Contents

Abstract

Our aim is to survey results in graph theory centered around four themes: hamiltonian graphs, pancyclic graphs, cycles through vertices and the cycle structure in a graph. We focus on problems related to the closure result of Bondy and Chvátal, which is a common generalization of two fundamental theorems due to Dirac and Ore. We also describe a number of proof techniques in this domain. Aside from the closure operation we give some applications of Ramsey theory in the research of cycle structure of graphs and present several methods used in the study of the structure of the set of cycle lengths in a hamiltonian graph.

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1. Introduction

The purpose of this paper is twofold: to survey the progress in results that deal with cycle structures of undirected graphs and to present several proof techniques related to this subject. The results discussed are centered around four themes: hamiltonian graphs, pancyclic graphs, cycles through specified vertices and the structure of the set of cycle lengths in graphs. Clearly, the choice of the subject is based on the author's recent results and his preferences. The paper will by no means exhaust the above topics; today there are too many directions of research on this subject covered by several thousands of papers. However, we try to survey the progress in the four themes mentioned above, made in the past twenty years.

The paper is organized as follows. In Section 2 we recall some basic concepts and terminology of graph theory.

Section 3 is devoted to hamiltonian graphs. The concept of a hamiltonian cycle for a long time played a fundamental role in the theory of graphs. Since the invention of Sir William Rowan Hamilton's game in 1856 (see [160]), many brilliant mathematicians have been occupied with the problem of hamiltonicity. Beginning with Dirac's theorem [101] in 1952 that stimulated a series of refinements culminating in the classical result of Bondy and Chvátal [63] on stability and closure, the main approach to the hamiltonian problem involved some vertex-degree conditions or some conditions on the degree sum of two nonadjacent vertices. The closure result is a common generalization of all classical sufficient conditions for hamiltonicity and provides easy proofs for these results.

There are two other approaches, the first one involving a condition on the cardinality of the union of the neighborhoods of k vertices forming an independent set, called a generalized degree condition, and the second one including a condition on degree sums of such vertices. Many generalizations of Dirac's and Ore's [227] theorems in terms of these conditions have been found.

In the same section we also treat Chvátal–Erdős-type conditions involving the connectivity and the independence number of a graph and we present forbidden-subgraph characterizations of hamiltonian graphs that were initiated by a result of Goodman and Hedetniemi [146]. This opened the door to the study of claw-free graphs where the Ryjáček [238] closure plays an important role. In Section 3 we also give several recent results, all generalizing Ore's theorem.

In Section 4 we concentrate around pancyclic graphs, those that contain cycles of all possible lengths. This concept was introduced by Bondy [56] in a paper published in 1971. He also formulated a "metaconjecture" stating that almost all nontrivial sufficient conditions for a graph to be hamiltonian also imply that it is pancyclic except for maybe a simple family of graphs. This metaconjecture was at the origin of a new direction in studying the cycles in graphs. All known sufficient conditions for a graph to be hamiltonian (especially in terms of vertex degrees) have been examined in light of this conjecture. In the same section we consider weakly pancyclic graphs, those that contain cycles of every length between their girth and circumference. It is shown that in many cases the conditions that ensure that a graph is weakly pancyclic are considerably weaker than those required to ensure that it is pancyclic.

The investigation of the pancyclicity of graphs satisfying the Chvátal–Erdős-type conditions, the problem which in general seems to be very difficult, is the main subject of this section. It is one of intriguing long-standing problems on pancyclicity. A partial solution of this problem is a beautiful result due to Erdős [111] stating that if the order of a hamiltonian graph G is sufficiently large compared with its stability number, then G is pancyclic. Its proof is based on the properties of cycle-complete graph Ramsey numbers.

In Section 5 we study the following question. Let G be a graph having a certain property and S a subset of $V(G)$. Does this property guarantee the existence of a cycle containing all vertices of S? The first and now well-known result of this type was obtained by Dirac [102]: if G is a k-connected graph, then every set of k vertices of G is contained in a cycle. We present several generalizations of this theorem and give a series of results analogous to those of hamiltonian graph theory. In particular, we give a result of Flandrin et al. [140] involving a Chvátal–Erdős-type condition. In the proof of this result we use a function depending on Ramsey numbers.

Section 6 is devoted to the cycle structure of graphs. We treat cycles of a given length and cycles of a given length modulo k, and we study the number of cycle lengths in a graph.

Bondy's "metaconjecture" initiated an investigation into the structure of the set of cycles lengths in hamiltonian graphs satisfying some constraints on the degree sum of two consecutive vertices on a hamiltonian cycle; as an example we can cite a well-known theorem of Schmeichel and Hakimi [248]. In that section we present several results of this type concerning the structure of this set in a hamiltonian graph with a given maximum degree or the degree sum of two fixed vertices. We point out the application of such results in studying the stability of the property of being pancyclic. We also present a theorem due to Woźniak and the author [219] that gives a lower bound on the number of different cycle lengths in a hamiltonian graph with a given maximum degree.

The second goal of this paper is to present selected proof techniques. We turn our attention to the following methods:

- application of a variety of closure operations (see for example Theorems 3.25, 3.26, 3.46 and 5.10);
- research on the stability of the Bondy–Chvátal closure (Theorems 3.27, 3.28, 3.30, 3.113 and Proposition 6.30);
- application of Ramsey theory to the investigation into the cycle structure of graphs (Theorems 4.40 and 5.30);
- investigation into the structure of the set of cycle lengths in hamiltonian graphs (Theorems 6.18, 6.35 and 6.36).

We also give proofs of fundamental results on the subject. Therefore, both beginners and graph theory experts may benefit from reading this paper.

We do not intend to give a complete survey of results related to the main topic. For the material not covered here, the reader is referred to the books by Walther and Voss [276], Voss [274], and the recent survey papers by Bondy [62], Gould [149, 150], Broersma [76], Bauer, Broersma and Schmeichel [22] and Broersma, Ryjáček and Schiermeyer [80]. We recommend the books by Bollobás [49], [48], Alon and Spencer [8] and the article by Gould [149] to the reader interested in probabilistic methods in the research on cycles.

We will not treat cycles in directed graphs, but we recommend the book by Bang-Jensen and Gutin [18] and an excellent survey due to Bermond and Thomassen [42].

2. Basic concepts

A graph G is a pair (V, E), where V is a finite nonempty set and E a subset of the set $V^{(2)}$ of all two-element subsets of V. The elements of $V = V(G)$ are called *vertices*, and those of $E = E(G)$ are called *edges*. An edge $\{x, y\}$ is said to *join* the vertices x and y and is denoted by xy. If xy is an edge of G, the vertices x and y are the *endvertices* of this edge and we say that x and y are *adjacent*. We say that the endvertices of an edge are *incident* with this edge.

We can draw a picture of a graph, a vertex being indicated by a point and an edge by a line joining its endvertices. Fig. 1 depicts the graph of the dodecahedron, one of the five Platonic graphs, the graphs of the Platonic solids.

The number of vertices of a graph G is called its *order* and denoted by n or $v(G)$. The number of edges of G is its *size*. It is denoted by $e(G)$. Clearly, the size of a graph of order n is between 0 and $\binom{n}{2}$.

A graph $G' = (V', E')$ is a *subgraph* of a graph $G = (V, E)$ if $V' \subset V$ and $E' \subset E$. If G' contains all edges of G that join two vertices in V', then G' is said to be *induced* by the set V' and is denoted by $G[V']$ or $\langle V' \rangle$. If $V = V'$ then G' is a *spanning* subgraph of G.

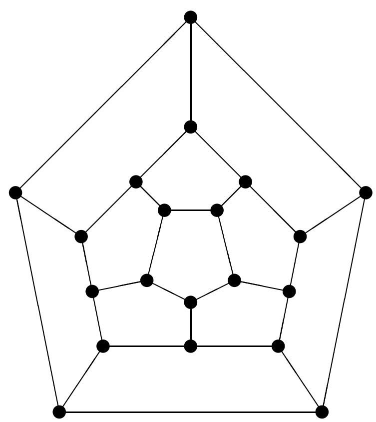

Fig. 1. Dodecahedron

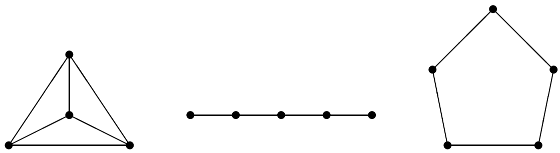

Fig. 2. K_4, P_5 and C_5

If $x \in V(G)$ then $G - x$ is the subgraph of G obtained by deleting x and all edges incident with x. Similarly, for $W \subset V(G)$ we denote by $G - W$ the subgraph $G[V(G) \setminus W]$ of G. If $xy \in E(G)$, then $G - xy$ is the spanning subgraph of G obtained by deleting the edge xy from G. Similarly, if x and y are nonadjacent vertices of G, then $G + xy$ is the graph obtained by adding the edge xy to G.

A graph of order n that possesses all possible edges is called the *complete graph* of order n and is denoted by K_n. Thus, the size of K_n is $\binom{n}{2}$. The graph K_4, depicted in Fig. 2, is the graph of tetrahedron; it belongs to the five Platonic graphs. A subgraph of a graph G which is complete is called a *clique* of G.

A graph $G = (V, E)$ is *isomorphic* to another graph $G' = (V', E')$ if there is a bijection $f : V \to V'$ such that $xy \in E$ iff $f(x)f(y) \in E'$. If G and G' are isomorphic graphs then we write either $G \cong G'$ or $G = G'$. The *complement* of a graph $G = (V, E)$ is the graph $\overline{G} = (V, V^{(2)} \setminus E)$. The complement $\overline{K}_n$ of the graph K_n is called the *empty* graph of order n and denoted by E_n.

Let x be a vertex of a graph G. The set of all vertices adjacent to x is called the *neighborhood* of x and denoted by $N_G(x)$ or simply $N(x)$, if no ambiguity can arise. The symbol $d_G(x)$ (or simply $d(x)$) stands for the cardinality of the set $N_G(x)$ and is called the *degree* of x. A vertex x with $d(x) = 0$ is an *isolated* vertex. The set $N_G(x) \cup \{x\}$, denoted by $N_G[x]$, is the *closed neighborhood* of x. We write $\delta(G)$ or simply δ for the *minimum* degree of the vertices of a graph G. We use the notation $\Delta(G)$ or Δ for the *maximum* degree of G.

If every vertex of G has degree k then G is said to be *k-regular*. 3-regular graphs are called *cubic*. The graph of the dodecahedron in Fig. 1 is an example of a cubic graph of order 20 and size 30.

Suppose $V(G) = \{x_1, \ldots, x_n\}$. Then

$$\sum_{i=1}^{n} d(x_i) = 2e(G),$$

because each edge has exactly two endvertices. The last formula is called the *handshaking lemma*. It follows immediately from this lemma that any graph has an even number of vertices of odd degree.

Let a be a real number. The symbol $\lfloor a \rfloor$ stands for the greatest integer not greater than a, and by $\lceil a \rceil = -\lfloor -a \rfloor$ we denote the smallest integer not less than a.

A *path of order* r is a graph $P = (V, E)$, where $V = \{x_1, \ldots, x_r\}$ and $E = \{x_1x_2, x_2x_3, \ldots, x_{r-1}x_r\}$. The number $r - 1 = |E|$ is the *length of the path*. A path P is also called a *path* from x_1 to x_r or an x_1-x_r *path* and denoted by $x_1, \ldots, x_r$. We frequently identify the path with this sequence of vertices. The vertex x_1 is the *initial* vertex of P, x_r a *terminal* vertex of P and the vertices $x_2, \dots, x_{r-1}$ are called *internal* vertices of P. Notice that the sequence $x_1, \dots, x_r$ gives a natural orientation of the path P. If x precedes y on P according to this orientation, we denote by $x\vec{P}y$ (or xPy) the sequence of consecutive vertices on P from x to y (x and y included) and by $y\overleftarrow{P}x$ the sequence of the same vertices but in reverse order. A path of order r is often denoted by P_r. We also use the notation $x^+ = x^{+1}$ and $x^- = x^{-1}$ for (if it exists) the successor and the predecessor of x on P with respect to a given orientation. We write x^{+k} for $(x^{+(k-1)})^+$, x^{-k} for $(x^{-(k-1)})^-$ and $x^{+0} = x^{-0} = x$.

A *cycle* is a graph $C = (V, E)$, where $V = \{x_1, \dots, x_p\}$ ($p \geq 3$) and $E = \{x_1x_2, x_2x_3, \dots, x_{p-1}x_p, x_px_1\}$. The number $p = |E| = |V|$ is the *length* of the cycle. Such a cycle is denoted by C_p or $x_1, \dots, x_p, x_1$ and called a *p-cycle*. We call C_3 a *triangle*, C_4 a *quadrilateral* and C_5 a *pentagon*.

The sequence $x_1, \dots, x_p$ of vertices of a cycle C defines a natural orientation of the cycle (clearly, this is not an order on the set $V(C)$). According to the natural orientation of C, x_1 succeeds x_p. We can also take a reverse orientation of C, such a cycle is denoted by $\overleftarrow{C}$. We define the symbols x^{+k} and x^{-k} just as for a path.

Assume that G contains a cycle C with a given orientation and let a and b be two distinct vertices of C. By aCb or $C[a,b]$, we will denote the path $a, a^+, a^{+2} \dots, b^-, b$ and we will call it a (closed) *segment* of C from a to b. Thus, aCb is the sequence of consecutive vertices of C from a to b (a and b included) in the direction specified by the orientation of C. If $a = b$, then by $aCb = [a,b]$ we mean the one-element set $\{a\}$. We shall consider $C[a,b]$ both as a path and a vertex set.

Throughout the paper the indices of a cycle $C = x_1, \dots, x_p$ are to be taken modulo p.

A path (cycle) that is a subgraph of a graph G is called a *path of* G (resp. a *cycle of* G).

A set S of vertices of a graph G is *independent* (or *stable*) if no two elements of S are adjacent; in other words, S is independent iff $G[S]$ is an empty graph. The *independence number* (or the *stability number*) $\alpha(G)$ of G is defined to be the maximum cardinality of an independent set.

A graph G is *connected* if any two distinct vertices of G are joined by a path. In other words, a graph is connected if its vertex set admits no partition into two nontrivial subsets such that vertices in different subsets are not adjacent. A *connected component* of a graph G is a maximal connected subgraph of G. Thus, a connected component is a connected subgraph of G which is a subgraph of no other connected subgraph of G. A graph G which is not connected is called *disconnected*.

Let G be a graph and let S be a subset (possibly empty) of $V(G)$. S is called a *vertex cut* of G if the graph $G - S$ (i.e., the graph obtained by removing from G all vertices of S) is not connected. Thus, the empty set is a vertex cut of any disconnected graph.

Let S be a vertex cut of G and Y, Z two connected components of $G - S$. If $x \in V(Y)$ and $y \in V(Z)$, we say that the vertex cut S *separates* x *and* y. Observe that only two nonadjacent vertices can be separated and the vertex cut S that separates x and y contains neither x nor y. We will also say that S *separates two vertices of* G.

If G is not a complete graph then the *connectivity of* G, denoted by $\kappa(G)$, is the smallest number $k \geq 0$ such that there exists in G a vertex cut of k vertices that separates two vertices of G. For the complete graph K_n we define $\kappa(K_n) = n-1$, $n \geq 2$, and $\kappa(K_1) = 1$. We clearly have $\kappa(G) = 0$ if G is disconnected. A graph G is *k-connected* ($k \geq 0$) if $k \leq \kappa(G)$. Furthermore, a graph is connected if and only if it is 1-connected. Notice that the degree of any vertex in a k-connected graph is at least k.

A *tree* is a connected graph without any cycle.

Now we will define the notion of a k-edge-connected graph. A set S of edges of a graph with at least two vertices is called an *edge cut* if the deletion of all the edges of S gives a disconnected graph (the endvertices of the edges of S remain in the graph). The *edge connectivity* of a connected graph G, denoted by $\kappa'(G)$, is the smallest number $s \geq 0$ such that there exist in G an edge cut of s edges. For the graph K_1, we define $\kappa'(K_1) = 1$. A connected graph is *k-edge-connected* if $k \leq \kappa'(G)$.

The *distance* $d(x, y)$ between two vertices x and y in a graph G is the length of a shortest $x - y$ path in G, if there is one; otherwise, their distance is ∞. The *diameter* of a graph is the maximum distance between its vertices.

A graph G is a *bipartite graph* with bipartition (X, Y) if $V(G) = X \cup Y$, $X \cap Y = \emptyset$ and X and Y are independent and nonempty sets. If $|X| = p$, $|Y| = r$ and G is a bipartite graph with bipartition (X, Y) containing all edges joining vertices in X and Y, then G is called the *complete bipartite graph $K_{p,r}$*.

In a graph $G = (V, E)$, a *hamiltonian path* is defined to be a path of G that includes every vertex of G. Similarly, a *hamiltonian cycle* is a cycle of G containing every vertex of the graph. A *hamiltonian graph* is one that contains a hamiltonian cycle. It is easy to check that K_4, C_5 and the graph of the dodecahedron (see Figs. 1 and 2) are hamiltonian. A graph that contains a hamiltonian path is *traceable*. Clearly, every hamiltonian graph is traceable, but the path of length p shows that the converse is not true. Observe also that if G is hamiltonian, then deleting a vertex results in a connected graph, therefore G is 2-connected.

A graph of order n is said to be *pancyclic* if it contains cycles of all lengths from 3 to n. Obviously, every pancyclic graph is hamiltonian. A deeper relation between these two notions is presented in Section 4 of this review. A graph G of order $n \geq 3$ is *vertex pancyclic* if every vertex of G is contained in a p-cycle for every p between 3 and n. Similarly, G is said to be *edge pancyclic* if every edge of G is contained in a p-cycle for each p between 3 and the order of the graph. Thus, every complete graph on $n \geq 3$ vertices is both vertex pancyclic and edge pancyclic (and of course, pancyclic).

A graph G is defined to be *hamiltonian-connected* if for each pair x, y of distinct vertices, there is a hamiltonian path with endvertices x and y. It is obvious that every hamiltonian-connected graph with at least three vertices is hamiltonian, but the converse is not true (see, for example, the graph C_5). A connected graph is *panconnected* if for each pair of distinct vertices x and y, there exist an $x - y$ path of length l, for each l such that $d(x, y) \leq l \leq |V(G)| - 1$. If G is panconnected, then it is hamiltonian-connected, so it is hamiltonian.

We write $G \cup H = (V(G) \cup V(H), E(G) \cup E(H))$ and kG for the union of k disjoint copies of G. The *join* of two vertex-disjoint graphs G and H is the graph denoted by $G \vee H$ obtained from $G \cup H$ by adding all edges between $V(G)$ and $V(H)$.

In this survey, we will not treat so-called general graphs or multigraphs, that may have multiple edges or loops (see for instance [62]). In the literature, the graphs studied herein are called finite, simple, undirected graphs.

The undefined terminology and notation as well as the elementary properties of the notions defined above can be found in the books by Bondy and Murty [66], Bollobás [49], Gould [148] and a survey paper by Bondy [62] in *Handbook of Combinatorics*. For elementary algorithmic background, we suggest the books by Gould [148] and Bang-Jensen and Gutin [18].

3. Hamiltonian graphs

3.1. Two fundamental theorems and their generalizations. A cycle containing all the vertices of a graph is called hamiltonian. This term is used in honor of Sir William Rowan Hamilton who, in 1865, invented a mathematical game on the dodecahedron (polyhedron with twelve pentagonal faces and 20 vertices), whose graph is depicted in Fig. 1. The vertices of the dodecahedron represented twenty cities in the world. The objective of the game was to travel to every city exactly once and return home using only the edges. In other words, this objective was to find a hamiltonian cycle in the graph of the dodecahedron. Since then, a lot of interesting works on spanning cycles and paths in graphs have appeared.

It is known that the problem of deciding whether a given graph is hamiltonian is NP-complete. Moreover, there exists no easily verifiable necessary and sufficient condition for the existence of a hamiltonian cycle in a graph. Nonetheless, there is a plethora of conditions that are either necessary or sufficient. Note that finding a hamiltonian cycle is a special case of a more general problem of finding a minimum-weight hamiltonian cycle in an edge-weighted complete graph. This question is known as the Traveling Salesman Problem and has been treated in numerous papers. In the present section we will study various approaches to the hamiltonian problem. The first one is based on the idea that a hamiltonian cycle is likely to exist if the graph has a sufficient number of edges properly distributed. In other words, such a graph must have a "high edge density". A quite natural measure of this density is the minimum degree of a graph. The first fundamental result using this measure is due to Dirac (1952).

Theorem 3.1 (Dirac [101]). *Let $G = (V, E)$ be a graph on n vertices, where $n \geq 3$. If $\delta(G) \geq n/2$, then G is hamiltonian.*

This result immediately follows from Ore's theorem presented below. For other proofs, see Bondy's chapter of *Handbook of Combinatorics* [62].

Corollary 3.2. *If $\delta(G) \geq (n-1)/2$, where n is the order of G, then G contains a hamiltonian path.*

Proof. For $n = 1$ the result is obvious. Suppose that G is a graph of order $n \geq 2$ satisfying the assumptions of the corollary and consider the graph H obtained by adding to G a new vertex x and joining x to all vertices of G. This graph H has $n+1$ vertices and minimum degree at least $(n+1)/2$. By Dirac's theorem, H contains a hamiltonian cycle C. Deleting x from H, we get the graph G with a hamiltonian path. ∎

The complete bipartite graph $K_{\lfloor (n-1)/2\rfloor,\lceil (n+1)/2\rceil}$ has minimum degree $\lfloor (n-1)/2\rfloor$ and no hamiltonian cycle, because the sets of the bipartition have different cardinalities. Thus Dirac's theorem is best possible.

In 1952 Dirac [101] generalized his own theorem to a result asserting the existence of a long cycle.

Theorem 3.3. *Let G be a 2-connected graph on $n \geq 3$ vertices. Then G contains either a cycle of length at least $2\delta(G)$ or a hamiltonian cycle.*

This result was improved by Voss and Zuluaga [275] for graphs with minimum degree at least three.

Theorem 3.4. *Let $r \geq 3$ be an integer and let G be a 2-connected nonbipartite graph on $n \geq 2r$ vertices such that $\delta(G) \geq r$. Then G contains both an odd cycle of length at least $2r-1$ and an even cycle of length at least $2r$.*

In 1975, Woodal [282] conjectured that every 2-connected graph of order n with at least $k+n/2$ vertices of degree at least k has a cycle of length at least $2k$ or a hamiltonian cycle. This is clearly a generalization of Dirac's Theorem 3.3. The conjecture was proved (only for $k \geq 431$) in 2002 by Li [194].

Theorem 3.5. *Let $k \geq 431$ and G be a 2-connected graph of order n with at least $\frac{n}{2}+k$ vertices of degree at least k. Then G contains either a cycle of length at least $2k$ or a hamiltonian cycle.*

It should be noted that Häggkvist and Li [156] proved the same conclusion holds for 3-connected graphs and $k \geq 25$.

A similar problem can be formulated for graphs which violate Dirac's condition ($d(x) \geq n/2$). The following result announced by Faudree et al. [131] is actually an immediate consequence of Theorem 3.11 (see the next subsection).

Theorem 3.6. *Let G be a graph of order n and minimum degree $\delta = \delta(G) < n/2$. If $|\{v \in V(G) \mid d(v) < n/2\}| \leq \delta - 1$, then G is hamiltonian.*

For $n \geq 3$ and $1 \leq k \leq (n-1)/2$, let $F(n,k)$ be the join $K_k \vee (K_{n-2k} \cup kK_1)$. This graph has k vertices of degree k and $n-k$ vertices of degree at least $(n-1)/2$; moreover, deleting k vertices belonging to K_k we get a graph having $k+1$ connected components, so $F(n,k)$ is not hamiltonian. Thus, the preceding theorem is best possible.

Another generalization of Dirac's theorem which concerns long cycles in G was given by Egawa and Miyamoto [105] and Bollobás and Brightwell [50].

Theorem 3.7. *Let s be a positive integer and G a simple graph of order $n \geq 3$ and minimum degree δ, where $\delta \geq n/(s+1)$. Then G contains a cycle of length at least n/s.*

The second fundamental result in hamiltonian graph theory was discovered by Ore [227] in 1960.

THEOREM 3.8 (Ore [227]). *If a graph G on $n \geq 3$ vertices is such that $d(x) + d(y) \geq n$ for every pair x, y of nonadjacent vertices, then G is hamiltonian.*

Proof. Suppose the assertion is not true. Thus, there exists a nonhamiltonian graph G of order $n \geq 3$ that satisfies the hypothesis of the theorem and such that for any pair x, y of nonadjacent vertices of G the graph $G + xy$ is hamiltonian. Clearly, the latter graph is not complete. Let x and y be two nonadjacent vertices of G and let $H = G + xy$. Clearly, H is hamiltonian and every hamiltonian cycle in H uses the edge xy. Therefore, there is a hamiltonian path $x = x_1, x_2, \ldots, x_n = y$ from x to y in G. Now, if $x_i \in N(x)$, then $x_{i-1} \notin N(y)$, since otherwise $x_1, x_i, x_{i+1}, \ldots, x_n, x_{i-1}, x_{i-2}, \ldots, x_1$ would be a hamiltonian cycle in G. Thus, for every vertex adjacent to x, there is a vertex of $V(G) - \{y\}$ which is not adjacent to y. Hence, $d(y) \leq n - 1 - d(x)$, a contradiction. ∎

The proof of the next corollary is almost the same as that of Ore's theorem.

COROLLARY 3.9. *Let G be a graph of order $n \geq 3$ and $x_1, \ldots, x_n$ a hamiltonian path in G such that $d(x_1) + d(x_n) \geq n$ and $x_1 x_n \notin E(G)$. Then G is hamiltonian.*

COROLLARY 3.10. *If G is a graph of order n such that $d(x) + d(y) \geq n - 1$ for any pair of nonadjacent vertices x and y, then G contains a hamiltonian path.*

Proof. For $n = 1$ the result is obvious. If $n \geq 2$, add a new vertex to G, say x, and join x to all vertices of G. The new graph is hamiltonian by Ore's theorem, hence G has a hamiltonian path. ∎

Ore's theorem was subject to many improvements and generalizations. Below we cite four well-known sufficient conditions for hamiltonicity, each of them being more general than its preceding one and more general than Ore's theorem. These results are due to Pósa [233], Bondy [54], Chvátal [93] and Las Vergnas [187], resp. In the first three theorems, by $d_1 \leq \cdots \leq d_n$, we shall denote the degree sequence of a graph G on n vertices.

THEOREM 3.11. *If $d_k > k$ for $1 \leq k < (n-1)/2$ and $d_{(n-1)/2} > (n-1)/2$, if n is odd, then G is hamiltonian.*

THEOREM 3.12. *If $d_j + d_k \geq n \geq 3$ for all pairs j, k with $j < k$, $d_j \leq j$, $d_k \leq k - 1$, then G is hamiltonian.*

THEOREM 3.13. *If $n \geq 3$ and $d_{n-k} \geq n - k$ for all k with $d_k \leq k < n/2$, then G is hamiltonian.*

THEOREM 3.14. *If there exists a labeling $v_1, \ldots, v_n$ of the vertices such that $j < k$, $k \geq n - j$, $v_k v_j \notin E(G)$, $d(v_j) \leq j$ and $d(v_{k-1}) \leq k - 1$ implies $d(v_j) + d(v_k) \geq n$, then G is hamiltonian.*

If we raise the degree bound in Ore's theorem, we obtain the hamiltonian-connectedness (see [229]).

THEOREM 3.15. *If a graph G of order $n \geq 3$ is such that $d(x) + d(y) \geq n + 1$ for every pair x, y of nonadjacent vertices, then G is hamiltonian-connected.*

As a corollary, we get a result of Erdős and Gallai [115].

Corollary 3.16. *If G is a graph of order $n \geq 3$ with $\delta(G) \geq (n+1)/2$, then G is hamiltonian-connected.*

Below we present two other generalizations of Ore's theorem. The first one is attributed to Pósa [233].

Theorem 3.17. *Let G be a connected graph of order $n \geq 3$ such that $d(x) + d(y) \geq d$ for any pair x, y of nonadjacent vertices. If $d < n$ then G contains a path of length at least d and if $d \geq n$, then G is hamiltonian.*

Proof. Let P be a longest path in G and l be the length of P. If $l < n-1$ and $l < d$, then we can apply Corollary 3.9 to the subgraph induced by $V(P)$ and find a cycle C of length $l+1$ with $V(C) = V(P)$. Since G is connected and there exists a vertex outside P, it also contains a path of length $l+1$, a contradiction. Hence $l \geq d$ and we are done. For $l = n-1$ and $d \geq n$, the assertion is true by Corollary 3.9. ∎

The second one was found independently by Bermond [40], Bondy [57] and Linial [201].

Theorem 3.18. *Let G be a 2-connected graph such that $d(x) + d(y) \geq d$ for any pair x, y of nonadjacent vertices. Then G contains either a cycle of length at least d or a hamiltonian cycle.*

Let us mention that there exists an $O(n^2)$ algorithm for finding a hamiltonian cycle in a graph satisfying Ore's condition (cf. [148]).

Faudree et al. [131] counted the number of pairs of nonadjacent vertices that can have the degree sum less than n but still implying that the graph is hamiltonian. They introduced the following function:

$$g(n,\delta) = \begin{cases} \infty & \text{if } n \leq 2\delta; \\ (n^2-1)/8 & \text{if } 2\delta+1 \leq n \leq 6\delta-3 \text{ and } n \text{ is odd;} \\ (n^2+2n-8)/8 & \text{if } 2\delta+2 \leq n \leq 6\delta-4 \text{ and } n \text{ is even;} \\ \delta n - \frac{3}{2}\delta^2 - \frac{1}{2}\delta & \text{if } n \geq 6\delta-2. \end{cases}$$

For a graph of order n, define $N_2(G)$ and $n_2(G)$ by

$$N_2(G) = \{(x,y) \mid xy \in E(\overline{G}) \text{ and } d(x)+d(y) < n\} \quad \text{and} \quad n_2(G) = |N_2(G)|.$$

Theorem 3.19. *Let G be a graph of order n with minimum degree δ. If $n_2(G) < g(n,\delta)$, then G is hamiltonian.*

There are examples showing that the theorem is sharp for $\delta \geq 2$.

Finally, note that a large number of edges does not ensure the existence of a hamiltonian cycle in a graph. Indeed, Ore [228] observed that if a graph on n vertices has more than $\binom{n-1}{2} + 1$ edges, then G is hamiltonian, and Bondy [58] showed that for each $n \neq 5$ the only nonhamiltonian graph with $\binom{n-1}{2} + 1$ edges is the graph $G(1,n)$ consisting of a complete graph K_{n-1} plus a vertex joined to a given vertex of this K_{n-1}.

A new and interesting direction in studying hamiltonian cycles was suggested by Fan [118]. He showed that we need not examine the degree sums of all pairs of nonadjacent vertices but only the degree sums of a particular subset of these pairs, namely the pairs of distance two. We will say that Fan localized the degree condition to this subset.

THEOREM 3.20. *Let G be a 2-connected graph of order n such that* $\max\{d(x), d(y)\} \geq n/2$ *for every pair of vertices x, y with $d(x, y) = 2$. Then G is hamiltonian.*

We present a proof of this theorem in the next subsection. Clearly, this result is a generalization of Ore's theorem. Bedrossian et al. [37] localized Fan's condition to the vertices belonging to the induced $K_{1,3}$ or the induced $K_{1,3}$ plus an edge. Other localizations of Fan's condition have been investigated by Mao and Liu [211] and Liu [204].

3.2. The idea of closure. Consider again Theorem 3.8 of Ore. It is easy to observe that Ore's theorem immediately implies the following result.

PROPOSITION 3.21. *Let G be a graph of order $n \geq 3$ and let x and y be nonadjacent vertices such that $d(x) + d(y) \geq n$. Then $G + xy$ has a hamiltonian cycle if and only if G has one.*

Proof. If $G+xy$ is hamiltonian but G is not, there is a hamiltonian path $x = x_1, x_2, \ldots, x_n = y$ from x to y in G. Now, if $x_i \in N_G(x)$, then $x_{i-1} \notin N_G(y)$, since otherwise $x_1, x_i, x_{i+1}, \ldots, x_n, x_{i-1}, x_{i-2}, \ldots, x_1$ would be a hamiltonian cycle in G. Thus, for every vertex adjacent to x, there is a vertex of $V(G) - \{y\}$ which is not adjacent to y. Hence, $d(y) \leq n - 1 - d(x)$, a contradiction. ■

Note that the proof of the proposition is almost the same as that of Ore's theorem.

This observation was a main idea of the notion of k-closure of a graph, which was introduced in the classical paper entitled "A method in graph theory" due to Bondy and Chvátal [63]. Namely, given a nonnegative integer k, we will call *k-closure of G* the graph obtained by recursively joining pairs x, y of nonadjacent vertices such that $d(x) + d(y) \geq k$ until no such pair remains. It will be denoted by $Cl_k(G)$. Now we must verify if this notion is well defined, that is, the resulting graph does not depend on the order of inserted edges.

PROPOSITION 3.22. *If G_1 and G_2 are two graphs obtained by recursively joining pairs of nonadjacent vertices whose degree sum is at least k until no such pair remains, then $G_1 = G_2$.*

Proof. Let $e_1, \ldots, e_p$ and $f_1, \ldots, f_r$ be two sequences of edges added to $E(G)$ to form G_1 and G_2, respectively. We will show that each edge e_i is in G_2 and each edge f_j is in G_1. Suppose that $uv = e_s$ is the first edge in the sequence $e_1, \ldots, e_p$ that does not belong to G_2 and consider the graph $H = G + \{e_1, \ldots, e_{s-1}\}$. Then, the definition of G_1 implies $d_H(u) + d_H(v) \geq k$. Since H is a subgraph of G_2, it follows that $d_{G_2}(u) + d_{G_2}(v) \geq k$. But this is a contradiction, because u and v are nonadjacent in G_2. In a similar way we can prove that each f_j is in G_1. Hence $G_1 = G_2$. ■

Obviously, any graph of order n satisfies the following relation:

$$G = Cl_{2n-3}(G) \subset Cl_{2n-4}(G) \subset \cdots \subset Cl_1(G) \subset Cl_0(G) = K_n,$$

where the relation $F \subset H$ means that F is a spanning subgraph of H.

Notice that the k-closure of a graph G of order n can be viewed as an intersection of all graphs H of order n such that $G \subset H$ and $d_H(u) + d_H(v) < k$ for all $uv \notin E(H)$.

Let k be a nonnegative integer and let P be a property defined on all graphs of order n. P is *k-stable* if whenever $G + xy$ has this property and $d_G(x) + d_G(y) \geq k$, then G has property P. For example, the property of containing a hamiltonian cycle is n-stable by Proposition 3.21. It is obvious that every k-stable property is $(k+1)$-stable and every property is $(2n-3)$-stable.

The definition of k-closure implies at once the following proposition.

PROPOSITION 3.23. *If P is k-stable and the k-closure of G has property P, then G itself has property P.*

The fundamental result of this work of Bondy and Chvátal is the following immediate consequence of Propositions 3.21 and 3.23 which is often called the Closure Lemma.

THEOREM 3.24 (Bondy and Chvátal [63]). *A graph of order n is hamiltonian if and only if its n-closure is hamiltonian.*

In particular, if the n-closure is complete then G is hamiltonian. For example, the closure of any graph satisfying Ore's or Dirac's condition is complete and this observation offers another proof of Ore's and Dirac's theorems. Below we show another application of the closure operation to prove the following result that implies Theorem 3.13 by Chvátal.

THEOREM 3.25. *Each graph that satisfies the hypothesis of Theorem* 3.13 *has the complete n-closure.*

Proof. Suppose that $d_1 \leq d_2 \leq \cdots \leq d_n$ is the degree sequence of G and $d_{n-k} \geq n-k$ for all k with $d_k \leq k < n/2$.

