [1, \gamma_N]$. Then taking account of (5.1.17) and the second relation in (5.1.8), we obtain, as $N \to \infty$,

$$I_{N,3}(B) \le C_8(B,k)(1+o(1)) \times \int_0^B \left|\frac{\beta_N^d P_{2N+d}(y/\beta_N) - \cos(y - d\pi/2)}{y^{kd}}\right|^p \left|\int_0^\infty \frac{t^{s+k}\,dt}{(t^2+y^2)h_d^k(t)}\right|^p dy, \tag{5.1.18}$$

where $C_8(B,1) = 1$ and $C_8(B,2) = C_7^p(B)$. Next, we apply again Lemma 5.1.2(b) to the integral on the right-hand side of (5.1.18) for $K = B$ and $H(y) = \beta_N^d P_{2N+d}(y/\beta_N) - \cos(y-d\pi/2)$, where H satisfies condition (5.1.4) with $A = C_6(B)\delta_N$, by estimate (5.1.16). Hence (5.1.5) yields

$$I_{N,3}(B) \le (1+o(1))C_9(B)\delta_N^p \le C_{10}(B)\delta_N^p. \tag{5.1.19}$$

Then combining (5.1.7), (5.1.15) and (5.1.19) with the right inequality in (5.1.11), we obtain

$$I(N) \le I^* + o(1)I^* + C_5/B^{(2+kd)p-1} + C_5/\beta_N^{(2+kd)p-1} + C_{10}(B)\delta_N^p.$$

Further, for any $\varepsilon \in (0,1)$ we choose $B = \varepsilon^{1/(1-(2+kd)p)}$ and then find $N_0 = N_0(\varepsilon)$ such that for all $N \ge N_0$,

$$\max\{o(1)I^*, (\min\{\gamma_N, a\beta_N/2\})^{1-(2+kd)p}, \beta_N^{1-(2+kd)p}, C_{10}(B)\delta_N^p\} < \varepsilon/4. \tag{5.1.20}$$

Hence $B \in [1, \min\{\gamma_N, a\beta_N/2\}]$ and we arrive at the upper estimate

$$I(N) \le I^* + \varepsilon, \qquad N \ge N_0(\varepsilon). \tag{5.1.21}$$

A similar lower estimate

$$I(N) \ge I^* - \varepsilon, \qquad N \ge N_0(\varepsilon), \tag{5.1.22}$$

follows from the inequality

$$I(N) \ge I_{N,1}(B) - I_{N,2}(B) - I_{N,3}(B),$$

estimates (5.1.15), (5.1.19), (5.1.20), and the left inequality in (5.1.11). Since ε is an arbitrary number from $(0,1)$, inequalities (5.1.21) and (5.1.22) yield (5.1.1) for $0 < p \le 1$.

CASE 2. Let $p \in (1,\infty)$. The proof in this case follows that of Case 1 if we use Minkowski's inequality instead of the integral triangle inequality for $0 < p \le 1$. To prove (5.1.1) in this case, we use the inequalities

$$I_{N,1}^{1/p}(B) - I_{N,2}^{1/p}(B) - I_{N,3}^{1/p}(B) \le I^{1/p}(N) \le I_{N,1}^{1/p}(B) + I_{N,2}^{1/p}(B) + I_{N,3}^{1/p}(B), \tag{5.1.23}$$

and an analogue of (5.1.11):

$$(1+o(1))I^{*1/p} - (1+o(1))/B^{2+kd-1/p} \le I_{N,1}^{1/p}(B) \le (1+o(1))I^{*1/p}, \tag{5.1.24}$$

which follows from (5.1.10). Analogues of (5.1.15) and (5.1.19) are also valid in the following form:

$$I_{N,2}^{1/p}(B) \le C_5(1/B^{2+kd-1/p} + 1/\beta_N^{2+kd-1/p}), \tag{5.1.25}$$

$$I_{N,3}^{1/p}(B) \le C_9(B)(1+o(1))\delta_N. \tag{5.1.26}$$

Then the estimates

$$I^{*1/p} - \varepsilon^{1/p} \le I^{1/p}(N) \le I^{*1/p} + \varepsilon^{1/p}, \qquad N \ge N_0(\varepsilon),$$

follow from (5.1.23) - (5.1.26), and (5.1.20). This completes the proof of Theorem 5.1.1. ∎

The following corollary is an immediate consequence of the theorem:

COROLLARY 5.1.3. *Let s, p, and $\{P_{2N+d}\}_{N=1}^{\infty}$ be as in Theorem* 5.1.1. *Then*

$$\left(\int_{-a}^{a} |I_{N,k}^{*}(y)w^k(y,N)|^p\,dy\right)^{1/p} \le C\beta_N^{-(\operatorname{Re} s + k(1-d) - 1 + 1/p)}, \tag{5.1.27}$$

where $\operatorname{Re} s + k(1-d) - 1 + 1/p > 0$.

REMARK 5.1.4. The constant $I^* = I^*_{d,k,s,p}$ defined by (5.1.2) can be evaluated explicitly in some cases of $p = 1$ and $s \in \mathbb{R}$. In particular,

$$\begin{aligned} I^*_{0,1,s,1} &= \int_{-\infty}^{\infty} \left|\cos y\right| \int_0^{\infty} \frac{t^{s+1}}{(y^2+t^2)\cosh t}\,dt\,dy \\ &= 4\Gamma(s+1)\beta(s+2), \qquad s > -1, \end{aligned} \tag{5.1.28}$$

$$\begin{aligned} I^*_{0,2,s,1} &= \int_{-\infty}^{\infty} \cos^2 y \int_0^{\infty} \frac{t^{s+2}}{(y^2+t^2)\cosh^2 t}\,dt\,dy \\ &= \pi 2^{-s-1}(1-2^{-s-1})\Gamma(s+2)\zeta(s+2), \qquad s > -2, \end{aligned} \tag{5.1.29}$$

$$\begin{aligned} I^*_{1,2,s,1} &= \int_{-\infty}^{\infty} \left(\frac{\sin y}{y}\right)^2 \int_0^{\infty} \frac{t^{s+2}}{(y^2+t^2)\sinh^2 t}\,dt\,dy \\ &= \pi 2^{-s+1}(s-1)\Gamma(s)\zeta(s), \qquad s > 0. \end{aligned} \tag{5.1.30}$$

Indeed, taking into account the integrals $(t > 0)$

$$\int_{-\infty}^{\infty} \frac{\cos^2 y}{y^2+t^2}\,dy = \frac{\pi(1+e^{-2t})}{2t}, \qquad \int_{-\infty}^{\infty} \frac{\sin^2 y}{y^2(y^2+t^2)}\,dy = \frac{\pi}{t^2}\left(1 - \frac{1-e^{-2t}}{2t}\right),$$

we arrive at (5.1.29) and (5.1.30). To prove (5.1.28), we first evaluate

$$\int_{-\infty}^{\infty}\frac{|\cos y|}{y^2+t^2}\,dy=\int_{-\pi/2}^{\pi/2}\cos y\sum_{k=-\infty}^{\infty}\frac{1}{y^2+(t+k\pi)^2}\,dy=\frac{\sinh 2t}{t}\int_{-\pi/2}^{\pi/2}\frac{\cos y}{\cosh 2t-\cos 2y}\,dy$$

$$=\frac{\sinh 2t\arctan(((\cosh 2t-1)/2)^{-1/2})}{t((\cosh 2t-1)/2)^{1/2}}=\frac{2\cosh t\arctan(1/\sinh t)}{t}.$$

Hence

$$I^*_{0,1,s,1}=2\int_0^{\infty}t^s\arctan(1/\sinh t)\,dt=\frac{2}{s+1}\int_0^{\infty}\frac{t^{s+1}}{\cosh t}\,dt.$$

Thus (5.1.28) is established.

In addition, we remark that the constant $I^*_{0,1,s,1}$ for $s>-1$, $s\neq 0,2,\dots$, was introduced by Nikol'skiĭ [42] who proved the asymptotic

$$\lim_{n\to\infty}n^{s+1}E_n(|a-x|^s)_{L_1[-1,1]}=2(|\sin s\pi/2|/\pi)I^*_{0,1,s,1}(1-a^2)^{(s+1)/2},\qquad a\in(-1,1),$$

where $E_n(f)_{L_1[-1,1]}$ is the error of best approximation to f by polynomials of degree n in the integral metric on $[-1,1]$. He also proved (5.1.28) for odd positive s. Bernstein [7] extended (5.1.28) to all $s>0$. Using Bernstein's idea from [7], we showed above that (5.1.28) holds for all $s>-1$.

5.2. L_p-asymptotic representations for zeta functions. In this section we discuss asymptotic representations for $|\zeta(s)|$ and $|\beta(s)|$ through the interpolation errors in the weighted L_p-metrics.

5.2.1. General theorem

THEOREM 5.2.1. *Let $\{P_{2N+d}\}_{N=1}^{\infty}\in\mathbb{P}^{**}_d(\beta,\gamma,\delta,W,k)$. Then the following statements hold:*

(a) *Let* $\operatorname{Re}s>-1-k(1-d)$, $s+k(1-d)-1\neq 0,2,\dots$, $C_{d.k}(s)\neq 0$, *and let*

$$p\in((kd+2)^{-1},(\max\{0,-\operatorname{Re}s-k(1-d)+1\})^{-1})\cap(p_0(k),p_1(k))\tag{5.2.1}$$

as in Theorem 5.1.1. *In addition, let*

$$\lim_{N\to\infty}\beta_N^{1/p}\left(\int_{-a}^{a}|y^{-kd}P^k_{2N+d}(y)w^k(y,N)|^p\,dy\right)^{1/p}=\infty.\tag{5.2.2}$$

If $\zeta_{d,k}(s)\neq 0$, then

$$|\zeta_{d,k}(s)|=\frac{\pi}{2|\sin((s+k(1-d)-1)\pi/2)||C_{d,k}(s)|}\lim_{N\to\infty}\beta_N^{\operatorname{Re}s+k(1-d)-1}$$
$$\times\frac{\left(\int_{-a}^{a}||y|^{s+k(1-d)-1}-L_{2kN}(y,|y|^{s+k(1-d)-1},y^{-kd+1}P^k_{2N+d}(y))|^p w^{kp}(y,N)\,dy\right)^{1/p}}{\left(\int_{-a}^{a}|y^{-kd}P^k_{2N+d}(y)w^k(y,N)|^p\,dy\right)^{1/p}}.\tag{5.2.3}$$

(b) *Let* $0<\operatorname{Re}s<1$ *and let p satisfy condition* (5.2.1). *In addition, let*

$$\lim_{N\to iy}\beta_N^{1/p}\mathcal{E}^*_{N,\operatorname{Re}s+(k-1)(1-d)}\left(\int_{-a}^{a}|y^{-kd}P^k_{2N+d}(y)w^k(y,N)|^p\,dy\right)^{1/p}=0.\tag{5.2.4}$$

If $\zeta_{d,k}(s) = 0$, *then*

$$\frac{\pi}{2|\sin((s+k(1-d)-1)\pi/2)|} \lim_{N\to\infty} \beta_N^{\mathrm{Re}\, s+k(1-d)-1+1/p}$$
$$\times \left(\int_{-a}^{a} \left||y|^{s+k(1-d)-1} - L_{2kN}(y, |y|^{s+k(1-d)-1}, y^{-kd+1}P_{2N+d}^k(y))\right|^p w^{kp}(y,N)\,dy\right)^{1/p}$$
$$= I_{d,k,s,p}^{*1/p}, \qquad (5.2.5)$$

where $I^*_{d,k,s,p}$ *is defined in* (5.1.2).

(c) *Let* $m = 1, 2, \ldots$, $m > (k-1)(1-d)$, *and let* $p \in ((kd+2)^{-1}, \infty) \cap (p_0(k), p_1(k))$. *If condition* (5.2.2) *is satisfied, then*

$$\zeta_{d,k}(2m+1-k(1-d)) = \frac{1}{C_{d,k}(2m+1-k(1-d))} \lim_{N\to\infty} \beta_N^{2m}$$
$$\times \frac{\left(\int_{-a}^{a} |y^{2m}\log|y| - L_{2kN}(y, y^{2m}\log|y|, y^{-kd+1}P_{2N+d}^k(y))|^p w^{kp}(y,N)\,dy\right)^{1/p}}{\left(\int_{-a}^{a} |y^{-kd}P_{2N+d}^k(y) w^k(y,N)|^p\,dy\right)^{1/p}}. \qquad (5.2.6)$$

Proof. We first note that $\int_{-a}^{a} |y^{-kd}P_{2N+d}^k(y)w^k(y,N)|^p\,dy < \infty$ by property (C5.4) of Definition 3.1.8. Next, Theorem 5.2.1 follows from Theorems 4.2.3 and 5.1.1 and Corollary 5.1.3. Indeed, relation (5.2.3) is an immediate consequence of relations (4.2.10), (4.2.11), (5.1.27), and (5.2.2); in turn, (5.2.5) follows from (4.2.10), (4.2.11), (5.1.1), (5.1.2), and (5.2.4); finally, (5.2.6) is a corollary of (4.2.13), (4.2.14), (5.1.27), and (5.2.2). ∎

Note that the condition $m > (k-1)(1-d)$ in Theorem 5.2.1(c) and subsequent corollaries (see also Theorem 4.4.1(d) and Corollary 4.4.4) rules out the case of $m = 1$, $d = 0$, $k = 2$ for the representation of $\zeta(1)$.

Further, we discuss special cases of Theorem 5.2.1 for polynomials P_{2N+d} studied in Sections 3.2 and 3.3.

