Introduction

This paper solves the functional inequality

$$af(s) + bf(t) \geq f(\alpha s + \beta t), \qquad s, t > 0, \tag{1}$$

with four positive parameters a, b, α, β arbitrarily fixed. The unknown function $f : (0, \infty) \to \mathbb{R}$ is assumed to satisfy the regularity condition

$$\limsup_{s \to 0+} f(s) \leq 0. \tag{2}$$

Two cases of inequality (1) have had a long history of research (cf. Section 1):

- Jensen-convexity, i.e., (1) with $a = b = \alpha = \beta = 1/2$.
- Subadditivity, i.e., (1) with $a = b = \alpha = \beta = 1$.

Recently, four other cases of (1) were analyzed (cf. Section 1):

- Generalized Jensen-convexity, i.e., (1) with $a + b = 1$, $\alpha + \beta = 1$. In this case the solutions to (1)–(2) are either convex or constant (N. Kuhn [1984, 1987], Z. Daróczy and Zs. Páles [1987], S. Kryński [1993], Z. Kominek [1992], J. Matkowski and M. Pycia [1995b]).
- Generalized subadditivity, i.e., (1) with $\beta = b = 1$. In this case the solutions to (1)–(2) share some regularity properties of subadditive functions (J. Matkowski [1994b], J. Matkowski and T. Świątkowski [1994]).
- (1) with $a = \alpha$, $b = \beta$, $\alpha < 1 < \alpha + \beta$. In this case the solutions to (1)–(2) are linear (J. Matkowski [1990b, 1992a, 1992b], Matkowski and Pycia [1995a]).
- (1) with $a = \alpha$, $b = \beta$, $\alpha + \beta < 1$. In this case there are special pairs of functions $f_1 < f_2$ such that every f with $f_1(t) < f(t) < f_2(t)$ for $t > 0$ satisfies (1)–(2) (this result is implied by K. Baron, J. Matkowski and K. Nikodem [1994]).

There are also quadruples of parameters such that

- (1)–(2) is satisfied only by the zero function (e.g. according to Corollary 7.4 this holds true for $\alpha = 1/2$, $\beta = 2$, $a = 2$, $b = 1/2$).

In this paper, we show which parameters α, β, a, b give classes of solutions of (1)–(2) that resemble the first five examples rather than the last two. The latter quadruples, and also some quadruples resembling the last example, are called critical (cf. Section 2). The paper gives precise descriptions of all critical classes of solutions, which consist of either convex, or constant nonpositive, or zero, or generalized subadditive, or homogeneous convex functions, or certain functions whose right-hand limit superior is subadditive and satisfies some homogeneity conditions.

This paper shows which parameters α, β, a, b, give classes of solutions of (1)–(2) that resemble the first five examples rather than the last two. Such quadruples of parameters are called critical (cf. Section 2). The paper gives precise descriptions of the classes of solutions for all the critical parameters.

The main results of the paper show that the classes of solutions of (1)–(2) for the critical parameters consist of either

- convex, or
- constant nonpositive, or
- zero, or
- generalized subadditive, or
- homogeneous convex functions, or
- certain functions whose right-hand side limit superior is subadditive and satisfies some homogeneity conditions.

For almost all parameters the last two classes are reduced to classes of power functions with fixed signs and exponents.

The paper also shows which quadruples of parameters are huge, i.e., resemble the sixth example, and shows which quadruples are trivial, i.e., have only the zero solution. In addition, the paper determines the sign of the solutions and estimates their rates of growth.

Section 1 presents a history of research on the family of inequalities (1).

Section 2 explains the notions specific to this paper, e.g. it defines huge and trivial classes of functions, and huge, trivial, and critical parameter quadruples (α, β, a, b) and pairs (a, b). It introduces the partial order $\lhd$ in the parameter space $(0, \infty)^4$ and the related notions of being above or below a set. It also presents some lemmas used throughout the paper.

Section 3 shows that the set of critical quadruples for positive solutions, as well as the one for negative solutions, is a three-dimensional topological manifold in the four-dimensional space of parameters α, β, a, b. Every quadruple of parameters (α, β, a, b) not belonging to the critical manifold for positive solutions is either above or below this manifold (with respect to the order $\lhd$); every quadruple above the manifold has a huge class of positive solutions; every quadruple below the manifold has a trivial class of positive solutions. Similarly, every quadruple of parameters (α, β, a, b) not belonging to the critical manifold for negative solutions is either above or below the manifold; every quadruple above the manifold is trivial; every quadruple below the manifold is huge. For fixed α, β, the same applies to the critical, huge, and trivial pairs (a, b). Section 3 discusses separately the positive and negative solutions and does not take into account the solutions which take both positive and negative values. This restriction causes no loss of generality because Theorems 4.1, 5.1, 5.2, 6.1, 6.3, 6.4, and 6.5, taken together, imply that if the class of solutions of (1)–(2) is huge then either the class of negative solutions is huge or the class of positive solutions is huge; moreover, if the class of negative solutions is trivial and the class of positive solutions is trivial then the zero function is the only real solution to (1)–(2).

Section 4 shows that if $\alpha < 1 < \alpha+\beta$, then every solution to (1)–(2) is either positive, or negative, or zero. It also shows that every solution to (1)–(2), with any parameters α, β, a, b, is bounded on finite intervals and gives power estimates on the rates of growth of the solutions at 0 and infinity ([1]) .

Section 5 deals with (1)–(2) for $\alpha+\beta \leq 1$. It establishes the set of parameters such that all solutions of (1)–(2) are nonpositive (henceforth referred to as the nonpositive region of parameters), and the set of parameters such that all solutions of (1)–(2) are nonnegative (the nonnegative region of parameters). The critical quadruples of the parameters analyzed by N. Kuhn [1984] have convex solutions. The critical quadruples analyzed by Z. Kominek [1992] have constant solutions. The remaining critical quadruples lead to (1)–(2) with either only the zero solution or constant nonpositive solutions.

Section 6 deals with (1)–(2) for either $\beta = 1$, or $\alpha = 1$, or $\alpha, \beta > 1$. It establishes the nonpositive and nonnegative regions of parameters. The critical quadruples with $\beta = 1$ or $\alpha = 1$ are described by J. Matkowski [1994b] and have one-sided limits. If $\alpha, \beta > 1$, then (1)–(2) with positive critical parameters have only the zero solution, and the set of negative critical parameters is empty.

Section 7 deals with (1)–(2) for $\alpha < 1 < \alpha + \beta$. It establishes the nonpositive and nonnegative regions of parameters in the case $\beta \leq 1$ (the next section characterizes the regions if $\beta > 1$). The main results show that the class of solutions for critical parameters consists of either constant and nonpositive functions, or convex functions that satisfy the homogeneity conditions

$$f(ut) = u^p f(t), \quad f(wt) = w^p f(t), \quad t > 0,$$

for fixed parameters u, w, p that depend on α, β, a, b; for almost all parameters α, β, the solutions in this class are power functions.

Section 8 deals with $\alpha < 1 < \beta$. It establishes the nonpositive and nonnegative regions of parameters. The right-hand-side limit superior of each solution for the critical parameters is subadditive and satisfies the homogeneity conditions

$$f(\alpha t) = af(t), \quad f(\beta t) = bf(t), \quad t > 0;$$

for almost all critical parameters the solutions are power functions. The section discusses also a counterpart of (1) on $\mathbb{R}_+$; then the class of solutions for critical parameters consists of either zero function or subadditive functions satisfying the above homogeneity conditions.

Section 9 contains a discussion of the role of condition (2) and its counterparts. Condition (2) excludes pathological solutions, e.g., nonmeasurable additive functions. For $a + b > 1$, it also excludes huge classes of positive solutions.

Section 10 deals with a multidimensional counterpart of the problem (1)–(2). It gives immediate extensions of the main results of Sections 7 and 8 onto cones in linear spaces and uses them to characterize F-pseudonorms, p-homogeneity, and p-convexity. It also

([1]) The analogues of results of Sections 3 and 4 hold true for functional inequalities

$$a_1 f(s_1) + \cdots + a_k f(s_k) \geq f(\alpha_1 s_1 + \cdots + \alpha_k s_k), \qquad s_1, \ldots, s_k > 0,$$

for any $k \geq 2$ and arbitrarily fixed positive parameters $a_1, \ldots, a_k, \alpha_1, \ldots, \alpha_k$.

proposes a simple proof of Matkowski's [1993] theorem that the axiom of homogeneity of norms might be considerably weakened.

The last section is followed by eight figures illustrating the sets of critical parameters and the nonnegative and nonpositive regions of parameters.

1. A review of related results

The subadditivity and Jensen-convexity have attracted most of the attention devoted to inequality (1) in the literature.

The basic results in the theory of subadditive functions are due to E. Hille [1948] and R. A. Rosenbaum [1950] who dealt with boundedness of subadditive functions, their rates of growth, continuity, and differentiability. R. Cooper [1927] dealt with measurable discontinuous subadditive functions; A. Bruckner [1960] constructed maximal subadditive extensions of subadditive functions; E. Berz [1975] considered subadditive functions satisfying the homogeneity condition $f(nt) = nf(t)$ for $n = 2, 3, \ldots$ and $t \in \mathbb{R}$. Some further references may be found in E. Hille and R. S. Phillips [1957, Chapter 7] and M. Kuczma [1985, Chapter XVI].

Recently, J. Matkowski and T. Świątkowski [1991, 1993a, 1993b] considered one-to-one subadditive functions, and established when they are homeomorphisms. This allowed them to prove some results on extensions of subadditive functions. Matkowski and Świątkowski [1994] also extended their results to one-to-one solutions of inequality (1) with $\beta = b = 1$, i.e., to the inequality

$$af(s) + f(t) \geq f(\alpha s + t), \qquad s, t > 0.$$

Matkowski [1994b] extended some of Hille's results on solutions of this inequality (cf. Section 6).

Jensen-convexity is named after J. L. W. V. Jensen who showed in [1906] that *an upper bounded Jensen-convex function is continuous on open subsets of the domain*, gave a differential condition of convexity, and derived some convexity-related inequalities. At the same time G. Hamel [1905] constructed an example of a discontinuous additive, hence Jensen-convex and subadditive, function. Then, F. Bernstein and G. Doetsch [1915] improved Jensen's main result by showing that *a Jensen-convex function is continuous on the interior of its domain if it is bounded above on an open set* (cf. Lemma 2A.6). W. Sierpiński [1920] showed that *a Jensen-convex function is continuous if it is measurable*, A. Ostrowski [1929] showed that *a Jensen-convex function bounded on a set of positive measure is continuous*, E. Mohr [1952] proved multidimensional analogues of the Bernstein–Doetsch and Hamel theorems, and M. R. Mehdi [1964] showed that *a Jensen-convex function bounded above on a set of second Baire category is continuous.* M. Kuczma [1985] discussed in detail the Bernstein–Doetsch-like conditions on continuity of Jensen-convex functions and results on Jensen-convexity on domains other than intervals in $\mathbb{R}$. An exposition of inequalities derived from Jensen-convexity may be found e.g. in G. H. Hardy, J. E. Littlewood, G. Pólya [1952] and M. Kuczma [1985].

Since Kuczma wrote his survey [1985] some other special cases of (1) have been analyzed. N. Kuhn [1984] dealt with solutions of the inequality

$$af(s) + (1-a)f(t) \geq f(as + (1-a)t),$$

where $a \in (0,1)$ is a fixed parameter, and showed that they are Jensen-convex. Z. Daróczy and Zs. Páles [1987] gave a short proof of this result (cf. proof of Lemma 5A.2). In [1987] Kuhn considered the inequality of the so-called (α, a)-convexity:

$$af(s) + (1-a)f(t) \geq f(\alpha s + (1-\alpha)t),$$

where $\alpha, a \in (0,1)$ are fixed, and proved that its solutions are Jensen-convex. Kuhn conjectured that (α, a)-convex functions are constant for some α, a. Independently of Kuhn, S. Kryński [1993] considered the (α, a)-convex functions in the context of metrical convexity and proved that a convex and (α, a)-convex function with $a \neq \alpha$ is nondecreasing, and consequently, constant. Answering a problem posed by S. Rolewicz, Z. Kominek [1992] showed that *if* $a \neq \alpha$ *then every* (α, a)*-convex function satisfying some weak regularity conditions* (*e.g. measurability*) *is constant.* J. Matkowski and M. Pycia [1995b] solved the Kuhn problem proving that (α, a)*-convex functions are constant if*, *and only if*, *the parameters* α *and* a *are not algebraically conjugate.*

K. Baron, J. Matkowski, and K. Nikodem [1994] showed that $f : (0, \infty) \to \mathbb{R}$ *satisfies*

$$af(s) + (\lambda - a)f(t) \geq f(as + (\lambda - a)t), \qquad s, t > 0,\ a \in (0, \lambda),$$

(*where* $\lambda \in (0,1)$) *if and only if there exists a convex function* $g : (0, \infty) \to \mathbb{R}$ *such that*

$$\lambda^{-1} g(\lambda t) \leq f(t) \leq g(t), \qquad t > 0.$$

This result means in particular that the classes of solutions to (1)–(2) are huge if $\alpha + \beta < 1$, $a = \alpha$, $b = \beta$.

Investigations of inequality (1) for $0 < \alpha < 1 < \alpha + \beta$, $a = \alpha$, $b = \beta$ were initiated by J. Matkowski [1990b, 1992a, 1992b] (cf. also Matkowski and Pycia [1995a]) who proved that solutions to (1)–(2) are linear. In [1992b] he applied this inequality to characterize L^p-norms in the class of L^p-like functionals, i.e. functionals of the form

$$\mathbf{P}_{\phi,\psi}(x) = \psi\Big(\int_{\Omega} \phi \circ x \, d\mu\Big)$$

for the argument x being a function defined on a measure space (Ω, Σ, μ) and the parameter functions ϕ, ψ satisfying some weak regularity conditions. Matkowski [1990a, 1992b, 1996] checked which L^p-like functionals satisfy the Minkowski inequality: if the measure space (Ω, Σ, μ) contains both small and large sets then the subadditivity of $\mathbf{P}_{\phi,\psi}$ implies that ϕ, ψ are power functions (cf. also A. C. Zaanen [1982], W. Wnuk [1984], Matkowski [1994], and Matkowski and Pycia [1996]). As a by-product Matkowski obtained a general inequality that contains the Hölder and Minkowski inequalities as special cases [1990b, 1991].

Matkowski [1994a] used inequality (1) to show that in a measure space containing both small and large sets the L^p-like functionals satisfying the Hölder inequality are simply L^p-norms. Matkowski and Pycia [1995a] proved that if $0 < \alpha < 1 < \alpha + \beta$, $a = \alpha$, $b = \beta$ then L^p-norms are the only L^p-like functionals which satisfy (1). Some results of Section 8 below were announced in Pycia [1993].

Let us note that the general linear equation

$$af(s) + bf(t) = f(\alpha s + \beta t), \qquad s, t \in \mathbb{R},$$

was studied by many authors. G. Hamel [1905] constructed nonmeasurable additive functions and M. Fréchet [1913, 1914] proved that *every measurable additive function is continuous, and consequently, linear*. Simple proofs and extensions of Fréchet's theorem were later proposed by many authors (cf. M. Kuczma [1985, Sec. IX.4] and J. Aczél [1966, Sec. 2.1]). The equation was solved in Z. Daróczy [1964] (cf. L. Losonczi [1964], and also Z. Daróczy [1961], J. Aczél [1948], [1966, Sec. 2.2.6], and M. Kuczma [1985, Sec. XIII.10]). *Every solution to the above equation is of the form* $f(t) = f(0) + g(t)$, *where* g *is an additive function. If there exists a nonconstant continuous solution, then* $a = \alpha$, $b = \beta$. *If the pairs* (α, β), (a, b) *are conjugate, i.e., if there exists an isomorphism* ϕ *from the field generated by* α, β *onto the field generated by* a, b *such that* $\phi(\alpha) = a$ *and* $\phi(\beta) = b$, *then every real function defined on a Hamel basis of* $\mathbb{R}$ *over the field generated by* α, β *may be extended to a function* f *satisfying the above equation.*

2. Preliminaries

Each section is divided into two parts: the expository and the auxiliary, which contains proofs. The labels of the auxiliary parts, their subsections, lemmas, and formulas contain the letter "A". Since we refer to auxiliary part formulas only within sections, the first auxiliary formula is labelled (A1) in each auxiliary section.

We denote by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{R}_+$ the nonnegative integers, integers, rationals, reals, and nonnegative reals, respectively. By *positive* we mean strictly greater than 0; by *negative* we mean strictly smaller than 0. By the *zero function* we mean $(0, \infty) \ni t \mapsto 0$. In this paper it is convenient to use the term *power function* for any function of the form $(0, \infty) \ni t \mapsto ct^p$ with parameters $c \in \mathbb{R}$ and $p > 0$.

Consider an arbitrary class $\mathcal{F}$ of functions $f : (0, \infty) \to \mathbb{R}$. A subclass G of $\mathcal{F}$ is called an *$\mathcal{F}$-strip* if there are $f_1, f_2 \in G$ such that $f_1(s) < f_2(s)$ for $s \in (0, \infty)$, and

$$G = \{g \in \mathcal{F} : \forall s \in (0, \infty),\ f_1(s) \le g(s) \le f_2(s)\}.$$

We say that a class of functions is *$\mathcal{F}$-huge* if it contains an $\mathcal{F}$-strip, and it is *$\mathcal{F}$-trivial* if its intersection with $\mathcal{F}$ is either empty, or contains only the zero function.

Remark 2.1. The class of solutions of (1)–(2) is huge with respect to the class of all real functions if it contains either a positive or a negative function f_0 such that

$$af_0(s) + bf_0(t) \ge Kf_0(\alpha s + \beta t),$$

where the real constant K is greater than 1 for positive f_0, and belongs to $(0, 1)$ for negative f_0. Indeed, under the above assumptions the class of solutions contains the strip of functions $f : (0, \infty) \to \mathbb{R}$ such that $f_0(t) \le f(t) \le Kf_0(t)$, $t > 0$. A strip constructed in this way is called a *strip around* f_0.

We say that a quadruple (α, β, a, b) is *$\mathcal{F}$-huge* if it leads to (1)–(2) with $\mathcal{F}$-huge class of solutions; we call it *$\mathcal{F}$-trivial* if it leads to (1)–(2) with $\mathcal{F}$-trivial class of solutions; and we call it *$\mathcal{F}$-critical* if in every neighborhood of (α,β,a,b) in $(0,\infty)^4$ there are $\mathcal{F}$-huge and $\mathcal{F}$-trivial quadruples of parameters.

Having fixed the parameters α,β, we apply the notions of $\mathcal{F}$-huge, $\mathcal{F}$-trivial, and $\mathcal{F}$-critical to the pairs of parameters (a,b). We say that a pair (a,b) is *$\mathcal{F}$-huge* if (α,β,a,b) is $\mathcal{F}$-huge, and we say that (a,b) is *$\mathcal{F}$-trivial* if (α,β,a,b) is $\mathcal{F}$-trivial. We say that a pair (a,b) is *$\mathcal{F}$-critical* if in every neighborhood of (a,b) in $(0,\infty)^2$ there are $\mathcal{F}$-huge and $\mathcal{F}$-trivial pairs of parameters.

Unless mentioned otherwise, in the following we take $\mathcal{F}$ equal to the class of either all, or all nonnegative, or all nonpositive real functions on $(0,\infty)$. For these classes $\mathcal{F}$, instead of writing $\mathcal{F}$-huge, $\mathcal{F}$-trivial, $\mathcal{F}$-critical we simply write *huge*, (or *positive huge*, or *negative huge*, respectively), *trivial* (or *positive trivial*, or *negative trivial*, respectively), *critical* (or *positive critical*, or *negative critical*, respectively). In Section 3 describing the three-dimensional critical manifolds we take $\mathcal{F}$ equal to the class of power functions on $(0,\infty)$, and we then write *power-huge* for $\mathcal{F}$-huge, etc. We also use the phrase "huge class of positive solutions" instead of "positive huge class of solutions", and "huge class of positive power solutions" instead of "positive power-huge class of solutions", etc.

By the *$\mathcal{F}$-critical set* we mean the set of $\mathcal{F}$-critical quadruples or pairs. It is proved in Section 3 that every nonempty critical set of pairs is a curve. Having proved this, we use the term *critical curve* to refer to the set of critical pairs.

It is clear that trivial points are not huge. A priori critical points might be trivial or huge. In fact, this paper shows that there are points that are simultaneously critical and trivial. It also shows that there are no points that are simultaneously critical and huge (though there are points which are simultaneously power-critical and power-huge).

It is also clear that every trivial pair corresponds to a trivial quadruple; every huge pair corresponds to a huge quadruple; and vice versa. Also, every critical pair corresponds to a critical quadruple. Not every critical quadruple corresponds to a critical pair, as shown in Remarks 5.3 and 6.6.

Let us introduce a partial order on the set $(0,\infty)^4$ by

$$(\widetilde{\alpha},\widetilde{\beta},\widetilde{a},\widetilde{b}) \lhd (\alpha,\beta,a,b) \quad \text{if } \widetilde{\alpha}\geq\alpha,\ \widetilde{\beta}\geq\beta,\ \widetilde{a}<a,\ \widetilde{b}<b.$$

Writing $x \rhd y$ we mean $y \lhd x$. Having fixed α,β we use the partial order induced by $\lhd$ on the quadrant $(0,\infty)^2$, i.e.,

$$(\widetilde{a},\widetilde{b}) \lhd (a,b) \quad \text{if } \widetilde{a}<a,\ \widetilde{b}<b.$$

Both in the two-dimensional and four-dimensional case we say that a point x is *above* a set if there is an element y of the set such that $y \lhd x$, and we say that a point x is *below* a set if there is an element y of the set such that $x \lhd y$. The relations of those notions to the problem (1)–(2) are explored in Section 3.

In the auxiliary part of this section and in Sections 8 and 9 we consider the counterpart of (1) on the closed half-line $\mathbb{R}_+$, i.e. the inequality

$$(1_{\mathbb{R}_+}) \qquad af(s)+bf(t)\geq f(\alpha s+\beta t), \qquad s,t\geq 0.$$

In the auxiliary part of this section and in Section 10 we consider the counterpart of (1) on a cone in a linear space over the real or complex numbers. A subset C of a linear space is called a *cone* if it contains at least two elements and satisfies $C + C \subseteq C$ and $tC \subseteq C$ for all $t > 0$. For a function $f : C \to \mathbb{R}$ the inequality

$$af(x) + bf(y) \geq f(\alpha x + \beta y), \qquad x, y \in C, \tag{1_C}$$

generalizes (1). Then, a counterpart of condition (2) is

$$\limsup_{s \to 0+} f(sx) \leq 0, \qquad x \in C. \tag{2_C}$$

2A. Auxiliary lemmas. The auxiliary part of this section provides lemmas which will be used throughout the other auxiliary parts. We formulate them in a general form for real functions defined on a cone C (in the following they will be used for C equal to either $(0, \infty)$ or $\mathbb{R}_+$).

