0. Introduction

Let (X, d) be a metric space. We shall denote by $\mathrm{CB}(X)$ the family of nonempty closed bounded subsets of X and by $K(X)$ the family of nonempty compact subsets of X. On $\mathrm{CB}(X)$ we have the Hausdorff metric H given by

$$H(A, B) := \max\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\}, \quad A, B \in \mathrm{CB}(X),$$

where for $x \in X$ and $C \subset X$, $d(x, C) := \inf\{d(x, y) : y \in C\}$ is the distance from the point x to the subset C. A multivalued mapping $T : X \to \mathrm{CB}(X)$ is said to be a *contraction* if there is a constant $k \in [0, 1)$ such that

$$H(Tx, Ty) \leq kd(x, y), \quad x, y \in X.$$

A multivalued mapping $T : X \to \mathrm{CB}(X)$ is said to be *nonexpansive* if

$$H(Tx, Ty) \leq d(x, y), \quad x, y \in X.$$

A point x is called a *fixed point* of a multivalued map T if $x \in Tx$.

In 1969, Nadler [38] established the multivalued version of Banach's contraction principle by proving the existence of a fixed point of a contraction $T : X \to \mathrm{CB}(X)$. Since then the metric fixed point theory of multivalued mappings has been rapidly developed. In 1972 Reich [41, 42] proved that if $T : X \to K(X)$ satisfies the contractive condition

$$H(Tx, Ty) \leq k(d(x, y))d(x, y), \quad x, y \in X, \ x \neq y,$$

where $k : (0, \infty) \to [0, 1)$ is a function with the property

$$\limsup_{r \to t^+} k(r) < 1 \quad \text{for all } t \in (0, \infty), \tag{0.1}$$

then T has a fixed point.

Reich [44, 48] raised the question: If T satisfies the same contractive condition as described above but takes values in $\mathrm{CB}(X)$, does T have a fixed point?

This question still remains open. However, the answer is yes provided either the inequality (0.1) holds also at $t = 0$ ([52], [37]) or k behaves near 0 like $1 - at^\sigma$ ([10]) for some $a > 0$ and $\sigma \in (0, 1)$ and for all $t > 0$ sufficiently small.

The fixed point theory for multivalued nonexpansive mappings is, however, more subtle. As a matter of fact, not many positive results have been obtained so far. In 1974, Lim [30], using Edelstein's [14] method of asymptotic centers, proved that a multivalued nonexpansive mapping $T : C \to K(C)$ has a fixed point, where C is a nonempty closed bounded convex subset of a uniformly convex Banach space. In 1990, Kirk and Massa [27] showed that if a closed bounded convex subset C of a Banach space X has the property that the asymptotic center in C of each bounded sequence of X is nonempty and compact, then every nonexpansive multivalued mapping $T : C \to \mathrm{KC}(C)$ has a fixed point, where $\mathrm{KC}(C)$ is the family of nonempty compact convex subsets of C.

Let C be a closed convex subset of a Banach space X. The *inward set* of C at $x \in C$ is given by

$$I_C(x) := \{x + \lambda(y - x) : \lambda \geq 0,\ y \in C\}.$$

A multivalued mapping $T : C \to 2^X$ is said to be *inward* (resp. *weakly inward*) on C if

$$Tx \subset I_C(x) \quad (\text{resp. } Tx \subset \overline{I_C(x)}) \quad \text{ for } x \in C.$$

Two remarkable theorems of Lim [32, 33] state that T has a fixed point either if T is a weakly inward nonexpansive mapping from C to $K(X)$, with X a uniformly convex Banach space, or if T is a weakly inward contraction from C to $\mathrm{CB}(X)$, with X a general Banach space. We shall extend the Kirk–Massa theorem to inward nonexpansive mappings. Also, we shall give another proof of Lim's theorem [32] by using an inequality characteristic of uniform convexity [56].

Random fixed point theory for multivalued mappings is quite a recent topic. In 1977, Itoh [22] established the random version of Nadler's multivalued version of Banach's contraction principle. We shall further show that the fixed point set function of a random contraction is measurable and thus the existence of a random fixed point is an immediate consequence of the Measurable Selection Theorem (cf. [1], [7], [51]). We shall present the random version of Lim's theorem [30]. We shall also show that if T is a single-valued random nonexpansive mapping in a uniformly smooth Banach space, then the fixed point set function of T is measurable and thus T has a random fixed point.

Multivalued mappings find applications in various fields, e.g., control theory (see the examples in Deimling [11]) and economics (see Yuan [60, Chapter 7]).

1. Multivalued contractions

In this chapter we first recall Nadler's multivalued version of Banach's contraction principle. We then discuss Reich's problem in Section 1.2. Partial answers to this problem will be presented. In Section 1.3, we prove some fixed point theorems for non-self-contractions which satisfy the weak inwardness condition. In the final Section 1.4, we discuss local contractions in an ε-chainable metric space.

1.1. Nadler's theorem. Let (X, d) be a metric space. Recall that $\mathrm{CB}(X)$ is the family of nonempty closed bounded subsets of X, $K(X)$ the family of nonempty compact subsets of X, and H the Hausdorff metric on $\mathrm{CB}(X)$; i.e.,

$$H(A, B) := \max\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\}, \quad A, B \in \mathrm{CB}(X).$$

DEFINITION 1.1.1. A multivalued mapping $T : X \to \mathrm{CB}(X)$ is said to be a *contraction* if there exists a constant $k \in [0, 1)$ such that

$$H(Tx, Ty) \leq k d(x, y), \quad x, y \in X.$$

For a point $x \in X$, a sequence $(x_n)_{n=0}^{\infty}$ in X is said to be an *orbit* of T at x if $x_0 = x$ and $x_n \in Tx_{n-1}$ for all $n \geq 1$.

Banach's contraction principle was extended by Nadler [38] to multivalued contractions. (See also Covitz and Nadler [9].)

THEOREM 1.1.2 (Nadler [38]). *Let (X, d) be a complete metric space and $T : X \to \mathrm{CB}(X)$ a contraction. Then T has a fixed point ξ and for any given x_0 and $\overline{k}$, $k < \overline{k} < 1$, there is an orbit (x_n) of T at x_0 converging to ξ with the estimate*

$$d(x_n, \xi) \leq \frac{\overline{k}^n}{1-\overline{k}} d(x_1, x_0), \quad n \geq 0. \tag{1.1}$$

Since we shall use the estimate (1.1) in Chapter 3, we include here a brief proof of Theorem 1.1.2.

PROOF. Set $\lambda := \overline{k}/k > 1$. Pick any $x_1 \in Tx_0$ and then recursively define $x_n \in Tx_{n-1}$ for $n \geq 2$ such that

$$d(x_n, x_{n+1}) \leq \lambda d(x_n, Tx_n) \leq \lambda H(Tx_n, Tx_{n-1}).$$

Since $H(Tx_n, Tx_{n-1}) \leq kd(x_n, x_{n-1})$, we see that

$$d(x_{n+1}, x_n) \leq \overline{k} d(x_n, x_{n-1}) \leq \ldots \leq \overline{k}^n d(x_1, x_0), \quad n \geq 0.$$

It then follows that

$$d(x_n, x_{n+p}) \leq \sum_{i=0}^{p-1} d(x_{n+i}, x_{n+i+1}) \leq \frac{\overline{k}^n}{1-\overline{k}} d(x_1, x_0), \quad n, p \geq 0. \tag{1.2}$$

Hence (x_n) is Cauchy and convergent as X is complete. Let ξ be the limit. Letting $p \to \infty$ in (1.2), we obtain the estimate (1.1). ■

1.2. Reich's problem. Let $k : (0, \infty) \to [0, 1)$ be a function with the property

$$\limsup_{s \to t^+} k(s) < 1 \quad \text{for all } t > 0. \tag{$*$}$$

THEOREM 1.2.1 (Reich [41, 42]). *Let (X, d) be a complete metric space and $T : X \to K(X)$ satisfy the condition*

$$H(Tx, Ty) \leq k(d(x, y))d(x, y), \quad x, y \in X,\ x \neq y, \tag{1.3}$$

where $k : (0, \infty) \to [0, 1)$ has property $(*)$. Then T has a fixed point.

In the above theorem, Reich weakened the contraction requirement. However, T is assumed to take compact values. Reich thus proposed the problem below.

PROBLEM. Does T have a fixed point if $T : X \to \mathrm{CB}(X)$ satisfies the condition (1.3) of Theorem 1.2.1 and k has property $(*)$?

Though Reich's problem remains unsolved, some partial answers have been obtained (see [52], [37], [8], [10]; see also [24]).

THEOREM 1.2.2 ([52], [37]). *Let (X, d) be a complete metric space and $T : X \to \mathrm{CB}(X)$ satisfy the condition*

$$H(Tx, Ty) \leq k(d(x, y))d(x, y), \quad x, y \in X,\ x \neq y, \tag{1.4}$$

where $k : (0, \infty) \to [0, 1)$ *is a function. If* k *has the property*

$$(**) \qquad \limsup_{s \to t^+} k(s) < 1 \quad \textit{for all } t \geq 0,$$

then T *has a fixed point.*

We here provide a simpler proof of Theorem 1.2.2 than those given in [52] and [37]. First we prove a lemma.

Lemma 1.2.3. *Assume* $T : X \to \mathrm{CB}(X)$ *satisfies* (1.3) *with* k *having property* $(*)$. *Then for any* $x_0 \in X$ *there exists an orbit* (x_n) *of* T *at* x_0 *such that the sequence* $(d(x_n, x_{n+1}))$ *is decreasing to* 0.

Proof. Take any $x_1 \in Tx_0$ and then pick, for $n = 1, 2, \ldots, x_{n+1} \in Tx_n$ such that

$$d(x_n, x_{n+1}) \leq [k(d(x_n, x_{n-1}))]^{-1/2} d(x_n, Tx_n). \tag{1.5}$$

(We have assumed $x_n \neq x_{n-1}$ without loss of generality, for otherwise $x_{n-1} = x_n$ and x_{n-1} is a fixed point of T.) Since

$$d(x_n, Tx_n) \leq H(Tx_{n-1}, Tx_n) \leq k(d(x_{n-1}, x_n)) d(x_{n-1}, x_n),$$

it follows from (1.5) that

$$d(x_n, x_{n+1}) \leq \sqrt{k(d(x_{n-1}, x_n))}\, d(x_{n-1}, x_n) < d(x_{n-1}, x_n). \tag{1.6}$$

So $(d(x_{n-1}, x_n))$ is decreasing. Let b be the limit of $(d(x_{n-1}, x_n))$. Taking limits in (1.6), we obtain $b \leq \sqrt{c} b$, where $c = \limsup_{r \to b^+} k(r)$. Hence by property $(*)$ we have $b = 0$. ∎

Proof of Theorem 1.2.2. Let (x_n) be as constructed in Lemma 1.2.3. By property $(**)$ we have a $\delta > 0$ and an $a \in (0, 1)$ such that

$$k(t) < a^2 \quad \text{for } t \in (0, \delta).$$

Let N be such that $d(x_{n-1}, x_n) < \delta$ for $n \geq N$. It then follows from the first inequality in (1.6) that for $n \geq N$,

$$d(x_n, x_{n+1}) \leq a d(x_{n-1}, x_n) \leq \ldots \leq a^{n-N+1} d(x_{N-1}, x_N).$$

This implies that (x_n) is Cauchy and hence convergent. Let $\xi = \lim x_n$. Since $x_n \in Tx_{n-1}$ for all n, taking the limit as $n \to \infty$ yields $\xi \in T\xi$ and ξ is a fixed point of T. ∎

Another partial answer to Reich's problem was given by Chen [8].

Theorem 1.2.4 (Chen [8]). *Let* (X, d) *be a complete metric space and* $T : X \to \mathrm{CB}(X)$ *satisfy the condition*

$$H(Tx, Ty) \leq k(d(x, y)) d(x, y), \quad x, y \in X, \ x \neq y,$$

where $k : (0, \infty) \to [0, 1)$ *has property* $(*)$. *Suppose in addition that* T *has the property*:

$(\diamond)$ *whenever* Y *is a closed subset of* X *such that* $Tx \cap Y \neq \emptyset$ *for all* $x \in Y$, *we have* $d(x, Tx \cap Y) = d(x, Tx)$ *for all* $x \in Y$.

Then T *has a fixed point.*

We divide the proof of Theorem 1.2.4 into two lemmas. Let $\varepsilon > 0$ be given. Let $l(\varepsilon)$ be such that $\widetilde{k}(\varepsilon) < l(\varepsilon) < 1$, where $\widetilde{k} : (0, \infty) \to [0, 1)$ is defined by

$$\widetilde{k}(t) := \limsup_{r \to t^+} k(r), \qquad t > 0.$$

By property $(*)$ we have a $\delta = \delta(\varepsilon) \in (0, 1)$ such that $k(t) < l(\varepsilon)$ for $\varepsilon \le t < \varepsilon + \delta$. Let $\widetilde{\varepsilon} = \varepsilon + 4/\delta$.

LEMMA 1.2.5. *Assume A is a closed subset of X such that*

$$A \cap Tx \neq \emptyset \quad \textit{for all } x \in A.$$

Then $\inf\{d(x, Tx) : x \in A\} = 0$.

PROOF. Pick any $x_0 \in A$ and $x_1 \in Tx_0 \cap A$. We then recursively define x_{n+1} for $n \ge 1$ as follows (we may assume that $x_n \neq x_{n+1}$ for all n): $x_{n+1} \in Tx_n \cap A$ is chosen in such a way that $d(x_n, x_{n+1}) < d(x_n, Tx_n) + \varepsilon_n$, where

$$0 < \varepsilon_n < \min\{d(x_{n-1}, x_n)[1 - k(d(x_{n-1}, x_n))], 1/n\}.$$

(This is possible since by assumption we have $d(x_n, Tx_n) = d(x_n, Tx_n \cap A)$ for all n.) It then follows that

$$d(x_n, x_{n+1}) \le H(Tx_{n-1}, Tx_n) + \varepsilon_n \le k(d(x_{n-1}, x_n))d(x_{n-1}, x_n) + \varepsilon_n < d(x_{n-1}, x_n).$$

Let $\lim d(x_{n-1}, x_n) = d$. If $d > 0$, we deduce by property $(*)$ that $d \le d\widetilde{k}(d) < d$, a contradiction. Hence $d = 0$. This implies $\inf\{d(x, Tx) : x \in A\} = 0$. ■

LEMMA 1.2.6. *Assume A is a closed subset of X such that*

$$A \cap Tx \neq \emptyset \quad \textit{for all } x \in A.$$

Then for $\varepsilon > 0$ there is $x_0 \in A$ so that

$$Ty \cap A \neq \emptyset \quad \textit{for all } y \in B,$$

where $B := \{y \in A : d(y, x_0) \le \widetilde{\varepsilon}\}$.

