1. Introduction

Classical Sobolev spaces, based upon the Lebesgue spaces L_p, have been widely accepted as one of the crucial instruments in Functional Analysis and have played a significant role in numerous parts of mathematics for many years; in particular, in connection with PDE's. Embeddings of the Sobolev spaces play a major role. Sobolev's classical embedding theorem [30] states that if $\Omega \subset \mathbb{R}^n$ is a domain with a sufficiently smooth boundary, then $W_p^k(\Omega) \hookrightarrow L_q(\Omega)$ continuously whenever $p < n/k$ and $p \leq q \leq np/(n - pk)$ (see also for instance [1, Lemmas 5.12 and 5.14]). In the limiting case, i.e., when $p = n/k$, this inclusion does not hold for $q = \infty$, unless $p = 1$ so that $k = n$. However, we do have

$$W_{n/k}^k(\Omega) \hookrightarrow L_q(\Omega) \quad \text{for all } q,\ p \leq q < \infty \tag{1.1}$$

(see for example [1, Corollary 5.13 and Lemma 5.14]). Therefore, the optimal integrability conditions satisfied by functions in $W_{n/k}^k$ cannot be specified as simple L_q conditions.

In 1967 Trudinger [33] (see also Pokhozhaev [29] and Yudovich [34]) found refinements of (1.1) expressed in terms of Orlicz spaces of exponential type. He was able to prove that a continuous embedding of the form

$$W_p^k(\Omega) \hookrightarrow L_\Phi(\Omega), \tag{1.2}$$

where $kp = n$ and Ω is a bounded domain in $\mathbb{R}^n$, $n > 1$, with a smooth boundary, holds for the Orlicz space $L_\Phi(\Omega)$ generated by the function $\Phi(t) = \exp t^\lambda$ for large t, where $\lambda = n/(n-1)$ for all $k \in \mathbb{N}$. Such an Orlicz space is clearly contained in $L_q(\Omega)$ for every $q < \infty$. Trudinger also showed that the value $\lambda = n/(n-1)$ is the best possible when $k = 1$. However, when $k \geq 2$, Strichartz [32] noted that Trudinger's result could be improved with the larger power $\lambda = p' = n/(n-k)$. The reason why Trudinger did not obtain the optimal power is that the case $k \geq 2$ was reduced to the case $k = 1$ by using a Sobolev result, namely if $u \in W_p^k(\Omega)$, $k \geq 2$, $kp = n$, then $u \in W_n^1(\Omega)$. Strichartz on the other hand used a direct argument. He also observed that $\lambda = p' = n/(n-k)$ is the best possible value of λ for any choice of $k \leq n - 1$. Note that in 1966, Peetre [28, Theorem 9.1] proved a limiting embedding concerning Besov spaces from which Trudinger's and Strichartz's limiting embeddings follow for $p = 2$.

To obtain further refinements of the limiting case of the Sobolev embedding theorem, it is necessary to work with a wider class of function spaces, such as the Lorentz–Zygmund spaces $L^{p,q}(\log L)^\alpha(\Omega)$ introduced by Bennett and Rudnick [2]. Let us just remark that the Orlicz space $L_\Phi(\Omega)$, defined above, coincides with the Lorentz–Zygmund space $L^{\infty,\infty}(\log L)^{-1/\lambda}(\Omega)$, also denoted by $E_\lambda(\Omega)$ in some literature.

In 1979 Hansson [19, pp. 96–101] and, independently, in 1980 Brézis and Wainger [5, Theorem 2, p. 781] proved the embedding

$$W_p^k(\Omega) \hookrightarrow L^{\infty,p}(\log L)^{-1}(\Omega),$$

where $kp = n$ and Ω is a bounded domain with smooth boundary.

As pointed out in both [19] and [5], the space $L^{\infty,p}(\log L)^{-1}(\Omega)$ is strictly smaller than the various versions of the space $L_\Phi(\Omega)$ which appear in (1.2). This fact also agrees with what was considered previously and with Theorem 9.5 of [2], where various inclusion relations among the Lorentz–Zygmund spaces were established.

In more recent times, Sobolev type embeddings in the limiting case have attracted some attention, mostly restricted to the case of classical Sobolev spaces where $k = n/p \in \mathbb{N}$, but in the context of general rearrangement-invariant spaces; see for instance Cwikel and Pustylnik [6] and Edmunds, Kerman and Pick [14]. We refer to Edmunds and Triebel [15] for embeddings of fractional Sobolev spaces and Besov spaces into rearrangement-invariant spaces. In particular, Cwikel and Pustylnik [6] showed that the space $L^{\infty,p}(\log L)^{-1}(\Omega)$ is the smallest rearrangement-invariant Banach function space into which $W_p^k(\Omega)$, with $kp = n$, can be continuously embedded. A more detailed description can be found in [6], [14] and [15].

When the space $L^n(\log L)^a$ is used instead of L^n as the underlying space, then the corresponding Sobolev space is embedded in another Orlicz space of single exponential type if $a < 0$ (see Fusco, Lions and Sbordone [17], Edmunds, Gurka and Opic [9, Remark 3.11(iv)], and [10, Section 6]), while if $a = (n-1)/n$ there is an embedding into an Orlicz space of double exponential type (see Edmunds, Gurka and Opic [9]–[11]). See Edmunds, Gurka and Opic [12] for the case when the Sobolev space is modelled upon a generalised Lorentz–Zygmund space.

In this paper we consider the Lorentz–Karamata spaces $L_{p,q;b}(R)$ where $p, q \in (0,\infty]$, b is a slowly varying function on $[1,\infty)$ and (R,μ) a measure space. With convenient choices of slowly varying functions these spaces give the generalised Lorentz–Zygmund (GLZ) spaces $L_{p,q;\alpha_1,\ldots,\alpha_m}(R)$ (introduced by Edmunds, Gurka and Opic [12]), Lorentz–Zygmund spaces $L^{p,q}(\log L)^\alpha(R)$ (introduced by Bennett and Rudnick [2]), Zygmund spaces $L^p(\log L)^\alpha(R)$, Lorentz spaces $L^{p,q}(R)$ and Lebesgue spaces $L^p(R)$.

When $1 < p < \infty$, $q \in [1,\infty]$, and (R,μ) is a resonant measure space, it is proved that $L_{p,q;b}(R)$, endowed with a convenient norm, is a rearrangement-invariant Banach function space with associate space $L_{p',q';1/b}(R)$. This result generalises Theorem IV.4.7 of [3], where the case of Lorentz spaces is considered, and also extends Lemma 3.4 of [12], where the case of generalised Lorentz–Zygmund spaces is considered.

Sufficient conditions on the indices p, q and on the slowly varying functions b are given in order to have embeddings between Lorentz–Karamata spaces. When p varies we consider the underlying measure space with finite measure. This condition is also necessary if the underlying measure space is resonant. When p is fixed the results are given for any measure space. These results extend and give the counterpart of the embedding results for the Lorentz–Zygmund spaces (see Bennett and Rudnick [2] and Bennett and Sharpley [3]) and generalised Lorentz–Zygmund spaces (see Evans, Opic and Pick [16] for GLZ spaces over a finite non-atomic measure space, case $m = 2$ and p fixed).

Embedding theorems for certain Bessel-potential spaces modelled upon Lorentz–Karamata spaces, referred to in what follows as Lorentz–Karamata–Bessel potential spaces, into Lorentz–Karamata spaces are given when the power exponent p is in the sublimiting case, i.e., $1 < p < n/\sigma$, where $\sigma \in (0, n)$, and when p has the limiting value n/σ, with $\sigma \in (0, n)$. These results generalise and improve (limiting case) those of Edmunds, Gurka and Opic [12], and refine the one of Hansson [19]. In order to do that, weighted Hardy-type inequalities involving slowly varying functions are considered.

A decomposition of the Luxemburg norm of the functions in an Orlicz space into two terms where one is of Marcinkiewicz type, provided the Young function satisfies a Lorentz type condition, is given and used to obtain embeddings of certain Lorentz–Karamata–Bessel potential spaces (limiting case) into Orlicz spaces, considered either on subsets of $\mathbb{R}^n$ with finite volume or on $\mathbb{R}^n$. These results extend those of Edmunds, Gurka and Opic [10], [12] and Gurka and Opic [18], and give refinements of those of Trudinger [33] and Strichartz [32]. The results of Gurka and Opic [18] concern embeddings of Bessel-potential spaces $H^\sigma Y(\mathbb{R}^n)$, modelled upon appropriate generalised Lorentz–Zygmund spaces $Y(\mathbb{R}^n)$, into Orlicz spaces $L_\Phi(\mathbb{R}^n)$, where $\Phi(t) = \exp(\exp(\ldots\exp(t^\nu)\ldots))$, for large t, $\nu > 0$, and $\Phi(t) = t^q$, for small t, with ν and q satisfying certain conditions. It was this modification of Φ near the origin that permitted the authors to consider the global embedding; see [7] for the case of fractional Sobolev spaces, and [1] and [8] for the case of Sobolev spaces.

We also present estimates for an appropriate norm of the convolution of a function f in a Lorentz space with one g in the intersection of a Lorentz–Karamata space with the Lebesgue L_1 space. In particular, we consider the case when f is the Riesz kernel I_σ, $0 < \sigma < n$. These results extend those of Edmunds, Gurka and Opic [9] on convolutions of functions in generalised Lorentz–Zygmund spaces which lead to double exponential integrability, and those of Brézis–Wainger [5] on convolution of functions in Lorentz spaces which lead to single exponential integrability. Furthermore, in some cases we improve the results of Edmunds, Gurka and Opic [9] by obtaining a triple exponential Orlicz space rather than a double exponential one. Moreover, we also obtain results related to that of Mizuta and Shimomura [22].

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2. Notation and preliminaries

As usual, $\mathbb{R}^n$ denotes Euclidean n-dimensional space. Let (R, Σ, μ), usually denoted by (R, μ), be a totally σ-finite measure space, referred to in what follows only as a measure space. A set $E \in \Sigma$ is called an *atom* of (R, Σ, μ) if $\mu(E) > 0$ and $F \subset E$, $F \in \Sigma$ implies either $\mu(F) = 0$ or $\mu(E \setminus F) = 0$. If there are no atoms, then (R, Σ, μ) is called *non-atomic*. A measure space (R, μ) is called *resonant* if it is one of the following two types: (i) non-atomic; (ii) completely atomic, with all atoms having equal measure. We refer to [3, pp. 45–51] for more details and for a different, but equivalent, definition. When $R = \mathbb{R}^n$ we always take μ to be Lebesgue measure μ_n, and write $|\Omega|_n = \mu_n(\Omega)$ for any measurable subset Ω of $\mathbb{R}^n$. The family of all extended scalar-valued (real or complex) μ-measurable functions on R will be denoted by $\mathcal{M}(R,\mu)$; $\mathcal{M}_0(R,\mu)$ will stand for the subset of $\mathcal{M}(R,\mu)$ consisting of all those functions which are finite μ-a.e. and $\mathcal{M}^+(R,\mu)$ (resp. $\mathcal{M}_0^+(R,\mu)$) will represent the subset of $\mathcal{M}(R,\mu)$ (resp. $\mathcal{M}_0(R,\mu)$) made up of all those functions which are non-negative μ-a.e.

Let $f \in \mathcal{M}_0(R,\mu)$. The *distribution function* μ_f of f is defined by

$$\mu_f(\lambda) = \mu\{x \in R : |f(x)| > \lambda\} \quad \text{for all } \lambda \ge 0,$$

the *non-increasing rearrangement* of f is the function $f^*_{(R,\mu)}$ defined on $[0,\infty)$ by

$$f^*_{(R,\mu)}(t) = \inf\{\lambda \ge 0 : \mu_f(\lambda) \le t\} \quad \text{for all } t \ge 0,$$

and the *maximal function* $f^{**}_{(R,\mu)}$ of $f^*_{(R,\mu)}$ is defined by

$$f^{**}_{(R,\mu)}(t) = \frac{1}{t}\int_0^t f^*_{(R,\mu)}(s)\,ds \quad \text{for all } t > 0.$$

If (R,μ) is a finite measure space, then the distribution function μ_f is bounded above by $\mu(R)$ and so $f^*_{(R,\mu)}(t) = 0$ for all $t \ge \mu(R)$. In this case we may regard $f^*_{(R,\mu)}$ as a function defined on the interval $[0,\mu(R))$; for more details we refer to [3]. If there is no danger of confusion, we write f^* (resp. f^{**}) or f^*_R (resp. f^{**}_R) instead of $f^*_{(R,\mu)}$ (resp. $f^{**}_{(R,\mu)}$).

Two functions $f \in \mathcal{M}_0(R,\mu)$ and $g \in \mathcal{M}_0(S,\nu)$ are said to be *equimeasurable* if they have the same distribution function, i.e., if $\mu_f(\lambda) = \nu_g(\lambda)$ for all $\lambda \ge 0$.

Although the non-increasing rearrangement does not preserve sums or products of functions, there are some basic inequalities that govern the process.

The next result concerns an inequality for sums [3, Theorem II.3.4].

Theorem 2.1. *If f and g belong to $\mathcal{M}_0(R,\mu)$, then*

$$(f+g)^{**}(t) \le f^{**}(t) + g^{**}(t) \quad \textit{for all } t > 0.$$

We also need the following Hardy–Littlewood inequality for products [3, Theorem II.2.2].

Theorem 2.2. *If f and g belong to $\mathcal{M}_0(R,\mu)$, then*

$$\int_R |fg|\,d\mu \le \int_0^\infty f^*(t)g^*(t)\,dt.$$

For general facts about Banach function spaces with Banach function norm (or simply a function norm) ϱ over a measure space (R,μ) we refer to [3, Chaps. 1, 2]. Nevertheless, let us recall a few concepts and results, for the convenience of the reader.

A function norm ϱ over a measure space (R,μ) is said to be *rearrangement-invariant* if $\varrho(f) = \varrho(g)$ for every pair of equimeasurable functions f and g in $\mathcal{M}_0^+(R,\mu)$.

Let (R,μ) be a measure space and let ϱ be a function norm. The *associate function norm* ϱ' of ϱ is defined on $\mathcal{M}^+(R,\mu)$ by

$$\varrho'(g)=\sup\Big\{\int_R fg\,d\mu : f\in\mathcal{M}^+(R,\mu),\ \varrho(f)\le 1\Big\}, \tag{2.1}$$

for each $g\in\mathcal{M}^+(R,\mu)$. The collection $X=X(\varrho)$ of all functions f in $\mathcal{M}(R,\mu)$ for which $\varrho(|f|)$ is finite is called a *Banach function space*. The norm of a function f in X is given by

$$\|f\|_X=\varrho(|f|). \tag{2.2}$$

The Banach function space $X=X(\varrho)$ generated by a rearrangement-invariant function norm ϱ is called a *rearrangement-invariant space*. The Banach function space $X(\varrho')$ determined by ϱ', where ϱ' is the associate norm of ϱ, is called the *associate space* of $X(\varrho)$ and is denoted by X'. It follows from (2.1) and (2.2) that the norm of a function g in the associate space X' is given by

$$\|g\|_{X'}:=\sup\Big\{\int_R |fg|\,d\mu : f\in X,\ \|f\|_X\le 1\Big\}.$$

The next result formulates the Hölder inequality in terms of the Banach function spaces X and X' generated by ϱ and ϱ', respectively [3, Corollary II.4.5].

Theorem 2.3. *Let X be a rearrangement-invariant space over a resonant measure space (R,μ). If f belongs to X and g to X', then*

$$\int_R |fg|\,d\mu\le\int_0^\infty f^*(t)g^*(t)\,dt\le\|f\|_X\|g\|_{X'}.$$

We shall also need the Lorentz–Luxemburg theorem [3, Theorem I.2.7].

Theorem 2.4. *Every Banach function space X coincides with its second associate space $X'':=(X')'$. In other words, a function f belongs to X if, and only if, it belongs to X'', and in that case $\|f\|_X=\|f\|_{X''}$.*

Remark 2.1. If X and Y are two Banach function spaces such that $Y=X'$, up to equivalence of norms, then it follows, by the Lorentz–Luxemburg theorem (Theorem 2.4), and by the definition of Y', that $Y'=X$, up to equivalence of norms. In other words, X and Y are *mutually associate*, up to equivalence of norms.

Now we recall the Luxemburg representation theorem [3, Theorem II.4.10].

Theorem 2.5. *Let ϱ be a rearrangement-invariant function norm over a resonant measure space (R,μ). Then there is a (not necessarily unique) rearrangement-invariant function norm $\overline{\varrho}$ over $(\mathbb{R}^+,\mu_1)$ such that $\varrho(f)=\overline{\varrho}(f^*)$ for all f in $\mathcal{M}_0^+(R,\mu)$.*

Furthermore, if σ is any rearrangement-invariant function norm over $(\mathbb{R}^+,\mu_1)$ which represents ϱ, in the sense that $\varrho(f)=\sigma(f^*)$ for all f in $\mathcal{M}_0^+(R,\mu)$, then the associate norm ϱ' of ϱ is represented in the same way by the associate norm σ' of σ, that is, $\varrho'(g)=\sigma'(g^*)$ for all g in $\mathcal{M}_0^+(R,\mu)$.

Let $p\in(0,\infty]$. We denote by $L_p(R)$ the Lebesgue space endowed with the (quasi-) norm $\|\cdot\|_{p;R}$.

Let X be a rearrangement-invariant Banach function space over $(\mathbb{R}^n, \mu_n)$. Then, by [3, Theorem II.6.6], $X \hookrightarrow L_1(\mathbb{R}^n) + L_\infty(\mathbb{R}^n)$. Therefore, if $f \in X$ and $g \in L_1(\mathbb{R}^n)$, $f * g = g * f \in \mathcal{M}_0(\mathbb{R}^n, \mu_n)$, where $f * g$ is the convolution of f and g.

By a *Young function* Φ we mean a continuous non-negative, strictly increasing and convex function on $[0, \infty)$ satisfying

$$\lim_{t \to 0^+} \Phi(t)/t = \lim_{t \to \infty} t/\Phi(t) = 0.$$

Given a Young function Φ, the *Orlicz space* $L_\Phi(R)$ is defined to be the collection of all functions $f \in \mathcal{M}_0(R, \mu)$ for which there is a $\lambda > 0$ such that

$$\int_R \Phi(|f|/\lambda)\, d\mu < \infty,$$

equipped with the *Luxemburg norm* $\|\cdot\|_{\Phi,R}$ given by

$$\|f\|_{\Phi,R} = \inf\Big\{\lambda > 0 : \int_R \Phi(|f|/\lambda)\, d\mu \le 1\Big\}.$$

We refer to [1, Chapter VIII] and [20, Chapter III] for more details.

Let Φ_1 and Φ_2 be Young functions. Recall that Φ_2 *dominates* Φ_1 *globally* if there is a positive constant κ such that

$$\Phi_1(t) \le \Phi_2(\kappa t) \tag{2.3}$$

for all $t \ge 0$. Similarly, Φ_2 *dominates* Φ_1 *near infinity* if there are positive constants κ and t_0 such that (2.3) holds for all $t \in [t_0, \infty)$. Two Young functions are said to be *equivalent globally* (resp. *near infinity*) if each dominates the other globally (resp. near infinity). From [1, Theorem 8.12, pp. 234–235] we have the following result: If Φ_1 and Φ_2 are equivalent globally (or near infinity and $\mu(R) < \infty$), then $L_{\Phi_1}(R) = L_{\Phi_2}(R)$ and the corresponding norms are equivalent.

Let Φ be a non-negative, non-decreasing, left-continuous function on $[0, \infty)$ with $\Phi(0+) = 0$ and $\Phi(\infty) = \infty$. Let (R, μ) be a measure space and let f belong to $\mathcal{M}_0(R, \mu)$. Then

$$\int_R \Phi(|f|)\, d\mu = \int_0^\infty \Phi(f_\mu^*(t))\, dt$$

(cf. [3, p. 87]). Therefore, if Φ is a Young function,

$$\|f\|_{\Phi,R} = \inf\Big\{\lambda > 0 : \int_0^\infty \Phi(f_\mu^*(t)/\lambda)\, dt \le 1\Big\}$$

for all $f \in L_\Phi(R)$.