5.2.2. Examples

Example 5.2.2 (Normalized Gegenbauer polynomials on $[-1,1]$). We have

$$P_{2N+d}(y) = C_{2N+d}^{\lambda}(y)/(C_{2N+d}^{\lambda})^{(d)}(0), \quad \beta_N = 2N+\lambda+d, \quad w(y,N) = (1-y^2)^{\lambda/2-1/4},$$
$$N = 1, 2, \ldots, \quad \lambda \ge 0, \quad a = 1, \quad p_0(k) = 0, \quad p_1(k) = 4/k.$$

For $N = 1, 2, \ldots$, we choose

$$\gamma_N := N^{\varepsilon}, \qquad \varepsilon \in (0, 2/3), \qquad (5.2.7)$$
$$\delta_N := \gamma_N^3/N^2 = N^{-2+3\varepsilon}, \qquad \varepsilon \in (0, 2/3). \qquad (5.2.8)$$

Then (5.2.7) and (5.2.8) show that conditions (3.2.5) and (3.2.6) are satisfied and by Proposition 3.2.1, $\{P_{2N+d}\}_{N=1}^{\infty} \in \mathbb{P}_d^{**}(\beta, C(\varepsilon)\gamma, \delta, W, k)$. In addition, (5.2.7) and (5.2.8) yield

$$N^{-2+3\varepsilon} \le \mathcal{E}^*_{N,\tau} = \delta_N + \gamma_N^{\tau} e^{-\gamma_N} \le C_1(\varepsilon) N^{-2+3\varepsilon}. \qquad (5.2.9)$$

Further, it follows from Theorem 3.3.4 that condition (5.2.2) is satisfied if and only if

$$\lim_{N\to\infty} N^{1/p}/\tau_{N,d,kp} = \infty, \qquad p \in (p_0(k), p_1(k)). \qquad (5.2.10)$$

In addition, combining Theorem 3.3.4 with (5.2.9), we conclude that condition (5.2.4) is satisfied if and only if there exists $\varepsilon = \varepsilon(p) \in (0, 2/3)$ small enough such that

$$\lim_{N\to\infty} N^{1/p-2+3\varepsilon(p)}/\tau_{N,d,kp} = 0, \qquad p \in (p_0(k), p_1(k)). \tag{5.2.11}$$

We recall that $\tau_{N,d,q}$ in (5.2.10) and (5.2.11) is defined by (3.3.15) for $q \in (0,4)$.

Next, it is easy to see that (5.2.10) holds if

$$p \in I^*_{d,k} := \begin{cases} (0,4), & d=0,\ k=1, \\ (0,1], & d=1,\ k=1, \\ (0,2), & d=0,1,\ k=2, \end{cases} \tag{5.2.12}$$

and (5.2.11) holds if

$$p \in I^{**}_{d,k} := \begin{cases} (1/2,4), & d=0,\ k=1, \\ (1/3,4) & d=1,\ k=1, \\ (1/2,2), & d=0,\ k=2, \\ (1/3,2), & d=1,\ k=2. \end{cases} \tag{5.2.13}$$

Therefore taking account of Theorem 3.3.4 again, we obtain the following special case of Theorem 5.2.1 for the Gegenbauer polynomials.

COROLLARY 5.2.3. (a) *Let* $\operatorname{Re} s > -1 - k(1-d)$, $s + k(1-d) - 1 \neq 0, 2, \ldots$, $C_{d,k}(s) \neq 0$, *and let*

$$p \in ((kd+2)^{-1}, (\max\{0, -\operatorname{Re} s - k(1-d) + 1\})^{-1}) \cap I^*_{d,k}, \tag{5.2.14}$$

where the interval $I^*_{d,k}$ *is defined in* (5.2.12). *If* $\zeta_{d,k}(s) \neq 0$, *then*

$$\begin{aligned} |\zeta_{d,k}(s)| = {} & \frac{\pi}{2|\sin((s+k(1-d)-1)\pi/2)||C_{d,k}(s)|\mathcal{G}_{d,kp}} \\ & \times \lim_{N\to\infty} (2N+\lambda+d)^{\operatorname{Re} s + k(1-d)-1} \tau_{N,d,kp} \\ & \times \Bigg(\int_{-1}^{1} \big| |y|^{s+k(1-d)-1} - L_{2kN}(y, |y|^{s+k(1-d)-1}, y^{-kd+1} \\ & \times (C^{\lambda}_{2N+d}(y)/(C^{\lambda}_{2N+d})^{(d)}(0))^k)\big|^p (1-y^2)^{(\lambda/2-1/4)kp} dy \Bigg)^{1/p}, \end{aligned} \tag{5.2.15}$$

where $\mathcal{G}_{d,q}$ *is defined in* (3.3.16) *for* $q \in (0,4)$.

(b) *Let* $0 < \operatorname{Re} s < 1$ *and let*

$$p \in \left((kd+2)^{-1}, (\max\{0, -\operatorname{Re} s - k(1-d) + 1\})^{-1}\right) \cap I^{**}_{d,k},$$

where the interval $I^{**}_{d,k}$ *is defined in* (5.2.13). *If* $\zeta_{d,k}(s) = 0$, *then*

$$\begin{aligned} & \frac{\pi}{2|\sin((s+k(1-d)-1)\pi/2)|} \lim_{N\to\infty} (2N+\lambda+d)^{\operatorname{Re} s + k(1-d) - 1 + 1/p} \\ & \quad \times \Bigg(\int_{-1}^{1} \big| |y|^{s+k(1-d)-1} - L_{2kN}(y, |y|^{s+k(1-d)-1}, y^{-kd+1} \\ & \quad \times (C^{\lambda}_{2N+d}(y)/(C^{\lambda}_{2N+d})^{(d)}(0))^k)\big|^p (1-y^2)^{(\lambda/2-1/4)kp}\, dy \Bigg)^{1/p} = I^{*1/p}_{d,k,s,p}, \end{aligned} \tag{5.2.16}$$

where $I^*_{d,k,s,p}$ *is defined in* (5.1.2).

(c) *Let* $m = 1, 2, \dots,$ $m > (k-1)(1-d)$, *and let* $p \in ((kd+2)^{-1}, \infty) \cap I^*_{d,k}$. *Then*

$$\zeta_{d,k}(2m+1-k(1-d)) = \frac{1}{C_{d,k}(2m+1-k(1-d))\mathcal{G}_{d,kp}} \lim_{N\to\infty} (2N+\lambda+d)^{2m}\tau_{N,d,kp}$$
$$\times \left(\int_{-1}^{1} |y^{2m}\log|y| - L_{2kN}(y, y^{2m}\log|y|, y^{-kd+1}\right.$$
$$\left.\times (C^\lambda_{2N+d}(y)/(C^\lambda_{2N+d})^{(d)}(0))^k)|^p (1-y^2)^{(\lambda/2-1/4)kp} dy\right)^{1/p}. \quad (5.2.17)$$

Example 5.2.4 (Normalized even Chebyshev polynomials of the first kind on $[-1,1]$). The corresponding asymptotics for the weight $(1-y^2)^{-1/4}$ follow from Corollary 5.2.3. Here, we consider the case of $w(y,N) = 1$ and $d = 0$, that is,

$$P_{2N}(y) = T_{2N}(y),\ \beta_N = 2N,\ w(y,N) = 1,\ N = 1, 2, \dots,\ a = 1,\ p_0(k) = 0,\ p_1(k) = \infty.$$

Choosing γ_N and δ_N as in (5.2.7) and (5.2.8), we see that

$$\{T_{2N}\}_{N=1}^\infty \in \mathbb{P}^{**}_d(\beta, C(\varepsilon)\gamma, \delta, W, k),$$

by Remark 3.2.2. Next, Theorem 3.3.6 shows that condition (5.2.2) holds for all $p \in (0,\infty)$. In addition, Theorem 3.3.6 and estimates (5.2.9) show that condition (5.2.4) holds for all $p \in (1/2, \infty)$.Therefore, the following corollary follows from Theorems 5.2.1 and 3.3.6:

Corollary 5.2.5. (a) *Let* $\operatorname{Re} s > -1-k$, $s+k-1 \neq 0, 2, \dots,$ $C_{0,k}(s) \neq 0$, *and let* $p \in (1/2, (\max\{0, -\operatorname{Re} s - k + 1\})^{-1})$. *If* $\zeta_{0,k}(s) \neq 0$, *then*

$$|\zeta_{0,k}(s)| = \frac{\pi}{2|\sin((s+k-1)\pi/2)|\,|C_{0,k}(s)|\mathcal{G}^*_{kp}}$$
$$\times \lim_{N\to\infty} (2N)^{\operatorname{Re} s+k-1}\left(\int_{-1}^{1} |\,|y|^{s+k-1} - L_{2kN}(y, |y|^{s+k-1}, yT^k_{2N}(y))|^p\,dy\right)^{1/p}, \quad (5.2.18)$$

where $\mathcal{G}^*_p$ *is the constant on the right-hand side of* (3.3.39).

(b) *Let* $0 < \operatorname{Re} s < 1$ *and let* $p \in (1/2, (\max\{0, -\operatorname{Re} s - k + 1\})^{-1})$. *If* $\zeta_{d,k}(s) = 0$, *then*

$$\frac{\pi}{2|\sin((s+k-1)\pi/2)|} \lim_{N\to\infty} (2N)^{\operatorname{Re} s+k-1+1/p}$$
$$\times \left(\int_{-1}^{1} |\,|y|^{s+k-1} - L_{2kN}(y, |y|^{s+k-1}, yT^k_{2N}(y))|^p\,dy\right)^{1/p} = I^{*1/p}_{0,k,s,p}. \quad (5.2.19)$$

(c) *Let* $m = 1, 2, \dots,$ $m > (k-1)(1-d)$, *and let* $p \in (1/2, \infty)$. *Then*

$$\zeta_{0,k}(2m+1-k) = \frac{1}{C_{0,k}(2m+1-k)\mathcal{G}^*_{kp}}$$
$$\times \lim_{N\to\infty} (2N)^{2m}\left(\int_{-1}^{1} |y^{2m}\log|y| - L_{2kN}(y, y^{2m}\log|y|, yT^k_{2N}(y))|^p dy\right)^{1/p}. \quad (5.2.20)$$

Example 5.2.6 (Normalized polynomials with equidistant zeros on $[-1,1]$). We have

$$P_{2N+d}(y) = y^d \prod_{p=1}^{N}\left(1 - \left(\frac{2N+d-1}{2p+d-1}\right)^2 y^2\right), \quad \beta_N = (2N+d-1)\pi/2,$$

$$w(y,N)=\left(\sqrt{(1-y^2)}((1+y)^{1+y}(1-y)^{1-y})^{(2N+d-1)/2}\right)^{-1},$$
$$N=1,2,\ldots,\qquad a=1,\qquad p_0(k)=0,\qquad p_1(k)=\infty.$$

Setting $\tau:=\operatorname{Re}s+(k-1)(1-d)$, we choose for $N>N_2$,

$$\gamma_N:=\log(N+1)+(\tau-2)\log\log(N+1), \tag{5.2.21}$$
$$CN^{-1}\log^2(N+1)\le\delta_N:=\gamma_N^2/N\le C_1N^{-1}\log^2(N+1). \tag{5.2.22}$$

Since $\gamma=\{\gamma_N\}_{N=N_2}^\infty$ is an increasing sequence and $\delta=\{\delta_N\}_{N=N_2}^\infty$ is a decreasing sequence for large enough N_2, we can choose γ_N and δ_N for $1\le N\le N_2$ such that γ and δ are increasing to ∞ and decreasing to 0, respectively. Next, (5.2.21) and (5.2.22) show that conditions (3.2.33) and (3.2.34) are satisfied and by Proposition 3.2.4, $\{P_{2N+d}\}_{N=1}^\infty\in\mathbb{P}_d^{**}(\beta,\gamma,\delta,W,k)$. In addition, (5.2.21) and (5.2.22) yield

$$CN^{-1}\log^2(N+1)\le\mathcal{E}_{N,\tau}^*=\delta_N+\gamma_N^\tau e^{-\gamma_N}\le C_1N^{-1}\log^2(N+1). \tag{5.2.23}$$

Further, similarly to the proof of Corollary 5.2.3, we note that it follows from Theorem 3.3.7 that condition (5.2.2) is satisfied if and only if

$$\lim_{N\to\infty}N^{1/p}/\tau_{N,d,kp}^*=\infty,\qquad p\in(0,\infty). \tag{5.2.24}$$

and it follows from Theorem 3.3.7 and (5.2.23) that condition (5.2.4) is satisfied if and only if

$$\lim_{N\to\infty}N^{1/p-1}\log^2(N+1)/\tau_{N,d,kp}^*=0,\qquad p\in(0,\infty). \tag{5.2.25}$$

We recall that $\tau_{N,d,q}^*$ in (5.2.24) and (5.2.25) is defined by (3.3.40) for $q\in(0,\infty)$.

Next, it is easy to see that (5.2.24) holds if

$$p\in J_{d,k}^*:=\begin{cases}(0,\infty), & d=0,\ k=1 \text{ or } d=0,1,\ k=2,\\ (0,1], & d=1,\ k=1,\end{cases} \tag{5.2.26}$$

and (5.2.25) holds if

$$p\in J_{d,k}^{**}:=\begin{cases}(1,\infty), & d=0,\ k=1,2,\\ (1/2,\infty), & d=1,\ k=1,2.\end{cases} \tag{5.2.27}$$

Therefore, the following corollary for polynomials with equidistant zeros follows from Theorems 5.2.1 and 3.3.7.

Corollary 5.2.7. (a) *Let* $\operatorname{Re}s>-1-k(1-d)$, $s+k(1-d)-1\ne 0,2,\ldots$, $C_{d,k}(s)\ne0$, *and let*

$$p\in\left((kd+2)^{-1},(\max\{0,-\operatorname{Re}s-k(1-d)+1\})^{-1}\right)\cap J_{d,k}^*, \tag{5.2.28}$$

where the interval $J_{d,k}^*$ *is defined in* (5.2.26). *If* $\zeta_{d,k}(s)\ne0$, *then*

$$\begin{aligned}|\zeta_{d,k}(s)|&=\frac{\pi}{2|\sin((s+k(1-d)-1)\pi/2)||C_{d,k}(s)|\mathcal{E}_{d,kp}}\\&\quad\times\lim_{N\to\infty}((2N+d-1)\pi/2)^{\operatorname{Re}s+k(1-d)-1}\tau_{N,d,kp}^*\\&\quad\times\left(\int_{-1}^{1}\left|\,|y|^{s+k(1-d)-1}-L_{2kN}(y,|y|^{s+k(1-d)-1},y^{-kd+1}P_{2N+d}^k)\right|^p\right.\\&\qquad\qquad\left.\times\, w^{kp}(y,N)\,dy\right)^{1/p},\end{aligned} \tag{5.2.29}$$

where $\mathcal{E}_{d,q}$ *is defined in* (3.3.41) *for* $q\in(0,\infty)$.