Let H be the n-closure of G. Suppose that H is not complete. Then the degree sum of any two nonadjacent vertices of H is at most $n-1$. Denote by x and y two vertices of the graph H whose degree sum is maximum. We may assume that $d_H(x) \leq d_H(y)$. Put $s = d_H(x)$. Thus $d_H(y) \leq n-1-s$ and $s \leq (n-1)/2 < n/2$. Let X be the set of vertices nonadjacent to y in H. Then $|X| = n-1-d_H(y) \geq s$. Moreover, by the choice of x and y, $d_G(u) \leq d_H(u) \leq s$ for every $u \in X$. Let Y be the set of vertices that are nonadjacent to x in H. Then $|Y| = n-1-s$ and $d_G(u) \leq d_H(u) \leq n-1-s$ for all $u \in Y$. Therefore, G contains at least s vertices of degree at most s and at least $n-s$ vertices (those of $Y \cup \{x\}$) of degree at most $n-1-s$. Hence, $d_s \leq s < n/2$, $d_{n-s} \geq n-s$, and, since (d_i) is not decreasing, $d_i \geq n-s$ for $i \geq n-s$. So we have at least $s+1$ vertices of degree at least $n-s$, a contradiction. ∎

Tian and Shi [267] and Veldman [271] applied the closure operation to prove a Fan type result (Theorem 3.20).

THEOREM 3.26. *The n-closure of every graph that satisfies the hypothesis of Theorem* 3.20 *is hamiltonian.*

Proof. Define $U = \{v \in V(G) \mid d_G(v) \geq n/2\}$ and let F be a connected component of $G - U$. Let u, v be two vertices of F and $P = x_1, \ldots, x_p$, where $u = x_1$ and $v = x_p$, a shortest path connecting u and v in F. If the length of P is at least two, then, because

$d_G(x_1) < n/2$ and $d_G(x_3) < n/2$, x_1 and x_3 are adjacent in G, a contradiction to the choice of P. Thus, u and v must be adjacent. It follows that every connected component of $G-U$ is complete. A similar argument shows that no two vertices belonging to distinct connected components of $G-U$ have a common neighbor in G. Since G is 2-connected, each connected component of $G-U$ of order at least two is joined to U by (at least) two independent edges and each such component of order one is joined to U by two edges. The set U induces a complete subgraph in the n-closure of G and this closure keeps the described structure of G. Such a graph is clearly hamiltonian. ∎

In [271] Veldman proved a number of Fan type results on hamiltonicity by applying the closure operation. Note that all hamiltonian graphs found by Theorems 3.11, 3.12 and 3.14 have a complete n-closure (cf. [188]). Bondy and Chvátal showed that n-closure can be constructed by a polynomial algorithm and, given a hamiltonian cycle in $Cl_n(G)$, we can find one in G again by a polynomial algorithm. Thus, for any graph G with $Cl_n(G) = K_n$, a hamiltonian cycle can be found in polynomial time, whereas this problem is generally NP-hard. Note also that Clark et al. [96] showed that if $Cl_n(G) = K_n$ then $|E(G)| \geq \lfloor (n+2)^2/8 \rfloor$.

Denote by $s(P)$ the smallest integer k for which P is k-stable. It will be called the *stability of* P. Consider now the following example. Let G be the graph obtained from a path $u_1, \ldots, u_n$ with $u_1 = u$ and $u_n = v$ by adding the edges uu_j for $j = 3, \ldots, n-1$. We have $d_G(u) + d_G(v) = n-1$, $G + uv$ is hamiltonian, but G is not. It follows that the property of being hamiltonian is not $(n-1)$-stable. Thus, the stability of this property is exactly n.

Bondy and Chvátal [63] investigated the stability of several properties of graphs. We here present four results concerning the properties of containing a cycle or a path.

THEOREM 3.27. *The stability of the property of having a C_r, $5 \leq r \leq n$, r odd, is $2n-r$.*

Proof. Suppose that $G + uv$ contains a C_r but G does not. Then G contains a path $u_1, \ldots, u_r$ with $u_1 = u$ and $u_r = v$. Let H denote the subgraph of G induced by the set $\{u_1, \ldots, u_r\}$. Then $H + uv$ is hamiltonian but H is not. Since the property of having a hamiltonian cycle defined on all graphs of order r is r-stable, we have $d_H(u)+d_H(v) < r$. It follows that $d_G(u) + d_G(v) \leq 2(n-r) + d_H(u) + d_H(v) < 2(n-r) + r = 2n-r$. Therefore, the property of having a C_r, $5 \leq r \leq n$, is $2n-r$ stable. We will see that the bound $2n-r$ cannot be improved for odd r. Indeed, take a path $u_1, \ldots, u_r$ with $u_1 = u$ and $u_r = v$ and join u and v to every vertex of the set $\{u_2, u_4, u_6, \ldots, u_{r-1}\}$ (if such an edge does not belong to the path). Now add $n-r$ new vertices $x_1, \ldots, x_{n-r}$ and join u and v to every new vertex x_i. Call the resulting graph G. Obviously, $d_G(u) + d_G(v) = 2(n-r) + 2(\frac{r-1}{2}) = 2n-r-1$, $G + uv$ has a C_r but G does not. ∎

THEOREM 3.28. *The stability of the property of having a C_r, $6 \leq r \leq n$, r even, is $2n-r-1$ if $r < n$ and n if $r = n$.*

Proof. Since the stability of being hamiltonian is n, we may assume that $4 \leq r \leq n-1$. If $G+uv$ contains a C_r but G does not, then G contains a path $u_1, u_2, \ldots, u_r$ with $u_1 = u$ and $u_r = v$. Let H denote the subgraph of G induced by the set $\{u_1, \ldots, u_r\}$. Using the same idea as in the proof of the previous theorem, we show that $d_H(u) + d_H(v) < r$. If

u and v have no common neighbor outside H, we are done, because

$$d_G(u) + d_G(v) \le (n-r) + d_H(u) + d_H(v) < n \le 2n - r - 1.$$

Therefore, we may assume that u and v have a common neighbor w outside H. Set

$$A = \{i \mid 2 \le i \le r,\ u \text{ is adjacent to } u_i\}, \qquad B = \{i \mid 2 \le i \le r,\ v \text{ is adjacent to } u_{i-1}\}.$$

We have $d_H(u) = |A|$ and $d_H(v) = |B|$. If $d_H(u) + d_H(v) < r - 1$, we are done, as

$$d_G(u) + d_G(v) \le 2(n-r) + d_H(u) + d_H(v) < 2n - r - 1.$$

Hence, we may suppose that

$$|A| + |B| \ge r - 1. \tag{1}$$

Observe that

$$A \cap B = \emptyset. \tag{2}$$

Indeed, if $i \in A \cap B$ then $u_1, u_i, u_{i+1}, \dots, u_r, u_{i-1}, \dots, u_1$ is a cycle of length r in G, a contradiction. Now, (1) and (2) imply

$$A \cup B = \{2, 3, \dots, r\}. \tag{3}$$

Obviously, $3 \notin A$, because otherwise $u_1, u_3, u_4, \dots, u_r, w, u_1$ would be a cycle of length r in G. Similarly, $r - 1 \notin B$, for otherwise $u_1, u_2, u_3, \dots, u_{r-2}, u_r, w, u_1$ would be a C_r. Hence (3) implies that $3 \in B$ and $r - 1 \in A$. It follows that u_r is adjacent to u_2 and u_1 is adjacent to u_{r-1}.

Observe that

$$j \in A \ \Rightarrow\ j + 1 \notin A, \tag{4}$$

because otherwise $u_1, u_{j+1}, u_{j+2}, \dots, u_r, u_2, u_3, \dots, u_j, u_1$ is a cycle of length r, a contradiction. We can show in a similar way that

$$j \in B \ \Rightarrow\ j + 1 \notin B. \tag{5}$$

Now, since $3 \in B$, from (3), (4) and (5) it follows that each odd j with $j \le r$ belongs to B. In particular, $r - 1 \in B$, a contradiction. Therefore, the property in question is $(2n - r - 1)$-stable.

Consider now a path $u_1, \dots, u_r$ with $u_1 = u$, $u_r = v$, and suppose r is an even number such that $6 \le r \le n - 1$. Join u to every vertex u_i with $i \in \{4, 5, \dots, r-1\}$. Add $n - r$ new vertices and join both u and v to each new vertex. This graph shows that the bound $2n - r - 1$ cannot be improved for $6 \le r \le n - 1$. ■

Observe that if $G + uv$ contains a C_4 but G does not, then u and v have at most one common neighbor in G. Thus, $d_G(u) + d_G(v) \le n - 2 + 1 = n - 1$. From this we conclude that the property of containing the cycle of length four is n-stable. To see that the stability of this property is also n, consider a graph G with vertex set $\{u_1, u_2, u_3, u_4 \dots, u_n\}$ where $u = u_1$, $v = u_4$, $u_2u_3 \in E(G)$, $vu_3 \in E(G)$ and u is adjacent to every vertex of G but itself and v. Thus, the following theorem holds.

THEOREM 3.29. *The stability of containing a C_4 is n.*

THEOREM 3.30. *The stability of the property of containing a P_r (a path on r vertices), $4 \le r \le n$, is $n - 1$.*

Proof. Suppose $G + uv$ contains a path $u_1, \ldots, u_r$ and G itself does not. Let H be the subgraph induced by the set $\{u_1, \ldots, u_r\}$. Then $(H \vee K_1) + uv$ is hamiltonian, but $H \vee K_1$ is not. Therefore,

$$(d_H(u) + 1) + (d_H(v) + 1) < r + 1.$$

Moreover, u and v have no common neighbors outside H (because otherwise G would have a P_{r+1}). So

$$d_G(u) + d_G(v) \leq (n - r) + d_H(u) + d_H(v) < n - 1.$$

To see that the stability of the property equals $n - 1$, consider the disjoint union of K_{r-2} and $K_{1,n-r+1}$ with two special vertices u and v, where $u \in V(K_{r-2})$ and v is the center of the star $K_{1,n-r+1}$. ∎

Recently Schiermeyer [245] investigated the stability of pancyclicity.

THEOREM 3.31. *The stability $s(P)$ for the property of being pancyclic satisfies*

$$\max(\lceil 6n/5 \rceil - 5, n + t) \leq s(P) \leq \max(\lceil 4n/3 \rceil - 2, n + t),$$

where $t = 0$ if n is odd and $t = 1$ if n is even. ∎

The proof of the theorem is not so easy as in the previous cases. The value of the upper bound (up to a constant) can also be obtained as a consequence of Theorem 6.29 in Section 6.

Since 1972, when Bondy and Chvátal introduced their closure, many other closure concepts have appeared (see for example the survey [80]). We here cite a very useful concept of the 0-*dual closure* introduced by Ainouche and Christofides [5]. This notion is based on the following result.

THEOREM 3.32. *Let u and v be two nonadjacent vertices of a 2-connected graph G, let $T := \{w \in V(G) \setminus \{u, v\} \mid u, v \notin N(w)\}$, $t := |T|$, $\lambda_{uv} := |N(u) \cap |N(v)|$, and let $d_1^T \leq \cdots \leq d_t^T$ be the degree sequence of the vertices of T (in G). If $d_i^T \geq t + 2$ for all i with $\max(1, \lambda_{uv} - 1) \leq i \leq t$, then G is hamiltonian if and only if $G + uv$ is hamiltonian.*

The 0-dual closure, denoted by C_0^*, is the graph obtained from G by recursively joining pairs of nonadjacent vertices that satisfy the hypothesis of the last theorem, until no such pair remains. It was shown in [5] that the 0-dual closure of a graph is unique and that $G \subset Cl_n(G) \subset C_0^*(G)$.

COROLLARY 3.33. *Let G be a 2-connected graph such that $C_0^*(G)$ is complete. Then G is hamiltonian.*

Several applications of the 0-dual closure are given in Subsection 3.5.

Faudree et al. [121] investigated the complete closure number $cc(G)$ of a graph G, i.e., the greatest integer $k \leq 2n-3$ such that the Bondy and Chvátal closure is complete. They obtained smaller values for the complete stability of properties related to the existence of cycles and paths than for the stability.

3.3. Connectivity of a set of vertices and Menger's theorem. Let X be a subset of the vertex set of a graph G such that $|X| \geq 2$ and X does not induce a complete subgraph of G. The *connectivity of X in G*, denoted by $\kappa(X)$, is the smallest number k

such that there exists in G a vertex cut of k vertices that separates two vertices of X. If X is a clique in G, then, by definition, $\kappa(X) = |X| - 1$ for $|X| \geq 2$ and $\kappa(X) = 1$ if $|X| = 1$. Thus, the number $\kappa(V(G))$ equals the connectivity $\kappa(G)$ of the graph G. Note that some authors (see for example [79]) define $\kappa(X) = \infty$ if X is a clique.

Consider now a set $\mathcal{P} = \{P_1, \ldots, P_s\}$ of paths of a graph G. These paths are *edge disjoint* if no two have a common edge, and *internally disjoint* if no two have an internal common vertex.

Menger's well-known Theorem [222] can be formulated as follows.

THEOREM 3.34. *Let x and y be two vertices of a graph G.*

(i) *The maximum number of edge-disjoint x-y paths in G is equal to the minimum number of edges in an edge cut separating x and y.*
(ii) *If x and y are not adjacent, then the maximum number of internally disjoint x-y paths is equal to the minimum number of vertices in a vertex cut separating x and y.*

The next lemma is an easy corollary of this theorem.

LEMMA 3.35. *Let $k \geq 1$ be an integer and let X be a subset of the vertex set of a graph G such that X is not a clique and $|X| > k$. The following two statements are equivalent:*

(i) *$\kappa(X) \geq k$;*
(ii) *any two vertices of X are connected by at least k internally disjoint paths (in G).*

Proof. For $k = 1$ our assertion is trivial. Suppose $k \geq 2$ and let X verify the assumption of the lemma. Clearly, if (ii) holds then we cannot separate two vertices of X by a vertex cut of at most $k-1$ vertices, so (i) is true. Assume that $\kappa(X) \geq k$ and let x and y be two vertices of X. If x and y are not adjacent, we can apply Menger's theorem and we are done. Assume that $xy \in E(G)$ and denote by G' the graph obtained from G by deleting the edge xy. Suppose there is a vertex cut S separating x and y in G' and such that $|S| \leq k-2$. Denote by U and V the connected components of $G' - S$ containing x and y, resp. Notice that no other connected component of $G' - S$ contains a vertex of X since otherwise S would separate two vertices of X in G. Therefore, because $|X| > k \geq 2$, U or V contains at least two vertices of X. Suppose for instance that $z \in U \cap X$ and $z \neq x$. Then $S \cup \{x\}$ separates z and y in the graph G and the cardinality of $S \cup \{x\}$ is $k-1$, which is a contradiction. Therefore, every vertex cut that separates x and y in G' has at least $k-1$ vertices. It follows from Menger's theorem that there are at least $k-1$ internally disjoint x-y paths in G'. Adding the edge xy we get the desired system of k internally disjoint x-y paths in G. ■

This lemma is best possible. Indeed, take three vertex-disjoint graphs: K_s, $\overline{K}_t$ and $\overline{K}_2$ where $t, s \geq 2$ and $V(\overline{K}_2) = \{x_1, x_2\}$. Denote by G the graph obtained by joining every vertex x_i, $i = 1, 2$, to each vertex of $V(K_s) \cup V(\overline{K}_t)$ and let $X = V(K_s) \cup \{x_1, x_2\}$. Clearly, $\kappa(X) = s + t > s + 1$, $|X| = s + 2 \leq \kappa(X)$ and no two vertices of the subgraph induced by $V(K_s)$ are connected by more than $s + 1$ internally disjoint paths. Observe also that for $s = 2$, $d(u) = 3 < \kappa(X)$ if $u \in K_s \subset X$. The well-known condition for a graph to be k-connected is an easy corollary of Lemma 3.35.

COROLLARY 3.36. *A graph G on at least two vertices is k-connected ($k \geq 0$) if and only if any two vertices of G are connected by at least k internally disjoint paths.*

Using the same method as in the proof of Lemma 3.35, we get the following

COROLLARY 3.37. *Let X be a subset of the vertex set of a graph G and let X not be a clique. Then any two vertices of X are connected by at least* $\min\{|X|-1, \kappa(X)\}$ *internally disjoint paths (in G).*

It easily follows from Menger's theorem that a graph G having at least two vertices is k-edge-connected if and only if any two vertices are connected by at least k edge-disjoint paths. Thus, the edge connectivity of a graph G is the greatest number k such that any two vertices of G are linked by at least k edge-disjoint paths. Since internally disjoint paths are edge-disjoint, every k-connected graph is k-edge connected.

There exist several versions of Menger's theorem. The following result, usually called the fan lemma, is a very useful tool in studying problems related to the connectivity of graphs. We present a version of this lemma involving the connectivity of a set of vertices.

LEMMA 3.38. *Let G be a graph and let X be a subset of $V(G)$ with $\kappa(X) \geq 1$. Let $\{x, x_1, \ldots, x_q\}$ be a subset of X with $q \leq \kappa(X)$, and let $Y \subset V(G)$ be another set of vertices such that $\{x_1, \ldots, x_q\} \subset Y$ and $x \notin Y$. Then there are different vertices $y_1, \ldots, y_q$ in Y and internally disjoint paths $P_1, \ldots, P_q$ such that*

(i) *P_i is an x-y_i path for $1 \leq i \leq q$,*
(ii) *$V(P_i) \cap Y = \{y_i\}$, $1 \leq i \leq q$.*

Proof. Consider a graph G' obtained by adding a new vertex y and joining it to all vertices $x_1, \ldots, x_q$. A set of $q-1$ vertices cannot separate y and x because the same set would separate x and a vertex belonging to the set $\{x_1, \ldots, x_q\}$, which contradicts the definition of $\kappa(X)$. Thus, by Menger's theorem, G' contains q internally disjoint x-y paths and the existence of the desired collection of paths is obvious. ∎

Let A, B and S be three sets of vertices of a graph G. An *A-B path* is a path from A to B such that $|A \cap V(P)| = |B \cap V(P)| = 1$. A common vertex of A and B is also an A-B path. We say that *S separates A and B* if every A-B path contains a vertex of S. If $\mathcal{P}$ is a collection of paths of G then by $V(\mathcal{P})$ we denote the union $\bigcup_{P \in \mathcal{P}} V(P)$. The following useful version of Menger's theorem is due to Böhme et al. [45].

THEOREM 3.39. *Let G be a graph, A, $B \subset V(G)$ such that A and B cannot be separated by a set of at most t vertices. Let $\mathcal{P}$ be a set of t disjoint A-B paths. Then there is a set $\mathcal{R}$ of $t+1$ disjoint A-B paths such that $A \cap V(\mathcal{P}) \subset A \cap V(\mathcal{R})$ and $B \cap V(\mathcal{P}) \subset B \cap V(\mathcal{R})$.*

Harant [162] used this theorem to study the cyclability of sets of vertices such that $|X| = \kappa(X) + 1$.

3.4. Chvátal–Erdős theorem. In 1972 Chvátal and Erdős [95] found a very simple connection between the stability number, the connectivity and the existence of a hamiltonian cycle in a graph.

THEOREM 3.40 (Chvátal and Erdős [95]). *Every k-connected graph on $n \geq 3$ vertices with stability $\alpha \leq k$ is hamiltonian.*

Proof. Let G satisfy the hypothesis of the theorem. We may assume that G is not complete. Thus $k \geq 2$ and by Corollary 3.36 it contains a cycle; let C be a longest one. If G has no hamiltonian cycle, there is a vertex x with $x \notin V(C)$. Since G is k-connected, it follows from the fan lemma (Lemma 3.38) that there are k paths starting at x and terminating in C which are pairwise disjoint apart from x and share with C just their terminal vertices $x_1, \ldots, x_k$. For each $i = 1, \ldots, k$ let y_i be the successor of x_i in a fixed orientation of C. No y_i is adjacent to x because otherwise we would replace the edge $x_i y_i$ in C by the path going from x_i to y_i outside C through x and obtain a longer cycle. Since the stability number of G is at most k, there is an edge of the form $y_i y_j$. Delete the edges $x_i y_i$ and $x_j y_j$ from C and add the edge $y_i y_j$ together with the path going from x_i to x_j outside C. We obtain a cycle longer than C, which is a contradiction. ∎

Observe that the complete bipartite graph $K_{r,r+1}$ is r-connected, its stability number is $r+1$, but it has no hamiltonian cycle. Similarly, the Petersen graph has stability number 4 and is 3-connected and nonhamiltonian. Therefore, the theorem is sharp.

The next corollary is an easy consequence of the Chvátal–Erdős theorem.

COROLLARY 3.41. *If G is a k-connected graph with stability number $\alpha \leq k+1$, then G is traceable.*

Proof. Suppose that G satisfies the hypothesis of the theorem and let G' be the graph obtained from G by adding a new vertex x and joining it to all the vertices of G. By Theorem 3.40, G' is hamiltonian, hence G has a hamiltonian path. ∎

The complete bipartite graph $K_{r,r+2}$ shows that the previous result is best possible. However, if $\alpha \leq k+1$ and the number of edges in the graph is sufficiently large, then the graph is hamiltonian (see [283]).

Applying the technique used in the proof of Theorem 3.40, Chvátal and Erdős [95] obtained the following result.

THEOREM 3.42. *Let G be a k-connected graph with $\alpha(G) \leq k-1$. Then G is hamiltonian-connected.*

The theorem is best possible, as shown by te example of the complete bipartite graph $K_{r,r}$. Note that by a result due to Bondy [59], the Chvátal–Erdős theorem implies Ore's theorem.

3.5. Further generalizations of fundamental results. Häggkvist and Nicoghossian [158] improved Dirac's theorem by including the connectivity into the degree bound.

THEOREM 3.43. *If G is a k-connected graph, $k \geq 2$, and $\delta(G) \geq \frac{1}{3}(n+k)$, then G is hamiltonian.*

Another interesting result including a degree condition was proposed by Zhu [288].

THEOREM 3.44. *Let $G = (V, E)$ be a 2-connected graph on n vertices with minimum degree δ. If $d(x) + d(y) \geq n/2 + \delta$ for all nonadjacent vertices x and y, then G is hamiltonian.*

This result was proved independently in [199]. In 2001 Flandrin et al. [137] introduced a new Ore-type condition.

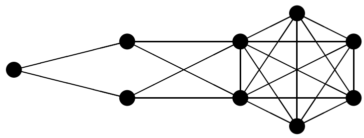

Fig. 3. A graph satisfying the hypothesis of Theorem 3.45

THEOREM 3.45. *Let $G = (V, E)$ be a 2-connected graph on n vertices with minimum degree δ and such that $xy \in E$ for every pair $x, y \in V$ such that $\delta = d(x)$ and $d(y) < n/2$. Then G is hamiltonian.*

The theorem is an immediate consequence of the following result by Skupień [254], involving the concept of Bondy–Chvátal closure.

THEOREM 3.46. *The Bondy–Chvátal n-closure of a graph satisfying the hypothesis of Theorem* 3.45 *is a complete graph.*

Proof. Let G be a graph of order n that satisfies the hypothesis of the theorem and let $K = Cl_n(G)$ denote the Bondy–Chvátal n-closure of G. Suppose that $K \neq K_n$. Then $2 \leq \delta < n/2$, whence $n \geq 5$. Let X and Y be the sets of vertices whose degrees in G are δ and in the interval $[\delta + 1, n/2)$, resp. Let $|X| = i$ $(i > 0)$ and $|Y| = j$. Then $i + j \leq \delta + 1$, because each vertex x of degree δ in G is adjacent to all vertices in $Y \cup X \setminus \{x\}$. Moreover, since $\delta + 1 \leq (n + 1)/2 \leq n - 2$, the complement of $X \cup Y$ in V is nonempty and induces in K a complete subgraph, say Q, all of whose vertices z have degrees $d_Q(z) = n-1-i-j \geq n-\delta-2$. If $\delta \geq i+j+1$, then in the graph K each vertex of $X \cup Y$ is adjacent to each vertex of $V(Q)$ and $d_K(y) \geq n - \delta > n/2$ for all $y \in Y$, so $K = K_n$, a contradiction. Hence $i + j - 1 \leq \delta \leq i + j$.

Suppose $i + j = \delta$. Then the set $V \setminus X$ is a clique in K with vertex degrees at least $n - 1 - \delta$, which are at least $n - \delta$ if $j > 0$, thus $K = K_n$, a contradiction. It follows that $j = 0$ and $i = \delta$. Then each of the i vertices in X has in G exactly one neighbor belonging to $V \setminus X$. However, because of 2-connectivity, in G there are two (or more) neighbors z_1, z_2 of the set X in $V(Q)$. Then $d_K(z_p) \geq d_Q(z_p) + 1 = n - \delta$, $p = 1, 2$. Hence both z_p's are adjacent in K to each $x \in X$. It follows that $d_K(x) \geq \delta + 1$ for each $x \in X$, whence $K = K_n$, a contradiction.

Therefore, $i + j = \delta + 1$. Then all neighbors of X in G are in $X \cup Y$. Because of 2-connectivity of G, $|Y| = j \geq 2$ and there are two (or more) neighbors z_1, z_2 of Y in $V(Q)$. Therefore, $d_K(z_p) \geq d_Q(z_p)+1 = n-1-\delta$, $p = 1, 2$, whence each z_p is adjacent in K to all vertices of Y, implying $d_K(z_p) \geq d_Q(z_p)+|Y| \geq n-\delta$. Consequently, $xz_p \in E(K)$ for all z_p's and all $x \in X$, therefore $d_K(x) \geq \delta + 2$. Hence the set $V \setminus Y$ is a clique in K with minimum degree at least $n - 1 - j \geq n - 1 - \delta$. This implies that each vertex y of Y

is adjacent to all vertices of Q in K and $d_K(y) \geq n-j \geq n-\delta > n/2$, whence $K = K_n$, a contradiction. ■

It is easy to see that the condition of Theorem 3.45 is weaker than both Ore's and Zhu's conditions mentioned above and is independent of Chvátal's [93] and Fan's [118] well known conditions (see Fig. 3). Clearly, the above result improves Theorems 3.8 and 3.44.

In Section 4 we will see that a graph satisfying the hypotheses of Theorem 3.45 is pancyclic or isomorphic to $K_{n/2,n/2}$.

For all n, δ with $2 \leq \delta \leq (n-1)/2$, define $\mathcal{F}_{n,\delta}$ as a graph of order n, minimum degree δ and vertex set $V(\mathcal{F}_{n,\delta}) = \{u_0, u_1, \ldots, u_\delta, w_1, \ldots, w_{n-\delta-1}\}$ such that $d(u_0) = \delta$, $N(u_0) = \{u_1, \ldots, u_\delta\}$, the vertices $u_1, \ldots, u_\delta$ are independent, the vertices $w_1, \ldots, w_{n-\delta-1}$ induce a clique and $u_i w_j \in E(G)$ for all $1 \leq i \leq \delta$ and $1 \leq j \leq \delta - 1$. Now, for $S = \{u_0, w_1, \ldots, w_{\delta-1}\}$ the number of connected components of the graph $\mathcal{F}_{n,\delta} - S$ equals $\delta + 1 > \delta = |S|$. This implies that $\mathcal{F}_{n,\delta}$ is not hamiltonian (it is not tough—see Subsection 3.6). Applying the Ainouche–Christofides 0-dual closure (see [5]) Schiermeyer and Woźniak [246] extended Theorem 3.45 in the following way.

THEOREM 3.47. *Let G be a 2-connected graph on n vertices with minimum degree δ. If there exists a vertex u with $d(u) = \delta$ such that any other vertex v with $d(v) < n/2$ is adjacent to u, then G is hamiltonian or $G \subset \mathcal{F}_{n,\delta}$.*

Note that Brandt and Veldman [74] investigated the graphs of order n satisfying the condition $d(x)+d(y) \geq n$ for every pair x, y of adjacent vertices. They proved that the circumference of such a graph G is $n - s(G)$, where $s(G) = \max\{0, \max_S(|S| - |N(S)| + 1\}$, where the inner maximum is taken over all nonempty sets S of independent vertices of G with $S \cup N(S) \neq V(G)$.

Denote by $\sigma_k(G)$ the minimum degree sum of an independent set of k ($k \geq 2$) vertices of G if $\alpha(G) \geq k$, and let $\sigma_k(G) = \infty$ otherwise. Ore's theorem states that if $\sigma_2(G) \geq n \geq 3$, where n is the order of G, then H is hamiltonian. Bondy [61] proposed the following conjecture that generalizes the Ore and Chvátal–Erdős theorems.

CONJECTURE 3.48. *Let G be a k-connected graph on n vertices with $\sigma_{k+1}(G) \geq n + k(k-1)$ and let C be a longest cycle in G. Then $G - C$ contains no path of length $k-1$.*

Observe that by Ore's theorem the conjecture is true for $k = 1$. For $k = 2$ it is a special case of the following theorem, which is a weaker form of this conjecture (see [61]).

THEOREM 3.49. *Let G be a k-connected graph on n vertices with $\sigma_{k+1}(G) \geq n + k(k-1)$ and let C be a longest cycle in G. Then $G - C$ contains no complete subgraph of order k.*

Bondy [61] also gave another strengthening of Ore's theorem.

THEOREM 3.50. *Let G be a k-connected graph on $n \geq 3$ vertices such that $\sigma_{k+1}(G) > \frac{1}{2}(k+1)(n-1)$. Then G is hamiltonian.*

Bauer et al. [22] generalized Dirac's theorem in the following way.

THEOREM 3.51. *Let G be a graph of order n and connectivity $\kappa \geq 2$ such that $\sigma_3(G) \geq n + \kappa$. Then G is hamiltonian.*

Theorem 3.51 is a consequence of the following result established by Bauer et al. [31].

THEOREM 3.52. *Let G be a 2-connected graph on n vertices with $\sigma_3(G) \geq n+2$. Then $c(G) \geq \min\{n, n+\sigma_3(G)/3 - \alpha(G)\}$.*

In [244] Schiermeyer proved that the 0-dual closure of a graph satisfying the hypothesis of Theorem 3.51 is complete. Fournier and Fraisse [144] deduced the following common generalization of the Chvátal–Erdős theorem and Theorem 3.18 by relating the k-connectivity and degree sums $\sigma_{k+1}(G)$.

THEOREM 3.53. *Let G be a k-connected graph, $k \geq 2$, such that $\sigma_{k+1}(G) \geq d$. Then G contains either a cycle of length at least $2d/(k+1)$ or a hamiltonian cycle.*

Another result involving a condition concerning three-element independent sets was discovered by Flandrin, Jung and Li [135].

THEOREM 3.54. *If G is a 2-connected graph of order n with $d(u)+d(v)+d(w) \geq n + |N(u) \cap N(v) \cap N(w)|$ for every set $\{u, v, w\}$ of three independent vertices, then G is hamiltonian.*

Recently a new concept has been used in the investigation of long cycles in graphs. Namely, if S is a set of vertices of G, then the degree of S is defined to be

$$\deg(S) = \Big|\bigcup_{v \in S} N(v)\Big|,$$

where $N(v)$ is the neighborhood of v.

Faudree et al. [127] were able to generalize Dirac's theorem by applying this notion to two-element sets.

THEOREM 3.55. *If G is a 2-connected graph of order $n \geq 3$ such that $\deg(S) \geq (2n-1)/3$ for each set $S = \{x, y\}$ of independent vertices, then G is hamiltonian.*

This result was extended in several ways. Using the 0-dual closure, Schiermeyer [244] obtained the following improvement.

THEOREM 3.56. *If G is a 2-connected graph of order $n \geq 3$ such that $\deg(S) \geq (2n-2)/3$ for each set $S = \{x, y\}$ of independent vertices, then $C_0^*(G)$ is complete.*

In [145] Fraisse considered k-connected graphs with larger independent sets of vertices and obtained the following strengthening of the previous two results.

THEOREM 3.57. *Let G be a k-connected graph of order n. Suppose there exists $t \leq k$ such that $\deg(S) \geq t(n-1)/(t+1)$ for every independent set S of t vertices. Then G is hamiltonian.*

Lindquester [200] localized a neighborhood union condition to the sets of two vertices at distance two and improved Theorem 3.55.

THEOREM 3.58. *If G is a 2-connected graph of order n such that $\deg(S) \geq (2n-1)/3$ for every set $S = \{x, y\}$ of vertices of distance 2 in G, then G is hamiltonian.*

Another result of this type was obtained by Song [256].

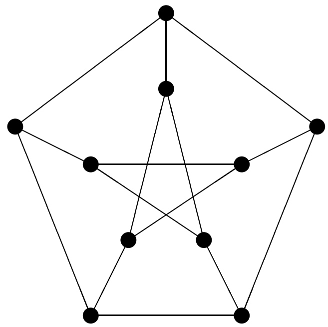

Fig. 4. Petersen graph

Theorem 3.59. *Let G be a 2-connected graph of order $n \geq 3$ with connectivity k. If there is an integer t such that $|N(u) \cup N(v)| \geq n - t$ for any different vertices u and v with $d(u, v) = 2$, and $\max\{d(u) \mid u \in S\} \geq t$ for any independent set S of cardinality $k + 1$, then G is hamiltonian.*

Note that Faudree et al. [128] investigated generalized degree conditions for graphs with bounded independence number and obtained some interesting results.

Another generalization of Theorem 3.55 was discovered by Chen [88]. He proved that if G is any graph of order n such that $2\deg(S) + d(x) + d(y) \geq 2n - 1$ for each set $S = \{x, y\}$ of nonadjacent vertices, then G is hamiltonian. Broersma et al. [78] showed that if G is a 2-connected graph and $|N(u) \cup N(v)| \geq n/2$ for every set $S = \{u, v\}$ of two nonadjacent vertices, then G is either hamiltonian, or the Petersen graph (see Fig. 4), or G is in one of three classes of exceptional graphs of connectivity 2. This result was extended by Liu and Wang [205] and Liu et al. [206]. A similar result for 3-connected graphs was obtained by Wei and Zhu [278].

If we assume that a graph G is k-regular and the order of G is at most $2k$, then, by Dirac's Theorem, the graph is hamiltonian. Jackson [177] showed that every 2-connected k-regular graph with $n \leq 3k$ is hamiltonian. This result has been extended in several papers. Here we give the strongest result due to Hilbig [171]. Denote by Π the Petersen graph and let Π^{Δ} be the graph obtained from Π by replacing one vertex by a triangle.

Theorem 3.60. *Let G be a 2-connected k-regular graph with at most $3k + 3$ vertices. Then G is hamiltonian if and only if $G \notin \{\Pi, \Pi^{\Delta}\}$.*

Jackson et al. [179] made the following conjecture for 3-connected graphs.

Conjecture 3.61. *For $k \geq 4$, every 3-connected k-regular graph on at most $4k$ vertices is hamiltonian.*

The strongest result related to this conjecture was found by Broersma et al. [77].

Theorem 3.62. *Every 3-connected k-regular graph on at most $\frac{7}{2}k - 7$ vertices is hamiltonian.*

For more detailed information we refer to an excellent survey by Broersma [76], where the author gives a description of a very useful proof technique based on a variation of Woodall's hopping lemma [281].

3.6. Hamiltonicity of tough graphs. In 1973 Chvátal [94] defined a noncomplete graph G to be *t-tough* ($t > 0$, t real) if for every vertex cut S of G, the number of components of $G - S$ is at most $|S|/t$. The complete graph is by definition *∞-tough*. The toughness measures not only "how connected" a graph G is but also "how tightly" the subgraphs of G are held together. Every hamiltonian graph is 1-tough because the number of components of $G - S$ is at most $|S|$ for every vertex cut S. Thus, if a graph G is hamiltonian, then G is both 1-tough and 2-connected. Clearly, both conditions are not sufficient for hamiltonicity. Marczyk and Skupień [218] described all nonhamiltonian 1-tough graphs of maximum size. In general, every t-tough graph is $\lceil 2t \rceil$-connected, but not necessarily conversely. In [94] Chvátal proposed the following conjecture.