LEMMA 2A.1 (Matkowski [1990b]). *Suppose* $a, b, \alpha, \beta \geq 0$. *If* $f : C \to \mathbb{R}$ *satisfies* (1_C) *then*

$$\sum_{i=0}^{N} a^{N-i} b^i \sum_{j=1}^{\binom{N}{i}} f(s_{i,j}) \geq f\Big(\sum_{i=0}^{N} \alpha^{N-i} \beta^i \sum_{j=1}^{\binom{N}{i}} s_{i,j}\Big), \qquad s_{i,j} \in C,\ N = 1, 2, \ldots.$$

Proof. Setting $x = s_{0,1}$, $y = s_{1,1}$ in (1_C) we obtain the inequality of the lemma for $N = 1$. Suppose that this inequality holds true for a positive $N \in \mathbb{N}$. Applying in turn the identity

$$\sum_{i=0}^{N+1} a^{N+1-i} b^i \sum_{j=1}^{\binom{N+1}{i}} s_{i,j} = a \sum_{i=0}^{N} a^{N-i} b^i \sum_{j=1}^{\binom{N}{i}} s_{i,j} + b \sum_{i=0}^{N} a^{N-i} b^i \sum_{j=1}^{\binom{N}{i}} s_{i+1, \binom{N}{i+1}+j}$$

(which holds true for all $N \in \mathbb{N}$, $s_{i,j} \in C$), inequality (1_C), the inductive hypothesis, and again the above identity, shows that the inequality holds true for $N+1$, which completes the proof. ∎

For given real α, β we put

$$D_{\alpha,\beta} = \left\{ k\alpha^n \beta^{m+1} : m, n \in \mathbb{N},\ k = 1, \ldots, \binom{n+m}{m} \right\}.$$

In connection with inequality (1), the numbers

$$\lambda_{\alpha,\beta} = k\alpha^n \beta^{m+1} \in D_{\alpha,\beta} \quad \text{and} \quad \lambda_{a,b} = ka^n b^{m+1}$$

are called *corresponding*. When referring to $D_{\alpha,\beta}$ and the corresponding elements $\lambda_{\alpha,\beta}, \lambda_{a,b}$, we always implicitly assume that $m, n \in \mathbb{N}$ and $k \in \{1, \ldots, \binom{n+m}{m}\}$.

LEMMA 2A.2. *Let* $a, b, \alpha, \beta > 0$. *For* $\lambda_{\alpha,\beta} = k\alpha^n \beta^m \in D_{\alpha,\beta}$ *and the corresponding* $\lambda_{a,b}$, *put*

$$c_{\alpha,\beta} = (\alpha + \beta)^{n+m} - \lambda_{\alpha,\beta}, \qquad c_{a,b} = (a + b)^{n+m} - \lambda_{a,b}.$$

If a function $f : C \to \mathbb{R}$ *satisfies inequality* (1_C) *then*

$$\lambda_{a,b} f(s) + c_{a,b} f(\delta / c_{\alpha,\beta}) \geq f(\lambda_{\alpha,\beta} s + \delta), \qquad s, \delta \in C.$$

Proof. Take $s, \delta \in C$ and apply Lemma 2A.1 with $N = n + m$, $s_{m,1} = \cdots = s_{m,k} = s$, and the remaining $s_{i,j} = \delta / c_{\alpha,\beta}$. ∎

We will use the following version of the well known result of Kronecker:

LEMMA 2A.3. *If* $0 < \alpha < 1 < \beta$ *and* $\log\beta/\log\alpha$ *is irrational, then the set* $\{\alpha^n\beta^m : n, m \in \mathbb{N}\}$ *is dense in* $(0,\infty)$. ■

LEMMA 2A.4 (Matkowski and Pycia [1995a]). *If* $0 < \alpha < 1 < \beta$ *then* $D_{\alpha,\beta}$ *is dense in* $(0,\infty)$.

Proof. If $\log\beta/\log\alpha$ is irrational then, by the preceding lemma, the subset $\{\alpha^n\beta^{m+1} : m, n \in \mathbb{N}\}$ of $D_{\alpha,\beta}$ is dense in $(0,\infty)$. In the remaining case there exist $n, m \in \mathbb{N}$ such that $\log\beta/\log\alpha = -n/m$, which means $\alpha^n\beta^m = 1$. Because $k \le \binom{kn+j+km}{km+1}$ for every $k, j \in \mathbb{N}$, $k > 0$, and

$$k\alpha^j\beta = k\alpha^{kn+j}\beta^{km+1},$$

we see that $k\alpha^j\beta \in D_{\alpha,\beta}$ and so $D_{\alpha,\beta}$ contains a dense subset $\{k\alpha^j\beta : k, j \in \mathbb{N}\}$. ■

The next lemma allows us to replace the assumption $\alpha + \beta > 1$ with $\beta > 1$ in some considerations.

LEMMA 2A.5 (Matkowski [1990b]). *If* $a, b, \alpha, \beta > 0$ *satisfy*

$$\alpha + \beta > 1,$$

then there exist $\beta_1 > 1$ *and* $b_1 > 0$ *such that every* $f : C \to \mathbb{R}$ *satisfying* (1_C) *satisfies*

$$af(x) + b_1 f(y) \ge f(\alpha x + \beta_1 y), \qquad x, y \in C.$$

If, moreover, $a + b > 1$ *then there exist* $\beta_1 > 1$ *and* $b_1 > 1$ *such that every* $f : C \to \mathbb{R}$ *satisfying* (1_C) *satisfies the above inequality.*

Proof. Choose $k \in \mathbb{N}$ such that $\beta(\alpha+\beta)^k > 1$, and put

$$b_1 = b(a+b)^k, \qquad \beta_1 = \beta(\alpha+\beta)^k.$$

Suppose that f satisfies (1_C). Then

$$f((\alpha+\beta)^k x) \le (a+b)^k f(x), \qquad x \in C,$$

and for all $x, y \in C$,

$$\begin{aligned} f(\alpha x + \beta_1 y) &= f(\alpha x + \beta(\alpha+\beta)^k y) \le af(x) + bf((\alpha+\beta)^k y) \\ &\le af(x) + b(a+b)^k f(y) = af(x) + b_1 f(y). \end{aligned}$$

If $a + b > 1$ then for large enough k we obtain $b_1 > 1$. ■

Finally, let us quote the famous

LEMMA 2A.6 (F. Bernstein and G. Doetsch [1915]). *If a Jensen-convex function defined on an open interval is bounded above on an open subset of the domain, then the function is continuous. In particular it is convex, i.e., for* x, y *from the domain of the function* f *we have*

$$tf(x) + (1-t)f(y) \ge f(tx + (1-t)y), \qquad t \in (0,1). \ ■$$

3. Parameter space and critical manifolds

In this section we investigate the relations between inequality (1) and the notions of being huge, trivial, or critical, the partial order $\lhd$, and the notions of being above or below a set. These notions were introduced in Section 2.

We discuss separately the positive and negative solutions and we do not take into account the solutions that take both positive and negative values. This approach will be justified by Theorems 4.1, 5.1, 5.2, 6.1, 6.3, 6.4, and 6.5, which show that if the class of solutions of (1)–(2) is huge, then either the class of negative solutions is huge or the class of positive solutions is huge; moreover, if the class of negative solutions is trivial and the class of positive solutions is trivial, then the zero function is the only real solution to (1)–(2).

Let us begin with the following simple, but important

LEMMA 3.1. *Let $\alpha, \beta > 0$ be fixed. If a positive f_0 satisfies* (1)–(2) *with positive real a_0, b_0 in place of a, b, then for every (a, b) above (a_0, b_0), i.e., $(a, b) \rhd (a_0, b_0)$, the class of positive solutions of* (1)–(2) *is huge and contains strips around f_0.*

As a consequence of the lemma we get

THEOREM 3.2. *Let $\alpha, \beta > 0$ be fixed.*

- *If the set of positive critical pairs (a, b) is empty then every pair $(a, b) \in (0, \infty)^2$ is huge.*
- *If the critical set is nonempty, then it is a curve and every pair $(a, b) \in (0, \infty)^2$ that does not belong to this curve is either above or below it. Moreover,*
  - *every pair (a, b) above the critical set leads to* (1)–(2) *with huge class of positive solutions;*
  - *every pair (a, b) below the critical set leads to* (1)–(2) *with trivial class of positive solutions.*

Now consider the whole parameter space $(0, \infty)^4$ and strips with respect to the class of positive increasing power functions. Theorem 3.4 will show that the set of positive power-critical quadruples (a, b, α, β) is a three-dimensional topological manifold. One of the goals of this paper is to show that this manifold coincides with the set of positive critical quadruples (a, b, α, β). This will follow when we prove Theorems 5.1, 5.2, 6.3, 6.4, and 6.5, and Corollaries 7.4, 7.5, and 8.5.

Let us start with an analogue of Lemma 3.1.

LEMMA 3.3. *If an increasing positive power function f_0 satisfies* (1)–(2) *with positive real $\alpha_0, \beta_0, a_0, b_0$ in place of α, β, a, b, then for every (α, β, a, b) above $(\alpha_0, \beta_0, a_0, b_0)$, i.e., $(\alpha, \beta, a, b) \rhd (\alpha_0, \beta_0, a_0, b_0)$, the class of positive solutions of* (1)–(2) *is power-huge and contains strips around f_0.*

THEOREM 3.4. *The set of positive power-critical quadruples $(\alpha, \beta, a, b) \in (0, \infty)^4$ is a three-dimensional topological manifold. Every quadruple $(\alpha, \beta, a, b) \in (0, \infty)^4$ that does not belong to this manifold is either above or below it. Moreover,*

- *every quadruple above the critical power set leads to* (1)–(2) *with huge class of positive power solutions;*
- *every quadruple below the critical power set leads to* (1)–(2) *with trivial class of positive power solutions.*

In a similar way to the results 3.3 and 3.4, one may prove their analogues for the class of increasing positive homeomorphisms (instead of the class of positive increasing power functions). Theorems 5.1, 5.2, 6.3, 6.4, and 6.5, and Corollaries 7.4, 7.5, and 8.5 will show that the critical set established in this way coincides with the critical set of Theorem 3.4.

In the second part of this section we consider negative solutions of (1)–(2). The results are analogous to those for positive solutions. First consider quadrants with fixed positive α, β.

LEMMA 3.5. *Let $\alpha, \beta > 0$ be fixed. If a negative f_0 satisfies* (1)–(2) *with positive real a_0, b_0 in place of a, b, then for every (a,b) below (a_0, b_0), i.e., $(a,b) \lhd (a_0, b_0)$, the class of negative solutions of* (1)–(2) *is huge and contains strips around f_0.*

THEOREM 3.6. *Let $\alpha, \beta > 0$ be fixed.*

- *If the set of negative critical pairs (a,b) is empty then every pair $(a,b) \in (0,\infty)^2$ is huge.*
- *If the critical set is nonempty, then it is a curve and every pair $(a,b) \in (0,\infty)^2$ that does not belong to this curve is either above or below it. Moreover,*
  - *every pair (a,b) above the critical set leads to* (1)–(2) *with trivial class of negative solutions;*
  - *every pair (a,b) below the critical set leads to* (1)–(2) *with huge class of negative solutions.*

Now consider the whole parameter space $(0,\infty)^4$ and strips with respect to the class of negative decreasing power functions. Theorem 3.8 shows that the set of negative power-critical quadruples (a, b, α, β) is a three-dimensional topological manifold. One of the goals of this paper is to show that this manifold coincides with the set of negative critical quadruples (a, b, α, β). As in the case of positive solutions, this will follow from Theorems 5.1, 5.2, 6.3, 6.4, and 6.5, and Corollaries 7.4, 7.5, and 8.5.

Let us start with an analogue of Lemmas 3.3 and 3.5:

LEMMA 3.7. *If a decreasing negative power function f_0 satisfies* (1)–(2) *with positive real $\alpha_0, \beta_0, a_0, b_0$ in place of α, β, a, b, then for every (α, β, a, b) below $(\alpha_0, \beta_0, a_0, b_0)$, i.e., $(\alpha, \beta, a, b) \lhd (\alpha_0, \beta_0, a_0, b_0)$, the class of negative solutions of* (1)–(2) *is power-huge and contains strips around f_0.*

THEOREM 3.8. *The set of negative power-critical quadruples $(\alpha, \beta, a, b) \in (0,\infty)^4$ is a three-dimensional topological manifold. Every quadruple $(\alpha, \beta, a, b) \in (0,\infty)^4$ that does not belong to this manifold is either above or below it. Moreover,*

- *every quadruple above the critical power set leads to* (1)–(2) *with trivial class of negative power solutions;*
- *every quadruple below the critical power set leads to* (1)–(2) *with huge class of negative power solutions.*

Since two-dimensional critical sets are curves and four-dimensional critical sets are three-dimensional manifolds, in the following we refer to them as *critical curves* and *critical manifolds*. From Lemmas 3.1 and 3.5 we infer

COROLLARY 3.9. *If positive* α, β, a, b *lead to the class of positive solutions of* (1)–(2) *that is neither huge nor empty, then* (a, b) *belongs to the positive critical curve for* α, β. *If positive* α, β, a, b *lead to the class of negative solutions that is neither huge nor empty, then* (a, b) *belongs to the negative critical curve for* α, β.

3A. Proofs

Proof of Lemma 3.1. Under the assumptions of the lemma every function $f : (0, \infty) \to (0, \infty)$ such that

$$f_0(t) \le f(t) \le \min\left(\frac{a}{a_0}, \frac{b}{b_0}\right) f_0(t), \qquad t > 0,$$

satisfies

$$\begin{aligned} af(s) + bf(t) &\ge af_0(s) + bf_0(t) \ge \min\left(\frac{a}{a_0}, \frac{b}{b_0}\right)(a_0 f_0(s) + b_0 f_0(t)) \\ &\ge \min\left(\frac{a}{a_0}, \frac{b}{b_0}\right) f_0(\alpha s + \beta t) \ge f(\alpha s + \beta t), \qquad s, t > 0, \end{aligned}$$

hence, it is a solution to (1)–(2). ■

Proof of Theorem 3.2. For brevity we write "critical" instead of "positive critical", "huge" instead of "positive huge", "trivial" instead of "positive trivial".

Consider the family of rays

$$L_x = \{(a, b) \in (0, \infty)^2 : a = x + b\}, \qquad x \in \mathbb{R}.$$

Lemma 3.1 implies the following trichotomy for each fixed $x \in \mathbb{R}$: either

- there are no positive solutions to (1)–(2) for $(a, b) \in L_x$, or
- for all $(a, b) \in L_x$ the class of positive solutions of (1)–(2) is huge, or
- there exists a critical point $(a(x), b(x)) \in L_x$ such that for all (a, b) below $(a(x), b(x))$ the class of positive solutions is trivial, and for all (a, b) above $(a(x), b(x))$ this class is huge.

If there are no critical pairs (a, b) then either all $(0, \infty)^2$ consists of huge pairs or it consists of trivial ones. Since for $a \ge \alpha$, $b \ge \beta$ all linear functions satisfy (1)–(2), all the pairs of $(0, \infty)^2$ are huge.

Now assume that the set of critical points is nonempty. First, we show that every critical point is of the form $(a(x), b(x))$ defined above. Indeed, take an arbitrary critical point $(a_0, b_0) \in (0, \infty)^2$. Then:

- There is a sequence of points $(a_n, b_n)_{n=1,2,\ldots}$ from $(0, \infty)^2$ which tends to (a_0, b_0) and such that every point (a_n, b_n) leads to a huge class of positive solutions to (1)–(2). By Lemma 3.1 every pair belonging to the quadrant $\{(a, b) : (a, b) \triangleright (a_n, b_n)\}$ leads to a huge class of positive solutions to (1)–(2). Consequently, every pair (a, b) from the set-theoretical sum of the quadrants does, and every $(a, b) \in \{(a, b) : (a, b) \triangleright (a_0, b_0)\}$ does.

- Similarly, there exists a sequence of points $(a_{-n}, b_{-n})_{n=1,2,...}$ from $(0,\infty)^2$ tending to (a_0, b_0) and such that every point (a_{-n}, b_{-n}) leads to a trivial class of positive solutions to (1)–(2). By Lemma 3.1 every pair from the rectangle $\{(a,b) : (0,0) \lhd (a,b) \lhd (a_{-n}, b_{-n})\}$ leads to a trivial class of solutions to (1)–(2). Thus every pair from the set-theoretical sum of the rectangles does, and every pair $(a,b) \in \{(a,b) : (0,0) \lhd (a,b) \lhd (a_0,b_0)\}$ does.

It follows that $(a_0, b_0) \in L_{a_0-b_0}$ is the unique critical point on the ray $L_{a_0-b_0}$.

Denote by I the set of reals x leading to critical points $(a(x), b(x))$. The set I is nonempty. In order to prove that the set of critical points is a curve we show that

- I is convex;
- $I \ni x \mapsto (a(x), b(x))$ is a homeomorphism between I and the critical set.

To show the convexity of I take $x, y \in I$, $x < y$, and z from the open interval (x, y). Then L_z has a nonempty intersection with the set of positive huge points $\{(a,b) : (a,b) \rhd (a(x), b(x))\}$. Since L_z also intersects $\{(a,b) \in (0,\infty)^2 : (a,b) \lhd (a(x),b(x)) \text{ or } (a,b) \lhd (a(y), b(y))\}$, it contains positive trivial points. Consequently, there is a positive critical point (a_z, b_z).

Now take $x, y \in I$ with $x < y$ and infer from Lemma 3.1 that $(a(y), b(y))$ lies on the closed segment with endpoints $(a(x), a(x) - y)$ and $(b(x) + y, b(x))$. So, the Euclidean $\mathbb{R}^2$ distance $\| \cdot - \cdot \|$ between $(a(y), b(y))$ and $(a(x), b(x))$ satisfies the estimates

$$\frac{1}{\sqrt{2}} |y - x| \leq \|(a(y), b(y)) - (a(x), b(x))\| \leq |y - x|,$$

and the mapping $I \ni x \mapsto (a(x), b(x))$ is a homeomorphism.

Thus, we have proved that the set of positive critical pairs is a curve. Applying now Lemma 3.1 we note that every pair above the positive critical curve gives (1)–(2) with huge classes of positive solutions and every pair below the positive critical curve gives (1)–(2) with no positive solutions. It remains to show that every point in $(0,\infty)^2$ that does not belong to the set of critical pairs is either above or below the above set (this will show in particular that I has more than one element, i.e., the critical curve is not degenerate). For an indirect proof let us take (a_0, b_0) that is neither critical, nor above the set of critical points, nor below it. Since $(a_0, b_0) \in L_{a_0-b_0}$ the trichotomy observed at the beginning of the proof shows that (a_0, b_0) is either huge or trivial.

First assume that (a_0, b_0) is huge. Since (a_0, b_0) is not above the set of critical points, no point below (a_0, b_0) is critical. Moreover, if a point (a,b) below (a_0, b_0) is trivial then Lemma 3.1 would imply that there exists a critical point on the segment with endpoints (a,b) and (a_0, b_0), so that (a_0, b_0) would be above a critical point. However, (a_0, b_0) is not above the critical set. Hence, every point below (a_0, b_0) is huge. Note that for every $(a,b) \in (0,\infty)^2$ there exists $(a_1, b_1) \in (0,\infty)^2$ such that

$$(a_1, b_1) \lhd (a,b), \qquad (a_1, b_1) \lhd (a_0, b_0).$$

We have just shown that (a_1, b_1) is huge, so Lemma 3.1 implies that (a,b) is huge. Thus the set of critical points would be empty, which is a contradiction.

Similarly, if (a_0, b_0) is trivial then every point above (a_0, b_0) would be trivial, and consequently, every point in $(0, \infty)^2$ would be trivial. ■

Proof of Lemma 3.3. Similarly to the proof of Lemma 3.1, every power function $f : (0, \infty) \to (0, \infty)$ such that

$$f_0(t) \le f(t) \le \min\left(\frac{a}{a_0}, \frac{b}{b_0}\right) f_0(t), \qquad t > 0,$$

is a solution of (1)–(2). ■

Proof of Theorem 3.4. Analogously to the proof of Theorem 3.2 define the rays

$$L_{x,y,z} = \{(\alpha, \beta, a, b) \in (0, \infty)^4 : a = x + b,\ \alpha = y - b,\ \beta = z - b\}, \qquad x, y, z \in \mathbb{R};$$

define the positive power-critical points

$$(\alpha(x, y, z), \beta(x, y, z), a(x, y, z), b(x, y, z)) \in L_{x,y,z};$$

use Lemma 3.3 instead of Lemma 3.1 and repeat the argument of that proof.

In this way we construct a convex subset $C \subseteq \mathbb{R}^3$ and a homeomorphism between C and the positive power-critical set. The critical set is hence a topological manifold. The argument from the proof of Theorem 3.2 shows that every point in $(0, \infty)^4$ that does not belong to the set of critical quadruples is either above or below that set. Lemma 3.3 says that points above the critical set are huge and points below it are trivial. ■

Proof of Lemmas 3.5 and 3.7. Similarly to the proof of Lemma 3.1, every function $f : (0, \infty) \to (-\infty, 0)$ such that

$$f_0(t) \le f(t) \le \min\left(\frac{a}{a_0}, \frac{b}{b_0}\right) f_0(t), \qquad t > 0,$$

is a solution to (1)–(2). ■

Proof of Theorems 3.6 and 3.8. One may use the rays L_x and $L_{x,y,z}$ defined in the previous proofs and follow the same argument as in the proofs of the positive counterparts of those results. ■

Proof of Corollary 3.9. For the proof of the positive part of the corollary, note that, by Lemma 3.1, for $(\widetilde{a}, \widetilde{b})$ above (a, b) the class of positive solutions to (1)–(2) (with $\widetilde{a}, \widetilde{b}$ substituted for a, b) is huge; for $(\widetilde{a}, \widetilde{b})$ below (a, b) the class of positive solutions to (1)–(2) is trivial. Hence (α, β, a, b) is critical.

For the proof of the negative part, use Lemma 3.5 to show that the class of negative solutions to (1)–(2) is trivial for $(\widetilde{a}, \widetilde{b})$ above (a, b) and that the class of negative solutions to (1)–(2) is huge for $(\widetilde{a}, \widetilde{b})$ below (a, b). ■

4. Sign, boundedness, rate of growth of solutions

The results of this section show that even huge classes of solutions of (1)–(2) have some special structure. The results will be used in Sections 5–8.

Let us start with a theorem that allows us to consider separately nonnegative and nonpositive solutions to (1)–(2).

THEOREM 4.1. *Let $\alpha, \beta, a, b > 0$ be such that $\alpha < 1 < \alpha + \beta$. If $f : (0, \infty) \to \mathbb{R}$ is a solution of* (1)–(2) *then f is either positive, or negative, or identically zero.*

Actually a stronger regularity is true, and Theorems 5.1, 5.2, 6.1, 6.3, 6.4, and 6.5, which we will prove in the following sections, imply that for positive α, β, a, b:

- if either ($\alpha, \beta, a, b > 1$), or ($\alpha + \beta, a + b < 1$), or ($\alpha = a$, $\beta = b$, $\alpha + \beta = 1$) then there are solutions to (1)–(2) that take both negative and positive values;
- otherwise, the solutions to (1)–(2) are either nonnegative or nonpositive; if moreover $a \neq \alpha$ or $b \neq \beta$, then the solutions of (1)–(2) are either all nonnegative or all nonpositive.