PROOF. Suppose there is no such $x_0 \in A$. Then for all $x \in A$, there exists some $y \in A$, $d(y, x) \le \widetilde{\varepsilon}$, such that $d(z, x) > \widetilde{\varepsilon}$ for all $z \in Ty$; in particular, $d(x, Ty) \ge \widetilde{\varepsilon}$.

CASE 1: $d(x, y) < \widetilde{\varepsilon} - 4/\delta$. In this case we have

$$d(x, Tx) \ge d(x, Ty) - H(Tx, Ty) \ge \widetilde{\varepsilon} - k(d(y, x))d(y, x) \ge \widetilde{\varepsilon} - d(y, x) \ge 4/\delta.$$

CASE 2: $d(x, y) \ge \widetilde{\varepsilon} - 4/\delta$. In this case we have

$$d(x, Tx) \ge d(x, Ty) - H(Tx, Ty) \ge \widetilde{\varepsilon} - k(d(x, y))d(x, y) \ge \widetilde{\varepsilon} - l(\varepsilon)\widetilde{\varepsilon} = \widetilde{\varepsilon}(1 - l(\varepsilon)).$$

We thus conclude that $\inf\{d(x, Tx) : x \in A\} > 0$. This contradicts Lemma 1.2.5. ■

PROOF OF THEOREM 1.2.4. Pick a sequence (ε_n) which is strictly decreasing to 0. Let $l_n = l(\varepsilon_n)$, $\delta_n = \delta(\varepsilon_n)$ and let $\widetilde{\varepsilon}_n$ be defined in the same way as before. Then by Lemma 1.2.6 we can construct a sequence of balls, (B_n), such that

(i) $Tx \cap B_n \neq \emptyset$ for all $x \in B_n$ and n;
(ii) B_n is a subball of B_{n-1} and the radius of B_n is ε_n.

Since $\operatorname{diam}(B_n) = 2\varepsilon_n \to 0$, we have $\bigcap_{n=1}^{\infty} B_n = \{\zeta\}$ for some $\zeta \in X$. By (i) this ζ is a fixed point of T. ■

Remark. Chen's condition ($\diamond$) is very restrictive. Indeed, even constant mappings do not always satisfy it, as shown in the example below.

Example. Let $X = [0,5]$ be equipped with the usual distance. Define a constant mapping F by

$$Fx := [0,1] \cup [4,5], \quad x \in X.$$

Let $Y = [1,3]$. Then we have $Fx \cap Y \neq \emptyset$ for all $x \in X$. But for $x = 3 \in Y$ we have $d(x, Fx) = 1$ while $d(x, Fx \cap Y) = 2$. So $d(x, Fx) \neq d(x, Fx \cap Y)$.

By Theorem 1.2.2 we know that to answer Reich's problem, we can assume that

$$\limsup_{t \to 0^+} k(t) = 1.$$

Daffer, Kaneko and Li [10] showed that if $k(t)$ behaves, near 0, like $1 - at^\sigma$ for some constants $a > 0$ and $\sigma \in (0,1)$, then T has a fixed point.

Theorem 1.2.7 ([10]). *Let (X,d) be a complete metric space and $T : X \to CB(X)$ satisfy the condition*

$$H(Tx, Ty) \le k(d(x,y))d(x,y), \quad x, y \in X,\ x \neq y, \tag{1.7}$$

where $k : (0,\infty) \to [0,1)$ has property $(*)$. If $k(t) \le 1 - at^\sigma$ for some constants $a > 0$ and $\sigma \in (0,1)$ and all $t > 0$ sufficiently small: $0 < t \le t_0$ for some $0 < t_0 < a^{-1/\sigma}$, then T has a fixed point.

Proof. We may assume that the inequality in (1.7) is strict. (Otherwise we can replace k with another function $k_1 > k$ which still satisfies property $(*)$ and which makes the inequality in (1.7) strict. For instance, taking any a' such that $0 < a' < a$, we have $1 - at^\sigma < 1 - a't^\sigma$ for $t > 0$.) Next pick any $x_0 \in X$ and construct an orbit (x_n) of T at x_0 such that

$$d(x_n, x_{n+1}) < \varphi(d(x_{n-1}, x_n)), \quad n \ge 1, \tag{1.8}$$

where $\varphi(t) := tk(t)$. This is possible since $d(x_n, Tx_n) \le H(Tx_{n-1}, Tx_n) < \varphi(d(x_{n-1}, x_n))$. (In the above construction of (x_n) we assumed that $x_{n-1} \neq x_n$ for all $n \ge 1$. Indeed, if $x_{n-1} = x_n$ for some $n \ge 1$, then x_{n-1} is a fixed point of T and we are done.) Since $\varphi(t) < t$ for $t > 0$, the sequence $(d(x_{n-1}, x_n))$ is decreasing. Repeat the proof of Lemma 1.2.3 to see that $\lim d(x_{n-1}, x_n) = 0$. Let $\varrho(t) = t(1 - at^\sigma)$. Then $\varphi(t) \le \varrho(t)$ on $t \in (0, t_0]$. By Lemma 4 of [10], for any fixed $\tau \in (0, t_0]$,

$$\sum_{n=0}^{\infty} \varrho^n(\tau) < \infty.$$

(Here ϱ^n is the nth iterate of ϱ.) Let N be large enough so that $d(x_{n-1}, x_n) < \tau$ for $n \ge N$. It then follows from the monotonicity of ϱ and (1.8) that

$$d(x_n, x_{n+1}) \le \varrho^{n-N+1}(d(x_{N-1}, x_N)) \le \varrho^{n-N+1}(\tau), \quad n > N.$$

This shows that

$$\sum d(x_n, x_{n+1}) < \infty.$$

Therefore (x_n) is convergent. The limit of (x_n) is obviously a fixed point of T. ∎

1.3. Weakly inward contractions. Let C be a nonempty closed subset of a Banach space X and $T : C \to 2^X \setminus \{\emptyset\}$ be a multivalued non-self-mapping with closed values.

DEFINITION 1.3.1. T is said to be *weakly inward* on C if

$$Tx \subset \overline{I_C(x)}, \quad x \in C, \tag{1.9}$$

where

$$I_C(x) := x + \{\lambda(y-x) : \lambda \geq 1,\ y \in C\}$$

is the inward set of C at x and $\overline{A}$ denotes the closure of a subset $A \subset X$.

REMARK. There is another kind of weak inwardness (see Deimling [11]): T is weakly inward on C if $Tx \subset x + T_C(x)$ for $x \in C$, where

$$T_C(x) := \left\{ y \in X : \liminf_{\lambda \to 0^+} \frac{d(x+\lambda y, C)}{\lambda} \right\} = 0.$$

If C is convex, the two concepts coincide; however, if C is not convex, they are different: $\overline{I_C(x)}$ is, in general, larger than $x + T_C(x)$.

THEOREM 1.3.2 (Deimling [11]). *Let C be a closed subset of a Banach space X and $T : C \to 2^X \setminus \{\emptyset\}$ be a contraction with closed values such that each $x \in C$ has a nearest point in Tx. Assume T is weakly inward in the sense that $Tx \subset x + T_C(x)$ for $x \in C$. Then T has a fixed point.*

Deimling [11] asked whether the weak inwardness in the above theorem can be weakened to the weak inwardness in the sense of Definition 1.3.1. Below is an affirmative answer. But first let us state Caristi's fixed point theorem.

LEMMA 1.3.3 (Caristi's fixed point theorem [6]). *Let (M,d) be a complete metric space and $f : M \to M$ be a mapping. If there exists a lower semicontinuous function $\varphi : M \to [0,\infty)$ such that*

$$d(x, f(x)) \leq \varphi(x) - \varphi(f(x)), \quad x \in M,$$

then f has a fixed point.

THEOREM 1.3.4. *Let C be a closed subset of a Banach space X and $T : C \to 2^X \setminus \{\emptyset\}$ be a contraction with closed values such that each $x \in C$ has a nearest point in Tx. Assume T is weakly inward in the sense that $Tx \subset \overline{I_C(x)}$ for $x \in C$. Then T has a fixed point.*

PROOF. First observe that the weak inwardness condition (1.9) that T satisfies is equivalent to the condition

$$Tx \subseteq x + \overline{\{\lambda(y-x) : \lambda > 1,\ y \in C\}} \quad \text{for all } x \in C. \tag{1.10}$$

Next choose $q \in (0,1)$ and $\varepsilon \in (0,1)$ such that

$$k < q < \frac{1-\varepsilon}{1+\varepsilon}.$$

Let

$$\varphi(x) = \frac{1}{q-k} d(x, Tx), \quad x \in C.$$

For $x \in C$, by assumption, we have some $f(x) \in Tx$ satisfying

$$\|x - f(x)\| = d(x, Tx).$$

It follows from (1.10) that there exist $\lambda_n > 1$ and $y_n \in C$ such that

$$\|f(x) - (x + \lambda_n(y_n - x))\| \to 0 \quad \text{as } n \to \infty.$$

Suppose T is fixed point free in C. Then we have an integer $N \geq 1$ large enough so that

$$\|f(x) - (x + \lambda_N(y_N - x))\| < \varepsilon d(x, Tx).$$

Let

$$z = \left(1 - \frac{1}{\lambda_N}\right)x + \frac{1}{\lambda_N}f(x)$$

and

$$g(x) := y_N.$$

This defines a mapping $g : C \to C$. We now claim that g satisfies

$$\|x - g(x)\| < \varphi(x) - \varphi(g(x)) \quad \text{for all } x \in C. \tag{1.11}$$

Lemma 1.3.3 then implies that g has a fixed point, which contradicts the strict inequality in (1.11) and finishes the proof.

So it remains to prove (1.11). As $f(x) \in T(x)$ we have

$$\begin{aligned} d(g(x), T(g(x))) &\leq \|g(x) - z\| + d(z, Tx) + H(Tx, T(g(x))) \\ &\leq \|g(x) - z\| + \|z - f(x)\| + k\|x - g(x)\| \\ &= \|y_N - z\| + \|z - f(x)\| + k\|x - g(x)\|. \end{aligned}$$

Since

$$\|f(x) - (x + \lambda_N(y_N - x))\| < \varepsilon d(x, Tx),$$

we have

$$\|y_N - z\| = \left\| y_N - \left(\frac{1}{\lambda_N}f(x) + \left(1 - \frac{1}{\lambda_N}\right)x\right)\right\| < \frac{\varepsilon}{\lambda_N}d(x, Tx).$$

Also,

$$\|z - f(x)\| = \left(1 - \frac{1}{\lambda_N}\right)\|x - f(x)\| = \left(1 - \frac{1}{\lambda_N}\right)d(x, Tx).$$

It follows that

$$\begin{aligned} \varphi(g(x)) &= \frac{1}{q-k}d(g(x), T(g(x))) \\ &\leq \varphi(x) + \frac{\varepsilon - 1}{\lambda_N(q-k)}d(x, Tx) + \frac{k}{q-k}\|x - g(x)\| \\ &= \varphi(x) - \frac{1-\varepsilon}{q-k}\|z - x\| + \frac{k}{q-k}\|x - g(x)\|. \end{aligned}$$

It remains to show

$$-\frac{1-\varepsilon}{q-k}\|z - x\| + \frac{k}{q-k}\|x - g(x)\| < -\|x - g(x)\|$$

or equivalently

$$\|x - g(x)\| < \frac{1-\varepsilon}{q}\|z - x\| = \frac{1-\varepsilon}{\lambda_N q}d(x, Tx).$$

In fact, by the choice of the integer N, we see that

$$\begin{aligned}\varepsilon d(x,Tx) &> \|(f(x)-x)-\lambda_N(y_N-x)\| \\ &\geq \|\lambda_N(y_N-x)\| - \|f(x)-x\| = \lambda_N\|y_N-x\| - d(x,Tx).\end{aligned}$$

Thus by the choice of q we have

$$\|g(x)-x\| = \|y_N-x\| < \frac{1+\varepsilon}{\lambda_N}d(x,Tx) < \frac{1-\varepsilon}{q\lambda_N}d(x,Tx),$$

as required. ∎

The above argument can be used to prove a fixed point theorem for a multivalued mapping which takes noncompact values. The result below is a slight improvement of Theorem 4 of Mizoguchi and Takahashi [37].

THEOREM 1.3.5. *Let E be a closed subset of a Banach space X and $T : E \to 2^X \setminus \{\emptyset\}$ a contraction with closed values. If, for any $x \in E$,*

$$\liminf_{\lambda\to 0^+} \frac{d((1-\lambda)x+\lambda z, E)}{\lambda} = 0 \quad \textit{uniformly in } z \in Tx, \tag{1.12}$$

then T has a fixed point.

PROOF. Take $\varepsilon \in (0,1)$ small enough so that $(1-\varepsilon)/(1+\varepsilon) > k$ and let

$$\varphi(x) = \left(\frac{1-\varepsilon}{1+\varepsilon} - k\right)^{-1} d(x,Tx), \quad x \in E.$$

Assume that T does not have a fixed point in E. Then $\mathrm{dist}(x,Tx) > 0$ for every $x \in E$. Hence from the assumption (1.12) we have a $\lambda = \lambda(x) \in (0,\varepsilon)$ such that

$$d((1-\lambda)x+\lambda z, E) < \tfrac{1}{2}\lambda\varepsilon d(x,Tx) \quad \text{for all } z \in Tx.$$

Now choose $z \in Tx$ such that

$$\|x-z\| < \frac{1}{1-\frac{1}{2}\varepsilon\lambda} d(x,Tx). \tag{1.13}$$

Next we take $y \in E$ such that

$$\|(1-\lambda)x+\lambda z - y\| < \tfrac{1}{2}\lambda\varepsilon d(x,Tx). \tag{1.14}$$

Define a mapping $f : E \to E$ by

$$f(x) := y.$$

It is easily seen that f is fixed point free on E; in fact, if $f(x) = x$ for some $x \in E$, then (1.14) implies that $d(x,Tx) \leq \|x-z\| < \frac{1}{2}\varepsilon d(x,Tx)$. Hence $\varepsilon > 2$, contradicting the assumption $\varepsilon \in (0,1)$.

We now show that

$$\|x-f(x)\| \leq \varphi(x) - \varphi(f(x)) \quad \text{for all } x \in E. \tag{1.15}$$

Indeed, setting $u = (1-\lambda)x+\lambda z$, we have

$$d(y,Ty) \leq \|y-u\| + d(u,Tx) + H(Tx,Ty).$$

As T is a contraction, we get

$$d(y,Ty) \leq \|y-u\| + d(u,Tx) + k\|x-y\|. \tag{1.16}$$

Since by (1.14),

$$\|u-y\| < \tfrac{1}{2}\lambda\varepsilon d(x,Tx) \le \tfrac{1}{2}\lambda\varepsilon\|x-z\| = \tfrac{1}{2}\varepsilon\|u-x\|$$

and

$$d(u,Tx) \le \|u-z\| = (1-\lambda)\|x-z\| = \|x-z\| - \lambda\|x-z\| = \|x-z\| - \|u-x\|$$

so that

$$\|x-y\| \le \|x-u\| + \|u-y\| \le \left(1+\tfrac{1}{2}\varepsilon\right)\|u-x\| < (1+\varepsilon)\|u-x\|,$$

it follows from (1.16) and (1.13) that

$$\begin{aligned} d(y,Ty) &\le \frac{1}{2}\varepsilon\|u-x\| + \|x-z\| - \|u-x\| + k\|x-y\| \\ &= -(1-\varepsilon)\|u-x\| + k\|x-y\| + \|x-z\| - \frac{1}{2}\varepsilon\|u-x\| \\ &\le -\left(\frac{1-\varepsilon}{1+\varepsilon} - k\right)\|x-y\| + \left(1 - \frac{1}{2}\lambda\varepsilon\right)\|x-z\| \\ &\le -\left(\frac{1-\varepsilon}{1+\varepsilon} - k\right)\|x-y\| + d(x,Tx), \end{aligned}$$

which implies (1.15). It then follows from Lemma 1.3.3 that f has a fixed point in E. This contradiction concludes the proof. ∎

Very recently T. C. Lim [33] removed the assumption that each x has a nearest point in Tx, thus answering another question of Deimling [11].