Conjecture 3.63. *There exists t_0 such that every t_0-tough graph is hamiltonian.*

The following result of Enomoto et al. [108] suggests that the above conjecture may hold even for $t_0 = 2$.

Theorem 3.64.

(a) *If G is a 2-tough graph with at least three vertices, then G contains a 2-factor (a 2-regular spanning subgraph).*
(b) *For every $\epsilon > 0$ there exists a $(2 - \epsilon)$-tough graph without a 2-factor.*

Therefore, the following conjecture also attributed to Chvátal seemed to be reasonable.

Conjecture 3.65. *Every 2-tough graph is hamiltonian.*

This had been a longstanding and intriguing open problem. In a paper published in 2000 Bauer, Broersma and Veldman [26] constructed $(9/4 - \epsilon)$-tough graphs without a hamiltonian path for an arbitrary $\epsilon > 0$, thereby refuting the conjecture. Notice that since every 2-tough graph is 4-connected, the conjecture is true for planar graphs by a result of Tutte [270]. Moreover, the well-known result of Fleischner [141] implies the conjecture is valid for squares of 2-connected graphs. However, Conjecture 3.63 remains still open.

A graph G is *chordal* if every cycle of length at least four has a chord. Chvátal [94] showed that for an arbitrary $\epsilon > 0$ there exist $(3/2 - \epsilon)$-tough graphs without a 2-factor. The graphs in these examples are all chordal. In [26] Bauer et al. studied t-tough chordal graphs with t close to $7/4$.

Theorem 3.66. *For every $\epsilon > 0$ there exists a $(7/4 - \epsilon)$-tough chordal nontraceable graph.*

However, Chen et al. [90] provided a positive result.

Theorem 3.67. *Every 18-tough chordal graph is hamiltonian.*

Numerous authors observed that the bounds in the classical results involving degree sums can be lowered if we add an additional assumption on the toughness of G. For example, Jung [181] lowered the bound in Ore's theorem by assuming that a graph is one-tough.

Theorem 3.68. *Let G be a 1-tough graph of order $n \geq 11$ such that $\sigma_2(G) \geq n - 4$. Then G is hamiltonian.*

This result was improved by Skupień [255].

THEOREM 3.69. *If G is a 1-tough graph of order $n \geq 11$ such that*

$$\sigma_2(G) \geq \begin{cases} n-4 & \text{for } n = 12 \text{ and for each odd } n \geq 11, \\ n-5 & \text{for each even } n \geq 14, \end{cases}$$

then G is hamiltonian.

The next result due to Bauer and Schmeichel [33] is a long cycle version of Jung's theorem. It was conjectured by Ainouche and Christofides.

THEOREM 3.70. *Let G be a 1-tough graph of order $n \geq 3$. Then the circumference of G is at least* $\min\{n, \sigma_2(G) + 2\}$.

The *toughness of* G, denoted $\tau(G)$, is the maximum value of t for which G is t-tough. Bauer et al. [27] showed that the degree bound in Jung's theorem can be lowered if $\tau(G) > 1$.

THEOREM 3.71. *Let G be a graph of order $n \geq 30$ with $\tau(G) > 1$. If $\sigma_2(G) \geq n-7$, then G is hamiltonian.*

This theorem is best possible with respect to the bound on σ_2. In [43] Bigalke and Jung included the independence number into the degree bound and proved the following theorem.

THEOREM 3.72. *If G is a 1-tough graph with $n \geq 3$ and $\delta(G) \geq \max\{n/3, \alpha(G) - 1\}$, then G is hamiltonian.*

This theorem was further extended by Bauer et al. [35]. Bauer et al. [31] added the assumption that a graph is 1-tough and lowered the bound on σ_3 in Theorem 3.52.

THEOREM 3.73. *Let G be a 1-tough graph on $n \geq 3$ vertices such that $\sigma_3(G) \geq n$. Then $c(G) \geq \min\{n, n + \sigma_3(G)/3 - \alpha(G)\}$.*

Observe that if S is an independent maximum-size set in G, then $A = V(G) \setminus S$ is a vertex cut of G such that $|A| = n - \alpha(G) \geq \omega(G - A)\tau(G) = \alpha(G)\tau(G)$. So, if G is 2-tough, then $\alpha(G) \leq n/3$. Therefore, the next theorem due to Bauer et al. [31] is an immediate corollary of Theorem 3.73.

THEOREM 3.74. *Let G be a 2-tough graph on $n \geq 3$ vertices with $\sigma_3(G) \geq n$. Then G is hamiltonian.*

It is easy to show that Theorem 3.73 is not best possible. Li [192] proposed the following conjecture.

CONJECTURE 3.75. *Let G be a 1-tough graph on $n \geq 3$ vertices with $\sigma_3(G) \geq n$. Then $c(G) \geq \min\{n, (3n+1)/4 + \sigma_3(G)/6\}$.*

Notice that the condition $\sigma_3 \geq n$ in Theorem 3.73 does not imply that the graph is hamiltonian. An appropriate bound for hamiltonicity was discovered by Fassbender [120].

THEOREM 3.76. *Let G be a 1-tough graph on $n \geq 13$ vertices with $\sigma_3(G) \geq (3n - 14)/2$. Then G is hamiltonian.*

Another result involving the toughness and minimum degree was obtained by Bauer et al. [22].

THEOREM 3.77. *Let G be a graph of toughness t and minimum degree δ on $n \geq 3$ vertices. If $n < (t+1)\delta + t + 1$, then G is hamiltonian.*

It follows from this result that Chvátal's conjecture is true for every t_0 and every graph with minimum degree at least $n/(t_0 + 1)$.

Jung and Wittmann [182] proved a long-cycle analogue of the theorem. It generalizes Theorems 3.3 and 3.77.

THEOREM 3.78. *If G is a 2-connected graph on n vertices of minimum degree δ and toughness t, then the circumference $c(G)$ is at least* $\min\{(t+1)\delta + t, n\}$.

Bauer, Broersma and Veldman [25] involved the notion of toughness in Fan's condition and obtained a generalization of Fan's theorem. It is worth noting that Bauer et al. [29] showed that the problem of recognizing tough graphs is NP-hard. For surveys joining hamiltonian properties and toughness we recommend the articles by Bauer, Schmeichel and Veldman [34], Bauer, Broersma and Schmeichel [22, 24] and Broersma [76].

3.7. Forbidden subgraph conditions and claw-free graphs. Let $\{H_1, \ldots, H_k\}$ be a set of graphs. A graph G will be called $\{H_1, \ldots, H_k\}$-*free* if G has no induced subgraph isomorphic to any H_i, $1 \leq i \leq k$. We say that the graphs $H_1, \ldots, H_k$ are *forbidden* for a given property $\mathcal{P}$ if every $\{H_1, \ldots, H_k\}$-free graph has property $\mathcal{P}$. A graph G is *claw-free* if it is $K_{1,3}$-free, where $K_{1,3}$ is the complete bipartite graph whose sets of bipartition are of cardinalities 1 and 3, respectively (see Figure 5). Presenting a list of forbidden subgraphs is a new way of looking for sufficient conditions for hamiltonicity. In Figs. 5 and 6 we illustrate seven typical examples of such forbidden subgraphs.

The first result of this type is due to Goodman and Hedetniemi [146].

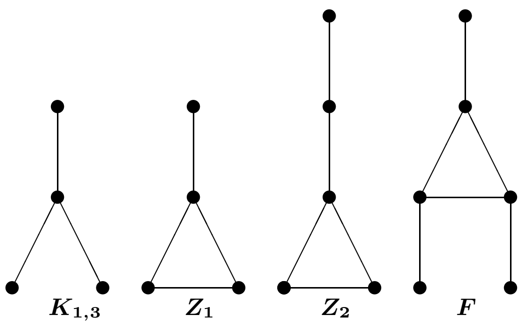

Fig. 5. Examples of forbidden subgraphs for hamiltonicity

THEOREM 3.79 (Goodman and Hedetniemi). *If G is a 2-connected, $\{K_{1,3}, Z_1\}$-free graph, then G is hamiltonian.*

This was strengthened by Duffus et al. [103].

THEOREM 3.80. *Let G be a $\{K_{1,3}, F\}$-free graph on $n \geq 3$ vertices. Then*

(i) *if G is connected, then G is traceable;*
(ii) *if G is 2-connected, then G is hamiltonian.*

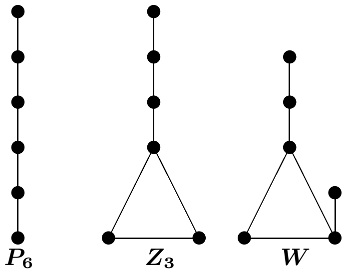

Fig. 6. P_6, Z_3 and W

Bedrossian [36] characterized the pairs (A, B) of connected graphs with the property that each 2-connected $\{A, B\}$-free graph G is hamiltonian. This result was extended by Faudree and Gould [124] (see also [150]) to a larger class of pairs of forbidden subgraphs.

Theorem 3.81. *Let A and B be connected graphs such that $A \neq P_3$, $B \neq P_3$ and G a 2-connected graph of order $n \geq 10$. Then G being $\{A, B\}$-free implies that G is hamiltonian if and only if $A = K_{1,3}$ and B is one of the graphs F, P_6, W, Z_2 or Z_3, or a connected induced subgraph of one of these graphs.*

It follows from this theorem that every forbidden pair contains the claw. This is not the case for forbidden triples. Faudree et al. [126] characterized those forbidden triples (X, Y, Z) of connected graphs, none of which is a generalized claw $K_{1,r}$, $r \geq 3$, for which each 2-connected $\{X, Y, Z\}$-free graph of sufficiently large order is hamiltonian. Brousek [75] considered all triples of forbidden connected graphs (X, Y, Z) which imply hamiltonicity and one of them is the claw. This result, together with the previous one, gives a full characterization of forbidden triples of subgraphs for hamiltonicity.

Note that the path P_3 (on three vertices) is the only nontrivial single graph which is forbidden for hamiltonicity (see [124]).

A vertex v in a graph G is *locally connected* if the subgraph of G induced by the neighborhood $N(v)$ of v is connected. A graph G is *locally connected* if every vertex x of G is locally connected. Relating these two notions, Oberly and Sumner [226] obtained the following result on claw-free graphs.

Theorem 3.82. *Every connected and locally connected claw-free graph of order $n \geq 3$ is hamiltonian.*

They also proposed the following conjecture.

Conjecture 3.83. *If G is a connected, locally k-connected, $K_{1,k+2}$-free graph on $n \geq 3$ vertices, then G is hamiltonian.*

In [81] Broersma and Veldman obtained generalizations to several results involving forbidden subgraph conditions.

It is a natural question whether the degree bounds in classical theorems on hamiltonicity can be lowered if we add the assumption that a graph is claw-free. The answer is positive in almost all cases. For instance, Zhang [286] investigated degree sums of $k + 1$ independent vertices in k-connected claw-free graphs.

THEOREM 3.84. *Let G be a k-connected claw-free graph on n vertices such that $k \geq 2$ and $\sigma_{k+1}(G) \geq n-k$. Then G is hamiltonian.*

Matthews and Sumner [221] studied claw-free graphs satisfying a Dirac-type condition. They obtained the following two results.

THEOREM 3.85. *If G is a 2-connected claw-free graph on $n \geq 3$ vertices such that $\delta(G) \geq (n-2)/3$, then G is hamiltonian.*

THEOREM 3.86. *If G is a connected claw-free graph on n vertices such that $\delta(G) \geq (n-2)/3$, then G is traceable.*

Liu and Wu [207] showed that one can lower the bound $(n-2)/3$ in Theorem 3.85 to $(n-1)/4$ if, additionally, G is regular, whereas Li [191] proved that we can reduce this bound to $n/4$ under the additional condition that G does not belong to a special family of graphs.

Claw-free graphs have been intensively studied during the last two decades and several sufficient conditions for a 2-connected claw-free graph have been found; for more information see the survey [123] by Faudree, Flandrin and Ryjáček. It is known that claw-freeness in a given graph can be tested in polynomial time.

For 3-connected graphs Li et al. [190] obtained the following result.

THEOREM 3.87. *Let G be a 3-connected claw-free graph on n vertices with $n \leq 6\delta(G)-7$. Then G is hamiltonian.*

Since the complete bipartite graph $K_{k,k+1}$ is k-connected and nonhamiltonian, there is no reasonable sufficient connectivity condition for a graph to be hamiltonian. However, the situation changes when we add the assumption that the graph is claw-free. Matthews and Sumner formulated then the following conjecture [220].

CONJECTURE 3.88. *If G is a 4-connected claw-free graph, then G is hamiltonian.*

Let G be a nonempty graph. The *line graph* $L(G)$ of G is a graph whose vertex set is in one-to-one correspondence with the set of edges of G, two vertices being adjacent in $L(G)$ if and only if the corresponding edges in G are adjacent. Since, by the well-known result due to Beineke [38], every line graph is claw-free, the following conjecture of Thomassen is weaker than the previous one.

CONJECTURE 3.89. *If G is a 4-connected line graph, then G is hamiltonian.*

Note that recognizing hamiltonian line graphs (and hamiltonian claw-free graphs) is an NP-complete problem (see [123]).

In studying the hamiltonicity of line graphs one often uses the notion of a dominating cycle. A cycle C in G is *dominating* if every vertex of $G-C$ has a neighbor on C. Harary and Nash-Williams [164] characterized hamiltonian line graphs in the following way.

THEOREM 3.90. *Let G be a connected graph of order $n \geq 3$. Then $L(G)$ is hamiltonian if and only if G has a dominating cycle or $G = K_{1,n-1}$.*

So Thomassen's conjecture is true when $G = L(H)$ and H is 4-connected.

In 1996 Ryjáček [238] introduced a new concept of closure that became a very useful tool in studying hamiltonicity of claw-free graphs.

Observe that if G is a claw-free graph and x any vertex of G then either x is locally connected or $\langle N(x)\rangle$ is the disjoint union of two complete graphs. Let x be a locally connected vertex with a noncomplete neighborhood and let G'_x be the graph obtained from G by adding to $\langle N(x)\rangle$ all missing edges (so in G'_x the neighborhood of x is a clique). G'_x will be called the *local completion of G at x*. The *Ryjáček closure $cl(G)$* of a claw-free graph G is the graph obtained from G by recursively repeating the local completion operation as long as possible. Thus, in $cl(G)$ the neighborhood of every vertex is a clique or the disjoint union of two cliques.

The following theorem is a basic result concerning Ryjáček's closure. Recall that $c(G)$ denotes the circumference of the graph G.

THEOREM 3.91. *Let G be a claw-free graph. Then*

(i) *the closure $cl(G)$ is uniquely determined,*
(ii) *there is a triangle-free graph H such that $cl(G) = L(H)$,*
(iii) $c(G) = c(cl(G))$.

The previous theorem implies immediately the following result of Ryjáček.

THEOREM 3.92. *Let G be a claw-free graph. Then G is hamiltonian if and only if $cl(G)$ is hamiltonian.*

COROLLARY 3.93. *The following statements are equivalent:*

(i) *every 4-connected claw-free graph is hamiltonian,*
(ii) *every 4-connected line graph is hamiltonian,*
(iii) *every 4-connected line graph of a triangle-free graph is hamiltonian.*

Proof. Clearly, (i) implies (ii) and (ii) implies (iii). Suppose (iii) holds and let G be a 4-connected claw-free graph. Then $cl(G)$ is also a 4-connected claw-free graph and, by Theorem 3.91, there is a triangle-free graph H such that $cl(G) = L(H)$. Therefore, $L(H)$ is a 4-connected line graph of a triangle-free graph and, by assumption, is hamiltonian. Thus $cl(G)$ is hamiltonian and by Theorem 3.92, G itself is hamiltonian. ∎

Thus, surprisingly, Conjectures 3.88 and 3.89 are equivalent.

Zhan [284] and, independently, Jackson [178], proved that every 7-connected line graph is hamiltonian. Therefore, if there exists a 7-connected nonhamiltonian claw-free graph G, then $cl(G)$ is a 7-connected nonhamiltonian line graph, which is impossible. Thus, the following theorem proved by Ryjáček [238] is true.

THEOREM 3.94. *Every 7-connected claw-free graph is hamiltonian.*

A more general result was obtained by Li [193].

THEOREM 3.95. *Every 6-connected claw-free graph with at most 33 vertices of degree 6 is hamiltonian.*

Note that another generalization of Theorem 3.94 was found by Fan [119].

Brandt [71] showed that if we raise the connectivity to 9, the claw-free graphs are hamiltonian connected.

Let us mention two interesting results on hamiltonicity of line graphs. The first one was obtained by Zhan [285].

Theorem 3.96. *Every line graph of a* 4*-edge connected graph is hamiltonian.*

The second one was proved by Kriesell [186].

Theorem 3.97. *Every* 4*-connected line graph of a claw-free graph is hamiltonian connected.*

Observe that if G is a connected and locally connected graph with at least three vertices, then $cl(G)$ is complete and, consequently, hamiltonian. By Theorem 3.92, also G is hamiltonian. Therefore, Theorem 3.82 of Oberly and Sumner is an easy corollary of Theorem 3.92.

Bollobás et al. [51] modified the notion of Ryjáček's closure by considering a local completion when a vertex was locally k-connected. Applying this new closure, denoted by $cl_k(G)$, they proved the following theorem.

Theorem 3.98. *Let G be a claw-free graph. Then*

(a) *$cl_k(G)$ is uniquely determined;*
(b) *G is hamiltonian connected if and only if $cl_3(G)$ is hamiltonian connected.*

For further information on claw-free graphs we recommend the survey paper [123] by Faudree, Flandrin and Ryjáček.

In [15] Asratian and Khachatrian introduced the following local Ore-type condition. For an integer $i \geq 0$, a graph G is an L_i-graph if $d(u) + d(v) \geq |N(u) \cup N(v) \cup N(w)| - i$ for each triple u, v, w of vertices with $d(u, v) = 2$ and $w \in N(u) \cap N(v)$. They were able to prove the following.

Theorem 3.99. *If G is a connected L_0-graph of order at least three, then G is hamiltonian.*

Clearly, the property of being in L_0 is weaker than that of satisfying the condition $\sigma_2(G) \geq |V(G)|$, thus this result is a generalization of Ore's theorem.

It is easy to verify that every claw-free graph is an L_1-graph. However, the converse is nor true. Define

$$\mathcal{K} = \{G \mid K_{p,p+1} \subseteq G \subseteq K_p \vee \overline{K}_{p+1}\}$$

for some $p \geq 2$. This family provides exceptions to several results on hamiltonian properties of L_i-graphs. Obviously, the members of this family are L_1-graphs but not claw-free.

Theorem 3.99 was further improved by Asratian et al. [14].

Theorem 3.100. *If G is a connected L_1-graph of order at least three such that $|N(u) \cap N(v)| \geq 2$ for each pair u, v of vertices with $d(u, v) = 2$, then G is hamiltonian.*

Li and Schelp [195] showed that Theorems 3.85 and 3.86 of Matthews and Sumner hold for the larger family of L_1-graphs.

Theorem 3.101. *Let G be a 2-connected L_1-graph of order n. If $\delta(G) \geq (n-2)/3$, then G is hamiltonian or $G \in \mathcal{K}$.*

Theorem 3.102. *Let G be a connected L_1-graph of order n. If $\delta(G) \geq (n-2)/3$, then G is traceable.*

In [240] Saito investigated hamiltonicity of L_1-graphs of diameter two.

Theorem 3.103. *Let G be a 2-connected L_1-graph of order n. Then either G is hamiltonian, or $G \in \mathcal{K}$.*

As a corollary we immediately get the following theorem by Gould [147].

Theorem 3.104. *Every 2-connected claw-free graph of diameter two is hamiltonian.*

Further information on L_i-graphs can be found in [195] and [13].

3.8. Hamiltonicity of powers of graphs. Let G be a graph and $k \geq 1$ an integer. The *kth power of G*, denoted by G^k, is the graph with the same set of vertices as G, two vertices being adjacent in G^k iff their distance in G is less than or equal to k. Denote by $\mathcal{G}_k$ the set of graphs which are the kth power of a graph. It was shown in [117] that there exists no characterization of $\mathcal{G}_k$ in terms of forbidden subgraphs. Bermond and Marczyk [41] compared the different classes $\mathcal{G}_p$ and $\mathcal{G}_k$ and proved that $\mathcal{G}_p \subset \mathcal{G}_k$ if and only if p is the multiple of k.

The investigation into hamiltonian properties of powers of graphs was initiated with a result of Sekanina [250] stating that the cube (i.e., G^3) of a connected graph is hamiltonian-connected. Sekanina also asked for graphs whose square is hamiltonian. Fleischner [141] gave a partial answer to this question proving his famous theorem.

Theorem 3.105. *The square of a 2-connected graph is hamiltonian.*

In [212] Marczyk gave a generalization of this result and showed that for every 2-connected graph G of circumference p there exists a hamiltonian cycle in G^2 that contains at least $\min\{\lceil p/2 \rceil + 4, p\}$ edges of G.

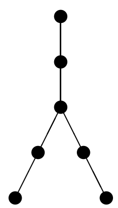

Fig. 7. $S(K_{1,3})$

Consider the subdivision graph $S(K_{1,3})$ of $K_{1,3}$ (see Figure 7). It is easily seen that the square of this graph is not hamiltonian, so we cannot lower the connectivity in Fleischner's theorem. Gould and Jacobson [151] proposed the conjecture that the square of any connected $S(K_{1,3})$-free graph is hamiltonian. This conjecture was proved by Hendry and Vogler [170]. Fleischner [142] also investigated other hamiltonian properties of squares of graphs.

Theorem 3.106. *For a connected graph G,*

(i) G^2 *is hamiltonian if and only if* G^2 *is vertex pancyclic, and*
(ii) G^2 *is hamiltonian-connected if and only if* G^2 *is panconnected.*

Further investigations into hamiltonian properties of powers of graphs can be found in [231].

3.9. Hamiltonicity of special classes of graphs. In this section we will present some examples of hamiltonicity problems in special classes of graphs. However, there is so large variety of results that we can mention here some directions of investigation only.

The hamiltonicity of planar graphs has been studied for a long time now. In 1969 Tutte [270] showed that every 4-connected planar graph is hamiltonian. Horton (see [66]) and Ellingham and Horton [106] constructed nonhamiltonian cubic 3-connected bipartite graphs. However, the following conjecture by Barnette (see [66]) still remains open.

Conjecture 3.107. *Every cubic* 3*-connected bipartite planar graph is hamiltonian.*

If $G = (V, E)$ and $H = (W, F)$ are two graphs, the *cartesian product* of G and H is defined to be a graph $G \times H$ whose vertex set is $V \times W$ and whose vertices (x, y) and (x', y') are adjacent if and only if either $x = x'$ and $yy' \in F$, or $y = y'$ and $xx' \in E$. The *lexicographic product* of G and H is defined to be a graph $G[H]$ whose vertex set is also $V \times W$ and whose vertices (x, y) and (x', y') are adjacent if and only if either $xx' \in E$ or $x = x'$ and $yy' \in F$. Notice that, in general, $G[H]$ is not isomorphic to $H[G]$.

Hamiltonian properties of different types of product of graphs have been studied by numerous authors: see for example [262], [263], [264] and [265]. An interesting class of graphs defined by the cartesian product are the *hypercubes* H_n, where $H_1 = K_2$ and $H_n = H_{n-1} \times K_2$. It follows from a result due to Aubert and Schneider [17] that the hypercube admits a hamiltonian decomposition.

A graph G is a *uniform subset graph* $G(n, k, t)$ if $V(G)$ is the set of all subsets of cardinality k of an n-set (i.e., a set with n elements) and two vertices of G are adjacent if and only if the corresponding k-subsets intersect in exactly t elements. This notion was introduced by Chen and Lih [87]. The special uniform subset graphs have been studied under different names. $G(2k - 1, k - 1, 0)$ are the odd graphs, $G(2n + k, n, 0)$ are called Kneser's graphs and $G(n, k, k-1)$ are Johnson's schemes $J(n, k)$. Chen and Lih proposed the following conjecture.

Conjecture 3.108. *The graph $G(n, k, t)$ is hamiltonian for any admissible triples (n, k, t) except $(5, 2, 0)$ and $(5, 3, 1)$.*

Chen and Lih [87], as well as Heinrich and Wallis [167] obtained some partial results relating to this conjecture.

There are a number of interesting results on hamiltonian properties of bipartite graphs. Obviously, the two sets of the bipartition of a hamiltonian bipartite graph have the same cardinality. Such a graph is called a *balanced bipartite graph*, and balanced bipartite graphs only will be considered in this section. Let us cite the bipartite version of Ore's theorem that was discovered by Moon and Moser [225] in 1963.

Theorem 3.109. *Let G be a balanced bipartite graph of order $2n$ with bipartition (X,Y). If $d(x)+d(y) \geq n+1$ for every pair of nonadjacent vertices $x \in X$ and $y \in Y$, then G is hamiltonian.*

Notice that if $\delta(G) \geq (n+1)/2$ or $|E(G)| \geq (n-1)^2+n+1$, then $\sigma_2(G) \geq n+1$. So the following two corollaries are true.

Corollary 3.110. *Let G be a balanced bipartite graph of order $2n$. If $\delta(G) \geq (n+1)/2$, then G is hamiltonian.*

Corollary 3.111. *If G is a balanced bipartite graph of order $2n$ with $|E(G)| \geq (n-1)^2+n+1$, then G is hamiltonian.*

We will see below that these three results are best possible. Let G and H be two bipartite graphs (not necessarily balanced) with bipartitions (X_1,Y_1) and (X_2,Y_2), resp. By $G \cup H$ we denote the disjoint union of G and H with bipartition $(X_1 \cup X_2, Y_1 \cup Y_2)$. The symbol $G+H$ stands for the graph obtained from $G \cup H$ by adding all the edges between X_1 and Y_2 and between X_2 and Y_1. Now consider the graph $G = \overline{K}_{2\lceil n/2\rceil} + K_{\lfloor n/2\rfloor,\lfloor n/2\rfloor}$. The minimum degree of G is $\lfloor n/2\rfloor$, but if we delete the vertices of one of the two sets of the bipartition of $K_{\lfloor n/2\rfloor,\lfloor n/2\rfloor}$, we get a graph with $\lceil n/2\rceil+1$ connected components. This means that G is not 1-tough, so it is not hamiltonian. It follows that Theorem 3.109 and Corollary 3.110 are best possible. The graph $\overline{K}_2 \cup K_{n-1,n-1}$ plus one edge between X_1 and Y_2 and all the edges between X_2 and Y_1 has $(n-1)^2+n$ edges and no hamiltonian cycle (it is not 2-connected). Hence, Corollary 3.111 is also best possible.

Bondy and Chvátal [63] introduced the notion of k-closure for bipartite graphs. Namely, let G be a balanced bipartite graph of order $2n$ with bipartition (X,Y). The *k-closure* BG_k of G with respect to the complete bipartite graph $K_{n,n}$ is the graph obtained from G by recursively joining with an edge every pair of non-adjacent vertices belonging, respectively, to the two classes of the bipartition and of degree sum at least k. It is easy to prove that the result does not depend on the order in which the edges are inserted. Obviously,

$$G = BG_{2\Delta+1} \subseteq \cdots \subseteq BG_{k+1} \subseteq BG_k \subseteq \cdots \subseteq BG_{2\delta} = K_{n,n}$$

for $0 \leq 2\delta \leq k \leq 2\Delta+1 \leq 2n+1$, and $BG_{2n-1} = G$.

Let P be a property defined on all the balanced bipartite graphs of order $2n$. The property P is *k-bistable* if the following implication holds: If G does not satisfy P, $x \in X$, $y \in Y$, $xy \notin E(G)$, $G+xy$ satisfies P, then $d(x)+d(y) < k$. In paper [63] by Bondy and Chvátal such a property is "k-stable relative to $K_{n,n}$". Clearly, every k-bistable property is $(k+1)$-bistable and every property is $(2n-1)$-bistable. The *bistability* $bs(P)$ of a property P is the smallest integer k for which P is k-bistable. Using the same idea as in the case of nonbipartite graphs, Bondy and Chvátal [63] proved the following two results.

Theorem 3.112. *If P is k-bistable, then BG_k has property P iff G has property P.*

Theorem 3.113. *The property of being hamiltonian is $(n+1)$-bistable.*

Proof. Suppose $G+uv$ is hamiltonian but G is not. Then G has a hamiltonian path $u=u_1,u_2,\ldots,u_{2n}=v$, where u and v belong to different classes of the bipartition, u can be adjacent to u_i's with even i only, while v can be adjacent to u_i's with odd i only. If $d_G(u) + d_G(v) \geq n + 1$, then there is an odd k such that u is adjacent to u_{k+1} and v is adjacent to u_k. Thus, G contains the hamiltonian cycle $u_1, u_{k+1}, u_{k+2}, \ldots, u_{2n}, u_k, u_{k-1}, \ldots, u_1$. ■

It follows at once that the $(n+1)$-biclosure of every balanced bipartite graph G such that $d(x) + d(y) \geq n + 1$ for each pair $x \in X$ and $y \in Y$ of nonadjacent vertices is isomorphic to the hamiltonian graph $K_{n,n}$. This proves Theorem 3.109. In [9] Amar et al. investigated the property: "G contains a C_{2s}".

THEOREM 3.114. *The property: "G contains a C_{2s}", $2 \leq s \leq n-1$, is $(2n-s+1)$-bistable.*

Proof. Suppose $G+xy$ contains a C_{2s}, but G does not. Then G contains a path $x_1, y_1, \ldots, x_s, y_s$ with $x_1 = x$ and $y_s = y$. The subgraph H induced in G by $\{x_i \mid 1 \leq i \leq s\} \cup \{y_i \mid 1 \leq i \leq s\}$ is not hamiltonian, but $H + xy$ is hamiltonian. By Theorem 3.113, $d_H(x) + d_H(y) < s + 1$ and thus $d_G(x) + d_G(y) < 2(n - s) + (s + 1) = 2n - s + 1$. ■

The graph $G(s) = K_{1,n-s} \cup K_{s-1,s-1} \cup K_{n-s,1}$ plus one edge between $K_{1,n-s} \cap X$ and $K_{s-1,s-1} \cap Y$ and all the edges between $K_{s-1,s-1} \cap X$ and $K_{n-s,1} \cap Y$ shows that the theorem is best possible. This result also implies that if G is a balanced bipartite graph such that $d_G(x) + d_G(y) \geq 2n - s + 1$ for every pair of nonadjacent vertices $x \in X$ and $y \in Y$, then G contains a C_{2s}. Surprisingly, this statement is not best possible—in the next section we will see that the bound $n + 1$ suffices for every s. This implies that if the number of edges in G is at least $(n - 1)^2 + n + 1$, then G contains a C_{2s} for every s between 2 and n. However, these results are probably not best possible. For example, determining how many edges in a bipartite graph guarantee the existence of a C_4 is a part of Zarankiewicz's problem (see [49]). It is known that if $|E(G)| > \frac{1}{2}(n + n\sqrt{4n-3})$, then G contains a C_4.

We recommend the paper by Amar et al. [9] on the biclosure and the bistability of bipartite graphs and application of these notions to the study of hamiltonian properties of such graphs.

By a balanced k-partite graph we mean the special case of a k-partite graph in which each partite set has the same number of vertices. Chen and Jacobson [89] obtained the following result.

THEOREM 3.115. *Let G be a balanced k-partite graph of order kn with $k \geq 2$. If for each pair of nonadjacent vertices u, v in different parts,*

$$d(u) + d(v) > \begin{cases} \left(k - \dfrac{2}{k+1}\right)n & \text{for } k \text{ odd,} \\ \left(k - \dfrac{4}{k+2}\right)n & \text{for } k \text{ even,} \end{cases}$$

then G is hamiltonian.

For more information on hamiltonicity of bipartite graphs, the reader is referred to [224], [268] and [287].

4. Pancyclic graphs

4.1. Metaconjecture of Bondy. Pancyclic graphs, those that contain cycles of all possible lengths, were introduced in 1971 by Bondy [55]. He noticed that sufficient conditions for hamiltonicity frequently imply pancyclicity and formulated his famous "metaconjecture " as follows:

> *Almost all nontrivial sufficient conditions for a graph to be hamiltonian also imply that it is pancyclic, except for maybe a simple family of graphs.*

This metaconjecture was at the origin of a new direction in studying the cycles in graphs. Firstly, all known sufficient conditions for a graph to be hamiltonian (especially in term of the vertex degrees) have been examined in respect of pancyclicity. Secondly, the metaconjecture initiated investigation on the structure of the set of cycle lengths in a hamiltonian graph satisfying some degree constraints.

The first fundamental result obtained in this direction is due to Bondy [55] and specifies how many edges are needed to guarantee the pancyclity in a hamiltonian graph.

Theorem 4.1. *Every hamiltonian graph of order n and size at least $n^2/4$ is either pancyclic or isomorphic to the complete bipartite graph $K_{n/2,n/2}$.*

Proof. The proof is by induction on n. Our assertion is evident for $n = 3$, so assume that it is true for every graph of order $n-1$, $n \geq 4$. Let G be a graph of order n that satisfies the hypothesis of the theorem.

Case 1: G contains a cycle of length $n-1$. Let $C := x_1, \ldots, x_{n-1}, x_1$ be such a cycle with the natural orientation and let x be the only vertex of $V(G) \setminus V(C)$. If $d(x) \leq (n-1)/2$, then

$$|E(G-x)| = |E(G)| - d(x) \geq \frac{n^2}{4} - \frac{n-1}{2} > \frac{(n-1)^2}{4}.$$

By the induction hypothesis, $G - x$ is pancyclic. Since G is hamiltonian, G is pancyclic too. If $d(x) > (n-1)/2$, there exists for each p, $3 \leq p \leq n$, an index i, such that both xx_i and xx_{i+p-2} are the edges of G. Then G contains the cycle $x, x_i, x_iCx_{i+p-2}, x_{i+p-2}, x$, of length p. Thus G is pancyclic in this case as well.

Case 2: G has no cycle of length $n-1$. Let $C := x_1, \ldots, x_n, x_1$ be a hamiltonian cycle in G. Then, for all i, j such that $1 \leq i, j \leq n$ and $j \neq i-1, i$, at most one of the pairs x_ix_j and $x_{i+1}x_{j+2}$ can be an edge of G, because otherwise G would have a cycle of length $n-1$. Therefore,

$$d(x_i) + d(x_{i+1}) \leq n.$$

Summing over all i, $1 \leq i \leq n$, we have

$$4|E(G)| = 2\sum_{i=1}^{n} d(x_i) \leq n^2,$$

whence $|E(G)| \leq n^2/4$.

Equality holds only for even n and, for all i, j such that $1 \leq i, j \leq n$ and $j \neq i-1, i$, exactly one of the pairs x_ix_j and $x_{i+1}x_{j+2}$ is an edge of G. It is easily seen that in this case G is isomorphic to the complete bipartite graph $K_{n/2,n/2}$. ■

This theorem yields at once the following improvement of Ore's condition for hamiltonicity that was discovered by Bondy [55].