The *Riesz kernel* I_σ, $0 < \sigma < n$, is defined by

$$I_\sigma(\xi) = |\xi|^{\sigma - n}, \qquad \xi \in \mathbb{R}^n.$$

The *Bessel kernel* g_σ, $\sigma > 0$, is defined by

$$g_\sigma(\xi) = \frac{1}{(4\pi)^{\sigma/2}\Gamma(\sigma/2)} \int_0^\infty e^{-\pi|\xi|^2/x}\, e^{-x/(4\pi)}\, x^{(\sigma-n)/2}\, \frac{dx}{x}, \qquad \xi \ne 0$$

(cf. [31, Chap. 5]). It is known that g_σ is a positive, integrable function which is analytic except at the origin, $\|g_\sigma\|_{1;\mathbb{R}^n} = 1$ and its Fourier transform is

$$\widehat{g_\sigma}(\xi) = (2\pi)^{-n/2}(1+|\xi|^2)^{-\sigma/2}, \qquad \xi \in \mathbb{R}^n,$$

where the Fourier transform $\widehat{f}$ of a function f is given by

$$\widehat{f}(\xi) = (2\pi)^{-n/2} \int\limits_{\mathbb{R}^n} e^{-i\xi\cdot x} f(x)\,dx, \qquad \xi \in \mathbb{R}^n.$$

Now let $m \in \mathbb{N}$ and $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^m$. We denote by $\vartheta^m_{\boldsymbol{\alpha}}$ the real function defined by

$$\vartheta^m_{\boldsymbol{\alpha}}(t) = \prod_{i=1}^{m} \ell_i^{\alpha_i}(t) \qquad \text{for all } t \in (0,\infty),$$

where $\ell_1, \ldots, \ell_m$ are positive functions defined on $(0,\infty)$ by

$$\ell_1(t) = 1 + |\log t|, \qquad \ell_i(t) = 1 + \log \ell_{i-1}(t), \quad i \in \{2, \ldots, m\},\ m \geq 2.$$

Let ℓ_0 be the function on $(0,\infty)$ defined by $\ell_0(t) = \max\{1/t, t\}$ for each $t > 0$. We define the numbers $exp_1, \ldots, exp_m$ by

$$exp_1 = e, \qquad exp_i = e^{exp_{i-1}}, \quad i \in \{2, \ldots, m\},\ m \geq 2.$$

Denote by $\mu^m_{\boldsymbol{\alpha}}$ the real function defined by

$$\mu^m_{\boldsymbol{\alpha}}(t) = \prod_{i=1}^{m} l_i^{\alpha_i}(t) \qquad \text{for all } t \in [exp_m, \infty),$$

where $l_1, \ldots, l_m$ are the positive functions defined by

$$\begin{aligned} l_1(t) &= \log t, && t \geq e, \\ l_i(t) &= \log l_{i-1}(t), && t \geq exp_i,\ i \in \{2, \ldots, m\},\ m \geq 2. \end{aligned}$$

For formal reasons, we put, if $m = 0$,

$$\vartheta^m_{\boldsymbol{\alpha}} = \prod_{i=1}^{m} \ell_i^{\alpha_i} = \mu^m_{\boldsymbol{\alpha}} = 1.$$

The symbol $\exp_m$ will represent the function $\underbrace{\exp \circ \exp \circ \ldots \circ \exp}_{m \text{ times}}$ and the symbol Exp_m will represent the positive function defined on $(0,\infty)$ by induction:

$$\mathrm{Exp}_m(x) = e^{x-1} \quad \text{if } m = 1, \qquad \mathrm{Exp}_m(x) = e^{\mathrm{Exp}_{m-1}(x)-1} \quad \text{if } m \geq 2.$$

Given $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_m), \boldsymbol{\beta} = (\beta_1, \ldots, \beta_m) \in \mathbb{R}^m$ and $\sigma \in \mathbb{R}$, we write $\boldsymbol{\alpha} + \boldsymbol{\beta} = (\alpha_1 + \beta_1, \ldots, \alpha_m + \beta_m)$, $\boldsymbol{\alpha} + \sigma = (\alpha_1 + \sigma, \ldots, \alpha_m + \sigma)$, $\sigma\boldsymbol{\alpha} = (\sigma\alpha_1, \ldots, \sigma\alpha_m)$. If $\boldsymbol{\alpha} = (0, \ldots, 0) \in \mathbb{R}^m$, we denote $\boldsymbol{\alpha}$ by $\mathbf{0}$. We write $\boldsymbol{\beta} \prec \boldsymbol{\alpha}$, or $\boldsymbol{\alpha} \succ \boldsymbol{\beta}$, if one of the following conditions is satisfied:

$$\beta_1 - \alpha_1 < 0; \qquad \begin{cases} \text{there exists } k \in \{2, \ldots, m\} \text{ such that} \\ \beta_j = \alpha_j \text{ for } j = 1, \ldots, k-1 \text{ and } \beta_k - \alpha_k < 0. \end{cases}$$

We use the symbol $\boldsymbol{\beta} \preceq \boldsymbol{\alpha}$, or $\boldsymbol{\alpha} \succeq \boldsymbol{\beta}$, to mean that either $\boldsymbol{\beta} \prec \boldsymbol{\alpha}$ or $\boldsymbol{\beta} = \boldsymbol{\alpha}$. Let $p \in [1,\infty]$, $k \in \{1, \ldots, m\}$. We denote by $\boldsymbol{\delta}_{p;m,k}$ the m-tuple $(\delta_1, \ldots, \delta_m) \in \mathbb{R}^m$, where $\delta_i = 1/p$, $i = 1, \ldots, k$, and, if $k + 1 \leq m$, $\delta_i = 0$, $i = k+1, \ldots, m$.

In what follows, we let c denote a positive constant. In a chain of inequalities c may stand for several different constants if it is not important to distinguish between them, otherwise we use c with subscripts. For two non-negative expressions (i.e. functions or functionals) $\mathcal{A}$, $\mathcal{B}$, the symbol $\mathcal{A} \precsim \mathcal{B}$ means that $\mathcal{A} \le c\mathcal{B}$ for some positive constant c independent of the variables in $\mathcal{A}$ and $\mathcal{B}$. If $\mathcal{A} \precsim \mathcal{B}$ and $\mathcal{B} \precsim \mathcal{A}$, we write $\mathcal{A} \approx \mathcal{B}$.

We adopt the convention that $a/\infty = 0$ and $a/0 = \infty$ for all $a > 0$. If $p \in [1, \infty]$, the conjugate number p' is given by $1/p + 1/p' = 1$.

3. Slowly varying functions and Lorentz–Karamata spaces

A positive and Lebesgue-measurable function b is said to be *slowly varying* (s.v.) on $[1, \infty)$ in the sense of Karamata if, for each $\varepsilon > 0$, $t^{\varepsilon} b(t)$ is equivalent to a non-decreasing function and $t^{-\varepsilon} b(t)$ is equivalent to a non-increasing function on $[1, \infty)$; see Chapter I in [4] for a detailed study of the Karamata theory.

Properties and examples of s.v. functions can be found in [36, Chapter V, p. 186], [4] and [14]. The following functions are s.v. on $[1, \infty)$:

(i) $b(t) = \vartheta^{m}_{\boldsymbol{\alpha}}(t)$ with $m \in \mathbb{N}$ and $\boldsymbol{\alpha} \in \mathbb{R}^m$;
(ii) $b(t) = \exp(\log^{\alpha} t)$ with $0 < \alpha < 1$;
(iii) $b_m(t) = \exp(\ell^{\alpha}_{m}(t))$ with $0 < \alpha < 1$ and $m \in \mathbb{N}$.

Note that if $m \ge 2$ in the last example, we may consider $\alpha = 1$. In this case $b_m \approx \ell_{m-1}$.

Given a slowly varying function b on $[1, \infty)$, we denote by γ_b the positive function defined by

$$\gamma_b(t) = b(\max\{t, 1/t\}) \quad \text{for all } t > 0.$$

It follows easily that the product of two slowly varying functions b_1 and b_2 on $[1, \infty)$ is still a slowly varying function on $[1, \infty)$ and

$$\gamma_{b_1 b_2}(t) = \gamma_{b_1}(t)\gamma_{b_2}(t) \quad \text{for all } t > 0.$$

Lemma 3.1. *Let b be a slowly varying function on $[1, \infty)$.*

(i) *Let $r \in \mathbb{R}$. Then b^r is a slowly varying function on $[1, \infty)$ and*

$$\gamma_{b^r}(t) = \gamma_b^r(t) \quad \textit{for all } t > 0.$$

(ii) *For each $\varepsilon > 0$, $t^{\varepsilon}\gamma_b(t)$ is equivalent to a positive non-decreasing function on $(0, \infty)$ and $t^{-\varepsilon}\gamma_b(t)$ is equivalent to a positive non-increasing function on $(0, \infty)$.*

(iii) *Let $\kappa > 0$. Then*

$$\gamma_b(\kappa t) \approx \gamma_b(t) \quad \textit{for all } t > 0.$$

(iv) *If*

$$\int_1^{\infty} \tau^{-1} b(\tau)\, d\tau < \infty,$$

then b_1 defined by

$$b_1(t) = \int_t^{\infty} \tau^{-1} b(\tau)\, d\tau, \qquad t \ge 1,$$

is a slowly varying function on $[1, \infty)$.

(v) *Let* $\alpha > 0$. *Then*

$$\int_0^t \tau^{\alpha-1}\gamma_b(\tau)\,d\tau \approx \sup_{0<\tau<t} \tau^{\alpha}\gamma_b(\tau) \approx t^{\alpha}\gamma_b(t) \quad \textit{for all } t>0; \tag{3.1}$$

$$\int_t^\infty \tau^{-\alpha-1}\gamma_b(\tau)\,d\tau \approx \sup_{t<\tau<\infty} \tau^{-\alpha}\gamma_b(\tau) \approx t^{-\alpha}\gamma_b(t) \quad \textit{for all } t>0. \tag{3.2}$$

(vi) *If* $\alpha < 0$, *then*

$$\begin{aligned}\int_0^t \tau^{\alpha-1}\gamma_b(\tau)\,d\tau &= \sup_{0<\tau<t} \tau^{\alpha}\gamma_b(\tau) = \int_t^\infty \tau^{-\alpha-1}\gamma_b(\tau)\,d\tau \\ &= \sup_{t<\tau<\infty} \tau^{-\alpha}\gamma_b(\tau) = \infty \quad \textit{for all } t>0.\end{aligned}$$

Proof. The easy proofs of (i), (iii) and (vi) are omitted. In (v), the estimates (3.2) follow from (3.1) by taking into account that $\gamma_b(t) = \gamma_b(1/t)$ for all $t > 0$.

For (ii), let $\varepsilon > 0$. We denote by f_ε the non-decreasing function equivalent to $t^\varepsilon b(t)$ on $[1,\infty)$, and by $f_{-\varepsilon}$ the non-increasing function equivalent to $t^{-\varepsilon}b(t)$ on $[1,\infty)$. Then it is easy to verify that $t^\varepsilon\gamma_b(t)$ is equivalent to the positive non-decreasing function Γ_ε on $(0,\infty)$ defined by

$$\Gamma_\varepsilon(t) = \frac{f_\varepsilon(1)}{f_{-\varepsilon}(1)} f_{-\varepsilon}(\max\{t,1/t\})\chi_{(0,1)}(t) + f_\varepsilon(\max\{t,1/t\})\chi_{[1,\infty)}(t),\ t>0,$$

and that $t^{-\varepsilon}\gamma_b(t)$ is equivalent to the positive non-increasing function $\Gamma_{-\varepsilon}$ on $(0,\infty)$ defined by

$$\Gamma_{-\varepsilon}(t) = f_\varepsilon(\max\{t,1/t\})\chi_{(0,1)}(t) + \frac{f_\varepsilon(1)}{f_{-\varepsilon}(1)} f_{-\varepsilon}(\max\{t,1/t\})\chi_{[1,\infty)}(t), \quad t>0.$$

To prove (iv), for each $\varepsilon > 0$, let f_ε and $f_{-\varepsilon}$ be as before. Then it is easy to verify that $t^\varepsilon b_1(t)$ is equivalent to the non-decreasing function g_ε on $[1,\infty)$ defined by

$$g_\varepsilon(t) = t^\varepsilon \int_t^\infty \tau^{-1+\varepsilon} f_{-\varepsilon}(\tau)\,d\tau, \quad t \ge 1,$$

and that $t^{-\varepsilon}b_1(t)$ is equivalent to the non-increasing function $g_{-\varepsilon}$ on $[1,\infty)$ defined by

$$g_{-\varepsilon}(t) = t^{-\varepsilon} \int_t^\infty \tau^{-1-\varepsilon} f_\varepsilon(\tau)\,d\tau, \quad t \ge 1.$$

Let us now prove the estimates (3.1) in (v). Let

$$\begin{aligned} g_1(t) &= \int_0^t \tau^{\alpha-1}\gamma_b(\tau)\,d\tau \quad \text{for all } t>0,\\ g_\infty(t) &= \sup_{0<\tau<t} \tau^\alpha\gamma_b(\tau) \quad \text{for all } t>0.\end{aligned}$$

Let $t > 0$. Then by (ii), we have

$$g_1(t) \gtrsim t^{-1}\gamma_b(t)\int_0^t \tau^\alpha\,d\tau \approx t^{-1}\gamma_b(t)t^{\alpha+1} = t^\alpha\gamma_b(t). \tag{3.3}$$

On the other hand, for each $t > 0$,

$$g_1(t) \precsim t^{\alpha/2}\gamma_b(t)\int_0^t \tau^{\alpha/2-1}\,d\tau \approx t^{\alpha/2}\gamma_b(t)t^{\alpha/2} = t^{\alpha}\gamma_b(t). \tag{3.4}$$

It now follows from (3.3) and (3.4) that

$$g_1(t) \approx t^{\alpha}\gamma_b(t) \quad \text{for all } t \in (0,\infty). \tag{3.5}$$

The estimate $g_\infty(t) \approx t^\alpha\gamma_b(t)$ for all $t > 0$, follows easily, and together with (3.5) gives (3.1). ■

Let $\alpha \in (0,1]$. Let $\mathcal{K}_\alpha$ be the class of all positive and Lebesgue-measurable functions b defined on $[1,\infty)$ such that, for each $\varepsilon > 0$, $\exp(\varepsilon\ell_1^\alpha(t))b(t)$ is equivalent to a non-decreasing function and $\exp(-\varepsilon\ell_1^\alpha(t))b(t)$ is equivalent to a non-increasing function on $[1,\infty)$.

Remark 3.1. Let $\alpha \in (0,1]$.

(i) If $\alpha = 1$, then $\mathcal{K}_\alpha$ coincides with the class of slowly varying functions.

(ii) Let $r \in \mathbb{R}$ and $b \in \mathcal{K}_\alpha$. Then $b^r \in \mathcal{K}_\alpha$.

(iii) Let $b_1, b_2 \in \mathcal{K}_\alpha$. Then $b_1 b_2 \in \mathcal{K}_\alpha$.

(iv) Let $a \in \mathbb{R}$, $\delta \in [0,\alpha]$ and let b_δ be the function defined by $b_\delta(t) = \exp(a\ell_1^{\alpha-\delta}(t))$ for $t \geq 1$. Then $b_0 \notin \mathcal{K}_\alpha$. However, $b_\delta \in \mathcal{K}_\alpha$ if $0 < \delta \leq \alpha$.

(v) Let $0 < \alpha < \beta \leq 1$. Then $\mathcal{K}_\alpha \subsetneq \mathcal{K}_\beta$.

In order to prove the results of Sections 5 and 6 and some of this section, we shall need weighted Hardy inequalities where the weights are slowly varying functions, and give general results below; the proofs are omitted since they simply involve checking well-known criteria (cf. e.g. [24, Theorems 5.9 & 5.10 & 6.2 & 6.3]).

Lemma 3.2. *Let $p, q \in [1,\infty]$, $\nu \neq 0$ and let b_1, b_2 be two slowly varying functions on $[1,\infty)$.*

(i) *The inequality*

$$\Big\|t^{\nu-1/q}\,\gamma_{b_2}(t)\int_0^t g(u)\,du\Big\|_{q;(0,\infty)} \precsim \|t^{\nu+1/p'}\,\gamma_{b_1}(t)\,g(t)\|_{p;(0,\infty)} \tag{3.6}$$

holds for all $g \in \mathcal{M}^+((0,\infty),\mu_1)$ if, and only if, $\nu < 0$ and one of the following conditions is satisfied:

$$1 \leq p \leq q \leq \infty, \qquad \sup_{0<x<1}\frac{b_2(1/x)}{b_1(1/x)} < \infty; \tag{3.7}$$

$$1 \leq q < p \leq \infty, \qquad \Big\|x^{-1/r}\frac{b_2(1/x)}{b_1(1/x)}\Big\|_{r;(0,1)} < \infty, \quad \textit{where} \quad \frac{1}{r} = \frac{1}{q} - \frac{1}{p}. \tag{3.8}$$

(ii) *The inequality*

$$\Big\|t^{\nu-1/q}\,\gamma_{b_2}(t)\int_t^\infty g(u)\,du\Big\|_{q;(0,\infty)} \precsim \|t^{\nu+1/p'}\,\gamma_{b_1}(t)\,g(t)\|_{p;(0,\infty)} \tag{3.9}$$

holds for all $g \in \mathcal{M}^+((0,\infty),\mu_1)$ *if, and only if,* $\nu > 0$ *and one of conditions* (3.7)–(3.8) *is satisfied.*

REMARK 3.2. Suppose $m \in \mathbb{N}$, $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_m), \boldsymbol{\beta} = (\beta_1, \dots, \beta_m) \in \mathbb{R}^m$, $b_1 = \vartheta^m_{\boldsymbol{\alpha}}$ and $b_2 = \vartheta^m_{\boldsymbol{\beta}}$.

(i) Let $1 \le p \le q \le \infty$. Then (3.7) is satisfied if, and only if, $\boldsymbol{\beta} \preceq \boldsymbol{\alpha}$. In this case, the previous lemma gives the result of [12, Lemma 3.1].

(ii) Let $1 \le q < p \le \infty$. Then (3.8) is satisfied if, and only if, $\boldsymbol{\beta} + 1/q \prec \boldsymbol{\alpha} + 1/p$.

(iii) See [16, Lemma 3.1], where the case $m = 2$ is considered, but with the interval $(0,\infty)$ in (3.6) and (3.9) replaced by the interval $(0,1)$.

REMARK 3.3. Let $m \in \mathbb{N}$, $p, q \in [1,\infty]$, $\alpha_1, \beta_1 \in \mathbb{R}$ and $0 < \alpha < 1$. Let $\phi_1, \phi_2 \in \mathcal{K}_\alpha$ and let b_1, b_2 be slowly varying functions on $[1,\infty)$ defined by

$$b_1(t) = \ell_m^{-(\alpha-1)/p'}(t) \prod_{i=1}^{m-1} \ell_i^{1/p'}(t) \exp(\alpha_1 \ell_m^\alpha(t))\, \phi_1(\ell_{m-1}(t)) \quad \text{for } t \ge 1,$$

$$b_2(t) = \ell_m^{(\alpha-1)/q}(t) \prod_{i=1}^{m-1} \ell_i^{-1/q}(t) \exp(\beta_1 \ell_m^\alpha(t))\, \phi_2(\ell_{m-1}(t)) \quad \text{for } t \ge 1.$$

(i) Let $1 \le p \le q \le \infty$. If either $\beta_1 < \alpha_1$ or $\beta_1 = \alpha_1$ and $\phi_2 \precsim \phi_1$, then (3.7) is satisfied.

(ii) Let $1 \le q < p \le \infty$. If $\beta_1 < \alpha_1$, then (3.8) is satisfied.

LEMMA 3.3. *Let* $p, q \in [1,\infty]$ *and let* b_1, b_2 *be two slowly varying functions on* $[1,\infty)$.

(i) *If* $1 \le p \le q \le \infty$, *then the inequality*

$$\Big\| t^{-1/q}\, b_2(1/t) \int_t^1 g(u)\, du \Big\|_{q;(0,1)} \precsim \| t^{1/p'}\, b_1(1/t) g(t) \|_{p;(0,1)} \tag{3.10}$$

holds for all $g \in \mathcal{M}^+((0,1),\mu_1)$ *if, and only if, there is a positive constant* c *such that*

$$\| t^{-1/q}\, b_2(1/t) \|_{q;(0,x)} \| (t^{1/p'}\, b_1(1/t))^{-1} \|_{p';(x,1)} \le c \quad \text{for all } x \in (0,1). \tag{3.11}$$

(ii) *If* $1 \le q < p \le \infty$, *then* (3.10) *holds for all* $g \in \mathcal{M}^+((0,1),\mu_1)$ *if, and only if,*

$$\int_0^1 [\| t^{-1/q}\, b_2(1/t) \|_{q;(0,x)} \| (t^{1/p'}\, b_1(1/t))^{-1} \|^{p'/q'}_{p';(x,1)}]^r (x^{1/p'}\, b_1(1/x))^{-p'}\, dx \tag{3.12}$$

is finite, where $1/r = 1/q - 1/p$.