THEOREM 1.3.6 ([33]). *Let C be a closed subset of a Banach space X and $T : C \to 2^X\backslash\{\emptyset\}$ be a contraction taking closed values. If T is weakly inward in the sense that $Tx \subset \overline{I_C(x)}$ for $x \in C$, then T has a fixed point.*

PROOF. Let $k \in [0,1)$ be the contraction constant of T. Pick l, $k < l < 1$, and $\varepsilon \in (0,1)$ so that $b := (1-\varepsilon)/(1+\varepsilon) - l > 0$. Assume on the contrary that T does not have fixed points. Take $z_0 \in C$ and $y_0 \in Tz_0$ arbitrarily. Let Ω be the first uncountable ordinal and γ an ordinal $< \Omega$. Suppose z_α, y_α have been defined for all $\alpha < \gamma$ such that

(i) $y_\alpha \in Tz_\alpha$ for $\alpha < \gamma$,
(ii) $z_\alpha \ne z_{\alpha+1}$ for $\alpha < \alpha+1 < \gamma$,
(iii) $b\max\{\|z_\beta - z_\alpha\|, (1/l)\|y_\beta - y_\alpha\|\} \le \|y_\alpha - z_\alpha\| - \|y_\beta - z_\beta\|$ for $\alpha, \beta < \gamma$.

We next define z_γ, y_γ so that (i)–(iii) remain valid for all $\alpha, \beta < \gamma+1$. We shall distinguish two cases.

CASE 1: γ has a predecessor $\gamma - 1$. In this case, since $y_{\gamma-1} \in Tz_{\gamma-1}$ and T is fixed point free, we see that $\|y_{\gamma-1} - z_{\gamma-1}\| > 0$. By the weak inwardness of T we have $z_\gamma \in C$ and $\lambda \ge 1$ such that

$$\|y_{\gamma-1} - (z_{\gamma-1} + \lambda_\gamma(z_\gamma - z_{\gamma-1}))\| \le \varepsilon\|y_{\gamma-1} - z_{\gamma-1}\|.$$

This clearly implies that $z_{\gamma-1} \ne z_\gamma$ and

$$\|z_\gamma - z_{\gamma-1}\| \le (1+\varepsilon)\mu_\gamma\|y_{\gamma-1} - z_{\gamma-1}\|, \quad \|z_\gamma - x_\gamma\| \le \varepsilon\mu_\gamma\|y_{\gamma-1} - z_{\gamma-1}\|,$$

where $\mu_\gamma = 1/\lambda_\gamma$ and $x_\gamma = \mu_\gamma y_{\gamma-1} + (1-\mu_\gamma)z_{\gamma-1}$. Since $H(Tz_\gamma, Tz_{\gamma-1}) \le k\|z_\gamma - z_{\gamma-1}\|$, there is some $y_\gamma \in Tz_\gamma$ such that

$$\|y_\gamma - y_{\gamma-1}\| \le l\|z_\gamma - z_{\gamma-1}\|. \tag{1.17}$$

Thus

$$\begin{aligned}\|y_\gamma - z_\gamma\| &\le \|y_\gamma - y_{\gamma-1}\| + \|y_{\gamma-1} - x_\gamma\| + \|x_\gamma - z_\gamma\| \\ &\le l\|z_\gamma - z_{\gamma-1}\| + (1-\mu_\gamma)\|y_{\gamma-1} - z_{\gamma-1}\| + \varepsilon\mu_\gamma\|y_{\gamma-1} - z_{\gamma-1}\| \\ &\le l\|z_\gamma - z_{\gamma-1}\| + \|y_{\gamma-1} - z_{\gamma-1}\| - \frac{1-\varepsilon}{1+\varepsilon}\|z_\gamma - z_{\gamma-1}\|.\end{aligned}$$

It then follows that

$$b\|z_\gamma - z_{\gamma-1}\| \le \|y_{\gamma-1} - z_{\gamma-1}\| - \|y_\gamma - z_\gamma\|$$

and, from (1.17),

$$\frac{b}{l}\|y_\gamma - y_{\gamma-1}\| \le \|y_{\gamma-1} - z_{\gamma-1}\| - \|y_\gamma - z_\gamma\|.$$

For any $\alpha < \gamma - 1$,

$$b\|z_\alpha - z_{\gamma-1}\| \le \|y_\alpha - z_\alpha\| - \|y_{\gamma-1} - z_{\gamma-1}\| \quad \text{by (iii)}.$$

So

$$b\|z_\gamma - z_\alpha\| \le b(\|z_\gamma - z_{\gamma-1}\| + \|z_{\gamma-1} - z_\alpha\|) \le \|y_\alpha - z_\alpha\| - \|y_\gamma - z_\gamma\|.$$

Similarly,

$$\frac{b}{l}\|y_\gamma - y_\alpha\| \le \|y_\alpha - z_\alpha\| - \|y_\gamma - z_\gamma\|.$$

So (i)–(iii) are valid for $\alpha, \beta < \gamma + 1$.

CASE 2: γ is a limit ordinal. We then have a strictly increasing sequence (γ_n) that converges to γ. Set $r_n = \|y_{\gamma_n} - z_{\gamma_n}\|$. Condition (iii) then implies that (z_{γ_n}) and (y_{γ_n}) are both Cauchy and hence convergent. Let z_γ and y_γ be their respective limits. Since $y_{\gamma_n} \in Tz_{\gamma_n}$, we have a $w_n \in Tz_\gamma$ such that $\|w_n - y_{\gamma_n}\| \le l\|z_{\gamma_n} - z_\gamma\|$. Thus $w_n - y_{\gamma_n} \to 0$. As $y_{\gamma_n} \to y_\gamma$, we get $w_n \to y_\gamma$ and thus $y_\gamma \in Tz_\gamma$ for Tz_γ is closed. Now for $\alpha < \gamma$, we have $\gamma_n > \alpha$ for sufficiently large n, so

$$b\|z_{\gamma_n} - z_\alpha\| \le \|y_\alpha - z_\alpha\| - \|y_{\gamma_n} - z_{\gamma_n}\|$$

and upon taking limits,

$$b\|z_\gamma - z_\alpha\| \le \|y_\alpha - z_\alpha\| - \|y_\gamma - z_\gamma\|.$$

Similarly,

$$\frac{b}{l}\|y_\gamma - y_\alpha\| \le \|y_\alpha - z_\alpha\| - \|y_\gamma - z_\gamma\|.$$

Therefore, (i)–(iii) remain valid for all $\alpha, \beta < \gamma + 1$. If $\alpha < \alpha + 1 < \gamma + 1$, then $\alpha < \gamma$. Since γ is a limit ordinal, $\alpha + 1 < \gamma$. So (ii) is also valid for $\alpha < \alpha + 1 < \gamma + 1$.

By transfinite induction, z_α, y_α for $\alpha < \Omega$ satisfying (i)–(iii) have been defined. Let $s_\alpha = \|y_\alpha - z_\alpha\|$. Since $(s_\alpha)_{\alpha<\Omega}$ is decreasing and bounded below by 0, it must eventually be constant. If $\gamma < \Omega$ is such that $s_\alpha = s_\beta$ for all $\alpha, \beta \ge \gamma$, then by (iii), $z_{\gamma+1} = z_\gamma$, contradicting (ii). Therefore, T must have a fixed point. ∎

Remarks. (i) In Theorems 1.3.4–1.3.6, the contraction T is not assumed to take compact values. The existence of fixed points of a compact-valued contraction T has been proved already (cf. Reich [46, Theorem 3.4]).

(ii) Theorem 1.3.6 actually contains Theorems 1.3.4 and 1.3.5 as special cases. We included the latter theorems to emphasize the role of Caristi's theorem in the fixed point theory for non-self-mappings. It is yet unclear whether Theorem 1.3.6 can be proved without using transfinite induction. A proof which uses Caristi's theorem would be of interest.

1.4. Local contractions. Let $\varepsilon > 0$ be given. Recall that a metric space (X, d) is *ε-chainable* if, for every pair $x, y \in X$, there exists a finite set of elements $x = x_0, x_1, \ldots, x_n = y$ in X such that $d(x_{i-1}, x_i) < \varepsilon$, $i = 1, \ldots, n$. Such a finite set is called an ε-chain linking x and y. The first result of this section shows that the ε-chainability of (X, d) implies the ε-chainability of $(K(X), H)$, where $K(X)$ is the collection of nonempty compact subsets of X and H is the Hausdorff distance on $K(X)$.

Theorem 1.4.1. *If (X, d) is an ε-chainable metric space, then so is $(K(X), H)$.*

Proof. Fix a $y \in X$ and let $Y = \{y\} \in K(X)$. Since ε-chainability is transitive, it suffices to show that each $A \in K(X)$ is ε-chainable in $K(X)$ to Y; i.e., there exists an ε-chain in $K(X)$ linking A and Y. We first show that this is true for a finite A. Towards this, we make induction on n, the number of elements that A contains. If $n = 1$, A is a singleton and the conclusion follows from the ε-chainability of X. Suppose now that the conclusion is valid for all finite subsets A of X consisting of not more than n elements. Let next A be a subset of X consisting of $n+1$ elements, say, $A = \{x_1, \ldots, x_{n+1}\}$. Since X is ε-chainable, there exists an ε-chain $x_1 = u_0, \ldots, u_m = x_2$ in X linking x_1 and x_2. It is easily seen that the finite set

$$A,\ \{u_1, x_2, \ldots, x_{n+1}\},\ \ldots,\ \{u_{m-1}, x_2, \ldots, x_{n+1}\},\ \{x_2, \ldots, x_{n+1}\}$$

forms an ε-chain in $K(X)$ linking A and $B := \{x_2, \ldots, x_{n+1}\}$. But by the induction assumption, B is ε-chainable in $K(X)$ to Y, and it follows that A is ε-chainable in $K(X)$ to Y. Now for a general compact A, we can find a finite family of subsets $\{A_k\}_{k=1}^n$ of A such that $A = \bigcup_{k=1}^n A_k$ and each A_k has diameter $< \varepsilon$. Pick for each k any $x_k \in A_k$ and put $C = \{x_1, \ldots, x_n\}$. It is then not hard to see that for each $z \in A$, $d(z, C) \le \operatorname{diam}(A_k)$ for some k, $1 \le k \le n$. It thus follows that

$$H(A, C) = \max\{\sup_{z \in A} d(z, C), \sup_{y \in C} d(y, A)\} = \sup_{z \in A} d(z, C) \le \max_{1 \le k \le n} \operatorname{diam}(A_k) < \varepsilon,$$

which shows that A is ε-chainable in $K(X)$ to C. However, we have shown that C is ε-chainable in $K(X)$ to Y. Hence A is ε-chainable in $K(X)$ to Y. ∎

Definition 1.4.2. Let (X, d) be an ε-chainable metric space. A mapping $T : X \to K(X)$ is said to be a *local contraction of Reich's type* if

$$H(Tx, Ty) \le k(d(x, y))d(x, y) \quad \text{for } x, y \in X \text{ such that } 0 < d(x, y) < \varepsilon,$$

where $k : (0, \infty) \to [0, 1)$ has property $(*)$ of Section 1.2.

Theorem 1.4.3 (Xu [54]). *Let (X,d) be a complete ε-chainable metric space and $T : X \to K(X)$ be a local contraction of Reich's type. Then T has a fixed point.*

Remark. Theorem 1.4.3, a special case of the next result, provides an affirmative answer to a question of Reich [41, p. 572]. It also improves upon Reich's theorem [41, p. 571].

Theorem 1.4.4 (Xu [54]). *Let (X,d) be a complete ε-chainable metric space and $T : X \to K(X)$ satisfy the following condition: for any η, $0 < \eta < \varepsilon$, there exists a $\delta > 0$ such that*

$$H(Tx,Ty) < \eta \quad \textit{whenever } \eta \le d(x,y) < \eta + \delta.$$

Then T has a fixed point.

Proof. Let $G : K(X) \to K(X)$ be defined by

$$G(A) := \bigcup_{a\in A} T(a), \quad A \in K(X).$$

Then it is not hard to see (cf. Reich [41]) that G has the following property: for $0 < \eta < \varepsilon$, there is $\delta > 0$ such that

$$H(G(A),G(B)) < \eta \quad \text{whenever } \eta \le H(A,B) < \eta + \delta.$$

Hence G has a fixed point $A \in K(X)$ by Proposition 1.4.5 below. Now since $A = G(A)$, T maps A into itself. It is easily seen that the contractive condition of T implies that $\inf\{d(x,Tx) : x \in A\} = 0$. Hence the compactness of A yields a point $x \in A$ for which $d(x,Tx) = 0$. Thus $x \in Tx$. ∎

Proposition 1.4.5. *Let (X,d) be a complete ε-chainable metric space and $F : X \to X$ satisfy the condition: for each η, $0 < \eta < \varepsilon$, there is a $\delta > 0$ such that*

$$x,y \in X,\ \eta \le d(x,y) < \eta + \delta \ \Rightarrow\ d(Fx,Fy) < \eta. \tag{4.1}$$

Then F has a fixed point.

Proof. We first show that

$$\lim_{n\to\infty} d(F^n x, F^n y) = 0 \quad \text{for all } x,y \in X \text{ with } d(x,y) < \eta. \tag{4.2}$$

In fact, since $d(x,y) < \eta$, it follows from (4.1) that the sequence $\{d(F^n x, F^n y)\}$ is nonincreasing and hence convergent. Let r be the limit. Suppose $r > 0$. Noting that $r \le d(x,y) < \varepsilon$, we have some $\delta > 0$ such that

$$u,v \in X,\ r \le d(u,v) < r + \delta \ \Rightarrow\ d(Fu,Fv) < r. \tag{4.3}$$

For this δ, we can take an integer N large enough so that $d(F^n x, F^n y) < r + \delta$ for $n \ge N$. Hence $d(F^n x, F^n y) < r$ for $n > N$ by (4.3). This contradicts the fact that $d(F^n x, F^y) \ge r$ for all $n \ge 0$. Hence (4.2) is proved.