THEOREM 4.2. *Let G be a graph of order n. If $d(x)+d(y) \geq n$ for each pair of nonadjacent vertices of G, then G is either pancyclic or isomorphic to the complete bipartite graph $K_{n/2,n/2}$.*

Proof. By Ore's theorem G is hamiltonian. Let $C := x_1, \dots, x_n, x_1$ be a hamiltonian cycle in G. Suppose G is not pancyclic. Therefore, there exists an integer p such that G contains no cycle of length p. It follows immediately that the vertices x_i and x_{i+p-1} are not adjacent ($i = 1, \dots, n$). By assumption, for each $i = 1, \dots, n$,

$$d(x_i) + d(x_{i+p-1}) \geq n.$$

Summing over all i, $1 \leq i \leq n$, we obtain

$$4|E(G)| = 2\sum_{i=1}^{n} d(x_i) \geq n^2,$$

whence $|E(G)| \geq n^2/4$. Since G is not pancyclic, it follows from Theorem 4.1 that G is isomorphic to the complete bipartite graph $K_{n/2,n/2}$. ∎

This result implies the following strengthening of Dirac's theorem.

COROLLARY 4.3. *If G has $n \geq 3$ vertices and $\delta(G) \geq n/2$, then G is either pancyclic or isomorphic to the complete bipartite graph $K_{n/2,n/2}$.*

In [6] Aldred et al. relaxed Ore's condition and considered graphs G with $\sigma_2(G) \geq n-1$.

THEOREM 4.4. *If G satisfies $\sigma_2(G) \geq n-1$, then G is pancyclic, unless G is isomorphic to one of the following graphs:*

- *a graph of order n consisting of two complete graphs sharing exactly one common vertex,*
- *a subgraph of the join of a complete graph of order $(n-1)/2$ and an empty graph of order $(n+1)/2$,*
- *$K_{n/2,n/2}$,*
- *C_5.*

Note that most of the results on pancyclicity are proved by starting with a hamiltonian cycle and by considering two consecutive vertices on the hamiltonian cycle. The first result of this type is due to Bondy [55].

PROPOSITION 4.5. *Let G be a hamiltonian graph of order n with a hamiltonian cycle $x_1, \dots, x_n, x_1$ such that $d(x_1) + d(x_n) \geq n+1$. Then G is pancyclic.*

Proof. Suppose that G contains no cycle of length p and define $A = N(x_1) \cap x_2Cx_{n-1}$, $B = N(x_n) \cap x_2Cx_{n-1}$ and

$$f(x_j) = \begin{cases} x_{j+p-3} & \text{if } 2 \leq j \leq n-p+2, \\ x_{p-n+j-1} & \text{if } n-p+2 < j \leq n-1. \end{cases}$$

If $x_j \in A$, then $f(x_j) \notin B$, because otherwise G would have a C_p passing through the edges x_1x_j and $x_nx_{f(x_j)}$. Obviously, f is a bijection from $C[x_2, x_{n-1}]$ onto $C[x_2, x_{n-1}]$ such that $f(A) \cap B = \emptyset$. Thus, $d(x_1) + d(x_n) = |A| + |B| + 2 = |f(A)| + |B| + 2 \leq n - 2 + 2 = n$, and we get a contradiction. ∎

A number of sufficient conditions for pancyclic graphs can be derived from the following extension of Proposition 4.5 by Schmeichel and Hakimi [248].

THEOREM 4.6 (Schmeichel–Hakimi [248]). *If G is a hamiltonian graph of order $n \geq 3$ with a hamiltonian cycle $x_1, \ldots, x_n, x_1$ such that $d(x_1) + d(x_n) \geq n$, then G is either*

- *pancyclic,*
- *bipartite, or*
- *missing only an $(n-1)$-cycle.*

Moreover, in the last case we have $d(x_{n-2}), d(x_{n-1}), d(x_2), d(x_3) < n/2$.

There are many others interesting results on pancyclic graphs that confirm Bondy's metaconjecture. All these results involve a minimum degree condition or another global condition for graph to be pancyclic. We below cite several results of importance. They generalize theorems of Chvátal (Theorem 3.13), Fan (Theorem 3.38), and Bondy (see [60]), respectively. The first and the third results are due to Schmeichel and Hakimi [247] and the second one to Benhocine and Wojda [39].

THEOREM 4.7. *Let G be a graph on $n \geq 3$ vertices with degree sequence $d_1 \leq \cdots \leq d_n$. If $d_{n-k} \geq n - k$ for all k with $d_k \leq k < n/2$, then G is pancyclic or bipartite.*

THEOREM 4.8. *Let G be a 2-connected graph of order n such that* $\max\{d(x), d(y)\} \geq n/2$ *for every pair of vertices x, y with $d(x, y) = 2$. Then G is either pancyclic, $K_{n/2,n/2}$, $K_{n/2,n/2} - e$, or the graph shown in Fig. 8.*

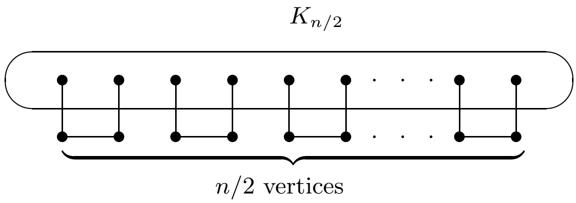

Fig. 8. An exceptional graph of Theorem 4.8

THEOREM 4.9. *Let G be a 2-connected graph on n vertices. If $\sigma_3(G) \geq \frac{3}{2}n - 1$, then G is either pancyclic, $K_{n/2,n/2}$, $K_{n/2,n/2} - e$, or C_5.*

Using Theorem 4.6, Bauer and Schmeichel [32] gave straightforward proofs of Theorems 4.7–4.9. Note that another generalization of Theorem 4.7 was found by Stacho [258].

Mitchem and Schmeichel [223] observed that in order to attain pancyclicity, the bounds appearing in the theorems on hamiltonicity might be lowered if we add the assumption that the graph is hamiltonian. As an example, we here give an improvement of Theorem 4.1 due to Faudree, Häggkvist and Schelp [130].

Theorem 4.10. *Every hamiltonian graph G of order n and size $e(G) > \lfloor (n-1)^2/4 \rfloor + 1$ is pancyclic or bipartite.*

Another example of this type is a result of Shi [252].

Theorem 4.11. *If G is a hamiltonian graph of order $n \geq 40$, and if $xy \notin E(G)$ implies $d(x) + d(y) > 4n/5$, then G is either pancyclic or bipartite.*

The lexicographic product $C_5[\bar{K_k}]$ is a hamiltonian, triangle-free $2k$-regular graph such that the degree sum of any two nonadjacent vertices equals $4n/5$. It shows that the theorem is best possible.

In [11] Amar et al. substantially improved Corollary 4.3.

Theorem 4.12. *Let G be a nonbipartite hamiltonian graph on $n \geq 102$ vertices such that $\delta(G) > 2n/5$. Then G is pancyclic.*

Over the years, the global conditions have been replaced by local ones that ensure the existence of cycles belonging to a specific interval of integers. For example, Zhang [287] extended Theorem 4.2 in the following way.

Theorem 4.13. *Let G be a hamiltonian graph on n vertices. If there is a vertex x such that $d(x) + d(y) \geq n$ for each y not adjacent to x, then G is either pancyclic or $K_{n/2,n/2}$.*

Gu, Song and Xu [153] investigated pancyclicity of graphs with a relaxation of the condition $\sigma_3 \geq \frac{3}{2}n - 1$ and obtained an extension of Theorem 4.9.

Theorem 4.14. *Let G be a 2-connected graph of order n. If for any three independent vertices in G there exist two vertices such that the sum of their degrees is at least n, then G is either pancyclic, or $G \cong K_{n/2,n/2}$, or $G \cong K_{n/2,n/2} - e$, or G is a cycle of length 5.*

Finally, we present a theorem of Faudree et al. [121] giving a relation between the Bondy–Chvátal closure and the pancyclism.

Theorem 4.15. *Let G be a graph of order n. If Cl_{n+1} is complete, then G is pancyclic.*

4.2. Weakly pancyclic graphs. Recall that the circumference of a graph G, denoted by $c(G)$, is the length of a longest cycle in G, and the girth of G, denoted $g(G)$, is the length of a shortest cycle in G. We say that a graph is *weakly pancyclic* if it contains cycles of every length between $g(G)$ and $c(G)$. Thus this generalizes the concept of pancyclic graphs. Throughout this section we will see that sufficient conditions for a graph to be weakly pancyclic are as a rule weaker than those for a graph to be pancyclic. Brandt, Faudree and Goddart [73] obtained the following sufficient degree condition for a nonbipartite graph to be weakly pancyclic.

Theorem 4.16. *Let G be a 2-connected nonbipartite graph of order n with $\delta(G) \geq n/4 + 250$. Then G is weakly pancyclic, unless G has odd girth 7, in which case it contains every cycle of length between 4 and $c(G)$ except C_5.*

It is easy to show that a shortest odd cycle in a graph G with $\delta(G) > 2n/7$ has length at most 5. Thus the next corollary is true.

Corollary 4.17. *If G is a 2-connected nonbipartite graph of sufficiently large order n with $\delta(G) > 2n/7$, then G is weakly pancyclic.*

If we add the requirement that the graph possesses a triangle and a hamiltonian cycle, we may conclude that the graph is pancyclic.

Corollary 4.18. *Let G be a 2-connected nonbipartite graph of order n with $\delta(G) \geq n/4 + 250$. If G contains a triangle and a hamiltonian cycle then G is pancyclic.*

Note that the bound $n/4 + 250$ is generally lower than the minimum degree for a nonbipartite graph to be pancyclic ($\delta(G) > 2n/5$, see Theorem 4.12 by Amar et al. [11]). Theorem 4.16 is best possible (up to a constant). Indeed, consider the graph G obtained from two copies of a $K_{m,m}$ that intersect at one vertex. Join one vertex on the opposite side of the intersection vertex in one $K_{m,m}$ to such a vertex in the other $K_{m,m}$. This graph is hamiltonian with $\delta(G) = m = (n+1)/4$ and contains a triangle, but it contains no even cycle of length larger than $(n+1)/2$. Corollary 4.17 is also best possible, because the lexicographic product $C_7[\bar{K}_r]$ for $r \geq 2$ has minimum degree $2n/7$, it contains C_4 and cycles of every length between 6 and $7r$ but it does not contain a C_5.

In [68] Brandt improved Theorem 4.16 for graphs of low order.

Theorem 4.19. *Every nonbipartite graph G of order n with $\delta(G) \geq (n+2)/3$ is weakly pancyclic (of girth 3 or 4).*

Clearly, Theorem 4.16 implies Theorem 4.19 for 2-connected graphs of a sufficiently large order. For triangle-free graphs with minimum degree greater than $n/3$ and a given independence number, Brandt [70] was able to prove what follows.

Theorem 4.20. *Let G be a nonbipartite triangle-free graph of order n. If $\delta(G) > n/3$, then G is weakly pancyclic with girth 4 and circumference $\min\{2(n - \alpha(G)), n\}$, unless G is a cycle on five vertices.*

This theorem is best possible (see [70]). Now we will give several corollaries of Theorems 4.16 and 4.19.

Corollary 4.21. *Let G be a 2-connected graph of order n. If $\delta(G) \geq \max\{\alpha(G), (n+2)/3\}$, then G contains cycles of every length between 4 and n, or $n = 2r$ and $G \cong K_{r,r}$.*

It follows from a result of Nash-Williams that the hypothesis of the corollary implies that the graph is hamiltonian. Observe that if G is bipartite with $\delta(G) \geq \alpha(G)$, then G is complete and balanced, so the corollary follows from Theorem 4.19. Since every graph G with $\delta(G) \geq \alpha(G) + 1$ is not bipartite and has a triangle, it follows from this corollary that every 2-connected graph G of order n such that $\delta(G) \geq \max\{\alpha(G) + 1, (n+2)/3\}$ is pancyclic.

A proof of the next corollary is presented in [73].

Corollary 4.22. *Let G be a graph of order n with connectivity $\kappa(G) \geq 2$. If $\delta(G) \geq (\kappa(G) + n)/3$, then G is pancyclic, or $n = 2r$ and $G = K_{r,r}$.*

Applying a result due to Broersma et al. [77] and Theorem 4.16, one can prove the following corollary for regular graphs.

Corollary 4.23. *Every 2-connected d-regular graph of a sufficiently large order n with $d \geq 2(n+7)/7$ is pancyclic, unless it is triangle-free or it contains three vertices which do not lie on a common cycle.*

In [69], Brandt generalized Theorem 4.10 and showed that the same size without the hamiltonicity requirement is sufficient for a graph to be weakly pancyclic with girth 3.

Theorem 4.24. *Every nonbipartite graph G of order n with more than $(n-1)^2/4+1$ edges contains cycles of every length l, where $3 \leq l \leq c(G)$.*

He also proposed the following conjecture.

Conjecture 4.25. *Every nonbipartite graph G of order n with more than $(n-1)(n-3)/4+4$ edges is weakly pancyclic.*

If true, this bound is best possible (for details see [69]). Bollobás and Thomason [53] obtained a result that came very closely to proving this conjecture.

Theorem 4.26. *Every nonbipartite graph G of order n and size at least $\lfloor n^2/4 \rfloor - n + 59$ contains cycles of every length l, where $4 \leq l \leq c(G)$.*

In [94], Chvátal conjectured that there is a constant t_1 such that every t_1-tough graph is pancyclic. Surprisingly, this conjecture is false, as shown by Bauer, van den Heuvel and Schmeichel [30].

Theorem 4.27. *For every t_1 there exists a t_1-tough triangle-free graph.*

In light of this result, one might conjecture the existence of a constant t_2 such that every t_2-tough graph is weakly pancyclic. However, the following result of Alon [7] implies that this conjecture is false.

Theorem 4.28. *For every t and g there exists a t-tough graph of girth strictly greater than g.*

Now take a t-tough graph with girth at least six and add to it one edge closing a triangle. The graph obtained in this way is still t-tough, contains a triangle, but no cycles of length four, thus it is not weakly pancyclic.

Note that Brandt [68] disproved this conjecture independently. Taking into account these results, Bauer et al. [30] conjectured the following.

Conjecture 4.29. *If G is a t-tough graph of order n and $\delta(G) > n/(t+1)$, then G is pancyclic.*

Bollobás [47] extended Theorem 4.1 and showed that every graph of order n, circumference $c(G)$ and size $e(G) > c(G)(2n - c(G))/4$ is weakly pancyclic (with girth 3).

Another question related to weakly pancyclic graphs was raised by Erdős. He asked how large the girth of a hamiltonian weakly pancyclic graph can be. In 1997 Bollobás and Thomason [53] proved that the girth of any hamiltonian graph G on n vertices containing also an $(n-1)$-cycle has an upper bound $2\sqrt{n}-1$. This immediately implies the following theorem.

Theorem 4.30. *Let G be a hamiltonian weakly pancyclic graph on n vertices. Then $g(G) \leq 2\sqrt{n}-1$.*

4.3. Chvátal–Erdős-type conditions. Complete bipartite graphs $K_{k,k}$ show that the classical Chvátal–Erdős condition "$\alpha \le \kappa$" (cf. [95]) does not ensure the existence of cycles of all possible lengths in a graph. The first result on pancyclicity of graphs satisfying this condition is due to Amar et al. [12]. It concerns graphs with a low independence number.

THEOREM 4.31. *Let G be a k-connected graph of order n and stability $\alpha \le k$.*

(1) *If $G \ne K_{k,k}$ and $G \ne C_5$, then G has a C_{n-1}.*
(2) *If $G \ne C_5$, then the girth of G is at most 4.*
(3) *If $G \ne C_5$, $G \ne C_4$ and $\alpha = 2$, then G is pancyclic.*
(4) *If $G \ne K_{3,3}$ and $\alpha = 3$, the G has cycles of all lengths between 4 and n.*

There is a large family of triangle-free graphs (see for example survey [73]) that satisfy the Chvátal–Erdős condition ($\alpha \le \kappa$) that are not pancyclic. This family contains the complete bipartite graphs as well as the Andrásfai graphs $G_i = \overline{C^i_{3i+2}}$, $i \ge 1$, i.e., each G_i is the complement of the ith power of the cycle C_{3i+2}. Thus $G_1 = C_5$ and G_8 is a cycle on 8 vertices with the longest chords. The lexicographic product $G_i[\bar{K}_s]$ $(s \ge 1)$ is a triangle-free $r = s(i+1)$-regular graph with stability number $\alpha = r$ and connectivity r, so it also satisfies the Chvátal–Erdős condition and is not pancyclic.

Statement (4) of Theorem 4.31 was improved by Chakroun and Sotteau [86] to read as follows.

THEOREM 4.32. *Suppose G is a 3-connected graph, $G \ne K_{3,3}$ and $G \ne G_8$. If $\alpha(G) \le 3$, then G is pancyclic.*

Thus we may formulate the immediate consequence of Theorems 4.31 and 4.32.

COROLLARY 4.33. *Let G be a k-connected graph with stability number $\alpha \le 3$. If $\alpha < k$, then G is pancyclic.*

In the light of the above results Amar et al. [12] proposed the following two conjectures.

CONJECTURE 4.34. *If a triangle-free graph G satisfies $\alpha(G) \le \kappa(G)$, then G has cycles of all length between 4 and the order of G, unless $G = K_{r,r}$ or $G = C_5$.*

CONJECTURE 4.35. *Let G be a graph of order n with $\alpha(G) \le \kappa(G)$. If G is not bipartite and $G \ne C_5$, then G has cycles of every length between 4 and n.*

Conjecture 4.34 was proved by Lou [208] and this result was generalized in many ways. Conjecture 4.35 seems to be still open.

Note that if a k-connected graph G satisfies $\alpha(G) < k$, then the neighborhood of any vertex of G is not a stable set, so G contains a C_3. Therefore, the following conjecture by Jackson and Ordaz [180] is an easy consequence of Conjecture 4.35.

CONJECTURE 4.36. *Let G be a k-connected graph with stability number α. If $\alpha < k$, then G is pancyclic.*

By Corollary 4.33, the above is true for $\alpha \le 3$. Note that the claim in [180] that Conjecture 4.34 implies Conjecture 4.36 is false.

In what follows we will need some elements of Ramsey theory (see for details [49]). The classical Ramsey theorem [235] can be formulated in the following way:

THEOREM 4.37. *For every pair $k, m \geq 2$ of integers there exists an integer $r(k, m)$ such that each graph of order $n \geq r(k, m)$ contains either a clique on k vertices, or a stable set of cardinality m.*

The Ramsey number $R(k, m)$ is defined to be the smallest number $r(k, m)$ with this property.

Very few of the nontrivial Ramsey numbers are known. There are several results giving a lower or an upper bound for these numbers. For example, assuming $3 \leq k \leq m$, we have (cf. [49])

$$\frac{k2^{k/2}}{e\sqrt{2}} < R(k,k) \leq R(k,m) \leq R(m,m) \leq \binom{2m-2}{m-1} \leq \frac{2^{2m-2}}{\sqrt{m}}.$$

Let H_1 and H_2 be arbitrary graphs of orders $k \geq 2$ and $m \geq 2$, resp. If $n \geq R(k, m)$ and G is a graph of order n, then, by Theorem 4.37, either G contains a subgraph ismorphic to H_1 or its complement $\overline{G}$ contains a subgraph isomorphic to H_2. Let $R(H_1, H_2)$ be the smallest value of n that ensures this property. The numbers $R(H_1, H_2)$ are called *generalized Ramsey numbers* or *graphical Ramsey numbers.* In [64], Bondy and Erdős studied the values of $R(C_m, K_p)$, called cycle-complete graph Ramsey numbers. They proved that these numbers are bounded from above by a polynomial depending on m and p.

THEOREM 4.38. *$R(C_m, K_p) \leq mp^2$ for each $m \geq 3$ and $p \geq 2$.*

The same authors gave a formula for exact values of infinitely many nontrivial cycle-complete graph Ramsey numbers.

THEOREM 4.39. *$R(C_m, K_p) = (p-1)(m-1)+1$ for each $m \geq p^2 - 2$ and $p \geq 2$.*

The most beautiful result related to both Bondy's "metaconjecture" and Jackson—Ordaz conjecture is due to Erdős [111]. Applying properties of cycle-complete graph Ramsey numbers presented above, he proved the following theorem conjectured by I. Zarins. Note that the original proof by Erdős has a small gap. Below we present a complete proof of this theorem (cf. [136]).

THEOREM 4.40. *Every hamiltonian graph with the independence number at most $p-1 \geq 2$ and the order greater than $4p^4$ is pancyclic.*

Proof. Let G be a graph of order n that satisfies the hypothesis of the theorem. Suppose that m is an integer such that $p^2 - 2 \leq m \leq n/p$ (by assumption, $p^2 - 2 < n/p$). Then $n \geq (m-1)(p-1)+1$ and, since G does not contain an independent set of cardinality p, it follows by Theorem 4.39 that G has a cycle of length m. Thus, it suffices to consider two cases.

CASE 1: $3 \leq m \leq p^2 - 3$. By Theorem 4.38 due to Bondy and Erdős, $R(C_m, K_p) \leq mp^2$ for all $m \geq 3$ and $p \geq 3$, so if $m \leq p^2 - 3$, then $mp^2 \leq p^4 - 3p^2 < 4p^4 < n$, so if the independence number is at most $p-1$, a C_m exists in G.

CASE 2: $n/p < m \leq n$. Since by assumption G is hamiltonian, it suffices to prove that if G contains a C_m, then it also contains a C_{m-1}. Let $C = x_1, \ldots, x_m, x_1$ be our cycle of length m. Obviously, $m > n/p > 4p^3$. Observe that for every i, the set $\{x_i, x_{i+2}, \ldots, x_{i+2(p-1)}\}$

of p vertices cannot be independent. Thus, the graph H induced by the set $V(C)$ contains at least $2p^2$ edges of the form

$$x_{i_r}x_{j_r},\ 2p > j_r - i_r \geq 2,\ 1 \leq i_1 < j_1 < \cdots < i_s < j_s \leq m,\ s \geq 2p^2. \tag{6}$$

In fact, we can assume $j_r - i_r > 2$, for otherwise we already have our cycle C_{m-1}. Moreover, we can assume that no two vertices x_u, x_v with

$$i_r \leq u < v \leq j_r,\ \{x_u x_v\} \neq \{x_{i_r}x_{j_r}\},\ v-u > 1 \tag{7}$$

are adjacent (otherwise we could replace $x_{i_r}x_{j_r}$ by $x_u x_v$).

An edge $x_{i_r}x_{j_r}$ is good if for every u, $i_r < u < j_r$, the degree in H of x_u is at least $p+2$. We claim that there is at least one good edge in H.

Indeed, if H has no good edge, it contains a set S of at least $2p^2$ vertices of degree at most $p+1$. It is well known that the independence number of the graph induced by S is at least $|S|/(\Delta_S+1)$, where Δ_S denote the maximum degree of this graph. But $|S|/(\Delta_S+1) \geq 2p^2/(p+2) \geq p$, a contradiction. Therefore, H contains a good edge.

Assume now that the edge $x_{i_r}x_{j_r}$ is good and label the vertices of C so that

$$y_1 = x_{i_r+1},\ y_2 = x_{i_r+2},\ \ldots,\ y_t = x_{j_r},\ \ldots,\ y_m = x_{i_r}, \qquad t = j_r - i_r.$$

Let C_l, $l < m$, be the longest cycle in H which contains all the vertices y_v with $t \leq v \leq m$ and perhaps some of the vertices y_v, $1 \leq v < t$. If $l = m-1$ our proof is finished (we have our C_{m-1}). Thus assume $l < m-1$ and this assumption will lead to a contradiction. Since the edge $y_m y_t = x_{i_r}x_{j_r}$ is good we have

$$d_H(y_v) \geq p+2, \qquad 1 \leq v < t. \tag{8}$$

Let y_u be a vertex of C which is not a vertex of C_l. By (7), y_u is joined only to y_{u-1} and y_{u+1} in the subgraph induced by the set $\{y_m, y_1, \ldots, y_t\}$. Thus, by (8), y_u is joined to at least p vertices of C_l. Orient now C_l in any way and let $A = N_{C_l}(y_u)$ be the set of neighbors of y_u on C_l. Note that no two neighbors of y_u are consecutive on C_l, for otherwise H would have a cycle of length $l+1$ containing the set $\{y_t, \ldots, y_m\}$. If x and y belong to A, then $x^+y^+ \notin E(G)$, because otherwise $x^+, y^+, y^{++}, \ldots, x^-, x, y_u, y, y^-, \ldots, x^+$ would be a cycle of length $l+1$ containing the set $\{y_t, \ldots, y_m\}$, contradicting our assumption. Thus the set A^+ is independent and contains p vertices and this contradiction proves that $l = m-1$. This finishes the proof. ∎

It immediately follows that every graph G of order $n > 4(\alpha(G)+1)^4$, which satisfies the Chvátal–Erdős condition $\alpha(G) \leq \kappa(G)$, is pancyclic. In his paper Erdős conjectured that the conclusion of Theorem 4.40 holds if we replace the bound $4p^4$ by Cp^2, where C is a constant (sufficiently large). He also wrote that a simple example showed that it certainly failed for $n < p^2/4$, but did not present it in the article. Consider now another example. Take $p-1$ disjoint copies $A_1, \ldots, A_{p-1}$ of the complete graph K_{2p-4}, where $p \geq 3$. Choose two vertices x_i, y_i in each copy A_i and add $p-1$ independent edges $x_i y_{i+1}$ (indices are taken modulo $p-1$). It is easily seen that the stability number of this hamiltonian graph is $p-1$ and there exist cycles C_m for every m except $m = 2p-3$, therefore, we cannot lower the bound of the theorem of Erdős below the number $(p-1)(2p-4) = 2p^2 - 6p + 4$.

However, for graphs satisfying the Chvátal–Erdős condition perhaps the following is true: there exist two constants c and C, $c < 2$, such that every graph G with $\alpha(G) = \alpha \leq \kappa(G)$ and $|V(G)| > C\alpha^c$ is pancyclic (see [72], [110]). The graphs $G_i[\bar{K_s}]$ show that such a constant c must be at least one.

A similar theorem, but weaker than that of Erdős, is an easy consequence of a more general result due to Flandrin et al. [140] concerning pancyclability of a set of vertices which is presented in Subsection 5.3. This result reads as follows:

THEOREM 4.41. *Let G be k-connected graph with stability number α. If $\alpha \leq k$ and the order of G is at least $2R(4\alpha, \alpha + 1)$, then G is pancyclic.*

Now we will show that the condition given in Conjecture 4.36 implies the existence of a C_5 (cf. [217]).

THEOREM 4.42. *Let G be a graph with connectivity $k \geq 5$ and independence number α. If $\alpha \leq k - 1$, then any vertex of G belongs to a cycle C_5.*

Proof. Let $x \in V(G)$ and $N(x) = \{y_1, \ldots, y_p\}$ be the neighborhood of x. Obviously, $p \geq k$. If $N[x] = N(x) \cup \{x\}$ does not contain a C_5 then the graph $\langle N(x) \rangle$ induced by $N(x)$ does not contain a P_4 and is a union of triangles, stars and isolated vertices.

For every y_i, $1 \leq i \leq p$, consider the set S_i of neighbors of y_i at distance two of x. Observe that $|S_i| \geq k - 1 - d_{N(x)}(y_i)$ for $i = 1, \ldots, p$. Suppose that $z_i \in S_i$ and $z_j \in S_j$ $(i \neq j)$ and note that if z_i and z_j are adjacent, then we have a C_5: x, y_i, z_i, z_j, y_j, x. It is then sufficient to find distinct vertices $z_1 \in S_1, \ldots, z_{k-1} \in S_{k-1}$ to get a C_5, for there is at least one edge in the previous set (otherwise $\{x, z_1, \ldots, z_{k-1}\}$ would be an independent set of cardinality k). Note that this is possible if we have $|S_1| \geq k - 1$, $|S_2| \geq k - 2$, …, $|S_{k-1}| \geq 1$.

Suppose that $\langle N(x) \rangle$ contains a triangle, say y_1, y_2, y_3. Observe that a common neighbor $z \in S_i \cap S_j$, $i = 1, 2, 3$, $j \neq i$, would give a C_5: x, y_j, z, y_i, y_w, x, where $w \in \{1, 2, 3\}$, $w \neq i$. So we may suppose that S_1, S_2, S_3 are mutually disjoint and all disjoint from S_j, $j \geq 4$. Let us index the y_j, $j \geq 4$, so that the cardinalities of the S_j are decreasing. Then, because $\langle N(x) \rangle$ is a union of triangles, stars and isolated vertices, we have $|S_4| \geq k - 3$, $|S_5| \geq k - 4$, …, $|S_{k-1}| \geq 2$ and we may find distinct vertices $z_j \in S_j$, $4 \leq j \leq k - 1$. Any three vertices $z_i \in S_i$, $1 \leq i \leq 3$, are necessarily distinct, and distinct from the previous ones, and the problem is solved.

Suppose now that $\langle N(x) \rangle$ does not contain any triangle. Since it is not an independent set, it consists of at least one star $K_{1,q}$, $q \geq 1$, and possibly other star-components and isolated vertices. Assume first $q \leq p - 2$, let y_1 be the center of the star, $y_2, \ldots, y_{q+1}$ its other vertices. Any $z \in S_i \cap S_j$ with $i \leq q + 1 < j$ would give a C_5 using an edge of the star. For $q \geq 2$, any $z \in S_1 \cap S_i$ with $2 \leq i \leq q + 1$ would give a C_5 using some edge of the star distinct from $y_1 y_i$.

In the case $q \geq 2$ we may assume $S_i \cap S_j = \emptyset$ if $i = 1 < j$ or $i \leq q + 1 < j$. Let us index the y_j, $j \geq q + 2$, as previously. Thus, $|S_2| \geq k - 2$, $|S_3| \geq k - 3$, …, $|S_{k-1}| \geq 1$ and we may find distinct vertices $z_j \in S_j$, $2 \leq j \leq k - 1$. Since S_1 is not empty, and any $z_1 \in S_1$ is distinct from the previous ones, our problem is solved.

In the case $q = 1$, it is possible that $S_1 \cap S_2 \neq \emptyset$, but $|S_1| \geq k-2$ and $S_1 \cap S_{k-1} = \emptyset$. We may choose z_j, $2 \leq j \leq k-1$ as before, and this set does not exhaust S_1 so we may complete it by a distinct vertex $z_1 \in S_1$.

It remains only to study the case $\langle N(x) \rangle = K_{1,p-1}$. Let y_1 be the center of this star. We have $|S_j| \geq k-2$ for $j \geq 2$ and our problem is solved if one of these S_j has cardinality at least $k-1$ or if two of them are distinct. Otherwise, we would have a set $S_2 = \cdots = S_p$ of cardinality $k-2$ giving together with y_1 a cut set of order $k-1$, a contradiction. ■

Let G be a graph with independence number $\alpha = \alpha(G)$ and connectivity $\kappa = \kappa(G) \geq 5$ such that $\alpha < \kappa$. By Theorem 4.42 it contains a C_5, and according to a result of Amar et al. [12], G has also a C_4. Let x be an arbitrary vertex of G. Set $G' = G - x$. Clearly, G' satisfies the inequality $\alpha(G') \leq \kappa(G')$, and since the degree of any vertex of G' is greater than or equal to 3, this graph is not isomorphic to C_5. Obviously, $G' \neq K_{\kappa,\kappa}$, because $\alpha < \kappa$. If $G' = K_{\kappa-1,\kappa-1}$, then x is joined to every vertex in G', because otherwise G would have a vertex cut of cardinality $\kappa - 1$. In such a graph we can easily find a C_{n-2}. Applying Theorem 4.31 for small α's and above remarks we get the following:

Corollary 4.43. *Let G be a k-connected graph of order n and stability number $\alpha \geq 4$. If $\alpha \leq k-1$, then G contains cycles of all lengths belonging to the set $\{3, 4, 5, n-2, n-1, n\}$.*

In [217] Marczyk and Saclé have verified that the Jackson–Ordaz conjecture is valid for all graphs with stability number at most four.

4.4. Further generalizations. In [199], the authors proved that for graphs satisfying Zhu's condition (see Theorem 3.44 in Subsection 3.5) Bondy's "metaconjecture" holds with complete bipartite graphs as exceptions. This result is in fact a corollary of the following strengthening of Theorem 3.45 due to Flandrin et al. [137].

Theorem 4.44. *Let $G = (V, E)$ be a 2-connected graph on n vertices with minimum degree δ and such that*

($*$) *$xy \in E$ for every pair $x, y \in V$ such that $\delta = d(x)$, $d(y) < n/2$.*

Then G is either pancyclic or isomorphic to $K_{n/2,n/2}$.

Proof. Let $G = (V, E)$ be a graph on n vertices, 2-connected with minimum degree δ such that the condition ($*$) holds. Suppose that G is not pancyclic.

By Theorem 3.45, G contains a hamiltonian cycle. Denote it by C and choose one of its orientations, say $\vec{C}$. Let u be a vertex of G such that $d(u) = \delta < n/2$.

We claim that in this case G cannot be bipartite. Suppose that G is bipartite. Since G is hamiltonian, it has to be a balanced bipartite graph, i.e., $G = (L, R, E)$ with $|L| = |R| = n/2$. Without loss of generality we may suppose that $u \in L$. Since $\delta < n/2$, there exists a vertex $b \in R$ nonadjacent to u. But then $d(b) < n/2$ and by ($*$) the edge ub has to belong to E, a contradiction.

Suppose now that there are two consecutive (with respect to the orientation $\vec{C}$) vertices x, y such that neither ux nor uy belongs to E. By ($*$) we have $d(x) \geq n/2$ and $d(y) \geq n/2$.

If $d(x) + d(y) \geq n + 1$, then G is pancyclic by Proposition 4.5. Thus we have

$$d(x) = d(y) = n/2.$$

Since, as claimed above, G is not bipartite, by applying Theorem 4.6 we conclude that, in particular, the degrees of the vertices $x^{-2}, x^{-}, y^{+}, y^{+2}$ are less than $n/2$. Therefore, the condition $(*)$ ensures that the edges $ux^{-2}, ux^{-}, uy^{+}, uy^{+2}$ belong to E. A simple counting argument shows that the degree of u has to be at least $n/2$. This contradiction finishes the proof of the theorem. ■

Corollary 4.45 ([199]). *Let $G = (V, E)$ be a 2-connected graph of order n with minimum degree δ. If $d(x) + d(y) \geq n/2 + \delta$ for all pairs x, y of nonadjacent vertices, then G is pancyclic or isomorphic to $K_{n/2,n/2}$.*

Recently Schiermeyer and Woźniak [246] strengthened Theorem 4.44 (cf. Subsection 3.5).

Theorem 4.46. *Let G be a 2-connected graph on n vertices with minimum degree δ. If there exists a vertex u with $d(u) = \delta$ such that any other vertex v with $d(v) < n/2$ is adjacent to u, then G is pancyclic, or $G \subset \mathcal{F}_{n,\delta}$, or $G \cong K_{n/2,n/2}$.*

Faudree et al. [125] and Bauer et al. [28] investigated the relation between cardinalities of neighborhood unions of two nonadjacent vertices and pancyclicity.

Theorem 4.47. *Let G be a 2-connected graph of order $n \geq 19$. If $|N(u) \cup N(v)| \geq (2n + 5)/3$ for each pair of nonadjacent vertices u and v of G, then G is pancyclic.*

This theorem was improved by Li and Wei [196] and Liu and Zhao [203]. Chu and Wang [92] localized the neighborhood union condition to vertices being at distance two.

Theorem 4.48. *Let G be a 2-connected graph of order n and minimum degree $\delta(G) \geq 4$. If $|N(u) \cup N(v)| \geq n - 4$ for all distinct vertices u and v with $d(u, v) = 2$, then G is pancyclic or $n = 8$ and $G \cong K_{4,4}$.*

Faudree and Gould [124] characterized the forbidden pairs of subgraphs for pancyclicity.

Theorem 4.49. *Let A and B be connected graphs ($A \neq P_3$, $B \neq P_3$), and let G ($G \neq C_n$) be a 2-connected graph of order $n \geq 10$. Then G being $\{A, B\}$-free implies that G is pancyclic if and only if $A \cong K_{1,3}$ and B is one of the graphs P_4, P_5, P_6, Z_1, or Z_2.*

They also exhibited the forbidden pairs for several hamiltonian properties in 2-connected graphs (see [124]). Note that there are several articles (cf. [124], [152], [251]) on forbidden pairs implying other hamiltonian properties in 3-connected graphs.