REMARK 3.4. Let $p, q \in [1,\infty]$, $m \in \mathbb{N}$, $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_m) \in \mathbb{R}^m$ and let $k \in \{1, \dots, m\}$ be such that $\alpha_k \ne 1/p'$ and, if $k \ge 2$, $\alpha_i = 1/p'$, $i = 1, \dots, k-1$. Let $\boldsymbol{\beta} = (\beta_1, \dots, \beta_m) \in \mathbb{R}^m$ with $\beta_k \ne -1/q$ and, if $k \ge 2$, $\beta_i = -1/q$, $i = 1, \dots, k-1$. Put $b_1 = \vartheta^m_{\boldsymbol{\alpha}}$ and $b_2 = \vartheta^m_{\boldsymbol{\beta}}$.

(i) Let $1 \le p \le q \le \infty$. Then (3.11) holds if, and only if,

$$\beta_k < -1/q \quad \text{and} \quad \boldsymbol{\beta} + \boldsymbol{\delta}_{q;m,k} \preceq \boldsymbol{\alpha} - \boldsymbol{\delta}_{p';m,k}.$$

If we omit the assumption $\beta_k \ne -1/q$ the assertion (3.11) will still hold provided that

$$1 \le p \le q = \infty, \quad \boldsymbol{\beta} = \mathbf{0}, \quad -\alpha_k + 1/p' < 0. \tag{3.13}$$

Moreover, with $k = m$, if we also omit the assumption $\alpha_m \neq 1/p'$, the assertion (3.11) will still hold provided that

$$p = 1, \quad q = \infty, \quad \boldsymbol{\alpha} = \boldsymbol{\beta} = \mathbf{0}.$$

(ii) Let $1 \leq q < p \leq \infty$. Then (3.12) holds if, and only if,

$$\beta_k < -1/q \quad \text{and} \quad \boldsymbol{\beta} + 1/q \prec \boldsymbol{\alpha} - \boldsymbol{\delta}_{1;m,k} + 1/p.$$

(iii) We refer to [16, Lemmas 3.2(ii), 3.3(ii) & Remark 3.4(iii)] for the case $m = 2$. In [16], only the case $k = m = 2$ in (3.13) is considered.

Remark 3.5. Let $m \in \mathbb{N}$, $p, q \in [1, \infty]$, $\alpha_1, \beta_1 \in \mathbb{R}$ and $0 < \alpha < 1$.

(i) Suppose $\beta_1 \neq 0$. Let b_1, b_2 be slowly varying functions on $[1, \infty)$ defined by

$$b_1(t) = \ell_m^{-(\alpha-1)/p'}(t) \prod_{i=1}^{m-1} \ell_i^{1/p'}(t) \exp(\alpha_1 \ell_m^{\alpha}(t)) \quad \text{for } t \geq 1,$$

$$b_2(t) = \ell_m^{(\alpha-1)/q}(t) \prod_{i=1}^{m-1} \ell_i^{-1/q}(t) \exp(\beta_1 \ell_m^{\alpha}(t)) \quad \text{for } t \geq 1.$$

(a) Let $1 \leq p \leq q \leq \infty$. Then (3.11) holds if, and only if,

$$\beta_1 < 0 \quad \text{and} \quad \beta_1 \leq \alpha_1.$$

Moreover, if we omit the assumption $\beta_1 \neq 0$ the assertion (3.11) will still hold provided that

$$\begin{array}{lllll} \text{either} & 1 \leq p \leq q = \infty, & \beta_1 = 0 & \text{and} & \alpha_1 > 0 \\ \text{or} & p = 1,\ q = \infty, & \beta_1 = 0 & \text{and} & \alpha_1 = 0. \end{array}$$

(b) Let $1 \leq q < p \leq \infty$. Then (3.12) holds if, and only if,

$$\beta_1 < 0 \quad \text{and} \quad \beta_1 < \alpha_1.$$

(ii) Suppose that $\alpha_1 \neq 0$ and $\beta_1 \neq 0$. Let $\phi_1, \phi_2 \in \mathcal{K}_\alpha$ and let b_1, b_2 be slowly varying functions on $[1, \infty)$ defined by

$$b_1(t) = \ell_m^{-(\alpha-1)/p'}(t) \prod_{i=1}^{m-1} \ell_i^{1/p'}(t) \exp(\alpha_1 \ell_m^{\alpha}(t))\, \phi_1(\ell_{m-1}(t)) \quad \text{for } t \geq 1,$$

$$b_2(t) = \ell_m^{(\alpha-1)/q}(t) \prod_{i=1}^{m-1} \ell_i^{-1/q}(t) \exp(\beta_1 \ell_m^{\alpha}(t))\, \phi_2(\ell_{m-1}(t)) \quad \text{for } t \geq 1.$$

(a) Let $1 \leq p \leq q \leq \infty$. If $\beta_1 < 0$, either $\beta_1 < \alpha_1$ or $\beta_1 = \alpha_1$ and $\phi_2 \precsim \phi_1$, then (3.11) is satisfied.

(b) Let $1 \leq q < p \leq \infty$. If $\beta_1 < 0$ and $\beta_1 < \alpha_1$, then (3.12) is satisfied.

Definition 3.1. Let $p, q \in (0, \infty]$ and let b be a slowly varying function on $[1, \infty)$. The *Lorentz–Karamata* (*LK*) *space* $L_{p,q;b}(R)$ is defined to be the set of all functions $f \in \mathcal{M}_0(R, \mu)$ such that

$$\|f\|_{p,q;b;R} := \|t^{1/p-1/q} \gamma_b(t) f^*(t)\|_{q;(0,\infty)} \tag{3.14}$$

is finite. Here $\|\cdot\|_{q;(0,\infty)}$ stands for the usual L_q (quasi-) norm over the interval $(0, \infty)$.

If $m \in \mathbb{N}$, $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^m$ and $b = \vartheta^m_{\boldsymbol{\alpha}}$, then $L_{p,q;b}(R)$ is precisely the generalised Lorentz–Zygmund (GLZ) space $L_{p,q;\boldsymbol{\alpha}}(R)$, introduced in [12], endowed with the (quasi-) norm $\|f\|_{p,q;\boldsymbol{\alpha};R}$. We remark that in [12], the GLZ space $L_{p,q;\boldsymbol{\alpha}}(R)$ and the quasi-norm $\|\cdot\|_{p,q;\boldsymbol{\alpha};R}$ defined above are denoted by $L_{p,q;\alpha_1,\ldots,\alpha_m}(R)$ and $\|\cdot\|_{p,q;\alpha_1,\ldots,\alpha_m;R}$, respectively. We use the notation of [12] only when we are considering particular cases. Let us observe that when we consider $\boldsymbol{\alpha} = (0, \ldots, 0)$, we obtain the Lorentz space $L_{p,q}(R)$ endowed with the (quasi-) norm $\|\cdot\|_{p,q;R}$, which is just the Lebesgue space $L_p(R)$ endowed with the (quasi-) norm $\|\cdot\|_{p;R}$ when $p = q$; if $p = q$, $m = 1$ and $(R, \mu) = (\Omega, \mu_n)$, we obtain the Zygmund space $L^p(\log L)^{\alpha_1}(\Omega)$ endowed with the (quasi-) norm $\|\cdot\|_{p;\alpha_1;\Omega}$.

When $0 < p < \infty$, the Lorentz–Karamata space $L_{p,q;b}(R)$ contains other functions than the null function; when $p = \infty$, it is different from the trivial space if, and only if, $\|t^{1/p-1/q}\gamma_b(t)\|_{q;(0,\infty)} < \infty$. Observe that, when $p = \infty$, $\|t^{1/p-1/q}\gamma_b(t)\|_{q;(0,\infty)} < \infty$ if, and only if, $\|t^{1/p-1/q}\gamma_b(t)\|_{q;(0,1)} < \infty$.

Lorentz–Karamata spaces with s.v. functions considered in Remark 3.5 have not been considered before in the literature, as far as we are aware.

Let $p, q \in (0, \infty]$ and let b be a slowly varying function on $[1, \infty)$. Let us introduce the functional $\|\cdot\|_{(p,q;b);R}$ defined by

$$\|f\|_{(p,q;b);R} := \|t^{1/p-1/q}\gamma_b(t)f^{**}(t)\|_{q;(0,\infty)}; \tag{3.15}$$

this is identical with that defined in (3.14) except that f^* has been replaced by f^{**}.

Lemma 3.4. *Suppose $1 < p \le \infty$, $1 \le q \le \infty$ and let b be a slowly varying function on $[1, \infty)$. Then*

$$\|f\|_{p,q;b;R} \le \|f\|_{(p,q;b);R} \precsim \|f\|_{p,q;b;R} \quad \text{for all } f \in \mathcal{M}_0(R, \mu). \tag{3.16}$$

In particular, the Lorentz–Karamata space $L_{p,q;b}(R)$ consists of all those functions f for which $\|f\|_{(p,q;b);R}$ is finite.

Proof. The first inequality follows immediately since $f^* \le f^{**}$ for all $f \in \mathcal{M}_0(R, \mu)$. As for the second, since $p > 1$, we see from Lemma 3.2(i) that

$$\begin{aligned}\|f\|_{(p,q;b);R} &= \Big\|t^{1/p-1-1/q}\gamma_b(t)\int_0^t f^*(s)\,ds\Big\|_{q;(0,\infty)} \\ &\precsim \|t^{1/p-1+1/q'}\gamma_b(t)f^*(t)\|_{q;(0,\infty)} = \|f\|_{p,q;b;R}. \ \blacksquare\end{aligned}$$

When $m \in \mathbb{N}$, $\boldsymbol{\alpha} \in \mathbb{R}^m$ and $b = \vartheta^m_{\boldsymbol{\alpha}}$, the previous lemma coincides with Lemma 3.2 of [12].

Since, by Theorem 2.1, $f \mapsto f^{**}$ is subadditive, it is easy to verify that $\|\cdot\|_{(p,q;b);R}$ is a norm provided that $q \ge 1$.

Lemma 3.5. *Let $1 < p, q < \infty$ and let b be a slowly varying function on $[1, \infty)$. Let $g \in \mathcal{M}_0(\mathbb{R}^+, \mu_1)$. Define Φ in $(0, \infty)$ by*

$$\Phi(s) = s^{q/p-1}(\gamma_b(s))^q\,(g^*(s))^{q-1}, \qquad s > 0.$$

Then

$$\Phi(s) \precsim \int_{s/2}^{s} \Phi(\tau)\tau^{-1}\,d\tau \quad \text{for } s>0.$$

Proof. Since g^* is non-increasing, we have

$$\int_{s/2}^{s} \Phi(\tau)\tau^{-1}\,d\tau \succsim (g^*(s))^{q-1} \int_{s/2}^{s} \tau^{q/p-1}(\gamma_b(\tau))^q\tau^{-1}\,d\tau. \tag{3.17}$$

Now let $\varepsilon > 1$. Then $q/p-1+\varepsilon>0$ and, by Lemma 3.1, $t^{q/p-1+\varepsilon}(\gamma_b(t))^q$ is equivalent to a positive non-decreasing function on $(0,\infty)$. Then, for each $s>0$,

$$\begin{aligned}\int_{s/2}^{s} \tau^{q/p-1}(\gamma_b(\tau))^q\tau^{-1}\,d\tau &= \int_{s/2}^{s} \tau^{q/p-1+\varepsilon}(\gamma_b(\tau))^q\tau^{-1-\varepsilon}\,d\tau \\ &\succsim (s/2)^{q/p-1+\varepsilon}(\gamma_b(s/2))^q\varepsilon^{-1}(2^\varepsilon-1)s^{-\varepsilon} \\ &\approx s^{q/p-1}(\gamma_b(s/2))^q \approx s^{q/p-1}(\gamma_b(s))^q.\end{aligned} \tag{3.18}$$

Now the result follows from (3.17) and (3.18). ■

THEOREM 3.1. *Let* $1<p<\infty$, $1\le q\le\infty$ *and let* b *be a slowly varying function on* $[1,\infty)$. *If* (R,μ) *is a resonant measure space*, *then*

$$X=(L_{p,q;b}(R), \|\cdot\|_{(p,q;b);R}) \quad \text{and} \quad Y=(L_{p',q';1/b}(R), \|\cdot\|_{(p',q';1/b);R})$$

are rearrangement-invariant Banach function spaces and they are mutually associate, *up to equivalence of norms.*

Proof. There is no difficulty in verifying that X and Y are Banach function spaces and the rearrangement-invariance is obvious, since two equimeasurable functions have the same non-increasing rearrangement.

Now we prove that X and Y are mutually associate.

Suppose $g\in Y$. Then for any $f\in X$ with $\|f\|_X\le 1$, by the Hardy–Littlewood inequality (cf. Theorem 2.2), Hölder's inequality, Lemma 3.1(i) and Lemma 3.4 we have

$$\int_R |fg|\,d\mu \le \int_0^\infty f^*(t)g^*(t)\,dt \le \|g\|_Y\,\|f\|_X.$$

Hence taking the supremum over all $f\in X$ with $\|f\|_X\le 1$, we get

$$\|g\|_{X'} = \sup\Big\{\int_R |fg|\,d\mu : f\in X,\ \|f\|_X\le 1\Big\} \le \|g\|_Y. \tag{3.19}$$

To establish an inequality reverse to (3.19), for all $g\in X'$, we follow the proof of Theorem IV.4.7 in [3] and the proof of Lemma 3.4 in [12], although with some technical differences which even simplify the proof of [12, Lemma 3.4]. By the Luxemburg representation theorem (cf. Theorem 2.5), it is sufficient to do so for the measure space $(R,\mu)=(\mathbb{R}^+,\mu_1)$ and functions g in $\mathbb{R}^+$ for which $g=g^*$. Let g be a simple function on $\mathbb{R}^+$ for which $g=g^*$; such a function belongs to the associate space X' of X.

Assume $1 < q \le \infty$, and define f by

$$f(s) = \int_{s/2}^{\infty} \Phi(\tau)\tau^{-1}\,d\tau \quad \text{for all } s > 0,$$

where

$$\Phi(s) = s^{q'/p'-1}(\gamma_{1/b}(s))^{q'} g^*(s)^{q'-1} \quad \text{for all } s > 0. \tag{3.20}$$

Since g is a simple function and $p' < \infty$, it follows from Lemma 3.1(i) and (3.1) that $\|g\|_{p',q';1/b;R} < \infty$. By (3.20),

$$\int_0^{\infty} \Phi(s)g^*(s)\,ds = \|g\|_{p',q';1/b;R}^{q'}. \tag{3.21}$$

Note that f is non-increasing and hence $f = f^*$. Moreover, $f \in X$. In fact, if $1 < q < \infty$, then Lemma 3.4, the change of variables $t = s/2$, properties (i) and (iii) of Lemma 3.1, and Lemma 3.2(ii) imply

$$\begin{aligned} \|f\|_X &\approx \Big\| s^{1/p-1/q}\gamma_b(s)\int_{s/2}^{\infty}\Phi(\tau)\tau^{-1}\,d\tau\Big\|_{q;(0,\infty)} \approx \Big\| t^{1/p-1/q}\gamma_b(t)\int_{t}^{\infty}\Phi(\tau)\tau^{-1}\,d\tau\Big\|_{q;(0,\infty)} \\ &\precsim \|t^{1/p-1/q}\gamma_b(t)\Phi(t)\|_{q;(0,\infty)} = \|g\|_{p',q';1/b;R}^{q'/q} < \infty. \end{aligned} \tag{3.22}$$

If $q = \infty$, then since $1/p' - 1 < 0$, by (3.2) we have

$$f^*(t) = f(t) = \int_{t/2}^{\infty} \tau^{1/p'-1}\gamma_{1/b}(\tau)\tau^{-1}\,d\tau \approx (t/2)^{1/p'-1}\gamma_{1/b}(t/2) \approx \Phi(t). \tag{3.23}$$

Hence, by Lemma 3.4 and (3.23),

$$\|f\|_X \approx \|f\|_{p,\infty;b;R} \approx \|t^{1/p}\gamma_b(t)t^{1/p'-1}\gamma_{1/b}(t)\|_{\infty;(0,\infty)} \approx 1. \tag{3.24}$$

If $1 < q < \infty$, then by Lemma 3.5 with the slowly varying function $1/b$, and with p and q replaced by p' and q', respectively, it follows that

$$\Phi(s) \precsim \int_{s/2}^{s} \Phi(\tau)\tau^{-1}\,d\tau \quad \text{for all } s > 0. \tag{3.25}$$

Now, by (3.21), (3.25) and Theorem 2.3, we have

$$\begin{aligned} \|g\|_{p',q';1/b;R}^{q'} &\precsim \int_0^{\infty}\Big(\int_{s/2}^{s}\Phi(\tau)\tau^{-1}\,d\tau\Big)g^*(s)\,ds \\ &\le \int_0^{\infty} f^*(s)g^*(s)\,ds \le \|f\|_X\,\|g\|_{X'}. \end{aligned} \tag{3.26}$$

Using (3.26) and (3.22), we see that $\|g\|_{p',q';1/b;R}^{q'} \precsim \|g\|_{p',q';1/b;R}^{q'/q}\,\|g\|_{X'}$, which gives

$$\|g\|_{p',q';1/b;R} \precsim \|g\|_{X'}. \tag{3.27}$$

If $q = \infty$, it follows from (3.21), (3.23), (3.24) and Theorem 2.3 that

$$\|g\|_{p',1;1/b;R} \precsim \|f\|_X\,\|g\|_{X'} \approx \|g\|_{X'}. \tag{3.28}$$

If $q = 1$, then by Theorem 2.3,

$$tg^{**}(t) = \int_0^\infty \chi_{(0,t)}(s)g^*(s)\,ds \le \|\chi_{(0,t)}\|_X\|g\|_{X'} \tag{3.29}$$

for each $t > 0$. On the other hand, by Lemma 3.4 and (3.1),

$$\|\chi_{(0,t)}\|_X \approx \|\chi_{(0,t)}(s)\|_{p,1;b;R} = \int_0^t s^{1/p-1}\gamma_b(s)\,ds \approx t^{1/p}\gamma_b(t). \tag{3.30}$$

It now follows from (3.29), (3.30) and Lemma 3.1(i) that

$$\begin{aligned}\|g\|_{p',\infty;1/b;R} &\le \sup_{t>0} t^{1/p'}\gamma_{1/b}(t)g^{**}(t) \\ &\precsim \sup_{t>0} t^{1/p'}\gamma_{1/b}(t)t^{-1/p'}\gamma_b(t)\|g\|_{X'} = \|g\|_{X'}.\end{aligned} \tag{3.31}$$

Therefore, Lemma 3.4, (3.27), (3.28) and (3.31) yield

$$\|g\|_Y \precsim \|g\|_{X'} \tag{3.32}$$

for all simple g such that $g^* = g$. Now it follows from the Fatou property (cf. [3, Property (P3) in Definition I.1.1]) and rearrangement-invariance of X' (cf. [3, Corollary I.I.4.4]) that (3.32) holds for all $g \in X'$.

The estimates (3.19) and (3.32) together show that Y coincides with the associate space X' of X, up to equivalence of norms, and hence (cf. Remark 2.1) the spaces X and Y are mutually associate. ■

The next lemma provides upper pointwise estimates of f^* and f^{**} when f belongs to an LK space, under certain conditions, which will be needed in Sections 5 and 6.