THEOREM 4.2. *Let $\alpha, \beta, a, b > 0$ be such that $\alpha < 1 < \alpha + \beta$, and define $A = \log a / \log \alpha$. Then there is a $B \in \mathbb{R}$ such that for every nonzero $f : (0, \infty) \to \mathbb{R}$ satisfying* (1)–(2) *there exists a real $M \neq 0$ such that*

$$\frac{1}{M} t^B \leq f(t) \leq M t^A, \quad t \in (0, 1]; \qquad \frac{1}{M} t^A \leq f(t) \leq M t^B, \quad t \geq 1.$$

REMARK 4.3. If $\beta > 1$ then the inequalities of Theorem 4.2 are true for $B = \log b / \log \beta$ and some $M \neq 0$. If $\beta \leq 1$ then an estimation of B may be obtained by inspecting the use of Lemma 2A.5 in the proof of the theorem. In particular, if $a + b > 1$ then the inequalities are true for some positive B.

THEOREM 4.4. *Let $\alpha, \beta, a, b > 0$. If $f : (0, \infty) \to \mathbb{R}$ satisfies* (1)–(2) *then f is bounded on every interval* $(0, T]$, $T > 0$.

PROPOSITION 4.5. *Let $\alpha, \beta, a, b > 0$ be such that $\alpha < 1$, and define $A = \log a / \log \alpha$. If $f : (0, \infty) \to \mathbb{R}$ satisfies* (1)–(2) *then there exists a positive constant M such that*

$$f(t) \leq M t^A, \quad t \in (0, 1]; \qquad -M t^A \leq f(t), \quad t \geq 1.$$

This section gives a complete account of power estimates on solutions of (1)–(2) that hold true without any assumptions on a, b. Note that by using Lemmas 4A.1–4A.6 one may additionally determine the sign of the constant M for some quadruples α, β, a, b. Moreover, Sections 5 and 7 improve the estimation of exponents in Theorem 4.2 and Proposition 4.5.

4A. Proofs. This subsection starts with six lemmas and the proof of Theorem 4.4. Then using Lemmas 4A.5, 4A.6 we prove Theorem 4.1; using Lemmas 4A.1, 4A.3 we prove Proposition 4.5; and, finally, using Theorems 4.1, 4.4 and Lemmas 4A.1–4A.6 we prove Theorem 4.2.

For brevity, this subsection skips explicit references to the global assumption that α, β, a, b are positive and f is a real function on $(0, \infty)$.

LEMMA 4A.1. *Suppose $\alpha < 1$, and define $A = \log a / \log \alpha$. If f satisfies* (1)–(2) *and there exists $M \in \mathbb{R}$ such that*

$$f(t) \geq M t^A, \qquad t \in [1, \alpha^{-1}),$$

then $f(t) \geq M t^A$ for every $t \geq 1$.

Proof. Take arbitrary $t \in [\alpha^{-1}, \alpha^{-2})$ and a sequence of positive reals $\varepsilon_n \to 0$. Then for large n we have $\alpha t + \beta\varepsilon_n \in [1, \alpha^{-1})$, and consequently, (1) implies

$$f(t) \geq a^{-1} f(\alpha t + \beta\varepsilon_n) - a^{-1} b f(\varepsilon_n) \geq a^{-1} M(\alpha t + \beta\varepsilon_n)^A - a^{-1} b f(\varepsilon_n).$$

Letting n tend to infinity and applying (2) we get $f(t) \geq Mt^A$ for $t \in [\alpha^{-1}, \alpha^{-2})$. By induction, $f(t) \geq Mt^A$ for $t \in [\alpha^{-n}, \alpha^{-n-1})$ with $n = 0, 1, 2, \ldots$. So, $f(t) \geq Mt^A$ for $t \geq 1$. ∎

Lemma 4A.2 *Suppose $\beta > 1$, and define $B = \log b / \log \beta$. If f satisfies* (1)–(2) *and there exists $M \in \mathbb{R}$ such that*

$$f(t) \geq Mt^B, \qquad t \in [\beta^{-1}, 1),$$

then $f(t) \geq Mt^B$ for every $t \in (0, 1)$.

Proof. Take arbitrary $t \in [\beta^{-2}, \beta^{-1})$ and a sequence of positive reals $\varepsilon_n \to 0$. Then for large n we have $\alpha\varepsilon_n + \beta t \in [\beta^{-1}, 1)$, and consequently, (1) implies

$$f(t) \geq b^{-1} f(\alpha\varepsilon_n + \beta t) - b^{-1} a f(\varepsilon_n) \geq b^{-1} M(\alpha\varepsilon_n + \beta t)^B - b^{-1} a f(\varepsilon_n).$$

Letting n tend to infinity and applying (2) we get $f(t) \geq Mt^B$ for $t \in [\beta^{-2}, \beta^{-1})$. By induction, $f(t) \geq Mt^B$ for $t \in [\beta^{-n-1}, \beta^{-n})$ with $n = 0, 1, 2, \ldots$. So, $f(t) \geq Mt^B$ for $t \in (0, 1)$. ∎

Lemma 4A.3. *Suppose $\alpha < 1$, and define $A = \log a / \log \alpha$. If f satisfies* (1)–(2) *and there exists $M \in \mathbb{R}$ such that*

$$f(t) \leq Mt^A, \qquad t \in (\alpha, 1],$$

then $f(t) \leq Mt^A$ for every $t \in (0, 1]$.

Proof. Take arbitrary $t \in (\alpha^2, \alpha]$ and a sequence of positive reals $\varepsilon_n \to 0$. Then for large n we have $(t - \beta\varepsilon_n)/\alpha \in (\alpha, 1]$, and consequently, (1) implies

$$f(t) \leq af\left(\frac{t - \beta\varepsilon_n}{\alpha}\right) + bf(\varepsilon_n) \leq aM\left(\frac{t - \beta\varepsilon_n}{\alpha}\right)^A + bf(\varepsilon_n).$$

Letting n tend to infinity and applying (2) we get $f(t) \leq Mt^A$ for $t \in (\alpha^2, \alpha]$. By induction, $f(t) \leq Mt^A$ for $t \in (\alpha^n, \alpha^{n-1}]$ with $n = 1, 2, \ldots$. So, $f(t) \leq Mt^A$ for $t \in (0, 1]$. ∎

Lemma 4A.4. *Suppose $\beta > 1$, and define $B = \log b / \log \beta$. If f satisfies* (1)–(2) *and there exists $M \in \mathbb{R}$ such that*

$$f(t) \leq Mt^B, \qquad t \in (1, \beta],$$

then $f(t) \leq Mt^B$ for every $t > 1$.

Proof. Take arbitrary $t \in (\beta^1, \beta^2]$ and a sequence of positive reals $\varepsilon_n \to 0$. Then for large n we have $(t - \alpha\varepsilon_n)/\beta \in (1, \beta]$, and consequently, (1) implies

$$f(t) \leq af(\varepsilon_n) + bf\left(\frac{t - \alpha\varepsilon_n}{\beta}\right) \leq af(\varepsilon_n) + bM\left(\frac{t - \alpha\varepsilon_n}{\beta}\right)^B.$$

Letting n tend to infinity and applying (2) we get $f(t) \leq Mt^B$ for $t \in (\beta^1, \beta^2]$. By induction, $f(t) \leq Mt^B$ for $t \in (\beta^n, \beta^{n+1}]$ with $n = 0, 1, 2, \ldots$. So, $f(t) \leq Mt^B$ for $t > 1$. ∎

LEMMA 4A.5. *If* $\alpha < 1$ *and* f *is a nonpositive solution to* (1)–(2) *that is not identically* 0, *then*

$$\sup\{f(s) : s > x\} < 0, \qquad x > 0.$$

Proof. For an indirect proof assume that there is a sequence (x_n), $x_n > x$, $n \in \mathbb{N}$, such that $\lim_{n\to\infty} f(x_n) = 0$. Fix $s \in (0, \alpha^{-1}x)$. Then $t_n = (x_n - \alpha s)\beta^{-1}$ is positive for all $n \in \mathbb{N}$ and by (1) we have

$$af(s) + bf(t_n) \geq f(\alpha s + \beta t_n) = f(x_n), \qquad n \in \mathbb{N}.$$

Letting n tend to the infinity we would get $f(s) \geq 0$ for every $s \in (0, \alpha^{-1}x)$. Hence Lemma 4A.1 with $M = 0$ and with the function $t \mapsto f(xt)$ in place of f would give $f(s) \geq 0$ for all $s \in (0, \infty)$. But f is nonpositive and nonzero. This contradiction completes the proof. ■

LEMMA 4A.6. *If* $\alpha < 1 < \alpha + \beta$ *and* f *is a nonnegative solution to* (1)–(2) *that is not identically* 0, *then for every compact interval* $I \subset (0, \infty)$,

$$\inf\{f(s) : s \in I\} > 0.$$

Proof. By Lemma 2A.5 we may assume that $\beta > 1$. For an indirect argument assume that there exists a sequence $(s_n)_{n=1,2,\dots}$, $s_n \in I$, such that $f(s_n) \to 0$. We may assume that s_n converges to an $s \in I$. Lemma 2A.2 and condition (2) show that for every $\lambda_{\alpha,\beta} \in D_{\alpha,\beta}$, for the corresponding $\lambda_{a,b}$, and for every positive sequence $\delta_n \to 0$, we have

$$0 \leq f(\lambda_{\alpha,\beta}s_n + \delta_n) \leq \lambda_{a,b}f(s_n) + c_{a,b}f(\delta_n/c_{\alpha,\beta}) \to 0.$$

Fix $t \in (0, \infty)$. Since $0 < \alpha < 1 < \beta$, the set $D_{\alpha,\beta}$ is dense in $(0, \infty)$ and we may choose $\lambda_{\alpha,\beta}$ and δ_n so that the sequence $t_n = \lambda_{\alpha,\beta}s_n + \delta_n$ is increasing and $t_n \to \beta^{-1}t$. Because $f(t_n) \to 0$, condition (2) leads to

$$0 \leq f(t) = f\left(\alpha\frac{t - \beta t_n}{\alpha} + \beta t_n\right) \leq af\left(\frac{t - \beta t_n}{\alpha}\right) + bf(t_n) \to 0,$$

and consequently, f would be the zero function. ■

Proof of Theorem 4.4. First let us prove that f is bounded above on $(0, T]$. For an indirect argument assume that there exists a sequence $t_n \in (0, T]$ such that $f(t_n) \to \infty$. By (2) there exists a positive δ such that f is bounded above on $(0, \delta)$. Consequently, we may assume that $t_n \in [\delta, T]$ and the sequence t_n converges to $t_\infty \in [\delta, T]$. Put $x = \left(t_\infty - \frac{1}{2}\alpha\delta\right)/\beta$ and note that for $n \in \mathbb{N}$ so large that $t_n > \beta x$, inequality (1) would give

$$af\left(\frac{t_n - \beta x}{\alpha}\right) + bf(x) \geq f(t_n) \to \infty.$$

For n large enough $(t_n - \beta x)/\alpha = (t_n - t_\infty)/\alpha + \delta/2 \in (0, \delta]$, so the values $f((t_n - \beta x)/\alpha)$ are bounded above uniformly with respect to n, and hence $f(x) = \infty$, which is a contradiction. Hence, there exists a positive M such that

$$f(t) \leq M, \qquad t \in (0, T].$$

To prove that f is bounded below on $(0, T]$ take an arbitrary $t \in (0, T)$. We have already proved that f is bounded above on finite intervals, so we may assume that f is bounded above by a constant $\widetilde{M}$ on $(0,(\alpha/\beta)T)$. The number $s=(\alpha/\beta)(T-t)$ belongs to $(0,(\alpha/\beta)T)$, so inequality (1) implies that

$$f(\alpha T)=f(\alpha t+\beta s)\le af(t)+bf(s),$$

and consequently,

$$f(t)\ge a^{-1}(f(\alpha T)-b\widetilde{M}),\qquad t\in(0,T).\ \blacksquare$$

Proof of Theorem 4.1. By Lemma 2A.5 we may assume that $\beta>1$. Define

$$\{f>0\}=\{t\in(0,\infty):f(t)>0\},\qquad \{f<0\}=\{s\in(0,\infty):f(s)<0\}.$$

From (1) we infer that

$$af(\alpha^{-1}t-\varepsilon)\ge f(t)-bf(\beta^{-1}\alpha\varepsilon),\qquad \varepsilon\in(0,\alpha^{-1}t),\ t>0,$$

and consequently, (2) gives

$$\text{(A1)}\qquad t\in\{f>0\}\ \Rightarrow\ \exists\varepsilon>0,\ (\alpha^{-1}t-\varepsilon,\alpha^{-1}t)\subseteq\{f>0\}.$$

Take $\lambda_{\alpha,\beta}\in D_{\alpha,\beta}$. Lemma 2A.2 gives

$$\lambda_{a,b}f(t)+c_{a,b}f(\delta/c_{\alpha,\beta})\ge f(\lambda_{\alpha,\beta}t+\delta),\qquad t,\delta>0,$$

where $\lambda_{a,b},c_{a,b},c_{\alpha,\beta}$ are corresponding positive constants. Hence, making use of (2) we obtain

$$t\in\{f<0\}\ \Rightarrow\ \exists\varepsilon>0,\ (\lambda_{\alpha,\beta}t,\lambda_{\alpha,\beta}t+\varepsilon)\subseteq\{f<0\}.$$

Now assume that $\{f<0\}$ is nonempty. According to Lemma 2A.4 the set $D_{\alpha,\beta}$ is dense in $(0,\infty)$. By the above implication $\{f<0\}$ is dense in $(0,\infty)$. Since $\{f<0\}$ and $\{f>0\}$ are disjoint there does not exist a nonempty open interval contained in $\{f>0\}$. Applying (A1) we conclude that $\{f<0\}$ is empty. This shows that f is either nonnegative or nonpositive, and Lemmas 4A.5 and 4A.6 complete the proof. ∎

Proof of Proposition 4.5. This proposition follows from Theorem 4.4 and Lemmas 4A.1 and 4A.3. ∎

Proof of Theorem 4.2. Recall that according to Lemma 2A.5 we may assume that $\beta>1$, and consequently, use all the preceding lemmas. Because of Theorem 4.1 it is enough to consider positive and negative solutions to (1)–(2).

First suppose that a nonzero function f is a negative solution to (1)–(2). To complete the proof in this case it is enough to find $m_A,M_A,m_B,M_B<0$ such that

$$m_Bt^B\le f(t)\le M_At^A,\quad 0<t\le 1;\qquad m_At^A\le f(t)\le M_Bt^B,\quad t\ge 1.$$

Theorem 4.4 and Lemmas 4A.1 and 4A.2 give the constants m_A, m_B and the relevant inequalities. Lemmas 4A.3, 4A.4, and 4A.5 give the constants M_A, M_B and the relevant inequalities.

Now suppose that a nonzero function f is a positive solution to (1)–(2). To complete the proof it is enough to find $m_A,M_A,m_B,M_B>0$ such that

$$m_Bt^B\le f(t)\le M_At^A,\quad 0<t\le 1;\qquad m_At^A\le f(t)\le M_Bt^B,\quad t\ge 1.$$

Theorem 4.4 and Lemmas 4A.3 and 4A.4 give the constants M_A, M_B and the relevant inequalities. Lemmas 4A.1, 4A.2, and 4A.6 give the constants m_A,m_B and the relevant inequalities. ∎

5. Convex and constant solutions

This section examines inequality (1) for α, β such that $\alpha + \beta \le 1$. We show that in this case the solutions on the critical curves are convex or constant. Recall that the precise meaning of being *above* or *below* a set is given in Section 2. In the following theorems, by *large* exponents we mean exponents from a neighborhood of infinity; by *small* ones we mean exponents from a neighborhood of 0.

All the figures appear after Section 10.

THEOREM 5.1 (cf. Figure 2). *Suppose that $\alpha, \beta, a, b > 0$ and $\alpha + \beta = 1$.*

- *If $a + b < 1$ then all the solutions to* (1)–(2) *are nonpositive; the class of negative solutions is huge and contains strips around power functions with small positive exponents.*
- *If $a + b = 1$ then*
  - *if $a = \alpha$, then the class of solutions of* (1)–(2) *consists of the convex functions satisfying* (2) (*N. Kuhn* [1984]);
  - *if $a \ne \alpha$ then the class of solutions to* (1)–(2) *consists of nonpositive constant functions* (*Z. Kominek* [1992]).
- *If $a+b > 1$ and $a^\alpha b^\beta < \alpha^\alpha \beta^\beta$ then the only solution to* (1)–(2) *is the zero function.*
- *If $a^\alpha b^\beta = \alpha^\alpha \beta^\beta$ then*
  - *if $a = \alpha$, then the class of solutions of* (1)–(2) *consists of the convex functions satisfying* (2) (*N. Kuhn* [1984]);
  - *if $a \ne \alpha$ then the only solution to* (1)–(2) *is the zero function.*
- *If $a^\alpha b^\beta > \alpha^\alpha \beta^\beta$ then all the solutions to* (1)–(2) *are positive; the class of positive solutions is huge and contains strips around power functions with large positive exponents.*

Because for any constants v_0, s_0 the function $s \mapsto g(s) = v_0 + f(s + s_0)$ satisfies (1) iff f does, we can replace assumption (2) with f being bounded from above on an open subset of the domain. Moreover, f can be defined on any convex set instead of $(0, \infty)$. Alternatively, one can use Sierpiński [1920] to replace assumption (2) by measurability of f.

Note the relation between the assumptions $a + b > 1$ and $a^\alpha b^\beta \ge \alpha^\alpha \beta^\beta$. Since the function $(0,1) \ni a \mapsto \alpha \log a + (1-\alpha)\log(1-a)$ has a unique global maximum at $a = \alpha$, the assumption

$$a^\alpha b^\beta \ge \alpha^\alpha \beta^\beta$$

implies

$$\begin{aligned}\alpha \log a + (1-\alpha)\log b \ge \alpha \log \alpha + (1-\alpha)\log \beta &= \alpha \log \alpha + (1-\alpha)\log(1-\alpha) \\ &\ge \alpha \log a + (1-\alpha)\log 1 - a,\end{aligned}$$

and consequently, $b \ge 1 - a$, that is,

$$a + b \ge 1$$

with equality only if $a = \alpha$.

Hence, the theorem implies that the positive critical curve for $\alpha+\beta=1$ is given by

$$b_{\text{positive critical}}(a)=\beta\left(\frac{\alpha}{a}\right)^{\alpha/\beta},\qquad a>0,$$

and the negative critical curve is given by

$$b_{\text{negative critical}}(a)=1-a,\qquad a\in(0,1).$$

THEOREM 5.2 (cf. Figure 1). *Suppose that $\alpha,\beta,a,b>0$ and $\alpha+\beta<1$. Then the class of positive solutions of* (1)–(2) *is huge and contains strips around power functions with large positive exponents. Moreover*,

- *if $a+b<1$, then the class of negative solutions of* (1)–(2) *is huge and contains strips around power functions with small positive exponents.*
- *if $a+b>1$, then all solutions to* (1)–(2) *are nonnegative*;
- *if $a+b=1$, then the class of negative solutions of* (1) *consists of constant functions.*

REMARK 5.3. Theorems 5.1 and 5.2 show that if $\alpha+\beta=1$, then the pairs (a,b) such that $a^\alpha b^\beta<\alpha^\alpha\beta^\beta$ do not belong to the positive critical curve for (α,β). However, the quadruples (α,β,a,b) such that $\alpha+\beta=1$, $a^\alpha b^\beta<\alpha^\alpha\beta^\beta$ are critical. See Figures 1–2.

5A. Proofs

LEMMA 5A.1. *Let a,b,α,β be positive. If there exists a positive p such that*

$$\frac{a}{\alpha^p}+\frac{b}{\beta^p}<1,$$

then (1) *is satisfied by functions from a strip around $(0,\infty)\ni t\mapsto -t^p$. In particular, the class of negative solutions to* (1)–(2) *is huge.*

Proof. Define

$$K=\frac{a}{\alpha^p}+\frac{b}{\beta^p}.$$

Then for $s,t>0$ we have

$$-as^p-bt^p=-(\alpha s+\beta t)^p\left(\frac{a}{\alpha^p}\left(\frac{\alpha s}{\alpha s+\beta t}\right)^p+\frac{b}{\beta^p}\left(\frac{\beta t}{\alpha s+\beta t}\right)^p\right)\ge -K(\alpha s+\beta t)^p,$$

and according to Remark 2.1, the class of solutions to (1)–(2) contains a strip around $(0,\infty)\ni t\mapsto -t^p$. ∎

Let us present two lemmas from the literature of the subject. The first one was proved by N. Kuhn [1987]; the second one is a particular case of the main result of Z. Kominek [1992].

LEMMA 5A.2 (Kuhn [1987]). *If $\alpha,\beta,a,b>0$ are such that $\alpha+\beta=1$, $a+b=1$, then each function f satisfying* (1) *is Jensen-convex.*

Following Matkowski and Pycia [1995b] we give a simple proof of this lemma based on an idea of Daróczy and Páles [1987]:

Proof. From the identity

$$\frac{s+t}{2}=\alpha\left[\alpha\frac{s+t}{2}+\beta t\right]+\beta\left[\alpha s+\beta\frac{s+t}{2}\right]$$

and (1) we have, for all $s,t>0$,

$$f\Big(\frac{s+t}{2}\Big)=f\Big(\alpha\Big[\alpha\frac{s+t}{2}+\beta t\Big]+\beta\Big[\alpha s+\beta\frac{s+t}{2}\Big]\Big)$$
$$\le af\Big(\alpha\,\frac{s+t}{2}+\beta t\Big)+bf\Big(\alpha s+\beta\,a\frac{s+t}{2}\Big)$$
$$\le a^2f\Big(\frac{s+t}{2}\Big)+abf(t)+baf(s)+b^2f\Big(\frac{s+t}{2}\Big),$$

which means that

$$a(1-a)f\Big(\frac{s+t}{2}\Big)\le a(1-a)\,\frac{f(s)+f(t)}{2},\qquad s,t>0,$$

and the lemma follows. ∎

LEMMA 5A.3 (Kominek [1992]). *If* $\alpha,\beta,a,b>0$ *are such that* $\alpha+\beta=1$, $a+b=1$, $\alpha\ne a$, *then each function* f *satisfying* (1)–(2) *is constant.*

Following Matkowski and Pycia [1995b] we give an elementary proof of this lemma:

Proof. According to Lemma 5A.2 the function f is Jensen-convex; Bernstein–Doetsch's Lemma 2A.6 implies that f is convex and continuous.

Let us fix an arbitrary point $t_0>0$ and define the function $g:(-t_0,\infty)\to\mathbb{R}$, $g(t)=f(t_0+t)-f(t_0)$. It can be checked that g is convex and satisfies

$$ag(s)+bg(t)\ge g(\alpha s+\beta t),\qquad s,t>-t_0.$$

Since $g(0)=0$, from this inequality we obtain

$$g(\alpha t)\le ag(t),\qquad g(\beta t)\le bg(t),\qquad t>-t_0.$$

Hence we get

$$\text{(A1)}\qquad \frac{g(\alpha t)}{\alpha t}\le\frac{a}{\alpha}\,\frac{g(t)}{t},\qquad \frac{g(\beta t)}{\beta t}\le\frac{b}{\beta}\,\frac{g(t)}{t},\qquad t>0,$$

$$\text{(A2)}\qquad \frac{g(\alpha t)}{\alpha t}\ge\frac{a}{\alpha}\,\frac{g(t)}{t},\qquad \frac{g(\beta t)}{\beta t}\ge\frac{b}{\beta}\,\frac{g(t)}{t},\qquad t\in(-t_0,0).$$

As a convex function, g has one-sided derivatives g'_+, g'_- at 0. So, letting t tend to 0 in inequalities (A1), we get

$$g'_+(0)\le a\alpha^{-1}g'_+(0),\qquad g'_+(0)\le b\beta^{-1}g'_+(0).$$

Since $a\ne\alpha$, it follows that $g'_+(0)=0$. In the same way, making use of (A2), we show that $g'_-(0)=0$. Thus we have proved that $g'(0)=0$. By the definition of g we hence get $f'(t_0)=g'(0)=0$. Since t_0 has been arbitrarily chosen, this completes the proof. ∎

In the proofs of the theorems we will also use Lemma 5A.4, which allows us to improve estimates of Section 4, and Lemma 5A.5, which allows us to estimate the parameters of Lemma 5A.4. Lemma 5A.5 is a particular case of the Hölder inequality.