(b) *Let* $0 < \operatorname{Re} s < 1$ *and let*

$$p \in \big((kd+2)^{-1}, (\max\{0, -\operatorname{Re} s - k(1-d) + 1\})^{-1}\big) \cap J^{**}_{d,k},$$

where the interval $J^{**}_{d,k}$ *is defined in* (5.2.27). *If* $\zeta_{d,k}(s) = 0$, *then*

$$\frac{\pi}{2|\sin((s+k(1-d)-1)\pi/2)|} \lim_{N\to\infty} ((2N+d-1)\pi/2)^{\operatorname{Re} s+k(1-d)-1+1/p}$$
$$\times \left(\int_{-1}^{1} \big| |y|^{s+k(1-d)-1} - L_{2kN}(y, |y|^{s+k(1-d)-1}, y^{-kd+1} P^k_{2N+d})\big|^p w^{kp}(y,N)\, dy\right)^{1/p}$$
$$= I^{*1/p}_{d,k,s,p}, \qquad (5.2.30)$$

where $I^*_{d,k,s,p}$ *is defined in* (5.1.2).

(c) *Let* $m = 1, 2, \dots$, $m > (k-1)(1-d)$, *and let* $p \in ((kd+2)^{-1}, \infty) \cap J^*_{d,k}$. *Then*

$$\zeta_{d,k}(2m+1-k(1-d)) = \frac{1}{C_{d,k}(2m+1-k(1-d))\mathcal{E}_{d,kp}} \lim_{N\to\infty} ((2N+d-1)\pi/2)^{2m} \tau^*_{N,d,kp}$$
$$\times \left(\int_{-1}^{1} |y^{2m}\log|y| - L_{2kN}(y, y^{2m}\log|y|, y^{-kd+1}P^k_{2N+d})|^p w^{kp}(y,N)\, dy\right)^{1/p}. \qquad (5.2.31)$$

Example 5.2.8 (Normalized Hermite polynomials on $(-\infty,\infty)$). We have

$$P_{2N+d}(y) = H_{2N+d}(y)/H^{(d)}_{2N+d}(0), \quad \beta_N = \sqrt{4N+2d+1}, \quad w(y,N) = e^{-y^2/2},$$
$$a_N = \sqrt{4N+2d}, \quad N = 1,2,\dots, \quad a = \infty, \quad p_0(k) = (2+kd+k/2)^{-1}, \quad p_1(k) = 4/k.$$

We recall that $\{a_N\}_{N=1}^\infty$ is a sequence of the Mhaskar–Rakhmanov–Saff numbers for the exponential weight $e^{-y^2/2}$. Next, we choose for $N = 1, 2, \dots$,

$$\gamma_N := N^\varepsilon, \qquad \varepsilon \in (0, 1/2), \qquad (5.2.32)$$
$$\delta_N := \gamma_N^2/N = N^{-1+2\varepsilon}, \qquad \varepsilon \in (0,1/2). \qquad (5.2.33)$$

Then (5.2.32) shows that conditions (3.2.47) and (3.2.48) are satisfied and by Proposition 3.2.6, $\{P_{2N+d}\}_{N=1}^\infty \in \mathbb{P}^{**}_d(\beta, C(\varepsilon)\gamma, \delta, W, k)$. In addition, (5.2.32) and (5.2.33) yield

$$N^{-1+2\varepsilon} \le \mathcal{E}^*_{N,\tau} = \delta_N + \gamma_N^\tau e^{-\gamma_N} \le C_1(\varepsilon) N^{-1+2\varepsilon}. \qquad (5.2.34)$$

Further, it follows from Theorem 3.3.8 that condition (5.2.2) is satisfied if and only if

$$\lim_{N\to\infty} N^{1/(2p)}/\tau^{**}_{N,d,kp} = \infty, \qquad p \in (p_0(k), p_1(k)). \qquad (5.2.35)$$

In addition, combining Theorem 3.3.8 with (5.2.34), we conclude that condition (5.2.4) is satisfied if and only if there exists $\varepsilon = \varepsilon(p) \in (0, 1/2)$ small enough such that

$$\lim_{N\to\infty} N^{1/(2p)-1+2\varepsilon(p)}/\tau^{**}_{N,d,kp} = 0, \qquad p \in (p_0(k), p_1(k)). \qquad (5.2.36)$$

We recall that $\tau^{**}_{N,d,q}$, $q \in (0, 8/3)$ in (5.2.35) and (5.2.36) is defined by (3.3.48).

Next, it is easy to see that (5.2.35) holds if

$$p \in K^*_{d,k} := \begin{cases} (2/5, 8/3), & d=0,\ k=1,\\ (2/7, 1], & d=1,\ k=1,\\ (1/3, 4/3), & d=0,\ k=2,\\ (1/5, 4/3), & d=1,\ k=2, \end{cases} \qquad (5.2.37)$$

and (5.2.36) holds if

$$p \in K_{d,k}^{**} := \begin{cases} (1, 8/3), & d=0,\ k=1, \\ (1/2, 8/3), & d=1,\ k=1, \\ (3/4, 4/3), & d=0,\ k=2, \\ (3/8, 4/3), & d=1,\ k=2. \end{cases} \tag{5.2.38}$$

Therefore, taking account of Theorem 3.3.8 again, we obtain the following special case of Theorem 5.2.1 for the Hermite polynomials.

COROLLARY 5.2.9. (a) *Let* $\operatorname{Re} s > -1-k(1-d)$, $s+k(1-d)-1 \neq 0, 2, \ldots$, $C_{d,k}(s) \neq 0$, *and let*

$$p \in \left((kd+2)^{-1}, (\max\{0, -\operatorname{Re} s - k(1-d)+1\})^{-1}\right) \cap K_{d,k}^{*}, \tag{5.2.39}$$

where the interval $K_{d,k}^{*}$ *is defined in* (5.2.37). *If* $\zeta_{d,k}(s) \neq 0$, *then*

$$\begin{aligned} |\zeta_{d,k}(s)| = {} & \frac{\pi}{2|\sin((s+k(1-d)-1)\pi/2)|\,|C_{d,k}(s)|\mathcal{H}_{d,kp}} \\ & \times \lim_{N\to\infty} (4N+2d+1)^{(\operatorname{Re} s+k(1-d)-1)/2} \tau_{N,d,kp}^{**} \biggl(\int_{\mathbb{R}} \bigl| |y|^{s+k(1-d)-1} - L_{2kN} \\ & \times (y, |y|^{s+k(1-d)-1}, y^{-kd+1}(H_{2N+d}(y)/H_{2N+d}^{(d)}(0))^k) \bigr|^p e^{-py^2/2} dy \biggr)^{1/p}, \end{aligned} \tag{5.2.40}$$

where $\mathcal{H}_{d,q}$ *is defined in* (3.3.49).

(b) *Let* $0 < \operatorname{Re} s < 1$ *and let*

$$p \in \left((kd+2)^{-1}, (\max\{0, -\operatorname{Re} s - k(1-d)+1\})^{-1}\right) \cap K_{d,k}^{**},$$

where the interval $K_{d,k}^{**}$ *is defined in* (5.2.38). *If* $\zeta_{d,k}(s) = 0$, *then*

$$\begin{aligned} & \frac{\pi}{2|\sin((s+k(1-d)-1)\pi/2)|} \lim_{N\to\infty} (4N+2d+1)^{(\operatorname{Re} s+k(1-d)-1+1/p)/2} \\ & \quad \times \biggl(\int_{\mathbb{R}} \bigl| |y|^{s+k(1-d)-1} - L_{2kN}(y, |y|^{s+k(1-d)-1}, y^{-kd+1}(H_{2N+d}(y)/H_{2N+d}^{(d)}(0))^k) \bigr|^p \\ & \qquad \times e^{-py^2/2} dy \biggr)^{1/p} = I_{d,k,s,p}^{*1/p}, \end{aligned} \tag{5.2.41}$$

where $I_{d,k,s,p}^{*}$ *is defined in* (5.1.2).

(c) *Let* $m = 1, 2, \ldots$, $m > (k-1)(1-d)$, *and let* $p \in ((kd+2)^{-1}, \infty) \cap K_{d,k}^{*}$. *Then*

$$\begin{aligned} \zeta_{d,k}(2m+1-k(1-d)) = {} & \frac{1}{C_{d,k}(2m+1-k(1-d))\mathcal{H}_{d,kp}} \\ & \times \lim_{N\to\infty} (4N+2d+1)^m \tau_{N,d,kp}^{**} \biggl(\int_{\mathbb{R}} \bigl| y^{2m} \log|y| - L_{2kN} \\ & \times (y, y^{2m}\log|y|, y^{-kd+1}(H_{2N+d}(y)/H_{2N+d}^{(d)}(0))^k) \bigr|^p e^{-py^2/2} dy \biggr)^{1/p}. \end{aligned} \tag{5.2.42}$$

EXAMPLE 5.2.10 (Normalized Williams–Apostol polynomials on $(-\infty, \infty)$). Polynomials of the first kind:

$$P_{2N+d}(y) = A_{2N+d}(y)/A^{(d)}_{2N+d}(0), \quad \beta_N = 2N+d,$$
$$w(y,N) = (1+y^2)^{-N-d/2-\alpha}, \quad \alpha \ge 0,\ a_N = 1,\ N = 1,2,\dots,\ a = \infty,$$
$$p_0(k) = \max\{(2\alpha k + kd)^{-1}, (kd+2)^{-1}\}, \quad p_1(k) = \infty.$$

Polynomials of the second kind:

$$P_{2N+d}(y) = W_{2N+d}(y)/W^{(d)}_{2N+d}(0), \quad \beta_N = 2N+d+1,$$
$$w(y,N) = (1+y^2)^{-N-d/2-\alpha-1/2}, \quad \alpha \ge 0,\ a_N = 1,\ N = 1,2,\dots,\ a = \infty,$$
$$p_0(k) = \max\{(2\alpha k + kd)^{-1}, (kd+2)^{-1}\}, \quad p_1(k) = \infty.$$

The proof of the following corollary for the Williams–Apostol polynomials P_{2N+d} of the first and second kinds follows that of Corollary 5.2.7 if we use Proposition 3.2.7 and Theorems 5.2.1 and 3.3.13.

Corollary 5.2.11. (a) *Let* $\operatorname{Re} s > -1-k(1-d)$, $s+k(1-d)-1 \ne 0,2,\dots$, $C_{d,k}(s) \ne 0$, $\alpha \ge 0$, *and let*

$$p \in ((kd+2)^{-1}, (\max\{0, -\operatorname{Re} s - k(1-d)+1\})^{-1}) \cap J^*_{d,k} \cap ((2\alpha k + kd)^{-1}, \infty),$$

where $J^*_{d,k}$ *is defined in* (5.2.26). *If* $\zeta_{d,k}(s) \ne 0$, *then*

$$|\zeta_{d,k}(s)| = \frac{\pi}{2|\sin((s+k(1-d)-1)\pi/2)||C_{d,k}(s)|\mathcal{D}_{d,kp,\alpha}} \lim_{N\to\infty} \beta_N^{\operatorname{Re} s+k(1-d)-1}$$
$$\times \tau^*_{N,d,kp}\biggl(\int_{\mathbb{R}} |\,|y|^{s+k(1-d)-1} - L_{2kN}(y, |y|^{s+k(1-d)-1}, y^{-kd+1}P^k_{2N+d})|^p$$
$$\times w^{kp}(y,N)\,dy\biggr)^{1/p}, \qquad (5.2.43)$$

where $\tau^*_{N,d,q}$, $q \in (0,\infty)$, *and* $\mathcal{D}_{d,q,\alpha}$ *are defined in* (3.3.40) *and* (3.3.86), *respectively.*

(b) *Let* $0 < \operatorname{Re} s < 1$, $\alpha \ge 0$ *and let*

$$p \in ((kd+2)^{-1}, (\max\{0, -\operatorname{Re} s - k(1-d)+1\})^{-1}) \cap J^{**}_{d,k} \cap ((2\alpha k + kd)^{-1}, \infty),$$

where the interval $J^{**}_{d,k}$ *is defined in* (5.2.27). *If* $\zeta_{d,k}(s) = 0$, *then*

$$\frac{\pi}{2|\sin((s+k(1-d)-1)\pi/2)|} \lim_{N\to\infty} \beta_N^{\operatorname{Re} s+k(1-d)-1+1/p}$$
$$\times \biggl(\int_{-1}^{1} |\,|y|^{s+k(1-d)-1} - L_{2kN}(y, |y|^{s+k(1-d)-1}, y^{-kd+1}P^k_{2N+d})|^p w^{kp}(y,N)\,dy\biggr)^{1/p}$$
$$= I^{*1/p}_{d,k,s,p}, \qquad (5.2.44)$$

where $I^*_{d,k,s,p}$ *is defined in* (5.1.2).