LEMMA 3.6. *Let $p \in (1,\infty)$, $q \in [1,\infty]$ or $p = q = \infty$, and let b be a slowly varying function on $[1,\infty)$. Then there exists a positive constant $c = c(p,q,b)$ such that for every $f \in L_{p,q;b}(R)$ and all $t > 0$,*

$$f^*(t) \le f^{**}(t) \le c\,\frac{t^{-1/p}}{\gamma_b(t)}\|f\|_{p,q;b;R}.$$

Proof. The first inequality is obvious. To prove the second inequality, we use the fact

$$\|f\|_{p,q;b;R} \approx \|t^{1/p-1/q}\gamma_b(t)f^{**}(t)\|_{q;(0,\infty)}, \tag{3.33}$$

according to Lemma 3.4. If $q = \infty$ the result follows immediately. If $q \in [1,\infty)$, then (3.33) and (3.1) give

$$\|f\|_{p,q;b;R} \succsim f^{**}(t)\Big[\int_0^t (s^{1/p-1/q}\gamma_b(s))^q\,ds\Big]^{1/q} \approx f^{**}(t)t^{1/p}\gamma_b(t)$$

for all $t > 0$, and the result now follows. ■

When $m = 2$, $\boldsymbol{\alpha} \in \mathbb{R}^m$, $b = \vartheta^m_{\boldsymbol{\alpha}}$, $p \in (1,\infty)$ and $q \in [1,\infty]$, we obtain the result of [9, Lemma 3.3].

Note that it can be proved as above that when $p \in (0,\infty)$, $q \in (0,\infty]$ or $p = q = \infty$, and b is a slowly varying function on $[1,\infty)$, there exists a positive constant $c = c(p,q,b)$

such that for every $f \in L_{p,q;b}(R)$ and all $t > 0$,

$$f^*(t) \le c\,\frac{t^{-1/p}}{\gamma_b(t)}\|f\|_{p,q;b;R}.$$

When $m \in \mathbb{N}$, $\boldsymbol{\alpha} \in \mathbb{R}^m$, $b = \vartheta^m_{\boldsymbol{\alpha}}$, $p \in (0,\infty)$, $q \in (0,\infty]$ or $p = q = \infty$, this gives [18, Lemma 4.4].

The remaining results in this section establish embeddings between Lorentz–Karamata spaces.

Theorem 3.2. *Let $0 < p \le \infty$, $0 < q_1, q_2 \le \infty$ and let b_1, b_2 be two slowly varying functions on $[1,\infty)$. Suppose that*

$$\|t^{1/p-1/q_1} b_1(1/t)\|_{q_1;(0,1)} < \infty \quad \text{if } p = \infty. \tag{3.34}$$

Then

$$L_{p,q_1;b_1}(R) \hookrightarrow L_{p,q_2;b_2}(R), \tag{3.35}$$

provided either

$$0 < q_1 \le q_2 \le \infty, \qquad \sup_{0<t<1} b_2(1/t)/b_1(1/t) < \infty, \tag{3.36}$$

or

$$0 < q_2 < q_1 \le \infty, \qquad \|t^{-1/r} b_2(1/t)/b_1(1/t)\|_{r;(0,1)} < \infty, \tag{3.37}$$

where $1/r = 1/q_2 - 1/q_1$.

Proof. (i) Let us first prove the case (3.36). Observe that under our conditions,

$$\gamma_{b_2}(t) \precsim \gamma_{b_1}(t) \quad \text{for all } t > 0.$$

Suppose $0 < q_1 < \infty$, otherwise the result follows trivially. Let $\varepsilon > 0$ and set $\varepsilon_p = 1/p + \varepsilon$. Let $f \in L_{p,q_1;b_1}(R)$. By (3.1) and the fact that f^* is non-increasing, we have, for each $t > 0$,

$$\begin{aligned} t^{1/p}\gamma_{b_2}(t) f^*(t) &= t^{\varepsilon_p}\gamma_{b_2}(t) t^{-\varepsilon} f^*(t) \approx t^{-\varepsilon} f^*(t)\Big(\int_0^t (\tau^{\varepsilon_p - 1/q_1}\gamma_{b_2}(\tau))^{q_1}\,d\tau\Big)^{1/q_1} \\ &= \Big(\int_0^t (\tau^{\varepsilon_p-1/q_1}\gamma_{b_2}(\tau) t^{-\varepsilon} f^*(t))^{q_1}\,d\tau\Big)^{1/q_1} \\ &\precsim \Big(\int_0^t (\tau^{1/p-1/q_1}\gamma_{b_1}(\tau) f^*(\tau))^{q_1}\,d\tau\Big)^{1/q_1} \precsim \|f\|_{p,q_1;b_1;R}. \end{aligned}$$

Hence,

$$\|f\|_{p,\infty;b_2;R} \precsim \|f\|_{p,q_1;b_1;R} \quad \text{for all } f \in L_{p,q_1;b_1}(R), \tag{3.38}$$

which establishes (3.35) in the case $q_2 = \infty$.

Suppose now that $0 < q_1 \le q_2 < \infty$ and let $f \in L_{p,q_1;b_1}(R)$. Then

$$\begin{aligned} \|f\|_{p,q_2;b_2;R} &= \Big(\int_0^\infty (\tau^{1/p}\gamma_{b_2}(\tau) f^*(\tau))^{q_2-q_1}(\tau^{1/p-1/q_1}\gamma_{b_2}(\tau) f^*(\tau))^{q_1}\,d\tau\Big)^{1/q_2} \\ &\precsim \|f\|^{1-q_1/q_2}_{p,\infty;b_2;R}\|f\|^{q_1/q_2}_{p,q_1;b_1;R}. \end{aligned}$$

When combined with (3.38), this gives $\|f\|_{p,q_2;b_2;R} \precsim \|f\|_{p,q_1;b_1;R}$ for all $f \in L_{p,q_1;b_1}(R)$, which establishes (3.35) in the case $0 < q_1 \le q_2 < \infty$.

(ii) Let us next prove the case (3.37). Since $0 < q_2 < q_1 \le \infty$, we deduce, by Hölder's inequality with exponents q_1/q_2 and $q_1/(q_1 - q_2)$ if $q_1 < \infty$, and immediately if $q_1 = \infty$, that

$$\|f\|_{p,q_2;b_2} \precsim \|f\|_{p,q_1;b_1} \left\| t^{-1/r} \frac{\gamma_{b_2}(t)}{\gamma_{b_1}(t)} \right\|_{r;(0,\infty)},$$

where $1/r = 1/q_2 - 1/q_1$. Since the last integral is finite if, and only if,

$$\|t^{-1/r} b_2(1/t)/b_1(1/t)\|_{r;(0,1)} < \infty,$$

the embedding (3.35) now follows. ∎

Remark 3.6. Let $m \in \mathbb{N}$, $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_m), \boldsymbol{\beta} = (\beta_1, \dots, \beta_m) \in \mathbb{R}^m$, $b_1 = \vartheta^m_{\boldsymbol{\alpha}}$ and $b_2 = \vartheta^m_{\boldsymbol{\beta}}$.

(i) Then (3.34) is satisfied if, and only if,

$$\begin{array}{llll} \text{either} & 0 < q_1 < \infty & \text{and} & \boldsymbol{\alpha} \prec -1/q_1 + \mathbf{0}, \\ \text{or} & q_1 = \infty & \text{and} & \boldsymbol{\alpha} \preceq \mathbf{0}. \end{array}$$

See [16, Lemma 6.1], where conditions are given in order to have the generalised Lorentz–Zygmund space, for the case $m = 2$, as the trivial space.

(ii) The second condition in (3.36) holds if, and only if, $\boldsymbol{\beta} \preceq \boldsymbol{\alpha}$.

(iii) The second condition in (3.37), is verified if, and only if, $\boldsymbol{\beta} + 1/q_2 \prec \boldsymbol{\alpha} + 1/q_1$.

When $p = \infty$, condition (3.36) can be weakened for some values of q_1, q_2. To this end we shall make use of the following simple lemma with $p = \infty$, which generalises [2, Lemma 9.2] and [16, Lemma 6.2].

Lemma 3.7. *Let $0 < p \le \infty$, $0 < q_1 < q_2 < \infty$ and let b_1, b_2 be slowly varying functions on $[1, \infty)$. Then*

$$\|f\|_{p,q_2;b_2;R} \le \|f\|^{q_1/q_2}_{p,q_1;b_1;R} \, \|f\|^{1-q_1/q_2}_{p,\infty;b_3;R} \tag{3.39}$$

for every $f \in \mathcal{M}_0(R, \mu)$, where b_3 is the slowly varying function on $[1, \infty)$ defined by

$$b_3(t) = \left[\frac{(b_2(t))^{q_2}}{(b_1(t))^{q_1}} \right]^{1/(q_2 - q_1)}, \qquad t \ge 1.$$

Proof. Let $f \in \mathcal{M}_0(R, \mu)$ and suppose that the right-hand side of (3.39) is finite, otherwise the result is trivial. Then

$$\|f\|^{q_2}_{p,q_2;b_2;R} = \int_0^\infty (t^{1/p} \gamma_{b_1}(t) f^*(t))^{q_1} (t^{1/p} \gamma_{b_3}(t) f^*(t))^{q_2 - q_1} \frac{dt}{t} \le \|f\|^{q_1}_{p,q_1;b_1;R} \|f\|^{q_2 - q_1}_{p,\infty;b_3;R},$$

and the result now follows. ∎

Theorem 3.3. *Let $0 < q_1 < \infty$, $0 < q_2 \le \infty$ and let b_1, b_2 be slowly varying functions on $[1, \infty)$. Suppose that*

$$\|t^{-1/q_1} b_1(1/t)\|_{q_1;(0,1)} < \infty.$$

Let $I_{(b_1,q_1)}$ be the slowly varying function on $[1,\infty)$ defined by

$$I_{(b_1,q_1)}(t) = \Big(\int_t^\infty (\tau^{-1/q_1} b_1(\tau))^{q_1}\, d\tau \Big)^{1/q_1}, \qquad t \ge 1.$$

Then

$$L_{\infty,q_1;b_1}(R) \hookrightarrow L_{\infty,q_2;b_2}(R), \tag{3.40}$$

provided either

$$0 < q_1 < q_2 = \infty, \qquad \sup_{0<t<1} \frac{b_2(1/t)}{I_{(b_1,q_1)}(1/t)} < \infty, \tag{3.41}$$

or

$$0 < q_1 < q_2 < \infty, \qquad \sup_{0<t<1} \left[\frac{(b_2(1/t))^{q_2}}{(b_1(1/t))^{q_1}} \right]^{1/(q_2-q_1)} \frac{1}{I_{(b_1,q_1)}(1/t)} < \infty. \tag{3.42}$$

In particular, if $0 < q_1 < \infty$,

$$L_{\infty,q_1;b_1}(R) \hookrightarrow L_{\infty,\infty;I_{(b_1,q_1)}}(R) \hookrightarrow L_{\infty,\infty;b_1}(R). \tag{3.43}$$

Proof. (i) Suppose (3.41) is satisfied. Let $f \in L_{\infty,q_1;b_1}(R)$. Then for each $t > 0$,

$$\gamma_{b_2}(t) f^*(t) \precsim \gamma_{I_{(b_1,q_1)}}(t) f^*(t) = f^*(t) \Big(\int_{\max\{t,1/t\}}^\infty (\tau^{-1/q_1} \gamma_{b_1}(\tau))^{q_1}\, d\tau \Big)^{1/q_1}$$

$$= f^*(t) \Big(\int_0^{\min\{t,1/t\}} (\tau^{-1/q_1} \gamma_{b_1}(\tau))^{q_1}\, d\tau \Big)^{1/q_1} \le f^*(t) \Big(\int_0^t (\tau^{-1/q_1} \gamma_{b_1}(\tau))^{q_1}\, d\tau \Big)^{1/q_1}$$

$$\le \Big(\int_0^t (\tau^{-1/q_1} \gamma_{b_1}(\tau) f^*(\tau))^{q_1}\, d\tau \Big)^{1/q_1} \le \|f\|_{\infty,q_1;b_1;R}.$$

Therefore, $\|f\|_{\infty,\infty;b_2;R} \precsim \|f\|_{\infty,q_1;b_1;R}$ for all $f \in L_{\infty,q_1;b_1}(R)$, which establishes (3.40) with $q_2 = \infty$.

(ii) Assume condition (3.42) holds. By Lemma 3.7, with $p = \infty$, we have

$$\|f\|_{\infty,q_2;b_2;R} \le (\|f\|_{\infty,q_1;b_1;R})^{q_1/q_2}\, (\|f\|_{\infty,\infty;b_3;R})^{1-q_1/q_2}, \tag{3.44}$$

where b_3 is the slowly varying function on $[1,\infty)$ defined by

$$b_3(t) = \left[\frac{(b_2(t))^{q_2}}{(b_1(t))^{q_1}} \right]^{1/(q_2-q_1)}, \qquad t \ge 1.$$

On the other hand, since condition (3.41) is satisfied with b_2 replaced by b_3, it follows from the previous case that

$$\|f\|_{\infty,\infty;b_3;R} \precsim \|f\|_{\infty,q_1;b_1;R}. \tag{3.45}$$

Now the result follows from (3.44) and (3.45).

(iii) The first embedding in (3.43) follows from (3.40), because $0 < q_1 < q_2 = \infty$ and condition (3.41), with $b_2 = I_{(b_1,q_1)}$, holds. The second embedding in (3.43) follows from Theorem 3.2, because

$$\sup_{0<t<1} \frac{b_1(1/t)}{I_{(b_1,q_1)}(1/t)} < \infty. \tag{3.46}$$

Indeed, let $\varepsilon > 0$. Since for each $t \geq 1$,

$$I_{(b_1,q_1)}(t) \gtrsim t^{\varepsilon/q_1} b_1(t) \left(\int_t^\infty \frac{1}{\tau^{1+\varepsilon}} \, d\tau \right)^{1/q_1} = \frac{1}{\varepsilon^{1/q_1}} b_1(t),$$

we have $b_1(t) \precsim I_{(b_1,q_1)}(t)$ for all $t \geq 1$, which entails (3.46). ■

REMARK 3.7. Let $0 < q_1 < \infty$, $0 < q_2 \leq \infty$, $m \in \mathbb{N}$, $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_m) \in \mathbb{R}^m$, $\boldsymbol{\beta} = (\beta_1, \dots, \beta_m) \in \mathbb{R}^m$, $b_1 = \vartheta^m_{\boldsymbol{\alpha}}$ and $b_2 = \vartheta^m_{\boldsymbol{\beta}}$. Let $k \in \{1, \dots, m\}$ be such that $\alpha_k < -1/q_1$, and, if $k > 1$, then $\alpha_i = -1/q_1$ for $i = 1, \dots, k-1$. Let $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_m) \in \mathbb{R}^m$ be defined by $\boldsymbol{\gamma} = \boldsymbol{\alpha} + \boldsymbol{\delta}_{q_1;m,k}$. Note that $\gamma_k < 0$ and, if $k > 1$, then $\gamma_j = 0$ for $j = 1, \dots, k-1$, and that

$$I_{(b_1,q_1)}(t) \approx \prod_{j=1}^m \ell_j^{\gamma_j}(t), \qquad t \geq 1.$$

Now, either (3.41) or (3.42) hold if, and only if, $\boldsymbol{\beta} + \boldsymbol{\delta}_{q_2;m,k} \preceq \boldsymbol{\alpha} + \boldsymbol{\delta}_{q_1;m,k}$.

From Theorems 3.2, 3.3 and Remarks 3.6, 3.7, we get the sufficiency part of [16, Theorem 6.3], where the case of a Lorentz–Zygmund space with $m = 2$ and (R, μ) a finite non-atomic measure space was considered.

The next theorem concerns embeddings between two Lorentz–Karamata spaces when their first indices are different.

THEOREM 3.4. *Let $0 < p_2 < p_1 \leq \infty$, $0 < q_1, q_2 \leq \infty$ and let b_1, b_2 be slowly varying functions on $[1, \infty)$. Suppose that*

$$\|t^{1/p_1 - 1/q_1} b_1(1/t)\|_{q_1;(0,1)} < \infty \qquad \text{if } p_1 = \infty.$$

Then

$$L_{p_1,q_1;b_1}(R) \hookrightarrow L_{p_2,q_2;b_2}(R), \tag{3.47}$$

provided (R, μ) is a finite measure space.

Proof. Since by (3.35), we have $L_{p_1,q_1;b_1}(R) \hookrightarrow L_{p_1,\infty;b_1}(R)$, in order to prove (3.47) it will be enough to prove the embedding

$$L_{p_1,\infty;b_1}(R) \hookrightarrow L_{p_2,q_2;b_2}(R), \tag{3.48}$$

with (R, μ) a finite measure space.

Suppose $0 < q_2 < \infty$ and let $f \in L_{p_1,\infty;b_1}(R)$. Since $p_2 < p_1$, by Lemma 3.1(v), we have

$$\begin{aligned} \|f\|_{p_2,q_2;b_2;R} &\leq \|f\|_{p_1,\infty;b_1;R} \left(\int_0^{\mu(R)} \left(t^{1/p_2 - 1/p_1 - 1/q_2} \frac{\gamma_{b_2}(t)}{\gamma_{b_1}(t)} \right)^{q_2} dt \right)^{1/q_2} \\ &\approx \|f\|_{p_1,\infty;b_1;R} (\mu(R))^{1/p_2 - 1/p_1} \frac{\gamma_{b_2}(\mu(R))}{\gamma_{b_1}(\mu(R))} \approx \|f\|_{p_1,\infty;b_1;R}, \end{aligned}$$

which establishes (3.48). The case $q_2 = \infty$ is proved similarly. Therefore the embedding (3.47) follows. ■

4. Decomposition of the Luxemburg norm

The next lemma indicates a relation between Orlicz spaces $L_{\Phi}(R)$ and Marcinkiewicz spaces, when Φ satisfies what we call a Lorentz-type condition (cf. [21, Theorem 2]).

LEMMA 4.1 [25, Lemma 8.5]. *Let $f \in \mathcal{M}_0(R,\mu)$ and let Φ be a Young function such that*

$$\int_0^\infty \Phi(\gamma\Phi^{-1}(1/t))\,dt < \infty \quad \text{for some } \gamma > 0. \tag{4.1}$$

Then $f \in L_\Phi(R)$ if, and only if, there exists a constant $K = K(f)$ such that

$$\sup_{0<t<\infty} \frac{f^*(t)}{\Phi^{-1}(1/t)} =: K < \infty.$$

It is clear that if $\Phi(t) = t^q$ for small positive t, with $q > 1$, then Φ does not satisfy (4.1), because

$$\int_1^\infty \Phi(\gamma\Phi^{-1}(1/t))\,dt = \infty$$

for any $\gamma > 0$. Therefore Lemma 4.1 does not hold for instance for the Orlicz space $L_\Phi(\mathbb{R}^n)$ generated by the Young function defined by $\Phi(t) = t^q$ for small enough $t > 0$, with $q > 1$, and defined by $\Phi(t) = \exp t^\lambda$ for large enough $t > 0$, with $\lambda > 0$, which appears as the target space of some global embeddings, as we will see in Section 5.

We aim to generalise the previous lemma, in order to include other cases. For this, we first need the following auxiliary result.

LEMMA 4.2. *Let $0 < t_0 < \infty$. Let $f \in \mathcal{M}_0(R,\mu)$ and let Φ be a Young function. Then*

$$\int_{t_0}^\infty \Phi(f^*(t)/\lambda)\,dt \le 1 \quad \text{for all } \lambda > 0 \tag{4.2}$$

if, and only if, $f^*(t) = 0$ for all $t \ge t_0$.

Proof. If $f^*(t) = 0$ for all $t \ge t_0$, then the result is trivial.

Conversely, suppose (4.2) holds but $f^*(t) \ge \varepsilon > 0$ in some interval $(t_0, t_0+\nu)$. Then, for each $\lambda > 0$,

$$1 \ge \int_{t_0}^\infty \Phi(f^*(t)/\lambda)\,dt \ge \int_{t_0}^{t_0+\nu} \Phi(f^*(t)/\lambda)\,dt \ge \Phi(\varepsilon/\lambda)\,\nu.$$

Since $\Phi(s) \uparrow \infty$ as $s \uparrow \infty$, we obtain a contradiction. Hence, $f^*(t) = 0$ for all $t > t_0$. By the right-continuity of f^*, it also follows that $f^*(t_0) = 0$. ∎

The next result gives us the generalisation of Lemma 4.1.