Next we take an arbitrary $z \in X$ and let $z = z_0, z_1, \dots, z_m = Fz$ be an ε-chain in X linking z and Fz. Since $d(F^n z, F^{n+1} z) \le \sum_{k=0}^{m-1} d(F^n z_k, F^n z_{k+1})$ and $d(z_k, z_{k+1}) < \varepsilon$ for $0 \le k \le m-1$, it follows from (4.2) that $\lim_{n\to\infty} d(F^n z, F^{n+1} z) = 0$. We now show that $\{F^n z\}$ is a Cauchy sequence. Suppose on the contrary that $\{F^n z\}$ is not a Cauchy sequence; then we can find two subsequences $\{n_i\}$ and $\{m_i\}$ of positive integers such that the sequence $\{(F^{n_i}z, F^{m_i}z)\}$ decreases to some $t > 0$. First choose $0 < a < 1$ such that $at < \varepsilon$ and then $0 < \delta < \frac{3}{4}at$ such that

$$x, y \in X,\ \tfrac{1}{2}at \le d(x,y) < \tfrac{1}{2}at + \delta \ \Rightarrow\ d(Fx, Fy) < \tfrac{1}{2}at. \tag{4.4}$$

Let N be large enough so that $d(F^n z, F^{n+1}z) < \frac{1}{3}\delta$ for all $n \ge N$. Assume that i is so large that $n_i > N$ and consider the finite sequence

$$d(F^{n_i}z, F^{n_i+1}z), d(F^{n_i}z, F^{n_i+2}z), \dots, d(F^{n_i}z, F_i^m z).$$

Since the first term is less than $\frac{1}{3}\delta$, the last term is larger than t, and any two adjacent terms differ by not more than $\frac{1}{3}\delta$, it follows that there exists some p_i, $n_i < p_i < m_i$, such that

$$\tfrac{1}{2}at + \tfrac{2}{3}\delta \le d(F^{n_i}z, F^{p_i}z) < \tfrac{1}{2}at + \delta$$

and hence by (4.4) we have

$$d(F^{n_i}z, F^{p_i}z) < \tfrac{1}{2}at. \tag{4.5}$$

But by the triangle inequality we get

$$d(F^{n_i+1}z, F^{p_i+1}z) \ge d(F^{n_i}z, F^{p_i}z) - d(F^{n_i}z, F^{n_i+1}z) - d(F^{p_i}z, F^{p_i+1}z) \ge \tfrac{1}{2}at,$$

contradicting (4.5). This shows that $\{F^n z\}$ is a Cauchy sequence. Set $w = \lim F^n z$. Then it is easily seen that w is a fixed point of F. ∎

REMARK. Proposition 1.4.5 is the local version of a theorem due to Meir and Keeler [36].

2. Multivalued nonexpansive mappings

This chapter is devoted to the fixed point theory for multivalued nonexpansive non-self-mappings in Banach spaces. In Section 2.1 we recall some facts on asymptotic centers and universal nets. A simpler and elementary proof to the Kirk–Massa fixed point theorem is included in Section 2.2. Section 2.3 contains some fixed point theorems for multivalued nonexpansive non-self-mappings. In particular, the Kirk–Massa theorem is extended to a non-self-inward-mapping and, using an inequality technique in a uniformly convex Banach space, another proof to Lim's theorem is given. A counterexample to a question of Downing and Kirk is also included.

2.1. Asymptotic centers. Let K be a weakly compact convex subset of a Banach space X and $\{x_n\}$ a bounded sequence in X. Define a function f on X by

$$f(x) := \limsup_{n\to\infty} \|x_n - x\|, \quad x \in X.$$

Let

$$r \equiv r_K(x_n) := \inf\{f(x) : x \in K\}$$

and

$$A \equiv A_K(x_n) := \{x \in K : f(x) = r\}.$$

Definition 2.1.1. We say that r and A are the *asymptotic radius* and *center* of $\{x_n\}$ relative to K, respectively. As K is weakly compact convex, $A_K(\{x_n\})$ is nonempty, weakly compact and convex.

Definition 2.1.2. Let $\{x_n\}$ and K be as above. Then $\{x_n\}$ is called *regular* with respect to K if $r_K(\{x_n\}) = r_K(\{x_{n_i}\})$ for all subsequences $\{x_{n_i}\}$ of $\{x_n\}$; further, $\{x_n\}$ is called *asymptotically uniform* if $A_K(\{x_n\}) = A_K(\{x_{n_i}\})$ for all subsequences $\{x_{n_i}\}$ of $\{x_n\}$.

The method of asymptotic centers plays an important role in the fixed point theory of both single-valued and multivalued nonexpansive mappings, due to the fundamental lemma below. (Some more information on uniform convexity and asymptotic centers can be found in Goebel and Reich [18].)

Lemma 2.1.3 (Goebel [16], Lim [31]). *Let $\{x_n\}$ and K be as above. Then*

(i) *there always exists a subsequence of $\{x_n\}$ which is regular with respect to K;*

(ii) *if K is separable, then $\{x_n\}$ contains a subsequence which is asymptotically uniform with respect to K.*

Remark. If X is uniformly convex in every direction (especially uniformly convex), then $A_K(\{x_n\})$ consists of exactly one point so every regular sequence in such a space is asymptotically uniform with respect to K.

We shall mainly work in the framework of a uniformly convex Banach space. Thus an inequality characteristic of uniform convexity is useful to us. Recall that a Banach space X is *uniformly convex* if

$$\delta_X(\varepsilon) := \inf\left\{1 - \frac{\|x+y\|}{2} : x, y \in B_X,\ \|x-y\| \geq \varepsilon\right\} > 0 \quad \text{for all } \varepsilon \in (0,2].$$

Proposition 2.1.4 (Xu [56]). *Let X be a Banach space and $r > 0$ be a given number. Then X is uniformly convex if and only if the norm $\|\cdot\|$ of X is uniformly convex on the closed ball $B_r := \{x \in X : \|x\| \leq r\}$. That is, there exists a continuous strictly increasing function $g : [0,\infty) \to [0,\infty)$, depending on r, with $g(t) = 0$ if and only if $t = 0$, such that*

$$\|tx + (1-t)y\| \leq t\|x\| + (1-t)\|y\| - t(1-t)g(\|x-y\|), \quad x, y \in B_r,\ t \in [0,1].$$

Proposition 2.1.5. *Let X be a uniformly convex Banach space and E a closed convex subset of X. Assume (x_n) is a bounded sequence in X and $z = A_E(x_n)$. Then $A_{\overline{I_E(z)}}(x_n) = z$ and $r_E(x_n) = r_{\overline{I_E(z)}}(x_n)$.*

Proof. Write $f(x) := \limsup_{n\to\infty} \|x_n - x\|$, $r_1 := \inf\{f(x) : x \in E\} = r_E(x_n)$ and $r_2 := \inf\{f(x) : x \in \overline{I_E(z)}\} = r_{\overline{I_E(z)}}(x_n)$. It suffices to show that $r_1 = r_2$. It is obvious that $r_2 \leq r_1$. So it remains to show that $r_1 \leq r_2$. To this end we apply Proposition 2.1.4 to get, for any $r > 0$, a continuous strictly increasing function g (depending on r) such that

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) - t(1-t)g(\|x-y\|), \quad x, y \in B_r,\ t \in [0,1]. \tag{2.1}$$

It is clear that $r_2 = \inf\{f(x) : x \in I_E(z)\}$; hence we have a sequence $w_n := z+\lambda_n(z_n - z)$, where $\lambda_n \geq 0$ and $z_n \in E$, such that $f(w_n) \to r_2$. If $\lambda_n \leq 1$ for infinitely many n, then $w_n \in E$ and thus $f(w_n) \geq r_1$ for these n. Upon taking the limit, we get $r_2 \geq r_1$ and we are done. So assume $\lambda_n > 1$ for all n. If (λ_n) has a bounded subsequence, still denoted by (λ_n), we may assume $\lambda_n \to \lambda \geq 1$. Since (z_n) is bounded, we may also assume $z_n \rightharpoonup \widetilde{z} \in E$. (Here "$\rightharpoonup$" denotes weak convergence) Thus, taking limits, we get $w_n \rightharpoonup \widetilde{w} := z+\lambda(\widetilde{z}-z)$. In other words, we can write

$$\widetilde{z} = \frac{1}{\lambda}\widetilde{w} + \left(1 - \frac{1}{\lambda}\right)z.$$

Since f is weakly lower semicontinuous, we get

$$f(\widetilde{w}) \leq \liminf f(w_n) = r_2.$$

On the other hand, by convexity of f we get

$$f(\widetilde{z}) \leq \frac{1}{\lambda}f(\widetilde{w}) + \left(1 - \frac{1}{\lambda}\right)f(z).$$

Since $f(\widetilde{z}) \geq r_1$, $f(z) = r_1$ and $f(\widetilde{w}) \leq r_2$, it follows from the last displayed inequality that $r_1 \leq \frac{1}{\lambda}r_2 + \left(1 - \frac{1}{\lambda}\right)r_1$ and hence $r_2 \geq r_1$. Finally, we assume $\lambda_n \to \infty$. In this case we can write

$$z_n = \frac{1}{\lambda_n}w_n + \left(1 - \frac{1}{\lambda_n}\right)z.$$

Let now r be large enough so that $B_r \supset \{w_n\} \cup \{z\}$. Use the inequality (2.1) to get

$$f(z_n) \leq \frac{1}{\lambda_n}f(w_n) + \left(1 - \frac{1}{\lambda_n}\right)f(z) - \frac{1}{\lambda_n}\left(1 - \frac{1}{\lambda_n}\right)g(\|w_n - z\|).$$

Since $f(z_n) \geq r_1$ and $f(z) = r_1$, from the last inequality we get

$$\left(1 - \frac{1}{\lambda_n}\right)g(\|w_n - z\|) \leq f(w_n) - r_1.$$

Let $n \to \infty$ to get

$$\limsup g(\|w_n - z\|) \leq r_2 - r_1 \leq 0.$$

Hence $w_n \to z$ and $r_2 = \lim f(w_n) = f(z) = r_1$. ∎

If T is a non-self-mapping from E to $K(X)$, we may not be able to assume the separability of E and hence Lemma 2.1.3(ii) is not applicable. We therefore need the tool of a universal net.

DEFINITION 2.1.6. A net $\{x_\alpha\}$ in a set S is called a *universal net* if for each subset U of S, either $\{x_\alpha\}$ is eventually in U or $\{x_\alpha\}$ is eventually in $S \setminus U$.

The following facts are relevant ([25, p. 81]):

(a) Every net in a set has a universal subnet.

(b) If $f : S_1 \to S_2$ is a map and if $\{x_\alpha\}$ is a universal net in S_1, then $\{f(x_\alpha)\}$ is a universal net in S_2.

(c) If S is compact and if $\{x_\alpha\}$ is a universal net in S, then $\lim_\alpha x_\alpha$ exists.

2.2. The Kirk–Massa theorem. Let E be a weakly compact convex subset of a Banach space X and $T : E \to K(E)$ a nonexpansive self-mapping. For a fixed element $x_0 \in E$ and an arbitrary integer $n \geq 1$, the contraction $T_n : E \to K(E)$ defined by

$$T_n(x) := \frac{1}{n}x_0 + \left(1 - \frac{1}{n}\right)Tx, \quad x \in E,$$

has a fixed point $x_n \in E$. It is easily seen that

$$d(x_n, Tx_n) \leq \frac{1}{n}\operatorname{diam}(E) \to 0.$$

Let r and A be the asymptotic radius and center of $\{x_n\}$ with respect to E, respectively. Since T is compact-valued, we can take $y_n \in Tx_n$ such that

$$\|y_n - x_n\| = d(x_n, Tx_n), \quad n \geq 1.$$

Since T is a self-mapping, we may assume that E is separable (otherwise, we can construct a closed convex subset of E that is invariant under T, see [28]). Then by Lemma 2.1.3 we may assume that $\{x_n\}$ is asymptotically uniform. Take any $z \in A$; as Tz is compact, we can find $z_n \in Tz$ satisfying

$$\|y_n - z_n\| = d(y_n, Tz) \leq H(Tx_n, Tz) \leq \|x_n - z\|.$$

Because of the compactness of Tz, we may also assume that $\{z_n\}$ (strongly) converges to a point $\widetilde{z} \in Tz$. It then follows that

$$\limsup \|x_n - \widetilde{z}\| = \limsup \|y_n - z_n\| \leq \limsup \|x_n - z\|.$$

This shows that $\widetilde{z} \in A$. Hence we can define a self-map $\widetilde{T} : A \to K(A)$ by setting

$$\widetilde{T}z := A \cap Tz, \quad z \in A.$$

This map $\widetilde{T}$ is in general neither nonexpansive nor lower semicontinuous. However, it is upper semicontinuous, which is observed by Kirk and Massa in [27]. With this observation they are able to prove Theorem 2.2.1 below by using the Bohnenblust–Karlin fixed point theorem (cf. [61]) that is of topological rather than metric nature. We shall now give an elementary proof in the sense that only the multivalued contraction principle (i.e., Nadler's theorem) is involved.

THEOREM 2.2.1 (Kirk–Massa [27]). *Let E be a nonempty closed bounded convex subset of a Banach space X and $T : E \to \mathrm{KC}(E)$ a nonexpansive mapping. Suppose that the asymptotic center in E of each bounded sequence of X is nonempty and compact. Then T has a fixed point.*

PROOF. Since T is a self-mapping, we may assume that E is separable. As before, we have an asymptotically uniform sequence (x_n) in E such that $\lim d(x_n, Tx_n) = 0$. We also have a sequence (y_n) satisfying $y_n \in Tx_n$ and $\|x_n - y_n\| = d(x_n, Tx_n)$ for all n. Set $A = A_E(x_n)$ and $r = r_E(x_n)$. We have already shown that

$$Tx \cap A \neq \emptyset \quad \text{for all } x \in A. \tag{2.2}$$

Our idea here is that we do not consider the self-mapping $\widetilde{T}x := Tx \cap A$ of A since $\widetilde{T}$ loses nonexpansivity. Instead, we view T as a non-self-mapping from A to $\mathrm{KC}(E)$. The advantage of this idea lies in that the nonexpansivity of T is kept and, moreover, a kind of (boundary) condition (e.g. (2.2)) is satisfied.

Now for each integer $m \geq 1$, define a contraction $S_m : A \to \mathrm{KC}(E)$ by

$$S_m(x) := \frac{1}{m}x_0 + \left(1 - \frac{1}{m}\right)Tx, \quad x \in A, \tag{2.3}$$

where $x_0 \in A$ is fixed. Then each S_m is a contraction which satisfies the (boundary) condition

$$S_m(x) \cap A \neq \emptyset \quad \text{for all } x \in A.$$

As S_m is compact and convex-valued, by Lemma 2.3.2 of the next section S_m has a fixed point $v_m \in A$. The sequence (v_m) satisfies

$$\lim_{m\to\infty} d(v_m, Tv_m) = 0. \tag{2.4}$$

Since A is compact, (v_m) has a convergent subsequence, whose limit is clearly a fixed point of T, due to (2.4). ■

2.3. Inwardness and weak inwardness. Let X be a Banach space and E a nonempty closed convex subset of X. Recall that the inward set of E at $x \in E$ is given by

$$I_E(x) = \{x + \lambda(y - x) : \lambda \geq 0,\ y \in E\}.$$

A multivalued mapping $T : E \to F(X)$ is *inward* (resp. *weakly inward*) if $Tx \subset I_E(x)$ (resp. $Tx \subset \overline{I_E(x)}$) for $x \in E$.