(c) *Let* $m = 1,2,\dots$, $m > (k-1)(1-d)$, $\alpha \ge 0$ *and let* $p \in (\max\{(2\alpha k+kd)^{-1}, (kd+2)^{-1}\}, \infty) \cap J^*_{d,k}$. *Then*

$$\zeta_{d,k}(2m+1-k(1-d)) = \frac{1}{C_{d,k}(2m+1-k(1-d))\mathcal{D}_{d,kp,\alpha}} \lim_{N\to\infty} \beta_N^{2m}\tau^*_{N,d,kp}$$
$$\times \biggl(\int_{-1}^{1} |y^{2m}\log|y| - L_{2kN}(y, y^{2m}\log|y|, y^{-kd+1}P^k_{2N+d})|^p w^{kp}(y,N)\,dy\biggr)^{1/p}. \qquad (5.2.45)$$

5.3. Interpolation L_∞-error criteria for $\zeta_{0,k}(s) \neq 0$ and $\zeta_{0,k}(s) = 0$. New criteria for zeros of $\zeta(s)$ and $\beta(s)$ to be in the critical strip are discussed in this section.

5.3.1. General L_∞-error criterion. Let $\Pi_0 = \{P_{2N}\}_{N=1}^\infty \in \mathbb{P}_0(\beta, \gamma, \delta)$ and let $W = \{w(y,N)\}_{N=1}^\infty$ be a sequence of even continuous weights on $[-1,1]$ or (∞, ∞). We consider the following conditions on Π_0 and W:

(i) $w(0,N) = 1$, $N = 1, 2, \dots$, and the sequence $\{w(y,N)\}_{N=1}^\infty$ converges uniformly to 1 in a neighborhood of zero.
(ii) $\sup_{|y|<a} |P_{2N}(y)| w(y,N) < C_1(w)$.
(iii) The following estimate for the error of best polynomial approximation in the weighted uniform metric holds:

$$\begin{aligned} E_{2kN}(|y|^{s+k-1}, (-a,a))_{w^k} &:= \inf_{Q_{2kN} \in \mathcal{P}_{2kN}} \sup_{|y|<a} ||y|^{s+k-1} - Q_{2kN}(y)| w^k(y,N) \\ &\geq C_2(w,k) \beta_N^{-(\operatorname{Re} s + k - 1)}. \end{aligned}$$

Here, $a = 1$ or $a = \infty$ and the constants C_1 and C_2 are independent of N. Then the following theorem holds:

THEOREM 5.3.1. *Let $0 < \operatorname{Re} s < 1$ and let Π_0 and W satisfy conditions* (i)–(iii). *Then $\zeta_{0,k}(s) \neq 0$ if and only if there exist a constant $C_3 = C_3(s, \Pi_0, W, k) \in (0,1]$ and a sequence of points $y_N \in (0,a)$, $N = 1, 2, \dots$, satisfying the properties:*

(A) $\lim_{N\to\infty} \beta_N y_N = \infty$.
(B) *We have*

$$\begin{aligned} &|y_N^{s+k-1} - L_{2kN}(y_N, |y|^{s+k-1}, yP_{2N}^k(y))| w^k(y_N, N) \\ &\qquad \geq C_3 \sup_{|y|<a} ||y|^{s+k-1} - L_{2kN}(y, |y|^{s+k-1}, yP_{2N}^k(y))| w^k(y,N). \end{aligned}$$

The following criterion for $\zeta_{0,k}(s) = 0$ is obviously equivalent to Theorem 5.3.1:

THEOREM 5.3.2. *Let $0 < \operatorname{Re}\ s < 1$ and let Π_0 and $\{w(y,N)\}_{N=1}^\infty$ satisfy conditions* (i)–(iii). *Then $\zeta_{0,k}(s) = 0$ if and only if for any constant $C_4 \in (0,1]$ there exists a constant C_5 such that the set*

$$\begin{aligned} S_{N,C_4} := \{y \in [0,a) : &|y^{s+k-1} - L_{2kN}(y, |y|^{s+k-1}, yP_{2N}(y))| w(y,N) \\ &\geq C_4 \sup_{y \in [0,a)} |y^{s+k-1} - L_{2kN}(y, |y|^{s+k-1}, yP_{2N}(y))| w(y,N)\} \end{aligned}$$

satisfies the property $S_{N,C_4} \subseteq [0, C_5/\beta_N]$.

Proof of Theorem 5.3.1. Necessity. Let Π_0 and W satisfy conditions (i) and (ii) and let $\zeta_{0,k}(s) \neq 0$. We first show that condition (i) implies the existence of a sequence $y_N \in (0,a)$, $N = 1, 2, \dots$, such that

$$\lim_{N\to\infty} \beta_N y_N = \infty, \qquad \lim_{N\to\infty} y_N = 0, \tag{5.3.1}$$

and, in addition, there exists $N_0 \in \mathbb{N}$ such that

$$\inf_{N \geq N_0} |P_{2N}(y_N)| w(y_N, N) > 1/4. \tag{5.3.2}$$

Indeed, setting

$$v_N := \min\{\beta_N^{1/2}, \gamma_N, (1/2)\log(1/\delta_N)\}, \qquad N = 1, 2, \ldots,$$

we see that by (3.1.1), the sequence

$$y_N := \begin{cases} 1/2, & 1 \le N \le N_1, \\ 2\pi\lfloor v_N/(2\pi)\rfloor/\beta_N, & N > N_1, \end{cases}$$

satisfies property (5.3.1). Here, N_1 is chosen so that $y_N \in [0, 1/2] \subseteq (0, a)$, $N = 1, 2, \ldots$. Moreover, by (3.1.3), there exists $N_2 > N_1$ such that

$$|P_{2N}(y_N)| \ge 1 - C\delta_N \exp(\beta_N y_N) \ge 1 - C\delta_N^{1/2} > 1/2, \qquad N \ge N_2. \tag{5.3.3}$$

Next, by condition (i) and (5.3.1), there exists a positive integer N_3 such that

$$w(y_N, N) \ge 1/2, \qquad N \ge N_3. \tag{5.3.4}$$

Then (5.3.2) follows from (5.3.3) and (5.3.4) for $N_0 = \max\{N_2, N_3\}$. Therefore, there exists a sequence $\{y_N\}_{N=1}^\infty$ with the required properties.

Next, using Corollary 4.3.1(a) and inequality (5.3.2), we obtain, as $N \to \infty$,

$$\begin{aligned} &\beta_N^{\mathrm{Re}\, s+k-1} |y_N^{s+k-1} - L_{2kN}(y_N, |y|^{s+k-1}, yP_{2N}^k(y))| w^k(y_N, N) \\ &\quad \ge C(s)|P_{2N}^k(y_N)| w^k(y_N, N) |C_{0,k}(s)\zeta_{0,k}(s) + o(1)| \ge C(s,k)|\zeta_{0,k}(s)| > 0. \end{aligned} \tag{5.3.5}$$

Further, we use statement (a) of Theorem 4.2.3 and condition (ii) to show that

$$\begin{aligned} &\beta_N^{\mathrm{Re}\, s+k-1} \sup_{|y|<a} ||y|^{s+k-1} - L_{2kN}(y, |y|^{s+k-1}, yP_{2N}^k(y))| w^k(y, N) \\ &\quad \le C(s) \sup_{|y|<a} |P_{2N}^k(y)| w^k(y, N) \sup_{|y|<a} (\beta_N y)^{-2} \int_0^\infty \frac{t^{\mathrm{Re}\, s+k}\, dt}{(1 + t^2/(\beta_N y)^2) h_d^k(t)} \\ &\quad \le C \int_0^\infty \frac{t^{\mathrm{Re}\, s+k-2}\, dt}{h_d^k(t)} < \infty. \end{aligned} \tag{5.3.6}$$

Finally, property (B) follows from (5.3.5) and (5.3.6), while property (A) follows from (5.3.1).

Sufficiency. Let Π_0 and W satisfy conditions (ii) and (iii). If a sequence $\{y_N\}_{N=1}^\infty$ has properties (A) and (B), then by Corollary 4.3.1(a) and by (ii),

$$\begin{aligned} |\zeta_{0,k}(s)| &\ge C(s,k) \lim_{N\to\infty} \beta_N^{\mathrm{Re}\, s+k-1} |y_N^{s+k-1} - L_{2kN}(y_N, |y|^{s+k-1}, yP_{2N}^k(y))| / |P_{2N}^k(y_N)| \\ &\ge C(s,k) \liminf_{N\to\infty} \beta_N^{\mathrm{Re}\, s+k-1} |y_N^{s+k-1} - L_{2kN}(y_N, |y|^{s+k-1}, yP_{2N}^k(y))| w^k(y_N, N). \end{aligned} \tag{5.3.7}$$

Using now property (B) and condition (iii), from (5.3.7) we obtain

$$\begin{aligned} |\zeta_{0,k}(s)| &\ge C \liminf_{N\to\infty} \beta_N^{\mathrm{Re}\, s+k-1} |y_N^{s+k-1} - L_{2kN}(y_N, |y|^{s+k-1}, yP_{2N}^k(y))| w^k(y_N, N) \\ &\ge C \liminf_{N\to\infty} \beta_N^{\mathrm{Re}\, s+k-1} E_{2kN}(|y|^{s+k-1}, (-a, a))_{w^k} > C_2(w, k) > 0. \end{aligned}$$

Thus $\zeta_{0,k}(s) \neq 0$. This completes the proof of Theorem 5.3.1. ∎

Remark 5.3.3. The above proof shows that the additional condition

$$\lim_{N\to\infty} y_N = 0$$

can be included into property (A) of Theorem 5.3.1.

5.3.2. Special L_∞-error criteria. To apply this theorem to special polynomials and weights, we consider the following simple condition that can replace condition (iii) in Theorem 5.3.1:

(iv) $\beta_N \geq CN$, $N = 1, 2, \ldots$.

PROPOSITION 5.3.4. *Condition* (iii) *follows from* (i) *and* (iv).

Proof. It follows from (i) that there exists $\delta \in (0, a)$, independent of N, such that $\inf_{y\in(-\delta,\delta)} w(y, N) \geq 1/2$. Then

$$\begin{aligned} E_{2kN}(|y|^{s+k-1}, (-a, a))_{w^k} &\geq 2^{-k} E_{2kN}(|y|^{s+k-1}, (-\delta, \delta))_1 \\ &= 2^{-k}\delta^{\operatorname{Re} s+k-1} E_{2kN}(|y|^{s+k-1}, (-1, 1))_1. \end{aligned} \tag{5.3.8}$$

If we use an estimate

$$E_{2kN}(|y|^{s+k-1}, (-1, 1))_1 \geq CN^{-(\operatorname{Re} s+k-1)} \tag{5.3.9}$$

(see [21, Lemma 2]), then (iii) follows from (5.3.8), (5.3.9), and condition (iv). ■

EXAMPLE 5.3.5 (Normalized Gegenbauer polynomials on $[-1, 1]$).

$P_{2N}(y) = C^\lambda_{2N}(y)/C^\lambda_{2N}(0)$, $\beta_N = 2N + \lambda$, $w(y, N) = (1 - y^2)^{\lambda/2}$, $N = 1, 2, \ldots, \lambda \geq 0$.

Conditions (i) and (iv) are satisfied. Condition (ii) is satisfied as well, by (3.2.7). Therefore, by Proposition 5.3.4, the criteria of Theorems 5.3.1 and 5.3.2 are valid for these polynomials.

EXAMPLE 5.3.6 (Normalized Chebyshev polynomials of the first kind on $[-1, 1]$).

$$P_{2N} = T_{2N}(y), \quad \beta_N = 2N, \quad w(y, N) = 1, \quad N = 1, 2, \ldots.$$

Conditions (i), (ii), and (iv) are satisfied and the criteria of Theorems 5.3.1 and 5.3.2 are valid for these polynomials.

EXAMPLE 5.3.7 (Normalized Hermite polynomials on $(-\infty, -\infty)$).

$P_{2N} = H_{2N}(y)/H_{2N}(0)$, $\beta_N = \sqrt{4N + 1}$, $w(y, N) = e^{-y^2/2}|1-y^2/4N|$, $N = 1, 2, \ldots$.

Conditions (i) and (ii) are satisfied; the latter follows from (3.2.54). To prove that condition (iii) is also satisfied, we use a special case of Theorem 8.1.1(a) from [22]:

$$\begin{aligned} 2^{(3/2)(\operatorname{Re} s+k-1)} \lim_{N\to\infty} N^{(\operatorname{Re} s+k-1)/2} E_{2N}(|t|^{s+k-1}, (-\infty, \infty))_{\exp(-t^2)} \\ = \inf_{g_1\in B_1} \sup_{t\in\mathbb{R}} \big||t|^{s+k-1} - g_1(t)\big| > 0, \end{aligned}$$

where B_1 is the class of all entire functions of exponential type ≤ 1. Hence by the substitution $t = \sqrt{k/2}\, y$, we arrive at the inequality

$$E_{2kN}(|y|^{s+k-1}, (-\infty, \infty))_{\exp(-ky^2/2)} \geq CN^{-(\operatorname{Re} s+k-1)/2} = C_1\beta_N^{-(\operatorname{Re} s+k-1)}.$$

Thus property (iii) is satisfied. Therefore, the criteria of Theorems 5.3.1 and 5.3.2 are valid for the Hermite polynomials.

REMARK 5.3.8. Note that the weights $w(y, N)$ for the Gegenbauer and Hermite polynomials in those examples are different from (3.2.4) and (3.2.46).

5.4. Interpolation L_p-error criteria for $\zeta_{d,k}(s) \neq 0$ and $\zeta_{d,k}(s) = 0$. In this section we discuss L_p-versions of the interpolation error criteria for zeros of the zeta functions to be in the critical strip.