THEOREM 4.1. *Let $0 < t_0 \le \infty$ and $0 \le L < \infty$. Let $f \in \mathcal{M}_0(R,\mu)$ and let Φ be a Young function which satisfies a Lorentz-type condition, i.e.,*

$$\int_0^{t_0} \Phi(\gamma\Phi^{-1}(1/t))\,dt < \infty \quad \text{for some } \gamma > 0.$$

Then $f \in L_\Phi(R)$ if, and only if,

$$\sup_{0<t<t_0} \frac{f^*(t)}{\Phi^{-1}(1/t)} < \infty \quad \textit{and} \quad \int_{t_0+L}^{\infty} \Phi(f^*(t)/\gamma_0)\, dt < \infty \quad \textit{for some } \gamma_0 > 0. \tag{4.3}$$

Moreover,

$$\|f\|_{\Phi,R} \approx \sup_{0<t<t_0} \frac{f^*(t)}{\Phi^{-1}(1/t)} + \inf\Big\{\lambda > 0 : \int_{t_0+L}^{\infty} \Phi(f^*(t)/\lambda)\, dt \le 1\Big\}. \tag{4.4}$$

Proof. Let f be a function, not identically null, that belongs to $\mathcal{M}_0(R,\mu)$ and for which (4.3) holds.

Let us just mention that since Φ is convex and $\Phi(0) = 0$, there is a positive constant c for which

$$\int_{t_0+L}^{\infty} \Phi\left(\frac{f^*(t)}{c\,\gamma_0}\right) dt \le 1.$$

Let

$$\alpha := \inf\Big\{\lambda > 0 : \int_{t_0+L}^{\infty} \Phi(f^*(t)/\lambda)\, dt \le 1\Big\}. \tag{4.5}$$

For $\alpha > 0$, the infimum is attained; in fact, letting λ decrease toward α in the inequality

$$\int_{t_0+L}^{\infty} \Phi(f^*(t)/\lambda)\, dt \le 1,$$

we obtain by the monotone convergence theorem

$$\int_{t_0+L}^{\infty} \Phi(f^*(t)/\alpha)\, dt \le 1.$$

Now, let $\lambda = K/\gamma + \alpha$, where

$$K := \sup_{0<t<t_0} \frac{f^*(t)}{\Phi^{-1}(1/t)}$$

and α is defined by (4.5). Then we have

$$\begin{aligned} \int_0^{\infty} \Phi(f^*(t)/\lambda)\, dt &\le \int_0^{t_0} \Phi(f^*(t)/\lambda)\, dt + L\Phi(f^*(t_0)/\lambda) + \int_{t_0+L}^{\infty} \Phi(f^*(t)/\lambda)\, dt \\ &\le (L/t_0 + 1) \int_0^{t_0} \Phi(f^*(t)/\lambda)\, dt + \int_{t_0+L}^{\infty} \Phi(f^*(t)/\lambda)\, dt \\ &\le (L/t_0 + 1) \int_0^{t_0} \Phi(\gamma\Phi^{-1}(1/t))\, dt + \int_{t_0+L}^{\infty} \Phi(f^*(t)/\lambda)\, dt. \end{aligned} \tag{4.6}$$

If $0 < t_0 < \infty$ and $\alpha > 0$, it follows from (4.6) that

$$\int_0^{\infty} \Phi(f^*(t)/\lambda)\, dt \le (L/t_0 + 1) \int_0^{t_0} \Phi(\gamma\Phi^{-1}(1/t))\, dt + \int_{t_0+L}^{\infty} \Phi(f^*(t)/\alpha)\, dt \tag{4.7}$$

$$\leq (L/t_0 + 1) \int_0^{t_0} \Phi(\gamma \Phi^{-1}(1/t))\, dt + 1 < \infty.$$

Hence, f belongs to $L_\Phi(R)$. Again, since Φ is convex and $\Phi(0) = 0$, there is a positive constant c such that

$$\int_0^\infty \Phi\left(\frac{f^*(t)}{c\lambda}\right) dt \leq 1,$$

and so $\|f\|_{\Phi,R} \precsim K + \alpha$.

If $t_0 = \infty$, then $\alpha = 0$, and by (4.6) it follows that

$$\int_0^\infty \Phi(f^*(t)/\lambda)\, dt \leq \int_0^\infty \Phi(\gamma \Phi^{-1}(1/t))\, dt < \infty.$$

As previously, it follows that $\|f\|_{\Phi,R} \precsim K = K + \alpha$.

If $0 < t_0 < \infty$ and $\alpha = 0$, by Lemma 4.2 we have $f^*(t) = 0$ for all $t \geq t_0 + L$. Then, by (4.6),

$$\int_0^\infty \Phi(f^*(t)/\lambda)\, dt \precsim \int_0^{t_0} \Phi(\gamma \Phi^{-1}(1/t))\, dt + 0 < \infty,$$

and as before $\|f\|_{\Phi,R} \precsim K = K + \alpha$.

Conversely, assume $f \in L_{\Phi(R)}$ and suppose f is not identically null. Then

$$\int_0^\infty \Phi(f^*(t)/\|f\|_{\Phi,R})\, dt \leq 1. \tag{4.8}$$

Let $t \in (0, t_0)$. Then by (4.8),

$$1 \geq \int_0^t \Phi\left(\frac{f^*(s)}{\|f\|_{\Phi,R}}\right) ds \geq \Phi\left(\frac{f^*(t)}{\|f\|_{\Phi,R}}\right) t,$$

which is equivalent to

$$f^*(t) \leq \|f\|_{\Phi,R}\, \Phi^{-1}(1/t). \tag{4.9}$$

Also by (4.8),

$$\int_{t_0+L}^\infty \Phi(f^*(t)/\|f\|_{\Phi,R})\, dt \leq 1,$$

and hence

$$\alpha \leq \|f\|_{\Phi,R}. \tag{4.10}$$

Therefore, by (4.9) and (4.10) it follows that $K + \alpha \precsim \|f\|_{\Phi,R}$. ∎

Remark 4.1. (i) If $t_0 = \infty$, we have $\int_{t_0+L}^\infty \Phi(f^*(t)/\lambda)\, dt = 0$, and we recover Lemma 4.1.

(ii) If $0 < t_0 < t_1 < \infty$, then

$$\sup_{0<t<t_0} \frac{f^*(t)}{\Phi^{-1}(1/t)} \approx \sup_{0<t<t_1} \frac{f^*(t)}{\Phi^{-1}(1/t)}.$$

Corollary 4.1. *Let (R,μ) be a finite measure space. Let $0 < t_0 \le \mu(R)$. Let $f \in \mathcal{M}_0(R,\mu)$ and let Φ be a Young function which satisfies a Lorentz-type condition, i.e.,*

$$\int_0^{t_0} \Phi(\gamma\Phi^{-1}(1/t))\,dt < \infty \quad \text{for some } \gamma > 0.$$

Then $f \in L_\Phi(R)$ if, and only if,

$$\sup_{0<t<t_0} \frac{f^*(t)}{\Phi^{-1}(1/t)} < \infty. \tag{4.11}$$

Moreover,

$$\|f\|_{\Phi,R} \approx \sup_{0<t<t_0} \frac{f^*(t)}{\Phi^{-1}(1/t)} \approx \sup_{0<t<\mu(R)} \frac{f^*(t)}{\Phi^{-1}(1/t)}. \tag{4.12}$$

Proof. This follows from Theorem 4.1, with $t_0 = \mu(R)$ and $L = \mu(R) - t_0$, because $f^*(t) = 0$ for all $t \ge \mu(R)$. ■

Example 4.1. Let $m \in \mathbb{N}$, $\boldsymbol{\alpha} = (\alpha_1,\ldots,\alpha_m) \in \mathbb{R}^m$ and let Ω be a measurable subset of $\mathbb{R}^n$ with finite volume. Let $k \in \{1,\ldots,m\}$ be such that $\alpha_k < 0$, and, if $k \ge 2$, then $\alpha_i = 0$ for $i = 1,\ldots,k-1$. If $k < m$, let $\boldsymbol{\beta} = (\beta_1,\ldots,\beta_{m-k}) \in \mathbb{R}^{m-k}$ with $\beta_i = -\alpha_{i+k}/\alpha_k$ for $i = 1,\ldots,m-k$.

(i) If $k = m$, let Ψ_m be the Young function defined by $\Psi_m(t) = \mathrm{Exp}_m(t^{-1/\alpha_m})$ for all large enough $t > 0$. Then $\Psi_m^{-1}(1/t) = \ell_m^{-\alpha_m}(t)$, for all small enough $t > 0$. It is now possible to check that Ψ_m satisfies the Lorentz-type condition with some $t_0 \in (0,\infty)$, for some $\gamma \in (0,1)$.

(ii) If $k < m$, let Ψ_k be the Young function defined by $\Psi_k(t) = \mathrm{Exp}_k(f_{m-k}(t))$ for all large enough $t > 0$, where f_{m-k} is the increasing function defined by

$$f_{m-k}(t) = t^{-1/\alpha_k}\vartheta_{\boldsymbol{\beta}}^{m-k}(t)$$

for all large enough $t > 0$. Since $f_{m-k}^{-1}(t) \approx t^{-\alpha_k}\vartheta_{\boldsymbol{\gamma}}^{m-k}(t)$ for all large enough $t > 0$, with $\boldsymbol{\gamma} = \alpha_k\boldsymbol{\beta}$, we have $\Psi_k^{-1}(1/t) \approx \vartheta_{-\boldsymbol{\alpha}}^m(t)$ for all small enough $t > 0$. By straightforward arguments it is now possible to check that Ψ_k satisfies the Lorentz-type condition with some $t_0 \in (0,\infty)$ for some $\gamma \in (0,1)$.

(iii) Let Ψ_k be the Young function defined in (i) if $k = m$ and defined in (ii) if $k < m$. Then (i), (ii) and the previous corollary entail

$$L_{\infty,\infty;\boldsymbol{\alpha}}(\Omega) = L_{\Psi_k}(\Omega), \tag{4.13}$$

with equivalent (quasi-) norms.

(iv) Let Φ_k be the Young function defined by $\Phi_k(t) = \exp_k(t^{-1/\alpha_k}\mu_{\boldsymbol{\beta}}^{m-k}(t))$ for all large enough $t > 0$, where, according to our conventions, $\mu_{\boldsymbol{\beta}}^{m-k} = 1$ if $m = k$. Since Ψ_k, defined in (i) if $k = m$ and defined in (ii) if $k < m$, and Φ_k are equivalent near infinity and Ω has finite volume, by [1, Theorem 8.12] and (4.13) we have $L_{\infty,\infty;\boldsymbol{\alpha}}(\Omega) = L_{\Psi_k}(\Omega) = L_{\Phi_k}(\Omega)$, with equivalent (quasi-) norms.

(v) We refer to [2, Theorem D] for the case $m = k = 1$, and to [10, Lemma 4.2] for the case $m = k = 2$; see also [13, Lemma 2.1] for the case $m = k = 1$ and $m = k = 2$, and [12] for the case $m = k$, although it is not explicitly proven there. The case $k = 1$ and $m = 2$ is given by Lemma 2.2(vi) of [16].

Note the streamlined appearance of the proofs of Theorem 4.1 and Corollary 4.1 as compared with the proofs of related results in the literature, such as [2, Theorem D] and [10, Lemma 4.2].

5. Bessel-potential-type embedding theorems

In this section we present some embedding results for certain Bessel-potential spaces modelled upon Lorentz–Karamata spaces either into Lorentz–Karamata spaces or Orlicz spaces. Namely when the power exponent p is in the sublimiting case, i.e., $1 < p < n/\sigma$, where $\sigma \in (0, n)$, and when p has the limiting value n/σ, with $\sigma \in (0, n)$.

Let $\sigma > 0$, $p \in (1, \infty)$, $q \in [1, \infty]$, and let b be a s.v. function on $[1, \infty)$. The *Lorentz–Karamata–Bessel* potential space $H^\sigma L_{p,q;b}(\mathbb{R}^n)$ is defined to be

$$\{u : u = g_\sigma * f,\ f \in L_{p,q;b}(\mathbb{R}^n)\}$$

and is equipped with the (quasi-) norm $\|u\|_{\sigma;p,q;b} := \|f\|_{p,q;b}$.

When we consider $m \in \mathbb{N}$, $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^m$ and $b = \vartheta^m_{\boldsymbol{\alpha}}$, we obtain the logarithmic Bessel potential space $H^\sigma L_{p,q;\boldsymbol{\alpha}}(\mathbb{R}^n)$, endowed with the (quasi-) norm $\|u\|_{\sigma;p,q;b}$, considered in [12]. Note that if $\boldsymbol{\alpha} = (0, \ldots, 0)$, then $H^\sigma L_{p,p;\boldsymbol{\alpha}}(\mathbb{R}^n)$ is simply the (fractional) Sobolev space of order σ.

Bessel potential spaces modelled upon Lorentz–Karamata spaces with s.v. functions b, where $b = b_1$ and b_1 considered in Remark 3.5, have not appeared before in the literature, as far as we are aware.

The next lemma, due to Edmunds, Gurka and Opic [10, Lemma 3.5], provides us the important estimate (5.1) for the non-increasing rearrangement of the Bessel kernel.

Lemma 5.1. *Let $0 < \sigma < n$. Then there exist constants $A, B \in (0, \infty)$ such that*

$$g^*_\sigma(t) \le A t^{\sigma/n-1} \exp(-B t^{1/n}) \quad \text{for all } t > 0, \tag{5.1}$$

$$g^{**}_\sigma(t) \le \frac{n}{\sigma} A t^{\sigma/n-1} \quad \text{for all } t > 0. \tag{5.2}$$

If $t \in (1, \infty)$, Gurka and Opic [18, Lemma 4.2] proved a better estimate for the maximal function of the non-increasing rearrangement of the Bessel kernel than that considered in Lemma 5.1, namely

$$g^{**}_\sigma(t) \precsim t^{-1} \quad \text{for } t \in (1, \infty). \tag{5.3}$$

The next result which considers the sublimiting case is an extension of [12, Theorem 4.8] and a refinement of [20, Theorem 5.7.7(i)].

Theorem 5.1. *Let $\sigma \in (0, n)$, $1 < p < n/\sigma$, $q \in [1, \infty]$ and let b be a slowly varying function on $[1, \infty)$. Then*

$$H^\sigma L_{p,q;b}(\mathbb{R}^n) \hookrightarrow L_{r,q;b}(\mathbb{R}^n), \tag{5.4}$$

where $1/r = 1/p - \sigma/n$.

Proof. Put $X = H^\sigma L_{n/\sigma,p;b}(\mathbb{R}^n)$. Let $u \in X$. Then $u = g_\sigma * f$, where $f \in L_{n/\sigma,p;b}(\mathbb{R}^n)$ and $\|f\|_{n/\sigma,p;b} = \|u\|_X$.

By O'Neil's inequality (cf. e.g. [35, Lemma 1.8.8]) we have

$$u^*(t) \le u^{**}(t) \le t g_\sigma^{**}(t) f^{**}(t) + \int_t^\infty g_\sigma^*(\tau) f^*(\tau)\, d\tau \quad \text{for all } t > 0. \tag{5.5}$$

The estimates (5.2) and (5.5) yield for every $t > 0$,

$$u^*(t) \precsim t^{\sigma/n-1} \int_0^t f^*(\tau)\, d\tau + \int_t^\infty \tau^{\sigma/n-1} f^*(\tau)\, d\tau. \tag{5.6}$$

Now from (5.6) we obtain

$$\|u\|_{r,q;b} \precsim N_1 + N_2, \tag{5.7}$$

where

$$N_1 = \Big\| t^{1/p-1-1/q}\, \gamma_b(t) \int_0^t f^*(\tau)\, d\tau \Big\|_{q;(0,\infty)},$$

$$N_2 = \Big\| t^{1/p-\sigma/n-1/q}\, \gamma_b(t) \int_t^\infty \tau^{\sigma/n-1} f^*(\tau)\, d\tau \Big\|_{q;(0,\infty)}.$$

Applying Lemma 3.2(i), we have

$$N_1 \precsim \|t^{1/p-1/q} \gamma_b(t)\, f^*(t)\|_{q;(0,\infty)} = \|f\|_{p,q;b}. \tag{5.8}$$

Finally, Lemma 3.2(ii) gives

$$N_2 \precsim \|t^{1/p-1/q}\, \gamma_b(t)\, f^*(t)\|_{q;(0,\infty)} = \|f\|_{p,q;b}. \tag{5.9}$$

The result now follows from inequalities (5.7)–(5.9). ∎

Next, we are going to investigate limiting embeddings. To this end we need the following lemma.

LEMMA 5.2. *Let $\sigma \in (0,n)$, $p,q \in [1,\infty]$ and let b_1, b_2 be slowly varying functions on $[1,\infty)$. Suppose that*

$$\|t^{-1/q} b_2(1/t)\|_{q;(0,1)} < \infty \tag{5.10}$$

and either conditions (3.7), (3.11) *or conditions* (3.8), (3.12) *are satisfied. Then*

$$\|u^*\|_{\infty,q;b_2;(0,1)} \precsim \|u\|_{\sigma;n/\sigma,p;b_1} \tag{5.11}$$

for all $u \in H^\sigma L_{n/\sigma,p;b_1}(\mathbb{R}^n)$.

Proof. Put $X = H^\sigma L_{n/\sigma,p;b_1}(\mathbb{R}^n)$. Let $u \in X$. Then $u = g_\sigma * f$, where $f \in L_{n/\sigma,p;b_1}(\mathbb{R}^n)$ and $\|f\|_{n/\sigma,p;b_1} = \|u\|_X$. Hence by O'Neil's inequality we have (5.5), which together with the estimate (5.2) yields, for every $t \in (0,1)$,

$$u^*(t) \le \frac{n}{\sigma} A t^{\sigma/n-1} \int_0^t f^*(\tau)\, d\tau + \int_t^1 g_\sigma^*(\tau) f^*(\tau)\, d\tau + \int_1^\infty g_\sigma^*(\tau) f^*(\tau)\, d\tau. \tag{5.12}$$

By Lemma 3.6, there is a positive constant c such that

$$f^*(t) \le f^{**}(t) \le c\, \frac{t^{-\sigma/n}}{\gamma_{b_1}(t)} \|f\|_{n/\sigma,p;b_1}, \qquad t > 0. \tag{5.13}$$

Using (5.1) and (5.13) we obtain

$$\int_1^\infty g_\sigma^*(\tau) f^*(\tau)\,d\tau \le C\|f\|_{n/\sigma,p;b_1} \int_1^\infty \tau^{\sigma/n-1}\exp(-B\tau^{1/n})\,\frac{\tau^{-\sigma/n}}{b_1(\tau)}\,d\tau = C_1\|f\|_{n/\sigma,p;b_1}. \tag{5.14}$$

The estimates (5.12) and (5.14) imply

$$\|u^*\|_{\infty,q;b_2;(0,1)} \precsim N_1 + N_2 + N_3\|f\|_{n/\sigma,p;b_1}, \tag{5.15}$$

where

$$N_1 = \Big\| t^{\sigma/n-1-1/q} b_2(1/t) \int_0^t f^*(\tau)\,d\tau \Big\|_{q;(0,1)},$$

$$N_2 = \Big\| t^{-1/q}\, b_2(1/t) \int_t^1 g_\sigma^*(\tau) f^*(\tau)\,d\tau \Big\|_{q;(0,1)},$$

$$N_3 = \|t^{-1/q}\, b_2(1/t)\|_{q;(0,1)}.$$

By hypothesis (5.10) we have $N_3 < \infty$. Applying Lemma 3.2, we obtain

$$N_1 \precsim \|t^{\sigma/n-1+1/p'}\gamma_{b_1}(t)\, f^*(t)\|_{p;(0,\infty)} = \|f\|_{n/\sigma,p;b_1}. \tag{5.16}$$

Finally, Lemma 3.3 and the estimate (5.1) yield

$$N_2 \precsim \|t^{1/p'}\,\gamma_{b_1}(t)\, t^{\sigma/n-1}\, f^*(t)\|_{p;(0,1)} \precsim \|f\|_{n/\sigma,p;b_1}. \tag{5.17}$$

Now the result follows from inequalities (5.15)–(5.17). ∎

Corollary 5.1. *Let $\sigma \in (0,n)$, $p,q \in [1,\infty]$, $m \in \mathbb{N}$, $\boldsymbol{\alpha} = (\alpha_1,\dots,\alpha_m) \in \mathbb{R}^m$ and let $k \in \{1,\dots,m\}$ be such that $\alpha_k \ne 1/p'$ and, if $k \ge 2$, then $\alpha_i = 1/p'$ for $i = 1,\dots,k-1$. Let $\boldsymbol{\beta} = (\beta_1,\dots,\beta_m) \in \mathbb{R}^m$ with $\beta_k \ne -1/q$ and, if $k \ge 2$, then $\beta_i = -1/q$ for $i = 1,\dots,k-1$. Then*

$$\|u^*\|_{\infty,q;\boldsymbol{\beta};(0,1)} \precsim \|u\|_{\sigma;n/\sigma,p;\boldsymbol{\alpha}} \tag{5.18}$$

for all $u \in H^\sigma L_{n/\sigma,p;\boldsymbol{\alpha}}(\mathbb{R}^n)$ provided one of the following conditions is satisfied:

$$1 \le p \le q \le \infty, \quad \beta_k < -1/q, \quad \boldsymbol{\beta} + \boldsymbol{\delta}_{q;m,k} \preceq \boldsymbol{\alpha} - \boldsymbol{\delta}_{p';m,k}; \tag{5.19}$$

$$1 \le q < p \le \infty, \quad \beta_k < -1/q, \quad \boldsymbol{\beta} + 1/q \prec \boldsymbol{\alpha} - \boldsymbol{\delta}_{1;m,k} + 1/p. \tag{5.20}$$

If we omit the assumption $\beta_k \ne -1/q$ the result will still hold provided that

$$1 \le p \le q = \infty, \quad \boldsymbol{\beta} = \mathbf{0}, \quad -\alpha_k + 1/p' < 0. \tag{5.21}$$

Moreover, for $k = m$, if we also omit the assumption $\alpha_m \ne 1/p'$, the result will still hold provided that

$$p = 1, \quad q = \infty, \quad \boldsymbol{\alpha} = \boldsymbol{\beta} = \mathbf{0}. \tag{5.22}$$

Proof. We consider $b_1 = \vartheta^m_{\boldsymbol{\alpha}}$ and $b_2 = \vartheta^m_{\boldsymbol{\beta}}$. Since $\beta_k < -1/q$ (or $\boldsymbol{\beta} = \mathbf{0}$ if $q = \infty$), by Remark 3.6(i) condition (5.10) is satisfied.