LEMMA 5A.4. *Suppose that* $\alpha, \beta, a, b > 0$, $q, M \in \mathbb{R}$, $s \in (0,1)$, *and*

(A3) $$m \le \min(\alpha s^{-1}, \beta(1-s)^{-1}), \quad 0 < m < 1,$$

(A4) $$\varrho = a\alpha^{-q}s^q + b\beta^{-q}(1-s)^q < 1.$$

If $f : (0,\infty) \to \mathbb{R}$ *satisfies* (1), *and*

(A5) $$f(t) \le Mt^q, \quad t \in (0,1],$$

then

(A6) $$f(t) \le \frac{M}{\varrho}\, t^{q+\log\varrho/\log m}, \quad t \in (0,1].$$

Proof. Put

$$x = s/\alpha, \quad y = (1-s)/\beta.$$

First, by induction, we show that

(A7) $$f(t) \le M\varrho^n t^q, \quad t \le m^n,$$

for all $n \in \mathbb{N}$. In view of (A5) this inequality is true for $n = 0$. For the inductive step fix $n \in \mathbb{N}$, suppose (A7) holds true, and take $t \le m^{n+1}$. By (A3) we have

$$xt \le m^{-1}t \le m^{-1}m^{n+1} = m^n, \quad yt \le m^{-1}t \le m^{-1}m^{n+1} = m^n.$$

Hence, by (1), the equality $\alpha x + \beta y = 1$, and the definition of ϱ,

$$f(t) \le af(xt) + bf(yt) \le aM\varrho^n(xt)^q + bM\varrho^n(yt)^q = M\varrho^n(ax^q + by^q)t^q = M\varrho^{n+1}t^q,$$

which proves (A7).

Rewrite inequality (A7) in the form

$$f(t) \le \frac{M}{\varrho}\,\varrho^{n+1}t^q, \quad t \in (m^{n+1}, m^n],\ n \in \mathbb{N}.$$

Conditions (A3) and (A4) give $\log\varrho/\log m > 0$. Consequently, for $t \in (m^{n+1}, m^n]$ we get

$$\begin{aligned} f(t) &\le \frac{M}{\varrho}\,\varrho^{n+1}t^q = \frac{M}{\varrho}\,(m^{\log\varrho/\log m})^{n+1}t^q = \frac{M}{\varrho}\,(m^{n+1})^{\log\varrho/\log m}t^q \\ &\le \frac{M}{\varrho}\,t^{\log\varrho/\log m}t^q = \frac{M}{\varrho}\,t^{q+\log\varrho/\log m}. \end{aligned}$$

Because $\bigcup_{n=0}^{\infty}(m^{n+1}, m^n] = (0,1]$, the above inequality yields (A6) and the proof is complete. ■

LEMMA 5A.5. *Suppose that* $\alpha, \beta, a, b > 0$ *and* $q > 1$. *Then there exists exactly one* $s_q \in (0,1)$ *such that*

$$\frac{a}{\alpha^q}\,(s_q)^q + \frac{b}{\beta^q}\,(1-s_q)^q = \min_{s\in(0,1)}\left(\frac{a}{\alpha^q}\,s^q + \frac{b}{\beta^q}\,(1-s)^q\right).$$

Moreover,

$$s_q = \frac{\alpha}{\alpha + \sqrt[q-1]{a\beta/\alpha b}\,\beta},$$

and

$$\min_{s\in(0,1)}\left(\frac{a}{\alpha^q}\,s^q + \frac{b}{\beta^q}\,(1-s)^q\right) = \left(\frac{1}{\sqrt[q-1]{\alpha/a}\,\alpha + \sqrt[q-1]{\beta/b}\,\beta}\right)^{q-1}.$$

Proof. Consider the function

$$[0,1] \ni s \mapsto \Phi_q(s) = \frac{a}{\alpha^q}\, s^q + \frac{b}{\beta^q}\,(1-s)^q.$$

Since Φ_q'' is positive, and $\Phi_q'(0) < 0$, $\Phi_q'(1) > 0$, the function Φ_q is strictly convex, decreasing on an interval $[0, s_q]$, and increasing on $[s_q, 1]$. The minimum $s_q \in (0,1)$ is uniquely determined by $\Phi_q'(s_q) = 0$. Put

$$\varrho(q) = \Phi_q(s_q),$$

and note that the definition of s_q means that

(A8) $$\frac{a}{\alpha^q}\, s_q^{q-1} = \frac{b}{\beta^q}\,(1-s_q)^{q-1},$$

so

$$\varrho(q) = s_q \frac{a}{\alpha^q}\, s_q^{q-1} + (1-s_q)\,\frac{b}{\beta^q}\,(1-s_q)^{q-1} = \frac{a}{\alpha^q}\, s_q^{q-1}.$$

Solving (A8) gives

$$s_q = \frac{\alpha}{\alpha + \sqrt[q-1]{a\beta/\alpha b}\,\beta}.$$

Hence,

$$\varrho(q) = \left(\frac{1}{\sqrt[q-1]{\alpha/a}\,\alpha + \sqrt[q-1]{\beta/b}\,\beta}\right)^{q-1},$$

which completes the proof. ∎

Remark 5A.6. For positive a, b, α, β define the function $F = F_{a,b,\alpha,\beta} : [0,1] \times \mathbb{R}_+ \to \mathbb{R}$ by the formula

$$F(s,r) = a\alpha^{-r}s^r + b\beta^{-r}(1-s)^r, \qquad s \in [0,1],\ r \geq 0.$$

Note that the function $(0,\infty) \ni t \mapsto t^r$ satisfies (1) if and only if $F(s,r) \geq 1$ for $s \in (0,1)$. Indeed, (1) means that

$$a\,\frac{s^r}{(\alpha s + \beta t)^r} + b\,\frac{t^r}{(\alpha s + \beta t)^r} \geq 1, \qquad s,t > 0,$$

and substitution of s for $\alpha s/(\alpha s + \beta t)$ completes the proof.

Similarly, the function $(0,\infty) \ni t \mapsto -t^r$ satisfies (1) if and only if $F(s,r) \leq 1$ for $s \in (0,1)$.

Proof of Theorem 5.1. Taking $s = t$ in (1) we note that

(A9) $$(a+b)f(t) \geq f(t), \qquad t > 0.$$

Hence, if $a + b < 1$ then each $f : (0,\infty) \to \mathbb{R}$ satisfying (1)–(2) is nonpositive. Moreover, the class of negative solutions to (1)–(2) is huge by Lemma 5A.1 applied with $p > 0$ such that $a + b < \alpha^p$, $a + b < \beta^p$.

The case $a + b = 1$, $a = \alpha$ (and hence $b = \beta$) follows from Kuhn's Lemma 5A.2 and Bernstein–Doetsch's Lemma 2A.6. The case $a + b = 1$, $a \neq \alpha$, follows from Kominek's Lemma 5A.3.

Taking into account the second remark following Theorem 5.1 we may assume in the following that $a+b>1$. By (A9) all solutions are nonnegative in this case. Define the function

$$(1,\infty)\ni q\mapsto \varrho(q)=\inf_{s\in(0,1)}\left(\frac{a}{\alpha^q}\,s^q+\frac{b}{\beta^q}\,(1-s)^q\right).$$

According to Lemma 5A.5 for every $q>1$ there is a uniquely determined number $s_q\in(0,1)$ such that

$$\varrho(q)=\left(\frac{1}{\sqrt[q-1]{\alpha/a}\,\alpha+\sqrt[q-1]{\beta/b}\,\beta}\right)^{q-1}=\frac{a}{\alpha^q}\,(s_q)^q+\frac{b}{\beta^q}\,(1-s_q)^q.$$

Define two auxiliary functions

$$\varphi(q)=\varrho(q)^{1/(1-q)}=\sqrt[q-1]{\frac{\alpha}{a}}\,\alpha+\sqrt[q-1]{\frac{\beta}{b}}\,\beta,\quad \psi(q)=\sqrt[q-1]{\frac{\alpha}{a}}\,\alpha\log\frac{\alpha}{a}+\sqrt[q-1]{\frac{\beta}{b}}\,\beta\log\frac{\beta}{b},\qquad q>1,$$

and note that

$$\varphi'(q)=-\frac{1}{(q-1)^2}\,\psi(q),\qquad \psi'(q)<0\quad\text{ for } q>1.$$

If $a^\alpha b^\beta>\alpha^\alpha\beta^\beta$ (the condition of the last item of Theorem 5.1) then

$$\lim_{q\to\infty}\psi(q)=\alpha\log\frac{\alpha}{a}+\beta\log\frac{\beta}{b}<0.$$

Since ψ is negative in a neighborhood of infinity the above formula on φ' implies that in the neighborhood of infinity φ increases to $\lim_{q\to\infty}\varphi(q)=1$. Hence $\varrho(q)=\varphi(q)^{1-q}$ is greater than 1 in the neighborhood of infinity. According to Remark 5A.6, the class of solutions to (1)–(2) is huge and for any $q>1$ such that $\varrho(q)>1$, the class contains strips around $(0,\infty)\ni t\mapsto t^q$.

Now it only remains to prove the theorem for $a+b>1$ and

$$a^\alpha b^\beta\le\alpha^\alpha\beta^\beta. \tag{A10}$$

The latter condition shows that either $a\le\alpha$, or $b\le\beta$. By symmetry it is enough to consider the case $a\le\alpha$. Since $a+b>1=\alpha+\beta$, it follows that $b>\beta$, and (A10) gives $a<\alpha$. Because of (A10) we also have

$$\lim_{q\to\infty}\psi(q)=\alpha\log\frac{\alpha}{a}+\beta\log\frac{\beta}{b}\ge 0.$$

So, the decreasing function ψ is strictly positive; and the formula on φ' implies that φ decreases to $\lim_{q\to\infty}\varphi(q)=1$. Hence

$$\varrho(q)=\varphi(q)^{1-q}<1,\qquad q>1.$$

The idea of the rest of the proof is to apply Proposition 4.5 and Lemma 5A.4 to estimate solutions of (1)–(2) from above. First put

$$q_1=\frac{\log a}{\log\alpha}>1,$$

and estimate $\log\varrho(q)$ for $q\ge q_1$. To this end note that there is a positive constant C

such that $\psi'(q) \le -C/(q-1)^2$ for $q > 1$. Hence

$$\psi(q) \ge \lim_{p\to\infty} \psi(p) - \int_q^\infty \left(-\frac{C}{(r-1)^2}\right) dr \ge 0 + \frac{C}{q-1} = \frac{C}{q-1}, \qquad q > 1.$$

Next,

$$\varphi'(q) \le -\frac{1}{(q-1)^2}\,\frac{C}{q-1} = -\frac{C}{(q-1)^3},$$

and

$$\varphi(q) \ge \lim_{p\to\infty} \varphi(p) - \int_q^\infty \left(-\frac{C}{(r-1)^3}\right) dr = 1 + \frac{C}{2(q-1)^2}, \qquad q > 1.$$

Consequently, there is a positive C_ϱ such that

$$\log \varphi(q) \ge \log\left(1 + \frac{C}{2(q-1)^2}\right) \ge \frac{C_\varrho}{(q-1)^2}, \qquad q \ge q_1,$$

and

(A11) $$\log \varrho(q) = (1-q)\log\varphi(q) \le (1-q)\,\frac{C_\varrho}{(q-1)^2} = -\frac{C_\varrho}{q-1}, \qquad q \ge q_1.$$

Now define

$$m(q) = \min\left(\frac{\alpha}{s_q}, \frac{\beta}{1-s_q}\right).$$

Inserting here the value of s_q calculated in Lemma 5A.5 gives

$$m(q) = \min\left(\alpha + \sqrt[q-1]{\frac{a\beta}{\alpha b}}\,\beta,\ \sqrt[q-1]{\frac{b\alpha}{\beta a}}\,\alpha + \beta\right) = \alpha + \sqrt[q-1]{\frac{a\beta}{\alpha b}}\,\beta \in (\alpha, 1),$$

and

$$m'(q) = -\frac{1}{(q-1)^2}\sqrt[q-1]{\frac{a\beta}{\alpha b}}\,\beta \log\frac{a\beta}{\alpha b} > 0.$$

For $C = -\beta \log \frac{a\beta}{\alpha b}$ we have

$$m'(q) \le \frac{C}{(q-1)^2}, \qquad q > 1,$$

and

$$m(q) \ge \lim_{p\to\infty} m(p) - \int_q^\infty \frac{C}{(r-1)^2}\, dr = 1 - \frac{C}{q-1}, \qquad q > 1.$$

This inequality (for large q) and $m(q) > \alpha > 0$ (for small q) give us a positive C_m such that

(A12) $$0 > \log m(q) \ge -\frac{C_m}{(q-1)}, \qquad q \ge q_1.$$

Finally, consider the increasing sequence $(q_n)_{n=1,2,\ldots}$ defined by

$$q_{n+1} = q_n + \frac{\log \varrho(q_n)}{\log m(q_n)}, \qquad n = 1, 2, \ldots.$$

By (A11) and (A12) there exists a constant $\Delta = C_\varrho/C_m > 0$ such that

(A13) $$q_{n+1} \ge q_n + \Delta, \qquad n = 1, 2, \ldots.$$

Consequently, for each f satisfying (1)–(2) Proposition 4.5 gives a constant $M > 0$ such that

$$f(t) \le M t^{q_1}, \qquad t \in (0,1],$$

and Lemma 5A.4, with $m(q_n)$ substituted for m, and $\varrho(q_n)$ substituted for ϱ, and s_{q_n} substituted for s, gives by induction

$$f(t) \le \frac{M}{\varrho(q_1)\cdots\varrho(q_n)}\, t^{q_{n+1}}, \qquad t \in (0,1],\ n = 1,2,\ldots.$$

Consequently, (A13) leads to

$$\text{(A14)} \qquad f(t) \le M t^{q_1}\, \frac{t^{\Delta}}{\varrho(q_1)} \cdots \frac{t^{\Delta}}{\varrho(q_n)}, \qquad t \in (0,1],\ n = 1,2,\ldots.$$

Consider $q_n > 2$. Then the concavity of the $(q_n - 1)$-root and the equality $\alpha + \beta = 1$ give

$$\begin{aligned}\varrho(q_n) &= \left(\frac{1}{\sqrt[q_n-1]{\alpha/a}\,\alpha + \sqrt[q_n-1]{\beta/b}\,\beta}\right)^{q_n-1}\\ &\ge \left(\frac{1}{\sqrt[q_n-1]{(\alpha/a)\alpha + (\beta/b)\beta}}\right)^{q_n-1} = \frac{1}{(\alpha/a)\alpha + (\beta/b)\beta} = \varrho(2).\end{aligned}$$

Hence, for positive $t < \varrho(2)^{1/\Delta}$ the estimate (A14) means that $f(t) \le 0$. Fix now an arbitrary $x > 0$, and consider the function

$$(0,\infty) \ni t \mapsto g_x(t) = f\left(\frac{2x}{\varrho(2)^{1/\Delta}}\, t\right).$$

Since the function g_x satisfies (1)–(2), we have

$$f(x) = g_x\left(\frac{1}{2}\varrho(2)^{1/\Delta}\right) \le 0,$$

which completes the proof of Theorem 5.1. ■

Proof of Theorem 5.2. The class of positive solutions contains strips around the power functions $t \mapsto t^p$ for large p. Indeed, take p such that

$$K = \frac{\min(a,b)}{(\alpha+\beta)^p} > 1,$$

note that for $s, t > 0$ we have

$$as^p + bt^p \ge K(\alpha+\beta)^p(s^p + t^p) \ge K(\alpha+\beta)^p(\max(s,t))^p \ge K(\alpha s + \beta t)^p,$$

and take into account Remark 2.1.

If $a + b < 1$ then

$$\frac{a}{\alpha^p} + \frac{b}{\beta^p} < 1$$

for small positive p. Thus Lemma 5A.1 shows that the class of negative solutions contains strips around $t \mapsto -t^p$ for small positive p.

Now suppose $a + b > 1$. If f is a solution to (1)–(2), and there exists $s > 0$ such that $f(s) < 0$, then

$$f((\alpha+\beta)^n s) \le (a+b)^n f(s), \qquad n \in \mathbb{N}.$$

Consequently, $(\alpha+\beta)^n s \to 0$ and $f((\alpha+\beta)^n s) \to -\infty$, which would contradict Theorem 4.4. So, for all $s > 0$ we have $f(s) \ge 0$.

Suppose now that $a + b = 1$ and f is a nonpositive solution to (1). Define

$$S = \liminf_{s\to 0} f(s),$$

and fix a decreasing sequence of positive reals $s_n \to 0$ such that $f(s_n) \to S$. By Theorem 4.4 we have $\liminf_{s\to 0} f(s) > -\infty$, so S is a nonpositive real. Take now arbitrary $t > 0$ and fix the sequence t_n such that $\alpha s_n + \beta t_n = t$. Note that for large n the numbers t_n are positive. For these n we have, from (1) and nonpositivity of f,

$$f(t) \le af(s_n) + bf(t_n) \le af(s_n) \to aS.$$

This is true for arbitrary positive t, so in particular $f(t_n) \le aS$, and we get

$$f(t) \le af(s_n) + bf(t_n) \le af(s_n) + baS \to (a + ab)S.$$

Repeating the procedure we obtain $f(t) \le (a + b(a + ab))S = a(1 + b + b^2)S$, etc. Consequently,

$$f(t) \le a(1 + b + b^2 + b^3 + \cdots)S = a\,\frac{1}{1-b}\,S = S,$$

which shows that $f(t) \le S$ for all $t > 0$. On the other hand, by (1) we have

$$f(t) = (a+b)^n f(t) \ge f((\alpha+\beta)^n t) \quad \text{for } n = 1, 2, \ldots.$$

Letting n tend to infinity we get $f(t) \ge S$. Thus, f is constant. ∎

6. Subadditive-like solutions

This section is devoted to the case of $\beta = 1$ or $\alpha = 1$. Additionally, the last theorem of the section deals with the case $\alpha, \beta > 1$. Along with Sections 7 and 8, this section provides a complete solution to (1)–(2) for parameters such that $\min(\alpha, \beta) < 1 < \alpha + \beta$. Theorems 6.1, 7.1, 7.3, and 8.2, and Proposition 8.1 give a complete description of the solutions to (1)–(2) for critical quadruples.

First consider the case $b = \beta = 1$, which reduces inequality (1) to

$$af(s) + f(t) \ge f(\alpha s + t), \qquad s, t > 0.$$

Solutions to this inequality were considered by J. Matkowski [1994b], who called them (α, a)-subadditive functions. He showed that (α, a)-subadditive functions satisfy properties originally proved by E. Hille [1948, Section 7] and R. A. Rosenbaum [1950] for subadditive functions. Among other results, Matkowski proved

THEOREM 6.1 (J. Matkowski [1994b]). *Suppose that* $a, \alpha > 0$ *and* $b = \beta = 1$. *If* $f : (0,\infty) \to \mathbb{R}$ *satisfies* (1)–(2) *then for every* $t > 0$, *the one-sided limits* $f_-(t) = \lim_{s\to t-} f(s)$ *and* $f_+(t) = \lim_{s\to t+} f(s)$ *exist, and*

$$f_+(t) \le f(t) \le f_-(t).$$

In particular, for every quadruple considered in this theorem the class of solutions of (1)–(2) is not huge. It is also not trivial because it contains either positive (if $a \ge \alpha$) or negative (if $a \le \alpha$) linear functions. Thus for every $(\alpha,\beta) \in (0,\infty)^2$ such that $\beta = 1$,

- the set $\{(a,b) \in (0,\infty)^2 : b = 1,\ a \ge \alpha\}$ is contained in the positive critical curve;
- the set $\{(a,b) \in (0,\infty)^2 : b = 1,\ a \le \alpha\}$ is contained in the negative critical curve.

REMARK 6.2. J. Matkowski and T. Świątkowski [1994] considered one-to-one (α,a)-subadditive functions. Among other results, they proved that *if* $a,\alpha > 0$, $b = \beta = 1$, *and* $f : (0,\infty) \to \mathbb{R}_+$ *is one-to-one and satisfies* (1)–(2), *then* f *is an increasing homeomorphism of* $(0,\infty)$ *onto the range of* f. They considered the case of subadditive functions in [1991].

As in Section 5, *large exponents* in the following theorems mean exponents from a neighborhood of infinity; *small exponents* are from a neighborhood of 0.

THEOREM 6.3 (cf. Figure 6). *Let* $\beta = \alpha = 1$.

- *If* $a,b > 1$ *then all solutions to* (1)–(2) *are nonnegative*; *the class of positive solutions is huge and contains strips around power functions with exponents from* $(0,1)$.
- *If* $a,b \in (0,1)$ *then all solutions to* (1)–(2) *are nonpositive*; *the class of negative solutions of* (1)–(2) *is huge and contains strips around power functions with exponents greater than* 1.
- *If* $0 < a < 1 < b$ *or* $0 < b < 1 < a$ *then the class of solutions of* (1)–(2) *is trivial.*

THEOREM 6.4 (cf. Figure 7). *Let* $\beta = 1$, $\alpha > 1$.

- *If* $a,b > 1$ *then all solutions to* (1)–(2) *are nonnegative*; *the class of positive solutions is huge and contains strips around power functions with small exponents.*
- *If* $b > 1$ *and* $0 < a \le 1$ *then* (1)–(2) *has only the zero solution.*
- *If* $0 < b < 1$ *then all solutions to* (1)–(2) *are nonpositive*; *the class of nonpositive solutions is huge and contains strips around power functions with large exponents.*

Symmetry gives the counterpart of Theorem 6.4 for $\beta > 1$, $\alpha = 1$. The case $\beta = 1$, $\alpha < 1$ (or $\beta < 1$, $\alpha = 1$) will be treated in Corollary 7.5.

THEOREM 6.5 (cf. Figure 8). *Let* $\alpha,\beta > 1$, $a,b > 0$. *Then the class of negative solutions to* (1)–(2) *is huge and contains strips around power functions with large exponents. Moreover*,

- *If* $a,b > 1$ *then there is a huge class of positive solutions of* (1)–(2) *which contains strips around power functions with small exponents.*
- *If* $a \le 1$ *or* $b \le 1$ *then all solutions to* (1)–(2) *are nonpositive.*

REMARK 6.6. Theorems 6.3 and 6.4 show that if $\alpha \ge \beta = 1$ then the pairs (a,b) such that $b > 1$ do not belong to the negative critical curve for (α,β). However, the quadruples (α,β,a,b) such that $\alpha \ge \beta = 1$, $b > 1$ are critical (this follows from comparison of Figures 6–8). Similarly, if $\beta \ge \alpha = 1$ then the pairs (a,b) such that $a > 1$ do not belong to the negative critical curve for (α,β), but the quadruples (α,β,a,b) are critical.