The first result of this section is an extension of the Kirk–Massa theorem to non-self-mappings.

THEOREM 2.3.1. *Let E be a nonempty closed bounded convex subset of a Banach space X and $T : E \to \mathrm{KC}(X)$ a nonexpansive non-self-mapping which satisfies the inwardness condition: $Tx \subset I_E(x)$ for $x \in E$. Suppose that the asymptotic center in E of each bounded sequence of X is nonempty and compact. Then T has a fixed point.*

PROOF. Fix $x_0 \in E$ and define for each integer $n \geq 1$ the contraction $T_n : E \to \mathrm{KC}(X)$ by

$$T_n(x) := \frac{1}{n}x_0 + \left(1 - \frac{1}{n}\right)Tx, \quad x \in E.$$

Then T_n satisfies the inwardness condition, i.e., $T_n x \subset I_E(x)$ for all $x \in E$. Thus by Theorem 1.3.4, T_n has a fixed point $x_n \in E$. By Lemma 2.1.3, we may assume that $\{x_n\}$ is regular. Let $y_n \in Tx_n$ be as constructed as before, i.e., $\|x_n - y_n\| = \mathrm{dist}(x_n, Tx_n)$. Let $\{x_{n_\alpha}\}$ be a universal subnet of $\{x_n\}$ and define a function g by

$$g(x) = \lim_{\alpha} \|x_{n_\alpha} - x\|, \quad x \in E.$$

Let

$$C := \{x \in E : g(x) = r\},$$

where $r = \inf_{x\in E} g(x)$. Then by assumption (see Proposition 6 of [27]), C is nonempty and compact. The key to the proof is that the inwardness of T on E implies that

$$Tx \cap I_C(x) \neq \emptyset, \quad x \in C. \tag{2.5}$$

Indeed, if $x \in C$, by compactness, we have, for each $n \geq 1$, some $z_n \in Tx$ such that

$$\|y_n - z_n\| = d(y_n, Tx) \leq H(Tx_n, Tx) \leq \|x_n - x\|.$$

Let $z = \lim_\alpha z_{n_\alpha} \in Tx$. It follows that

$$g(z) = \lim_\alpha \|x_{n_\alpha} - z\| = \lim_\alpha \|y_{n_\alpha} - z_{n_\alpha}\| \leq \lim_\alpha \|x_{n_\alpha} - x\|.$$

Hence

$$g(z) \leq g(x) = r. \tag{2.6}$$

It remains to show $z \in I_C(x)$. As $Tx \subset I_E(x)$, we have some $\lambda \geq 0$ and $v \in E$ such that

$$z = x + \lambda(v - x).$$

If $\lambda \leq 1$, then $z \in E$ by the convexity of E, and hence, by (2.6), $z \in C \subset I_C(x)$ and we are done. So assume $\lambda > 1$. Then we can write

$$v = \mu z + (1-\mu)x \quad \text{with } \mu = 1/\lambda \in (0,1).$$

By the convexity of g and by (2.6), we have

$$g(v) \leq \mu g(z) + (1-\mu) g(x) \leq r.$$

Since $v \in E$, it follows that $v \in C$ and thus $z = x + \lambda(v - x)$ belongs to $I_C(x)$. Now we have a nonexpansive mapping $T : C \to \mathrm{KC}(X)$ which satisfies the (boundary) condition (2.5). The lemma below shows that T has a fixed point in C. ∎

LEMMA 2.3.2. *If C is a compact convex subset of a Banach space X and $T : C \to \mathrm{KC}(X)$ is a nonexpansive mapping satisfying the* (*boundary*) *condition*

$$Tx \cap \overline{I_C(x)} \neq \emptyset \quad \textit{for all } x \in C.$$

Then T has a fixed point.

PROOF. Fix an $x_0 \in C$ and define for each integer $n \geq 1$ a mapping $T_n : C \to \mathrm{KC}(X)$ by

$$T_n(x) := \frac{1}{n} x_0 + \left(1 - \frac{1}{n}\right) Tx, \quad x \in C.$$

Then T_n is a contraction satisfying the same (boundary) condition as T does, i.e.,

$$T_n(x) \cap \overline{I_C(x)} \neq \emptyset \quad \text{for all } x \in C.$$

Hence by Theorem 11.5 of Deimling [11], T_n has a fixed point $x_n \in C$. Since C is compact, we may assume $x_n \to x \in C$. Also, it is easily seen that

$$d(x_n, Tx_n) \leq \frac{1}{n} \operatorname{diam} C \to 0 \quad \text{as } n \to \infty.$$

Taking the limit as $n \to \infty$ yields $d(x, Tx) = 0$ and hence $x \in Tx$. ∎

REMARKS. (i) Lemma 2.3.2 is a special case of Halpern [19, Theorem 2] and of Reich [43, Corollary 2.2]. Moreover, if T satisfies the stronger condition $Tx \cap I_C(x) \neq \emptyset$ for all $x \in C$, Lemma 2.3.2 follows from a fixed point theorem of F. E. Browder (cf. [11]). However, in our nonexpansive case, the proof is constructive.

(ii) Theorem 2.3.1 applies to Banach spaces which are uniformly convex or more general k-uniformly rotund Banach spaces ([50]) since the asymptotic center of a bounded sequence with respect to a bounded closed convex subset of such spaces is compact ([26]).

However, Theorem 2.3.1 does not apply to a nearly uniformly convex Banach space since in such a space, the asymptotic center of a bounded sequence with respect to a closed bounded convex subset is not necessarily compact (cf. [28]). (Recall that a Banach space X is said to be *nearly uniformly convex* (NUC) [21] if X is reflexive and if for any $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon) \in (0,1)$ such that

$$\{x_n\} \subset B_X,\ x_n \rightharpoonup x,\ \mathrm{sep}(x_n) \geq \varepsilon \ \Rightarrow\ \|x\| \leq 1-\delta,$$

where B_X is the closed unit ball of X and $\mathrm{sep}(x_n) := \inf\{\|x_n - x_m\| : n \neq m\}$.)

The following result answers a question of Deimling [11, p. 161] in the positive.

THEOREM 2.3.3. *Assume X is a uniformly convex Banach space, E is a closed bounded convex subset of X, and $T : E \to K(X)$ is a nonexpansive mapping satisfying the weak inwardness condition*:

$$Tx \subset \overline{I_E(x)}, \quad x \in E.$$

Then T has a fixed point.

PROOF. As before, we fix an $x_0 \in E$ and define, for each integer $n \geq 1$, the contraction $T_n : E \to K(X)$ by

$$T_n(x) = \frac{1}{n}x_0 + \left(1 - \frac{1}{n}\right)Tx, \quad x \in E.$$

As it is easily seen that T_n also satisfies the weak inwardness condition: $T_n(x) \subset \overline{I_E(x)}$ for all $x \in E$, we deduce by Theorem 1.3.4 that T_n has a fixed point, denoted by x_n. It is also easily seen that

$$d(x_n, Tx_n) \leq \frac{1}{n}\,\mathrm{diam}\,E \to 0 \quad (n \to \infty).$$

We may assume that $\{x_n\}$ is regular and hence asymptotically uniform as the space X is uniformly convex. Let z be the unique element of the asymptotic center of $\{x_n\}$ in E; that is, $z \in E$ is the unique minimizer in E of the function

$$f(x) := \limsup_{n\to\infty} \|x_n - x\|.$$

Let

$$r = f(z) = \inf_{x \in E} f(x).$$

Choose $z_n \in Tz$ satisfying

$$\|y_n - z_n\| = d(y_n, Tz).$$

From the nonexpansiveness of T it follows that

$$\|y_n - z_n\| \leq H(Tx_n, Tz) \leq \|x_n - z\|.$$

Since Tz is compact, we may assume that $\{z_n\}$ strongly converges to a point $\widetilde{z} \in Tz$. It follows that

$$f(\widetilde{z}) = \limsup \|x_n - \widetilde{z}\| = \limsup \|y_n - z_n\| \leq \limsup \|x_n - z\|;$$

that is,

$$f(\widetilde{z}) \leq f(z). \tag{2.7}$$

We shall show $\widetilde{z} = z$, which then finishes the proof. Towards this end we first note that $\widetilde{z} \in \overline{I_E(z)}$. Then by Proposition 2.1.5 and (2.7) we get $f(z) = f(\widetilde{z})$. This means that z and $\widetilde{z}$ are both minimizers of f over $\overline{I_E(z)}$ and hence $z = \widetilde{z}$ by uniqueness. This is the proof of Lim [32]. Below we give another proof which uses the inequality characteristic of uniform convexity, i.e., Proposition 2.1.4.

As $\widetilde{z} \in \overline{I_E(z)}$, we can find a sequence $\{\lambda_k\}$ of nonnegative numbers and a sequence $\{u_k\}$ of elements of E such that

$$w_k := z + \lambda_k(u_k - z) \to \widetilde{z}. \tag{2.8}$$

If $\lambda_k \le 1$ for infinitely many k, then $w_k \in E$ for these k and hence $\widetilde{z} \in E$. Therefore, (2.7) shows that $\widetilde{z} \in E$ is also a minimizer of f in E and hence $\widetilde{z} = z$ by uniqueness. Thus we may assume $\lambda_k > 1$ for all k. If $\{\lambda_k\}$ has a bounded subsequence, then as $\{u_k\}$ is bounded, we have, for some subsequence $\{k_i\}$ of positive integers, $\lambda_{k_i} \to \lambda$ and $u_{k_i} \rightharpoonup u \in E$. It follows that

$$\widetilde{z} = z + \lambda(u - z)$$

reducing the case to the inwardness case that has been treated in Downing–Kirk [13], Reich [44], and Deimling [11]. So we assume $\lambda_k \to \infty$. We then rewrite (2.8) as

$$u_k = \mu_k w_k + (1 - \mu_k)z, \quad \text{where } \mu_k = \frac{1}{\lambda_k} \to 0.$$

Now let r be a number large enough so that the closed ball B_r contains the sequences $\{x_n - w_k\}$ and $\{x_n - z\}$. Apply Proposition 2.1.4 to get

$$\begin{aligned} f(z) &\le f(u_k) \quad \text{as } u_k \in E \\ &= f(\mu_k w_k + (1 - \mu_k)z) \\ &= \limsup_{n\to\infty} \|\mu_k(x_n - w_k) + (1 - \mu_k)(x_n - z)\| \\ &\le \mu_k f(w_k) + (1 - \mu_k)f(z) - \mu_k(1 - \mu_k)g(\|w_k - z\|). \end{aligned}$$

Hence

$$(1 - \mu_k)g(\|w_k - z\|) \le f(w_k) - f(z).$$

Taking the limit as $k \to \infty$ yields

$$g(\|\widetilde{z} - z\|) \le f(\widetilde{z}) - f(z) \le 0 \quad \text{as } f(\widetilde{z}) \le f(z).$$

Since g is strictly increasing with $g(0) = 0$, we must have $\widetilde{z} = z$. ∎

We conclude this section with a counterexample to a question of Downing and Kirk [13] where they proved the following result.

Theorem D-K. *Let E be a nonempty closed convex subset of a Banach space X and let $T : E \to F(X)$ be an upper semicontinuous mapping satisfying the conditions*:

(a) *For $x \in E$, there exists $\delta = \delta(x) > 0$ such that, for $k \in (0, 1)$,*

$$y \in B_\delta(x) \cap E \ \Rightarrow\ d(y, Ty) \le d(y, Tx) + k\|x - y\|.$$

(b) *$T_1(x) \cap \overline{I_E(x)} \ne \emptyset$ for $x \in E$, where $T_1(x) = \{z \in Tx : \|x - z\| = \operatorname{dist}(x, Tx)\}$.*

Then T has a fixed point.

They asked the following

QUESTION. If the inwardness assumption in the condition (b) of Theorem D-K above is altered to $T(x) \cap I_E(x) \neq \emptyset$ for $x \in E$, does T have a fixed point?

The simple example below answers the question in the negative.

EXAMPLE. Let $X = \mathbb{R}$ and $E = [0,1]$. Define $T : E \to K(X)$ by

$$T(x) = \{-1, 2\} \quad \text{for all } x \in E.$$

Then T is a constant and $Tx \cap I_E(x) \neq \emptyset$ for all $x \in E$ as $I_E(x) = \mathbb{R}$ for $x \in (0,1)$, or $(-\infty, 1]$ for $x = 1$, or $[0, \infty)$ for $x = 0$.

2.4. Open problems. This section contains some open questions on fixed points of multivalued nonexpansive mappings in Banach spaces.

PROBLEM 1. Let X be a uniformly smooth Banach space, E a nonempty closed bounded convex subset of X, and $T : E \to K(E)$ a nonexpansive mapping. Does T have a fixed point?

PROBLEM 2. In Kirk–Massa's theorem (Theorem 2.2.1), T is assumed to take compact and convex values. Does T have a fixed point if T is only assumed to take compact values?

PROBLEM 3. Can Theorem 2.3.1 be extended to weakly inward nonexpansive mappings $T : E \to \mathrm{KC}(X)$?

PROBLEM 4. Characterize those Banach spaces X for which the asymptotic center of each bounded sequence in X with respect to every closed bounded convex subset of X is compact.

PROBLEM 5. Let X be a uniformly convex Banach space and E a closed bounded convex subset of X. Suppose $T : E \to \mathrm{KC}(X)$ is a nonexpansive mapping satisfying the condition $Tx \cap \overline{I_E(x)} \neq \emptyset$ for all $x \in E$ (or even $Tx \cap I_E(x) \neq \emptyset$ for all $x \in E$). Does T have a fixed point?

PROBLEM 6. Let X be a nearly uniformly convex (NUC) Banach space and E a closed bounded convex subset of X. Assume $T : E \to \mathrm{KC}(E)$ is nonexpansive. Does T have a fixed point?

REMARKS. (i) Problems 1 and 6 are indeed special cases of Problem 8 of Reich [48] which asked if every nonexpansive $T : E \to K(E)$ has a fixed point, where E, a weakly compact convex subset of a Banach space X, has the fixed point property for (single-valued) nonexpansive mappings.

(ii) Problem 5 is related to Theorem 5.4 of Reich [42] in which the Banach space X satisfies *Opial's condition*: if a sequence $(x_n) \subset X$ weakly converges to x, then $\liminf \|x_n - x\| < \liminf \|x_n - y\|$ for $y \in X \setminus \{x\}$.