By Remarks 3.2 and 3.4, either conditions (3.7), (3.11) or conditions (3.8), (3.12) are satisfied. Now the result follows from Lemma 5.2. ∎

The situation when $\alpha_1 = \ldots = \alpha_m = 1/p'$ is also covered by the previous corollary, by using $\widetilde{\boldsymbol{\alpha}} = (\widetilde{\alpha}_1, \ldots, \widetilde{\alpha}_{m+1}) \in \mathbb{R}^{m+1}$ in place of $\boldsymbol{\alpha} \in \mathbb{R}^m$, where $\widetilde{\alpha}_j = 1/p'$ for $j = 1, \ldots, m$ and $\widetilde{\alpha}_{m+1} = 0$.

We remark that when $k = m$, $q = \infty$ and $\beta_m = \alpha_m - 1/p' < 0$ in Corollary 5.1, we obtain

$$\sup_{t\in(0,1)} [\ell_m^{\alpha_m - 1/p'}(t) u^*(t)] \precsim \|u\|_{\sigma;n/\sigma,p;\boldsymbol{\alpha}} \tag{5.23}$$

for all $u \in H^\sigma L_{n/\sigma,p;\boldsymbol{\alpha}}(\mathbb{R}^n)$, which is precisely a result due to Edmunds, Gurka and Opic [12, Lemma 4.10]; see [10, Lemma 4.1] for the case $m = 2$.

Observe that Corollary 5.1 gives a better estimate than [12, Lemma 4.10]. In fact, suppose we are under the conditions of Corollary 5.1 with $k = m$, $q \in [p, \infty)$ and $\alpha_m < 1/p'$. Let $\beta_m = \alpha_m - 1/q - 1/p'$. Then

$$\sup_{t\in(0,1)} [\ell_m^{\alpha_m - 1/p'}(t) u^*(t)] \precsim \|u^*\|_{\infty,q;\boldsymbol{\beta};(0,1)} \precsim \|u\|_{\sigma;n/\sigma,p;\boldsymbol{\alpha}}$$

for all $u \in H^\sigma L_{n/\sigma,p;\boldsymbol{\alpha}}(\mathbb{R}^n)$ (for the first estimate see the proof of (3.43) in Theorem 3.2).

THEOREM 5.2. *Let $\sigma \in (0, n)$, $p, q \in [1, \infty]$ and let b_1, b_2 be slowly varying functions on $[1, \infty)$. Suppose that*

$$\|t^{-1/q} b_2(1/t)\|_{q;(0,1)} < \infty$$

and either conditions (3.7), (3.11) or conditions (3.8), (3.12) are satisfied. Let $I_{(b_2,q)}$ be the s.v. function on $[1, \infty)$ defined by

$$I_{(b_2,q)}(t) = \Big(\int_t^\infty (\tau^{-1/q} b_2(\tau))^q \, d\tau \Big)^{1/q}, \qquad t \geq 1,$$

if $1 \leq q < \infty$, and by $I_{(b_2,q)}(t) = b_2(t)$, $t \in [1, \infty)$, if $q = \infty$. Let Ω be a measurable subset of $\mathbb{R}^n$ with finite volume. Then

$$H^\sigma L_{n/\sigma,p;b_1}(\mathbb{R}^n) \hookrightarrow L_{\infty,q;b_2}(\Omega) \hookrightarrow L_{\infty,\infty;I_{(b_2,q)}}(\Omega). \tag{5.24}$$

Proof. First we remark that by Lemma 3.1, $I_{(b_2,q)}$ is a slowly varying function on $[1, \infty)$.

With no loss of generality we shall assume that $|\Omega|_n = 1$. Then by (5.11) of Lemma 5.2, it follows that

$$\|u\|_{\infty,q;b_2;\Omega} = \|t^{-1/q} \gamma_{b_2}(t) u^*_\Omega(t)\|_{q;(0,1)} \leq \|t^{-1/q} \gamma_{b_2}(t) u^*(t)\|_{q;(0,1)} \precsim \|u\|_{\sigma;n/\sigma,p;b_1}$$

for all $u \in H^\sigma L_{n/\sigma,p;b_1}(\mathbb{R}^n)$, which gives $H^\sigma L_{n/\sigma,p;b_1}(\mathbb{R}^n) \hookrightarrow L_{\infty,q;b_2}(\Omega)$. On the other hand by Theorem 3.3 we have the embedding

$$L_{\infty,q;b_2}(\Omega) \hookrightarrow L_{\infty,\infty;I_{(b_2,q)}}(\Omega), \tag{5.25}$$

and the result follows. ∎

If we consider $q = \infty$ in the previous theorem, the second embedding in (5.24) is trivial.

COROLLARY 5.2. *Let $\sigma \in (0, n)$, $p, q \in [1, \infty]$ and $m \in \mathbb{N}$. Let $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{R}^m$ and $k \in \{1, \ldots, m\}$ as in Corollary 5.1. Let $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_m) \in \mathbb{R}^m$ with $\nu_k = \beta_k + 1/q$ and, if*

$k \geq 2$, $\nu_j = 0$ for $j = 1, \ldots, k-1$, and, if $k+1 \leq m$, $\nu_j = \beta_j$ for $j = k+1, \ldots, m$. Let Ω be a measurable subset of $\mathbb{R}^n$ with finite volume. Then

$$H^\sigma L_{n/\sigma,p;\boldsymbol{\alpha}}(\mathbb{R}^n) \hookrightarrow L_{\infty,q;\boldsymbol{\beta}}(\Omega) \hookrightarrow L_{\infty,\infty;\boldsymbol{\nu}}(\Omega), \tag{5.26}$$

provided one of the conditions (5.19)–(5.22) *is satisfied.*

Proof. We consider $b_1 = \vartheta^m_{\boldsymbol{\alpha}}$ and $b_2 = \vartheta^m_{\boldsymbol{\beta}}$. Since $\beta_k < -1/q$ (or $\boldsymbol{\beta} = \mathbf{0}$ if $q = \infty$), by Remark 3.6(i) condition (5.10) is satisfied.

By Remarks 3.2 and 3.4, either conditions (3.7), (3.11) or conditions (3.8), (3.12) are satisfied. Now the result follows from Theorem 5.2 and Remark 3.7. ■

If we consider $q = \infty$ in the previous theorem, the second embedding in (5.26) is trivial.

Remark 5.1. Assume that the conditions of the previous corollary hold.

Let $1 \leq p, q \leq \infty$, $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^m$ and let $k \in \{1, \ldots, m\}$ be such that $\alpha_k \neq 1/p'$ and, if $k \geq 2$, then $\alpha_i = 1/p'$ for $i = 1, \ldots, k-1$. Suppose additionally that $\alpha_k > 1/p'$ (or $\boldsymbol{\alpha} = \mathbf{0}$, $k = m$ and $p = 1$). Then

$$H^\sigma L_{n/\sigma,p;\boldsymbol{\alpha}}(\mathbb{R}^n) \hookrightarrow L_\infty(\Omega) \hookrightarrow L_{\infty,q;\boldsymbol{\beta}}(\Omega)$$

with $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_m) \in \mathbb{R}^m$, $\beta_m < -1/q$ ($\beta_m \leq 0$ if $q = \infty$) and, if $m > 1$, then $\beta_j = -1/q$ for $j = 1, \ldots, m-1$. The first embedding follows from the previous corollary, because of (5.21)–(5.22), and the second embedding follows from Theorem 3.2 and Remark 3.6.

As we shall see in Remark 5.3, there is a better result than the previous one.

Let $1 \leq p, q \leq \infty$, $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{R}^m$ and $k \in \{1, \ldots, m\}$ as in Corollary 5.1. Suppose additionally that $\alpha_k < 1/p'$ and that one of conditions (5.19)–(5.20) is satisfied. Then

$$H^\sigma L_{n/\sigma,p;\boldsymbol{\alpha}}(\mathbb{R}^n) \hookrightarrow L_{\infty,p;\boldsymbol{\nu}}(\Omega) \hookrightarrow L_{\infty,q;\boldsymbol{\beta}}(\Omega),$$

where $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_m) \in \mathbb{R}^m$ with $\nu_k = \alpha_k - 1$ and, if $k \geq 2$, then $\nu_j = -1/p$ for $j = 1, \ldots, k-1$, and, if $k+1 \leq m$, then $\nu_j = \alpha_j$ for $j = k+1, \ldots, m$. The first embedding follows from the previous corollary with $q = p$. Since $\boldsymbol{\delta}_{1;m,k} = \boldsymbol{\delta}_{p;m,k} + \boldsymbol{\delta}_{p';m,k}$ and $\boldsymbol{\nu} = \boldsymbol{\alpha} - \boldsymbol{\delta}_{1;m,k}$, the second embedding follows from Theorem 3.2 and Remark 3.6 if $q \leq p$, and from Theorem 3.3 and Remark 3.7 if $q > p$.

Remark 5.2. The case $m = k = 1$ of inequality (5.18) in Corollary 5.1 and of the first embedding in (5.26) of Corollary 5.2, with $1 \leq p \leq q \leq \infty$, $\alpha_m < 1/p'$ ($\alpha_m = \beta_m = 0$ if $p = 1$, $q = \infty$) are contained in [27, Theorem 5.1(a)], which is a consequence of [26, Theorem 3.2]. Nevertheless, we remark that we only had access to these articles after our results had been proved.

We refer to Remark 3.5 for other examples of s.v. for which Lemma 5.2 and Theorem 5.2 hold.

Corollary 5.3. *Let $\sigma \in (0, n)$, $p \in [1, \infty]$ and $m \in \mathbb{N}$. Let $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^m$ and let $k \in \{1 \ldots, m\}$ be such that $\alpha_k \neq 1/p'$ and, if $k > 1$, then $\alpha_j = 1/p'$ for $j = 1, \ldots, k-1$. Assume $\alpha_k < 1/p'$. Let $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_m) \in \mathbb{R}^m$ with $\beta_k = \alpha_k - 1/p'$*

and, if $k > 1$*, then* $\beta_j = 0$ *for* $j = 1, \ldots, k-1$*, and, if* $k+1 \le m$*, then* $\beta_j = \alpha_j$ *for* $j = k+1, \ldots, m$*. Suppose that* $\Omega \subset \mathbb{R}^n$ *is such that* $|\Omega|_n < \infty$*. Then*

$$H^\sigma L_{n/\sigma,p;\boldsymbol{\alpha}}(\mathbb{R}^n) \hookrightarrow L_{\Phi_k}(\Omega),$$

where the Young function Φ_k *is given by*

$$\Phi_k(t) = \exp_k(t^{-1/\beta_k} \mu_{\boldsymbol{\gamma}}^{m-k}(t)) \quad \textit{for all large enough } t > 0,$$

with, if $k < m$*,* $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_{m-k}) \in \mathbb{R}^{m-k}$ *and* $\gamma_i = -\beta_{i+k}/\beta_k$ *for* $i = 1, \ldots, m-k$*.*

Proof. Since $\boldsymbol{\alpha}, \boldsymbol{\beta}$ satisfy condition (5.19), with $q = \infty$, we deduce from Corollary 5.2 that $H^\sigma L_{n/\sigma,p;\boldsymbol{\alpha}}(\mathbb{R}^n) \hookrightarrow L_{\infty,\infty;\boldsymbol{\beta}}(\Omega)$. Now, by Example 4.1, $L_{\Phi_k}(\Omega) = L_{\infty,\infty;\boldsymbol{\beta}}(\Omega)$, with equivalent (quasi-) norms, and the result follows. ∎

When $k = m$, the previous corollary gives a result due to Edmunds, Gurka and Opic [12, Theorem 4.3].

The next result gives us a natural generalisation of the previous one.

THEOREM 5.3. *Let* $\sigma \in (0, n)$*,* $p \in [1, \infty]$ *and let* b *be a slowly varying function on* $[1, \infty)$*. Let* Φ *be a Young function for which the restriction of* Φ^{-1} *to* $[1, \infty)$ *is a slowly varying function on* $[1, \infty)$*. Suppose that*

$$\int_0^1 \Phi(\gamma \Phi^{-1}(1/t))\, dt < \infty \quad \textit{for some } \gamma > 0, \tag{5.27}$$

$$\sup_{0<x<1} \frac{1}{\Phi^{-1}(1/x) b(1/x)} < \infty, \tag{5.28}$$

$$\sup_{0<x<1} \frac{1}{\Phi^{-1}(1/x)} \|(t^{1/p'} b(1/t))^{-1}\|_{p';(x,1)} < \infty. \tag{5.29}$$

Let Ω *be a measurable subset of* $\mathbb{R}^n$ *with finite volume. Then*

$$H^\sigma L_{n/\sigma,p;b}(\mathbb{R}^n) \hookrightarrow L_\Phi(\Omega).$$

Proof. With no loss of generality we assume that $|\Omega|_n = 1$. Let $b_1 = b$ and $b_2(t) = 1/\Phi^{-1}(t)$, $t \ge 1$. Note that b_2 is a s.v. function on $[1, \infty)$. Since Φ^{-1} is an increasing function and $\Phi^{-1}(t) > 0$, $t > 0$, it follows that

$$\sup_{0<t<1} b_2(1/t) = \frac{1}{\Phi^{-1}(1)} < \infty,$$

and condition (5.10) with $q = \infty$ is then satisfied. Condition (3.7) with $q = \infty$ is precisely (5.28). Since

$$\sup_{0<t<x} b_2(1/t) = \frac{1}{\Phi^{-1}(1/x)},$$

condition (3.11), with $q = \infty$, is precisely (5.29). Therefore, the conditions of Theorem 5.2 are satisfied and we have the embedding

$$H^\sigma L_{n/\sigma,p;b}(\mathbb{R}^n) \hookrightarrow L_{\infty,\infty;b_2}(\Omega).$$

Since Φ satisfies a Lorentz-type condition, i.e. satisfies condition (5.27), it follows from Corollary 4.1 that $L_{\infty,\infty;b_2}(\Omega) = L_\Phi(\Omega)$ with equivalent (quasi-) norms, and the result is established. ∎

The previous results concern either local estimates or local embeddings. However, following the ideas of [18], we are also able to consider global ones.

THEOREM 5.4. *Let* $\sigma \in (0,n)$, $p,q,s \in [1,\infty]$ *and let* b_1, b_2, b_3 *be slowly varying functions on* $[1,\infty)$.

(i) *Suppose that* $\|t^{-1/q} b_2(1/t)\|_{q;(0,1)} < \infty$ *and either conditions* (3.7), (3.11) *or conditions* (3.8), (3.12) *are satisfied. Then*

$$\|u^*\|_{\infty,q;b_2;(0,1)} \precsim \|u\|_{\sigma;n/\sigma,p;b_1} \tag{5.30}$$

for all $u \in H^\sigma L_{n/\sigma,p;b_1}(\mathbb{R}^n)$.

(ii) *Suppose that either condition* (3.7) *or condition* (3.8) *is satisfied, with* q *replaced by* s *and* b_2 *replaced by* b_3. *Then*

$$\|u^*\|_{n/\sigma,s;b_3;(1,\infty)} \precsim \|u\|_{\sigma;n/\sigma,p;b_1} \tag{5.31}$$

for all $u \in H^\sigma L_{n/\sigma,p;b_1}(\mathbb{R}^n)$. *Moreover, if either* $n/\sigma < s \le \infty$, *or* $s = n/\sigma$ *and* $\sup_{1<x<\infty} 1/b_3(x) < \infty$, *then*

$$\|u^*\|_{s;(1,\infty)} \precsim \|u\|_{\sigma;n/\sigma,p;b_1} \tag{5.32}$$

for all $u \in H^\sigma L_{n/\sigma,p;b_1}(\mathbb{R}^n)$.

Proof. Put $X = H^\sigma L_{n/\sigma,p;b_1}(\mathbb{R}^n)$. Let $u \in X$. Then $u = g_\sigma * f$, where $f \in L_{n/\sigma,p;b_1}(\mathbb{R}^n)$ and $\|f\|_{n/\sigma,p;b_1} = \|u\|_X$. The estimate (5.30) is precisely (5.11) of Lemma 5.2. To prove (5.31) we follow the proof of the estimate (4.9) in [18, Theorem 3.1], which is the counterpart of (5.32) for logarithmic Bessel potential spaces.