6A. Proofs

Proof of Theorem 6.1 (Matkowski [1994b]). Fix $t > 0$ and choose two sequences $(t_n), (s_n)$ such that $t < t_n < s_n$, $n \in \mathbb{N}$, $\lim_{n\to\infty} s_n = t$, and

$$\liminf_{s\to t+} f(s) = \lim_{n\to\infty} f(t_n), \qquad \limsup_{s\to t+} f(s) = \lim_{n\to\infty} f(s_n).$$

Theorem 4.4 implies that $\lim_{n\to\infty} f(t_n)$ and $\lim_{n\to\infty} f(s_n)$ are finite. From (1) we have

$$f(s_n) = f(\alpha(\alpha^{-1}(s_n - t_n)) + t_n) \le af(\alpha^{-1}(s_n - t_n)) + f(t_n), \qquad n \in \mathbb{N}.$$

Letting n tend to infinity and making use of (2) we get

$$\limsup_{s\to t+} f(s) \le \liminf_{s\to t+} f(s),$$

which proves that the right limit $f_+(t)$ exists. Similarly,

$$f(s_n) = f(\alpha(\alpha^{-1}(s_n - t)) + t) \le af(\alpha^{-1}(s_n - t)) + f(t), \qquad n \in \mathbb{N},$$

and, letting $n \to \infty$ in this inequality, we obtain $f_+(t) \le f(t)$.

In a similar way, choosing $(t_n), (s_n)$ such that $t_n < s_n < t$, $n \in \mathbb{N}$, $\lim_{n\to\infty} t_n = t$ we can prove that the left limit $f_-(t)$ exists and $f(t) \le f_-(t)$. ∎

LEMMA 6A.1. *If* $0 < a \le 1 < \alpha$ *and* $b, \beta > 0$, *then all solutions to* (1)–(2) *are nonpositive.*

Proof. Take $f : (0,\infty) \to \mathbb{R}$ which satisfies (1)–(2) and fix a positive ε. By (2) there is δ such that ε is an upper bound of f on $(0,\delta)$. For $s \in (0,\delta)$, $t > 0$, inequality (1) gives

$$f(\alpha s + \beta t) \le af(s) + bf(t) \le \varepsilon + bf(t).$$

Letting $t \to 0$ we find that ε is an upper bound of f on $(0, \alpha\delta)$. By induction, ε is an upper bound of f on $(0, \alpha^n\delta)$ for $n = 1, 2, \dots$. Consequently, ε is an upper bound of f on $(0,\infty)$. Now, since ε was fixed arbitrarily, f is nonpositive. ∎

LEMMA 6A.2. *If* $\beta = 1 < b$ *and* $\alpha, a > 0$ *then every solution of* (1)–(2) *is nonnegative.*

Proof. For an indirect argument suppose that $f(t_0) < 0$ for some $t_0 > 0$. By (2) there exists a positive δ such that

$$af(s) < \tfrac{1}{2}(1-b)f(t_0), \qquad s \in (0,\delta],$$

and consequently, by (1),

$$f(\alpha s + t_0) \le af(s) + bf(t_0) < \tfrac{1}{2}(1+b)f(t_0), \qquad s \in (0,\delta],$$

and, for $s \in (0,\delta]$,

$$\begin{aligned} f(\alpha s + t_0) &= f\big(\alpha\tfrac{1}{2}s + \alpha\tfrac{1}{2}s + t_0\big) \le af\big(\tfrac{1}{2}s\big) + bf\big(\alpha\tfrac{1}{2}s + t_0\big) \\ &< \tfrac{1}{2}(1-b)f(t_0) + b\tfrac{1}{2}(1+b)f(t_0) = \tfrac{1}{2}(1+b^2)f(t_0). \end{aligned}$$

By induction, for all $n \in \mathbb{N}$,

$$f(\alpha s + t_0) = f\left(\alpha\,\frac{s}{n} + \dots + \alpha\,\frac{s}{n} + t_0\right) < \frac{1}{2}\,(1+b^n)f(t_0), \qquad s \in (0,\delta].$$

Letting n tend to ∞ gives $f(\alpha s + t_0) = -\infty$ for $s \in (0,\delta]$, and completes the proof. ∎

LEMMA 6A.3. *If* $\beta = 1 > b$ *and* $\alpha, a > 0$ *then every solution of* (1)–(2) *is nonpositive.*

Proof. The indirect proof is analogous to that of Lemma 6A.2. Assume $f(t_0) > 0$. By (2) there exists $\delta \in (0, t_0)$ such that

$$\frac{a}{b}\, f(s) < \frac{1}{2}\left(\frac{1}{b} - 1\right) f(t_0), \qquad s \in (0, \delta].$$

Consequently, for $s \in (0, \delta]$ inequality (1) implies that

$$f(t_0 - \alpha s) \geq \frac{1}{b}\, f(t_0) - \frac{a}{b}\, f(s) > \frac{1}{b}\, f(t_0) - \frac{1}{2}\left(\frac{1}{b} - 1\right) f(t_0) = \frac{1}{2}\left(1 + \frac{1}{b}\right) f(t_0).$$

and by induction, for all $n = 1, 2, \ldots$, we obtain

$$f(t_0 - \alpha s) \geq \frac{1}{2}\left(1 + \frac{1}{b^n}\right) f(t_0), \qquad s \in (0, \delta].$$

Letting n tend to ∞ gives $f(t_0 - \alpha s) = \infty$ for $s \in (0, \delta]$ and completes the proof. ∎

Proof of Theorem 6.3. If $a, b > 1$ then by Lemma 6A.2 all solutions to (1)–(2) are nonnegative. Since positive power functions with exponents from $(0, 1)$ are positive subadditive, Lemma 3.1 implies that the class of positive solutions to (1)–(2) is huge and contains strips around power functions.

If $a, b \in (0, 1)$ then by Lemma 6A.3 all solutions to (1)–(2) are nonpositive. Since negative power functions with exponents greater than 1 are subadditive, Lemma 3.5 implies that the class of negative solutions to (1)–(2) is huge and contains strips around power functions.

If $0 < a < 1 < b$ or $0 < b < 1 < a$ then each solution is nonnegative (by Lemma 6A.2 and its counterpart for $\alpha = 1 < a$) and at the same time nonpositive (by Lemma 6A.3 and its counterpart for $\alpha = 1 > a$). Hence the class of solutions of (1)–(2) is trivial. ∎

Proof of Theorem 6.4. If $a, b > 1$ then Lemma 6A.2 shows that all solutions to (1)–(2) are nonnegative. Take $K \in (1, \min(a, b))$ and a positive $p < 1$ satisfying $K\alpha^p < a$. Then

$$K(\alpha s + t)^p \leq K\alpha^p s^p + Kt^p < as^p + bt^p,$$

so, according to Remark 2.1, the class of positive solutions to (1)–(2) contains strips around $t \mapsto t^p$.

If $a \leq 1$ then Lemma 6A.1 shows that every solution of (1)–(2) is nonpositive. Since Theorem 6.1 says that the class of solutions to (1)–(2) is not huge for $b = 1$, Lemma 3.5 implies that the class of solutions to (1)–(2) is trivial if $b > 1$.

If $b < 1$ then Lemma 6A.3 shows that all solutions to (1)–(2) are nonpositive. Take a real $p > 1$ such that $\alpha^p > a$. Then

$$(\alpha s + t)^p \geq \alpha^p s^p + t^p \geq \min(\alpha^p/a, 1/b)(as^p + bt^p),$$

so, according to Remark 2.1, the class of negative solutions of (1)–(2) contains strips around $t \mapsto -t^p$. ∎

Proof of Theorem 6.5. The class of negative solutions of (1)–(2) is huge because large positive p satisfy the condition of Lemma 5A.1.

In order to show for $a, b > 1$ that the class of positive solutions is huge take a positive $p < 1$ such that $a > \alpha^p$, $b > \beta^p$. By the subadditivity and homogeneity of the power function $(0,\infty) \ni t \mapsto f_0(t) = t^p$ we have $f_0(\alpha s + \beta t) \le \alpha^p f_0(s) + \beta^p f_0(t)$, and consequently, every function $f : (0,\infty) \to \mathbb{R}$ such that

$$f_0(t) \le f(t) \le \min(a/\alpha^p, b/\beta^p) f_0(t), \qquad t > 0,$$

satisfies (1)–(2).

The remaining statement of the theorem follows from Lemma 6A.1. ■

7. Convex homogeneous solutions

In this section we consider (1)–(2) with parameters a, b, α, β such that $\alpha < 1 < \alpha + \beta$. Theorems 7.1 and 7.3 give a complete description of the solutions to (1)–(2) for critical quadruples.

The auxiliary Subsection 7A.1 shows that for each pair (α, β) such that $\alpha < 1 < \alpha+\beta$ there exists a unique pair $(u_{\alpha,\beta}, w_{\alpha,\beta})$ of positive reals such that

$$w_{\alpha,\beta} < 1 < u_{\alpha,\beta}, \qquad \alpha u_{\alpha,\beta} + \beta w_{\alpha,\beta} = 1, \qquad \alpha u_{\alpha,\beta} \log u_{\alpha,\beta} = -\beta w_{\alpha,\beta} \log w_{\alpha,\beta}.$$

We use the name *scaling factors* of the pair (α, β) when referring to $(u_{\alpha,\beta}, w_{\alpha,\beta})$. For brevity we will write u and w whenever it is clear that the symbols refer to the scaling factors of the pair (α, β). The pair of scaling factors (u, w) depends continuously on (α, β). This notion is used in the formulation of the theorems of this section.

THEOREM 7.1. *Suppose that* $0 < \alpha < 1 < \alpha + \beta$, *and denote by* (u, w) *the pair of scaling factors for* (α, β). *If* $a, b, p > 0$ *are such that*

$$\alpha u = a u^p, \qquad \beta w = b w^p,$$

then a function $f : (0,\infty) \to \mathbb{R}$ *satisfies* (1)–(2) *if and only if it is convex and*

$$f(ut) = u^p f(t), \qquad f(wt) = w^p f(t), \qquad t > 0.$$

Observe that (cf. Subsection 7A.5):

- if $a \in (\alpha, \alpha u)$ then $p \in (0, 1)$, and consequently, solutions to (1)–(2) are nonpositive;
- if $a < \alpha$ then $p > 1$, and consequently, solutions to (1)–(2) are nonnegative.

For almost all α, β considered in the above theorem $\log u / \log w$ is irrational; then the solutions of (1)–(2) are power functions. If $a = \alpha$, $b = \beta$ then the solutions are linear.

COROLLARY 7.2. *Let* a, b, α, β *and* u, w, p *be as in Theorem* 7.1.

(1) *If* $\log u / \log w$ *is irrational then* f *satisfies* (1)–(2) *if, and only if,*

$$f(t) = f(1) t^p, \qquad t > 0,$$

where $f(1)$ *is nonnegative for* $p > 1$, *and nonpositive for* $p \in (0, 1)$.

(2) *If* $p = 1$ *then* f *satisfies* (1)–(2) *if, and only if,*

$$f(t) = f(1) t, \qquad t > 0.$$

The second part of this corollary was proved in a different way by J. Matkowski (cf. Matkowski [1990b, 1992a] and Matkowski and Pycia [1995a]).

THEOREM 7.3. *Suppose that* $0 < \alpha < 1 < \alpha + \beta$, *and denote by* (u, w) *the pair of scaling factors for* (α, β). *If* $a \in [\alpha u, 1]$ *and* $b = 1 - a$, *then* $f : (0, \infty) \to \mathbb{R}$ *satisfies* (1)–(2) *if and only if* f *is constant and nonpositive.*

The above two theorems completely describe the critical curves in the half-plane $\{(a, b) : a \geq \alpha\}$ and the classes of solutions of (1)–(2) for the parameters from the critical curves. The description of the remaining part of the critical curves depends on whether β is smaller than, equal to, or greater than 1. If $\beta < 1$ then α and β play symmetric roles, and the above theorems may be applied to describe solutions and critical curves. Recall that the precise meaning of being below, or above, a set is given in Section 2.

COROLLARY 7.4 (cf. Figure 3). *Suppose that* $0 < \alpha, \beta < 1 < \alpha + \beta$, *and denote by* $(u_{\alpha,\beta}, w_{\alpha,\beta})$ *the scaling factors for* (α, β), *and by* $(u_{\beta,\alpha}, w_{\beta,\alpha})$ *the scaling factors for* (β, α). *Then the positive critical curve is given by*

$$b_{\text{positive critical}}(a) = \begin{cases} \beta^{1/\beta w_{\alpha,\beta}} a^{\alpha u_{\alpha,\beta}/\beta w_{\alpha,\beta}}, & a \leq \alpha, \\ \beta^{1/\beta w_{\beta,\alpha}} a^{\alpha u_{\beta,\alpha}/\beta w_{\beta,\alpha}}, & a \geq \alpha, \end{cases}$$

and the negative critical curve is given by

$$b_{\text{negative critical}}(a) = \begin{cases} 1 - a, & a \in [0, \alpha u_{\beta,\alpha}], \\ \beta^{1/\beta w_{\beta,\alpha}} a^{\alpha u_{\beta,\alpha}/\beta w_{\beta,\alpha}}, & a \in [\alpha u_{\beta,\alpha}, \alpha], \\ \beta^{1/\beta w_{\alpha,\beta}} a^{\alpha u_{\alpha,\beta}/\beta w_{\alpha,\beta}}, & a \in [\alpha, \alpha u_{\alpha,\beta}], \\ 1 - a, & a \in [\alpha u_{\alpha,\beta}, 1]. \end{cases}$$

- *If* (a, b) *is above the positive critical curve then all solutions to* (1)–(2) *are positive, and the class of positive solutions is huge.*
- *If* (a, b) *is below the positive critical curve and above the negative critical curve then the only solution to* (1)–(2) *is the zero function.*
- *If* (a, b) *is below the negative critical curve then all solutions are nonpositive, and the class of nonpositive solutions is huge.*

The huge classes of this corollary contain strips around power functions.

The description of the case $\beta = 1$, begun in Section 6, is also completed by:

COROLLARY 7.5 (cf. Figure 4). *Suppose that* $0 < \alpha < 1$, $\beta = 1$, *and denote by* (u, w) *the scaling factors for* (α, β). *Then the positive critical curve is given by*

$$b_{\text{positive critical}}(a) = \begin{cases} \beta^{1/\beta w} a^{\alpha u/\beta w}, & a \leq \alpha, \\ 1, & a \geq \alpha, \end{cases}$$

and the negative critical curve is given by

$$b_{\text{negative critical}}(a) = \begin{cases} 1, & a \in [0, \alpha], \\ \beta^{1/\beta w} a^{\alpha u/\beta w}, & a \in [\alpha, \alpha u_{\alpha,\beta}], \\ 1 - a, & a \in [\alpha u_{\alpha,\beta}, 1]. \end{cases}$$

- *If* (a, b) *is above the positive critical curve then all solutions to* (1)–(2) *are positive, and the class of positive solutions is huge.*

- *If (a, b) is below the positive critical curve and above the negative critical curve then the only solution to* (1)–(2) *is the zero function.*
- *If (a, b) is below the negative critical curve then all solutions are nonpositive*, *and the class of nonpositive solutions is huge.*

The huge classes of this corollary contain strips around power functions.

The case $\beta > 1$ requires some new results and will be discussed in Section 8.

7A. Proofs

7A.1. *Scaling factors.* We show the existence and uniqueness of the scaling factors (u, w) for the pair (α, β) such that $0 < \alpha < 1 < \alpha+\beta$. The scaling factors are defined as positive reals such that:

$$0 < w < 1 < u,$$

$$\alpha u + \beta w = 1, \tag{A1}$$

$$\alpha u \log u = -\beta w \log w. \tag{A2}$$

Condition (A1) will be satisfied if $w = (1 - \alpha u)/\beta$. Given this, the inequalities will be satisfied if $u \in (1, \alpha^{-1})$. Indeed, first note that $\alpha+\beta > 1$ implies that $w = (1 - \alpha u)/\beta < 1$ if $u > 1$. Moreover, $w = (1 - \alpha u)/\beta > 0$ if $u < \alpha^{-1}$.

Consequently, it is enough to prove that there exists a unique $u \in (1, \alpha^{-1})$ such that

$$\psi(u) = \alpha u \log u + \beta \frac{1 - \alpha u}{\beta} \log \frac{1 - \alpha u}{\beta} = 0.$$

The existence of u follows from the continuity of ψ and the relation

$$\psi(1) < 0 < \lim_{s \to \alpha^{-1}-} \psi(s),$$

which is a consequence of the inequality $\alpha < 1 < \alpha + \beta$. The uniqueness of u is a consequence of the monotonicity of ψ; the monotonicity follows from

$$\psi'(u) = \alpha \log u - \alpha \log \frac{1 - \alpha u}{\beta} > 0.$$

Finally, the monotonicity of ψ and its continuous dependence on the parameters α, β implies the continuous dependence of u (and hence w) on α and β.

7A.2. *Basic lemma.* The proofs of the results of Section 7 are based on Lemma 7A.2. In its proof we apply

LEMMA 7A.1. *Suppose $p \in (0, 1)$ and $a, b \in \mathbb{R}$ satisfy $pa + (1 - p)b = 0$. Let E be the closed additive subgroup of $\mathbb{R}$ generated by a, b. Then every upper bounded and upper semicontinuous function $\phi : E \to \mathbb{R}$ satisfying*

$$\phi(t) \le p\phi(t + a) + (1 - p)\phi(t + b), \qquad t \in E, \tag{A3}$$

is constant.

Proof (suggested by S. Kwapień). If ϕ is not constant on E then there exist $x, y \in E$ such that $\phi(x) > \phi(y)$. By the upper semicontinuity of ϕ there exist a neighborhood V

of $y-x$ and an $\varepsilon>0$ such that

$$\phi(x)-\varepsilon>\phi(s),\qquad s\in x+V=\{x+t:t\in V\}.$$

Inequality (A3) gives, by induction with respect to $N=1,2,\ldots$ (in the sums below, the numbers $a_1,\ldots,a_N$ are taken from the set $\{a,b\}$; if $a_i=a$ then $p_i=p$; if $a_i=b$ then $p_i=1-p$; in the inductive step, (A3) is applied to the second summand),

$$\begin{aligned}\phi(t)\le\sum_{n=1}^{N}\ \sum_{a_1\notin V,\,\ldots,\,a_1+\cdots+a_{n-1}\notin V,\ a_1+\cdots+a_n\in V} & p_1\cdots p_n\phi(t+a_1+\cdots+a_n)\\ +\sum_{a_1\notin V,\,\ldots,\,a_1+\cdots+a_{N-1}\notin V,\,a_1+\cdots+a_N\notin V} & p_1\cdots p_N\phi(t+a_1+\cdots+a_N).\end{aligned}$$

Let M be a global upper bound of ϕ. The above inequalities give

$$\begin{aligned}\phi(x)\le\Big(\sum_{n=1}^{N}\ \sum_{a_1\notin V,\,\ldots,\,a_1+\cdots+a_{n-1}\notin V,\,a_1+\cdots+a_n\in V} & p_1\cdots p_n\Big)(\phi(x)-\varepsilon)\\ +\Big(\sum_{a_1\notin V,\,\ldots,\,a_1+\cdots+a_{N-1}\notin V,\,a_1+\cdots+a_N\notin V} & p_1\cdots p_N\Big)M.\end{aligned}$$

Consider the probability measure μ on E given by $\mu(a)=p$ and $\mu(b)=1-p$. In the lemma we assumed that $\mu(a)a+\mu(b)b=0$, so the random walk on E governed by the measure μ is recurrent (cf. W. Feller [1971, Th. VI.10.4]). Thus the probability that the random walk starting from 0 reaches V in the 1st, 2nd, …, or Nth step tends to 1 as $N\to\infty$, that is,

$$\sum_{n=1}^{N}\ \sum_{a_1\notin V,\,\ldots,\,a_1+\cdots+a_{n-1}\notin V,\,a_1+\cdots+a_n\in V} p_1\cdots p_n\to 1,$$

and

$$\sum_{a_1\notin V,\,\ldots,\,a_1+\cdots+a_{N-1}\notin V,\,a_1+\cdots+a_N\notin V} p_1\cdots p_N\to 0.$$

Consequently, $\phi(x)\le\phi(x)-\varepsilon$, which is a contradiction. ∎

The above lemma is a particular case of an equivalence between the structure of the class of superharmonic functions on a Markov chain and the recurrence of the chain. J. Azéma, M. Kaplan-Duflo and D. Revuz [1966, 1969], R. K. Getoor [1990], and M. Pycia [1997] study this equivalence for various classes of Markov chains.

As a corollary we obtain the following crucial

LEMMA 7A.2. *Let $a,b,\alpha,\beta>0$, $u,w>0$, $p\ge 0$ be such that $\alpha<1<\alpha+\beta$ and* (A1) *holds true. Suppose that*

$$au^p+bw^p=1,\tag{A4}$$

$$au^p\log u=-bw^p\log w.\tag{A5}$$

If $f:(0,\infty)\to\mathbb{R}$ satisfies (1) *and the function $(0,\infty)\ni t\mapsto f(t)/t^p$ is bounded above, then f is convex and*

$$f(ut)=u^pf(t),\quad f(wt)=w^pf(t),\qquad t>0.\tag{A6}$$

If, *in addition*, $\log u/\log w$ *is irrational then*

$$f(t) = f(1)t^p, \qquad t > 0.$$

Proof. Define $\varphi : \mathbb{R} \to \mathbb{R}$ by the formula

$$\varphi(t) = f(e^t)/e^{pt}, \qquad t \in \mathbb{R},$$

and note that it is bounded above. Substituting ut for s, wt for t, taking into account (A1), and writing (1) in terms of the function φ gives the following inequality of the form (A3):

$$au^p\varphi(t + \log u) + bw^p\varphi(t + \log w) \geq \varphi(t), \qquad t \in \mathbb{R}.$$

Put

$$E = \text{closure of } \{j \log w + i \log u : j, i \in \mathbb{Z}\} \text{ in } \mathbb{R},$$

and adopt the notation $t + E = \{t + s : s \in E\}$. Now if $\varphi|_{t+E}$ is upper semicontinuous then, by (A4), (A5), and boundedness above of φ, the conditions of Lemma 7A.1 are satisfied. Hence, if $\varphi|_{t+E}$ is upper semicontinuous then it is constant. The rest of the proof is divided into two independent parts.

CASE $\log u/\log w \in \mathbb{Q}$. Take $t \in \mathbb{R}$. Then the sets E and $t+E$ are discrete. Hence $\varphi|_{t+E}$ is continuous, and consequently, constant. Thus f satisfies (A6).

To show the convexity of f, apply (A6) and (1) to get

$$au^pf(s) + bw^pf(t) = af(us) + bf(wt) \geq f(\alpha us + \beta wt), \qquad s, t > 0.$$

Since αu, $au^p \in (0, 1)$, this inequality means that f is $(\alpha u, au^p)$-convex; and Kuhn's Lemma 5A.2 says that f is Jensen-convex. Since f is locally bounded above and Jensen-convex, by the Bernstein–Doetsch Lemma 2A.6, the function f is convex, which completes the proof in the first case.