5.4.1. General L_p-error criterion

THEOREM 5.4.1. *Let* $\{P_{2N+d}\}_{N=1}^{\infty} \in \mathbb{P}_d^{**}(\beta,\gamma,\delta,W,k)$ *and* $0 < \operatorname{Re} s < 1$. *In addition, let* p *satisfy condition* (5.2.1) *and let both conditions* (5.2.2) *and* (5.2.4) *be satisfied. Then the following statements hold:*

(a) $\zeta_{d,k}(s) \neq 0$ *if and only if*

$$C_1\beta_N^{-(\operatorname{Re} s+k(1-d)-1)} \le \frac{\left(\int_{-a}^{a}\big||y|^{s+k(1-d)-1} - L_{2kN}(y,|y|^{s+k(1-d)-1}, y^{-kd+1}P_{2N+d}^k(y))\big|^p w^{kp}(y,N)\,dy\right)^{1/p}}{\left(\int_{-a}^{a}|y^{-kd}P_{2N+d}^k(y)w^k(y,N)|^p\,dy\right)^{1/p}} \le C_2\beta_N^{-(\operatorname{Re} s+k(1-d)-1)}. \tag{5.4.1}$$

(b) $\zeta_{d,k}(s) = 0$ *if and only if*

$$C_3\beta_N^{-(\operatorname{Re} s+k(1-d)-1+1/p)} \le \frac{\left(\int_{-a}^{a}\big||y|^{s+k(1-d)-1} - L_{2kN}(y,|y|^{s+k(1-d)-1}, y^{-kd+1}P_{2N+d}^k(y))\big|^p w^{kp}(y,N)\,dy\right)^{1/p}}{\left(\int_{-a}^{a}|y^{-kd}P_{2N+d}^k(y)w^k(y,N)|^p\,dy\right)^{1/p}} \le C_4\beta_N^{-(\operatorname{Re} s+k(1-d)-1+1/p)}. \tag{5.4.2}$$

Proof. Theorem 5.4.1 follows immediately from Theorem 5.2.1. Indeed, if $\zeta_{d,k}(s) \neq 0$, then (5.4.1) follows from Theorem 5.2.1(a). If inequalities (5.4.1) hold, then assuming that $\zeta_{d,k}(s) = 0$ and using Theorem 5.2.1(b), we see that relation (5.2.5) contradicts (5.4.1) because of condition (5.2.2). This proves (a). Statement (b) can be proved similarly. ∎

5.4.2. Special L_p-error criteria. Special cases of Theorem 5.2.1 are discussed below.

EXAMPLE 5.4.2 (Normalized Gegenbauer polynomials on $[-1,1]$). Let $0 < \operatorname{Re} s < 1$ and

$$P_{2N+d}(y) = C_{2N+d}^{\lambda}(y)/(C_{2N+d}^{\lambda})^{(d)}(0), \quad \beta_N = 2N+\lambda+d, \quad w(y,N) = (1-y^2)^{\lambda/2-1/4},$$
$$N = 1,2,\ldots, \quad \lambda \ge 0, \quad a = 1, \quad p_0(k) = 0, \quad p_1(k) = 4/k.$$

It is established in Section 5.2 that for the Gegenbauer polynomials, all three conditions (5.2.1), (5.2.2), and (5.2.4) are satisfied for

$$p \in I_{d,k,s} := ((kd+2)^{-1}, (\max\{0, -\operatorname{Re} s - k(1-d)+1\})^{-1}) \cap I_{d,k}^{*} \cap I_{d,k}^{**},$$

where the intervals $I_{d,k}^{*}$ and $I_{d,k}^{**}$ are defined in (5.2.12) and (5.2.13), respectively. It is easy to see that

$$I_{d,k,s} = \begin{cases} (1/2,4), & d=0,\ k=1,\\ (1/3,1], & d=1,\ k=1,\\ (1/2,2), & d=0,\ k=2,\\ (1/3,\min\{2,(1-\operatorname{Re} s)^{-1}\}), & d=1,\ k=2. \end{cases} \tag{5.4.3}$$

Then the following special case of Theorem 5.4.1 holds for the Gegenbauer polynomials:

Corollary 5.4.3. *Let $0 < \operatorname{Re} s < 1$ and $p \in I_{d,k,s}$, where $I_{d,k,s}$ is defined by (5.4.3), and let $\tau_{N,d,kp}$ is defined by (3.3.15). Then:*

(a) $\zeta_{d,k}(s) \neq 0$ *if and only if*

$$
\begin{aligned}
&C_1 N^{-(\operatorname{Re} s+k(1-d)-1)}\tau_{N,d,kp}^{-1}\\
&\le \left(\int_{-1}^{1} \big||y|^{s+k(1-d)-1} - L_{2kN}(y, |y|^{s+k(1-d)-1}, y^{-kd+1}P_{2N+d}^{k}(y))\big|^{p}(1-y^2)^{(\lambda/2-1/4)kp}\,dy\right)^{1/p}\\
&\le C_2 N^{-(\operatorname{Re} s+k(1-d)-1)}\tau_{N,d,kp}^{-1}.
\end{aligned}
$$

(b) $\zeta_{d,k}(s) = 0$ *if and only if*

$$
\begin{aligned}
&C_3 N^{-(\operatorname{Re} s+k(1-d)-1+1/p)}\\
&\le \left(\int_{-1}^{1} \big||y|^{s+k(1-d)-1} - L_{2kN}(y, |y|^{s+k(1-d)-1}, y^{-kd+1}P_{2N+d}^{k}(y))\big|^{p}(1-y^2)^{(\lambda/2-1/4)kp}\,dy\right)^{1/p}\\
&\le C_4 N^{-(\operatorname{Re} s+k(1-d)-1+1/p)}.
\end{aligned}
$$

Example 5.4.4 (Normalized even Chebyshev polynomials of the first kind on $[-1,1]$). Let $0 < \operatorname{Re} s < 1$ and

$$
\begin{gathered}
P_{2N}(y) = T_{2N}(y), \quad \beta_N = 2N, \quad w(y,N) = 1,\\
N = 1,2,\dots, \quad a = 1, \quad p_0(k) = 0, \quad p_1(k) = \infty.
\end{gathered}
$$

It is established in Section 5.2 that for these polynomials, all three conditions (5.2.1), (5.2.2), and (5.2.4) are satisfied for $p \in (1/2, \infty)$.

Corollary 5.4.5. *Let $0 < \operatorname{Re} s < 1$ and $p \in (1/2, \infty)$. Then:*

(a) $\zeta_{d,k}(s) \neq 0$ *if and only if*

$$
\begin{aligned}
C_1 N^{-(\operatorname{Re} s+k-1)} &\le \left(\int_{-1}^{1} \big||y|^{s+k-1} - L_{2kN}(y, |y|^{s+k-1}, yT_{2N}^{k}(y))\big|^{p}\,dy\right)^{1/p}\\
&\le C_2 N^{-(\operatorname{Re} s+k-1)}.
\end{aligned}
$$

(b) $\zeta_{d,k}(s) = 0$ *if and only if*

$$
\begin{aligned}
C_3 N^{-(\operatorname{Re} s+k-1+1/p)} &\le \left(\int_{-1}^{1} \big||y|^{s+k-1} - L_{2kN}(y, |y|^{s+k-1}, yT_{2N}^{k}(y))\big|^{p}\,dy\right)^{1/p}\\
&\le C_4 N^{-(\operatorname{Re} s+k-1+1/p)}.
\end{aligned}
$$

Example 5.4.6 (Normalized polynomials with equidistant zeros on $[-1,1]$). Let $0 < \operatorname{Re} s < 1$ and

$$
\begin{aligned}
&P_{2N+d}(y) = y^d \prod_{p=1}^{N}\left(1 - \left(\frac{2N+d-1}{2p+d-1}\right)^2 y^2\right), \qquad \beta_N = (2N+d-1)\pi/2,\\
&w(y,N) = \left(\sqrt{(1-y^2)}((1+y)^{1+y}(1-y)^{1-y})^{(2N+d-1)/2}\right)^{-1}, \qquad N = 1,2,\dots,\\
&\qquad\qquad\qquad\qquad\qquad\qquad a = 1, \quad p_0(k) = 0, \quad p_1(k) = \infty.
\end{aligned}
$$

It is established in Section 5.2 that for these polynomials, all three conditions (5.2.1), (5.2.2), and (5.2.4) are satisfied for

$$p \in J_{d,k,s} := ((kd+2)^{-1}, (\max\{0, -\operatorname{Re} s - k(1-d)+1\})^{-1}) \cap J^*_{d,k} \cap J^{**}_{d,k},$$

where the intervals $J^*_{d,k}$ and $J^{**}_{d,k}$ are defined in (5.2.26) and (5.2.27), respectively. It is easy to see that

$$J_{d,k,s} = \begin{cases} (1,\infty), & d=0,\ k=1, \\ (1/2,1], & d=1,\ k=1, \\ (1,\infty), & d=0,\ k=2, \\ (1/2,(1-\operatorname{Re} s)^{-1}), & d=1,\ k=2. \end{cases} \tag{5.4.4}$$

Then the following special case of Theorem 5.4.1 holds for the polynomials with equidistant zeros:

Corollary 5.4.7. *Let $0 < \operatorname{Re} s < 1$ and $p \in J_{d,k,s}$, where $J_{d,k,s}$ is defined by (5.4.4), and let $\tau^*_{N,d,kp}$ be defined by (3.3.40). Then:*

(a) $\zeta_{d,k}(s) \neq 0$ *if and only if*

$$\begin{aligned} &C_1 N^{-(\operatorname{Re} s+k(1-d)-1)}(\tau^*_{N,d,kp})^{-1} \\ &\quad\le \left(\int_{-1}^{1} \big||y|^{s+k(1-d)-1} - L_{2kN}(y, |y|^{s+k(1-d)-1}, y^{-kd+1}P^k_{2N+d}(y))\big|^p w^{kp}(y,N)\,dy\right)^{1/p} \\ &\quad\le C_2 N^{-(\operatorname{Re} s+k(1-d)-1)}(\tau^*_{N,d,kp})^{-1}. \end{aligned}$$

(b) $\zeta_{d,k}(s) = 0$ *if and only if*

$$\begin{aligned} &C_3 N^{-(\operatorname{Re} s+k(1-d)-1+1/p)} \\ &\quad\le \left(\int_{-1}^{1} \big||y|^{s+k(1-d)-1} - L_{2kN}(y, |y|^{s+k(1-d)-1}, y^{-kd+1}P^k_{2N+d}(y))\big|^p w^{kp}(y,N)\,dy\right)^{1/p} \\ &\quad\le C_4 N^{-(\operatorname{Re} s+k(1-d)-1+1/p)}. \end{aligned}$$

Example 5.4.8 (Normalized Hermite polynomials on $(-\infty,\infty)$). Let $0 < \operatorname{Re} s < 1$ and

$$P_{2N+d}(y) = H_{2N+d}(y)/H_{2N+d})^{(d)}(0), \quad \beta_N = \sqrt{2N+2d+1}, \quad w(y,N) = e^{-y^2/2},$$
$$N = 1,2,\ldots, \quad a=\infty, \quad p_0(k) = 1/2, \quad p_1(1) = 4, \quad p_1(2) = 2.$$

It is established in Section 5.2 that for the Hermite polynomials, all three conditions (5.2.1), (5.2.2), and (5.2.4) are satisfied for

$$p \in K_{d,k,s} := ((kd+2)^{-1}, (\max\{0, -\operatorname{Re} s - k(1-d)+1\})^{-1}) \cap K^*_{d,k} \cap K^{**}_{d,k},$$

where the intervals $K^*_{d,k}$ and $K^{**}_{d,k}$ are defined in (5.2.37) and (5.2.38), respectively. It is easy to see that

$$K_{d,k,s} = \begin{cases} (1,8/3), & d=0,\ k=1, \\ (1/2,1], & d=1,\ k=1, \\ (3/4,4/3), & d=0,\ k=2, \\ (3/4,\min\{4/3,(1-\operatorname{Re} s)^{-1}\}), & d=1,\ k=2. \end{cases} \tag{5.4.5}$$

Then the following special case of Theorem 5.4.1 holds for the Hermite polynomials:

Corollary 5.4.9. *Let* $0 < \operatorname{Re} s < 1$ *and* $p \in K_{d,k,s}$, *where* $K_{d,k,s}$ *is defined by* (5.4.5), *and let* $\tau^{**}_{N,d,kp}$ *is defined by* (3.3.48). *Then:*

(a) $\zeta_{d,k}(s) \neq 0$ *if and only if*

$$
\begin{aligned}
&C_1 N^{-(\operatorname{Re} s+k(1-d)-1)/2}(\tau^{**}_{N,d,kp})^{-1}\\
&\qquad\le \left(\int_{\mathbb{R}} \big||y|^{s+k(1-d)-1} - L_{2kN}(y,|y|^{s+k(1-d)-1}, y^{-kd+1}P^k_{2N+d}(y))\big|^p e^{-kpy^2/2}\,dy\right)^{1/p}\\
&\qquad\le C_2 N^{-(\operatorname{Re} s+k(1-d)-1)/2}(\tau^{**}_{N,d,kp})^{-1}.
\end{aligned}
$$

(b) $\zeta_{d,k}(s) = 0$ *if and only if*

$$
\begin{aligned}
&C_3 N^{-(\operatorname{Re} s+k(1-d)-1+1/p)/2}\\
&\qquad\le \left(\int_{\mathbb{R}} \big||y|^{s+k(1-d)-1} - L_{2kN}(y,|y|^{s+k(1-d)-1}, y^{-kd+1}P^k_{2N+d}(y))\big|^p e^{-kpy^2/2}dy\right)^{1/p}\\
&\qquad\le C_4 N^{-(\operatorname{Re} s+k(1-d)-1+1/p)/2}.
\end{aligned}
$$

Example 5.4.10 (Normalized Williams–Apostol polynomials on $(-\infty,\infty)$). Polynomials of the first kind:

$$
\begin{aligned}
&P_{2N+d}(y) = A_{2N+d}(y)/A_{2N+d})^{(d)}(0), \qquad \beta_N = 2N+2d,\\
&w(y,N) = (1+y^2)^{-N-d/2-\alpha}, \qquad \alpha \ge 0, \qquad a_N = 1, \qquad N = 1,2,\dots,\\
&a = \infty, \qquad p_0(k) = \max\{(2\alpha k + kd)^{-1}, (kd+2)^{-1}\}, \qquad p_1(k) = \infty.
\end{aligned}
$$

Polynomials of the second kind:

$$
\begin{aligned}
&P_{2N+d}(y) = W_{2N+d}(y)/W_{2N+d})^{(d)}(0), \qquad \beta_N = 2N+2d+1,\\
&w(y,N) = (1+y^2)^{-N-d/2-\alpha-1/2}, \qquad \alpha \ge 0, \qquad a_N = 1, \qquad N = 1,2,\dots,\\
&a = \infty, \qquad p_0(k) = \max\{(2\alpha k + kd)^{-1}, (kd+2)^{-1}\}, \qquad p_1(k) = \infty.
\end{aligned}
$$

It is established in Section 5.2 that for the Williams–Apostol polynomials, all three conditions (5.2.1), (5.2.2), and (5.2.4) are satisfied for $p \in J_{d,k,s} \cap ((2\alpha + d)^{-1}, \infty)$, where $J_{d,k,s}$ is defined in (5.4.4).