3. Random multivalued mappings

The main purpose of this chapter is to investigate the measurability of the fixed point set function F_T of a random multivalued mapping T. The measurability of F_T is proved in Section 3.2 for a random multivalued contraction T and in Section 3.3 for a random multivalued nonexpansive T provided either $I-T$ is demiclosed at 0 or the underlying space X is uniformly smooth and T is single-valued. The existence of a random fixed point of a random multivalued nonexpansive mapping is also proved in a uniformly convex Banach space in Section 3.3.

3.1. Introduction and preliminaries. Let (Ω, Σ) be a a measurable space. This means that Ω is a nonempty set and Σ is a sigma-algebra of subsets of Ω. (Throughout this chapter we do not assume the existence of any probability on (Ω, Σ).) Let (X, d) be a metric space. A multivalued mapping $T : \Omega \to 2^X \setminus \{\emptyset\}$ is said to be (Σ)-*measurable* ("weakly measurable" in Himmelberg's terminology [20]) if, for any open set $B \subset X$, we have

$$T^{-1}(B) := \{\omega \in \Omega : T(\omega) \cap B \neq \emptyset\} \in \Sigma.$$

Note that if $T(\omega) \in K(X)$ for all $\omega \in \Omega$, then T is measurable if and only if $T^{-1}(F) \in \Sigma$ for all closed subsets $F \subset X$. A measurable (single-valued) function $x : \Omega \to X$ is called a *measurable selection* of a measurable multivalued mapping $T : \Omega \to 2^X \setminus \{\emptyset\}$ if, for each $\omega \in \Omega$, $x(\omega) \in T(\omega)$. Let M be a subset of X. An operator $T : \Omega \times M \to 2^X \setminus \{\emptyset\}$ is said to be a *random operator* if, for each $x \in M$, the mapping $T(\cdot, x) : \Omega \to 2^X \setminus \{\emptyset\}$ is measurable. A measurable function $x : \Omega \to M$ is called a *random fixed point* of a random operator T if $x(\omega) \in T(\omega, x(\omega)) \cap M$ for all $\omega \in \Omega$. The *fixed point set function* F_T of a random multivalued mapping $T : \Omega \times M \to 2^X \setminus \{\emptyset\}$ is given by

$$F(\omega) \equiv F_T(o) := \{x \in M : x \in T(\omega, x)\}, \quad \omega \in \Omega.$$

Recall that when X is a normed space and M is a convex subset of X, a multivalued mapping $T : M \to 2^X \setminus \{\emptyset\}$ is *convex* if its graph

$$\mathrm{Gr}(T) := \{(x, y) \in M \times X : y \in Tx\}$$

is a convex subset of $M \times X$. Recall also that we say a multivalued mapping $T : M \to 2^X \setminus \{\emptyset\}$ is *closed*- (*convex*-, *closed convex*-, etc.) *valued* if, for each $x \in M$, the image Tx has that particular property.

A multivalued mapping $T : M \to \mathrm{CB}(X)$ is said to be *demiclosed* at 0 if, for any sequences (x_n), (y_n) such that $y_n \in Tx_n$ for each n, the conditions $x_n \rightharpoonup x$ and $y_n \to 0$ imply that $x \in M$ and $0 \in Tx$.

The *Kuratowski measure of noncompactness* of a nonempty bounded subset E of X is defined as the number

$$\alpha(E) := \inf\{\varepsilon > 0 : E \text{ can be covered by a finite number of subsets of } X \text{ of diameter less than } \varepsilon\}.$$

A multivalued mapping $T : M \to 2^X \setminus \{\emptyset\}$ is called *condensing* if, for each bounded subset E of M with $\alpha(E) > 0$,

$$\alpha(T(E)) < \alpha(E).$$

Here $T(E) := \bigcup\{Tx : x \in E\}$.

Recall also that a set-valued operator $T : M \to 2^X \setminus \{\emptyset\}$ is said to be *upper semicontinous* on M if the set $\{x \in M : Tx \subset V\}$ is open whenever $V \subset X$ is open, while T is *lower semicontinuous* provided $T^{-1}(V) = \{x \in M : Tx \cap V \neq \emptyset\}$ is open in M for every open set V in X. Next, T is *continuous* if T is both upper and lower semicontinuous. Note that there is another kind of continuity for multivalued mappings $T : M \to \mathrm{CB}(X)$. Namely, T is said to be *Hausdorff continuous* if $H(Tx_n, Tx) \to 0$ whenever $x_n \to x$. It is known that these two continuity concepts coincide if T is compact-valued. But in general, Hausdorff continuity implies continuity, not vice versa. We say that a random operator $T : \Omega \times M \to 2^X \setminus \{\emptyset\}$ is *continuous* (*contraction*, *nonexpansive*, *condensing*, etc.) if, for each fixed $\omega \in \Omega$, the (deterministic) multivalued mapping $T(\omega, \cdot) : M \to 2^X \setminus \{\emptyset\}$ is continuous (contraction, nonexpansive, condensing, etc.) We need the following propositions, where X is a complete separable metric space.

Proposition 3.1.1 (Measurable Selection Theorem (cf. [1], [7], [51])). *If $T : \Omega \to 2^X \setminus \{\emptyset\}$ is a measurable closed-valued mapping, then T has a measurable selection* (*that is, there exists a measurable* (*single-valued*) *fucntion $x : \Omega \to X$ such that $x(\omega) \in T(\omega)$ for $\omega \in \Omega$*).

Proposition 3.1.2 (Castaing's characteristic theorem [7]). *If $f : \Omega \to 2^X \setminus \{\emptyset\}$ is a closed-valued mapping, then the following are equivalent*:

(a) *f is measurable.*
(b) *For each $x \in X$, the function $\omega \mapsto d(x, f(\omega))$ is measurable.*
(c) *There exists a sequence $\{f_n(\omega)\}$ of measurable selections of f such that*

$$\mathrm{cl}\{f_n(\omega)\} = f(\omega) \quad \textit{for all } \omega \in \Omega,$$

where $\mathrm{cl}\, A$ *denotes the closure of A in X.*

Since $|d(y, A) - d(y, B)| \leq H(A, B))$ for all $y \in X$ and $A, B \subset X$, we deduce the following result (see also Itoh [22]).

Proposition 3.1.3. *Assume $f(\omega)$ is a closed-valued mappping and $\{f_n(\omega)\}$ is a sequence of measurable mappings. If* $\lim H(f_n(\omega), f(\omega)) = 0$ *for all $\omega \in \Omega$, then f is measurable.*

For the next two propositions (cf. [2, 22]), M is a closed bounded convex separable subset of a Banach space X.

Proposition 3.1.4. *A mapping $f : \Omega \to 2^X \setminus \{\emptyset\}$ is measurable if and only if it is weakly measurable, i.e., for each $x^* \in X^*$, the numerically-valued mapping $x^* f : \Omega \to 2^{\mathbb{R}} \setminus \{\emptyset\}$ is measurable.*

Proposition 3.1.5. *Let $T : \Omega \times M \to \mathrm{CB}(X)$ be a random continuous* (*i.e., upper and lower semicontinuous*) *operator. Then for any $s > 0$, the operator $G : \Omega \to 2^M$ given by*

$$G(\omega) := \{x \in M : d(x, T(\omega, x)) < s\}, \quad \omega \in \Omega,$$

is measurable and so is the operator $\mathrm{cl}\{G(\omega)\}$, *where the closure is taken under either the strong or weak topology of the space X.*

In recent years a lot of efforts have been made (cf. [3], [12], [15], [22], [23], [34], [40], [45], [49], [53], [55], [56], [59] and references therein) to show the existence of random fixed points of certain single-valued and set-valued random operators. The main objective of this chapter is to study the measurability of the fixed point set function of a multivalued measurable mapping. More precisely, let $T : \Omega \times M \to 2^X \setminus \{\emptyset\}$ be a measurable multivalued mapping. Let $F(\omega) := \{x \in M : x \in T(\omega, x)\}$ be the fixed point set function of the mapping $T(\omega, \cdot) : M \to 2^X \setminus \{\emptyset\}$. We shall show in the next sections that $F : \Omega \to 2^M \setminus \{\emptyset\}$ will be measurable either if T is a random contraction or if T is a random nonexpansive mapping satisfying certain conditions.

If the fixed point set function F of T is measurable, then the existence of a random fixed point of T is an immediate consequence of the Measurable Selection Theorem (Proposition 3.1.1). (Obviously, we need some mild conditions (e.g. continuity) to ensure that the fixed point sets $F(\omega)$ of $T(\omega, \cdot)$ are closed.) From this point of view it is harder to show the measurability of the fixed point set function F of T than to show the existence of merely a random fixed point of T. Indeed, in Section 3.3 we shall prove the existence of a random fixed point of a random nonexpansive multivalued mapping T in a uniformly convex Banach space. However, we do not know if the fixed point set function F of T is measurable.

The technique used to show the existence of random fixed points seems to be unsuitable for verifying whether the fixed point set function is measurable. Indeed, there exists a measurable multivalued mapping which has random fixed points but whose fixed point set function fails to be measurable, as shown in the next example due to Christoph Bandt (private communication).

Example. Let $M = \Omega = [0, 1]$ and Σ be the σ-algebra of Lebesgue measurable subsets of $[0, 1]$. Let $T : [0, 1] \times [0, 1] \to K([0, 1])$, the family of nonempty compact subsets of $[0, 1]$, be given by

$$T(\omega, x) = \begin{cases} [x, 1] & \text{if } \omega = x \in E, \\ \{1\} & \text{otherwise,} \end{cases}$$

where E is a non-Lebesgue measurable set in M. Then, for any $x \in M$ and any interval $I \subset M$,

$$T(\cdot, x)^{-1}(I) = \{\omega \in \Omega : T(\omega, x) \cap I \neq \emptyset\}$$

is either $[0, 1]$ or at most a singleton; hence $T(\cdot, x)$ is measurable. Similarly, for each $\omega \in \Omega$, $T(\omega, \cdot)$ is upper semicontinuous. Also, the fixed point set of $T(\omega, \cdot)$ is

$$F(\omega) = \begin{cases} \{\omega, 1\} & \text{if } \omega \in E, \\ \{1\} & \text{if } \omega \notin E. \end{cases}$$

It follows that $x(\omega) \equiv 1$ is a random fixed point of T, while F is not measurable as $F^{-1}([0, 1)) = E$ is not measurable.

Suppose that M is a weakly compact convex separable subset of a Banach space. One of the approaches to show the existence of a fixed point for a random nonexpansive multivalued mapping $T : \Omega \times M \to \mathrm{CB}(M)$ is to approximate T by the multivalued random contractions $T_n : \Omega \times M \to \mathrm{CB}(M)$ given by

$$T_n(\omega, x) = \frac{1}{n}x_0 + \left(1 - \frac{1}{n}\right)T(\omega, x), \qquad \omega \in \Omega,\ x \in M,$$

where x_0 is a fixed element of M. Obviously, $\lim_{n\to\infty} H(T_n(\omega, x), T(\omega, x)) = 0$ for all

$\omega \in \Omega$ and $x \in M$. However, the sequence $\{F_n(\omega)\}$ of fixed point sets of $\{T_n(\omega, x)\}$ does not necessarily converge (in appropriate sense) to the fixed point set $F(\omega)$ of $T(\omega, x)$; see [35]. This indicates that we cannot use the measurability of the fixed point sets of the approximants to T to deduce the measurability of the fixed point set of T. Another approach to tackle this problem is this: Set

$$F_n(\omega) = \mathrm{cl}\left\{x \in M : d(x, T(\omega, x)) < \frac{1}{n}\right\}.$$

Then by Proposition 3.1.5, the continuity of T implies that F_n is measurable for each n. Moreover, it is immediately clear that

$$\bigcap_{n=1}^{\infty} F_n(\omega) = F(\omega).$$

So one can try to prove the measurability of F from this countable intersection of measurable sets. We have

PROPOSITION 3.1.6. *If $F_n(\omega)$ is compact in (M, d) for all n and ω, then F is measurable.*

PROOF. The compactness of $F_n(\omega)$ implies that $\lim d(y, F_n(\omega)) = d(y, F(\omega))$ for all $y \in M$ (and even the stronger result: $\lim H(F_n(\omega), F(\omega)) = 0$), which in turn implies the measurability of F by Propositions 3.1.2 or 3.1.3. ∎

However, if the $F_n(\omega)$'s are not compact, the above argument would fail; it is unclear if Proposition 3.1.6 is valid without assuming compactness of the $F_n(\omega)$'s.

3.2. Random contractions. Let us first recall Nadler's theorem, the multivalued version of Banach's contraction principle.

THEOREM 3.2.1. *Let (M, d) be a complete metric space and $T : M \to \mathrm{CB}(M)$ be a multivalued contraction, i.e., there is a constant $k \in [0, 1)$ such that*

$$H(Tx, Ty) \le kd(x, y), \quad x, y \in M.$$

Then T has a fixed point ξ, and moreover, for any $x_0 \in M$ and $\overline{k}$, $k < \overline{k} < 1$, there exists an orbit (x_n) of T at x_0 which converges to ξ with the estimate

$$d(x_n, \xi) \le \frac{\overline{k}^n}{1 - \overline{k}} d(x_1, x_0), \quad n \ge 0.$$

LEMMA 3.2.2. *Let (M, d) be a complete metric space and $S : M \to \mathrm{CB}(M)$ be a contraction: $H(Sx, Sy) \le kd(x, y)$ for all $x, y \in M$, where $k \in [0, 1)$ is a constant. For each $\alpha > 0$ set*

$$F_\alpha = \mathrm{cl}\{x \in M : d(x, Sx) < \alpha\} \quad \text{and} \quad F = \{x \in M : x \in Sx\}.$$

Then

$$H(F_\alpha, F) \le \frac{\alpha}{1 - k}.$$

PROOF. Since $F_\alpha \supseteq F$, we have

$$H(F_\alpha, F) = \sup_{x \in F_\alpha} d(x, F).$$

For an arbitrary $x \in F_\alpha$ and $\varepsilon > 0$, we can choose an $x_1 \in Sx$ satisfying $d(x, x_1) < (1+\varepsilon)\alpha$. Starting from $x_0 = x$, $x_1 \in Sx$ and $k < \overline{k} < 1$, we can construct an orbit (x_n) of S at x such that

$$d(x_n, \xi) \le \frac{\overline{k}^n}{1-\overline{k}} d(x_1, x_0) \quad \text{for all } n \ge 0,$$

where $\xi \in F$ is the limit of $\{x_n\}$; in particular, we have

$$d(x, F) \le d(x, \xi) = d(x_0, \xi) \le \frac{1}{1-\overline{k}} d(x_1, x_0) \le \frac{(1+\varepsilon)\alpha}{1-\overline{k}}.$$

Since $\varepsilon > 0$ and $\overline{k} \in (k, 1)$ are arbitrary, it follows that $d(x, F) \le \alpha/(1-k)$, which implies that $H(F_\alpha, F) \le \alpha/(1-k)$ as $x \in F_\alpha$ is arbitrary. ∎

THEOREM 3.2.3. *Suppose that (M, d) is a complete separable metric space, (Ω, Σ) is a measurable space with Σ a σ-algebra of subsets of Ω, and $T : \Omega \times M \to \mathrm{CB}(M)$ a random contraction, that is, for each $x \in M$, $T(\cdot, x)$ is measurable, and for each $\omega \in \Omega$, there exists a number $k(\omega) \in [0, 1)$ such that*

$$H(T(\omega, x), T(\omega, y)) \le k(\omega) d(x, y), \quad x \in M.$$

Then the fixed point set function F of T given by $F(\omega) := \{x \in M : x \in T(\omega, x)\}$ is measurable (and hence T admits a random fixed point).