Faudree, Gould and Ryjáček [129] gave a complete characterization of all 2-connected claw-free graphs which are Z_3-free, B-free, W-free, or HP_7-free graphs and which are not pancyclic (HP_7, the hourglass, is the graph obtained by identifying a vertex in two distinct copies of a triangle).

To close this subsection we present a closure result on pancyclicity due to Faudree et al. [121].

Theorem 4.50. *If $Cl_{n+1}(G)$ is complete, then the graph G is pancyclic.*

4.5. Vertex pancyclic graphs. A vertex of a graph G on n vertices is *r-pancyclic* if it is contained in p-cycle for every p between r and n. G is *r-pancyclic* if every vertex of G is r-pancyclic. A 3-pancyclic vertex is called *pancyclic.* Clearly, a 3-pancyclic graph is vertex pancyclic. The investigation of vertex pancyclic graphs is closely related to Bondy's metaconjecture, since vertex pancyclicity implies pancyclicity and pancyclicity implies hamiltonicity. We start with some results involving Dirac-type or Ore-type conditions. The first one is due to Hendry [168].

Theorem 4.51. *Let G be a graph of order $n \geq 3$ such that $\delta(G) \geq (n+1)/2$. Then G is vertex pancyclic.*

The next theorem, due to Randerath et al. [236], is based on a result of Hendry [168].

Theorem 4.52. *Let G be a graph of order $n \geq 4$ such that $\sigma_2(G) \geq n$. Then G is vertex 4-pancyclic, unless n is even and $G \cong K_{n/2,n/2}$.*

The same authors [236] established a Dirac-type condition for edge pancyclicity.

Theorem 4.53. *Let G be a graph of order $n \geq 3$ such that $\delta(G) \geq (n+2)/2$. Then G is edge pancyclic.*

Liu et al. [202] and Wei and Zhu [279] investigated neighborhood unions for vertex pancyclicity. They were able to prove a conjecture of Faudree et al. [125].

Theorem 4.54. *Let G be a 2-connected graph. If $|N(u) \cup N(v)| \geq n - \delta(G) + 1$ for each pair of nonadjacent vertices u and v of G, then G is vertex pancyclic.*

Lin and Song [197] showed that if we raise the bound in the last theorem to $n-\delta(G)+2$, then the graph is edge pancyclic, with few exceptions. They also investigated [198] vertex pancyclicity of graphs satisfying a similar condition where the bound $n - \delta(G) + 1$ was replaced by $n - \delta(G)$.

Asratian and Sarkisian [16] investigated vertex-pancyclicity of a L_0-graphs (see Subsection 3.7).

Theorem 4.55. *Let G be a connected L_0-graph of order $n \geq 4$. Then G is vertex 4-pancyclic, unless n is even and $G \cong K_{n/2,n/2}$.*

This implies the following Corollary due to Song and Zhang [257].

Corollary 4.56. *Let G be a graph of order $n \geq 4$ such that $d(u)+d(v) \geq n$ for any path u, w, v with $uv \notin E$. Then G is vertex 4-pancyclic, unless n is even and $G \cong K_{n/2,n/2}$.*

In [170] Hendry and Vogler proved a conjecture due to Gould and Jacobson [151] that the square of a connected $S(K_{1,3})$-free graph is vertex pancyclic.

Finally, we present two closure results on vertex and edge pancyclity, sourced from [236].

Theorem 4.57. *If $Cl_{\lceil (4n-3)/3 \rceil}(G)$ is complete, then G is vertex pancyclic.*

Theorem 4.58. *If $Cl_{\lceil (3n-3)/2 \rceil}(G)$ is complete, then G is edge pancyclic.*

For more information on vertex pancyclic graphs, the reader is referred to a nice article by Randerath, Schiermeyer, Tewes and Volkmann [236].

4.6. Pancyclicity of different classes of graphs. Since a bipartite graph contains no odd cycle, it is not pancyclic. Therefore, if we restrict ourselves to bipartite graphs only, we study the existence of cycles of all possible even lengths. Namely, a bipartite graph G of order $2n$ is called *bipancyclic* if for every k, $2 \leq k \leq n$, G contains a cycle of length $2k$. Clearly, every bipancyclic graph is hamiltonian and balanced.

Mitchem and Schmeichel [224] investigated the number of edges that guarantees the bipancyclicity.

THEOREM 4.59. *Let G be a balanced bipartite graph of order $2n$. If $|E(G)| > n(n-1)+1$, then G is bipancyclic.*

Entringer and Schmeichel [109] observed that the bound in the previous theorem can be lowered if one assumes that the graph is hamiltonian.

THEOREM 4.60. *Let G be a hamiltonian bipartite graph of order $2n$, $n > 3$. If $|E(G)| > n^2/2$, then G is bipancyclic.*

Mitchem and Schmeichel [224] suggested that the bound on the number of edges is not best possible and proposed the following conjecture.

CONJECTURE 4.61. *Let G be a hamiltonian bipartite graph on $2n$ vertices and at least $n^2/4+n+1$ edges. Then G is bipancyclic.*

The following result due to Schmeichel and Mitchem [249] is similar to the well-known Schmeichel–Hakimi theorem [248] on pancyclicity of a graph with two consecutive vertices on a hamiltonian cycle with a large degree sum. It is a very useful tool in the investigation of bipancyclity.

THEOREM 4.62. *Let G be such a balanced bipartite graph of order $2n$ with bipartition (X, Y) which contains a hamiltonian cycle $C = 1, 2, \dots, 2n, 1$.*

- *If $d(1)+d(2n) > n+1$, then G is bipancyclic.*
- *If $d(1)+d(2n) = n+1$, then either G is bipancyclic or—if G is missing a $2l$-cycle—for each odd integer k, $3 \leq k \leq 2n-1$, exactly one of the two pairs $(2n, k)$ and $(1, f_{2l}(k))$ is an edge of G, where $f_{2l}(k) = 2n-2l+k+1$, if $3 \leq k \leq 2l-3$, and $f_{2l}(k) = k-2l+3$, if $2l-1 \leq k \leq 2n-1$.*

In [249] the same authors studied the Dirac-type condition for bipancyclicity.

THEOREM 4.63. *Let G be a balanced bipartite graph of order $2n$. If $\delta(G) \geq (n+1)/2$, then G is bipancyclic.*

Tian and Zang [268] proved a Dirac-type theorem for bipancyclability in hamiltonian graphs.

THEOREM 4.64. *Let G be a hamiltonian balanced bipartite graph of order $2n$. If $\delta(G) > 2n/5+2$ and $n \geq 60$, then G is bipancyclic.*

A very interesting theorem involving a degree sum condition has been discovered by Zhang [287]. Observe that in the statement of this theorem the author required that the Ore-type condition should be satisfied only for one vertex of a given set of bipartition.

THEOREM 4.65. *Let G be a balanced bipartite graph of order $2n > 6$ and bipartition (X, Y). If there exists a vertex $x \in X$ such that $d(x) + d(y) \geq n + 1$ for each $y \in Y$ nonadjacent to x, then G is bipancyclic.*

Schmeichel and Mitchem [249] considered a Chvátal-type condition (see Theorem 3.13) for bipartite graphs.

THEOREM 4.66. *Let G be a balanced bipartite graph of order $2n \neq 6$ with bipartition (X, Y). Let the degree sequences of the vertices in X and Y, resp., be ordered as follows: $d(x_1) \leq \cdots \leq d(x_n)$ and $d(y_1) \leq \cdots \leq d(y_n)$. If $d(x_k) \leq k < n$ implies that $d(y_{n-k}) \geq n - k + 1$, then G is bipancyclic.*

Recall that the property "G is hamiltonian" is $(n+1)$-bistable. The following example shows that it is not the case for the property "G is bipancyclic" (here n is half the order of the graph). For n odd let G be the graph of order $2n$ defined by the cycle $x_1, y_1, x_2, y_2, \ldots, x_n, y_n, x_1$ with additional edges $x_1 y_j$ for j odd, $3 \leq j \leq n-2$, and $x_i y_2$ for i odd, $5 \leq i \leq n$. G has no cycle of length 4. On the other hand, the graph $G + x_1 y_2$ is bipancyclic and $d(x_1) + d(y_2) = n + 1$.

Note the exact value of the bistability of the property "G is bipancyclic" is unknown. However, the following closure result due to Amar et al. [9] holds.

THEOREM 4.67. *If G is a balanced bipartite graph with $BG_{n+2} = K_{n,n}$, then G is bipancyclic.*

The same authors proposed the following conjecture.

CONJECTURE 4.68. *If G is a balanced bipartite graph with $BG_{n+1} = K_{n,n}$, then G is bipancyclic.*

For further information on hamiltonian properties of bipartite graphs, we recommend a survey paper by Amar et al. [9].

Pancyclicity of line graphs was studied by van Blanken and al. [44]. They considered the function $f(n)$, the smallest integer such that for every graph G of order n with minimum degree $\delta(G) > f(n)$, the line graph $L(G)$ of G is pancyclic whenever $L(G)$ is hamiltonian. They proved that $f(n) = \Theta(n^{1/3})$.

Flandrin et al. [134] investigated pancyclicity of 2-connected claw-free graphs.

THEOREM 4.69. *Let G be a 2-connected claw-free graph on n vertices, where $n \geq 35$. If $\delta(G) > (n-2)/3$, then G is pancyclic.*

Trommel et al. [269] found a minimum degree condition for a claw-free graph to be weakly pancyclic with girth 3.

THEOREM 4.70. *Let G be a claw-free graph on n vertices where $n \geq 5$. If $\delta(G) > \sqrt{3n+1} - 2$, then G is weakly pancyclic with girth 3.*

This degree bound is best possible. Indeed, consider the graph G_p, $p \geq 2$, defined as follows. Let $H_1, \ldots, H_p$ be p disjoint copies of K_{3p-2}, and $u_i v_i$ an edge of H_i $(i = 1, \ldots, p)$. G_p is obtained from $\bigcup_{i=1}^{p} (H_i - u_i v_i)$ by adding the edges $v_1 u_2, v_2 u_3, \ldots, v_{p-1} u_p$ and $v_p u_1$. The graph G_p is both hamiltonian and claw-free. Moreover, $\delta(G_p) = 3p - 3$ and

$n = |V(G_p)| = p(3p-2)$, implying $\delta(G_p) = \sqrt{3n+1} - 2$. Clearly, G_p does not contain C_{3p-1} and hence is not weakly pancyclic.

From Theorem 4.70, we immediately get the following corollary.

COROLLARY 4.71. *Let G be a claw-free hamiltonian graph on n vertices where $n \geq 5$. If $\delta(G) > \sqrt{3n+1} - 2$, then G is pancyclic.*

In [221], Matthew and Sumner proved that every 2-connected nonhamiltonian claw-free graph contains a cycle of length at least $2\delta + 4$. Thus, the following result is true.

COROLLARY 4.72. *Let G be a 2-connected claw-free graph of order $n \geq 5$. If $\delta(G) > \sqrt{3n+1} - 2$, then G contains cycles of every length p with $3 \leq p \leq \min\{2\delta + 4, n\}$.*

5. Cycles through specified vertices

5.1. Cycles containing a given subset of vertices. The investigation into cycles passing through a given set of vertices in k-connected graphs was initiated by Dirac [102].

THEOREM 5.1. *Let G be a k-connected graph, where $k \geq 2$, and let X be a set of k vertices of G. Then there is in G a cycle containing every vertex of X.*

Watkins and Mesner [277] characterized the k-connected graphs in which some set of $k+1$ vertices is included in no cycle. Bondy and Lovász [65] proved that a $(k+1)$-connected nonbipartite graph has an odd cycle containing any k specified vertices. Recently Häggkvist and Mader [157] showed that every set of $k + \lfloor \frac{1}{3}\sqrt{k} \rfloor$ vertices in a k-connected k-regular graph belongs to some cycle. Egawa et al. [104] proved the following common generalizations of Theorem 5.1 and Dirac's classical theorem [101] on hamiltonicity.

THEOREM 5.2. *Let G be a k-connected graph, where $k \geq 2$, and let X be a set of k vertices of G. Then G contains either a cycle of length at least $2\delta(G)$ including every vertex of X or a hamiltonian cycle.*

There exists a generalization of the notion of a cycle passing through a given set of vertices. Namely, let $m \geq 0$ and $n \geq 0$ be two integers. We say that a graph G is in $\mathcal{C}(m,n)$ if for every pair M, N of disjoint subsets of the vertex set of G, with the cardinalities of M and N being m and n, resp., there is in G a cycle C containing all the vertices of M and none of the vertices in N. For example, by Theorem 5.1, every k-connected graph is in $\mathcal{C}(k,0)$. Another example: every 3-connected cubic graph is in $\mathcal{C}(9,0)$ (Holton et al. [173]). A survey paper on this class of graphs was written by Holton [172].

Consider a subset X of the vertex set of a graph G. By $\alpha(X)$, we denote the maximum number of pairwise nonadjacent vertices in the subgraph of G induced by X, while the symbol $\delta(X)$ stands for the minimum degree (in G) of the vertices of X. For $2 \leq k \leq \alpha(X)$, we denote by $\sigma_k(X)$ the minimum degree sum (in G) of any k independent vertices of X; for $k > \alpha(X)$, we put $\sigma_k(X) = k(n - \alpha(X))$. We say that X is *cyclable in G* if G has a cycle containing all the vertices of X. Notice that putting $X = V(G)$ in the above definitions concerning X, we clearly recover the usual notions of independence number, minimum degree, $\sigma_k(G)$ and hamiltonicity.

In [139], Flandrin et al. gave a generalization of Theorem 5.1 (due to Dirac) involving the notion of the connectivity of a set of vertices.

THEOREM 5.3. *Let G be a graph and Y a subset of $V(G)$ with $\kappa(Y) \geq 2$. Let X be a subset of Y with $|X| \leq \kappa(Y)$. Then X is cyclable in G.*

Proof. Let $X = \{x_1, \ldots, x_q\}$ be a subset of Y, where $q \leq \kappa(Y)$ and $|Y| \geq 2$. We may assume that Y is not a clique in G.

The proof is by induction on q. Assume $q = 2$ and $X = \{x_1, x_2\} \subset Y$. If x_1 and x_2 are not adjacent we apply Menger's theorem and we are done. Otherwise, since Y is not a clique, there exists another vertex u in Y, so we may use the fan lemma (Lemma 3.38) and find a cycle containing x_1, x_2 (and u). Suppose the assertion is true for every set Z of p vertices, $p \leq q-1 < \kappa(Y)$ and let $X = \{x_1, \ldots, x_q\}$. By the induction hypothesis there is a cycle that contains the vertices $x_1, \ldots, x_{q-1}$. Denote by C such a cycle with a given orientation. By Lemma 3.38 there is a collection $\mathcal{P} = P_1, P_2, \ldots, P_{q-1}$ of $q-1$ internally disjoint paths and there are $q-1$ different vertices $y_1, \ldots, y_{q-1}$ in C such that for each i, $1 \leq i \leq q-1$, P_i is an x_q-y_i path with $V(P_i) \cap V(C) = \{y_i\}$. We may assume without loss of generality that the vertices y_i and x_j appear on C in the order indicated by C and the paths P_i are oriented from x_q to y_i. If there is a cycle in G containing X we are done. So suppose the contrary. If y_i and y_{i+1} belong to $x_j\vec{C}x_{j+1}$ for some i and j (indices are taken mod $q-1$), then the cycle $y_{i+1}\vec{C}y_i\overleftarrow{P}_i x_q P_{i+1} y_{i+1}$ contains all the vertices of X, a contradiction. Moreover, if $x_s = y_t$ for some s, t or $x_j\vec{C}x_{j+1}$ does not contain any vertex of X, then, by the pigeonhole principle, there are two indices i and j such that y_i and y_{i+1} belong to $x_j\vec{C}x_{j+1}$, which leads to a contradiction. Thus, in each segment $x_j^+\vec{C}x_{j+1}^-$ there is exactly one y_j and we may assume that $x_q x_i \notin E(G)$ for $1 \leq i \leq q-1$. Since x_1 and x_q are not adjacent, it follows from Menger's theorem that there is a collection $\mathcal{Q} = Q_1, Q_2, \ldots, Q_{\kappa(Y)}$ of $\kappa(Y)$ internally disjoint x_q-x_1 paths. We may assume that these paths are oriented from x_q to x_1. Denote by $f(Q_j)$ the first vertex of Q_j on C $(j = 1, \ldots, \kappa(Y))$. We claim that

$$f(Q_j) \neq x_1$$

for each $j = 1, \ldots, \kappa(Y)$. Indeed, suppose that for some s, $f(Q_s) = x_1$. If Q_s is internally disjoint from any path P_i, then the cycle $y_j\vec{C}x_1\overleftarrow{Q}_s x_q P_j y_j$, where y_j is the only vertex of $x_1^+\vec{C}x_2^-$, contains all the vertices of X, a contradiction. Thus Q_s contains an internal vertex of a path of $\mathcal{P}$. Let $l(Q_s)$ be the last vertex of Q_s belonging to $\bigcup V(P_i) \setminus (\{x_q\} \cup V(C))$. Assume $l(Q_s) \in P_r$ and let $P'_r = x_q P_r l(Q_s) Q_s x_1$. Therefore, the collection $P_1, \ldots, P_{r-1} P'_r, P_{r+1}, \ldots, P_{q-1}$ of $q-1$ paths satisfies the condition of Lemma 3.38. Moreover, the terminal vertex of the path P'_r belongs to X and we can easily find a cycle in G passing through all the vertices of X, which is a contradiction. This prove our claim.

Set $z_j = V(Q_j) \cap V(C)$, $j = 1, \ldots, \kappa(Y)$. Since $q-1 < \kappa(Y)$, it follows by the pigeonhole principle that there are two vertices z_i and z_r belonging to the same segment of the form $x_j\vec{C}x_{j+1}$, whence there is a cycle containing all the vertices of X, a contradiction. ∎

Note that Harant [163] independently proved Theorem 5.3. Bollobás and Brightwell [50] and, independently, Shi [253] obtained an extension of Dirac's theorem on hamiltonian graphs.

THEOREM 5.4. *Let G be a 2-connected graph of order n and let X be a set of vertices of G. If $d_G(x) \geq n/2$ for each $x \in X$, then X is cyclable in G.*

Shi [253] improved both Ore's theorem and the previous one in the following way.

THEOREM 5.5. *Let G be a graph of order n and let X be a subset of its vertex set such that $\kappa(X) \geq 2$. If $d_G(x) + d_G(y) \geq n$ for each pair x, y of nonadjacent vertices of X, then X is cyclable in G.*

Notice that if $|X| \geq 3$ the Ore type condition implies $\kappa(X) \geq 2$ and the assumptions on the connectivity in the last two theorems can be dropped. It is easy to see that the latter result follows from the following theorem due to Ota [230].

THEOREM 5.6. *Let G be a graph of order n, and let X be a set of vertices of G with $\kappa(X) \geq k \geq 2$. If for any $s \geq k$ and for any independent subset S of X of $s+1$ vertices we have*

$$\sum_{x \in S} d_G(x) \geq n + s^2 - s,$$

then X is cyclable in G.

Note that in the original paper [230] the author used internally disjoint paths in order to define the connectivity of a set of vertices. However, from Theorem 5.3 his assertion is obvious if $|X| \leq \kappa(X)$, and by Lemma 3.35 the two conditions used to define the connectivity are equivalent if $|X| > \kappa(X)$.

Broersma et al. [79] studied cyclability of sets of vertices of graphs satisfying a local Chvátal–Erdős type condition that involves above defined parameters. They obtained a generalization of a result of Fournier [143] and of the Chvátal–Erdős theorem.

We present this result with a weaker hypothesis on the connectivity and give an alternative proof (there is a gap in the original one).

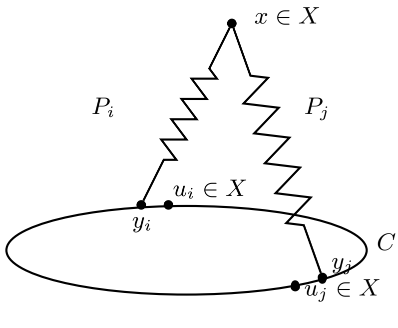

Fig. 9. The cycle C in the proof of Theorem 5.7

THEOREM 5.7. *Let G be a graph and let $X \subset V(G)$ with $\kappa(X) \geq 2$. If $\alpha(X) \leq \kappa(X)$, then X is cyclable in G.*

Proof. By Theorem 5.3 the assertion is true if $|X| \le \kappa(X)$. Suppose $\alpha(X) \le \kappa(X)$ and X is not cyclable in G. Hence $|X| > \kappa(X) \ge 2$. Let C be a cycle containing as many vertices of X as possible and let $\vec{C}$ denote this cycle with a given orientation. Again by Theorem 5.3, C contains at least $\kappa(X)$ vertices of X. Since X is not cyclable, there exists at least one vertex of X, say x, that does not belong to $V(C)$. By Lemma 3.38 there is a collection $\mathcal{P} = P_1, \dots, P_{\kappa(X)}$ of $\kappa(X)$ internally disjoint paths and there are $\kappa(X)$ different vertices $y_1, y_2, \dots, y_{\kappa(X)}$ in C such that for each i, $1 \le i \le \kappa(X)$, P_i is an x-y_i path with $V(P_i) \cap V(C) = \{y_i\}$. Using the same argument as in the proof of Theorem 5.3 we conclude that in each segment $y_j^+\vec{C}y_{j+1}^-$ (indices are taken mod $\kappa(X)$) there is at least one vertex of X. Let u_i be the first vertex of X in $y_i^+\vec{C}y_{i+1}^-$ $(i = 1, \dots, \kappa(X))$. Suppose, for instance, that $u_iu_j \in E$. Let us consider the cycle

$$u_iu_j\vec{C}y_i\overleftarrow{P}_ix\vec{P}_jy_j\overleftarrow{C}u_i$$

(see Figure 9).

Since, by definition of u_i, the segments $y_j^+\vec{C}u_j^-$ and $y_i^+\vec{C}u_i^-$ contain no vertex of X, the above cycle contains more vertices of X than C, a contradiction.

Also in the case $xu_i \in E$, it is easy to show a cycle with more vertices of X than C, which is a contradiction.

Therefore, $\{x, u_1, \dots, u_{\kappa(X)}\}$ is an independent set of $\kappa(X)+1$ vertices, again a contradiction. ■

Observe that this theorem is true for $\kappa(X) = 1$ provided $|X| \ge 3$.

The X-*length* of a cycle in G is the number of vertices of X contained in this cycle. The X-*circumference* of G, denoted by $c_X(G)$, is the X-length of a cycle in G that contains as many vertices of X as possible. Čada et al. [84] studied the property of cyclability under several closure concepts. They proved the following proposition closely related to the Bondy–Chvátal closure.

Proposition 5.8. *Let G be a graph of order n. Let X be a nonempty subset of $V(G)$ and let k be an integer, $1 \le k \le |X|$. Let u, v be two vertices of G such that $uv \notin E(G)$ and $d(u) + d(v) \ge n$. Then G contains a cycle C with $|V(C) \cap X| \ge k$ if and only if the graph $G' = G + uv$ contains a cycle C' with $|V(C') \cap X| \ge k$.*

Proof. A cycle C in G with $|V(C) \cap X| \ge k$ is also contained in G' and satisfies the same condition. Suppose now that C' is a cycle in G' with $|V(C') \cap X| \ge k$. We may assume $uv \in E(C')$ (otherwise there is nothing to do). Denote by P the uv-path $P = C' - uv$ in G and set $t = |V(P)|$ and $R = V(G) \setminus V(P)$.

Suppose first that $d_P(u) + d_P(v) \ge t$. Then using the classical Ore's argument we can show that G has a cycle C with $V(C) = V(P)$. Clearly, C is the required cycle.

Assume $d_P(u) + d_P(v) \le t - 1$. Then

$$d_R(u) + d_R(v) = d(u) + d(v) - (d_P(u) + d_P(v)) \ge n - (t-1).$$

Since $|R| = n - t$, there is a vertex $y \in N_R(u) \cap N_R(v)$ and $C = uPvyu$ is a cycle in G with $|V(C) \cap X| \ge k$. ■

The next theorem is an easy consequence of the last proposition.

THEOREM 5.9. *Let G be a graph of order n and let $X \subset V(G)$, $X \neq \emptyset$. Then*

(i) $c_X(G) = c_X(Cl_n(G))$;
(ii) *X is cyclable in G if and only if X is cyclable in $Cl_n(G)$.*

This implies that the property of cyclability of a given set of vertices is n-stable. The example below shows that the stability of this property is exactly n.

Let G and $X \subseteq V(G)$ satisfy the following conditions:

(i) $X = \{x_1, x_2, x_3, x_4, x_5\}$,
(ii) $V(G) = V_1 \cup V_2 \cup \{x_3\}$, where $V_1 \cap V_2 = \emptyset$, $\{x_1, x_2\} \subseteq V_1$, $\{x_4, x_5\} \subseteq V_2$,
(iii) $N(x_1) = (V_1 \setminus \{x_1\}) \cup \{x_4\}$, $N(x_5) = (V_2 \setminus \{x_5\}) \cup \{x_2\}$, $N(x_3) = \{x_2, x_4\}$.

We have $x_1x_5 \notin E(G)$, $d(x_1)+d(x_5) = |V_1|+|V_2| = |V(G)|-1$, X is cyclable in $G+x_1x_5$ but not in G. Hence, the property of cyclability of a given set X is not $(n-1)$-stable.

In [139] Flandrin et al. studied the problem of cyclability under the condition called a *regional Ore's condition.* We present below a very short proof of this result based on the Bondy–Chvátal closure operation.

THEOREM 5.10. *Let $G = (V, E)$ be a graph of order n. Let $X_1, \ldots, X_q$ be subsets of the vertex set V such that the union $X = X_1 \cup \cdots \cup X_q$ satisfies $2 \leq q \leq \kappa(X)$. If for each $i = 1, \ldots, q$, and for any pair of nonadjacent vertices $x, y \in X_i$, we have $d(x) + d(y) \geq n$, then X is cyclable in G.*

Proof. Consider the Bondy–Chvátal closure $Cl_n(G)$. By assumption, every set X_i, $i = 1, \ldots, q$, induces a clique in $Cl_n(G)$. Let A be an independent set in the subgraph induced by X. It is obvious that A can have at most one vertex in each clique X_i, so $|A| \leq q$. Hence, $\alpha(X) \leq q \leq \kappa(X)$. Now, by Theorem 5.7, X is cyclable in $Cl_n(G)$ and, by Theorem 5.9, X is cyclable in G. ∎

In particular, if $X = V$ (cf. [138]) we get

THEOREM 5.11. *Let $G = (V, E)$ be a k-connected graph, $k \geq 1$, of order n and let $V = X_1 \cup \cdots \cup X_k$. If for each $i = 1, \ldots, k$, and for any pair of nonadjacent vertices $x, y \in X_i$, we have $d(x) + d(y) \geq n$, then G is hamiltonian.*

So, we get the hamiltonicity of a graph for which the Ore condition holds in each of parts separately (regionally) provided that the graph is k-connected. For $k = 1$ we get the classical Ore theorem. Notice that in this case the connectivity (even 2-connectivity) is implied by the condition itself. It is also clear that Theorem 5.10 improves Theorem 5.5 and implies the following generalization of Theorem 5.1 due to Dirac.

THEOREM 5.12. *Let $G = (V, E)$ be a k-connected graph, $k \geq 2$. For any k cliques $X_1, \ldots, X_k$ of G, there exists a cycle of G containing all vertices of these cliques.*

Another example of application of Theorem 5.10 is the following classical result which can be also viewed as a corollary of Menger's theorem.

THEOREM 5.13. *Let $G = (V, E)$ be a 2-connected graph and let e, f be two edges of G. Then G contains a cycle passing through e and f.*

Proof. We insert a new vertex on each of both edges, e and f. Observe that the graph obtained in this way, say G', remains 2-connected. By applying Theorem 5.10 in G' with respect to these new vertices, we get a cycle containing them. This cycle passes evidently through e and f in G. ∎

In order to compare the regional Ore condition with other ones, consider the graph G on n vertices ($n \geq 12$, $n \equiv 0 \pmod 4$) with $X = V(G) = X_1 \cup X_2 \cup X_3$, where $|X_1| = n/2 + 2$, $|X_i| = n/4 - 1$ for $i = 2, 3$, and such that X_2 and X_3 induce a clique of G and X_1 induces a clique without one edge. Moreover, there is one edge between X_1 and X_2 and two independent edges between X_1, X_3 and X_2, X_3 (all these edges are independent), so G is 3-connected. It is easy to see that this graph satisfies no one of the well-known conditions implying hamiltonicity as for instance the conditions of Ore, Chvátal, Fan, Chvátal–Erdős, etc. but it is hamiltonian by Theorem 5.11.

In [84] Čada et al. introduced the notion of (k, X)-closure of G. Namely, for $X \subset V(G)$ and an integer k the (k, X)-closure of G is the graph obtained by recursively adding all missing edges uv with $d(u) + d(v) \geq k$, $u, v \in X$. The (k, X)-closure of G will be denoted by $Cl_k^X(G)$. It is well-defined and has similar properties to the Bondy–Chvátal closure: if $|X| \geq 3$ then $c_X(G) = c_X(Cl_n^X(G))$; and X is cyclable in G if and only if X is cyclable in $Cl_n^X(G)$.

The following theorem generalizes an analogous result for hamiltonicity that can be found in [4].

Theorem 5.14. *Let G be a 2-connected graph of order n and $X \subset V(G)$ be such that $d_G(x) + d_G(y) \geq n - 1$ for each pair $x, y \in X$, with $x,y \notin E(G)$. Then either X is cyclable in G, or n is odd and G contains an independent set $X_1 \subset X$ such that $|X_1| = (n+1)/2$ and every vertex of X_1 is adjacent to all vertices in $G - X_1$.*

Corollary 5.15. *Let G be a 1-tough graph on n vertices and let X be a subset of $V(G)$ with at least three vertices. If $d_G(x) + d_G(y) \geq n - 1$ for each pair x,y of nonadjacent vertices of X, then X is cyclable in G.*

Proof. Suppose X is not cyclable in G and let X_1 be a subset of X introduced in the previous theorem. Set $R = G - X_1$. Thus $X_1 = G - R$ and X_1 has $|X_1| = |R| + 1$ connected components. So G is not 1-tough, which is a contradiction. ∎

Note that Stacho [261] obtained a similar result on cyclability in 1-tough graphs. Using another closure concept Čada et al. [84] investigated cyclability of graphs which are locally claw-free and satisfy the σ_3-condition.

Theorem 5.16. *Let G be a 2-connected graph of order $n \geq 33$ and let $\emptyset \neq X \subset V(G)$ be such that*

(i) *no vertex in $X \cup N(X)$ is a claw center;*
(ii) $\sigma_3(X) \geq n - 2$.

Then X is cyclable in G.

This theorem immediately implies the following corollary.

Corollary 5.17. *Let G be a 2-connected graph of order $n \geq 33$ and let $\emptyset \neq X \subset V(G)$. If*

(i) *no vertex in $X \cup N(X)$ is a claw center;*
(ii) *$\delta(X) \geq (n-2)/3$,*

then X is cyclable in G.

Notice that Theorem 5.16 implies for $n \geq 33$ Theorem 3.85 due to Matthews and Sumner and the following result by Broersma [75] and Zhang [286].

Corollary 5.18. *Every 2-connected claw-free graph of order $n \geq 3$ such that $\sigma_3(G) \geq n-2$ is hamiltonian.*

Note that other results on cycles through subsets of vertices were discovered by Egawa et al. [104]. Broersma et al. [79] extended Theorem 3.51 by Bauer et al. as follows.

Theorem 5.19. *Let G be a 2-connected graph on n-vertices and let $X \subset V(G)$. If $\sigma_3(X) \geq n + \min\{\kappa(X), \delta(X)\}$, then X is cyclable.*

The next result involving a condition on independent sets of three elements extends Theorem 3.54. It was found by Favaron et al. [132].

Theorem 5.20. *If G is a 2-connected graph and X a subset of $V(G)$ such that*

$$d(u) + d(v) + d(w) \geq n + |N(u) \cap N(v) \cap N(w)|$$

for every set $\{u, v, w\} \subset X$ of three independent vertices, then X is cyclable in G.

Harkat-Benhamdine et al. [165] studied the cyclability of sets with large σ_4.

Theorem 5.21. *Let G be a 3-connected graph and X a subset of $V(G)$.*

(a) *If $\sigma_4(X) \geq n + 2\alpha(X) - 2$, then X is cyclable in G.*
(b) *If $\sigma_4(X) \geq n + \delta(X)$ and $d(v) \geq n/2$ for every $v \in X \setminus N[w]$, where $w \in X$ and $d(w) = \delta(X)$, then X is cyclable in G.*

Polický [232] introduced a parameter $\omega(u, v)$ denoting the number of components of the graph $G[N(u)]$ containing no neighbor of v. He obtained the following Ore-type result for hamiltonicity.

Theorem 5.22. *If G is a graph of order n such that $d(u)+d(v)+\max\{\omega(u,v), \omega(v,u)\} \geq n$ for each pair of nonadjacent vertices u and v, then G is hamiltonian.*

Subsequently, Stacho [259, 261] applied the parameter $\omega(u, v)$ to obtain several new sufficient conditions for hamiltonicity and cyclability.

Note also that Abderrezzak et al. [2] studied cyclability in bipartite graphs.

Theorem 5.23. *Let G be a 2-connected bipartite balanced graph of order $2n$ with bipartition (Y, Z). Let X be a subset of Y of cardinality at least 3. If $d(x) + d(y) \geq n + 1$ for every pair of nonadjacent vertices (x, y), $x \in X$, $y \in Z$, then X is cyclable in G.*

5.2. Cycles through edges. Another problem that appears in hamiltonian graph theory is the problem of the existence of cycles through specified edges in a graph. Clearly, such a set of edges must form a *path system* in the graph, i.e., a union of disjoint paths. For instance, let us quote an old result of Pósa [234] that provides another strengthening of Dirac's theorem.

THEOREM 5.24. *Let s be an nonnegative integer and G a graph of order $n \geq 3$ with $\delta(G) \geq (n+s)/2$. Let S be a set of s edges of G that induce a path system. Then G contains a hamiltonian cycle that includes all edges of S.*

Observe that this results implies Corollary 3.16 due to Erdős and Gallai. Indeed, if two vertices are adjacent, then the assertion follows directly from this theorem. Otherwise we can add an edge e joining them and apply the same theorem to the graph $G+e$ and the set $S = \{e\}$.

Lovász [209] conjectured that if G is k-connected ($k \geq 2$), $F = \{e_1, \ldots, e_k\}$ is a set of k independent edges of G and $G - \{e_1, \ldots, e_k\}$ is connected when k is odd, then G has a cycle passing through all the edges of F. He showed that his conjecture is true for $k = 3$. Häggkvist and Thomassen [159] proved a weaker version of Lovász's conjecture.

THEOREM 5.25. *If L is a set of k independent edges in G such that any two vertices incident with L are connected by $k+1$ internally disjoint paths, then G has a cycle containing all edges of L.*

THEOREM 5.26. *If G is a $(\alpha(G)+k)$-connected graph, then any set of k independent edges of G is contained in a cycle.*

They also made the following conjecture.

CONJECTURE 5.27. *If G is an $\alpha(G)$-connected graph and L a set of independent edges such that $G - L$ is connected, then G has a cycle containing all edges of L.*

Let F be a path system in G. Then G is said to be F-hamiltonian if F is contained in a hamiltonian cycle. Häggkvist [155] proved the following.