Corollary 5.4.11. *Let* $0 < \operatorname{Re} s < 1$ *and* $p \in J_{d,k,s} \cap ((2\alpha+d)^{-1}, \infty)$. *Then statements* (a) *and* (b) *of Corollary* 5.4.7 *with* $\int_{-1}^{1}$ *replaced by* $\int_{\mathbb{R}}$ *hold for the polynomials of Example* 5.4.10.

6. Other applications

In this chapter we present three more applications of asymptotic relations for the zeta functions.

6.1. Universal exponential sums. A number of universal functions and sequences (mostly in the form of divergent power series) that allow approximation of functions from a variety of classes have been introduced in analysis for the last 100 years (see the survey [25]). However, all these universal objects have not been given in the explicit form, with one exception. In 1975 Voronin [53] discovered the following remarkable result on universal translates of $\zeta_{d,k}(s)$ along the imaginary axis:

THEOREM 6.1.1. *Let* $0<r<1/4$ *and let* $g(s)$ *be continuous on the disk* $\bar{D}_r(3/4) := \{s\in\mathbb{C} : |s-3/4| \le r\}$ *and analytic in the interior of* $\bar{D}_r(3/4)$. *If* $g(s) \neq 0$ *in the interior of* $\bar{D}_r(3/4)$, *then for any* $\varepsilon > 0$ *there exists a real* $T = T(\varepsilon, g, r)$ *such that*

$$\max_{s\in\bar{D}_r(3/4)} |g(s) - \zeta_{d,k}(s+iT)| < \varepsilon. \tag{6.1.1}$$

Theorem 6.1.1 for $\zeta_{d,k}(s) = \zeta(s)$ was proved in [53]. Voronin mentions in [53] that the analogue of this result holds for any Dirichlet L-function (in particular, for $\zeta_{d,k}(s) = \beta(s)$). The proof of this extension of Voronin's theorem can be found in [4]. Further extensions and improvements of Voronin's theorem can be found in [4, 28, 32, 39, 47, 33, 41]. In particular, Theorem 6.1.1 is valid if $\bar{D}_r(3/4)$ is replaced with a compact $K \subset \{s \in \mathbb{C} : \operatorname{Re} s \in [3/4-r, 3/4+r]\}$ [4, 39, 47].

The following real version of this result easily follows from Theorem 6.1.1.

COROLLARY 6.1.2. *Let* $0 < r < 1/4$ *and let* $f : \mathbb{R} \to \mathbb{R}$ *be a continuous, nontrivial, and* 2π*-periodic function on* $\mathbb{R}$. *If* f *does not have sign changes* (*that is,* $f(t) \ge 0$ *or* $f(t) \le 0$ *on* $\mathbb{R}$), *then for any* $\varepsilon > 0$ *there exists a real* $T = T(\varepsilon, f, r)$ *such that*

$$\max_{t\in[0,2\pi)} |f(t) - \operatorname{Re}\zeta_{d,k}(3/4 + re^{it} + iT)| < \varepsilon. \tag{6.1.2}$$

Proof. We first assume that $f(t) = a_0/2 + \sum_{k=1}^{n}(a_k \cos kt + b_k \sin kt)$ is a nontrivial trigonometric polynomial of degree at most n with real coefficients. Then the function

$$g(s) := a_0/2 + \sum_{k=1}^{n}(a_k - ib_k)(s-3/4)^k r^{-k}, \qquad s = 3/4 + \rho e^{it},\ 0 \le \rho \le r,$$

is continuous on $\bar{D}_r(3/4)$ and analytic in the interior of $\bar{D}_r(3/4)$. In addition,

$$f(t) = \operatorname{Re} g(3/4 + re^{it}). \tag{6.1.3}$$

Next, $g(s) \neq 0$ in the interior of $\bar{D}_r(3/4)$ since $\operatorname{Re} g(s) \neq 0$ due to (6.1.3) and the integral representation for the harmonic function $\operatorname{Re} g(s)$. Therefore, g satisfies all conditions of Theorem 6.1.1, so (6.1.2) follows from (6.1.1) and (6.1.3).

To prove the corollary for any continuous, nontrivial, and 2π-periodic function f that does not have sign changes on $[0, 2\pi)$, it is sufficient to approximate f by a trigonometric polynomial that does not have sign changes on $[0, 2\pi)$. For example if $f(t) \geq 0$ on $[0, 2\pi)$, then for any $\varepsilon > 0$ there exist $n = n(\varepsilon, f)$ and a trigonometric polynomial Q of degree at most n such that $Q(t) \geq f(t)$ and $\max_{t\in[0,2\pi)} |f(t) - Q(t)| < \varepsilon/2$ (see [36, p. 96]). ■

Combining Theorem 6.1.1 with asymptotic representations for $\zeta_{d,k}(s)$, it is possible to find several families of universal sequences of exponential sums whose shifts along the imaginary axis can approximate a continuous function on $\bar{D}_r(3/4)$ that is analytic and nonvanishing in the interior of the disk. Real versions of these results follow from Corollary 6.1.2. In particular, two of these universal sequences, generated by

$$\zeta_{d,1}(s) = \begin{cases} \beta(s), & d = 0, \\ \zeta(s), & d = 1, \end{cases}$$

are presented below.

Corollary 6.1.3. *Let us set* ($N = 1, 2, \ldots,\ d = 0, 1$)

$$R_{N,d}(s) := \frac{\pi^{s+1/2}\sqrt{N}}{\sin((s-d)\pi/2)(1-2^{-s})^d \Gamma(s) 2^{2N+s+d-1}} \times \sum_{p=1}^{N} (-1)^{p+1} \binom{2N+d-1}{N-p} (2p+d-1)^{s-1}.$$

If $0 < r < 1/4$ and $g(s)$ is a continuous function on $\bar{D}_r(3/4)$ that is analytic and nonvanishing in the interior of the disk, then for any $\varepsilon > 0$ there exist $T = T(\varepsilon, g, r) \in \mathbb{R}$ and $N_0 = N_0(\varepsilon, g, r)$ such that for any $N \geq N_0$,

$$\max_{s\in\bar{D}_r(3/4)} |g(s) - R_{N,d}(s + iT)| < \varepsilon. \tag{6.1.4}$$

Proof. Using formula (4.4.38) from Corollary 4.4.4(c), we obtain

$$\zeta_{d,1}(s) = R_{N,d}(s) + O(N^{-1}\log^2 N), \qquad 3/4 - r \leq \operatorname{Re} s \leq 3/4 + r, \tag{6.1.5}$$

where the constant $C(s)$ in $O(N^{-1}\log^2 N)$ is independent of N and satisfies the property $\sup_{3/4-r\leq \operatorname{Re} s\leq 3/4+r} C(s) < \infty$, by (4.4.29). In other words, asymptotic (6.1.5) holds uniformly on the strip $\{s \in \mathbb{C} : \operatorname{Re} s \in [3/4-r, 3/4+r]\}$. Then choosing $T(\varepsilon/2, g, r)$ from Theorem 6.1.1 such that (6.1.1) holds with ε replaced by $\varepsilon/2$, we can find $N_0(\varepsilon/2, g, r)$ from (6.1.5) such that

$$|\zeta_{d,1}(s + iT) - R_{N,d}(s + iT)| < \varepsilon/2, \qquad N \geq N_0.$$

Hence (6.1.4) follows from (6.1.1). ■

Using Corollary 6.1.2 instead of Theorem 6.1.1, we can similarly prove the following real result:

Corollary 6.1.4. *Let $0 < r < 1/4$ and let $f : \mathbb{R} \to \mathbb{R}$ be a continuous, nontrivial, and 2π-periodic function on $\mathbb{R}$. If f does not have sign changes on $\mathbb{R}$, then for any $\varepsilon > 0$ there* *exist* $T = T(\varepsilon, f, r)$ *and* $N_0 = N_0(\varepsilon, f, r)$ *such that for any* $N \geq N_0$,

$$\max_{0 \leq t < 2\pi} |f(t) - \mathrm{Re}(R_N(3/4 + re^{it} + iT))| < \varepsilon.$$

Making the substitution $x = \cos t$ and using Corollary 6.1.4, we immediately obtain the following result:

COROLLARY 6.1.5. *Let* $0 < r < 1/4$ *and let* $h : [-1, 1] \to \mathbb{R}$ *be a nontrivial continuous function on* $[-1, 1]$. *If* h *does not have sign changes on* $[-1, 1]$, *then for any* $\varepsilon > 0$ *there exist* $T = T(\varepsilon, h, r)$ *and* $N_0 = N_0(\varepsilon, h, r)$ *such that for any* $N \geq N_0$,

$$\max_{|x| \leq 1} |h(x) - \mathrm{Re}(R_N(3/4 + rx + ir\sqrt{1 - x^2} + iT))| < \varepsilon.$$

REMARK 6.1.6. Using other formulae for $\zeta_{d,k}(s)$, $0 < \mathrm{Re}\, s < 1$, from Corollary 4.4.4, it is possible to find other special universal sums, while the general ones can be obtained from statements (a) and (b) of Theorem 4.4.1. One more source for universal sums is asymptotic (4.2.10), which generates the universal exponential sums for each fixed $y \in \mathbb{R} \setminus \{0\}$ and $\{P_{2N+d}\}_{N=1}^{\infty} \in \mathbb{P}_d(\beta, \gamma, \delta)$. An important role in all these processes is played by the uniform estimates (4.2.12), (4.4.2), and (4.4.30) for the constant $C(s)$ in the corresponding remainder term.

In addition, we note that Corollary 6.1.3 is valid in the more general case of the disk $\bar{D}_r(3/4)$ being replaced with a compact $K \subset \{s \in \mathbb{C} : \mathrm{Re}\, s \in [3/4 - r, 3/4 + r]\}$.

6.2. Functional equations for zeta functions. Here, we show that functional equations for zeta functions $\zeta(s)$ and $\beta(s)$ follow from asymptotic representations for these functions.

THEOREM 6.2.1. (a) *For* $s \in \mathbb{C} \setminus \{0, 1\}$, *the following Riemann's equation holds:*

$$\zeta(1 - s) = 2^{1-s}\pi^{-s}\Gamma(s)\cos(s\pi/2)\zeta(s). \tag{6.2.1}$$

(b) *For* $s \in \mathbb{C}$, *the following functional equation holds:*

$$\beta(1 - s) = (2/\pi)^s \Gamma(s)\sin(s\pi/2)\beta(s). \tag{6.2.2}$$

Proof. We first need some technical estimates. For $a > b > 0$ and $\alpha > 0$, $\beta \in \mathbb{R}$, the following inequality is valid:

$$|a^{\alpha+i\beta} - b^{\alpha+i\beta}| \leq a^{\alpha}(a/b - 1)(|\beta| + \alpha). \tag{6.2.3}$$

Indeed,

$$|a^{\alpha+i\beta} - b^{\alpha+i\beta}| \leq a^{\alpha}|a^{i\beta} - b^{i\beta}| + (a^{\alpha} - b^{\alpha}) \leq a^{\alpha}|\beta|\log(a/b) + \alpha a^{\alpha}(a/b - 1). \tag{6.2.4}$$

Then (6.2.3) follows from (6.2.4). Next setting $a = x$, $b = \sin x$, $x \in (0, \pi/2]$, we obtain from (6.2.3), for $\alpha > 0$, $\beta \in \mathbb{R}$,

$$|x^{\alpha+i\beta} - (\sin x)^{\alpha+i\beta}| \leq Cx^{\alpha}(x/\sin x - 1) \leq C_1 x^{2+\alpha}, \tag{6.2.5}$$

where C and C_1 are independent of x. Next, (6.2.5) immediately implies the estimate

$$|x^{-(\alpha+i\beta)} - (\sin x)^{-(\alpha+i\beta)}| \leq Cx^{2-\alpha}, \qquad x \in (0, \pi/2],\ \alpha > 0,\ \beta \in \mathbb{R}, \tag{6.2.6}$$

where C is independent of x. Further, setting

$$\alpha + i\beta = 1 - s, \quad \mathrm{Re}\, s < 1, \quad x = (2p - 1)\pi/(4N), \quad p = 1, \ldots, N,$$

from (6.2.6) we obtain

$$\left|\frac{1}{(2N)^{1-s}\big(\sin\frac{(2p-1)\pi}{4N}\big)^{1-s}}-\frac{(2/\pi)^{1-s}}{(2p-1)^{1-s}}\right|\le C\frac{(2p-1)^{1+\mathrm{Re}\,s}}{N^2},\tag{6.2.7}$$

where C is independent of p and N.