CASE $\log u/\log w \notin \mathbb{Q}$. The auxiliary function $g : (0, \infty) \to \mathbb{R}$ given by

$$g(t) = \lim_{\delta \to 0} \sup_{s \in (t-\delta, t+\delta)} f(s)$$

is well defined. Note that it is upper semicontinuous and $g(t)/t^p$ is bounded above. In order to show that g satisfies (1), fix arbitrary $s, t > 0$ and take $s_n \to s$ and $t_n \to t$ such that $f(\alpha s_n + \beta t_n) \to g(\alpha s + \beta t)$. Then

$$ag(s) + bg(t) \geq \lim_{n\to\infty} \sup(af(s_n) + bf(t_n)) \geq \lim_{n\to\infty} f(\alpha s_n + \beta t_n) = g(\alpha s + \beta t).$$

Now, substituting g for f in the preparatory reasoning of the proof, we infer that, being upper semicontinuous on $E = \mathbb{R}$, the function $\varphi(t) = g(e^t)/e^{pt}$ is constant. Consequently,

$$\text{(A7)} \qquad g(t) = g(1)t^p, \qquad t > 0.$$

Fix $t \in (0, \infty)$. If $s_n \to uw^{-1}t$ then equality (A1) gives

$$\alpha s_n + \beta t \to (1 - \beta w)w^{-1}t + \beta t = w^{-1}t.$$

Thus, taking a sequence (s_n) such that $s_n \to uw^{-1}t$ and $f(\alpha s_n + \beta t) \to g(w^{-1}t)$, we have, by (1),

$$f(t) \geq b^{-1}[f(\alpha s_n + \beta t) - af(s_n)] \geq b^{-1}[f(\alpha s_n + \beta t) - ag(s_n)].$$

Letting n tend to ∞, using (A7) and (A4), we get

$$f(t) \geq b^{-1}[g(w^{-1}t) - ag(uw^{-1}t)] = g(1)t^p.$$

On the other hand, $f(t) \leq g(t) = g(1)t^p$. Thus $f(t) = g(1)t^p$, which ends the proof of the lemma. ∎

7A.3. *Proof of Theorem 7.1 and Corollary 7.2.* Lemma 7A.2 easily yields the proof of Theorem 7.1 for nonpositive f. However, the case of nonnegative solutions needs some additional preparatory considerations.

For positive a, b, α, β define the function $F = F_{a,b,\alpha,\beta} : [0,1] \times \mathbb{R}_+ \to \mathbb{R}$ by

$$F(s,r) = a\alpha^{-r}s^r + b\beta^{-r}(1-s)^r, \qquad s \in [0,1],\ r \geq 0.$$

In Remark 5A.6 we have noted that the function $(0,\infty) \ni t \mapsto t^r$ satisfies (1) if and only if $F(s,r) \geq 1$ for $s \in (0,1)$, and the function $(0,\infty) \ni t \mapsto -t^r$ satisfies (1) if and only if $F(s,r) \leq 1$ for $s \in (0,1)$.

We start with three lemmas.

LEMMA 7A.3. *Let* $0 < \alpha < 1 < \alpha + \beta$, $a, b > 0$, $u > 1 > w > 0$ *and* $p > 1$ *satisfy* (A1), (A2), *and*

$$\text{(A8)} \qquad \alpha u = au^p, \qquad \beta w = bw^p.$$

Then:

- *The function* $(0,\infty) \ni t \mapsto t^p$ *satisfies* (1) *and*

$$\inf_{s\in(0,1)} F(s,p) = F(\alpha u, p) = 1.$$

- *If* $q > 1$ *and the function* $(0,\infty) \ni t \mapsto t^q$ *satisfies* (1) *then* $q = p$. *Moreover, for every* $\varepsilon \in (0, \min\{\alpha u, 1-\alpha u\})$ *there exists a* $\delta > 0$ *such that*

$$0 < \inf_{s\in(\alpha u-\varepsilon,\alpha u+\varepsilon)} F(s,r) < 1, \qquad r \in [p-\delta, p) \cup (p, p+\delta].$$

Proof. First note that the function $(0,\infty) \ni t \mapsto t^p$ satisfies (1). Indeed, it is convex, so by (A8) and (A1),

$$\begin{aligned} as^p + bt^p &= au^p(s/u)^p + bw^p(t/w)^p \\ &\geq (au^p(s/u) + bw^p(t/w))^p = (\alpha u(s/u) + \beta w(t/w))^p = (\alpha s + \beta t)^p. \end{aligned}$$

Consequently, $\inf_{s\in(0,1)} F(s,p) \geq 1$ and direct computation shows that $F(\alpha u, p) = 1$.

To prove the second bullet point of the lemma define

$$\varrho(q) = \inf_{s\in(0,1)} F(s,q) = \inf_{s\in(0,1)} \left(\frac{a}{\alpha^q}\, s^q + \frac{b}{\beta^q}\,(1-s)^q\right), \qquad q > 1.$$

According to Lemma 5A.5 we have

$$\varrho(q)^{1/(1-q)} = \sqrt[q-1]{\frac{\alpha}{a}}\,\alpha + \sqrt[q-1]{\frac{\beta}{b}}\,\beta,$$

so

$$\frac{d}{dq}\varrho(q)^{1/(1-q)} = -\frac{1}{(q-1)^2}\left(\sqrt[q-1]{\frac{\alpha}{a}}\,\alpha\log\frac{\alpha}{a} + \sqrt[q-1]{\frac{\beta}{b}}\,\beta\log\frac{\beta}{b}\right).$$

Hence, (A8) gives

$$\frac{d}{dq}\varrho(q)^{1/(1-q)} = -\frac{p-1}{(q-1)^2}(u^{(p-1)/(q-1)}\alpha\log u + w^{(p-1)/(q-1)}\beta\log w).$$

Since u, w are positive and different from 1, the function $(0,\infty) \ni x \mapsto u^x\alpha\log u + w^x\beta\log w$ is strictly increasing. Since (A2) implies that it is equal to 0 for $x = 1$, this function is positive for $x > 1$ and negative for $x < 1$. Consequently, $\frac{d}{dq}\varrho(q)^{1/(1-q)}$ is negative for $q \in (1,p)$, equal to 0 for $q = p$, and positive for $q > p$, which proves that

$$\varrho(q)^{1/(1-q)} > \varrho(p)^{1/(1-p)} = 1, \qquad q \in (1,p) \cup (p,\infty).$$

Because $q > 1$, we hence get

$$\text{(A9)} \qquad \inf_{s\in(0,1)}\left(\frac{a}{\alpha^q}\,s^q + \frac{b}{\beta^q}\,(1-s)^q\right) = \varrho(q) < 1, \qquad q \in (1,p) \cup (p,\infty).$$

Thus the first part of the second bullet point of the lemma is proved.

To prove the "moreover" part, fix positive $\varepsilon < \min\{\alpha u, 1-\alpha u\}$ and a compact interval $I \subset (1,\infty)$ such that $p \in I$. Since $F(\cdot,p)$ is strictly concave with minimum $\varrho(p) = 1$ attained at αu, we have

$$\inf_{s\in[0,\alpha u-\varepsilon]\cup[\alpha u+\varepsilon,1]} F(s,p) > 1.$$

By the uniform continuity of F on $([0,\alpha u-\varepsilon]\cup[\alpha u+\varepsilon,1]) \times I$ there is a $\delta > 0$ such that

$$\inf_{s\in[0,\alpha u-\varepsilon]\cup[\alpha u+\varepsilon,1]} F(s,r) > 1, \qquad r \in [p-\delta, p+\delta] \subseteq I.$$

Hence, making use of (A9),

$$\inf_{s\in[\alpha u-\varepsilon,\alpha u+\varepsilon]} F(s,r) < 1, \qquad r \in [p-\delta,p) \cup (p,p+\delta]. \ \blacksquare$$

The next lemma is an analogue of Lemma 5A.4.

LEMMA 7A.4. *Suppose that* $\alpha, \beta, a, b > 0$, $q, M, m \in \mathbb{R}$, $s \in (0,1)$,

$$\text{(A10)} \qquad m \ge \max(\alpha s^{-1}, \beta(1-s)^{-1}), \qquad m > 1,$$

and

$$\text{(A11)} \qquad \varrho = a\alpha^{-q}s^q + b\beta^{-q}(1-s)^q < 1.$$

If $f : (0,\infty) \to \mathbb{R}$ *satisfies* (1) *and*

$$\text{(A12)} \qquad f(t) \le Mt^q, \qquad t \ge 1,$$

then

$$\text{(A13)} \qquad f(t) \le \frac{M}{\varrho}\,t^{q+\log\varrho/\log m}, \qquad t \ge 1.$$

Proof. Define

$$x = s/\alpha, \qquad y = (1-s)/\beta.$$

First, by induction, we show that

$$\text{(A14)} \qquad f(t) \le M\varrho^n t^q, \qquad t \ge m^n,\ n \in \mathbb{N}.$$

In view of (A12) this inequality is true for $n = 0$. For the inductive step fix $n \in \mathbb{N}$, suppose (A14), and take $t \ge m^{n+1}$. By (A10) we have

$$xt \ge m^{-1}t \ge m^{-1}m^{n+1} = m^n, \qquad yt \ge m^{-1}t \ge m^{-1}m^{n+1} = m^n.$$

Hence, by (1), the equality $\alpha x + \beta y = 1$, and the definition of ϱ,

$$f(t) \le af(xt) + bf(yt) \le aM\varrho^n(xt)^q + bM\varrho^n(yt)^q = M\varrho^n(ax^q + by^q)t^q = M\varrho^{n+1}t^q,$$

which completes the proof of (A14).

Rewrite inequality (A14) in the form

$$\text{(A15)} \qquad f(t) \le \frac{M}{\varrho}\,\varrho^{n+1}t^q, \qquad t \in [m^n, m^{n+1}),\ n \in \mathbb{N}.$$

The conditions $\varrho < 1$ and (A10) give $\log\varrho/\log m < 0$. Consequently, for $t \in [m^n, m^{n+1})$ we get

$$\begin{aligned} f(t) &\le \frac{M}{\varrho}\,\varrho^{n+1}t^q = \frac{M}{\varrho}\,(m^{\log\varrho/\log m})^{n+1}t^q = \frac{M}{\varrho}\,(m^{n+1})^{\log\varrho/\log m}t^q \\ &\le \frac{M}{\varrho}\,t^{\log\varrho/\log m}t^q = \frac{M}{\varrho}\,t^{q+\log\varrho/\log m}. \end{aligned}$$

Because $\bigcup_{n=0}^{\infty}[m^n, m^{n+1}) = [1,\infty)$, the above inequality yields (A13) and the proof is complete. ∎

Now we present the main lemma that reduces Theorem 7.1 to Lemma 7A.2.

LEMMA 7A.5. *Let* $0 < \alpha < 1 < \alpha + \beta$, $a, b > 0$, $0 < w < 1 < u$ *and* $p > 1$ *satisfy* (A1), (A2), (A8). *If* $f : (0,\infty) \to \mathbb{R}$ *satisfies* (1)–(2), *then* $\sup_{t>0} f(t)/t^p < \infty$.

Proof. According to Theorem 4.1, the function f is either nonpositive or nonnegative. In the first case there is nothing to prove. So, assume that f is nonnegative. First let us show that $\sup_{t\ge1} f(t)/t^p < \infty$.

Let us start with a few definitions. Let ε, δ be as in Lemma 7A.3. For every $r \in [p-\delta, p+\delta]$ choose $s_r \in [\alpha u - \varepsilon, \alpha u + \varepsilon]$ such that

$$F(s_r, r) = \min_{s\in[\alpha u-\varepsilon,\alpha u+\varepsilon]} F(s, r).$$

The numbers s_r are uniquely determined by the strict convexity of $F(\cdot, r)$. Uniqueness of s_r and uniform continuity of F on $[\alpha u-\varepsilon, \alpha u+\varepsilon] \times [p-\delta, p+\delta]$ imply that s_r depends continuously on r. Now according to Lemma 7A.3,

$$F(s_p, p) = 1, \qquad 0 < F(s_r, r) < 1, \qquad r \in [p-\delta, p) \cup (p, p+\delta].$$

Take a number

$$\text{(A16)} \qquad m > \max(1, \alpha(\alpha u-\varepsilon)^{-1}, \beta(1-\alpha u-\varepsilon)^{-1}),$$

and note that m satisfies (A10) with $s \in [\alpha u - \varepsilon, \alpha u + \varepsilon]$. We do not repeat this while referring to Lemma 7A.4. Finally, define

$$\mathbf{L} = \{r \in R : \exists M > 0,\ f(t) \le Mt^r,\ t \ge 1\}.$$

According to Theorem 4.2 the set $\mathbf{L}$ is nonempty. In order to complete the proof, it is enough to show that $p \in \mathbf{L}$.

Because $(\inf \mathbf{L}, \infty) \subseteq \mathbf{L}$, $p \notin \mathbf{L}$ implies $p \le \inf \mathbf{L}$. If $\inf \mathbf{L} > p$ then, by Lemma 7A.3, the function $(0,\infty) \ni t \mapsto t^{\inf \mathbf{L}}$ would not satisfy (1). Consequently, according to Remark 5A.6 one could find an $s \in (0,1)$ such that the number ϱ defined by (A11) is less than 1; and the assertion (A13) of Lemma 7A.4 with $q = \inf \mathbf{L}$ would show that $\inf \mathbf{L} = \mathbf{q} > \inf \mathbf{L}$. This contradiction proves that $p \in \mathbf{L}$ or $p = \inf \mathbf{L}$. So, in the following we may assume that

$$p = \inf \mathbf{L}.$$

Let us introduce the sequence (q_n). First, using the numbers s_r fixed for every $r \in (p, p+\delta]$ define an auxiliary function

$$\widetilde{\varrho} : [p, p+\delta] \to (0,1], \qquad \widetilde{\varrho}(r) = F(s_r, r),$$

whose range is determined according to (A15). Then put

$$q_0 = p + \delta, \qquad q_{n+1} = q_n + \frac{1}{\log m} \log \widetilde{\varrho}(q_n).$$

The definition of (q_n) is correct because $q_0 \in (p, p+\delta] \subset \mathbf{L}$ and, by induction, if $q_n \in [p, p+\delta] \cap \mathbf{L}$ then also $q_{n+1} \in (p, p+\delta] \cap \mathbf{L}$. Indeed, Lemma 7A.4 gives $q_{n+1} \in \mathbf{L} \subset [p, \infty)$; the definition of $\widetilde{\varrho}$ and inequality (A16) give $q_{n+1} \le q_n \le p + \delta$.

The sequence (q_n) is decreasing and bounded below. Let

$$\text{(A17)} \qquad q_{\lim} = \lim_{n\to\infty} q_n = q_0 + \frac{1}{\log m} \sum_{n=0}^{\infty} \log \widetilde{\varrho}(q_n).$$

To observe that $p = q_{\lim}$ note that otherwise $q_{\lim} > p$, and there would exist a positive ε_1 such that

$$F(s_{q_{\lim}}, q_{\lim}) \le 1 - 4\varepsilon_1.$$

Since F is continuous and s_r depends continuously on r we would have

$$F(s_{q_n}, q_n) \le 1 - 2\varepsilon_1 \qquad \text{for suitable large } n \in \mathbb{N}.$$

Hence, the series $\sum_{n=0}^{\infty} \log \widetilde{\varrho}(q_n)$ would be divergent to $-\infty$, contrary to (A17).

Note that the convergence of the above series implies the convergence of the infinite product $\prod_{n=0}^{\infty} \widetilde{\varrho}(q_n)$, and set

$$M = \prod_{n=0}^{\infty} \widetilde{\varrho}(q_n) > 0.$$

Take a real M_0 such that

$$f(t) \le M_0 t^{q_0}, \qquad t \ge 1.$$

Induction, based on Lemma 7A.4, gives

$$f(t) \le \frac{M_0}{\prod_{i=0}^{n-1} \widetilde{\varrho}(q_i)} t^{q_n}, \qquad n \in \mathbb{N}.$$

Letting n tend to infinity and applying the definition of M, we obtain $f(t) \le (M_0/M)t^p$.

In a similar way one may show that $\sup_{t\in(0,1]} f(t)/t^p < \infty$: use Lemma 5A.4 instead of Lemma 7A.4, take

$$m' < \min(1, \alpha(\alpha u + \varepsilon)^{-1}, \beta(1 - \alpha u + \varepsilon)^{-1}),$$

define

$$\mathbf{L}' = \{r : \exists M > 0,\ f(t) \le Mt^r,\ t \in (0,1]\},$$

show that $p = \sup \mathbf{L}'$, and construct the sequence

$$q'_0 = p - \delta, \qquad q'_{n+1} = q'_n + \frac{1}{\log m'} \log \widetilde{\varrho}(q'_n). \ \blacksquare$$

We are now prepared to prove Theorem 7.1. In its proof we refer to the conditions of the theorem by labels introduced in the proof-part of this section.

Proof of Theorem 7.1. First let us show that $f(t)/t^p$ is bounded above for $p \in (0,1]$. We consider separately $t \in (0,1]$ and $t > 1$.

To prove the estimate for $t \le 1$ put $A = \log a / \log \alpha$ and observe that (A1) and (A8), which are assumed in Theorem 7.1, imply $(\alpha u)^p \ge \alpha u = au^p$. So $\alpha^p \ge a$, i.e. $A \ge p$. According to Theorem 4.2, there is $M > 0$ such that

$$f(t) \le Mt^A, \qquad t \in (0,1].$$

Hence

$$f(t)t^{-p} \le f(t)t^{-A} \le M, \qquad t \in (0,1].$$

To prove the estimate for $t \ge 1$ fix $n \in \mathbb{N} \setminus \{0\}$ such that $(\alpha+\beta)^n - \alpha^n > 1$ and put

$$\alpha_0 = \alpha^n, \qquad \beta_0 = (\alpha+\beta)^n - \alpha^n, \qquad a_0 = a^n, \qquad b_0 = (a+b)^n - a^n, \qquad B_0 = \frac{\log b_0}{\log \beta_0}.$$

By Lemma 2A.1 every solution g to (1) satisfies

$$\text{(A18)} \qquad a_0 g(s) + b_0 g(t) \ge g(\alpha_0 s + \beta_0 t), \qquad s,t > 0.$$

The negative function $(0,\infty) \ni t \mapsto -t^p$ satisfies (1), because of its convexity and relations (A1), (A8) between parameters of (1). Consequently, this function satisfies (A18); so Remark 4.3 and Theorem 4.2 give $M_0 > 0$ such that

$$-t^p \le -M_0 t^{B_0}, \qquad t > 1.$$

Hence, $p \ge B_0$. Since f satisfies (A18), Theorem 4.2 and Remark 4.3 give a positive M such that

$$f(t) \le Mt^{B_0} < Mt^p, \qquad t > 1.$$

Thus we have showed that $f(t)t^{-p}$ is uniformly bounded above for $t > 0$ in the case $p \le 1$. For $p > 1$ the estimate is established by Lemma 7A.5. Consequently, Lemma 7A.2 completes the proof of the first implication of Theorem 7.1.

To prove the converse implication, assume that f satisfies (A6) and it is convex. Taking into account the assumptions (A1) and (A8) of Theorem 7.1 we get

$$\begin{aligned} af(s) + bf(t) &= au^p f(u^{-1}s) + bw^p f(w^{-1}t) \\ &\ge f(\alpha u u^{-1} s + \beta w w^{-1} t) = f(\alpha s + \beta t), \qquad s,t > 0, \end{aligned}$$

so (1) holds true. Since $p > 0$, the equations (A6) of Theorem 7.1 imply that $\lim_{s\to 0+} f(s) = 0$, and consequently, (2) is fulfilled. $\blacksquare$

Proof of Corollary 7.2. Both "if" parts are easy. To prove the "only if" parts note that according to Theorem 7.1 a function f fulfilling (1)–(2) satisfies

$$f(u^n w^m t) = (u^n w^m)^p f(t), \qquad t > 0,\ n, m = 0, 1, \ldots.$$

If $\log u / \log w$ is irrational then by the Kronecker Lemma 2A.3 the set $\{u^n w^m : n, m = 0, 1, \ldots\}$ is dense in $(0, \infty)$ and the continuity of f, shown in Theorem 7.1, completes the proof of the first item of the corollary.

Suppose now that $p = 1$. Fix arbitrary $s, t > 0$ and take $n \in \mathbb{N}$ such that $s \in (w^n t, u^n t)$. Then there exists $\lambda \in [0, 1]$ such that $s = \lambda w^n t + (1 - \lambda) u^n t$ and

$$\begin{aligned} tf(s) &= tf(\lambda w^n t + (1 - \lambda) u^n t) \\ &\le t(\lambda f(w^n t) + (1 - \lambda) f(u^n t)) = (\lambda w^n t + (1 - \lambda) u^n t) f(t) = sf(t), \end{aligned}$$

which shows that $f(s) \le sf(1)$ and $tf(1) \le f(t)$. ∎

7A.4. *Proof of Theorem 7.3*

LEMMA 7A.6. *Let* $\alpha, \beta > 0$ *and* (u, w) *be the scaling factors for* (α, β)*. Suppose* $a \in [\alpha u, 1)$*, and* $b = 1 - a$*,* $p = 0$*. Then there exist* $\widetilde{u}$*,* $\widetilde{w}$ *such that* $0 < \widetilde{w} < 1 < \widetilde{u}$ *and* (A1), (A4), (A5) *with* $\widetilde{u}$*,* $\widetilde{w}$ *in place of* u*,* w*, respectively, hold true.*

Proof. Since $p = 0$, the relation (A4) holds automatically for all $\widetilde{u}, \widetilde{w} > 0$. Equation (A5) means that $\widetilde{w} = \widetilde{u}^{a/(a-1)}$, so to establish (A1) it is enough to show that there exists $\widetilde{u} > 1$ such that

$$\alpha \widetilde{u} + \beta \widetilde{u}^{a/(a-1)} = 1.$$

This follows from the Darboux property of the continuous function

$$[1, u] \ni s \mapsto \psi(s) = \alpha s + \beta s^{a/(a-1)}$$

and the inequalities

$$\begin{aligned} \psi(1) &= \alpha + \beta > 1, \\ \psi(u) &= \alpha u + \beta u^{a/(a-1)} \le \alpha u + \beta u^{\alpha u/(\alpha u - 1)} = 1. \ \blacksquare \end{aligned}$$

Proof of Theorem 7.3. First consider the case $a = 1$, $b = 0$. Take $s > 0$ and $t \in (\alpha s, \alpha^{-1} s)$. Then $(t - \alpha s)/\beta$, $(s - \alpha t)/\beta$ are positive and (1) gives

$$\begin{aligned} f(s) &\ge f(\alpha s + \beta(t - \alpha s)/\beta) = f(t), \\ f(t) &\ge f(\alpha t + \beta(s - \alpha t)/\beta) = f(s). \end{aligned}$$

Hence, f is locally constant, and consequently, constant.