By O'Neil's inequality we have (5.5). Consequently

$$\begin{aligned} \|u^*\|_{n/\sigma,s;b_3;(1,\infty)} &\le \|t^{\sigma/n+1-1/s}\, b_3(t)\, g_\sigma^{**}(t) f^{**}(t)\|_{s;(1,\infty)} \\ &\quad + \Big\| t^{\sigma/n-1/s}\, b_3(t) \int_t^\infty g_\sigma^*(\tau) f^*(\tau)\, d\tau \Big\|_{s;(1,\infty)} \\ &=: N_1 + N_2 \end{aligned} \tag{5.33}$$

The estimate (5.3) yields

$$\begin{aligned} N_1 &\precsim \|t^{\sigma/n-1/s}\, b_3(t)\, f^{**}(t)\|_{s;(1,\infty)} \\ &= \Big\| t^{\sigma/n-1/s-1}\, b_3(t) \Big(\int_0^1 f^*(\tau)\, d\tau + \int_1^t f^*(\tau)\, d\tau \Big) \Big\|_{s;(1,\infty)} \\ &\le \Big(\int_0^1 f^*(\tau)\, d\tau \Big) \|t^{\sigma/n-1/s-1}\, b_3(t)\|_{s;(1,\infty)} \\ &\quad + \Big\| t^{\sigma/n-1/s-1}\, b_3(t) \int_1^t f^*(\tau)\, d\tau \Big\|_{s;(1,\infty)} \\ &=: N_{11} + N_{12}. \end{aligned} \tag{5.34}$$

Since $n/\sigma > 1$, by (3.2) we have $\|t^{\sigma/n-1/s-1}\, b_3(t)\|_{s;(1,\infty)} \approx b_3(1) \approx 1$, and thus, by

Hölder's inequality and (3.1),

$$N_{11} \precsim \int_0^1 f^*(\tau)\,d\tau = \int_0^1 (\tau^{\sigma/n-1/p} b_1(1/t) f^*(\tau))(\tau^{1/p-\sigma/n}(b_1(1/t))^{-1})\,d\tau \tag{5.35}$$
$$\precsim \|f\|_{n/\sigma,p;b_1} \Big(\int_0^1 \Big(\tau^{1/p-\sigma/n} \frac{1}{b_1(1/t)} \Big)^{p'} d\tau \Big)^{1/p'}$$
$$\approx \|f\|_{n/\sigma,p;b_1} (b_1(1))^{-1} \approx \|f\|_{n/\sigma,p;b_1}.$$

Applying Lemma 3.2 (the estimate (3.6) with $\nu = \sigma/n - 1 < 0$), we have

$$N_{12} \le \Big\| t^{\sigma/n-1/s-1} \gamma_{b_3}(t) \int_0^t f^*(\tau)\,d\tau \Big\|_{s;(0,\infty)} \tag{5.36}$$
$$\precsim \|t^{\sigma/n-1+1/p'} \gamma_{b_1}(t)\, f^*(t)\|_{p;(0,\infty)} = \|f\|_{n/\sigma,p;b_1}.$$

Together with (5.35), this yields

$$N_1 \precsim \|f\|_{n/\sigma,p;b_1}. \tag{5.37}$$

Using Lemma 3.2 (the estimate (3.9) with $\nu = \sigma/n > 0$), the estimate (5.1), and the fact that $t^{\sigma/n}\exp(-Bt^{1/n}) \precsim 1$ for all $t \in (0,\infty)$, we arrive at

$$N_2 \le \Big\| t^{\sigma/n-1/s} \gamma_{b_3}(t) \int_t^\infty g_\sigma^*(\tau) f^*(\tau)\,d\tau \Big\|_{s;(0,\infty)} \tag{5.38}$$
$$\precsim \|t^{\sigma/n+1/p'} \gamma_{b_1}(t)\, g_\sigma^*(t)\, f^*(t)\|_{p;(0,\infty)}$$
$$\precsim \|t^{\sigma/n+1/p'} \gamma_{b_1}(t) t^{\sigma/n-1} \exp(-Bt^{1/n}) f^*(t)\|_{p;(0,\infty)}$$
$$\precsim \|t^{\sigma/n-1/p} \gamma_{b_1}(t) f^*(t)\|_{p;(0,\infty)} = \|f\|_{n/\sigma,p;b_1}$$

and (5.31) now follows from inequalities (5.33), (5.37) and (5.38).

Let us prove (5.32). If $n/\sigma < s \le \infty$, then $t^{\sigma/n-1/s} b_3(t)$ is equivalent to a non-decreasing function on $[1,\infty)$. Hence,

$$\|u^*\|_{s;(1,\infty)} \le \|u^*\|_{n/\sigma,s;b_3;(1,\infty)}. \tag{5.39}$$

If $s = n/\sigma$ and $\sup_{1<x<\infty} 1/b_3(x) < \infty$, then (5.39) also holds. Now, (5.32) follows from (5.31) and (5.39). ∎

The first version of the previous theorem did not contain the estimate (5.31), only the estimate (5.32) with $s \ge p$. We are grateful to the Referee for suggesting this improvement to us.

See [23, Theorem 4.3] ([27, Theorem 5.1]) for the estimate near infinity, estimate (5.31), where the case of Bessel potential spaces (with logarithmic smoothness) modelled upon Lorentz–Zygmund spaces is considered.

Corollary 5.4. *Let $\sigma \in (0,n)$, $p,q \in [1,\infty]$ and $m \in \mathbb{N}$. Let $\boldsymbol{\alpha},\boldsymbol{\beta} \in \mathbb{R}^m$ and $k \in \{1,\dots,m\}$ be as in Corollary* 5.1.

(i) *Suppose that one of conditions* (5.19)–(5.22) *is satisfied. Then*

$$\|u^*\|_{\infty,q;\boldsymbol{\beta};(0,1)} \precsim \|u\|_{\sigma;n/\sigma,p;\boldsymbol{\alpha}} \tag{5.40}$$

for all $u \in H^\sigma L_{n/\sigma,p;\boldsymbol{\alpha}}(\mathbb{R}^n)$.

(ii) *Let* $s \in [1,\infty]$ *and let* $\boldsymbol{\nu} \in \mathbb{R}^m$. *Suppose that*

$$\text{either} \quad 1 \le p \le s \le \infty \quad \text{and} \quad \boldsymbol{\nu} \preceq \boldsymbol{\alpha}$$
$$\text{or} \quad 1 \le s < p \le \infty \quad \text{and} \quad \boldsymbol{\nu} + 1/s \prec \boldsymbol{\alpha} + 1/p.$$

Then

$$\|u^*\|_{n/\sigma,s;\boldsymbol{\nu};(1,\infty)} \precsim \|u\|_{\sigma;n/\sigma,p;\boldsymbol{\alpha}} \tag{5.41}$$

for all $u \in H^\sigma L_{n/\sigma,p;\boldsymbol{\alpha}}(\mathbb{R}^n)$. *Moreover, if either* $n/\sigma < s \le \infty$, *or* $s = n/\sigma$ *and* $\mathbf{0} \preceq \boldsymbol{\nu}$, *then*

$$\|u^*\|_{s;(1,\infty)} \precsim \|u\|_{\sigma;n/\sigma,p;\boldsymbol{\alpha}} \tag{5.42}$$

for all $u \in H^\sigma L_{n/\sigma,p;\boldsymbol{\alpha}}(\mathbb{R}^n)$.

Proof. We consider $b_1 = \vartheta^m_{\boldsymbol{\alpha}}$, $b_2 = \vartheta^m_{\boldsymbol{\beta}}$ and $b_3 = \vartheta^m_{\boldsymbol{\nu}}$. Part (i) is precisely Corollary 5.1. To prove (ii), we deduce by Remark 3.2 that either (3.7) or (3.8) holds. By taking into consideration Remark 3.6(ii), the result now follows from Theorem 5.4. ■

Remark 5.3. Let $\sigma \in (0,n)$, $p \in [1,\infty]$ and $m \in \mathbb{N}$. Let $\boldsymbol{\alpha} = (\alpha_1,\dots,\alpha_m) \in \mathbb{R}^m$ and let $k \in \{1,\dots,m\}$ be such that $\alpha_k \ne 1/p'$ and, if $k \ge 2$, then $\alpha_i = 1/p'$ for $i = 1,\dots,k-1$. Suppose additionally that $\alpha_k > 1/p'$. Then the previous corollary with $q = s = \infty$ and $\boldsymbol{\beta} = \mathbf{0}$ gives

$$H^\sigma L_{n/\sigma,p;\boldsymbol{\alpha}}(\mathbb{R}^n) \hookrightarrow L_\infty(\mathbb{R}^n),$$

which is a result due to Edmunds, Gurka and Opic [12, Lemma 4.5 & Corollary 4.6 & Remark 4.7]. It also follows that the previous embedding holds with $p = 1$ and $\boldsymbol{\alpha} = \mathbf{0}$.

Theorem 5.5. *Let* $\sigma \in (0,n)$, $p \in [1,\infty)$ *and let* b *be a slowly varying function on* $[1,\infty)$. *Let* $s \in [p,\infty)$ *and suppose that either* $n/\sigma < s < \infty$, *or* $s = n/\sigma$ *and* $\sup_{1<x<\infty} 1/b(x) < \infty$. *Let* Φ *be a Young function such that* $\Phi(t) = t^s$ *for all small enough* $t \ge 0$, *and for which the restriction of* Φ^{-1} *to* $[1,\infty)$ *is a slowly varying function on* $[1,\infty)$. *Suppose that conditions* (5.27)–(5.29) *are also satisfied. Then*

$$H^\sigma L_{n/\sigma,p;b}(\mathbb{R}^n) \hookrightarrow L_\Phi(\mathbb{R}^n). \tag{5.43}$$

Proof. Let $b_1 = b$ and $b_2(t) = 1/\Phi^{-1}(t)$, $t \ge 1$. Note that b_2 is an s.v. function on $[1,\infty)$. As in the proof of Theorem 5.3, it follows that the conditions of Theorem 5.4 are satisfied with $q = \infty$. Therefore the estimate (5.30) gives

$$\sup_{0<t<1} \frac{u^*(t)}{\Phi^{-1}(1/t)} \precsim \|u\|_{\sigma;n/\sigma,p;b} \quad \text{for all } u \in H^\sigma L_{n/\sigma,p;b}(\mathbb{R}^n). \tag{5.44}$$

Let $b_3 = b_1 = b$. Then Theorem 5.4(ii) (estimate (5.32)) gives us the two estimates

$$\|u^*\|_{s;(1,\infty)} \precsim \|u\|_{\sigma;n/\sigma,p;b} \quad \text{for all } u \in H^\sigma L_{n/\sigma,p;b}(\mathbb{R}^n), \tag{5.45}$$

and

$$u^*(1) = \|u^*\|_{\infty;(1,\infty)} \le \kappa \|u\|_{\sigma;n/\sigma,p;b} \quad \text{for all } u \in H^\sigma L_{n/\sigma,p;b}(\mathbb{R}^n), \tag{5.46}$$

where κ is a positive constant.

Now we follow the end of Step 3 in the proof of [18, Theorem 3.1].

Let $t_0 \in (0,1)$ be such that

$$\Phi(t) = t^s \quad \text{for all } 0 \le t < t_0. \tag{5.47}$$

If $\kappa \le t_0$, we deduce from (5.45)–(5.47) that

$$\int_1^\infty \Phi\left(\frac{u^*(t)}{\|u\|_{\sigma;n/\sigma,p;b}}\right) dt \lesssim 1 \quad \text{for all } u \in H^\sigma L_{n/\sigma,p;b}(\mathbb{R}^n). \tag{5.48}$$

If $\kappa > t_0$, then taking $x \in (t_0, \kappa]$ and $c_1 = \Phi(\kappa)/t_0^s$ we have $\Phi(x) \le \Phi(\kappa) \le c_1 x^s$. Together with (5.47), this implies that $\Phi(x) \le cx^s$ for all $x \in [0, \kappa]$, where $c = \max\{1, c_1\}$. Then, by (5.45),

$$\int_1^\infty \Phi\left(\frac{u^*(t)}{\|u\|_{\sigma;n/\sigma,p;b}}\right) dt \le c \int_1^\infty \left(\frac{u^*(t)}{\|u\|_{\sigma;n/\sigma,p;b}}\right)^s dt \lesssim 1 \tag{5.49}$$

for all $u \in H^\sigma L_{n/\sigma,p;b}(\mathbb{R}^n)$.

Since Φ is convex with $\Phi(0) = 0$, it follows from (5.48) and (5.49) that

$$\inf\left\{\lambda > 0 : \int_1^\infty \Phi(u^*(t)/\lambda)\, dt \le 1\right\} \lesssim \|u\|_{\sigma;n/\sigma,p;b} \tag{5.50}$$

for all $u \in H^\sigma L_{n/\sigma,p;b}(\mathbb{R}^n)$.

Now, because Φ satisfies the Lorentz-type condition (5.27), it follows from Theorem 4.1, (5.44) and (5.50) that $\|u\|_\Phi \lesssim \|u\|_{\sigma;n/\sigma,p;b}$ for all $u \in H^\sigma L_{n/\sigma,p;b}(\mathbb{R}^n)$, which gives the embedding (5.43). ∎

Corollary 5.5. *Let $\sigma \in (0, n)$, $p \in [1, \infty]$ and $m \in \mathbb{N}$. Let $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^m$ and let $k \in \{1\ldots, m\}$ be such that $\alpha_k \ne 1/p'$ and, if $k > 1$, then $\alpha_j = 1/p'$ for $j = 1, \ldots, k-1$. Assume $\alpha_k < 1/p'$ and let $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_m) \in \mathbb{R}^m$ be defined by $\boldsymbol{\beta} = \boldsymbol{\alpha} - \boldsymbol{\delta}_{p';m,k}$. Suppose that $s \in [p, \infty)$ and that either $s > n/\sigma$, or $s = n/\sigma$ and $p > 1$. Then*

$$H^\sigma L_{n/\sigma,p;\boldsymbol{\alpha}}(\mathbb{R}^n) \hookrightarrow L_{\Phi_k}(\mathbb{R}^n),$$

where the Young function Φ_k is given by

$$\Phi_k(t) = \begin{cases} t^s & \text{for all small enough } t \ge 0, \\ \exp_k(t^{-1/\beta_k} \mu_\gamma^{m-k}(t)) & \text{for all large enough } t > 0, \end{cases}$$

with, if $k < m$, $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_{m-k}) \in \mathbb{R}^{m-k}$ and $\gamma_i = -\beta_{i+k}/\beta_k$, $i = 1, \ldots, m-k$.

Proof. Let Ψ_k be the Young function defined by

$$\Psi_k(t) = \begin{cases} t^s, & 0 \le t < t_0 \le 1, \\ \mathrm{Exp}_k(t^{-1/\beta_k} \vartheta_\gamma^{m-k}(t)), & 1 < t_\infty < t < \infty, \end{cases} \tag{5.51}$$

where t_0 is small enough and t_∞ is large enough. Let $T_\infty = \Psi_k(t_\infty) > 1$. Then

$$\Psi_k^{-1}(t) \approx \vartheta_{-\boldsymbol{\beta}}^m(t), \qquad t > T_\infty,$$

and it follows that the restriction of Ψ^{-1} to $[1, \infty)$ is a s.v. function on $[1, \infty)$.

From Example 4.1, Ψ_k satisfies the Lorentz-type condition (5.27). Let us consider $b = \vartheta_{\boldsymbol{\alpha}}^m$. Then conditions (5.28) and (5.29) are also easily verified. Taking into account Remark 3.6(ii), it now follows from Theorem 5.5 that $H^\sigma L_{n/\sigma,p;\boldsymbol{\alpha}}(\mathbb{R}^n) \hookrightarrow L_{\Psi_k}(\mathbb{R}^n)$. Since Φ_k and Ψ_k are equivalent globally, we have $L_{\Phi_k}(\mathbb{R}^n) = L_{\Psi_k}(\mathbb{R}^n)$, with equivalent norms, and the result follows. ∎

The previous corollary with $k = m$ coincides with [18, Theorem 3.1].

6. Riesz-potential-type embedding theorems

In [9], the authors proved the following result which characterises boundedness of convolution operators in generalised Lorentz–Zygmund spaces.

THEOREM 6.1 [9, Theorem 2.1]. *Let $s, q \in (1,\infty)$, $\gamma, \delta \in \mathbb{R}$. Let $q_1, q_2 \in (1,\infty]$ be such that $0 < 1/q_1 + 1/q_2 < 1$ and set $1/p = 1/q_1 + 1/q_2$. Let $\boldsymbol{\alpha} = (1/p', \delta/p)$ and $\boldsymbol{\beta} = (-1/q, \gamma/q)$. Assume $f \in L_{s,q_1}(\mathbb{R}^n)$, $g \in L_{s',q_2;\boldsymbol{\alpha}}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$, $u = f * g$ and suppose that*

$$\begin{cases} \textit{either} & p \le q, \gamma < -1,\ \frac{\gamma+1}{q} - \frac{\delta+1}{p} + 1 \le 0; \\ \textit{or} & p > q,\ \gamma < -1,\ \frac{\gamma+1}{q} - \frac{\delta+1}{p} + 1 < 0. \end{cases} \tag{6.1}$$

Then

$$\|u^*\|_{\infty,q;\boldsymbol{\beta};(0,1)} \precsim \|f\|_{s,q_1}(\|g\|_{s',q_2;\boldsymbol{\alpha}} + \|g\|_1). \tag{6.2}$$

The next result, which also characterises boundedness of convolution operators in generalised Lorentz–Karamata spaces, generalises and extends Theorem 6.1, as we will see later on.

THEOREM 6.2. *Let $s \in (1,\infty)$, $q \in [1,\infty]$. Let $q_1, q_2 \in [1,\infty]$ be such that $1/q_1 + 1/q_2 \le 1$ and set $1/p = 1/q_1 + 1/q_2$. Let b_1, b_2 be slowly varying functions on $[1,\infty)$. Suppose that*

$$\|t^{-1/q} b_2(1/t)\|_{q;(0,1)} < \infty, \qquad \left\| t^{-1/q} \frac{b_2(1/t)}{b_1(1/t)} \right\|_{q;(0,1)} < \infty. \tag{6.3}$$

Suppose that either $1 \le p \le q \le \infty$ and (3.11) is satisfied, or $1 \le q < p \le \infty$ and (3.12) is satisfied. Assume $f \in L_{s,q_1}(\mathbb{R}^n)$, $g \in L_{s',q_2;b_1}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$, $u = f * g$. Then

$$\|u^*\|_{\infty,q;b_2;(0,1)} \precsim \|f\|_{s,q_1}(\|g\|_{s',q_2;b_1} + \|g\|_1). \tag{6.4}$$

Moreover, if Ω is a measurable subset of $\mathbb{R}^n$ with finite volume, then $u \in L_{\infty,q;b_2}(\Omega)$ and

$$\|u\|_{\infty,q;b_2;\Omega} \precsim \|f\|_{s,q_1}(\|g\|_{s',q_2;b_1} + \|g\|_1). \tag{6.5}$$

Proof. Since $u = f * g$, by O'Neil's inequality (cf. e.g. [35, Lemma 1.8.8]) we have

$$u^*(t) \le u^{**}(t) \le t f^{**}(t) g^{**}(t) + \int_t^\infty f^*(\tau) g^*(\tau)\, d\tau. \tag{6.6}$$

By Lemma 3.6, for all $t > 0$ we have

$$f^*(t) \le f^{**}(t) \precsim t^{-1/s}\|f\|_{s,q_1}, \tag{6.7}$$

$$g^*(t) \le g^{**}(t) \precsim \frac{t^{-1/s'}}{\gamma_{b_1}(t)}\|g\|_{s',q_2;b_1}. \tag{6.8}$$

Using (6.6) and the previous estimates, for $t \in (0,1)$, we obtain

$$u^*(t) \le u^{**}(t) \precsim \frac{1}{\gamma_{b_1}(t)}\|f\|_{s,q_1}\|g\|_{s',q_2;b_1} + \int_t^1 f^*(\tau) g^*(\tau)\, d\tau + \int_1^\infty f^*(\tau) g^*(\tau)\, d\tau. \tag{6.9}$$

The estimate (6.7) together with the obvious inequality

$$g^*(\tau) \le g^{**}(\tau) \le \tau^{-1}\|g\|_1, \qquad \tau > 0,$$

gives

$$\int_1^\infty f^*(\tau)g^*(\tau)\,d\tau \precsim \|f\|_{s,q_1}\|g\|_1 \int_1^\infty \tau^{-(1+1/s)}\,d\tau \approx \|f\|_{s,q_1}\|g\|_1. \tag{6.10}$$

Therefore, the estimates (6.9) and (6.10) imply

$$\|u^*\|_{\infty,q;b_2;(0,1)} \precsim \|f\|_{s,q_1}\|g\|_{s',q_2;b_1}N_1 + N_2 + \|f\|_{s,q_1}\|g\|_1 N_3, \tag{6.11}$$

where

$$N_1 = \left\| t^{-1/q}\frac{\gamma_{b_2}(t)}{\gamma_{b_1}(t)}\right\|_{q;(0,1)},$$
$$N_2 = \left\| t^{-1/q}\gamma_{b_2}(t)\int_t^1 f^*(\tau)g^*(\tau)\right\|_{q;(0,1)},$$
$$N_3 = \|t^{-1/q}\gamma_{b_2}(t)\|_{q;(0,1)}.$$

By hypothesis $N_1 < \infty$ and $N_3 < \infty$. Finally, Lemma 3.3 and Hölder's inequality yield

$$\begin{aligned} N_2 &\precsim \|t^{1/p'}\gamma_{b_1}(t)f^*(t)g^*(t)\|_{p;(0,1)} = \|t^{1/s-1/q_1}f^*(t)t^{1/s'-1/q_2}\gamma_{b_1}(t)g^*(t)\|_{p;(0,1)} \\ &\le \|t^{1/s-1/q_1}f^*(t)\|_{q_1;(0,1)}\|t^{1/s'-1/q_2}\gamma_{b_1}(t)g^*(t)\|_{q_2;(0,1)} = \|f\|_{s,q_1}\|g\|_{s',q_2;b_1}. \end{aligned} \tag{6.12}$$

Now (6.4) follows from inequalities (6.11)–(6.12).