To prove statement (a), we use relation (4.4.32) for $d=0$:

$$\zeta(s)=\frac{\pi 2^{s-1}}{\cos(s\pi/2)(1-2^{1-s})\Gamma(s+1)}\lim_{N\to\infty}(2N)^{s-1}H_N(s),\qquad -1<\mathrm{Re}\,s<0,\tag{6.2.8}$$

where

$$\begin{aligned}H_N(s)&:=\sum_{p=1}^{N}\Big(\cos\frac{(2p-1)\pi}{4N}\Big)^{s-1}\Big((s+1)\cos^2\frac{(2p-1)\pi}{4N}-s\Big)\\&=-\sum_{p=1}^{N}\Big(\cos\frac{(2p-1)\pi}{4N}\Big)^{s-1}\Big((s+1)\sin^2\frac{(2p-1)\pi}{4N}-1\Big)\\&=-\sum_{p=1}^{N}\Big(\sin\frac{(2p-1)\pi}{4N}\Big)^{s-1}\Big((s+1)\cos^2\frac{(2p-1)\pi}{4N}-1\Big).\end{aligned}\tag{6.2.9}$$

Next, we approximate $(2N)^{s-1}H_N(s)$ by the sum

$$I_N(s):=-(2/\pi)^{1-s}s\sum_{p=1}^{N}(2p-1)^{s-1},\qquad -1<\mathrm{Re}\,s<0.\tag{6.2.10}$$

Then using (6.2.7), we obtain

$$\begin{aligned}&\left|\frac{(s+1)\cos^2\frac{(2p-1)\pi}{4N}-1}{(2N)^{1-s}\big(\sin\frac{(2p-1)\pi}{4N}\big)^{1-s}}-\frac{s(2/\pi)^{1-s}}{(2p-1)^{1-s}}\right|\\&\le|s|\left|\frac{1+O\big(\big(\frac{(2p-1)\pi}{4N}\big)^2\big)}{(2N)^{1-s}\big(\sin\frac{(2p-1)\pi}{4N}\big)^{1-s}}-\frac{(2/\pi)^{1-s}}{(2p-1)^{1-s}}\right|+N^{\mathrm{Re}\,s-1}\Big(\sin\frac{(2p-1)\pi}{4N}\Big)^{1+\mathrm{Re}\,s}\\&\le C\frac{(2p-1)^{1+\mathrm{Re}\,s}}{N^2},\end{aligned}\tag{6.2.11}$$

where C is independent of p and N. It follows from (6.2.9)–(6.2.11) that for $-1<\mathrm{Re}\,s<0$,

$$|(2N)^{s-1}H_N(s)-I_N(s)|\le C\sum_{p=1}^{N}(2p-1)^{1+\mathrm{Re}\,s}/N^2\le CN^{\mathrm{Re}\,s}.\tag{6.2.12}$$

Therefore combining (6.2.8) with (6.2.12), for $-1<\mathrm{Re}\,s<0$ we obtain

$$\begin{aligned}\zeta(s)&=\frac{-\pi 2^{s-1}(2/\pi)^{1-s}s}{\cos(s\pi/2)(1-2^{1-s})\Gamma(s+1)}\lim_{N\to\infty}\sum_{p=1}^{N}(2p-1)^{s-1}\\&=-\frac{\pi^s}{\cos(s\pi/2)(1-2^{1-s})\Gamma(s)}\sum_{p=1}^{\infty}\frac{1}{(2p-1)^{1-s}}=\frac{\pi^s}{\cos(s\pi/2)2^{1-s}\Gamma(s)}\zeta(1-s).\end{aligned}$$

Thus equation (6.2.1) is valid for $-1<\mathrm{Re}\,s<0$. To complete the proof of statement (a), it is sufficient to use analyticity of $\zeta(s)$ for $s\in\mathbb{C}\setminus\{1\}$.

The proof of statement (b) is similar, if we use relation (4.4.30) for $d = 0$. We have

$$\beta(s) = \frac{\pi}{2\sin(s\pi/2)\Gamma(s)} \lim_{N\to\infty} (2N)^{s-1} H_N^*(s), \qquad -1 < \operatorname{Re} s < 0,$$

where

$$\begin{aligned} H_N^*(s) &:= (-1)^{N+1} \sum_{p=1}^{N} (-1)^{p+1} \left(\cos \frac{(2p-1)\pi}{4N}\right)^{s-1} \sin \frac{(2p-1)\pi}{4N} \\ &= \sum_{p=1}^{N} (-1)^{p+1} \left(\sin \frac{(2p-1)\pi}{4N}\right)^{s-1} \cos \frac{(2p-1)\pi}{4N}. \end{aligned}$$

Next, we approximate $(2N)^{s-1} H_N^*(s)$ by the sum

$$I_N^*(s) := (2/\pi)^{1-s} \sum_{p=1}^{N} (-1)^{p+1} (2p-1)^{s-1}, \qquad -1 < \operatorname{Re} s < 0.$$

Then using (6.2.7), we obtain

$$\left| \frac{\cos \frac{(2p-1)\pi}{4N}}{(2N)^{1-s}\left(\sin \frac{(2p-1)\pi}{4N}\right)^{1-s}} - \frac{(2/\pi)^{1-s}}{(2p-1)^{1-s}} \right| \le C \frac{(2p-1)^{1+\operatorname{Re} s}}{N^2}.$$

Hence

$$|(2N)^{s-1} H_N^*(s) - I_N^*(s)| \le C N^{\operatorname{Re} s}, \qquad -1 < \operatorname{Re} s < 0,$$

and

$$\beta(s) = \frac{\pi (2/\pi)^{1-s}}{2\sin(s\pi/2)\Gamma(s)} \lim_{N\to\infty} \sum_{p=1}^{N} (-1)^{p+1} (2p-1)^{s-1} = \frac{(\pi/2)^s}{\sin(s\pi/2)\Gamma(s)} \beta(1-s).$$

Therefore, (6.2.2) is valid for $-1 < \operatorname{Re} s < 0$, and due to analyticity of $\beta(s)$ on $\mathbb{C}$, statement (b) is established. ∎

6.3. Combinatorial representations for Bernoulli and Euler numbers. Asymptotic formulae for $\zeta(2n)$ and $\beta(2n+1)$ lead to new combinatorial representations for the Bernoulli numbers

$$B_{2n} := 2(-1)^{n+1}(2n)!(2\pi)^{-2n}\zeta(2n), \qquad n = 0, 1, \ldots, \tag{6.3.1}$$

and the Euler numbers

$$E_{2n} := 2(-1)^n (2n)!(2/\pi)^{2n+1}\beta(2n+1), \qquad n = 0, 1, \ldots. \tag{6.3.2}$$

THEOREM 6.3.1. *For $n = 0, 1, \ldots$, the following formulae hold:*

$$\begin{aligned} B_{2n} = {} & \frac{(-1)^{n+1}}{(1-2^{1-2n})(2n+1)2^{4n}\prod_{j=0}^{n}(2j+1)} \\ & \times \sum_{m=0}^{n} (-1)^m \binom{2n+1}{n-m} (2m+1) \sum_{\sum_{r=0}^{n} \alpha_{r,m} = n} \prod_{r=0}^{n} \frac{(2r+1)^{2\alpha_{r,m}+1}}{(2\alpha_{r,m}+1)!}, \end{aligned} \tag{6.3.3}$$

$$E_{2n} = \frac{(-1)^n}{(2n+1)2^{2n}} \sum_{m=0}^{n} (-1)^m \binom{2n+1}{n-m} \sum_{\sum_{r=0}^{n} \alpha_{r,m} = n} \prod_{r=0}^{n} \frac{(2r+1)^{2\alpha_{r,m}}}{(2\alpha_{r,m})!}, \tag{6.3.4}$$

where the second summation in (6.3.3) *and* (6.3.4) *is taken over all nonnegative integral solutions* $\alpha_{0,m}, \dots, \alpha_{n,m}$ *to the equation* $\sum_{r=0}^{n} \alpha_{r,m} = n$, *provided that* $\alpha_{m,m} := 0$, $0 \le m \le n$.

Proof. To prove (6.3.3), we use (6.3.1) and the limit relation (4.4.32) for $d = 0$:

$$B_{2n} = \frac{\pi^{1-2n}}{1 - 2^{1-2n}} \lim_{N\to\infty} (2N)^{2n-1} S_{N,n}, \tag{6.3.5}$$

where

$$S_{N,n} := 2n \sum_{p=1}^{N} \cos^{2n-1} \frac{(2p-1)\pi}{4N} - (2n+1) \sum_{p=1}^{N} \cos^{2n+1} \frac{(2p-1)\pi}{4N}.$$

Using the Fourier expansion for $(\cos y)^{2q-1}$, $q \in \mathbb{N}$, we obtain by straightforward calculation

$$\begin{aligned}
S_{N,n} &= \frac{n}{2^{2n-3}} \sum_{p=1}^{N} \sum_{m=1}^{n} \binom{2n-1}{n-m} \cos \frac{(2m-1)(2p-1)\pi}{4N} \\
&\quad - \frac{2n+1}{2^{2n}} \sum_{p=1}^{N} \sum_{m=1}^{n+1} \binom{2n+1}{n-m+1} \cos \frac{(2m-1)(2p-1)\pi}{4N} \\
&= \frac{n}{2^{2n-2}} \sum_{m=1}^{n} (-1)^{m+1} \binom{2n-1}{n-m} \frac{1}{\sin \frac{(2m-1)\pi}{4N}} \\
&\quad - \frac{2n+1}{2^{2n+1}} \sum_{m=1}^{n+1} (-1)^{m+1} \binom{2n+1}{n-m+1} \frac{1}{\sin \frac{(2m-1)\pi}{4N}} \\
&= \frac{1}{(2n+1)2^{2n+1}} \sum_{m=0}^{n} (-1)^{m+1} \binom{2n+1}{n-m} \frac{(2m+1)^2}{\sin \frac{(2m+1)\pi}{4N}}.
\end{aligned} \tag{6.3.6}$$

Then it follows from (6.3.5) and (6.3.6) that

$$\begin{aligned}
&(1 - 2^{1-2n})(2n+1)2^{4n} B_{2n} = \lim_{y\to 0} y^{1-2n} \sum_{m=0}^{n} (-1)^{m+1} \binom{2n+1}{n-m} \frac{(2m+1)^2}{\sin(2m+1)y} \\
&= \Big(\prod_{j=0}^{n} (2j+1) \Big)^{-1} \lim_{y\to 0} y^{-3n} \sum_{m=0}^{n} (-1)^{m+1} \binom{2n+1}{n-m} (2m+1)^2 \\
&\times \prod_{0\le p\le n,\, p\ne m} \sin(2p+1)y = \Big((3n)! \prod_{j=0}^{n} (2j+1) \Big)^{-1} \\
&\times \sum_{m=0}^{n} (-1)^{m+1} \binom{2n+1}{n-m} (2m+1)^2 \Big(\prod_{0\le p\le n,\, p\ne m} \sin(2p+1)y \Big)^{(3n)}_{y=0}.
\end{aligned} \tag{6.3.7}$$

It remains to use the generalized Leibniz theorem

$$\begin{aligned}
&\Big(\prod_{0\le p\le n,\, p\ne m} \sin(2p+1)y \Big)^{(3n)}_{y=0} \\
&\quad = \frac{1}{2m+1} \sum_{\sum_{r=0}^{n}(2\alpha_{r,m}+1)=3n} \frac{(3n)!}{\prod_{r=0}^{n}(2\alpha_{r,m}+1)!} \prod_{p=0}^{n} (\sin(2p+1)y)^{(2\alpha_{r,m}+1)}_{y=0} \\
&\quad = \frac{(-1)^n (3n)!}{2m+1} \sum_{\sum_{r=0}^{n} \alpha_{r,m}=n} \prod_{r=0}^{n} \frac{(2r+1)^{2\alpha_{r,m}+1}}{(2\alpha_{r,m}+1)!},
\end{aligned} \tag{6.3.8}$$

where $\alpha_{m,m} := 0$, $0 \le m \le n$. Thus (6.3.3) follows from (6.3.7) and (6.3.8).

We prove (6.3.4) similarly. Indeed, using (6.3.2) and (4.4.30) for $d = 0$, we obtain

$$E_{2n} = \pi(2/\pi)^{2n+1} \lim_{N\to\infty} (-1)^{N+1}(2N)^{2n} S^*_{N,n}, \tag{6.3.9}$$

where

$$\begin{aligned}
S^*_{N,n} &:= \sum_{p=1}^{N} (-1)^{p+1} \cos^{2n} \frac{(2p-1)\pi}{4N} \sin \frac{(2p-1)\pi}{4N} \\
&= 2^{-2n} \sum_{p=1}^{N} (-1)^{p+1} \left(\binom{2n}{n} + 2 \sum_{m=1}^{n} \binom{2n}{n-m} \cos \frac{2m(2p-1)\pi}{4N} \right) \sin \frac{(2p-1)\pi}{4N} \\
&= 2^{-2n} \left(\binom{2n}{n} \sum_{p=1}^{N} (-1)^{p+1} \sin \frac{(2p-1)\pi}{4N} \right. \\
&\quad + \sum_{m=1}^{n} \binom{2n}{n-m} \sum_{p=1}^{N} (-1)^{p+1} \sin \frac{(2m+1)(2p-1)\pi}{4N} \\
&\quad \left. - \sum_{m=1}^{n} \binom{2n}{n-m} \sum_{p=1}^{N} (-1)^{p+1} \sin \frac{(2m-1)(2p-1)\pi}{4N} \right) \\
&= \frac{(-1)^{N+1}}{2^{2n+1}} \sum_{m=0}^{n} (-1)^m \frac{\binom{2n}{n-m}(2m+1)}{(n+m+1)\cos \frac{(2m+1)\pi}{4N}}.
\end{aligned}$$

Then using this relation and (6.3.9), we obtain

$$\begin{aligned}
E_{2n} &= 2^{-2n} \lim_{y\to 0} y^{-2n} \sum_{m=0}^{n} (-1)^m \binom{2n}{n-m} \frac{(2m+1)}{(n+m+1)\cos(2m+1)y} \\
&= \frac{1}{(2n+1)2^{2n}} \lim_{y\to 0} y^{-2n} \sum_{m=0}^{n} (-1)^m \binom{2n+1}{n-m} (2m+1) \prod_{0\le p\le n,\, p\ne m} \cos(2p+1)y \\
&= \frac{1}{(2n+1)2^{2n}(2n)!} \sum_{m=0}^{n} (-1)^m \binom{2n+1}{n-m} (2m+1) \\
&\quad \times \left(\prod_{0\le p\le n,\, p\ne m} \cos(2p+1)y \right)^{(2n)}_{y=0}.
\end{aligned} \tag{6.3.10}$$