THEOREM 5.28. *If F is a 1-factor of of G and $\sigma_2(G) \geq n+1$, then G is F-hamiltonian.*

5.3. Pancyclability. Let G be a graph and S a subset of $V(G)$. A vertex of S is called an *S-vertex* and a cycle of G that contains exactly p S-vertices is said to have *S-length* p; such a cycle will be denoted by C_p^S. A set S having at least three vertices is said to be *pancyclable in* G if G contains cycles of every S-length p with $3 \leq p \leq |S|$.

Favaron et al. [123] and, independently, Stacho [260] investigated an Ore-type condition for pancyclability of sets of vertices.

THEOREM 5.29. *Let G be a graph on n vertices and let S be a subset of $V(G)$. If $d(x) + d(y) \geq n$ for every pair of nonadjacent vertices x and y of S, then either S is pancyclable in G or else n is even, $S = V(G)$ and $G = K_{n/2,n/2}$, or $G[S] = C_4 = x_1, x_2, x_3, x_4, x_1$ and the structure of G is as follows: $V(G)$ is partitioned into $S \cup V_1 \cup V_2 \cup V_3 \cup V_4$; for any i, $1 \leq i \leq 4$, $G[V_i]$ is any graph on $|V_i|$ vertices with $|V_i| \geq 0$, and each vertex x_i is adjacent to all the vertices of V_{i+1} and V_i, where the index i is taken modulo 4.*

In [140] Flandrin et al. considered a Chvátal–Erdős type condition "$\alpha(S) \le \kappa(S)$" and proved that this condition also implies that S is pancyclable in G provided the cardinality of S is large enough with respect to $\alpha(S)$. In the proof they needed the following notion. Consider a subset S of $V(G)$, a cycle C of G with a given orientation and two S-vertices s_1 and s_2 on C. Then s_2 is said to be the *S-vertex following s_1 on C* if $C[s_1, s_2] \cap S = \{s_1, s_2\}$.

THEOREM 5.30. *Let G be a graph and $S \subset V$. If $\alpha(S) \le \kappa(S)$ and $|S| \ge 2R(4\alpha(S), \alpha(S)+1)$, then S is pancyclable in G.*

Proof. Suppose that G is a graph, S a subset of $V(G)$ such that $\alpha(S)$ and $\kappa(S)$ satisfy $\alpha(S) \le \kappa(S)$ and $|S| \ge 2R(4\alpha(S), \alpha(S) + 1)$. Notice that if $\alpha(S) = 1$, then S is a clique and we are done, therefore we can assume $2 \le \alpha(S) \le \kappa(S)$ and $|S| \ge 2R(8, 3) > 46$. Now we shall show that G contains a C_p^S for each p, $3 \le p \le |S|$.

The proof will be divided into two parts, depending on the S-length of the cycles that we want to obtain.

CASE 1: $p \ge |S|/2 - 1$. Observe that, by Theorem 5.7, this statement is evident for $p = |S|$ and suppose that G contains a cycle C_p^S with $p \ge |S|/2$. We shall prove that G also contains a C_{p-1}^S.

Let $a_1, \ldots, a_p$ be the vertices of $C_p^S \cap S$ appearing in that order on C_p^S, where the indices are considered modulo p. Since $p \ge |S|/2 \ge R(4\alpha(S), \alpha(S) + 1)$, and the graph induced by $C_p^S \cap S$ has no independent set of cardinality $\alpha(S)+1$, it follows from Ramsey's theorem that it contains a clique, say K, having $4\alpha(S)$ S-vertices. Assume that among the cycles of S-length p passing through $\{a_1, \ldots, a_p\}$, C_p^S is chosen so that it contains as many edges of K as possible and fix an arbitrary orientation of C_p^S.

Suppose now that G does not contain any cycle with $p - 1$ S-vertices. Clearly a_i cannot be adjacent to a_{i+2} for $1 \le i \le p$ and, consequently, if a_i belongs to K, a_{i+2} is not in K.

Let $d_1, \ldots, d_r$ be the vertices of K, appearing in that order on C_p^S, such that for $1 \le i \le r$, the S-vertex following d_i on C_p^S is not in K.

From the above remark, there are at least $2\alpha(S)$ such vertices d_i, and we shall denote by b_i the S-vertex following d_i on C_p^S, $1 \le i \le r$, $r \ge 2\alpha(S)$. Since $2\alpha(S) > \alpha(S)$, there are necessarily two vertices b_{i_1} and b_{i_2} that are adjacent.

Using the edges $b_{i_1}b_{i_2}$ and $d_{i_1}d_{i_2}$, we easily obtain a cycle with exactly the same S-vertices as C_p^S and that contains more edges of K than C_p^S, contrary to the choice of C_p^S. This implies the existence of a cycle of S-length $p - 1$ as soon as $p \ge |S|/2$. Hence, by induction, G contains cycles C_p^S for each $p \ge |S|/2 - 1$.

CASE 2: $p < |S|/2 - 1$. Since $|S| \ge 2R(4\alpha(S), \alpha(S) + 1)$ and S has no independent set of cardinality $\alpha(S) + 1$, it follows from Ramsey's theorem that S contains a clique on $4\alpha(S)$ vertices. Thus, our statement is evident for $3 \le p \le 4\alpha(S)$. Suppose G has a C_p^S for some p satisfying $p < |S|/2 + 1 - 4\alpha(S)$. We claim that it also contains a cycle with exactly $p + 4\alpha(S) - 2$ S-vertices.

Since $p = |C_p^S \cap S| < |S|/2$, the graph $G - C_p^S$ contains at least $|S|/2 \ge R(4\alpha(S), \alpha(S) + 1)$ S-vertices, hence also contains a clique, say K, on $4\alpha(S)$ vertices.

Since we cannot separate two vertices, the first one of K and the second one of $C_p^S \cap S$, by deletion of fewer than $\kappa(S)$ vertices, it follows from Menger's theorem that there are at least $\min(\kappa(S), p, 4\alpha(S))$ vertex-disjoint paths between the vertices of K and the vertices of $C_p^S \cap S$. Consequently, using the assumption $\alpha(S) \leq \kappa(S)$, there exist $r = \min(\alpha(S), p)$ vertex-disjoint paths that join C_p^S to K. Fix an arbitrary orientation of C_p^S, and denote by x_i and y_i, $i = 1, \ldots, r$ the endvertices of those paths belonging to $V(C_p^S)$ and $V(K)$, respectively. We assume that the vertices $x_1, \ldots, x_r$ appear on the cycle C_p^S in the order of their indices. Let P_i $(i = 1, \ldots, r)$ be the path of endvertices x_i and y_i. Notice that x_i does not belong necessarily to S. We will assume that every path P_i has minimum S-length, hence, from the definition of $\alpha(S)$, $|V(P_i) \cap S| \leq 2\alpha(S)$ for every P_i, $1 \leq i \leq r$. Set $l_i = |(V(P_i) - \{x_i\}) \cap S| \leq 2\alpha(S)$, $i = 1, \ldots, r$.

Claim 1. *Assume that for some i, $1 \leq i \leq r$, we have $C_p^S[x_i, x_{i+1}] \cap S \subset \{x_i, x_{i+1}\}$. Then G contains a $C_{p+4\alpha(S)-2}^S$.*

Proof. Suppose first that $l_i + l_{i+1} \leq 4\alpha(S) - 2$. Delete the interior vertices and the edges of the segment $C_p^S[x_i, x_{i+1}]$ and add the paths P_i, P_{i+1} and Q_i, where Q_i is a path from y_i to y_{i+1} in K with $4\alpha(S) - 2 - l_i - l_{i+1} \geq 0$ interior vertices. In this way we obtain a cycle with $p + 4\alpha(S) - 2$ vertices of S.

Suppose now that $4\alpha(S) - 1 \leq l_i + l_{i+1} \leq 4\alpha(S)$ and consider the case $l_i = 2\alpha(S)$ and $l_{i+1} = 2\alpha(S) - 1$. Let $s_1, \ldots, s_{2\alpha(S)} = y_i$ be the S-vertices of the directed path $P_i[x_i, y_i]$ appearing on $P_i[x_i, y_i]$ in the order of their indices. Obviously, $s_1 \notin V(C_p^S)$. Because of the choice of P_i, the set $s_2, s_4, s_6, \ldots, s_{2\alpha(S)}$ is independent. Denote now by z the last S-vertex on C_p^S (according to the orientation of C_p^S) before x_i. From the definition of $\alpha(S)$, z must be adjacent to a vertex s_{2j} for some $j \leq \alpha(S)$. Delete the interior vertices and the edges of the segment $C_p^S[z, x_{i+1}]$ and add the edge zs_{2j} and the paths $P_i[s_{2j}, y_i]$, P_{i+1} and Q_i, where Q_i is a path from y_i to y_{i+1} in K with $4\alpha(S) - 2 - (2\alpha(S) - 2j + 1) - (2\alpha(S) - 1) \geq 0$ interior vertices. In this way we get a cycle having $p + 4\alpha(S) - 2$ S-vertices, as required. Considering, if necessary, the first S-vertex on C_p^S after x_{i+1}, we proceed in a similar way in other subcases of the case $4\alpha(S) - 1 \leq l_i + l_{i+1} \leq 4\alpha(S)$. ■

Consequently, we assume that any two vertices x_i and x_{i+1} are separated by at least one S-vertex on C_p^S. There are two possibilities, depending on the value of p with respect to $\alpha(S)$.

Case 2.1: $\alpha(S) \leq p$. We have $r = \alpha(S)$. For $1 \leq i \leq \alpha(S)$, let v_i be the S-vertex following x_i on C_p^S, which is, from our hypothesis, interior to the segment $C_p^S[x_i, x_{i+1}]$. Let x be any vertex of $K \setminus \{y_1, \ldots, y_r\}$. Then $A = \{v_1, \ldots, v_{\alpha(S)}, x\}$ is a subset of S with $\alpha(S) + 1$ vertices and so the subgraph $G\langle A \rangle$ contains at least one edge. Suppose first that $xv_i \in E$ for some i. Then we apply Claim 1, where the path P_{i+1} is replaced by the path x, v_i of length one, and we obtain a cycle having $p + 4\alpha(S) - 2$ vertices of S.

So we may now assume that such an edge joins two vertices of the cycle C_p^S, say v_i and v_j (see Figure 10). Suppose that $l_i + l_j \leq 4\alpha(S) - 2$. Delete the interior vertices and the edges of the segments $C_p^S[x_i, v_i]$, $C_p^S[x_j, v_j]$ and add the paths P_i, P_j and Q_{ij}, where Q_{ij} is a path from y_i to y_j in K with $4\alpha(S) - 2 - l_i - l_j \geq 0$ interior vertices. In this way we obtain a cycle with $p + 4\alpha(S) - 2$ vertices of S. There remains the case

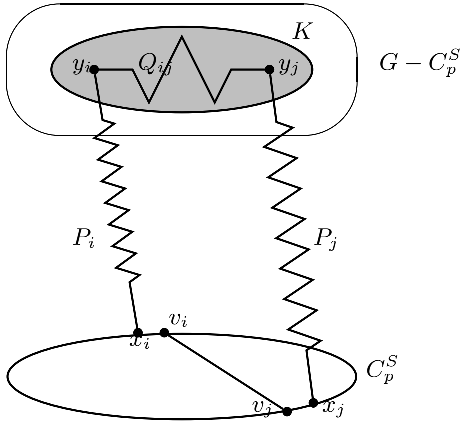

Fig. 10. Case 2.1 of the proof of Theorem 5.30

where $4\alpha(S) - 1 \leq l_i + l_j \leq 4\alpha(S)$. Suppose $l_i = 2\alpha(S)$ and $l_j = 2\alpha(S) - 1$ and let $s_1, \ldots, s_{2\alpha(S)} = y_i$ be the S-vertices of the directed path $P_i[x_i, y_i]$ appearing on $P_i[x_i, y_i]$ in the order of their indices. Clearly, $s_1 \notin V(C_p^S)$. Denote now by z the last S-vertex on C_p^S (according to the orientation of C_p^S) before x_i. We can show, as in the proof of Claim 1, that z must be adjacent to a vertex s_{2m}, for some $m \leq \alpha(S)$. Delete the interior vertices and the edges of the segments $C_p^S[z, v_i]$, $C_p^S[x_j, v_j]$ and add the edge zs_{2m} and the paths $P_i[s_{2m}, y_i]$, P_j and Q_{ij}, where Q_{ij} is a path from y_i to y_j in K with $4\alpha(S) - 2 - (2\alpha(S) - 2m + 1) - (2\alpha(S) - 1) \geq 0$ interior vertices. Thus, we get a cycle having $p + 4\alpha(S) - 2$ vertices of S as required. We proceed in a similar way in other subcases of the case $4\alpha(S) - 1 \leq l_i + l_j \leq 4\alpha(S)$.

Case 2.2: $p < \alpha(S)$. We have $r = p$. If one of the segments $C[x_i, x_{i+1}]$ has no interior vertex in S then, by Claim 1, we are done. Otherwise, there is exactly one vertex of S interior to the segment $C[x_i, x_{i+1}]$ for $1 \leq i \leq p$. If $l_i + l_{i+1} \leq 4\alpha(S) - 1$ for some i, then the cycle $x_i^-, x_i, P_i[x_i, y_i], Q_i[y_i, y_{i+1}], P_{i+1}[y_{i+1}, x_{i+1}], x_{i+1}, x_{i+1}^+, \ldots, x_i^-$ has S-length $p+4\alpha(S)-2$, where Q_i is a path from y_i to y_{i+1} in K with $4\alpha(S)-2-l_i-l_{i+1}+1 \geq 0$ interior vertices. If $l_i + l_{i+1} = 4\alpha(S)$ we proceed as in the proof of Claim 1.

From the existence of C_p^S for $3 \leq p \leq 4\alpha(S)$ and the fact that for every cycle of S-length p, $p < |S|/2 + 1 - 4\alpha(S)$, we obtain a cycle of S-length $p + 4\alpha(S) - 2$, we deduce, by induction, that G contains C_p^S for $3 \leq p < |S|/2 - 1$. This concludes the study of Case 2.

Putting together the results in Cases 1 and 2 completes the proof of Theorem 5.30. ∎

Obviously, the last result implies Theorem 4.41.

In [84] Čada et al. investigated the stability of the property of pancyclability of a given set of vertices. They proved an anologue of Theorem 5.16 for cyclability.

Theorem 5.31. *Let G be a graph of order n and let S be a subset of $V(G)$ having at least three elements. Let u, $v \in V(G)$ be such that $uv \notin E(G)$ and $d(u) + d(v) \geq n + |S| - 3$. Then S is pancyclable in G if and only if S is pancyclable in $G + uv$.*

Proof. Suppose that for some k, $3 \le k \le |S|$ there exists a cycle C' in $G' = G + uv$ with $|V(C') \cap S| = k$ and there is no cycle C in G with $|V(C) \cap S| = k$. Obviously, $uv \in E(C')$. Denote by P the uv-path $C' - uv$ in G (with an orientation from u to v) and set $t = |V(P)|$ and $R = V(G) \setminus V(P)$. Clearly, $|V(P) \cap S| = k$. If $d_P(u) + d_P(v) \ge t$, then using the classical Ore argument we can show that G has a cycle C with $V(C) = V(P)$, so $|V(C) \cap S| = k$, a contradiction. Hence, $d_P(u) + d_P(v) \le t - 1$. Then

$$\begin{aligned} d_R(u) + d_R(v) &= d(u) + d(v) - (d_P(u) + d_P(v)) \ge n + |S| - 3 - (t-1) \\ &= n - t + |S| - 2 \ge n - t + |S| - 2 - k + 3 = n - t + |S| - k + 1 \end{aligned}$$

(because $k \ge 3$). Since $|R| = n - t$ and $|R \cap S| = |S| - k$, the vertices u and v have a common neighbor $y \in R \setminus S$ and $C = uPvyu$ is a cycle in G with $|V(C) \cap S| = k$, a contradiction. ∎

COROLLARY 5.32. *Let G be a graph of order n and let S be a subset of $V(G)$ with at least three vertices. Then S is pancyclable in G if and only if S is pancyclable in the Bondy–Chvátal closure $Cl_{n+|S|-3}(G)$.*

Note that using a similar idea to the proof of Theorem 5.31 we can show that if G has n vertices then a set S is pancyclable in G iff S is pancyclable in the closure $Cl^S_{n+|S|-3}(G)$.

It is worth noting that Abderrezzak et al. [2] studied pancyclability in bipartite graphs. They showed that if a graph G and a set X satisfy the hypothesis of Theorem 5.23, where $n+1$ is replaced by $n+3$, then X is pancyclable in G.

6. The structure of the set of cycle lengths in a graph

6.1. Cycles of a given length. It is well-known that a graph on n vertices having at least n edges has a cycle. Moreover, the bound on the number of edges is best possible because a tree has $n-1$ edges and no cycle. Let $\mathcal{P}$ be a property of graphs. The graphs without the property $\mathcal{P}$ and with maximum number of edges will be called extremal for the property in question. Thus, trees are extremal for the property of being acyclic. Observe that the length of a cycle of a graph on n vertices may be any integer between 3 and n. However, given an integer p one can ask how many edges are needed to guarantee the existence of a cycle of length p. The following result due to Mantel [210] provides an answer for a triangle.

THEOREM 6.1. *Let G be a graph of order n. If $|E(G)| > n^2/4$, then G contains a triangle.*

It can be easily seen that this bound is best possible and that the extremal graphs in this case are complete bipartite graphs $K_{n/2,n/2}$. In 1971 Bondy [57] showed that any graph G with $|E(G)| > n^2/4$ contains cycles of all lengths p with $3 \le p \le \lfloor (n+3)/2 \rfloor$. This number of edges is extremal only for odd p. If p belongs to the interval $[\lfloor (n+3)/2 \rfloor, n]$, then the extremal number of edges is $\binom{p-1}{2} + \binom{n-p+2}{2}$ (Woodall [280]). The connected graph with two blocks K_{p-1} and K_{n-p+2} intersecting in one vertex has this number of edges and no cycle of length p. Surprisingly, the number of needed edges decreases if we consider short even cycles. For example, Reiman [237] showed that if the size of a graph is at least $n^{3/2}/2+n/4$ then the graph has a cycle of length 4. Bondy and Simonovits [67] proved that a graph of order n and more than $c(p)n^{1+1/p}$ edges, where $c(p)$ is a function in p, has a cycle of length $2p$. This result was slightly improved by Verstraëte in [272]. On the other hand, the following problem has not been solved yet.

PROBLEM 1. *For every $p \geq 2$ and for infinitely many n, find a graph of order n and $c(p)n^{1+1/p}$ edges containing no cycle of length $2p$.*

In 1995 Hendry and Brandt [169] proposed to study extremal graphs with the additional assumption that the graphs in question are hamiltonian. For integers n and p with $3 \leq p \leq n-1$, let $f(n,p)$ denote the maximum number of edges in a hamiltonian graph of order n which does not contain a cycle of length p. According to Theorem 4.1 due to Bondy, every hamiltonian graph of order n with at least $n^2/4$ edges is pancyclic or isomorphic to a complete bipartite graph $K_{n/2,n/2}$. From this fact it follows that $f(n,p) \leq n^2/4$ for all p, with equality holding if and only if n is even and p is odd. Häggkvist, Faudree and Schelp [130] extended Theorem 4.1 by proving that every hamiltonian graph with more than $(n-1)^2/4+1$ edges is pancyclic or bipartite. This implies that $f(n,p) \leq (n-1)^2/4+1$ for all p if n is odd. Observe that the graph obtained by subdividing an edge of $K_{(n-1)/2,(n-1)/2}$ has no triangle. Therefore, $f(n,3) = (n-1)^2/4+1$ if n is odd.

Now consider a cycle $C = x_1, \ldots, x_9$ of length 9 and let $F = \{x_3x_5, x_6x_8, x_9x_2\}$ and $H_9 = C^2 - F$. Hendry and Brandt [169] found the numbers $f(n,5)$ for odd n.

THEOREM 6.2. *For odd $n \geq 7$,*

$$f(n,5) = (n-3)^2/4+5$$

unless $n = 9$, in which case $f(9,5) = 15$ and H_9 is the unique extremal graph which does not contain C_5.

According to an unpublished result of Erdős, $f(n,2k) = \mathcal{O}(n^{1+1/k})$ for each $k \geq 2$.

Other results and questions related to extremal graphs are treated in the book of Bollobás [47] and the survey paper by Bondy [62].

6.2. Cycles of a given length modulo k. Let $k > 0$ and $s \geq 0$ be two integers. A cycle of length l is called an $(s \bmod k)$-*cycle* if $l = s \bmod k$. For any natural number k, we define a graph to be *pancyclic mod k* if it contains a cycle of every length modulo k. It is well-known (see König [184]) that a graph is bipartite if and only if it contains no odd cycle. By Theorem 3.4 due to Voss and Zuluaga (see [275]), every 2-connected nonbipartite graph G with $3 \leq \delta(G) = \delta \leq |V(G)|/2$ contains both an odd cycle of length at least $2\delta - 1$ and an even cycle of length at least 2δ, i.e., it contains a cycle of every length modulo 2. Chen and Saito [91] obtained the following result on cycles of length 0 mod 3, proving a conjecture of Barefoot et al. [19].

THEOREM 6.3. *Every graph of minimum degree at least three contains a cycle of length $0 \bmod 3$.*

Dean et al. [98] showed that every 2-connected graph G with $\delta(G) \geq 3$ contains a (1 mod 3)-cycle and a (2 mod 3)-cycle except the Petersen graph, K_4 and $K_{3,n}$. It follows that every graph G with $\delta(G) \geq 4$ is pancyclic mod 3. Moreover, it is easy to show that every graph with minimum degree at least $k+1$ contains a (2 mod k)-cycle.

Observe that a bipartite graph has no cycle of odd length mod k if k is even; therefore, a high minimum degree does not guarantee the pancyclicity mod k. Taking into account this remark, Thomassen [266] proposed the following conjecture.

Conjecture 6.4. *Every graph G with $\delta(G) \geq k+1$ contains a $(2s \bmod k)$-cycle, where s and k are natural numbers.*

Thus, the conjecture is true for $k = 3$. Moreover, it follows by a result of Dean et al. [99] that it also holds for $k = 4$. Dean (see [91]) conjectured that a graph with $\delta(G) \geq k \geq 3$ has a (0 mod k)-cycle. Clearly, by Theorem 6.3, this holds for $k = 3$ and was verified for $k = 4$ by Dean et al. [99].

Erdős and Burr [112] conjectured that for any odd number k, there exists a constant c_k such that every graph of average degree at least c_k is pancyclic modulo k. This conjecture was resolved by Bollobás [46] in 1977 with $c_k = 2[(k+1)^k - 1]/k$. Recently Verstraëte [272] showed that the conjecture holds with $c_k = 8k$. This is a consequence of the following theorem on the existence of an arithmetic progression of cycle lengths in graphs (see [272]).

Theorem 6.5. *Let $k \geq 2$ be a natural number and G a bipartite graph of average degree at least $4k$ and girth g. Then there exist cycles of $(g/2 - 1)k$ consecutive even lengths in G.*

Thomassen [266] formulated another conjecture on pancyclicity mod k of 2-connected, nonbipartite graphs.

Conjecture 6.6. *Every 2-connected nonbipartite graph G with $\delta(G) \geq k+2$ is pancyclic mod k.*

A graph is said to be *vertex-pancyclic mod k* if, for every vertex v and every integer s, $0 \leq s < k$, G has an (s mod k)-cycle containing v. Cai and Shreve [85] investigated pancyclicity mod k of claw-free graphs and $K_{1,4}$-free graphs. They obtained the following three results.

Theorem 6.7. *Every 2-connected claw-free graph G with $\delta(G) \geq k+1$ is vertex-pancyclic mod k.*

Theorem 6.8. *Every claw-free graph G with $\delta(G) \geq k+1$ is pancyclic mod k.*

Theorem 6.9. *Every $K_{1,4}$-free graph G with $\delta(G) \geq k+3$ is pancyclic mod k.*

They also conjectured that every $K_{1,4}$-free graph G with $\delta(G) \geq k+1 \geq 4$ is pancyclic mod k.

Observe that Theorem 6.8 implies that Conjecture 6.4 is true for claw-free graphs. The complete graph K_{k+1} shows that this result is best possible. Note also that Dean [97] proved that every 3-connected planar graph (except K_4) with minimum degree at least k is pancyclic modulo k.

6.3. The number of cycle lengths in a graph. Consider a graph G of minimum degree k and let x be an end vertex of a longest path in G. Clearly, G contains $k-1$ cycles of different lengths formed by the path segments from x to the $k-1$ neighbors of x at greatest distance along the path. The graphs K_{k+1} and $K_{k,k}$ show that the number $k-1$ is best possible. Of course G has also a cycle of length at least $k+1$ and a path of length at least k. Thus the investigation of the number of cycle lengths in a graph is more interesting if we add other requirements, for example a minimum girth requirement. Let $n(g,d)$ be the minimum number of different cycle lengths in a graph with girth at least g and minimum degree at least d. In 1999 Erdős et al. [114] proved the following two theorems.

THEOREM 6.10.

(i) $n(5,k) \le k^2 - k$ *if* $k-1$ *is a prime power,*
(ii) $n(5,k) \ge (k^2+k-2)/4$ *for all* $k \ge 2$,
(iii) $n(5,k) = \Theta(k^2)$.

THEOREM 6.11. *There exist constants* c_1, c_2 *and* c_3 *such that*

(1) $n(9,k) \ge c_1 k^3$,
(2) $n(7,k) \ge c_2 k^{5/2}$,
(3) $n(4t-1,k) \ge c_3 k^{t/2}$ *for* $t \ge 2$.

In 1966 Erdős and Hajnal [116] introduced the following interesting measure of the variety of cycle lengths in a graph:

$$\mathcal{L}(G) := \sum \{1/p : G \text{ has a cycle of length } p\}.$$

Note that $\mathcal{L}(K_{k+1}) \approx \log k + \gamma - 3/2$ and $\mathcal{L}(K_{k,k}) \approx 1/2(\log k + \gamma - 3/2)$, where $\gamma \approx 0.5772$ is the Mascheroni constant. Erdős and Hajnal asked if there is a constant c such that $\mathcal{L}(G) \ge c \log k$ for every graph G of minimum degree k. Gyárfás et al. gave an affirmative answer in [154]. Note that the same property holds (with a different constant) in every graph with average degree k.

There is another interesting approach to the problem of the cycle structure of a graph. A set S of integers is called a *cycle set* on $\{1,\dots,n\}$ if there exists a graph of order n such that the set of cycle lengths in G is S. Erdős [113] conjectured that the number of cycle sets on $\{1,\dots,n\}$ is $o(2^n)$. In [273] Verstraëte corrected this conjecture and proved that this number is $o(2^{n-n^c})$ for some $c>0$. He obtained this result by combining graph theory and additive number theory in a very original way.

6.4. The number of cycle lengths in a hamiltonian graph with a given maximum degree. Let G be a hamiltonian graph of order n and maximum degree Δ. Denote by $C(G)$ the set of integers p, $3 \le p \le n$, such that G contains a cycle of length p. The purpose of the present subsection is the study of the cardinality of the set $C(G)$.

In [219] Marczyk and Woźniak expressed a lower bound for this number as a function in the maximum degree Δ and the order of the graph. In particular, they observed a "jump" of this bound in the neighborhood of the value $\Delta = n/2$.

Below we give some definitions and lemmas which are necessary for the presentation of this result.

The symbol G stands for a hamiltonian graph of order n with vertex set $[1,n] = \{1,\ldots,n\}$ and edge set E. By $C = 1,\ldots,n,1$ we denote a hamiltonian cycle of G. The edges of a complementary graph of the graph G are referred to as *red edges*. The degree of the vertex 1 is Δ, the maximum degree of G. The set of neighbors of 1 will be denoted by X. Note that with this notation, if $p \in X$ and $2 < p < n$, then $p \in C(G)$.

It is easily checked that for a hamiltonian graph G of maximum degree $\Delta \le n/2$, $|C(G)| \ge \Delta - 1$. This is, in a sense, best possible in view of the construction below.

Let $k \ge 4$ and $q \ge 0$ be two integers. Define G as follows. The order of G equals $(q+2)(k-2)+2$ and the edge-set of G consists of the hamiltonian cycle $1,\ldots,n,1$ and of the edges joining 1 to every vertex of the form $k+x(k-2)$, where $0 \le x \le q$. It is easy to see that G has maximum degree $\Delta = q+3$ and that the cycles of G may only have $q+1$ lengths of the form $k+x(k-2)$, $0 \le x \le q$ and, of course, one cycle of length n. Thus $|C(G)| = q+2 = \Delta - 1$.

For a given $A \subset V$, we denote by $f(A)$ the number of neighbors of 1 in A, i.e.,

$$f(A) = |X \cap A|.$$

Proposition 6.12. *If $k \notin C(G)$, then $k \notin X$ and $n-k+2 \notin X$.* ∎

Proposition 6.13. *If $k \notin C(G)$ and $a \in X$ and $a+k-2 < n$, then $a+k-2 \notin X$.* ∎

Corollary 6.14. *Let A and B be two disjoint subsets of $[1,n]$ with $B = A+(k-2)$. If $k \notin C(G)$ then*

$$f(A \cup B) = f(A) + f(B) \le |A| = \tfrac{1}{2}(|A|+|B|).$$

Proof. The proof follows from the observation that if $x \in A \cap X$ then, by Proposition 6.13, the vertex $x+(k-2)$ belonging to B is not in X . ∎

In particular, we will use the last corollary when A and B are two consecutive segments (i.e., their union is also a segment) each containing $k-2$ elements. However, in this case we will need a more general result.

Lemma 6.15. *Let $B_1,\ldots,B_{2t}$ be $2t$ disjoint, consecutive segments of $[1,n]$, each of length $k-2$. If $k \notin C(G)$, then*

$$f\Big(\bigcup_{i=1}^{2t} B_i\Big) \le \frac{1}{2}\sum_{i=1}^{2t} |B_i|.$$

Proof. Since the number of segments B_i is $2t$, we can divide the segments into t pairs $(B_1,B_2),(B_3,B_4),\ldots,(B_{2t-1},B_{2t})$ and apply Corollary 6.14 to each pair separately. By adding the resulting inequalities we get the conclusion. ∎

Since, for $t > 1$, B_1 and B_{2t} are not consecutive, the value of $f(B_1 \cup B_{2t})$ may be greater than $|B_1|$. However, in this case we have the following estimate.

Lemma 6.16. *With the same notation as in the previous lemma suppose that $k \notin C(G)$. Then*

$$f(B_1 \cup B_{2t}) \le |B_1| + \xi,$$

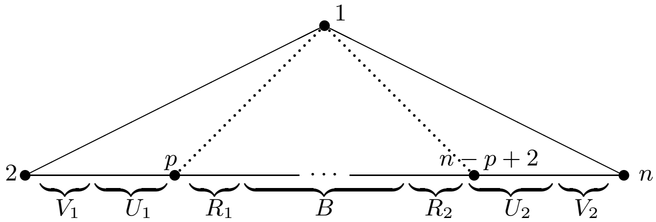

Fig. 11. Segments V_1, V_2, U_1, U_2 R_1, R_2 and B of the proof of Theorem 6.18

where ξ is defined by

$$f\Big(\bigcup_{i=2}^{2t-1} B_i\Big) = \frac{1}{2}\sum_{i=2}^{2t-1} |B_i| - \xi.$$

Proof. The assertion follows from Corollary 6.14 in the case $t = 1$. Therefore, suppose $t > 1$. Applying the previous lemma to the sequence $B_2, \ldots, B_{2t-1}$ we deduce that $\xi \geq 0$. Now it suffices to apply Lemma 6.15 again to $B_1, \ldots, B_{2t}$. ∎

The key lemma is the following:

LEMMA 6.17. *If $|C(G)| < n/2 + \Delta/2 - 3/2$, then there exists an integer $p \leq (n+2)/2$ such that $p \notin C(G)$ and $n - p + 2 \notin C(G)$.*

Proof. Let $C(G)^c = [3, n]\backslash C(G)$. We note that if $(n+2)/2 \in C(G)^c$, then $(n+2)/2$ is the desired integer p. Assume that $(n+2)/2 \notin C(G)^c$. If $k \in C(G)^c$, then Proposition 6.12 states that $k \notin X$ and $n - k + 2 \notin X$. Now if there exist distinct $k, l \in C(G)^c$ such that $\{k, n-k+2\}$ intersects $\{l, n-l+2\}$, then $l = n - k + 2 \in C(G)^c$, and k is the desired integer p. So, we suppose the sets $\{k, n-k+2\}$ with $k \in C(G)^c$ are pairwise disjoint. By Proposition 6.12, this implies that the number of red edges incident with vertex 1 is at least $2|C(G)^c|$. The degree of vertex 1 is then at most $n - 1 - 2|C(G)^c| < n - 1 - (n - \Delta - 1) = \Delta$, a contradiction. ∎

In [219] Marczyk and Woźniak proved the following theorem.

THEOREM 6.18. *Let G be a hamiltonian graph of order n and maximum degree Δ. If $\Delta \leq n/2$, then $|C(G)| \geq \Delta - 1$. Moreover, for every $\Delta \geq 2$ there exist a graph G for which this bound is attained. If $\Delta > n/2$, then $|C(G)| \geq n/2 + \Delta/2 - 3/2$. This bound is best possible.*

Proof. Clearly, by the remark given at the beginning of this subsection, the theorem holds for small values of Δ.

Suppose, contrary to our claim, that there is a graph G of order n and $\Delta > n/2$ such that $|C(G)| < n/2 + \Delta/2 - 3/2$. Let p with $3 \leq p \leq (n+2)/2$ be an integer satisfying the following property:

$$p \notin C(G) \quad \text{and} \quad n - p + 2 \notin C(G). \tag{$*$}$$

The existence of p is guaranteed by Lemma 6.17. If $p < (n+2)/2$ we have $n - 2(p-2) - 3 \geq 0$ vertices between p and $n - p + 2$ on C. Let t and r be the quotient and the remainder when $n - 2p + 1$ is divided by $2(p-2)$, i.e.,

$$n - 2p + 1 = 2t(p-2) + r \tag{$**$}$$

with $0 \leq r < 2(p-2)$. If $p = (n+2)/2$ we put $t = r = 0$.

Let r_1, r_2 be two integers such that $r_1 + r_2 = r$, $0 \le r_1 \le r_2 \le r_1 + 1$. For $r_1 \ge 1$ we define two segments on C,

$$R_1 = [p+1, p+r_1], \qquad R_2 = [n-p-r_2+2, n-p+1].$$

In other words R_1 is the segment having r_1 vertices with first vertex $p+1$ and R_2 is the segment having r_2 vertices with last vertex $n-p+1$. For $r_1 = 0$ or $r_2 = 0$ the corresponding set R_i is, by definition, empty.

Denote by B the segment $[p+r_1+1, n-p-r_2+1]$. By the construction, the segment B consists of an even number of segments, each of length $p-2$.

We put $V_1 = \{i-(p-2) : i \in R_1\}$ and $V_2 = \{i+(p-2) : i \in R_2\}$. Hence $V_1 = [3, r_1+2]$ and $V_2 = [n-r_2, n-1]$. Of course, if R_i is empty, then the set V_i is also empty.

Finally, denote by U_1, U_2 the remaining parts of the segments $[3,p]$ and $[n-p+2, n-1]$, respectively. In other words $U_1 = [r_1+3, p]$ and $U_2 = [n-p+2, n-r_2-1]$. Observe that the segments $U_1 \cup R_1$ and $U_2 \cup R_2$ are both of length $p-2$ (see Figure 11). By Lemma 6.15 we know that $f(B) \le \frac{1}{2}|B|$.