Now assume $a \in [\alpha u, 1)$. By (2) the function f is bounded above on an interval $(0, \delta]$ for small enough positive δ, i.e. there exists $M > 0$ such that

$$f(t) \le M, \qquad t \in (0, \delta].$$

Using (1) we obtain by induction

$$f((\alpha + \beta)^n t) \le (a + b)^n M = M, \qquad n \in \mathbb{N},\ t \in (0, \delta],$$

that is,

$$f(t) \le M, \qquad t \in \bigcup_{n=0}^{\infty} (\alpha + \beta)^n (0, \delta] = (0, \infty).$$

Consequently, the above lemma and Lemma 7A.2 with $p = 0$ and $\widetilde{u}$, $\widetilde{w}$ substituted for u, w, respectively, imply that f is convex and satisfies (A6), i.e.

$$f(t) = f(\widetilde{u}^n t), \quad f(t) = f(\widetilde{w}^n t), \qquad t > 0,\, n \in \mathbb{N}.$$

Fix arbitrary $s, t > 0$ and take $n \in \mathbb{N}$ such that $s \in (\widetilde{w}^n t, \widetilde{u}^n t)$. Thus, there exists $\lambda \in [0,1]$ such that $s = \lambda \widetilde{w}^n t + (1-\lambda)\widetilde{u}^n t$ and

$$f(s) = f(\lambda \widetilde{w}^n t + (1-\lambda)\widetilde{u}^n t) \le \lambda f(\widetilde{w}^n t) + (1-\lambda) f(\widetilde{u}^n t) = f(t),$$

which proves that f is constant. Condition (2) implies that f is nonpositive. On the other hand, all nonpositive constant functions satisfy (1) and (2). ∎

7A.5. *Proofs of the remark following Theorem 7.1 and of Corollaries 7.4, 7.5*

Proof of the remark following Theorem 7.1. From convexity and the homogeneity property of f we get

$$\left(\frac{u-1}{u-u^{-1}}\, u^{-p} + \frac{1-u^{-1}}{u-u^{-1}}\, u^{p}\right) f(t) = \frac{u-1}{u-u^{-1}}\, f(u^{-1}t) + \frac{1-u^{-1}}{u-u^{-1}}\, f(ut) \ge f(t), \qquad t > 0.$$

Now, if $p > 1$ then

$$\frac{u-1}{u-u^{-1}}\, u^{-p} + \frac{1-u^{-1}}{u-u^{-1}}\, u^{p} > \left(\frac{u-1}{u-u^{-1}}\, u^{-1} + \frac{1-u^{-1}}{u-u^{-1}}\, u\right)^{p} = 1,$$

and f is nonnegative. If $p < 1$ then

$$\frac{u-1}{u-u^{-1}}\, u^{-p} + \frac{1-u^{-1}}{u-u^{-1}}\, u^{p} < \left(\frac{u-1}{u-u^{-1}}\, u^{-1} + \frac{1-u^{-1}}{u-u^{-1}}\, u\right)^{p} = 1,$$

and f is nonpositive. ∎

Proof of Corollaries 7.4 and 7.5. The critical curves of Corollary 7.4 are established by Theorems 7.1 and 7.3. The critical curves of Corollary 7.5 are established by Theorems 6.1, 7.1, and 7.3. The remaining statements follow from Theorem 4.1 and Lemmas 3.1 and 3.5. ∎

8. Subadditive homogeneous solutions

This section is devoted to (1)–(2) with $\alpha < 1 < \beta$.

PROPOSITION 8.1. *Suppose* $0 < \alpha < 1 < \beta$, $a \ge 1$.

- *If* $b = 1$ *then* $f : (0,\infty) \to \mathbb{R}$ *satisfies* (1)–(2) *if and only if* $f \equiv 0$.
- *If* $b > 1$ *then all solutions of* (1)–(2) *are nonnegative*; *the class of nonnegative solutions is huge*, *and contains strips around power functions with exponents from a neighborhood of* 0.

In particular, $\{(a,b) : b = 1,\ a \ge 1\}$ is contained in the positive critical curve for (α,β). Other parts of critical curves are described by

THEOREM 8.2. *Let* $\alpha, \beta, a, b > 0$, $\alpha < 1 < \beta$, *be such that*

$$\frac{\log a}{\log \alpha} = \frac{\log b}{\log \beta} > 0. \tag{3}$$

Suppose that a function $f : (0,\infty) \to \mathbb{R}$ *satisfies* (2).

If $\log\beta/\log\alpha$ *is irrational, then* f *satisfies* (1) *if and only if* $f(t) = f(1)t^p$, $t > 0$, *where* $p = \log a/\log\alpha$ *and* $f(1) \le 0$ *if* $p > 1$ *and* $f(1) \ge 0$ *if* $p < 1$.

If f *is subadditive and*

$$f(\alpha t) = af(t), \quad f(\beta t) = bf(t), \qquad t > 0, \tag{4}$$

then f *satisfies* (1).

If f *satisfies* (1) *then the function* $f_{\sup+} : \mathbb{R}_+ \to \mathbb{R}$ *given by* $f_{\sup+}(t) = \limsup_{s\to t+} f(s)$ *is subadditive and satisfies* (4).

As a consequence of J. Matkowski's Corollary 7.2(2), *under the assumptions of the above theorem, if* $p = 1$ (*i.e.* $a = \alpha$, $b = \beta$), *then the function* f *is linear.*

The description of solutions of (1) given by Theorem 8.2 is not complete. However, the description is a complete characterization of solutions of a counterpart of (1) for functions defined on $\mathbb{R}_+$, as shown by

THEOREM 8.3. *Let* $\alpha, \beta, a, b > 0$, $\alpha < 1 < \beta$, *satisfy* (3). *Then* $f : \mathbb{R}_+ \to \mathbb{R}$ *satisfies* $(1_{\mathbb{R}_+})$, *i.e.*

$$af(s) + bf(t) \ge f(\alpha s + \beta t), \qquad s, t \ge 0,$$

and satisfies $(2_{\mathbb{R}_+})$, *i.e.*

$$f(0) \le 0 \quad \text{and} \quad \lim_{s\to 0+} f(s) \le 0,$$

if and only if f *is subadditive and* (4) *holds.*

In particular, if $\log\beta/\log\alpha$ *is irrational and* $(2_{\mathbb{R}_+})$ *holds, then each* f *satisfying* $(1_{\mathbb{R}_+})$ *is of the form* $f(t) = f(1)t^p$, $t \ge 0$, *where* $p = \log a/\log\alpha$.

The characterization of solutions of (1) becomes complete also when we impose an additional weak regularity condition—for example the right-continuity of f. The difference between Theorems 8.2 and 8.3 is inherent to the problem. In the case $\log\beta/\log\alpha \in \mathbb{Q}$ there are solutions of (1)–(2) which do not satisfy (4). The following example illustrates this situation:

EXAMPLE 8.4. Assume $0 < \alpha < 1 < \beta$, $a = \alpha^p$, $b = \beta^p$, $p \in (0, 1)$, and $\log\beta/\log\alpha \in \mathbb{Q}$, i.e. there exist relatively prime integers n, m such that $\log\beta/\log\alpha = -n/m$. Put $\nu = \beta^{1/n}$ and note that $\nu > 1$. Finally, take a real $M \in [p, 1)$, a sequence $(f_k)_{k\in\mathbb{Z}}$ such that $f_k \in [1, M\nu + (1 - M)]$, and define the function $f : (0, \infty) \to \mathbb{R}$ by

$$f(t) = \begin{cases} \nu^{pk}(M\nu^{-k}t + (1 - M)), & t \in (\nu^k, \nu^{k+1}), \\ \nu^{pk} f_k, & t = \nu^k, \end{cases} \qquad k \in \mathbb{Z}.$$

Then f is subadditive and satisfies (1); the proof is given at the end of the proof-part of this section. On the other hand, we may choose f_k so that f does not satisfy (4).

Since linear combinations with positive coefficients of solutions of (1)–(2) also satisfy (1)–(2), the above example allows us to find even wilder solutions of (1)–(2). However, the author does not know whether every $f : (0, \infty) \to \mathbb{R}$ satisfying (1)–(2), with parameters considered in Theorem 8.2, must be subadditive.

Making use of Theorems 7.1 and 7.3 and the results of this section we may now completely describe the critical curves for $\alpha < 1 < \beta$.

Corollary 8.5 (cf. Figure 3). *Suppose* $0 < \alpha < 1 < \beta$ *and denote by* (u, w) *the pair of scaling factors for* (α, β). *Then the positive critical curve is given by*

$$b_{\text{positive critical}}(a) = \begin{cases} \beta^{1/\beta w} a^{\alpha u/\beta w}, & a \le \alpha, \\ \beta^{\log a/\log \alpha}, & a \in [\alpha, 1], \\ 1, & a \ge 1, \end{cases}$$

and the negative critical curve is given by

$$b_{\text{negative critical}}(a) = \begin{cases} \beta^{\log a/\log \alpha}, & a \in [0, \alpha], \\ \beta^{1/\beta w} a^{\alpha u/\beta w}, & a \in [\alpha, \alpha u_{\alpha,\beta}], \\ 1 - a, & a \in [\alpha u_{\alpha,\beta}, 1]. \end{cases}$$

- *If* (a, b) *is above the positive critical curve then all solutions to* (1)–(2) *are nonnegative, and the class of positive solutions is huge.*
- *If* (a, b) *is below the positive critical curve and above the negative critical curve then the only solution to* (1)–(2) *is the zero function.*
- *If* (a, b) *is below the negative critical curve then all solutions to* (1)–(2) *are nonpositive, and the class of negative solutions is huge.*

The huge classes of this corollary contain strips around power functions.

Corollary 8.6. *Let* $\alpha, \beta, a, b > 0$ *satisfy* $\alpha < 1 < \beta$ *and* (3).

- *If* $b < \beta$ *then every function* $f : (0, \infty) \to \mathbb{R}$ *satisfying* (1)–(2) *is nonnegative.*
- *If* $b > \beta$ *then every function* $f : (0, \infty) \to \mathbb{R}$ *satisfying* (1)–(2) *is nonpositive.*

8A. Proofs. The proofs of Theorems 8.2 and 8.3 are based on ideas of J. Matkowski, who considered the case $0 < \alpha < 1 < \alpha + \beta$, $a = \alpha$, $b = \beta$. Following Matkowski [1992a] we prove Theorem 8.2 via a right-hand limit function satisfying $(1_{\mathbb{R}_+})$–$(2_{\mathbb{R}_+})$. Lemmas 8A.2 and 8A.3 are simple extensions of the results of Matkowski [1990b].

First, let us introduce the condition

$$\text{(A1)} \qquad a, b > 0, \quad 0 < \alpha < 1 < \beta, \quad p = \frac{\log a}{\log \alpha} = \frac{\log b}{\log \beta} > 0,$$

and, given $f : (0, \infty) \to \mathbb{R}$, define $f_{\sup +} : \mathbb{R}_+ \to \mathbb{R}$ by

$$f_{\sup +}(t) = \limsup_{s \to t+} f(s).$$

If f satisfies (1)–(2) then, according to Theorem 4.4, it is locally bounded, and consequently, $f_{\sup +}$ is well defined, and satisfies $(1_{\mathbb{R}_+})$–$(2_{\mathbb{R}_+})$.

Lemma 8A.1. *Assume* (A1). *Let* $f : (0, \infty) \to \mathbb{R}$ *satisfy* (1)–(2). *If*

$$f_{\sup +}(t) = f_{\sup +}(1) t^p, \qquad t \ge 0,$$

then also

$$f(t) = f_{\sup +}(1) t^p, \qquad t > 0.$$

Proof. Fix $t > 0$, take an increasing sequence $t_n \to \alpha^{-1} t$, and for each n choose an $s_n \in (t_n, \alpha^{-1} t)$ such that

$$f(s_n) \le f_{\sup +}(t_n) + 1/n = f_{\sup +}(1)(t_n)^p + 1/n, \qquad n \in \mathbb{N}.$$

Hence, we have

$$f(t) \le af(s_n) + bf\Big(\frac{t-\alpha s_n}{\beta}\Big) \le af_{\sup +}(1)(t_n)^p + \frac{a}{n} + bf\Big(\frac{t-\alpha s_n}{\beta}\Big).$$

By (2), letting $n \to \infty$, one gets

$$f(t) \le af_{\sup +}(1)(\alpha^{-1}t)^p = f_{\sup +}(1)t^p.$$

Similarly, take a decreasing sequence $t_n \to \alpha t$ and for each n choose an $s_n \in (t_n, t_n + 1/n)$ such that

$$f(s_n) \ge f_{\sup +}(t_n) - 1/n = f_{\sup +}(1)(t_n)^p - 1/n, \qquad n \in \mathbb{N}.$$

Then

$$f(t) \ge a^{-1}\Big(f(s_n) - bf\Big(\frac{s_n - \alpha t}{\beta}\Big)\Big) \ge a^{-1}\Big(f_{\sup +}(1)(t_n)^p - \frac{1}{n} - bf\Big(\frac{s_n - \alpha t}{\beta}\Big)\Big),$$

which for $n \to \infty$ yields $f(t) \ge f_{\sup +}(1)t^p$, and ends the proof. ∎

LEMMA 8A.2. *Suppose* (A1) *and* $\log\beta/\log\alpha \notin \mathbb{Q}$. *If* $f : \mathbb{R}_+ \to \mathbb{R}$ *satisfies* $(1_{\mathbb{R}_+})$–$(2_{\mathbb{R}_+})$ *then* $f(t) = f(1)t^p$, $t \ge 0$.

Proof (cf. Matkowski [1990b]). By Theorem 4.2 and Remark 4.3 there exists a positive constant M such that

$$f(t) \le Mt^p, \qquad t > 0.$$

Applying Lemma 2A.1 with $N = n + m + 1$, $s_{m+1,1} = s$, $s_{m+1,2} = \varepsilon/(\alpha^n\beta^{m+1})$, $s_{i,j} = 0$ for the remaining i, j, we get

$$a^n b^{m+1} f(s) + a^n b^{m+1} f(\varepsilon/(\alpha^n\beta^{m+1})) \ge f(\alpha^n\beta^{m+1}s + \varepsilon)$$

for all $n, m \in \mathbb{N}$ and $s, \varepsilon > 0$. Combining the above two inequalities gives

$$(\alpha^n\beta^{m+1})^p f(s) + M\varepsilon^p \ge f(\alpha^n\beta^{m+1}s + \varepsilon), \qquad n, m \in \mathbb{N},\ s, \varepsilon > 0.$$

According to Lemma 2A.3, the set $D = \{\alpha^n\beta^{m+1} : n, m \in \mathbb{N}\}$ is dense in $(0, \infty)$, and we can write the preceding inequality in the form

$$\text{(A2)} \qquad \lambda^p f(s) + M\varepsilon^p \ge f(\lambda s + \varepsilon), \qquad \lambda \in D,\ s, \varepsilon > 0.$$

Now fix arbitrary $s, t > 0$ and take a sequence (λ_n) such that

$$\lambda_n \in D, \quad \lambda_n < t \quad (n \in \mathbb{N}), \qquad \lim_{n\to\infty} \lambda_n = t.$$

Setting $\varepsilon = (t - \lambda_n)s$ in (A2), we get $\varepsilon > 0$ and

$$(\lambda_n)^p f(s) + M((t-\lambda_n)s)^p \ge f(\lambda_n s + (t-\lambda_n)s) = f(ts), \qquad n \in \mathbb{N}.$$

By (A1) we know that $p > 0$. Letting n tend to infinity we obtain $t^p f(s) \ge f(ts)$, which implies that $t^p f(s) = f(ts)$. Consequently, $f(t) = f(1)t^p$, $t > 0$. ∎

LEMMA 8A.3. *Suppose* (A1) *and* $\log\beta/\log\alpha \in \mathbb{Q}$. *A function* $f : \mathbb{R}_+ \to \mathbb{R}$ *such that* $f(0) \le 0$ *satisfies* $(1_{\mathbb{R}_+})$ *if and only if* f *is subadditive and satisfies* (4).

Proof (cf. Matkowski [1990b]). The "if" part is obvious. In order to prove the "only if" part let us assume that f satisfies (1) and $n, m \in \mathbb{N}$ satisfy $-n/m = \log\beta/\log\alpha$. Let

$s, t \in \mathbb{R}_+$. Substituting $N = n + m$, $s_{m,1} = s$, $s_{m,2} = t$, and $s_{i,j} = 0$ for the remaining indices in Lemma 2A.1, and making use of $\alpha^n \beta^m = 1$, we get

$$f(s) + f(t) \geq f(s+t),$$

i.e., the function f is subadditive.

Similarly, for $t \geq 0$ we have

$$af(t) \geq f(\alpha t),$$

and

$$f(\alpha t) = aa^{n-1}b^m f(\alpha t) \geq af(\alpha^{n-1}\beta^m \alpha t) = af(t),$$

so $af(t) = f(\alpha t)$. Analogously, $bf(t) = f(\beta t)$ for all $t \geq 0$. ∎

Proof of Proposition 8.1. Suppose $\alpha < 1 < \beta$ and $a \geq 1$. If $f(t) < 0$ for some $t > 0$ then, according to Theorem 4.1, the function f would be nonpositive. Since $a \geq 1$, we would get

$$f\left(\frac{\beta}{1-\alpha}t\right) \geq af\left(\frac{\beta}{1-\alpha}t\right) > af\left(\frac{\beta}{1-\alpha}t\right) + bf(t) \geq f\left(\frac{\beta}{1-\alpha}t\right),$$

which is a contradiction. Thus, f is nonnegative.

If $b = 1$ then (1) with $s = 0$ gives

$$f_{\sup +}(t) \geq f_{\sup +}(\beta t), \qquad t \geq 0.$$

The nonnegativity of f and (2) imply that

$$\lim_{s \to 0+} f_{\sup +}(s) = 0,$$

and thus $f_{\sup +} \equiv 0$. Applying Lemma 8A.1, we obtain $f \equiv 0$.

If $b > 1$, then take $p \in (0,1)$ such that $\beta^p < b$. Since also $\alpha^p < 1 \leq a$, we have

$$as^p + bt^p \geq \min(a/\alpha^p, b/\beta^p)(\alpha^p s^p + \beta^p t^p) \geq \min(a/\alpha^p, b/\beta^p)(\alpha s + \beta t)^p, \qquad s, t > 0,$$

and Remark 2.1 shows that the class of solutions of (1)–(2) contains strips around $(0,\infty) \ni t \mapsto t^p$. ∎

Proof of Theorem 8.2. The function $f_{\sup +}$ satisfies $(1_{\mathbb{R}_+})$–$(2_{\mathbb{R}_+})$. So, in the case $\log\beta/\log\alpha \notin \mathbb{Q}$, it is enough to apply consecutively Lemmas 8A.2 and 8A.1. In the case $\log\beta/\log\alpha \in \mathbb{Q}$ it is enough to apply Lemma 8A.3. ∎

Proof of Theorem 8.3. If $\log\beta/\log\alpha \in \mathbb{Q}$ then Lemma 8A.3 gives the assertion. If $\log\beta/\log\alpha \notin \mathbb{Q}$, then Lemma 8A.2 proves one of the implications of the theorem. The remaining implication is obvious. ∎

Proof of Example 8.4. By construction f satisfies (2). In order to prove that f satisfies (1) take $s, t > 0$. Because of the homogeneity-like behavior of f we may assume that $\alpha s + \beta t \in (1, \nu)$. Note that

$$f(t) \geq t^p, \qquad t > 0.$$

Using this observation and the definition of f let us note that:

- If $\alpha s, \beta t \in (1, \nu)$, then

$$\begin{aligned} af(s) + bf(t) &= a\alpha^{-p} f(\alpha s) + b\beta^{-p} f(\beta t) \\ &\ge \alpha^p \alpha^{-p}(M\alpha s + (1-M)) + \beta^p \beta^{-p}(M\beta t + (1-M)) \\ &= M(\alpha s + \beta t) + 2(1-M) > f(\alpha s + \beta t). \end{aligned}$$

- If $\alpha s, \beta t \le 1$, then

$$\begin{aligned} af(s) + bf(t) &\ge as^p + bt^p = (\alpha s)^p + (\beta t)^p \ge (\alpha s) + (\beta t) \\ &= M(\alpha s + \beta t) + (\alpha s + \beta t)(1-M) \\ &\ge M(\alpha s + \beta t) + (1-M) = f(\alpha s + \beta t). \end{aligned}$$

- If $\alpha s \in (1, \nu)$, $\beta t \le 1$, then

$$\begin{aligned} af(s) + bf(t) &\ge \alpha^p \alpha^{-p}(M\alpha s + (1-M)) + \beta^p t^p \ge M\alpha s + (1-M) + \beta t \\ &> M(\alpha s + \beta t) + (1-M) = f(\alpha s + \beta t). \end{aligned}$$

- If $\alpha s \le 1$, $\beta t \in (1, \nu)$, then we may proceed analogously to the preceding case.

This proves (1). The proof of subadditivity of f may be divided into four cases depending on whether s, t are greater or smaller than 1, and carried out analogously. ∎

Proof of Corollary 8.5. The critical curves are established by Theorems 7.1, 7.3, Proposition 8.1, and Theorem 8.2. The remaining statements follow from Theorem 4.1 and Lemmas 3.1 and 3.5. ∎

Proof of Corollary 8.6. Suppose that f satisfies (1)–(2), fix $t > 0$, and take $n \in \mathbb{N}$ and $k \in \mathbb{N}$ such that $\beta^n - k \in [0, 1)$. Then, by Theorem 8.2,

$$b^n f_{\sup +}(t) = f_{\sup +}(\beta^n t) \le k f_{\sup +}(t) + f_{\sup +}((\beta^n - k)t).$$

According to Theorem 4.4, the values $f_{\sup +}((\beta^n - k)t)$ are uniformly bounded above with respect to $n, k \in \mathbb{N}$, and the displayed inequality gives

- nonnegativity of $f_{\sup +}(t)$ if $b < \beta$,
- nonpositivity of $f_{\sup +}(t)$ if $b > \beta$.

Now, the argument used to prove Lemma 8A.1 completes the proof. ∎

9. Remarks about regularity assumptions

This section is devoted to a discussion of the role of the regularity assumption (2) and its counterparts. The examples and remarks build on J. Matkowski [1990b]. The role of regularity assumptions is basically the same in our general case as in the case $0 < \alpha < 1 < \alpha + \beta$, $a = \alpha$, $b = \beta$, considered by Matkowski.

Some regularity conditions imposed on solutions of (1) are necessary to avoid discontinuous additive solutions. This is explained by the results of G. Hamel [1905], M. Fréchet [1913, 1914], J. Aczél [1948], Z. Daróczy [1961, 1964], and L. Losonczi [1964], presented at the end of Section 1. The irregular behavior of discontinuous additive functions is well known.

EXAMPLE 9.1. Let α, β be positive, $p \le 1$, and let $\gamma : \mathbb{R} \to \mathbb{R}$ be a discontinuous additive function homogeneous with respect to α and β. Put $a = \alpha^p$, $b = \beta^p$. Then $f = |\gamma|^p_{|(0,\infty)}$ satisfies (1).

One may be interested in solutions of (1) satisfying

$$\lim_{s\to 0+} f(s) = 0. \tag{5}$$

Roughly speaking, this condition leads to the same theory as the weaker condition (2). If the solutions to (1)–(2) are zero, power, convex, or subadditive, then the solutions to (1) and (5) are zero, power, convex, or subadditive. Condition (5) is satisfied by all critical solutions other than the nonpositive constant solutions of Theorems 5.1, 5.2, and 7.3, and some convex functions of Theorem 5.1. Moreover, all the arguments of Section 3 remain valid for problem (1) and (5). Thus, there is a natural question if substitution of (5) for (2) changes the critical curves, enlarging the trivial regions. It does not, because of Theorems 5.1, 5.2, and 6.5, and the following consequence of Theorem 4.4 and Lemmas 2A.5 and 4A.2:

COROLLARY 9.2. *If* $\alpha+\beta > 1$ *and* $a+b > 1$ *then every solution of* (1)–(2) *satisfies* (5). ■

Another natural question concerns inequality $(1_{\mathbb{R}_+})$. If 0 belongs to the domain of f then assumption $(2_{\mathbb{R}_+})$ can be weakened. We have the following

REMARK 9.3. Let $\alpha, \beta > 0$, $a, b \ge 0$ and $a, \alpha < 1$. Suppose that $f : \mathbb{R}_+ \to \mathbb{R}$ satisfies $(1_{\mathbb{R}_+})$. Then (2) is easily implied by

(i) $f(0) \le 0$;
(ii) f is bounded above in a neighborhood of 0.