It remains to note that by the generalized Leibniz theorem,

$$\left(\prod_{0\le p\le n,\, p\ne m} \cos(2p+1)y \right)^{(2n)}_{y=0} = (-1)^n (2n)! \sum_{\sum_{r=0}^{n} \alpha_{r,m} = n} \prod_{r=0}^{n} \frac{(2r+1)^{2\alpha_{r,m}}}{(2\alpha_{r,m})!}, \tag{6.3.11}$$

where $\alpha_{m,m} := 0$, $0 \le m \le n$. Thus (6.3.4) follows from (6.3.10) and (6.3.11). This completes the proof of Theorem 6.3.1. ∎

Remark 6.3.2. Note that the following well-known relations between the Bernoulli and the Euler numbers:

$$B_{2n} = \frac{2n}{4^{2n} - 2^{2n}} \sum_{j=0}^{n-1} \binom{2n-1}{2j} E_{2j}, \qquad n = 1, 2, \ldots, \tag{6.3.12}$$

$$E_{2n} = 1 - \sum_{j=1}^{n} \binom{2n}{2j-1} \frac{4^{2j} - 2^{2j}}{2j} B_{2j}, \quad n = 1, 2, \ldots, \tag{6.3.13}$$

can be proved by representation (4.4.38). Indeed, it follows from (6.3.1), (6.3.2), and (4.4.38) for $d = 1$ and $d = 0$ that

$$B_{2n} = \frac{\sqrt{\pi}\, n}{2^{4n-2}(1 - 2^{-2n})} \lim_{N\to\infty} \frac{\sqrt{N}}{2^{2N}} \sum_{p=1}^{N} (-1)^{p+1} \binom{2N}{N-p} (2p)^{2n-1}, \tag{6.3.14}$$

$$E_{2n} = 4\sqrt{\pi} \lim_{N\to\infty} \frac{\sqrt{N}}{2^{2N}} \sum_{p=1}^{N} (-1)^{p+1} \binom{2N-1}{N-p} (2p-1)^{2n}. \tag{6.3.15}$$

Next,

$$\begin{aligned}
&\sum_{j=1}^{n-1} \binom{2n-1}{2j} \sum_{p=1}^{N} (-1)^{p+1} \binom{2N-1}{N-p} (2p-1)^{2j} \\
&\quad = \sum_{p=1}^{N} (-1)^{p+1} \binom{2N-1}{N-p} \sum_{j=1}^{n-1} \binom{2n-1}{2j} (2p-1)^{2j} \\
&\quad = \frac{1}{2} \sum_{p=1}^{N} (-1)^{p+1} \binom{2N-1}{N-p} ((2p)^{2n-1} - (2p-2)^{2n-1}) \\
&\quad = \frac{1}{2} \left(\sum_{p=1}^{N} (-1)^{p+1} \binom{2N-1}{N-p} (2p)^{2n-1} + \frac{1}{2} \sum_{p=1}^{N-1} (-1)^{p+1} \binom{2N-1}{N-p-1} (2p)^{2n-1} \right) \\
&\quad = \frac{1}{2} \sum_{p=1}^{N} (-1)^{p+1} \binom{2N}{N-p} (2p)^{2n-1}.
\end{aligned} \tag{6.3.16}$$

Further, combining (6.3.14) and (6.3.15) with (6.3.16), we obtain

$$\begin{aligned}
&\frac{2n}{4^{2n} - 2^{2n}} \sum_{j=0}^{n-1} \binom{2n-1}{2j} E_{2j} \\
&\qquad = \frac{8\sqrt{\pi}\, n}{4^{2n} - 2^{2n}} \lim_{N\to\infty} \frac{\sqrt{N}}{2^{2N}} \sum_{j=1}^{n-1} \binom{2n-1}{2j} \sum_{p=1}^{N} (-1)^{p+1} \binom{2N-1}{N-p} (2p-1)^{2j} \\
&\qquad = \frac{4\sqrt{\pi}\, n}{4^{2n} - 2^{2n}} \lim_{N\to\infty} \frac{\sqrt{N}}{2^{2N}} \sum_{p=1}^{N} (-1)^{p+1} \binom{2N}{N-p} (2p)^{2n-1} = B_{2n}.
\end{aligned}$$

This proves (6.3.12). Relation (6.3.13) can be proved similarly.

References

[1] N. I. Akhiezer, *The Classical Moment Problem, and Some Related Questions in Analysis*, Hafner, New York, 1965.

[2] T. M. Apostol, *Elementary proof of Euler's formula for* $\zeta(2n)$, Amer. Math. Monthly 80 (1973), 425–431.

[3] A. I. Aptekarev, V. S. Buyarov and I. S. Degeza, *Asymptotic behavior of the* L^p*-norms and the entropy for general orthogonal polynomials*, Mat. Sb. 185 (1994), no. 8, 3–30 (in Russian); English transl.: Russian Akad. Sci. Sb. Math. 82 (1995), 373–395.

[4] B. Bagchi, *A joint universality theorem for Dirichlet L-functions*, Math. Z. 181 (1982), 319–334.

[5] S. N. Bernstein, *Extremal Properties of Polynomials and the Best Approximation of Continuous Functions of a Single Real Variable*, State United Sci. Tech. Publ. House, Moscow, 1937 (in Russian).

[6] S. Bernstein, *Sur la meilleure approximation de* $|x|^p$*, par des polynômes de degrés très élevés*, Bull. Acad. Sci. USSR Sér. Math. 2 (1938), no. 2, 181–190.

[7] S. N. Bernstein, *On the best approximation of* $|x-c|^p$, in: Collected Works, Vol. II, Akad. Nauk SSSR, Moscow, 1954, 273–280 (in Russian).

[8] D. Borwein, J. M. Borwein and I. E. Leonard, L_p *norms and the sinc function*, Amer. Math. Monthly 117 (2010), 528–539.

[9] T. J. Bromwich, *An Introduction to the Theory of Infinite Series*, 2nd ed., Blackic, Glasgow, 1926.

[10] G. Byrne, T. M. Mills and S. J. Smith, *On Lagrange interpolation with equidistant nodes*, Bull. Austral. Math. Soc. 42 (1990), 81–89.

[11] K. Chandrasekharan, *Introduction to Analytic Number Theory*, Springer, Berlin, 1969.

[12] B. Conrey, *The Riemann Hypothesis*, Notices Amer. Math. Soc. 50 (2003), 341–353.

[13] D. Cvijovič and J. Klinowski, *An application for the Chebyshev polynomials*, Mat. Vesnik 50 (1998), 105–110.

[14] D. Cvijovič and J. Klinowski, *Finite cotangent sums and the Riemann zeta function*, Math. Slovaca 50 (2000), 149–157.

[15] D. Cvijovič and J. Klinowski, *Polynomials with easily deducible zeros*, Atti Sem. Mat. Fis. Univ. Modena 48 (2000), 317–333.

[16] A. Erdélyi, W. Magnus, F. Oberhettinger and F. G. Tricomi, *Higher Transcendental Functions*, Vol. I, McGraw-Hill, New York, 1953.

[17] A. Erdélyi, W. Magnus, F. Oberhettinger and F. G. Tricomi, *Higher Transcendental Functions*, Vol. II, McGraw-Hill, New York, 1953.

[18] L. Fejér, *Lebesguesche Konstanten und divergente Fourierreihen*, J. Reine Angew. Math. 138 (1910), 22–53.

[19] M. I. Ganzburg, *The Bernstein constant and polynomial interpolation at the Chebyshev nodes*, J. Approx. Theory 119 (2002), 193–213.

[20] M. I. Ganzburg, *Strong asymptotics in Lagrange interpolation with equidistant nodes*, J. Approx. Theory 122 (2003), 224–240.

[21] M. I. Ganzburg, *Polynomial interpolation, an L-function, and pointwise approximation of continuous functions*, J. Approx. Theory 153 (2008), 1–18.

[22] M. I. Ganzburg, *Limit theorems of polynomial approximation with exponential weights*, Mem. Amer. Math. Soc. 192 (2008), no. 897.

[23] M. I. Ganzburg, *Lagrange interpolation formulae and new asymptotic formulae for the Riemann zeta function*, in: Approximation Theory XIII: San Antonio 2010, M. Neamtu and L. Schumaker (eds.), Springer Proc. Math., Springer, New York, 2012, 79–93.

[24] I. S. Gradshtein and I. M. Ryzhik, *Tables of Integrals, Series, and Products*, 5th ed., Academic Press, San Diego, 1994.

[25] K.-G. Grosse-Erdmann, *Universal families and hypercyclic operators*, Bull. Amer. Math. Soc. 36 (1999), 345–381.

[26] C. Hermite, *Sur la formule d'interpolation de Lagrange*, J. Reine Angew. Math. 84 (1878), 70–79.

[27] A. Ivić, *The Riemann Zeta-Function: Theory and Applications*, Wiley, New York, 1985.

[28] A. A. Karatsuba and S. M. Voronin, *The Riemann Zeta-Function*, de Gruyter, Berlin, 1982.

[29] B. A. Kats, *The refined metric dimension with applications*, Comput. Methods Function Theory 7 (2007), 77–89.

[30] B. A. Kats, *The Cauchy integral over non-rectifiable paths*, in: Contemp. Math. 455, Amer. Math. Soc., Providence, RI, 2008, 183–196.

[31] M. G. Krein and A. A. Nudelman, *The Markov Moment Problem and Extremal Problems*, Amer. Math. Soc., Providence, RI, 1977.

[32] A. Laurinčikas, *Limit Theorems for the Riemann Zeta-Function*, Kluwer, Dordrecht, 1996.

[33] A. Laurinčikas, *Universality of the Riemann zeta-function*, J. Number Theory 130 (2010), 2323–2331.

[34] A. L. Levin and D. S. Lubinsky, *Christoffel functions, orthogonal polynomials, and Nevai's conjecture for Freud weights*, Constr. Approx. 8 (1992), 463–535; Erratum, ibid. 11 (1995), 417–418.

[35] A. L. Levin and D. S. Lubinsky, *Orthogonal Polynomials for Exponential Weights*, Springer, New York, 2001.

[36] G. G. Lorentz, M. von Golitschek and Y. Makovoz, *Constructive Approximation: Advanced Problems*, Springer, New York, 1996.

[37] D. S. Lubinsky, *An update on orthogonal polynomials and weighted approximation on the real line*, Acta Appl. Math. 33 (1993), 121–164.

[38] D. S. Lubinsky, *Best approximation and interpolation of* $(1+(ax)^2)^{-1}$ *and its transforms*, J. Approx. Theory 125 (2003), 106–115.

[39] K. Matsumoto, *Probabilistic value-distribution theory of zeta-functions*, Sūgaku 53 (2001), 279–296 (in Japanese); English transl.: Sūgaku Expo. 17 (2004), 51–71.

[40] P. Mattila and M. S. Melnikov, *Existence and weak type inequalities for Cauchy integrals of general measures on rectifiable curves and sets*, Proc. Amer. Math. Soc. 120 (1994), 143–149.

[41] T. Meyrath, *On the universality of derived functions of the Riemann zeta-function*, J. Approx. Theory 163 (2011), 1419–1426.

[42] S. M. Nikolskiĭ, *On the best approximation in the mean to the function* $|a-x|^s$ *by polynomials*, Izv. Akad. Nauk SSSR 11 (1947), 139–180 (in Russian).

[43] K. Prachar, *Primzahlverteilung*, Springer, Berlin, 1957.

[44] O. I. Prudnikov, A. P. Brychkov and Yu. A. Marichev, *Integrals and Series, Vol. I: Elementary Functions*, CRC Press, Boca Raton, FL, 1998.

[45] M. Revers, *On Lagrange interpolation with equally spaced nodes*, Bull. Austral. Math. Soc. 62 (2000), 357–368.

[46] M. Revers, *On the asymptotics of polynomial interpolation to* $|x|^\alpha$ *at the Chebyshev nodes*, J. Approx. Theory 165 (2013), 70–82.

[47] J. Steuding, *Value Distribution of L-Functions*, Lecture Notes in Math. 1877, Springer, Berlin, 2007.

[48] G. Szegö, *Orthogonal Polynomials*, 4th ed., Amer. Math. Soc. Colloq. Publ. 23, Amer. Math. Soc., Providence, RI, 1975.

[49] E. C. Titchmarsh, *The Zeta-Function of Riemann*, Cambridge Univ. Press, Cambridge, 1930.

[50] E. C. Titchmarsh, *The Theory of Functions*, 2nd ed., Oxford University Press, London, 1939.

[51] X. Tolsa, *Bilipschitz maps, analytic capacity, and the Cauchy integral*, Ann. of Math. 162 (2005), 1243–1304.

[52] I. N. Vekua, *Generalized Analytic Functions*, 2nd ed., Nauka, Moscow, 1988 (in Russian).

[53] S. M. Voronin, *Theorem on the "universality" of the Riemann zeta-function*, Izv. Akad. Nauk SSSR Ser. Mat. 39 (1975), 475–486 (in Russian); English transl.: Mat. USSR-Izv. 9 (1975), 443–453.

[54] J. L. Walsh, *Interpolation and Approximation by Rational Functions in the Complex Domain*, 2nd ed., Amer. Math. Soc. Colloq. Publ. 20, Amer. Math. Soc., Providence, RI, 1960.

[55] G. N. Watson, *A Treatise on the Theory of Bessel Functions*, 2nd ed., Cambridge Univ. Press, Cambridge, 1966.

[56] K. S. Williams, *On* $\sum_{n=1}^{\infty}(1/n^{2k})$, Math. Mag. 44 (1971), 273–276.

[57] A. Zygmund, *Trigonometric Series*, Vol. I, Cambridge Univ. Press, Cambridge, 1959.

Index