Let us put

$$f(B) = \tfrac{1}{2}|B| - \xi \tag{9}$$

Applying Lemma 6.16 to the sequence of segments $U_1 \cup R_1, B, R_2 \cup U_2$ and using (9) we get

$$f(U_1 \cup R_1) + f(R_2 \cup U_2) \le p - 2 + \xi \tag{10}$$

Applying Corollary 6.14 to the sets V_1 and R_1, as well as to the sets V_2 and R_2, we get

$$f(V_1) + f(R_1) \le |R_1| = r_1, \tag{11}$$

$$f(V_2) + f(R_2) \le |R_2| = r_2. \tag{12}$$

Consider the set $V_1 \cup U_1 \cup U_2 \cup V_2$. Suppose that there exists $x \in [3, p-1]$ with $x \in X$. Then $n - p + x \notin X$, for otherwise we would have a cycle of length $n-p+2$ defined by $1, x, x+1, \dots, n-p+x, 1$, which contradicts $(*)$. By symmetry, we therefore obtain

$$f(V_1) + f(U_1) + f(U_2) + f(V_2) \le p - 3. \tag{13}$$

Set $A = V_1 \cup U_1 \cup R_1 \cup R_2 \cup U_2 \cup V_2$. Observe that $|A| = 2(p-2) + r_1 + r_2$. Moreover, $n = 3 + |A| + |B|$. Adding the inequalities (10)–(13) we get

$$2f(A) \le p - 2 + \xi + r_1 + r_2 + p - 3.$$

Hence

$$2f(A) \le |A| + \xi - 1.$$

Thus

$$f(A) \le \frac{|A|}{2} + \frac{\xi}{2} - \frac{1}{2}.$$

Using the last inequality, (9) and the fact that the edges $(1,2)$ and $(1,n)$ are in E we get

$$\Delta = 2 + f(A) + f(B) \le 2 + \frac{|A|}{2} + \frac{\xi}{2} - \frac{1}{2} + \frac{|B|}{2} - \xi \le \frac{|A| + |B| + 3}{2} = \frac{n}{2},$$

a contradiction.

Finally, for given n define a graph G of order n and maximum degree Δ as follows: the edge-set of G consists of the hamiltonian cycle $1, 2, 3, \dots, n-1, n, 1$ and of the edges joining 1 to every vertex x where $(n-\Delta+5)/2 \le x \le (n+\Delta-1)/2$ if $n-\Delta$ is odd and $(n-\Delta+4)/2 \le x \le (n+\Delta-2)/2$ if $n-\Delta$ is even.

It is easy to see that G has indeed maximum degree Δ and that G have no cycle of length greater than $(n+\Delta-1)/2$ if $n-\Delta$ is odd and $(n+\Delta)/2$ if $n-\Delta$ is even (except for the cycle of length n). Thus $|C(G)| = (n+\Delta-3)/2$ if $n-\Delta$ is odd and $|C(G)| = (n+\Delta-2)/2$ if $n-\Delta$ is even. This finishes the proof of the theorem. ∎

6.5. The structure of the set of cycle lengths in a hamiltonian graph

6.5.1. Hamiltonian graphs with one vertex of a large degree. Let n and Δ be two integers such that $2 \le \Delta \le n-1$. In this subsection we describe the set $D(n,\Delta)$ of cycle lengths occurring in any hamiltonian graph G of order n and maximum degree Δ. Clearly, $D(n,\Delta) = \bigcap C(G)$, where the intersection is taken over all hamiltonian graphs of order n and maximum degree Δ. The vertices of a graph G of order n will be denoted by integers $0,1,\ldots,n-1$ and considered modulo n. The symbol $C = 0,1,\ldots,n-1,0$ is used for a hamiltonian cycle in G with a natural orientation. G is said to be $[a,b]$-*pancyclic* if for every p, $a \le p \le b$, it contains a cycle of length p. Clearly, a $[3,n]$-pancyclic graph is pancyclic.

Let n and p be two integers such that $n \ge 3$ and $3 \le p \le n-1$. Now, we shall construct a hamiltonian graph of order n not containing C_p and with maximum degree Δ as large as possible. Denote by t and r the quotient and the remainder when $n-1$ is divided by $p-2$, i.e., $n-1 = (p-2)t + r$ and $0 \le r < p-2$.

Suppose $t = 2k+1$ is an odd number and let $G_1(n,p)$ be the simple graph obtained by adding to the cycle C all edges of the form $(0,j)$, where $j \in \{1,2,\ldots,p-2, 2(p-2)+1, 2(p-2)+2,\ldots,3(p-2), 4(p-2)+1,\ldots,2k(p-2)+1, 2k(p-2)+2,\ldots,(2k+1)(p-2)\} \setminus \{2k(p-2)+r\}$. Clearly, the maximum degree of this graph equals $(k+1)(p-2) = n/2 + ((p-2)-r-1)/2 \ge n/2$. Moreover, it is easy to check that this graph contains no cycle of length p.

Now assume $t = 2k$ is an even number and add to C every edge of the form $(0,j)$ for $j \in \{1,\ldots,p-2, 2(p-2)+1, 2(p-2)+2,\ldots,3(p-2), 4(p-2)+1,\ldots,(2k-2)(p-2)+1, \ldots,(2k-1)(p-2), 2k(p-2)+1, 2k(p-2)+2,\ldots,n-2\} \setminus \{(2k-1)(p-2)+r\}$ (if $(0,j) \notin E(C)$). The graph obtained in this way will be denoted by $G_0(n,p)$. Obviously, it contains no cycle of length p and its maximum degree is $k(p-2)+r = n/2 + (r-1)/2 \ge (n-1)/2$.

Therefore, the following proposition is true.

Proposition 6.19. *For any two integers n and p, $n \ge 3$, there exists a hamiltonian graph G of order n with $\Delta(G) \le n/2$ which does not contain any cycle of length p.*

The following two propositions were established by Marczyk [213].

Proposition 6.20. *Let G be a hamiltonian graph of order n with maximum degree $\Delta > n/2$ and let p, k and r be three positive integers with $3 \le p \le n-1$, $n-1 = (2k+1)(p-2)+r$ and $r < p-2$. If $p < r + 2\Delta - n + 3$, then G contains a C_p.*

This result is best possible. In fact, consider a graph G isomorphic to $G_1(n,p)$ for and $n-1 = (p-2)(2k+1)+r$. We have

$$\Delta = d(0) = \frac{2k+2}{2}\,(p-2) = \frac{(2k+1)(p-2)+(p-2)}{2} = \frac{n-1+p-2-r}{2}.$$

Thus, $p = r + 2\Delta - n + 3$ but G has no C_p.

PROPOSITION 6.21. *Let G be a hamiltonian graph of order n with maximum $\Delta > n/2$ and let p, k and r be three positive integers with $3 \le p \le n-1$, and $n-1 = 2k(p-2)+r$, $r < p-2$. If $r < 2\Delta - n + 1$, then G contains a C_p.*

Consider now a graph G isomorphic to $G_0(n,p)$ with $n-1 = (p-2)2k + r$ We have

$$\Delta = d_G(0) = \frac{2k(p-2)+2r}{2} = \frac{n-1+r}{2}.$$

Hence $r = 2\Delta - n + 1$ and G has no cycle C_p. Proposition 6.21 is best possible.

We can summarize the last two propositions as follows.

THEOREM 6.22. *Let G be a hamiltonian graph of order n and maximum degree Δ and let p be an integer, $3 \le p \le n-1$. If*

$$\Delta > (p-2)\left\lfloor \frac{n-1}{2(p-2)} \right\rfloor + \min\{(n-1) \bmod 2(p-2), p-2\},$$

then a cycle of length p is guaranteed in G.

The graphs $G_0(n,p)$ and $G_1(n,p)$ show that this result is best possible.

The last theorem implies the following result obtained by Marczyk [213].

THEOREM 6.23. *Let G be a hamiltonian graph of order n and maximum degree $\Delta > n/2$. If p is an integer such that*

$$p \in [3, 2\Delta - n + 2] \cup [n - \Delta + 2, \Delta + 1] \cup \bigcup_{2 \le s < \Delta/(2\Delta - n)} \left(\frac{n-1-\Delta}{s} + 2, \frac{\Delta}{s} + 2 \right),$$

then G contains a cycle of length p.

This result implies (cf. [213]) the following extension of a theorem due to Kouider and Marczyk [185].

THEOREM 6.24. *Let G be a hamiltonian graph of order n and $\Delta(G) = \Delta > n/2$. Then G contains a cycle C_p for every integer p belonging to the union*

$$[3, 2\Delta - n + 2] \cup [n - \Delta + 2, \Delta + 1].$$

Moreover, if $\Delta > \frac{2}{3}(n-1)$, then G is $[3, \Delta+1]$-pancyclic.

For $n \ge 3$ and $\Delta < n$ we denote by $\psi = \psi(n,\Delta)$ the maximum integer k such that every hamiltonian graph G of order n and $\Delta(G) = \Delta$ is $[3,k]$-pancyclic.

The next corollary is an easy consequence of the previous results (cf. [213]).

COROLLARY 6.25. *The function $\psi(n,\Delta)$ has the following properties:*

(i) $\psi(n,\Delta) \le \Delta + 1$ *if* $n/2 < \Delta$;
(ii) $\psi(n,\Delta) = \Delta + 1$ *if* $n - 2 > \Delta > \frac{2}{3}(n-1)$;
(iii) $\psi(n,\Delta) = n$, *if* $\Delta \in \{n-2, n-1\}$;
(iv) $\psi(n,\Delta) \ge 2\Delta - n + 3$ *if* $n/2 < \Delta \le \frac{2}{3}(n-1)$ *and* $\frac{n-1-\Delta}{2\Delta-(n-1)} \notin \mathbb{N}$;
(v) $\psi(n,\Delta) \ge 2\Delta - n + 2$ *if* $n/2 < \Delta \le \frac{2}{3}(n-1)$ *and* $\frac{n-1-\Delta}{2\Delta-(n-1)} \in \mathbb{N}$.

Observe that the function $\psi(n, \Delta)$ presented above has a big jump for $\Delta = 2(n-1)/3$. This number cannot be lowered. In fact, for $\Delta = 2(n-1)/3$ and $p = (n-1)/3 + 2 \in \mathbb{N}$ we have $n - 1 = 2(p-2) + r$, where $r = (n-1)/3$, and the graph $G_0(n, p)$ has no cycle of length $p = 2\Delta - n + 3 < \Delta + 1$. However, for $\Delta > 2(n-1)/3$ G is $[3, \Delta + 1]$-pancyclic.

Now we will give an example of application of Theorem 6.23. Consider a hamiltonian graph G on $n = 44$ vertices with $\Delta = 24$. We have $\Delta/(2\Delta - n) = 6$. By Theorem 6.23, G has a cycle C_p for every p belonging to the set

$$\begin{aligned}
&[3, 2\Delta - n + 2] \cup [n - \Delta + 2, \Delta + 1] \cup \bigcup_{2 \le s < \Delta/(2\Delta - n)} \left(\frac{n - 1 - \Delta}{s} + 2, \frac{\Delta}{s} + 2 \right) \\
&\qquad = [3, 6] \cup [22, 25] \cup \bigcup_{s=2}^{5} \left(\frac{19}{s} + 2, \frac{24}{s} + 2 \right) \\
&\qquad = [3, 6] \cup [22, 25] \cup (23/2, 14) \cup (25/3, 10) \cup (27/4, 8) \cup (29/5, 34/5)
\end{aligned}$$

Therefore, if an integer p belongs to this union $[3, 7] \cup \{9, 12, 13\} \cup [22, 25]$, then the graph G has a cycle C_p. Therefore, the solution of the problem is easily found.

6.5.2. Hamiltonian graphs with two vertices of a large degree sum. Let us recall that Theorem 4.1 due to Bondy ensures the existence of all cycles of even lengths in a hamiltonian graph of order n provided its size is at least $n^2/4$. In a proof of this result Bondy applied Proposition 4.5 stating that if the degree sum of two fixed consecutive vertices on a hamiltonian cycle is at least $n + 1$, then the graph is pancyclic. This result was further generalized by Schmeichel and Hakimi (Theorem 4.6).

The purpose of the present subsection is a description of the set of cycle lengths occurring in any hamiltonian graph of order n having two fixed vertices x and y with $d(x)+d(y) \ge n+z$, where z is an integer satisfying $0 \le z \le n-2$. Clearly, these vertices are not necessarily consecutive on a hamiltonian cycle. We start with the following theorem of Schelten and Schiermeyer [243] which is an improvement of an earlier result of Faudree et al. [122].

THEOREM 6.26. *Let G be a hamiltonian graph of order $n \ge 32$ and x and y two nonadjacent vertices of G with $d(x) + d(y) \ge n + z$, where $z = 0$ if n is odd, and $z = 1$ if n is even. Then G contains cycles of every length p, where $3 \le p \le (n + 13)/5$.*

Note that there is a similar result of Han [161] which concerns two vertices at distance two on a hamiltonian cycle.

The important context in which this type of investigation is of interest is the Bondy–Chvátal closure. Recall that by a result of Faudree et al. [121], if the $(n + 1)$-closure is complete, then the graph is pancyclic. The authors of that paper asked about the lengths of cycles the graph G must contain provided the closure $Cl_n(G)$ is complete. By Ore's theorem this graph is hamiltonian. Apart from the bipartite case, Theorem 6.26 is the first step in this direction.

Below we present several results obtained by the author (Theorems 6.29, 6.35 and 6.36, see also [213] and [214]) that are natural generalizations of Theorem 4.6 by Schmeichel and Hakimi and Theorem 6.26.

We shall need two auxiliary results which are due to Faudree et al. [122].

THEOREM 6.27. *If G has a hamiltonian (u,v)-path for a pair of nonadjacent vertices u, v such that $d(u)+d(v) \geq n$, then G is pancyclic.*

THEOREM 6.28. *Let G contain a hamiltonian path $P = v_1, \ldots, v_n$ such that $v_1v_n \notin E$ and $d(v_1)+d(v_n) \geq n+d$, for some d, $0 \leq d \leq n-4$. Then for any m, $2 \leq m \leq d+3$ there is a (v_1,v_n)-path of length m.*

In [213] the author proved the following theorem.

THEOREM 6.29. *Let G be a hamiltonian graph of order n and let z be an integer with $\frac{1}{3}(n+8) < z \leq n-4$. If x and y are two vertices of G that satisfy $d(x)+d(y) \geq n+z$, then G contains a cycle C_p for every p such that $3 \leq p < \frac{2}{3}n + z/3 - 1/3$.*

Recall that Schiermeyer [245] showed that the stability $s(P)$ for the property of being pancyclic satisfies $\max(\lceil 6n/5 \rceil - 5, n+t) \leq s(P) \leq \max(\lceil 4n/3 \rceil - 2, n+t)$, where $t=0$ if n is odd and $t=1$ if n is even.

Now we shall show that Theorem 6.29 immediately implies a similar (up to a constant) result.

COROLLARY 6.30. *The stability $s(P)$ for the property P of being pancyclic satisfies*

$$s(P) \leq \lfloor 4n/3 \rfloor + 8/3.$$

Proof. Let G be a graph on n vertices and x, y two nonadjacent vertices of G such that $d(x)+d(y) \geq \lfloor 4n/3 \rfloor + 8/3$, $G+xy$ is pancyclic and G is not. It follows from Theorems 3.27 and 3.28 that G contains a cycle C_p for every $p \geq \lceil 2n/3 \rceil$, in particular G is hamiltonian. By assumption, $d(x)+d(y) \geq n+z$, where $z = \lfloor n/3 \rfloor + 8/3 > (n+5)/3$. By Theorem 6.29, G contains a cycle C_p for every p such that $3 \leq p < 2n/3 + z/3$ and $2n/3 + z/3 > 2n/3 + 1 > \lceil 2n/3 \rceil$ $(n \geq 4)$. So G is pancyclic, a contradiction. ■

Four auxiliary lemmas presented below are needed in the proofs of further results. In their formulations we will use the following notation. For a subgraph D of G and a vertex x not belonging to D we denote by $N_D(x)$ the set $\{y \in V(D) \mid xy \in E = E(G)\}$ and by $d_D(x)$ the number $|N_D(x)|$. We will write $]a,b[\, = C]a,b[$ for an open segment of a cycle C with a given orientation. The segments $[a,b]$, $]a,b]$ and $[a,b[$ are defined in a similar way.

LEMMA 6.31. *Let D be a path of G and x a vertex of G that does not belong to D. If $d_D(x) > p-2$ and $|V(D)| \leq 2(p-2)$, then G has a C_p.*

Proof. Let $D = x_1, \ldots, x_q$. By assumption, $p-1 \leq q \leq 2(p-2)$. Put $P = \{x_1, \ldots, x_{q-p+2}\}$ and let t denote the cardinality of the set $N(x) \cap P$. Suppose G has no cycle of length p. Therefore, if $x_k \in N(x) \cap P$, then $(x, x_{k+p-2}) \notin E$, because otherwise $x, x_k, x_{k+1}, \ldots, x_{k+p-2}, x$ would be a cycle of length p. So we have $d_D(x) = |N(x) \cap P| + |N(x) \cap (D \setminus P)| \leq t + (p-2) - t = p-2$, which is a contradiction. ■

LEMMA 6.32. *Let $C = 0, 1, \ldots, n-1, 0$ be a hamiltonian cycle in a graph G and suppose that the vertices 0 and l are not adjacent. Let D be a segment of C contained in $[1, l-1]$*

and let $p \geq 3$ be an integer such that $|V(D)| \geq 2(p-2)$ and $d_D(0) + d_D(l) \geq |V(D)| + 2$. Then G contains a cycle of length p.

Proof. Suppose that $a, b \in N_D(0)$, $f, g \in N_D(l)$, $N_D(0) \subset [a, b]$ and $N_D(l) \subset [f, g]$. Clearly, $[a, b] \cap [f, g] \neq \emptyset$, $N_D(0) \neq \emptyset$ and $N_D(l) \neq \emptyset$.

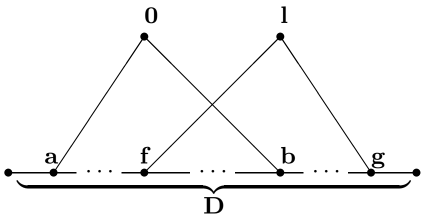

Fig. 12. Case 1 of the proof of Lemma 6.32

Case 1: $[a, b]$ is not contained in $[f, g]$ and $[f, g]$ is not contained in $[a, b]$. Assume without loss of generality that $a < f < b < g$ (see Figure 12). By assumption, $|[a, g]| \geq p - 2$. Let F be the graph induced by the $0, a, a + 1, \ldots, f, f + 1, \ldots, b, b + 1, \ldots, g, l$. We have $d_F(0) + d_F(l) = d_D(0) + d_D(l) \geq |V(D)| + 2 \geq |V(F)|$. By Theorem 6.27, F is pancyclic and contains a C_p.

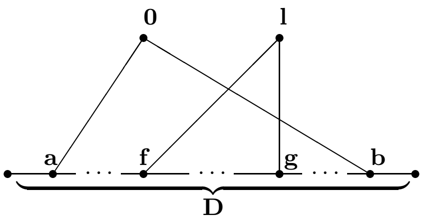

Fig. 13. Case 2 of the proof of Lemma 6.32

Case 2: $[f, g] \subset [a, b]$ or $[a, b] \subset [f, g]$. We may assume without loss of generality that the first relation holds (see Figure 13). Let F be the graph induced by the path $0, a$, $a + 1, \ldots, f, f + 1, \ldots, g, l$. Clearly, $|V(D)| \geq |[a, b]| = |[a, g]| + |]g, b]|$, therefore, $d_F(0) + d_F(l) \geq d_D(0) + d_D(l) - |]g, b]| \geq |V(D)| + 2 - |]g, b]| \geq |[a, g]| + 2 = |V(F)|$. If $|[a, g]| \geq p - 2$ then, by Theorem 6.27, F is pancyclic, so it contains a C_p. Using a similar argument we can show the existence of C_p if $|[f, b]| \geq p - 2$. Observe that if $|[a, g]| < p - 2$ and $|[f, b]| < p - 2$ then $|[a, f[| + |[f, g][| < p - 2$, $|[f, g][| + |]g, b]| < p - 2$, hence $d_D(0) + d_D(l) \leq |[a, f[| + |]g, b]| + 2|[f, g][| < 2(p - 2) < |V(D)| + 2$, and we get a contradiction. ■

Lemma 6.33. *Let $C = 0, 1, \ldots, n - 1, 0$ be a hamiltonian cycle in a graph G and 0, l two nonadjacent vertices on C. Let p, $p > 4$, be an integer and let D be a segment of C contained in $[1, l - 1]$ such that $|V(D)| \geq 7p/4$, $p > n - l + 1 \geq 3p/4$ and $d_D(0) + d_D(l) \geq |V(D)| + 2$. Then G contains a cycle of length p.*

Proof. We may assume $|V(D)| < 2(p - 2)$ because, by Lemma 6.32, our assertion is true for $|V(D)| \geq 2(p - 2)$. Moreover, by Lemma 6.31, our assertion is valid if $d_D(0) > p - 2$. So we shall assume $d_D(0) \leq p - 2$.

Suppose as in the proof of Lemma 6.32 that $a, b \in N_D(0)$, $f, g \in N_D(l)$, $N_D(0) \subset [a, b]$ and $N_D(l) \subset [f, g]$. Clearly, $[a, b] \cap [f, g] \neq \emptyset$, $N_D(0) \neq \emptyset$ and $N_D(l) \neq \emptyset$.

Case 1: $[a, b]$ is not contained in $[f, g]$ and $[f, g]$ is not contained in $[a, b]$. Assume without loss of generality that $a < f < b < g$ and denote by F the graph induced by $0, a, a+1, \ldots, f, \ldots, b, \ldots, g-1, g, l$ (see Figure 12). If $|[a, g]| \leq (|V(D)|+2)/2$ then $d_D(0)+d_D(l) < 2(|V(D)|+2)/2 = |V(D)|+2$, which contradicts our assumption. So we shall assume $|[a, g]| > (|V(D)|+2)/2 > \frac{7}{8}p$.

Case 1.1: $|[a, g]| \geq p-2$. By using the same method as in the proof of Lemma 6.32 we can show that F is pancyclic and contains a cycle of length p.

Case 1.2: $7p/8 < |[a, g]| \leq p-3$. Now we have $d_F(0) + d_F(l) = d_D(0) + d_D(l) \geq |V(D)| + 2 \geq \frac{7}{4}p + 2 = (p-1) + 3p/4 + 3 \geq |V(F)| + 3p/4 + 3$. From Theorem 6.28, for any m, $2 \leq m \leq 3p/4 + 6$, there exists a $(0, l)$-path of length m. Thus we can extend the path $l, l+1, \ldots, n-1, 0$ of length $n-l$, where $3p/4 - 1 \leq n - l \leq p - 2$, by a $(0, l)$-path of length $p-(n-l)$ and we obtain a cycle of length p.

Case 2: $[f, g] \subset [a, b]$ or $[a, b] \subset [f, g]$. We may assume without loss of generality that $[f, g] \subset [a, b]$ (see Figure 13). If $|[a, g]| \geq p-2$ or $|[f, b]| \geq p-2$ we proceed as in the proof of Lemma 6.32 and we show that G has a C_p. Therefore, we may assume $|[a, g]| < p-2$ and $|[f, b]| < p-2$. Obviously, $\frac{7}{8}p + 1 \leq \frac{1}{2}(|V(D)|+2) \leq \frac{1}{2}(d_D(0)+d_D(l)) \leq |[a, b]|$.

Case 2.1: $|[a, b]| \geq \frac{5}{4}p$. Since $|[a, b]| = |[a, f[| + |[f, b]| < |[a, f[| + p - 2$, we have $|[a, f[| > \frac{1}{4}p + 2$. It follows that $d_D(l) \leq |[f, g]| = |[a, g]| - |[a, f[| < (p-2) - (\frac{1}{4}p + 2) = \frac{3}{4}p - 4$. Since $d_D(0) \leq p-2$, we have $d_D(0) + d_D(l) < \frac{7}{4}p - 6 < \frac{7}{4}p + 2 \leq |V(D)| + 2$, a contradiction.

Case 2.2: $\frac{7}{8}p + 1 \leq |[a, b]| < \frac{5}{4}p$. Since $d_D(0) + d_D(l) \geq |V(D)| + 2 \geq \frac{7}{4}p + 2$ and $d_D(0) \leq p-2$ it follows that $|[f, g]| \geq d_D(l) \geq \frac{7}{4}p + 2 - (p-2) = \frac{3}{4}p + 4$. Assume without loss of generality that $|]g, b]| \leq |[a, f[|$. Therefore, $2(|]g, b]|) \leq (|]g, b]|) + |[a, f[| = (|[a, b]|) - |[f, g][| < \frac{5}{4}p - (\frac{3}{4}p + 4) = p/2 - 4$. Hence $|]g, b]| < p/4 - 2$. Furthermore, $d_{[a,g]}(0) + d_{[a,g]}(l) \geq \frac{7}{4}p + 2 - |]g, b]| > \frac{7}{4}p + 2 - (p/4 - 2) = \frac{3}{2}p + 4$. Denote by F the graph induced by the path $0, a, (a+1), \ldots, f, \ldots, g, l$. We have $d_F(0) + d_F(l) = d_{[a,g]}(0) + d_{[a,g]}(l) \geq (p-1) + \frac{1}{2}p + 5 \geq |V(F)| + \frac{1}{2}p + 5$. By Theorem 6.28, for every m, $2 \leq m \leq \frac{1}{2}p + 8$, F contains a $(0, l)$-path of length m. Since the length of the path $l, l+1, \ldots, n-1, 0$ is $n-l$ and $3p/4 - 1 \leq n - l \leq p - 2$, we can choose a $(0, l)$-path of length $p - n + l$. This gives a cycle of length p. ■

Lemma 6.34. *Let $C = 0, 1, \ldots, n-1, 0$ be a hamiltonian cycle in a graph G and 0, l two nonadjacent vertices on C. Let $p \geq 5$ and $d \geq 0$ be two integers with $0 \leq d \leq n - l - 3 < p - 4$ and such that $d_{[l,0]}(0) + d_{[l,0]}(l) \geq (n - l + 1) + d$. Suppose D is a segment contained in $[1, l-1]$ such that $d_D(0) > 0$, $d_D(l) > 0$, $|V(D)| = 2p - d - 9$ and $d_D(0) + d_D(l) \geq 2p - 2d - 9$. Then G contains a C_p.*

Proof. Since $p > n - l + 1 \geq d + 4 \geq 4$, it follows that $p - d - 5 \geq 0$ and $|V(D)| = 2p - d - 9 \geq 1$. Thus, $D = [a, a + 2p - d - 10]$, where a is an integer, $1 \leq a \leq l - 2p + d + 9$. However, in order to simplify the calculations, we will assume that $D = [1, 2p - d - 9]$, and, though 0 and 1 are adjacent, we will also consider the case when $(0, 1) \notin E(G)$, since D is supposed to be any segment in $[1, l-1]$.

If $p-d-5=0$, then $p=d+5$, $n-l+1=d+4$, $d_{[l,0]}(0)+d_{[l,0]}(l)=2(n-l+1)-4$ and $|V(D)|=d+1=p-4$. Because $d_D(0)>0$ and $d_D(l)>0$, we can easily find a p-cycle formed by a $(0,l)$-path, a segment contained in D and two edges of G. So we shall assume $p-d-5\geq 1$. Now we may write $D=F_1\cup F_2\cup F_3$, where $F_1=[1,p-d-5]$, $F_2=[p-d-4,p-4]$ and $F_3=[p-3,2p-d-9]$. Suppose, contrary to our claim, that G has no cycle of length p.

Case 1: $F_1\cap N(0)\neq\emptyset$ and $F_1\cap N(l)\neq\emptyset$ (or $F_3\cap N(0)\neq\emptyset$ and $F_3\cap N(l)\neq\emptyset$, but in this case the proof is analogous). Choose $k\in F_1$ such that $k\in N(0)$ and $(0,j)\notin E$ for $j<k$ and $j\in F_1$. If $(l,s)\in E$ with $k+p-d-5\leq s\leq k+p-4$, then G has the cycle $0,u_1,u_2,\ldots,u_r,l,s,s-1,\ldots,k+1,k,0$, where $0,u_1,\ldots,u_r,l$ is a $(0,l)$-path of length $r+1=p-s+k-2$, $2\leq p-s+k-2\leq d+3$. Theorem 6.28 ensures the existence of such a path. This gives a cycle of length p, a contradiction. Therefore, the vertices $k+p-d-5,k+p-d-4,\ldots,k+p-4$ do not belong to $N(l)$. Suppose now $j>k$ and $j\in F_1$. If $(0,j)\in E$, then $(l,j+p-4)\notin E$, $(j+p-4\in F_3)$, because otherwise $0,j,j+1,\ldots,j+p-4,l,u_1,0$ (where $0,u_1,l$ is a $(0,l)$-path of length 2) would be a cycle of length p. Now, if $d_{F_1}(0)=t>0$, then $d_{F_2\cup F_3}(l)\leq 2p-d-9-(p-d-5)-(t-1+d+2)=p-d-5-t$. Thus

$$d_{F_1}(0)+d_{F_2\cup F_3}(l)\leq p-d-5.$$

By symmetry,

$$d_{F_1}(l)+d_{F_2\cup F_3}(0)\leq p-d-5,$$

hence $d_D(0)+d_D(l)\leq 2p-2d-10$ and we get a contradiction.

Case 2: $F_1\cap N(0)\neq\emptyset$, $F_1\cap N(l)=\emptyset$, $F_3\cap N(0)=\emptyset$ and $F_3\cap N(l)\neq\emptyset$ (we omit the case $F_1\cap N(l)\neq\emptyset$, $F_1\cap N(0)=\emptyset$, $F_3\cap N(l)=\emptyset$ and $F_3\cap N(0)\neq\emptyset$). By assumption, there is $s\in F_1$ such that 0 and s are adjacent. Then for every $j\in[s+p-d-5,s+p-4]\subset F_2\cup F_3$ we have $(l,j)\notin E$, because otherwise G would contain a C_p. Furthermore, there is $s_1\in F_3$ with $(l,s_1)\in E$. Thus, for any $j\in[s_1-(p-4),s_1-(p-d-5)]\subset F_1\cup F_2$, $(0,j)\notin E$. Therefore, $d_D(0)+d_D(l)\leq 2((p-4)-(d+2))=2p-2d-12$, a contradiction.

Case 3: $F_1\cap N(0)=\emptyset$ and $F_3\cap N(0)=\emptyset$ (or $F_1\cap N(l)=\emptyset$ and $F_3\cap N(l)=\emptyset$). Hence there is $s\in F_2$ with $(0,s)\in E$. We assume that for $j<s$, $(0,j)\notin E$. Therefore, by the same argument as in the previous case, $[s+p-d-5,2p-d-9]\cap N(l)=\emptyset$ and $[1,s-(p-d-5)]\cap N(l)=\emptyset$. Note that $|[s+p-d-5,2p-d-9]\cup[1,s-(p-d-5)]|=d+2$. Furthermore, there is $s_1\in[s-(p-d-5)+1,s+(p-d-5)-1]$ with $(l,s_1)\in E$. Suppose $F_2\cap N(l)=\emptyset$ and s_1 belongs to (for example) F_1. Using the same argument as in Case 1 we get

$$d_{F_1}(l)+d_{F_2\cup F_3}(0)\leq p-d-5,$$

and

$$d_{F_1}(0)+d_{F_2\cup F_3}(l)=0+d_{F_3}(l)\leq p-d-5.$$

Thus, $d_D(0)+d_D(l)\leq 2p-2d-10$, a contradiction. So assume $s_1\in F_2$. Clearly, $[s_1+p-d-5,2p-d-9]\cap N(0)=\emptyset$ and $[1,s_1-(p-d-5)]\cap N(0)=\emptyset$. Now if $w\in\,]s_1,s+p-d-5[$ and w,l are adjacent, then $w-(p-d-5)\in\,]s_1-(p-d-5),s[$ and $(w-(p-d-5),0)\notin E$. Similarly, if $w\in\,]s-(p-d-5),s_1[\,\cap N(l)$ then $w+p-d-5\in\,]s,s_1+p-d-5[$ and

$w+p-d-5$ and 0 are not adjacent. Therefore, as in the previous cases, $d_D(0)+d_D(l) \le 2(2p-d-9-(d+2))-(2p-2d-10-2)) = 2p-2d-10$, a contradiction. ■

THEOREM 6.35. *Let G be a hamiltonian graph of order n and x, y two nonadjacent vertices of G with $d(x)+d(y) \ge n+z$, where $1 \le z \le (n+8)/3$. Then G contains all cycles of length p with $3 \le p \le \frac{4}{19}(n+z+8)$.*

Proof. Let $C = 0, 1, \ldots, n-1, 0$ be a hamiltonian cycle in G. We may assume without loss of generality that $x = 0$, $y = l$ and $n/2 \le l \le n-2$. Put $A = G([l, 0])$ and $B = G([0, l])$. It is easy to verify that the theorem is true for $n \le 7$. Therefore, we shall assume $n \ge 8$. Thus, $(4n+4z+32)/19 \le n/2+1$ for $z \le \frac{1}{3}(n+8)$.

Suppose, contrary to our assertion, that G does not contain any cycle C_p, for some $p \le (4n+4z+32)/19$. So if $d_B(0)+d_B(l) \ge l+1 \ge n/2+1$, then, by Theorem 6.27, G contains cycles of all lengths between 3 and $l+1$ and we get a contradiction. Suppose then $d_B(0)+d_B(l) \le l$. Thus $d_A(0)+d_A(l) \ge n+z-l$. Define $d := d_A(0)+d_A(l)-(n-l+1)$. Obviously, $0 \le z-1 \le d \le (n-l+1)-4$ and $d(0)+d(l) \le l+n-l+1+d = n+d+1$. It follows from Theorem 6.27 that G contains a C_m for any m between 3 and $n-l+1$. Thus $p > n-l+1 \ge d+4 \ge 4$ and $p-d-5 \ge 0$.

CASE 1: $[1, 2p-d-9] \cap N(l) = \emptyset$ and $[l-(2p-d-9), l-1] \cap N(0) = \emptyset$. Observe that $2p-d-9 < n/2 \le l$ if $p \le (4n+4z+32)/19$. Thus $l-1 = k(2p-d-9)+r$, where $k \ge 1$ and r are the integers such that $0 \le r < 2p-d-9$. We can write

$$[1, l-1] = H_{k+1} \cup \bigcup_{i=1}^{k} H_i,$$

where $H_i = [(i-1)(2p-d-9)+1, i(2p-d-9)]$ for $i = 1, 2, \ldots, k$ and $H_{k+1} = [k(2p-d-9)+1, l-1]$ ($H_{k+1} = \emptyset$ if $r = 0$). Suppose there is $j \le k$ with $d(0) \cap H_j \ne \emptyset$ and $d(l) \cap H_j \ne \emptyset$. Then, by Lemma 6.34, $p-d-5 > 0$ and $d_{H_j}(0)+d_{H_j}(l) \le 2p-2d-10$. Moreover, since $2p-2d-10 < 2p-d-9$, we have $d_{H_i}(0)+d_{H_i}(l) \le |V(H_i)|$ for all $i \ne j$ (this is also true for $i = k+1$ because $H_{k+1} \subset [l-(2p-d-9), l-1]$ and $[l-(2p-d-9), l-1] \cap N(0) = \emptyset$). Therefore, $d(0)+d(l) \le (n-l+1+d)+2p-2d-10+(l-1-(2p-d-9)) = n-1 < n+z$, a contradiction. Thus, for every i, $N(0) \cap H_i = \emptyset$ or $N(l) \cap H_i = \emptyset$. Hence, there is j such that $N(l) \cap H_j = \emptyset$ and $N(0) \cap H_{j+1} = \emptyset$. Suppose now $N(0) \cap H_j \ne \emptyset$. Then we can find $s \in V(H_j)$ such that $(0, s) \in E$ and $(0, r) \notin E$ for $r > s$ and $r \in V(H_j)$. By assumption, $s < l-(2p-d-9)$. Set $D = [s, s+2p-d-10]$. If $N(l) \cap D \ne \emptyset$, then $p-d-5 > 0$, $d_D(0)+d_D(l) \le 2p-2d-10$ and, consequently, $d(0)+d(l) \le n-1$, which is a contradiction. Therefore, $N(l) \cap D = \emptyset$, $d_D(0)+d_D(l) = 1 \le 2p-2d-9$ and we get a contradiction as in the previous cases.