Now with no loss of generality we shall assume that $|\Omega|_n = 1$. Then, from (6.4), we obtain

$$\|u\|_{\infty,q;b_2;\Omega} \le \|t^{-1/q}\gamma_{b_2}(t)u^*(t)\|_{q;(0,1)} \precsim \|f\|_{s,q_1}(\|g\|_{s',q_2;b_1} + \|g\|_1),$$

which gives (6.5). ■

Note that if Ω is a measurable subset of $\mathbb{R}^n$ with finite volume and g is a measurable function on $\mathbb{R}^n$ with $\operatorname{supp} g \subset \Omega$, then $g \in L_{p,q;b}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ if, and only if, $g \in L_{p,q;b}(\mathbb{R}^n)$, where $p \in (1,\infty)$, $q \in [1,\infty]$ and b is a slowly varying function on $[1,\infty)$.

Corollary 6.1. *Let $s \in (1,\infty)$, $q \in [1,\infty]$ and $m \in \mathbb{N}$. Let $q_1, q_2 \in [1,\infty]$ be such that $1/q_1 + 1/q_2 \le 1$ and set $1/p = 1/q_1 + 1/q_2$. Let $\boldsymbol{\alpha} = (\alpha_1,\dots,\alpha_m) \in \mathbb{R}^m$ and let $k \in \{1,\dots,m\}$ be such that $\alpha_k \ne 1/p'$ and, if $k \ge 2$, then $\alpha_i = 1/p'$ for $i = 1,\dots,k-1$. Let $\boldsymbol{\beta} = (\beta_1,\dots,\beta_m) \in \mathbb{R}^m$ with $\beta_k \ne -1/q$ and, if $k \ge 2$, then $\beta_i = -1/q$ for $i = 1,\dots,k-1$. Assume $f \in L_{s,q_1}(\mathbb{R}^n)$, $g \in L_{s',q_2;\boldsymbol{\alpha}}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$, $u = f * g$. Suppose one of the following conditions is satisfied:*

$$1 < p \le q \le \infty, \quad \beta_k < -1/q, \quad \boldsymbol{\beta} + \boldsymbol{\delta}_{q;m,k} \preceq \boldsymbol{\alpha} - \boldsymbol{\delta}_{p';m,k}; \tag{6.13}$$
$$1 \le q < p \le \infty, \quad \beta_k < -1/q, \quad \boldsymbol{\beta} + 1/q \prec \boldsymbol{\alpha} - \boldsymbol{\delta}_{1;m,k} + 1/p; \tag{6.14}$$
$$p = 1,\ q = \infty, \quad \beta_k < 0, \quad \boldsymbol{\beta} \preceq \boldsymbol{\alpha}. \tag{6.15}$$

Then

$$\|u^*\|_{\infty,q;\boldsymbol{\beta};(0,1)} \precsim \|f\|_{s,q_1}(\|g\|_{s',q_2;\boldsymbol{\alpha}} + \|g\|_1). \tag{6.16}$$

Moreover, if Ω is a measurable subset of $\mathbb{R}^n$ with finite volume, then $u \in L_{\infty,q;\boldsymbol{\beta}}(\Omega)$ and

$$\|u\|_{\infty,q;\boldsymbol{\beta};\Omega} \precsim \|f\|_{s,q_1}(\|g\|_{s',q_2;\boldsymbol{\alpha}} + \|g\|_1). \tag{6.17}$$

If we omit the assumption $\beta_k \neq -1/q$ *the result will still hold provided that*

$$1 \le p \le q = \infty, \quad \boldsymbol{\beta} = \mathbf{0}, \quad -\alpha_k + 1/p' < 0. \tag{6.18}$$

Moreover, for $k = m$, *if we also omit the assumption* $\alpha_m \neq 1/p'$, *the result will still hold provided that*

$$p = 1, \quad q = \infty, \quad \boldsymbol{\beta} = \mathbf{0}, \quad \boldsymbol{\alpha} = \mathbf{0}. \tag{6.19}$$

Proof. Let $b_1 = \vartheta^m_{\boldsymbol{\alpha}}$ and $b_2 = \vartheta^m_{\boldsymbol{\beta}}$. Then under our conditions (note that $p > 1$ in (6.13)), condition (6.3) is satisfied. According to Lemma 3.3 and Remark 3.4, either $1 \le p \le q \le \infty$ and (3.11) is satisfied, or $1 \le q < p \le \infty$ and (3.12) is satisfied. Hence the result follows from Theorem 6.2. ∎

Remark 6.1. Assume that the conditions of the previous corollary hold.

When $1 \le p \le \infty$, $\alpha_k > 1/p'$ (or $\boldsymbol{\alpha} = \mathbf{0}$, if $p = 1$), we have

$$\|u^*\|_{\infty;(0,1)} \precsim \|f\|_{s,q_1}(\|g\|_{s',q_2;\boldsymbol{\alpha}} + \|g\|_1).$$

In particular, if Ω is a measurable subset of $\mathbb{R}^n$ with finite volume, it follows that

$$\|u\|_{\infty;\Omega} \precsim \|f\|_{s,q_1}(\|g\|_{s',q_2;\boldsymbol{\alpha}} + \|g\|_1).$$

Let $1 < p \le \infty$, $1 \le q \le \infty$. Take $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{R}^m$ and $k \in \{1, \dots, m\}$ as in Corollary 6.1. Suppose additionally that $\alpha_k < 1/p'$ and that one of conditions (6.13)–(6.14) is satisfied. Then

$$\|u^*\|_{\infty,p;\boldsymbol{\nu};(0,1)} \precsim \|f\|_{s,q_1}(\|g\|_{s',q_2;\boldsymbol{\alpha}} + \|g\|_1),$$

where $\boldsymbol{\nu} = \boldsymbol{\alpha} - \boldsymbol{\delta}_{1;m,k}$, which is better than the estimate

$$\|u^*\|_{\infty,q;\boldsymbol{\beta};(0,1)} \precsim \|f\|_{s,q_1}(\|g\|_{s',q_2;\boldsymbol{\alpha}} + \|g\|_1).$$

In particular, if Ω is a measurable subset of $\mathbb{R}^n$ with finite volume, it follows that

$$\|u\|_{\infty,p;\boldsymbol{\nu};\Omega} \precsim \|f\|_{s,q_1}(\|g\|_{s',q_2;\boldsymbol{\alpha}} + \|g\|_1).$$

Remark 6.2. Under the conditions of Theorem 6.1, if we choose $\delta = p - 1$, we have

$$\|u^*\|_{\infty,q;\boldsymbol{\beta};(0,1)} \precsim \|f\|_{s,q_1}(\|g\|_{s',q_2;\boldsymbol{\alpha}} + \|g\|_1)$$

for all $f \in L_{s,q_1}(\mathbb{R}^n)$ and $g \in L_{s',q_2;\boldsymbol{\alpha}}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$, where $u = f * g$, $\boldsymbol{\alpha} = (1/p', 1/p')$, $\boldsymbol{\beta} = (-1/q, \gamma/q)$ and $\gamma < -1$.

However, from Corollary 6.1, with $m = k = 3$ and $\alpha_3 = 0$, it follows that

$$\|u^*\|_{\infty,q;\boldsymbol{\beta}_1;(0,1)} \precsim \|f\|_{s,q_1}(\|g\|_{s',q_2;\boldsymbol{\alpha}_1} + \|g\|_1) \tag{6.20}$$

for all $f \in L_{s,q_1}(\mathbb{R}^n)$ and $g \in L_{s',q_2;\boldsymbol{\alpha}_1}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$, where $u = f * g$, $\boldsymbol{\alpha}_1 = (1/p', 1/p', 0)$, $\boldsymbol{\beta}_1 = (-1/q, -1/q, \beta_3)$, with β_3 satisfying either (6.13) or (6.14).

Since $L_{s',q_2;\boldsymbol{\alpha}}(\mathbb{R}^n) = L_{s',q_2;\boldsymbol{\alpha}_1}(\mathbb{R}^n)$ and $\|u^*\|_{\infty,q;\boldsymbol{\beta};(0,1)} \precsim \|u^*\|_{\infty,q;\boldsymbol{\beta}_1;(0,1)}$, Corollary 6.1 gives us the better estimate (6.20).

In the special case when the function u is a Riesz-potential of the function g, i.e. $u = I_\sigma * g$, where I_σ, $0 < \sigma < n$, is the Riesz kernel, we have the following corollary, which generalises and extends the sufficiency part of Theorem 2.2 and Remark 3.11(iv) in [9].

Corollary 6.2. *Let $\sigma \in (0,n)$, $p,q \in [1,\infty]$. Let b_1, b_2 be slowly varying functions on $[1,\infty)$ such that condition* (6.3) *holds and either $1 \le p \le q \le \infty$ and* (3.11) *is satisfied, or $1 \le q < p \le \infty$ and* (3.12) *is satisfied. Assume $g \in L_{n/\sigma,p;b_1}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$, $u = I_\sigma * g$. Then*

$$\|u^*\|_{\infty,q;b_2;(0,1)} \precsim \|g\|_{n/\sigma,p;b_1} + \|g\|_1. \tag{6.21}$$

Moreover, if Ω is a measurable subset of $\mathbb{R}^n$ with finite volume, then $u \in L_{\infty,q;b_2}(\Omega)$ and

$$\|u\|_{\infty,q;b_2;\Omega} \precsim \|g\|_{n/\sigma,p;b_1} + \|g\|_1. \tag{6.22}$$

Proof. One can easily compute that $(I_\sigma)^*(t) = (t/\omega_n)^{\sigma/n-1}$, $t > 0$ (see [35, pp. 97–98]), where ω_n is the volume of the unit ball in $\mathbb{R}^n$. Therefore, $I_\sigma \in L_{n/(n-\sigma),\infty}(\mathbb{R}^n)$, and the result now follows from Theorem 6.2 on putting $s = n/(n-\sigma)$ and $q_1 = \infty$. ∎

Corollary 6.3. *Let $\sigma \in (0,n)$, $p,q \in [1,\infty]$ and $m \in \mathbb{N}$. Let $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{R}^m$ and $k \in \{1,\dots,m\}$ as in Corollary* 6.1. *Assume $g \in L_{n/\sigma,p;\boldsymbol{\alpha}}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$, $u = I_\sigma * g$. Suppose that one of conditions* (6.13)–(6.15), (6.18)–(6.19) *is satisfied. Then*

$$\|u^*\|_{\infty,q;\boldsymbol{\beta};(0,1)} \precsim \|g\|_{n/\sigma,p;\boldsymbol{\alpha}} + \|g\|_1. \tag{6.23}$$

Moreover, if Ω is a measurable subset of $\mathbb{R}^n$ with finite volume, then $u \in L_{\infty,q;\boldsymbol{\beta}}(\Omega)$ and

$$\|u\|_{\infty,q;\boldsymbol{\beta};\Omega} \precsim \|g\|_{n/\sigma,p;\boldsymbol{\alpha}} + \|g\|_1. \tag{6.24}$$

Proof. The result follows from Corollary 6.1 on putting $s = n/(n-\sigma)$, $q_1 = \infty$ and $f = I_\sigma \in L_{n/(n-\sigma),\infty}(\mathbb{R}^n)$. ∎

Remark 6.3. Assume that the conditions of the previous corollary hold.

When $1 \le p \le \infty$, $\alpha_k > 1/p'$ (or $\boldsymbol{\alpha} = \mathbf{0}$ if $p = 1$), we have

$$\|u^*\|_{\infty;(0,1)} \precsim \|g\|_{n/\sigma,p;\boldsymbol{\alpha}} + \|g\|_1.$$

In particular, if Ω is a measurable subset of $\mathbb{R}^n$ with finite volume, it follows that

$$\|u\|_{\infty;\Omega} \precsim \|g\|_{n/\sigma,p;\boldsymbol{\alpha}} + \|g\|_1.$$

Let $1 < p \le \infty$, $1 \le q \le \infty$. Take $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{R}^m$ and $k \in \{1,\dots,m\}$ as in Corollary 6.3. Suppose additionally that $\alpha_k < 1/p'$ and that one of conditions (6.13)–(6.14) is satisfied. Then

$$\|u^*\|_{\infty,p;\boldsymbol{\nu};(0,1)} \precsim \|g\|_{n/\sigma,p;\boldsymbol{\alpha}} + \|g\|_1,$$

where $\boldsymbol{\nu} = \boldsymbol{\alpha} - \boldsymbol{\delta}_{1;m,k}$, which is better than the estimate

$$\|u^*\|_{\infty,q;\boldsymbol{\beta};(0,1)} \precsim \|g\|_{n/\sigma,p;\boldsymbol{\alpha}} + \|g\|_1.$$

In particular, if Ω is a measurable subset of $\mathbb{R}^n$ with finite volume, it follows that

$$\|u\|_{\infty,p;\boldsymbol{\nu};\Omega} \precsim \|g\|_{n/\sigma,p;\boldsymbol{\alpha}} + \|g\|_1.$$

Remark 6.4. Following the reasoning of Remark 6.2, the previous corollary with $m = k = 3$, $\alpha_3 = 0$ and $1 \le q < \infty$ gives a better estimate than the one given by [9, Theorem 2.2]. More precisely, we obtain

$$\|u^*\|_{\infty,q;\boldsymbol{\beta};(0,1)} \precsim \|g\|_{n/\sigma,p;\boldsymbol{\alpha}} + \|g\|_1,$$

for all $g \in L_{n/\sigma,p;\boldsymbol{\alpha}}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$, where $u = I_\sigma * g$, $\boldsymbol{\alpha} = (1/p', 1/p', 0)$ and $\boldsymbol{\beta} = (-1/q, -1/q, \beta_3)$, with β_3 satisfying either (6.13) or (6.14), rather than the estimate

$$\|u^*\|_{\infty,q;\boldsymbol{\beta}_1;(0,1)} \precsim \|g\|_{n/\sigma,p;\boldsymbol{\alpha}} + \|g\|_1$$

for all $g \in L_{n/\sigma,p;\boldsymbol{\alpha}}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$, where $u = I_\sigma * g$, $\boldsymbol{\alpha} = (1/p', 1/p', 0)$ and $\boldsymbol{\beta}_1 = (-1/q, \gamma/q)$, with $\gamma < -1$.

The following results extend and improve [9, Corollary 2.3(ii), Corollary 2.4(ii)].

THEOREM 6.3. *Let $s \in (1,\infty)$ and let $q_1, q_2 \in [1,\infty]$ be such that $1/q_1 + 1/q_2 \le 1$ and set $1/p = 1/q_1 + 1/q_2$. Let b be a slowly varying function on $[1,\infty)$. Let Φ be a Young function for which the restriction of Φ^{-1} to $[1,\infty)$ is a slowly varying function on $[1,\infty)$. Suppose that*

$$\int_0^1 \Phi(\gamma\Phi^{-1}(1/t))\,dt < \infty \quad \text{for some } \gamma > 0, \tag{6.25}$$

$$\sup_{0<x<1} \frac{1}{\Phi^{-1}(1/x)b(1/x)} < \infty, \tag{6.26}$$

$$\sup_{0<x<1} \frac{1}{\Phi^{-1}(1/x)} \|(t^{1/p'}b(1/t))^{-1}\|_{p';(x,1)} < \infty. \tag{6.27}$$

Let Ω be a measurable subset of $\mathbb{R}^n$ with finite volume. If $f \in L_{s,q_1}(\mathbb{R}^n)$, $g \in L_{s',q_2;b}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ and $u = f * g$, then $u \in L_\Phi(\Omega)$ and

$$\|u\|_{\Phi,\Omega} \precsim \|f\|_{s,q_1}(\|g\|_{s',q_2;b} + \|g\|_1).$$

Proof. With no loss of generality we assume that $|\Omega|_n = 1$. Let $b_1 = b$ and $b_2(t) = 1/\Phi^{-1}(t)$, $t \ge 1$. Note that b_2 is a s.v. function on $[1,\infty)$. Since Φ^{-1} is an increasing function and $\Phi^{-1}(t) > 0$, $t > 0$, it follows that

$$\sup_{0<t<1} b_2(1/t) = \frac{1}{\Phi^{-1}(1)} < \infty,$$

and the first condition in (6.3), with $q = \infty$, is satisfied. The second condition in (6.3), with $q = \infty$, is precisely (6.26). Since

$$\sup_{0<t<x} b_2(1/t) = \frac{1}{\Phi^{-1}(1/x)},$$

it follows from (6.27) that (3.11) with $q = \infty$ is satisfied. Therefore, the conditions of Theorem 6.2 are satisfied and the estimate $\|u\|_{\infty,\infty;b_2;\Omega} \precsim \|f\|_{s,q_1}(\|g\|_{s',q_2;b_1} + \|g\|_1)$ follows. Since Φ satisfies a Lorentz-type condition, i.e. satisfies (6.25), we see from Corollary 4.1 that $L_{\infty,\infty;b_2}(\Omega) = L_\Phi(\Omega)$ with equivalent (quasi-) norms, and the result is established. ∎

Now it is easy to verify the next three results.

COROLLARY 6.4. *Let $\sigma \in (0,n)$, $p \in [1,\infty]$. Let b be a slowly varying function on $[1,\infty)$. Let Φ be a Young function for which the restriction of Φ^{-1} to $[1,\infty)$ is a slowly varying function on $[1,\infty)$. Suppose that conditions (6.25)–(6.27) are also satisfied. Let Ω be a measurable subset of $\mathbb{R}^n$ with finite volume. Then $I_\sigma * g \in L_\Phi(\Omega)$ and*

$$\|I_\sigma * g\|_{\Phi,\Omega} \precsim \|g\|_{n/\sigma,p;b} + \|g\|_1$$

for all $g \in L_{n/\sigma,p;b}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$.

Corollary 6.5. *Let $s \in (1,\infty)$ and $m \in \mathbb{N}$. Let $q_1, q_2 \in [1,\infty]$ be such that $1/q_1 + 1/q_2 \le 1$ and set $1/p = 1/q_1 + 1/q_2$. Let $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^m$ and let $k \in \{1, \ldots, m\}$ be such that $\alpha_k \neq 1/p'$ and, if $k > 1$, then $\alpha_j = 1/p'$ for $j = 1, \ldots, k-1$. Assume $\alpha_k < 1/p'$ and let $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_m) \in \mathbb{R}^m$ be defined by $\boldsymbol{\beta} = \boldsymbol{\alpha} - \boldsymbol{\delta}_{p';m,k}$. Let Ω be a measurable subset of $\mathbb{R}^n$ with finite volume. Let Φ_k be the Young function defined by*

$$\Phi_k(t) = \exp_k(t^{-1/\beta_k} \mu_\gamma^{m-k}(t)) \quad \text{for all large enough } t > 0,$$

with, if $k < m$, $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_{m-k}) \in \mathbb{R}^{m-k}$ and $\gamma_i = -\beta_{i+k}/\beta_k$ for $i = 1, \ldots, m-k$. Then $u \in L_{\Phi_k}(\Omega)$ and

$$\|u\|_{\Phi_k,\Omega} \precsim \|f\|_{s,q_1}(\|g\|_{s',q_2;\boldsymbol{\alpha}} + \|g\|_1)$$

for all $f \in L_{s,q_1}(\mathbb{R}^n)$ and $g \in L_{s',q_2;\boldsymbol{\alpha}}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$.

Corollary 6.6. *Let $\sigma \in (0,n)$, $p \in [1,\infty]$ and $m \in \mathbb{N}$. Let $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{R}^m$, $k \in \{1, \ldots, m\}$ and Φ_k as in Corollary 6.5. Let Ω be a measurable subset of $\mathbb{R}^n$ with finite volume. Then $u = I_\sigma * g \in L_{\Phi_k}(\Omega)$ and*

$$\|I_\sigma * g\|_{\Phi_k,\Omega} \precsim \|g\|_{n/\sigma,p;\boldsymbol{\alpha}} + \|g\|_1$$

for all $g \in L_{n/\sigma,p;\boldsymbol{\alpha}}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$.

Remark 6.5. Following the same line of reasoning as in Remarks 6.2 and 6.4, and Corollaries 6.5 and 6.6, with $m = 3$ and $\alpha_3 = 0$, we arrive at a triple exponential Orlicz space as a target space, improving in this way [9, Corollary 2.3(ii), Corollary 2.4(ii)], with $\delta = p-1$ (see Theorem 6.1), which give a double exponential Orlicz space.

Corollary 6.6 with $k = 1$, $m = 2$ and $p = n/\sigma$ gives a result related to the one of Mizuta and Shimomura [22, Theorem A].

The previous remark extends [9, Remark 3.11(iv)].

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