PROOF. By Theorem 3.2.1, $F(\omega)$ is nonempty for every $\omega \in \Omega$. For each integer $n \ge 1$, let

$$F_n(\omega) = \mathrm{cl}\{x \in M : d(x, T(\omega, x)) < 1/n\}.$$

It follows from Proposition 3.1.5 and Lemma 3.2.2 that each $F_n(\omega)$ is measurable and $H(F_n(\omega), F(\omega)) \le 1/((1-k(\omega))n) \to 0$ as $n \to \infty$. So F is measurable by Proposition 3.1.3. ∎

REMARK. The existence of a random fixed point for a multivalued random contraction was proved by Itoh [22]. But he required that the function k of the Lipschitz constants of $T(\omega, \cdot)$ be measurable. Here we have proved the measurability of the fixed point set function F without assuming the measurability of k.

We conclude this section with a random version of Theorem 1.2.2.

THEOREM 3.2.4. *Let (Ω, Σ) be a measurable space with Σ a σ-algebra of subsets of Ω, let (X, d) be a complete separable metric space, and let $T : \Omega \times X \to \mathrm{CB}(X)$ be a random mapping such that*

$$d(T(\omega, x), T(\omega, y)) \le k(\omega, d(x, y)) d(x, y), \quad \omega \in \Omega,\ x, y \in X,\ x \ne y,$$

where $k : \Omega \times (0, \infty) \to (0, 1)$ is a measurable function satisfying

$$\limsup_{r \to t^+} k(\omega, r) < 1, \quad \omega \in \Omega,\ t \in [0, \infty).$$

Then T has a random fixed point.

To prove the theorem we need the following lemma, which can be proved in the same way as the proof of Proposition 4 in [22].

LEMMA 3.2.5. *Let Ω and X be as in Theorem 3.2.4. Assume $v : \Omega \to X$ and $S : \Omega \to \mathrm{CB}(X)$ are measurable. Then, for any measurable function $r : \Omega \to (1, \infty)$, there exists* *a measurable selection $w(\cdot)$ for $S(\cdot)$ such that*

$$d(v(\omega), w(\omega)) \le r(\omega) d(v(\omega), S(\omega)), \quad \omega \in \Omega.$$

Proof of Theorem 3.2.4. Pick any measurable function $x_0 : \Omega \to X$. Consider the measurable mapping $\omega \mapsto T(\omega, x_0(\omega))$ from Ω to $\mathrm{CB}(X)$. Apply Proposition 3.1.1 to get a measurable selection $x_1(\cdot)$ of $T(\cdot, x_0(\cdot))$. Next apply Lemma 3.2.5 to get a measurable selection $x_2(\cdot)$ for $T(\cdot, x_1(\cdot))$ satisfying

$$d(x_1(\omega), x_2(\omega)) \le [k(\omega, d(x_0(\omega), x_1(\omega)))]^{-1/2} d(x_1(\omega), T(\omega, x_1(\omega))), \quad \omega \in \Omega.$$

Continuing this way we construct a sequence $\{x_n(\omega)\}$ of measurable functions $x_n : \Omega \to X$ such that

(i) $x_{n+1}(\cdot)$ is a measurable selection of $T(\cdot, x_n(\cdot))$,
(ii) $d(x_n(\omega), x_{n+1}(\omega)) \le [k(\omega, d(x_{n-1}(\omega), x_n(\omega)))]^{-1/2} d(x_n(\omega), T(\omega, x_n(\omega)))$, $\omega \in \Omega$.

According to the proof of the deterministic case (Theorem 1.2.2) we see that for each fixed $\omega \in \Omega$, $\{x_n(\omega)\}$ converges to some fixed point $x(\omega)$ of $T(\omega, x(\omega))$. Thus we have a mapping $x : \Omega \to X$. This x is measurable since it is the pointwise limit of the sequence $\{x_n(\omega)\}$ of measurable functions. Hence x is a random fixed point of T. ∎

3.3. Random nonexpansive mappings. In this section we first show the existence of random fixed points for a random nonexpansive multivalued mapping in a uniformly convex Banach space. We then show some results on the measurability of the fixed point set function F of either a random multivalued nonexpansive mapping or a random condensing multivalued mapping.

Theorem 3.3.1. *Let (Ω, Σ) be a measurable space with Σ a σ-algebra of subsets of Ω. Let C be a nonempty, bounded, closed, convex and separable subset of a uniformly convex Banach space X and $T : \Omega \times C \to K(C)$ a random nonexpansive multivalued mapping. Then T has a random fixed point.*

Proof. Let $v \in C$ be fixed and define, for each $n \ge 1$, $T_n : \Omega \times C \to K(C)$ by

$$T_n(\omega, x) := \left(1 - \frac{1}{n}\right) T(\omega, x) + \frac{1}{n} v, \quad \omega \in \Omega,\ x \in C.$$

Then T_n is a random contraction and hence has a random fixed point x_n. Let (n_α) be a universal subnet of the net of positive integers (n). We then define a function $f : \Omega \times C \to [0, \infty)$ by

$$f(\omega, x) := \lim_\alpha \|x_{n_\alpha}(\omega) - x\|.$$

Since $\{x_{n_\alpha}(\omega)\}$ is countable, we see that f is measurable. On the other hand, since X is uniformly convex, C is also weakly compact. Thus there exists a unique point $x(\omega) \in C$ which minimizes $f(\omega, \cdot)$ over C, that is,

$$f(\omega, x(\omega)) = \inf_{x \in C} f(\omega, x) =: r(\omega).$$

(This point $x(\omega)$ is referred to as the *asymptotic center* of the net $(x_{n_\alpha}(\omega))$ in C.) According to the deterministic case, for each fixed $\omega \in \Omega$, $x(\omega)$ is actually a fixed point of

$T(\omega,\cdot)$. So it remains to verify that x is measurable. To this end, let (u_n) be a countable dense subset of C. Thus we have

$$r(\omega) = \inf_{n\geq 1} f(\omega, u_n), \quad \omega \in \Omega.$$

This indicates that $r : \Omega \to \mathbb{R}$ is measurable since $f(\cdot, u_n)$ is measurable for each n.

Next set, for each integer $k \geq 1$,

$$A_k(\omega) := \{x \in C : f(\omega, x) \leq r(\omega) + 1/k\}.$$

It follows that $A_k : \Omega \to 2^C \setminus \{\emptyset\}$ is measurable and for each $\omega \in \Omega$, $A_k(\omega)$ is a weakly compact convex subset of C. It is also readily seen that

$$\bigcap_{k=1}^{\infty} A_k(\omega) = \{x(\omega)\}. \tag{3.1}$$

Since C is separable, the weak topology on C is metrizable. Let d_w be the metric on C which is induced by the weak topology of C and let H_w be the Hausdorff metric induced by d_w. We now show that

$$\lim_{k\to\infty} H_w(A_k(\omega), x(\omega)) = 0. \tag{3.2}$$

Indeed, since $\bigcap_{k=1}^{\infty} A_k(\omega) = \{x(\omega)\}$, the limit above exists; we denote it by $h(\omega)$. If $h(\omega) > 0$, then observing

$$H_w(A_k(\omega), x(\omega)) = \sup\{d_w(y, x(\omega)) : y \in A_k(\omega)\},$$

we have for each $k \geq 1$ some $y_k \in A_k(\omega)$ such that

$$d_w(y_k, x(\omega)) > \tfrac{1}{2}h(\omega). \tag{3.3}$$

Since C is weakly compact, there is a subsequence $(y_{k'})$ of (y_k) converging weakly to some $y \in C$; i.e., we have $d_w(y_{k'}, y) \to 0$ as $k' \to \infty$. This contradicts (3.3) and (3.2) is thus verified. By Propositions 3.1.3 and 3.1.4, $x(\omega)$ is measurable. ∎

THEOREM 3.3.2. *Let (Ω, Σ) be a measurable space with Σ a σ-algebra of subsets of Ω, let C be a weakly compact convex subset of a Banach space X, and let $T : \Omega \times C \to \mathrm{CB}(X)$ be a random continuous mapping. If, in addition, for each fixed $\omega \in \Omega$, $I - T(\omega,\cdot)$ is demiclosed at* 0, *where I is the identity operator, and for each $\omega \in \Omega$, the (deterministic) mapping $T(\omega,\cdot) : C \to \mathrm{CB}(X)$ has a fixed point, then the fixed point set function F of T is measurable (and hence T has a random fixed point).*

PROOF. Set for each $n \geq 1$,

$$A_n(\omega) = \{x \in C : d(x, T(w, x)) < 1/n\}.$$

Then $A_n(\omega)$ is measurable by Proposition 3.1.5. For a fixed $\omega \in \Omega$, from the demiclosedness of $I - T(\omega,\cdot)$ it follows that the fixed point set $F(\omega)$ of $T(\omega,\cdot)$ is weakly closed. Thus

$$F(\omega) = \bigcap_{n=1}^{\infty} \text{w-cl}\, A_n(\omega). \tag{3.4}$$

(Here w-cl A denotes the closure of A under the weak topology.) Indeed, as $A_n(\omega) \supseteq F(\omega)$ for each n, it follows that $\bigcap_{n=1}^{\infty} \text{w-cl}\, A_n(\omega) \supseteq F(\omega)$. To show the reverse inclusion, suppose that there exists some $x \in \bigcap_{n=1}^{\infty} \text{w-cl}\, A_n(\omega)$, but $x \notin F(\omega)$. Let d_{w} be a distance on C that produces the weak topology on C. Then we have $d_{\text{w}}(x, A_n(\omega)) = 0$ for each n. So we can find for each n an $x_n \in A_n(\omega)$ such that $d_{\text{w}}(x_n, x) < 1/n$. Furthermore, by the definition of $A_n(\omega)$, we can select $y_n \in T(\omega, x_n)$ for which $\|x_n - y_n\| < 1/n$ for $n \geq 1$. Now the sequence $\{x_n\}$ weakly converges to x, and $\{x_n - y_n\}$, with $y_n \in T(\omega, x_n)$ for each n, converges strongly to 0. It then follows from the demiclosedness of $I - T(\omega, \cdot)$ at 0 that $0 \in (I - T(\omega, \cdot))x$. Hence $x \in T(\omega, x)$. This contradicts the assumption that $x \notin F(\omega)$. Thus (3.4) is proved. Since C is weakly compact, $\text{cl}\, A_n(\omega)$ is weakly compact for each n. An application of Propositions 3.1.6 and 3.1.4 yields the measurability of F. ■

REMARK. If the space X satisfies Opial's property [39] (i.e., $x_n \rightharpoonup x \Rightarrow \limsup \|x_n - x\| < \limsup \|x_n - y\|$ for all $y \in X$, $y \neq x$), then $I - f$ is demiclosed at 0 (cf. [29], see also [58]) provided $f : C \to K(C)$ is nonexpansive. So Theorem 3.3.2 applies to such a space. All Hilbert spaces and l^p spaces $(1 < p < \infty)$ have Opial's property, but it remains an open question whether $I - f$ is demiclosed at 0 if the space X is uniformly convex (e.g. $L^p[0,1]$, $1 < p < \infty$, $p \neq 2$) and $f : C \to K(C)$ is nonexpansive. (Note: The answer for a single-valued nonexpansive mapping f is yes, which is the famous theorem of Browder [4].) A remarkable fixed point theorem for multivalued mappings is Lim's result: If C is a nonempty closed bounded convex subset of a uniformly convex Banach space X and $f : C \to K(C)$ is nonexpansive, then f has a fixed point. Theorem 3.3.1 is the randomization of Lim's theorem. However, it is unknown whether the fixed point set function F in this case is measurable.

COROLLARY 3.3.3. *Suppose X is a Banach space with Opial's property, C is a weakly compact convex separable subset of X, and $T : \Omega \times C \to K(C)$ is a random nonexpansive mapping. Then the fixed point set function F of T is measurable and hence T has a random fixed point.*

REMARK. The existence of a random fixed point of T under the assumptions of Corollary 3.3.3 was proved by Itoh [23]. Here we proved the stronger measurability result for the random fixed point set function F of T.

Recall now that a multivalued mapping $S : C \to 2^X$ is convex if the graph of S,

$$\text{Gr}(S) := \{(x, y) \in C \times X : y \in Sx\},$$

is convex. If S is convex, then it is easily seen that for any number r, the "level" set $\{x \in C : d(x, Sx) < r\}$ is a convex set in C. The following improves upon Theorem 3.2 of [12]. Also, the proof given here is simpler.