CASE 2: $3p/4 \le n-l+1 < p$ and ($[1, 2p-d-9] \cap N(l) \ne \emptyset$ or $[l-(2p-d-9), l-1] \cap N(0) \ne \emptyset$). Hence $p-d-5 > 0$. We shall assume without loss of generality $[1, 2p-d-9] \cap N(l) \ne \emptyset$. Thus, $|[2p-d-8, l-1]| = n-(2p-d-9)-(n-l+1) > n-2p+d+9-p \ge (\frac{19}{4}p-z-8)-3p+d+9 \ge \frac{7}{4}p$ if $p \le (4n+4z+32)/19$ and $z \le d+1$. Set $D = [2p-d-8, l-1]$. Since G does not contain any cycle of length p we have, by Lemmas 6.33 and 6.34,

$$\begin{aligned} d(0)+d(l) &\leq (|V(D)|+1)+(2p-2d-10)+(n-l+1+d) \\ &= |V(D)|+(2p-d-9)+(n-l+1) = n < n+z, \end{aligned}$$

which is a contradiction.

CASE 3: $3p/4 > n-l+1$ and $([1, 2p-d-9] \cap N(l) \neq \emptyset$ or $[l-(2p-d-9), l-1] \cap N(0) \neq \emptyset)$. Hence $p-d-5 > 0$. We assume as in the previous case that $[1, 2p-d-9] \cap N(l) \neq \emptyset$. Set $D = [2p-d-8, l-1]$. Clearly,

$$\begin{aligned} |V(D)| &= n-(2p-d-9)-(n-l+1) \\ &> \tfrac{19}{4}p-(d+1)-8-2p+d+9-\tfrac{3}{4}p = 2p > 2p-4. \end{aligned}$$

By Lemmas 6.32 and 6.34, $d(0)+d(l) \leq (|V(D)|+1)+(2p-2d-10)+(n-l+1+d) = n < n+z$, which contradicts our assumption. ■

THEOREM 6.36. *Let G be a hamiltonian graph of order n having two nonadjacent vertices x and y satisfying $d(x)+d(y) \geq n+z$, where $z \geq \frac{5}{18}n$ is an integer. Then there are cycles in G of all lengths p with $3 \leq p \leq n/2+z/2+1$.*

Proof. I. First we shall prove that G contains cycles of every length between 3 and $\frac{11}{30}n+2$. It is a simple matter to check that this is true for $n \leq 7$. So we shall assume that $n \geq 8$. Observe also that $z \geq \frac{5}{18}n \geq 2$ and $\frac{11}{30}n+2 \leq n/2+1$ for $n \geq 8$.

Suppose, contrary to our assertion, that G does not contain any cycle of length p, for some $p \leq \frac{11}{30}n+2$. Using the notation and the same argument as in the proof of Theorem 6.35 we conclude that $d_B(0)+d_B(l) \leq l$, $d_A(0)+d_A(l) = n-l+1+d$, $(0 \leq z-1 \leq d \leq n-l+1-4)$ and $p > n-l+1 \geq d+4 \geq 4$. Moreover, the following inequalities hold:

(1) $2(p-d-5) < d+1$,
(2) $4(p-d-5)+d+1 < l-1$.

Indeed, by assumption, $d \geq z-1 \geq \frac{5}{18}n-1$, so $2p-2d-10 \leq \frac{11}{15}n+4-\frac{5}{9}n+2-10 = \frac{8}{45}n-4 < \frac{5}{18}n-1+1 \leq d+1$. Furthermore, $4(p-d-5)+d+1 = 4p-3d-19 \leq \frac{22}{15}n+8-\frac{5}{6}n+3-19 = \frac{19}{30}n-8$. On the other hand, $l > n-p+1 \geq n-\frac{11}{30}n-2+1 = \frac{19}{30}n-1$, thus the second inequality is true.

Denote by k the quotient and by r the remainder when $l-1$ is divided by $2p-d-9$. By (2), $k \geq 1$. Thus,

$$[1, l-1] = H_{k+1} \cup \bigcup_{i=1}^{k} H_i,$$

where $H_i = [(i-1)(2p-d-9)+1, i(2p-d-9)]$ for $i = 1, 2, \ldots, k$ and $H_{k+1} = [k(2p-d-9)+1, l-1]$ ($H_{k+1} = \emptyset$ if $r = 0$).

CASE 1: $[1, 2p-d-9] \cap N(l) = \emptyset$ and $[l-(2p-d-9), l-1] \cap N(0) = \emptyset$. Proceeding in the same manner as in Case 1 of the proof of Theorem 6.35 we get a contradiction.

CASE 2: $[1, 2p-d-9] \cap N(l) \neq \emptyset$ or $[l-(2p-d-9), l-1] \cap N(0) \neq \emptyset$. Hence, by Lemma 6.34, $p-d-5 > 0$. We shall assume without loss of generality $[1, 2p-d-9] \cap N(l) \neq \emptyset$.

Case 2.1: $k \geq 2$.

Case 2.1.1: For every $i \geq 2$, $N(0) \cap H_i = \emptyset$ or $N(l) \cap H_i = \emptyset$. Applying the same method as in the proof of Theorem 6.35 (Case 1) we obtain a contradiction.

Case 2.1.2: There is i, $2 \leq i \leq k$, such that $N(0) \cap H_i \neq \emptyset$ and $N(l) \cap H_i \neq \emptyset$. We may assume without loss of generality that $i = 2$, $d_{H_{k+1}}(0) > 0$ and $d_{H_{k+1}}(l) > 0$. Therefore, $d_{H_{k+1}}(0) + d_{H_{k+1}}(l) \leq 2p - 2d - 10$. Applying (1) we have

$$\begin{aligned} d_{[1,l-1]}(0) + d_{[1,l-1]}(l) &= \sum_{j=1}^{k+1} (d_{H_j}(0) + d_{H_j}(l)) \\ &\leq 2(p-d-5) + 2(p-d-5) + \sum_{j=3}^{k} |V(H_j)| + 2(p-d-5) \\ &< (2(2p-d-9) + \sum_{j=3}^{k} |V(H_j)|) - (d+1) \leq l - 1 - (d+1). \end{aligned}$$

(We put $\sum_{j=3}^{k} |V(H_j)| = 0$ if $k < 3$.) Hence $d(0) + d(l) \leq n - l + 1 + d + l - 1 - d - 1 = n - 1$, a contradiction.

Case 2.1.3: $N(0) \cap H_{k+1} \neq \emptyset$ and $N(l) \cap H_{k+1} \neq \emptyset$. We may assume that $N(0) \cap H_k = \emptyset$ or $N(l) \cap H_k = \emptyset$, because otherwise we have the previous case. Suppose then $N(l) \cap H_k = \emptyset$. Choose $s \in H_{k+1}$ satisfying $(l, s) \in E$ and $(l, j) \notin E$ for $j < s$ and $j \in H_{k+1}$. Therefore, for $i \in [s - (p-4), s - (p-d-5)]$ we have $(0, i) \notin E$ (since otherwise G would have a C_p). Moreover, for $j > s$, $j - (p-d-5)$ is not adjacent to 0. Hence,

$$\begin{aligned} d_{H_k \cup H_{k+1}}(0) + d_{H_k \cup H_{k+1}}(l) &\leq |V(H_k)| + |V(H_{k+1})| + d_{H_{k+1}}(l) - (d + 2 + d_{H_{k+1}}(l) - 1) \\ &= |V(H_k)| + |V(H_{k+1})| - (d+1). \end{aligned}$$

Thus, $d_{[1,l-1]}(0) + d_{[1,l-1]}(l) \leq l - 1 - (d+1)$, and consequently $d(0) + d(l) \leq n - l + 1 + d + l - 1 - d - 1 = n - 1$, a contradiction.

Case 2.2: $k = 1$. By (2), $2p - 2d - 10 < |V(H_{k+1})| = |V(H_2)| < 2p - d - 9$. Moreover, since $d_{H_2}(0) + d_{H_2}(l) \leq 2p - 2d - 10$ if $N(0) \cap H_2 \neq \emptyset$ and $N(l) \cap H_2 \neq \emptyset$, it follows that $d_{H_2}(0) + d_{H_2}(l) \leq \max(|V(H_2)|, 2p - 2d - 10) = |V(H_2)|$. Therefore, $d_{[1,l-1]}(0) + d_{[1,l-1]}(l) \leq 2p - 2d - 10 + |V(H_2)| = l - 1 - (d+1)$, and $d(0) + d(l) \leq n - l + 1 + d + l - 1 - d - 1 = n - 1$, which is a contradiction.

II. Since $d(0) + d(l) \geq n + z$, the maximum degree Δ satisfies $\Delta \geq n/2 + z/2 \geq n/2 + 5n/36 = \frac{23}{36}n$. From Theorem 6.23, G contains a cycle C_p for every p belonging to $[n - \Delta + 2, \Delta + 1]$. Since $n - \Delta + 2 \leq n - \frac{23}{36}n + 2 = \frac{13}{36}n + 2 < \frac{11}{30}n + 2$, it follows that G has cycles of every length between 3 and $\Delta + 1$. Therefore, G contains a cycle C_p for every p between 3 and $n/2 + z/2 + 1$, which finishes the proof of the theorem.

The following corollary is an easy consequence of Theorems 6.26, 6.29, 6.35 and 6.36.

Corollary 6.37. *Let G be a hamiltonian graph of order n having two nonadjacent vertices x and y with the degree sum $d(x) + d(y) \geq n + z$, for some integer $z \geq 0$. Then G has cycles of every length p, $3 \leq p \leq \phi(n, z)$, where*

$$\phi(n,z) = \begin{cases} \dfrac{n+13}{5} & \textit{if } z = 0,\ n \textit{ is odd and } n \geq 33; \\ \dfrac{4n+4z+32}{19} & \textit{if } 1 \leq z < \dfrac{5}{18}n; \\ \dfrac{n+z+2}{2} & \textit{if } \dfrac{5}{18}n \leq z \leq \dfrac{n+8}{3}; \\ \dfrac{2n+z-1}{3} & \textit{if } \dfrac{n+8}{3} < z \leq n-4. \end{cases}$$

Consider now the following example (see also [243]). Let $C = 0, 1, \ldots, n-1, 0$ be a cycle in G with $n = 6s + 2$ and let all vertices in A be adjacent to 0 and $l = n - s - 1$, where $A = \{1, \ldots, s+x, n-2s-1, n-2s, \ldots, n-s-2, n-s, \ldots, n-1\}$ for some x, $x \leq n - 3s - 2$. Clearly in such a graph $d(0) + d(l) = 6s + 2x = n - 2 + 2x$. Any cycle in G either contains the path $s + x + 1, \ldots, n - 2s - 2$ or avoids it, so the graph G has every cycle C_p for $3 \leq p \leq 2s + x + 2$ and for $3s - x + 3 \leq p \leq n$. A cycle may be missing in G if $2s + x + 2 + 1 < 3s - x + 3$, i.e., $x < (n-2)/12$. Setting $2x - 2 = z$ we obtain a graph containing cycles of length p for every p such that $3 \leq p \leq (n-2)/3 + \frac{1}{2}z + 3$ and we cannot increase the upper bound for $z < (n-14)/6$.

In [185] we constructed a hamiltonian graph G of order n with two vertices x and y satisfying $d(x) + d(y) \geq 2n - 4r - 2$ $(3 \leq 3r < n-1)$ that contains no cycle of length $n - r + 1$. In the light of the two examples we conjecture the following.

Conjecture 6.38. *Let G be a hamiltonian graph of order n and let x and y be two distinct vertices of G that satisfy $d(x) + d(y) \geq n + z$ for some integer $z \geq 0$. Then there exist two constants c_1 et c_2 such that*

(i) *G contains a cycle C_p for every p with $3 \leq p \leq n/3 + z/2 + c_2$ if $0 \leq z < n/6 + c_1$;*
(ii) *G contains all cycles C_p with $3 \leq p \leq (3n + z + 2)/4$ if $z \geq n/6 + c_1$.*

Clearly, Theorems 6.26, 6.29, 6.35 and 6.36 are not best possible but they make the first steps towards the solution of the conjecture.

The following theorem of Kouider and Marczyk [185] is another generalization of Proposition 4.5 due to Bondy [55].

Theorem 6.39. *Let $C = 0, 1, \ldots, n-1, 0$ be a hamiltonian cycle in a graph G and $(0, l)$ an edge of G $(n/2 \leq l \leq n-1)$. If $d(0) + d(l) > \min(2l, 2n - l - 1)$, then G contains a cycle C_p for every integer p with $3 \leq p \leq l + 1$.*

7. Further extensions of classical results on cycles

7.1. Partitioning the vertices into cycles. In [3], Aigner and Brandt proved an interesting Dirac-type result, conjectured in a weaker form by Sauer and Spencer [241].

Theorem 7.1. *Every graph of order n with $\delta(G) \geq (2n-1)/3$ contains any graph with at most n vertices and maximum degree two.*

Corollary 7.2. *Let $n, n_1, n_2, \ldots, n_k$ be integers such that $n_i \geq 3$ for all i's and $n \geq n_1 + \cdots + n_k$. If G is a graph of order n with $\delta(G) \geq (2n-1)/3$, then G contains the vertex-disjoint union of the cycles $C_{n_1} \cup \cdots \cup C_{n_k}$.*

However, the bound $(2n-1)/3$ can be lowered. In [107] El-Zahar proposed the following conjecture.

Conjecture 7.3. *Let G be a graph of order $n = n_1 + \cdots + n_k$ with $n_i \geq 3$ for $i = 1, \ldots, k$, and*

$$\delta(G) \geq \sum_{i=1}^{k} \lceil n_i/2 \rceil.$$

Then G contains the vertex-disjoint union of the cycles $C_{n_1} \cup \cdots \cup C_{n_k}$.

The graph $K_{s-1} \vee K_{\lceil (n-s+1)/2 \rceil, \lfloor (n-s+1)/2 \rfloor}$ has minimum degree $\lfloor (n+s-1)/2 \rfloor < (n+s)/2$ and contains no s vertex-disjoint odd length cycles. This means that this conjecture is best possible. It is obvious that for $k = 1$ this conjecture is true by Dirac's theorem. The case $k = 2$ was solved by El-Zahar [107]. In [1] Abbasi proved this conjecture for large n using Szemerédi's regularity lemma (cf. the book of Bollobás [49]).

7.2. Arbitrarily vertex decomposable graphs. According to Corollary 7.2, if the minimum degree of a graph G of order n is at least $(2n-1)/3$, then we can partition the vertex set of G into 2-connected subgraphs of prescribed sizes. Clearly, one can consider a similar condition that guarantees the existence of such partitions into connected subgraphs. In order to present some results related to this topic, we need several definitions formulated below.

A sequence $\tau = (n_1, \ldots, n_k)$ of positive integers is called *admissible* for a graph G of order n if $n_1 + \cdots + n_k = n$. If $\tau = (n_1, \ldots, n_k)$ is an admissible sequence for G and there exists a partition $(V_1, \ldots, V_k)$ of the vertex set V such that for each $i \in \{1, \ldots, k\}$, $|V_i| = n_i$ and a subgraph induced by V_i is connected, then τ is called *realizable in G* and the sequence $(V_1, \ldots, V_k)$ is said to be a *G-realization of τ* or a *realization of τ in G*. A graph G is *arbitrarily vertex decomposable* (*avd* for short) if for each admissible sequence τ for G there exists a G-realization of τ. It is clear that each avd graph admits a perfect matching or a matching that omits exactly one vertex (called a quasi-perfect matching). Note also that if G_1 is a spanning subgraph of a graph G_2 and G_1 is avd, then so is G_2.

Arbitrarily vertex decomposable trees have been investigated in several papers. In [176] Horňák and Woźniak conjectured that if T is a tree with maximum degree $\Delta(T)$ at least five, then T is not avd. This conjecture was proved by Barth and Fournier [21].

Let $S(2, a, b)$ denote a tree obtained from a path $x_1, \ldots x_a, x_{a+1}, \ldots, x_{a+b-1}$ of order $a + b - 1$ by adding a single vertex x and joining it to x_a. It will be called a *caterpillar with one single leg* or simply a *caterpillar $S(2, a, b)$*.

The first result characterizing such simple trees was found by Barth et al. [20] and, independently, by Horňák and Woźniak [175].

Proposition 7.4. *A caterpillar $S(2, a, b)$ is avd if and only if the integers a and $n = a+b$ are coprime. Moreover, each admissible and nonrealizable sequence in $S(2, a, b)$ is of the form $(d, \ldots, d)$, where $a \equiv n \equiv 0 \pmod{d}$ and $d > 1$.*

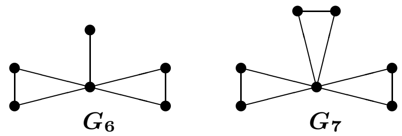

Fig. 14. Two non-avd graphs

The next proposition was presented in [215].

PROPOSITION 7.5. *Let G be the graph of order $n \geq 4$ obtained by taking a path $P = x_1, \ldots, x_{n-1}$, a single vertex x and by adding the edges $xx_{i_1}, xx_{i_2}, \ldots, xx_{i_p}$, where $1 < i_1 < \cdots < i_p < n-1$ and $p \geq 1$. Then G is not avd if and only if there are integers $d > 1$, $\lambda, \lambda_1, \ldots, \lambda_p$ such that $n = \lambda d$ and $i_j = \lambda_j d$ for $j = 1, \ldots, p$. Moreover, each admissible and nonrealizable sequence in G is of the form $(d, d, \ldots, d)$, where $i_j \equiv n \equiv 0 \pmod d$ $(j = 1, \ldots, p)$ and $d > 1$.*

Proof. Suppose that the integers $d > 1$, λ, $\lambda_1, \ldots, \lambda_p$ satisfy the conditions $n = \lambda d$ and $i_j = \lambda_j d$ for $j = 1, \ldots, p$ and consider the admissible sequence $\tau = (\underbrace{d, \ldots, d}_{\lambda})$ for G. Observe that if G' is a connected subgraph of G of order d which contains the vertex x, then the connected component of $G - V(G')$ containing the vertex x_1 is a path P' such that d does not divide the order of P'. Thus, τ is not realizable in G. Conversely, if $\tau = (n_1, \ldots, n_\lambda)$ is an admissible sequence for G that is not realizable in G, then τ is also not realizable in the caterpillar $S(2, i_1, n - i_1)$. By Proposition 7.4, there are two integers $d > 1$ and λ_1 such that $n_1 = \cdots = n_\lambda = d$ and $i_1 = \lambda_1 d$. The sequence τ cannot be realizable in the caterpillar $S(2, i_2, n - i_2)$, therefore, again by Proposition 7.4, $i_2 = \lambda_2 d$ for some integer λ_2. Repeating the same argument we prove that the conditions of the proposition hold. ∎

In [20] and [21] Barth et al. and Barth and Fournier studied a family of trees each of them being homeomorphic to $K_{1,3}$ or $K_{1,4}$ (they call them tripods or 4-pods) and showed that determining if such a tree is avd can be done using a polynomial algorithm.

Observe that any necessary condition for a graph to contain a perfect matching (or a quasi-perfect matching) is a necessary condition for a graph to be arbitrarily vertex decomposable. Thus we will assume that a graph under consideration contains a perfect matching or a quasi-perfect matching. Clearly, the condition that the independence number of an n-vertex graph G is at most $\lceil n/2 \rceil$ is a necessary condition for G to contain such a subgraph.

However, it is obvious that each graph having a hamiltonian path is avd. Thus, any graph G of order n with $\delta(G) \geq (n-1)/2$ or $\sigma_2(G) \geq n - 1$ is avd. In the next results we will lower these two bounds.

Let $n \geq 4$ be an integer. Consider the disjoint union $K_p \cup K_q \cup K_r$ of three complete graphs such that $p + q + r = n - 1$ with numbers p, q, r as equal as possible. Denote by G_n the join $K_1 \vee (K_p \cup K_q \cup K_r)$. We may assume that $p \leq q \leq r \leq p+1$, hence $n-1 = 3p+d$, where $0 \leq d \leq 2$ (in Fig. 14 we show G_6 and G_7). Observe that $\sigma_2(G_n) = \lfloor 2n/3 \rfloor$ for $n \equiv 1 \pmod 3$ and $\sigma_2(G_n) = \lfloor 2n/3 \rfloor - 1$ otherwise. Every connected subgraph of $K_p \cup K_q \cup K_r$

contains at most $p+1$ vertices and the integer $w = n-2(p+2) = p+d-3$ is nonnegative for $p \geq 3$. This implies that the sequence $(w, p+2, p+2)$ ($(p+2, p+2)$ for $w = 0$) is admissible and nonrealizable in G_n for $n \geq 10$. It is easy to check that for each $4 \leq n \leq 9$, $n \neq 5$ ($p \leq 2$) there is an admissible sequence of the form $(w, p+2, p+2)$, $(p+2, p+2)$, $(w, p+1, p+1)$ or $(p+1, p+1)$ which is not realizable in the graph G_n.

In [216] Marczyk obtained the following result on avd graphs that involves an Ore-type condition.

THEOREM 7.6. *Let G be a connected graph of order n such that $\sigma_2(G) \geq n-3$, $\alpha(G)$ is at most $\lceil n/2 \rceil$ and G is isomorphic neither to G_6 nor to G_7. Then G is avd.*

Proof. Suppose G is not avd and satisfies the hypothesis of our theorem. Then G is not traceable, so $n \geq 4$, and by Theorem 3.17 due to Pósa, there exists in G a path of length at least $n-3$.

CASE 1: The length of a longest path is $n-3$. Let $P = x_1, \ldots, x_{n-2}$ be such a path and let x and y be two vertices outside P such that $d_P(x) \geq d_P(y)$. Denote by $A = N_P(x)$ the set of neighbors of x on P and let $p := d_P(x) = |A|$.

CASE 1.1: x and y are not adjacent. Hence $d_P(x) = d(x) \geq (n-3)/2$ and, since G is connected and the length of the longest path equals $n-3$, we have $p \geq 1$, $x_1 x \notin E$, $x_{n-2} x \notin E$ and $x_1 x_{n-2} \notin E(G)$. Furthermore, there is at least one vertex between any two consecutive neighbors of x on P, i.e., $A \cap A^+ = \emptyset$ and $A \cup A^+ \subseteq \{x_2, x_3, \ldots, x_{n-2}\}$. It follows that $d(x) = |A| \leq (n-3)/2$, so $d(x) = (n-3)/2$, $n \geq 5$ is odd and $A = \{x_2, x_4, \ldots, x_{n-3}\}$.

Since x and y are not adjacent, we have $d(y) \geq (n-3)/2$ and using a similar argument we can show that $d(y) = (n-3)/2$ and $N(y) = A$. Observe now that $x_1 u \notin E(G)$ for each $u \in A^+$, for otherwise $x, u^-, u^{-2}, \ldots, x_1, u, \ldots, x_{n-2}$ is a path of length $n-2$ in G, a contradiction. Using a similar argument we can show that $x_{n-2} u \notin E(G)$ for each $u \in A^+ \setminus \{x_{n-2}\}$. It is obvious that any edge of the form $x_{2i-1} x_{2j-1}$ would create a path of length at least $n-2$ in G, so the set $\{x, y, x_1, x_3, \ldots, x_{n-4}, x_{n-2}\}$ of $(n+3)/2$ vertices is independent and we obtain a contradiction.

CASE 1.2: x and y are adjacent. Obviously, the vertices $x_1, x_2, x_{n-3}, x_{n-2}$ do not belong to $N(x) \cup N(y)$, since otherwise G would contain a path of length $n-2$. We have by assumption $p + 1 + d(x_1) = d(x) + d(x_1) \geq n-3$, thus $d(x_1) \geq n-4-p$. On the other hand, we can show as in the previous case that if $u \in A^+$ then $x_1 u \notin E$, and, because $x_1 x_{n-2} \notin E$, $x x_{n-2} \notin E$ and $x x_{n-3} \notin E$, we have $A^+ \subseteq \{x_4, \ldots, x_{n-3}\}$ and $d(x_1) \leq n-4-p$. This means that x_1 is adjacent to each vertex of $V(P) \setminus (A^+ \cup \{x_{n-2}\})$. If $x_r x \in E(G)$ and $r < n-4$, then x_r^{+2} is adjacent to x_1 and it is easy to check that G contains a path of length $n-2$, a contradiction. Hence x_{n-4} is the only neighbor of x. Thus, $p = 1$, and, by symmetry, $x x_3 \in E$, so $n-4 = 3$, x_1 and x_5 are adjacent to x_3. Thus, $n = 7$, $d(x_1) = d(x_5) = n-p-4 = 2$ and $\{x_1, x_5\} \subset N(x_3)$. Therefore, since x_1 and y are not adjacent and $d(x_1) = 2$, we have $d(y) = 2$ and $N(y) = \{x, x_3\}$. Since x_2 and x_4 cannot be adjacent, G is isomorphic to G_7, which contradicts our assumption.

Case 2: The length of a longest path equals $n-2$. Let $Q = x_1, \dots, x_{n-1}$ be a path of length $n-2$ and x the unique vertex outside Q. Let $A = N(x) = \{x_{i_1}, \dots, x_{i_p}\}$, $1 \le i_1 < \dots \le i_p \le n-1$, be the set of neighbors of x. Since G is connected and nontraceable, we have $p \ge 1$, $i_1 > 1$, $i_p < n-1$ and $x_1x_{n-1} \notin E(G)$. By Proposition 7.5, there are integers $d > 1$, λ, $\lambda_1, \dots, \lambda_p$ such that $n = \lambda d$ and $i_j = \lambda_j d$ for $j = 1, \dots, p$. Hence, there is at least one vertex between any two consecutive neighbors of x on Q.

Since $x_1x \notin E(G)$ and $x_{n-1}x \notin E(G)$, it follows by assumption that $d(x_1) \ge n-3-p$ and $d(x_{n-1}) \ge n-3-p$. We can show as in the previous case that if $u \in A^+$, then $x_1u \notin E(G)$. Therefore, $d(x_1) \le n-2-p$, hence $d(x_1) \in \{n-3-p, n-2-p\}$.

Case 2.1: $x_{n-2}x \in E$, i.e., $i_p = n-2$. Thus, using Proposition 7.5, $d = 2$, n is even and $\tau = (2, \dots, 2)$ is the only nonrealizable sequence for G. Moreover, every path $x_{i_j}Qx_{i_{j+1}}$ is of even length, i.e., contains an odd number of vertices.

Case 2.1.1: There is some integer s such that $|V(x_{i_s}Qx_{i_{s+1}})| \ge 5$. Set $u = x_{i_s}^+$ and $v = x_{i_{s+1}}^-$. Notice that $x_{n-1}u \notin E$ and $x_1v \notin E$ because G is not traceable. Thus $N(x_{n-1}) \subseteq V(Q) \setminus (\{x_{n-1}, u\} \cup A^-)$ and $N(x_1) \subseteq V(Q) \setminus (\{x_1, v\} \cup A^+)$, so $d(x_{n-1}) \le n-3-p$ and $d(x_1) \le n-3-p$, therefore $d(x_1) = d(x_{n-1}) = n-3-p$. If $i_1 \ge 4$ then $x_1 \notin A^-$, so $d(x_{n-1}) \le n-1-3-p$ and we obtain a contradiction. Therefore, $xx_2 \in E$. Similarly, if for some integer $q \ne s$ we have $|V(x_{i_q}Qx_{i_{q+1}})| \ge 5$, then also $d(x_{n-1}) \le n-4-p$, and we get a contradiction. Hence, s is the unique integer j such that $|V(x_{i_j}Qx_{i_{j+1}})| \ge 5$. Now, if $|V(x_{i_s}Qx_{i_{s+1}})| > 5$, all the vertices of the path u^+Qv^- are adjacent to x_1 and x_{n-1}, so $x_1u^{+3} \in E(G)$ and $x_{n-1}u^{+2} \in E(G)$. Then $C = x_1, u^{+3}, u^{+4}, \dots, x_{n-1}, u^{+2}, u^+, \dots, x_1$ is a cycle with $V(C) = V(Q)$. Hence G is traceable, which contradicts our assumption. Suppose then $|V(x_{i_s}Qx_{i_{s+1}})| = 5$. If $uv \notin E$ then the set $\{x_1, v, x\} \cup A^+$ of $(n+2)/2$ vertices is independent, a contradiction. Assume that u and v are adjacent. Then the vertex $u^+ = v^-$ is connected to both x_1 and x_{n-1} and it can be easily seen that $G - \{u, v, x, x_{i_s}, x_1, u^+\}$ is the vertex-disjoint union of two traceable subgraphs of even order (possibly one of them is empty), thus G admits a perfect matching. But we have assumed that $\tau = (2, \dots, 2)$ is a nonrealizable sequence for G, a contradiction.

Case 2.1.2: Every path $x_{i_j}Qx_{i_{j+1}}$ contains exactly three vertices. Suppose first that $i_1 = 4$. Clearly, $N(x_{n-1}) \subseteq \{x_2, \dots, x_{n-2}\} \setminus A^-$, so $d(x_{n-1}) = n-p-3$ and $x_2x_{n-1} \in E$. Now, if $x_1x_3 \in E$, then G contains a cycle $x_1, x_3, x_4, \dots, x_{n-1}, x_2, x_1$ and G is traceable, a contradiction. Therefore, $A^- \cup \{x_1, x_{n-1}, x\}$ is an independent set of cardinality $(n+2)/2$ and we get a contradiction. Notice that the same set is independent if $i_1 = 2$. Suppose then $i_1 \ge 6$. It follows that $x_{n-1}x_2 \in E(G)$ and $x_{n-1}x_4 \in E(G)$, because $d(x_{n-1}) = n-p-3$ and x_{n-1} is adjacent to each vertex of $V(Q) \setminus (A^- \cup \{x_1, x_{n-1}\})$. Now, if $x_1x_3 \in E(G)$ or $x_1x_5 \in E(G)$, then we can easily find a cycle C with $V(C) = V(Q)$. Hence G is traceable, a contradiction. So $N(x_1) \subseteq V(Q) \setminus (A^+ \cup \{x_1, x_3, x_5\})$ and $d(x_1) \le n-p-4$, again a contradiction.

Case 2.2: $i_p \le n-3$. By the same argument as in the previous cases, $d(x_1) = n-3-p$. If $d = 2$, then we can assume $x_2x \notin E(G)$ (and also $x_3x \notin E(G)$), for otherwise we have the situation described in Case 2.1. Hence, $N(x_{n-1}) \subseteq V(Q) \setminus (A^- \cup \{x_{n-1}, x_1\})$ and

$d(x_{n-1}) = n-3-p$, whence $x_{n-1}x_2 \in E(G)$ and $x_1x_3 \in E(G)$, and we can easily find a cycle with $V(C) = V(Q)$. It follows that G is traceable, a contradiction. Therefore, $d \geq 3$. By Proposition 7.5, there are at least two vertices between any two consecutive neighbors of x on Q. It follows that for $p \geq 2$, x_1 is not adjacent to $x_{i_2}^-$ (otherwise G would have a hamiltonian path: $x_{i_1}^+, x_{i_1}^{+2}, \ldots, x_{i_2}^-, x_1, \ldots, x_{i_1}, x, x_{i_2}, \ldots, x_{n-1}$), so $N(x_1) \subseteq V(Q) \setminus (A^+ \cup \{x_1, x_{i_2}^-, x_{n-1}\}$ and $d(x_1) \leq n-4-p$, a contradiction. Thus $p = 1$, $d(x_1) = d(x_{n-1}) = n-4$, so, if $n \geq 7$, then $d(x_1) + d(x_{n-1}) = 2(n-4) \geq n-1$ and by Proposition 3.9 there is a cycle C with $V(C) = V(Q)$. Hence G is traceable, a contradiction. It follows from Proposition 7.5 that $n = 6$ and $d = 3$, furthermore, since $d(x_1) = d(x_5) = 2$, we have $x_1x_3 \in E$ and $x_5x_3 \in E$. Clearly, x_2 and x_4 are not adjacent, so G is isomorphic to G_6 and we get a contradiction. ■

The next corollary is an easy consequence of Theorem 7.6 (cf. [216]).

COROLLARY 7.7. *Let G be a 2-connected graph of order n with $\sigma_2(G) \geq n-3$. Then either*

- *G is avd,*
- *$n \geq 7$ is odd and $K_{(n+3)/2,(n-3)/2} \subseteq G \subseteq \overline{K}_{(n+3)/2} \vee K_{(n-3)/2}$,*
- *$n \geq 6$ is even, $K_{(n+2)/2,(n-2)/2} \subseteq G \subseteq \overline{K}_{(n+2)/2} \vee K_{(n-2)/2}$, or*
- *$n \geq 8$ is even, $K_{(n+2)/2,(n-2)/2} - e \subseteq G \subseteq (\overline{K}_{(n+2)/2} \vee K_{(n-2)/2}) - e$, where e is an arbitrary edge of $K_{(n+2)/2,(n-2)/2}$.*

We can also formulate an immediate corollary of Theorem 7.6 involving a Dirac-type condition.

COROLLARY 7.8. *If G is a connected graph on n vertices such that $\alpha(G) \leq \lceil n/2 \rceil$, $G \notin \{G_6, G_7\}$ and minimum degree $\delta(G) \geq (n-3)/2$, then G is avd.*

The graphs G_n show that we cannot lower the bound $n-3$ of Theorem 7.6 below $\sigma_2(G_n)$. However, we believe that if $\sigma_2(G) \geq n-6$ and n is large enough, then G is avd provided it admits a perfect matching or a quasi-perfect matching.

Notice that the problem of deciding whether a given graph is arbitrarily vertex decomposable is NP-complete [20] but we do not know if this problem is NP-complete when restricted to trees.

Another interesting problem relating to the notion of avd graphs is the characterization of on-line arbitrarily vertex decomposable graphs. The complete characterization of on-line avd trees has been recently found by Horňák et al. [174], whereas Kalinowski [183] discovered an on-line version of Theorem 7.6.

7.3. Maximal common subgraphs of the Dirac family of graphs. Denote by $\mathcal{DF}^n$ the set of graphs of order n and minimum degree $\delta \geq n/2$. It will be called the *Dirac family of graphs*. By Dirac's theorem each graph of $\mathcal{DF}^n$ has a subgraph isomorphic to C_n. Such a subgraph is said to be a *common subgraph* of the family $\mathcal{DF}^n$. One can ask a quite natural question: is C_n the only common subgraph of Dirac's family? Surprisingly, the answer is negative. In [83] Bucko et al. found another extension of Dirac's theorem.

THEOREM 7.9. *Let $G = (V, E)$ be a graph of order $|V| = n$, with $n \geq 7$. If G satisfies the Dirac's condition $\delta(G) \geq n/2$, then G contains as a subgraph a hamiltonian cycle with a chord that skips two vertices on this cycle.*

A *maximal common subgraph* of the graphs of the set $\mathcal{DF}^n$ is a common subgraph F of order n of each member of $\mathcal{DF}^n$, that is not properly contained in any larger common subgraph of each member of $\mathcal{DF}^n$. By the last result the hamiltonian cycle C_n is not a maximal common subgraph of $\mathcal{DF}^n$. Bucko et al. [83] showed that C_4 is the unique maximal common subgraph of $\mathcal{DF}^4$, and there are exactly two maximal common subgraphs of $\mathcal{DF}^6$ and three for $\mathcal{DF}^8$.

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