THEOREM 3.3.4. *Let (Ω, Σ) be a measurable space with Σ a σ-algebra of subsets of Ω, let C be a weakly compact convex subset of a Banach space X, and let $T : \Omega \times C \to \text{CB}(X)$ be a random multivalued continuous (i.e., u.s.c. and l.s.c.) convex mapping such that for each $\omega \in \Omega$, the (deterministic) mapping $T(\omega, \cdot)$ has a fixed point. Then the fixed point set function F of T is measurable.*

PROOF. For each $n \geq 1$ the set

$$B_n(\omega) := \{x \in M : d(x, T(\omega, x)) < 1/n\}, \quad \omega \in \Omega,$$

is convex. Noting that the fixed point set $F(\omega)$ of T is closed and convex, we have

$$\bigcap_{n=1}^{\infty} \text{w-cl}\, B_n(\omega) = F(\omega), \quad \omega \in \Omega.$$

Now by weak compactness and Propositions 3.1.6 and 3.1.4, it follows that $F(\omega)$ is measurable. ∎

Theorem 3.3.5. *Let C be a closed convex separable subset of a Banach space X and $T : \Omega \times C \to \mathrm{CB}(X)$ be a continuous condensing random mapping such that for each $\omega \in \Omega$, $T(\omega, C)$ is bounded. Suppose also that for each $\omega \in \Omega$, the deterministic mapping $T(\omega, \cdot) : C \to \mathrm{CB}(X)$ has a fixed point. Then the fixed point set function F of T is measurable and hence T has a random fixed point.*

Proof. As before, we set for each $\omega \in \Omega$ and integer $n \geq 1$,

$$F(\omega) := \{x \in C : x \in T(\omega, x)\} \quad \text{and} \quad F_n(\omega) := \mathrm{cl}\{x \in C : d(x, T(\omega, x)) < 1/n\}.$$

Then $F(\omega)$ is closed and nonempty since $T(\omega, \cdot)$ is upper semicontinuous. Also, each F_n is measurable by Proposition 3.1.5. We next show that for each $\omega \in \Omega$,

$$\lim_{n\to\infty} H(F_n(\omega), F(\omega)) = 0. \tag{3.5}$$

Since $\{F_n(\omega)\}$ is decreasing and $\bigcap_{n=1}^{\infty} F_n(\omega) = F(\omega)$, the limit on the left hand side of (3.5) exists. Denote this limit by $b(\omega)$. Then we have

$$H(F_n(\omega), F(\omega)) = \sup_{y \in F_n(\omega)} d(y, F(\omega)) \geq b(\omega).$$

Thus for each $n \geq 1$, one can pick a $y_n \in F_n(\omega)$ such that

$$d(y_n, F(\omega)) > b(\omega) - 1/n. \tag{3.6}$$

Set $D := \{y_n\}$. Since each y_n lies in $F_n(\omega)$, i.e.,

$$d(y_n, T(\omega, y_n)) \leq 1/n \to 0 \quad \text{as } n \to \infty, \tag{3.7}$$

it follows that

$$\alpha(T(\omega, D)) \leq \alpha(D).$$

Therefore, $\alpha(D) = 0$ for $T(\omega, \cdot)$ is condensing. This implies that (y_n) admits a subsequence $(y_{n'})$ converging to some $y \in C$. By the upper semicontinuity of T, we deduce from (3.7) that $d(y, T(\omega, y)) = 0$. Hence $y \in F(\omega)$, which together with (3.6) yields $b(\omega) \leq d(y, F(\omega)) = 0$ and (3.5) is verified. Now by Proposition 3.1.3, $F(\cdot)$ is measurable and each measurable selection is a random fixed point of T. ∎

Corollary 3.3.6 (A partial random version of Browder's fixed point theorem [5]). *Suppose C is a nonempty compact convex subset of a Banach space X and $T : \Omega \times C \to \mathrm{KC}(X)$ is a random continuous mapping. Suppose in addition that either of the following boundary conditions is satisfied*:

(i) *For each $x \in \partial C$, the boundary of C, and each $\omega \in \Omega$, there exist $y \in T(\omega, x)$, $u \in C$, and $\lambda > 0$ such that*

$$y = x + \lambda(u - x).$$

(ii) *For each* $x \in \partial C$ *and* $\omega \in \Omega$, *there exist* $y \in T(\omega, x)$, $u \in C$, *and* $\lambda < 0$ *such that*

$$y = x + \lambda(u - x).$$

Then the random fixed point set function F *of* T *is measurable and thus* T *has a random fixed point.*

PROOF. For each $\omega \in \Omega$, under either of the boundary conditions above, the deterministic mapping $T(\omega, \cdot) : C \to \mathrm{KC}(X)$ has a fixed point by Browder's fixed point theorem [5]. The conclusion then follows from Theorem 3.3.5. ■

REMARK. We do not know if the conclusions of Theorem 3.3.5 and Corollary 3.3.6 remain valid if the continuity of T is weakened to the upper semicontinuity of T. (Note that the answer is yes for a deterministic mapping T.)

We conclude this section with a result for a single-valued mapping in a uniformly smooth Banach space. Recall that a Banach space X is said to be *uniformly smooth* if, for any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that

$$\|x + y\| + \|x - y\| < 2 + \varepsilon\|y\| \quad \text{for all } x, y \in X,\ \|x\| = 1,\ 0 < \|y\| < \varepsilon.$$

It is known that X is uniformly smooth if and only if the norm $\|\cdot\|$ of X is uniformly Fréchet differentiable; namely, the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists and is attained uniformly for $x, y \in S_X$, the unit sphere of X.

Assume X is a uniformly smooth Banach space and C a nonempty closed bounded convex subset of X. Let $S : C \to C$ be a single-valued nonexpansive mapping. For each $u \in C$ and integer $n \geq 1$ we define the contraction $S_n : C \to C$ by

$$S_n x = \frac{1}{n} u + \left(1 - \frac{1}{n}\right) S x, \quad x \in C.$$

Let x_n be the unique fixed point of S_n. A result of Reich [47] states that the strong $\lim x_n$ exists and

$$P(u) := \lim x_n, \quad u \in C,$$

defines a nonexpansive retraction from C onto $F(S)$, the set of fixed points of S.

LEMMA 3.3.7. *If* C *is separable and* $\{u_n\} \subset C$ *is a countable set dense in* C, *then* $\{P(u_n)\}$ *is dense in* $F(S)$.

PROOF. Let $y \in F(S)$. Since the retraction $P : C \xrightarrow{\text{onto}} F(S)$ is nonexpansive, it follows that $\|y - P(u_n)\| \leq \|y - u_n\|$ for all n, which implies the density of $\{P(u_n)\}$ in $F(S)$ as $\{u_n\}$ is dense in C. ■

THEOREM 3.3.8. *Let* C *be a closed bounded convex separable subset of a uniformly smooth Banach space, let* (Ω, Σ) *be a measurable space with* Σ *a* σ*-algebra of subsets of* Ω, *and let* $T : \Omega \times C \to C$ *be a* (*single-valued*) *random nonexpansive mapping. Then the fixed point set function* F *of* T *is measurable and hence* T *has a random fixed point.*

PROOF. Let $\{u_n\}$ be a countable subset of C which is dense in C. For each pair of integers n, k, let $x_{n,k}(\omega)$ be the unique random fixed point of the random contraction

$$T_n^k(\omega) := \frac{1}{n}u_k + \left(1 - \frac{1}{n}\right)T(\omega, x), \quad \omega \in \Omega,\ x \in C.$$

Let $x_k(\omega) \in F(\omega)$ be the strong limit of the sequence $\{x_{n,k}(\omega)\}$ as $n \to \infty$. By Lemma 3.3.7, $\{x_k(\omega)\}$ is dense in $F(\omega)$, i.e., $\mathrm{cl}\{x_k(\omega)\} = F(\omega)$ for every $\omega \in \Omega$. Therefore, F is measurable by Proposition 3.1.2. ∎

Acknowledgements. The author thanks the referee for his comments and suggestions which improved the presentation of the article. This work was partially supported by the National Research Foundation of South Africa.

Bibliography

[1] J.-P. Aubin and H. Frankowska, *Set-Valued Analysis*, Birkhäuser, Boston, 1990.

[2] A. T. Bharucha-Reid, *Random Integral Equations*, Academic Press, New York and London, 1972.

[3] —, *Fixed point theorems in probabilistic analysis*, Bull. Amer. Math. Soc. 82 (1976), 641–645.

[4] F. E. Browder, *Nonexpansive nonlinear operators in a Banach space*, Proc. Nat. Acad. Sci. U.S.A., 54 (1965), 1041–1044.

[5] —, *The fixed point theory of multi-valued mappings in topological vector spaces*, Math. Ann. 177 (1968), 283–301.

[6] J. Caristi, *Fixed point theorems for mappings satisfying inwardness conditions*, Trans. Amer. Math. Soc. 215 (1976), 241–251.

[7] C. Castaing and M. Valadier, *Convex Analysis and Multifunctions*, Lecture Notes in Math. 560, Springer, Berlin, 1977.

[8] Y. Q. Chen, *On a fixed point problem of Reich*, Proc. Amer. Math. Soc. 124 (1996), 3085–3088.

[9] H. Covitz and S. B. Nadler, Jr., *Multi-valued contraction mappings in generalized metric spaces*, Israel J. Math. 8 (1970), 5–11.

[10] P. D. Daffer, H. Kaneko and W. Li, *On a conjecture of S. Reich*, Proc. Amer. Math. Soc. 124 (1996), 3159–3162.

[11] K. Deimling, *Multivalued Differential Equations*, Walter de Gruyter, Berlin and New York, 1992.

[12] T. Dominguez Benavides, G. Lopez Acedo and H. K. Xu, *Random fixed points of set-valued operators*, Proc. Amer. Math. Soc. 124 (1996), 831–838.

[13] D. Downing and W. A. Kirk, *Fixed point thorems for set-valued mappings in metric and Banach spaces*, Math. Japon. 22 (1977), 99–112.

[14] M. Edelstein, *The construction of an asymptotic center with a fixed point property*, Bull. Amer. Math. Soc. 78 (1972), 206–208.

[15] H. W. Engl, *Random fixed point theorems for multivalued mappings*, Pacific J. Math. 76 (1978), 351–360.

[16] K. Goebel, *On a fixed point theorem for multivalued nonexpansive mappings*, Ann. Univ. Mariae Curie-Skłodowska Sect. A 29 (1975), 69–72.

[17] K. Goebel and W. A. Kirk, *Topics in Metric Fixed Point Theory*, Cambridge Univ. Press, 1990.

[18] K. Goebel and S. Reich, *Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings*, Monogr. Textbooks Pure Appl. Math. 83, Marcel Dekker, New York and Basel, 1984.

[19] B. Halpern, *Fixed point theorems for set-valued maps in infinite-dimensional spaces*, Math. Ann. 189 (1970), 87–98.

[20] C. J. Himmelberg, *Measurable relations*, Fund. Math. 87 (1975), 53–72.

[21] R. E. Huff, *Banach spaces which are nearly uniformly convex*, Rocky Mountain J. Math. 10 (1980), 743–749.

[22] S. Itoh, *A random fixed point theorem for a multivalued contraction mapping*, Pacific J. Math. 68 (1977), 85–90.

[23] —, *Random fixed-point theorems with an application to random differential equations in Banach spaces*, J. Math. Anal. Appl. 67 (1979), 261–273.

[24] J. R. Jachymski, *Caristi's fixed point theorem and selections of set-valued contractions*, ibid. 227 (1998), 55–67.

[25] J. L. Kelley, *General Topology*, Van Nostrand, Princeton, NJ, 1955.

[26] W. A. Kirk, *Nonexpansive mappings in product spaces, set-valued mappings and k-uniform rotundity*, in: Nonlinear Functional Analysis and Applications, F. E. Browder (ed.), Proc. Sympos. Pure Math. 45, part 2, Amer. Math. Soc., Providence, RI, 1986, 51–64.

[27] W. A. Kirk and S. Massa, *Remarks on asymptotic and Chebyshev centers*, Houston J. Math. 16 (1990), 357–364.

[28] T. Kuczumow and S. Prus, *Compact asymptotic centers and fixed points of multivalued nonexpansive mappings*, ibid., 465–468.

[29] E. Lami Dozo, *Multivalued nonexpansive mappings and Opial's condition*, Proc. Amer. Math. Soc. 38 (1973), 286–292.

[30] T. C. Lim, *A fixed point theorem for multivalued nonexpansive mappings in a uniformly convex Banach space*, Bull. Amer. Math. Soc. 80 (1974), 1123–1126.

[31] —, *Remarks on some fixed point theorems*, Proc. Amer. Math. Soc. 60 (1976), 179–182.

[32] —, *On asymptotic centers and fixed points of nonexpansive mappings*, Canad. J. Math. 32 (1980), 421–430.

[33] —, *A fixed point theorem for weakly inward multivalued contractions*, J. Math. Anal. Appl., to appear.

[34] T. C. Lin, *Random approximations and random fixed point theorems for non-self-maps*, Proc. Amer. Math. Soc. 103 (1988), 1129–1135.

[35] G. Lopez Acedo and H. K. Xu, *Remarks on multivalued nonexpansive mappings*, Soochow J. Math. 21 (1995), 107–115.

[36] A. Meir and E. Keeler, *A theorem on contraction mappings*, J. Math. Anal. Appl. 28 (1969), 326–329.

[37] N. Mizoguchi and W. Takahashi, *Fixed point theorems for multivalued mappings on complete metric spaces*, J. Math. Anal. Appl. 141 (1989), 177–188.

[38] S. B. Nadler, Jr., *Multi-valued contraction mappings*, Pacific J. Math. 30 (1969), 475–488.

[39] Z. Opial, *Weak convergence of the sequence of successive approximations for nonexpansive mappings*, Bull. Amer. Math. Soc. 73 (1967), 595–597.

[40] N. S. Papageorgiou, *Random fixed point theorems for measurable multifunctions in Banach spaces*, Proc. Amer. Math. Soc. 32 (1987), 507–514.

[41] S. Reich, *A fixed point theorem for locally contractive multivalued functions*, Rev. Roumaine Math. Pures Appl. 17 (1972), 569–572.

[42] —, *Fixed points of contractive functions*, Boll. Un. Mat. Ital. (4) 5 (1972), 26–42.

[43] —, *Fixed points in locally convex spaces*, Math. Z. 125 (1972), 17–31.

[44] —, *Some fixed point theorems*, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) 57 (1974), 194–198.

[45] S. Reich, *A random fixed point theorem for set-valued mappings*, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) 64 (1978), 65–66.

[46] —, *Approximate selections, best approximations, fixed points, and invariant sets*, J. Math. Anal. Appl. 62 (1978), 104–113.

[47] —, *Strong convergence theorems for resolvents of accretive operators in Banach spaces*, ibid. 75 (1980), 287–292.

[48] —, *Some problems and results in fixed point theory*, in: Contemp. Math. 21, Amer. Math. Soc., 1983, 178–187.

[49] V. M. Sehgal and S. P. Singh, *On random approximations and a random fixed point theorem for set-valued mappings*, Proc. Amer. Math. Soc. 95 (1985), 91–94.

[50] F. Sullivan, *A generalization of uniformly rotund Banach spaces*, Canad. J. Math. 31 (1979), 628–636.

[51] D. H. Wagner, *Survey of measurable selection theorems*, SIAM J. Control Optim. 19 (1977), 859–903.

[52] H. K. Xu, *Fixed point theorems for single-valued and set-valued mappings*, thesis, Zhejiang Univ., 1985.

[53] —, *A random fixed point theorem for multivalued nonexpansive operators in a uniformly convex Banach space*, Proc. Amer. Math. Soc. 117 (1993), 1089–1092.

[54] —, *ε-chainability and fixed points of set-valued mappings*, Math. Japon. 39 (1994), 353–356.

[55] —, *Random fixed point theorems for nonlinear uniformly Lipschitzian mappings*, Nonlinear Anal. 26 (1996), 1301–1311.

[56] —, *Inequalities in Banach spaces with applications*, ibid. 16 (1991), 1127–1138.

[57] —, *Measurability of fixed point sets of multivalued random operators*, J. Math. Anal. Appl. 225 (1998), 62–72.

[58] K. Yanagi, *On some fixed point theorems for multivalued mappings*, Pacific J. Math. 87 (1980), 233–240.

[59] H. W. Yi and Y. C. Zhao, *Fixed point theorems for weakly inward multivalued mappings and their randomizations*, J. Math. Anal. Appl. 183 (1994), 613–619.

[60] G. X.-Z. Yuan, *The study of minimax inequalities and applications to economies and variational inequalities*, Mem. Amer. Math. Soc. 625 (1998).

[61] E. Zeidler, *Nonlinear Functional Analysis and its Applications I: Fixed-Point Theorems*, Springer, New York, 1986.