Both assumptions (i) and (ii) are necessary as shown in the following two examples.

EXAMPLE 9.4. The function $f : \mathbb{R}_+ \to \mathbb{R}$ given by $f(t) = t^{-1}$, $t > 0$, and $f(0) = 0$ satisfies inequality $(1_{\mathbb{R}_+})$ for positive a, b, α, β such that $a = \alpha$, $b = \beta$ and $0 < \min(a, b) < 1 < a+b$.

EXAMPLE 9.5. Let $a = \alpha = 1/2$, $b = \beta = 2$. The function $f : [0,\infty) \to \mathbb{R}$ given by $f(t) = 1$, $t \in [0, 1)$, and $f(t) = -t$, $t \ge 1$, satisfies inequality (1).

The last example shows that for functions defined on $(0,\infty)$ assumption (2) cannot be replaced by condition (ii) of Remark 9.3. This example also shows that assumption (2) cannot be easily released in Theorem 4.1 about the sign of solutions to (1)–(2).

Also in Theorems 4.2 and 4.4 and Proposition 4.5 condition (2) is indispensable. This is demonstrated by

EXAMPLE 9.6. The following functions satisfy (1) and do not satisfy (2):

- $f : (0,\infty) \to \mathbb{R}_+$ satisfying $(a + b)^{-1}c \le f(t) \le c$, where $c > 0$ is a constant (in the case $a + b \ge 1$);
- $(0,\infty) \ni t \mapsto t^p$, where $p < 0$ is a constant (in the case $a = \alpha^p$, $b = \beta^p$).

10. Multidimensional inequality

This section applies the main results of the paper to study an analogue of inequality (1) for functions defined on cones in linear spaces. Note that the main results of this paper were formulated for functions on $(0,\infty)$. This is the minimal set on which we can study (1) with arbitrary parameters. Indeed, this is the minimal subset of the one-dimensional linear space $\mathbb{R}$ such that, for arbitrary fixed positive α, β, if s, t belong to the subset then $\alpha s + \beta t$ belongs to it. The results formulated on $(0,\infty)$ are easily applicable across many richer contexts. This section will give examples of such applications.

First note that we could take $[0,\infty)$ to be the set of arguments for f without changing much in the theory (cf. Section 9). Moreover, an analogue of Theorem 5.1 remains true on any interval while analogues of Theorems 4.4 (with $\alpha + \beta \leq 1$) and 5.2 are true on intervals that have 0 as an endpoint.

Now, let X denote an arbitrary real (or complex) linear space. A set $C \subset X$ is said to be a *cone* if $C + C = \{x + y : x, y \in C\} \subset C$ and $tC = \{tx : x \in C\} \subset C$ for all $t > 0$. For a function $f : C \to \mathbb{R}$, a counterpart of inequality (1) is

$$(1_C) \qquad af(x) + bf(y) \geq f(\alpha x + \beta y), \qquad x,\, y \in C,$$

and a counterpart of condition (2) is

$$(2_C) \qquad \limsup_{s\to 0+} f(sx) \leq 0, \qquad x \in C.$$

The qualitative behavior of solutions of (1_C)–(2_C) in this general multidimensional case is similar to that of (1)–(2). The argument leading to the division of the space of parameter quadruples (α, β, a, b) into critical manifolds and regions of huge and empty classes of solutions remains valid. The regions of parameters with trivial classes of multidimensional solutions contain the regions of parameters with trivial classes of solutions to (1)–(2).

Theorems 7.1 and 8.3 allow us to characterize p-homogeneous F-pseudonorms and p-convex functions. Before formulating these results let us recall some definitions (cf. S. Rolewicz [1984, pp. 4, 15, 90]).

Let X be a real or complex linear space. By t, t_n we denote scalars, by x, x_k vectors from X, and by p a real number. A functional $f : X \to \mathbb{R}_+$ is called an *F-pseudonorm* if it satisfies the following regularity conditions:

- $f(t_k x) \to 0$ provided $t_k \to 0$,
- $f(t x_k) \to 0$ provided $x_k \to 0$,
- $f(t_k x_k) \to 0$ provided $t_k \to 0$, $x_k \to 0$,
- f is subadditive,

and

$$(6) \qquad f(tx) = f(x) \qquad \text{for } |\, t \,| = 1.$$

An F-pseudonorm $f : X \to \mathbb{R}$ is called a *p-pseudonorm* if it is p-homogeneous, i.e.

- $f(tx) = t^p f(x)$, $t > 0$.

An F-pseudonorm is an *F-norm* if

- $f(x) = 0$ if and only if $x = 0$.

Let $p \in (0,1]$ and C be a convex subset of X. A function $f: C \to \mathbb{R}$ is called *p-convex* if it satisfies

$$t^p f(x) + (1-t)^p f(y) \geq f(tx + (1-t)y), \qquad x, y \in C,\ t \in [0,1].$$

This notion was introduced by W. W. Breckner [1978] who proved that every locally upper bounded p-convex function defined on an open convex subset of a linear topological space is continuous. The relations of p-convexity to other types of convexity were analyzed by H. Hudzik and L. Maligranda [1994]. In the definition of p-pseudonorm the condition of subadditivity may be replaced by p-convexity. Note that for $p \notin (0,1]$ the p-pseudonorm becomes the trivial 0 functional.

Recall that the scaling factors were defined at the beginning of Section 7.

THEOREM 10.1. *Suppose that X is a linear space and C is a cone in X. Let $0 < \alpha < 1 < \alpha + \beta$, and denote by (u, w) the pair of scaling factors for (α, β). If there exists $p > 0$ such that $a = \alpha u^{1-p}$, $b = \beta w^{1-p}$, then $f: C \to \mathbb{R}$ satisfies (1_C) –(2_C) if and only if it is convex, i.e.*

$$tf(x) + (1-t)f(y) \geq f(tx + (1-t)y), \qquad x, y \in C,\ t \in [0,1],$$

and

$$f(ux) = u^p f(x), \qquad f(wx) = w^p f(x), \qquad x \in C.$$

Moreover:

- *If $p \in (0,1)$ then f is nonpositive.*
- *If $p > 1$ then f is nonnegative.*
- *If $p \in (0,1]$, $\log w / \log u \notin \mathbb{Q}$, $C = X$, and f additionally satisfies* (6) *and the regularity conditions of F-pseudonorms, then f is a p-pseudonorm.*

THEOREM 10.2. *Suppose that X is a linear space and C is a cone in X such that $0 \in C$. If $0 < \alpha < 1 < \beta$ and there exists a $p > 0$ such that $a = \alpha^p$, $b = \beta^p$, then $f: C \to \mathbb{R}$ satisfies (1_C)–(2_C) if and only if it is subadditive and satisfies*

$$f(\alpha x) = \alpha^p f(x), \qquad f(\beta x) = \beta^p f(x), \qquad x \in C.$$

Moreover:

- *If $p \in (0,1)$, then f is nonnegative.*
- *If $p > 1$, then f is nonpositive.*
- *If $\log \beta / \log \alpha \notin \mathbb{Q}$, then f is p-convex and*

$$f(tx) = t^p f(x), \qquad x \in C,\ t \in (0, \infty).$$

- *If $\log \beta / \log \alpha \notin \mathbb{Q}$, $p \in (0,1)$, $C = X$, and f additionally satisfies* (6) *and the regularity conditions of F-pseudonorms, then f is a p-pseudonorm.*

THEOREM 10.3 (Matkowski and Pycia [1995a]). *Suppose that X is a linear space and C is a cone in X. If $0 < \alpha < 1 < \alpha + \beta$, $a = \alpha$, $b = \beta$, then $f: C \to \mathbb{R}$ satisfies (1_C)–(2_C) if and only if it is subadditive and positive homogeneous, i.e. $f(tx) = tf(x)$ for $t > 0$, $x \in C$.*

THEOREM 10.4. *Suppose that X is a linear space and $p \in (0,1)$. If a p-convex $f: X \to \mathbb{R}$ satisfies* (6) *and the regularity conditions of F-pseudonorms, and for every $x \in X$ there* *exists a real* $\beta_x > 1$ *such that*

$$f(\beta_x y) \le \beta_x^p f(y), \qquad y \in \{tx : t \in \mathbb{R}\},$$

then f *is a* p*-pseudonorm.*

The idea of Theorem 10.4 comes from J. Matkowski [1993], who showed that *if* X *is a real linear space,* f *is subadditive and satisfies* (6), *for every* $x \in X$ *the function* $(0,\infty) \ni t \mapsto f(tx)$ *is bounded above on an open set, and* $f(\beta x) \le \beta f(x)$ *for some* $\beta_x \in (0,1)$, *then* f *is a seminorm.* Matkowski's proof is based on the theory of subadditive functions. Note that his result is an immediate consequence of Corollary 7.2(2).

10A. Proofs

Proof of Theorem 10.1. For every fixed $x \in C$, Theorem 7.1, applied to the function f restricted to the ray $\{tx : t > 0\}$, gives

$$u^p f(x) = f(ux), \qquad w^p f(x) = f(wx), \qquad x \in C.$$

Thus from (1) we have

$$au^p f(x) + bw^p f(y) = af(ux) + bf(wy) \ge f(\alpha ux + \beta wy), \qquad x, y \in C.$$

Fix $z_1, z_2 \in C$ and consider the function $[0,1] \ni t \mapsto \varphi(t) = f(tz_1 + (1-t)z_2)$. Because u, w are the scaling factors for α, β, we have $\alpha u + \beta w = 1$. Moreover, $au^p = \alpha u$, $bw^p = \beta w$, and the above inequality implies the $(\alpha u, au^p)$-convexity of φ. Now, according to the Kuhn Lemma 5A.2 the function φ is Jensen-convex, hence f is Jensen-convex, and

$$\varphi(t) = f(tz_1 + (1-t)z_2) \le \tfrac{1}{2} f(2tz_1) + \tfrac{1}{2} f(2(1-t)z_2), \qquad t \in (0,1).$$

The functions $(0,\infty) \ni t \mapsto f(2tz_1)$, $(0,\infty) \ni t \mapsto f(2tz_2)$ satisfy (1)–(2), and according to Theorem 4.4, they are bounded on $(0,1)$. Consequently, φ is bounded above and Jensen-convex. It follows from the Bernstein–Doetsch Lemma 2A.6 that φ is convex. Hence, f is convex. Thus the "only if" part is proved. The "if" part is easy, so we omit it.

The first two "moreover" statements follow from the remark after Theorem 7.1, the last one follows from Corollary 7.2(1). ■

Proof of Theorem 10.2. Taking $x \in C$ and restricting the function f to $\{tx : t > 0\}$, we deduce from Theorem 8.3 that

$$af(x) = f(\alpha x), \quad bf(x) = f(\beta x), \qquad x \in C.$$

So, (1) gives

$$f(x) + f(y) = af(x/\alpha) + bf(y/\beta) \ge f(x+y), \qquad x, y \in C,$$

i.e. f is subadditive, and the "only if" statement is proved. The "if" statement is obvious.

The first two "moreover" statements follow from Corollary 8.6. If $\log\beta/\log\alpha \notin \mathbb{Q}$, then Theorem 8.3 gives

$$t^p f(x) = f(tx), \qquad t > 0,\ x \in C,$$

and subadditivity implies that

$$t^p f(x) + (1-t)^p f(y) = f(tx) + f((1-t)y) \ge f(tx + (1-t)y), \qquad t \in (0,1),\ x, y \in C,$$

i.e., f is p-convex. Now, the last two "moreover" statements follow. ■

Proof of Theorem 10.3. The homogeneity of f follows from J. Matkowski's Corollary 7.2(2). Then the subadditivity is a consequence of (1). ■

Proof of Theorem 10.4. Fix an arbitrary $x \in X$ and put $f_x(t) = f(tx)$. Then p-convexity and the condition involving β_x give

$$\lambda^p f_x(s) + \beta_x^p (1-\lambda)^p f_x(t) \geq \lambda^p f_x(s) + (1-\lambda)^p f_x(\beta_x t) \geq f_x(\lambda s + (1-\lambda)\beta_x t)$$

for all $s, t > 0$, $\lambda \in (0,1)$. Taking λ such that $\beta_x \lambda < 1$ and $\beta_x/\lambda \notin \mathbb{Q}$, noting that the first regularity condition of F-pseudonorms implies (2_C), we get from Theorem 7.1 the p-homogeneity of f. Now the subadditivity is a consequence of p-convexity. ■

Figures of critical curves

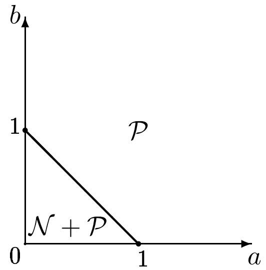


Fig. 1. Dependence of the class of solutions of (1)–(2) on the parameters a, b for fixed $\alpha + \beta < 1$. Cf. Theorem 5.2.

- In the region $\mathcal{P}$ all solutions are nonnegative, and the class of positive solutions is huge.
- In the region $\mathcal{N} + \mathcal{P}$ there are huge classes of both positive and negative solutions; there are also solutions taking both positive and negative values.

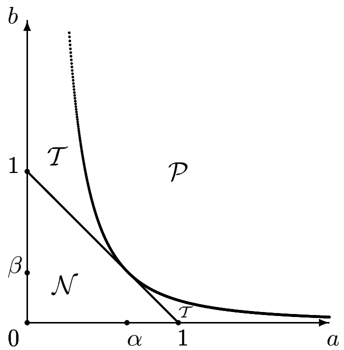


Fig. 2. Dependence of the class of solutions of (1)–(2) on the parameters a, b for fixed $\alpha = 2/3$, $\beta = 1/3$. Cf. Theorem 5.1.

- In $\mathcal{P}$ all solutions are nonnegative, and the class of positive solutions is huge.
- In $\mathcal{N}$ all solutions are nonpositive, and the class of negative solutions is huge.
- In $\mathcal{T}$ only the zero function satisfies (1)–(2).

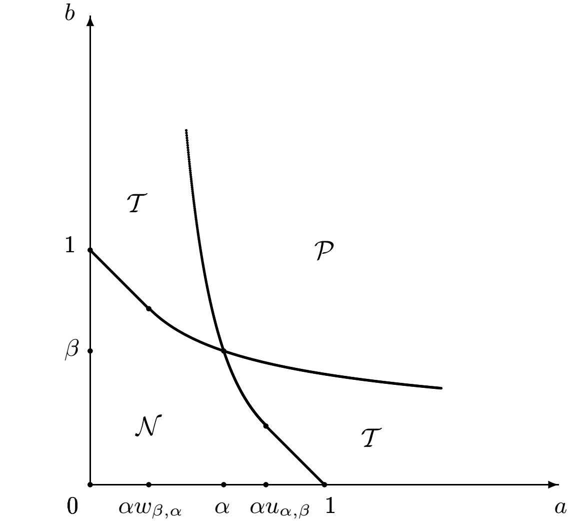


Fig. 3. Dependence of the class of solutions of (1)–(2) on the parameters a, b for fixed $\alpha = 0.57$, $\beta = 0.57$. The parameter $u_{\alpha,\beta}$ is the first scaling factor for (α, β), the parameter $w_{\beta,\alpha}$ is the second scaling factor for (β, α). Cf. Corollary 7.4 and Theorems 7.1, and 7.3.

- In $\mathcal{P}$ all solutions are nonnegative, and the class of positive solutions is huge.
- In $\mathcal{N}$ all solutions are nonpositive, and the class of negative solutions is huge.
- In $\mathcal{T}$ only the zero function satisfies (1)–(2).

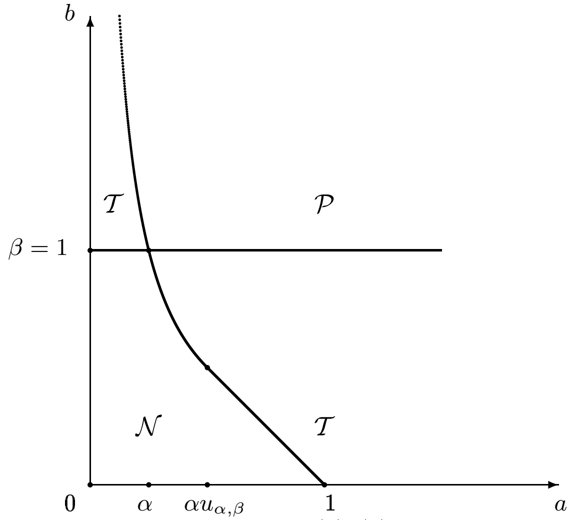
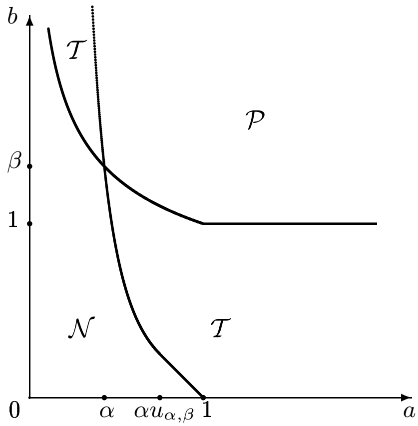


Fig. 4. Dependence of the class of solutions of (1)–(2) on the parameters a, b for fixed $\alpha = 0.25$, $\beta = 1$. The parameter $u_{\alpha,\beta}$ is the first scaling factor for (α, β). Cf. Corollary 7.5 and Theorems 6.1, 7.1, and 7.3.

- In $\mathcal{P}$ all solutions are nonnegative, and the class of positive solutions is huge.
- In $\mathcal{N}$ all solutions are nonpositive, and the class of negative solutions is huge.
- In $\mathcal{T}$ only the zero function satisfies (1)–(2).

Fig. 5. Dependence of the class of solutions of (1)–(2) on the parameters a, b for fixed $\alpha = 0.43$,



$\beta = 1.33$. The parameter $u_{\alpha,\beta}$ is the first scaling factor for (α, β). Cf. Corollary 8.5, Theorems 7.1, 7.3, 8.2, and Proposition 8.1.

- In $\mathcal{P}$ all solutions are nonnegative, and the class of positive solutions is huge.
- In $\mathcal{N}$ all solutions are nonpositive, and the class of negative solutions is huge.
- In $\mathcal{T}$ only the zero function satisfies (1)–(2).

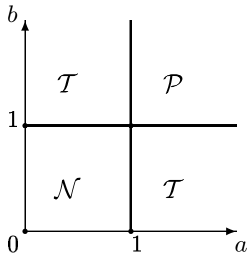


Fig. 6. Dependence of the class of solutions of (1)–(2) on the parameters a, b for fixed $\alpha = \beta = 1$. Cf. Theorems 6.1 and 6.3.

- In $\mathcal{P}$ all solutions are nonnegative, and the class of positive solutions is huge.
- In $\mathcal{N}$ all solutions are nonpositive, and the set of negative solutions is huge.
- In $\mathcal{T}$ the zero function is the only solution to (1)–(2).

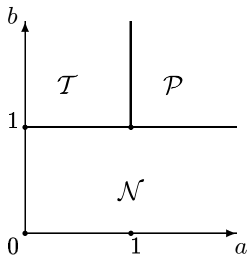


Fig. 7. Dependence of the class of solutions of (1)–(2) on the parameters a, b for fixed $\alpha > \beta = 1$. Cf. Theorems 6.1 and 6.4.

- In $\mathcal{P}$ all solutions are nonnegative, and the class of positive solutions is huge.
- In $\mathcal{N}$ all solutions are nonpositive, and the class of negative solutions is huge.
- In $\mathcal{T}$ the zero function is the only solution to (1)–(2).

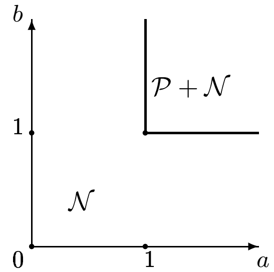


Fig. 8. Dependence of the class of solutions of (1)–(2) on the parameters a, b for fixed $\alpha, \beta > 1$. Cf. Theorem 6.5.

- In $\mathcal{N}$ all solutions are nonpositive, and the class of negative solutions is huge.
- In $\mathcal{N}+\mathcal{P}$ there are huge classes of both positive and negative solutions; there are also solutions taking both positive and negative values.

References

J. Azéma, M. Kaplan-Duflo, D. Revuz

[1966] *Récurrence fine des processus de Markov,* Ann. Inst. H. Poincaré Sect. B (N.S.) 2, 185–220.

[1969] *Propriétés relatives des processus de Markov récurrents*, Z. Wahrsch. Verw. Gebiete 13, 286–314.

J. Aczél

[1948] *Über eine Klasse von Funktionalgleichungen*, Comment. Math. Helv. 21, 247–252.

[1966] *Lectures on Functional Equations and Their Applications*, Academic Press, New York.

K. Baron, J. Matkowski, K. Nikodem

[1994] *A sandwich with convexity*, Math. Pannon. 5, 139–144.

F. Bernstein, G. Doetsch

[1915] *Zur Theorie der konvexen Funktionen*, Math. Ann. 76, 514–526.

E. Berz

[1975] *Sublinear functions on* $\mathbb{R}$, Aequationes Math. 12, 200–206.

W. W. Breckner

[1978] *Stetigkeitsaussagen für eine Klasse verallgemeinerter konvexer Funktionen in topologischen linearen Räumen*, Publ. Inst. Math. (Beograd) 23, 13–20.

A. Bruckner

[1960] *Minimal superadditive extensions of superadditive functions*, Pacific J. Math. 10, 1155–1162.

R. Cooper

[1927] *The converses of the Cauchy–Hölder inequality and the solutions of the inequality* $g(x+y) \leq g(x)+g(y)$, Proc. London Math. Soc. (2) 26, 415–432.

Z. Daróczy

[1961] *Notwendige und hinreichende Bedingungen für die Existenz von nichtkonstanten Lösungen linearer Funktionalgleichungen*, Acta Sci. Math. (Szeged) 22, 31–41.

[1964] *On a class of bilinear functional equations*, Mat. Lapok 15, 52–86 (in Hungarian).

Z. Daróczy, Zs. Páles

[1987] *Convexity with given infinite weight sequences*, Stochastica 11, 5–12.

W. Feller

[1971] *An Introduction to Probability Theory and its Applications*, Vol. II, 2nd ed., Wiley, New York.

M. Fréchet

[1913] *About the functional equation* $f(x+y) = f(x) + f(y)$, Enseign. Math. 15, 390–393 (in Esperanto).

[1914] *À propos d'un article sur l'équation fonctionnelle* $f(x+y) = f(x)+f(y)$, Enseign. Math. 16, 136.

R. K. Getoor

[1990] *Excessive Measures*, Birkhäuser, Boston.

G. Hamel

[1905] *Eine Basis aller Zahlen und die unstetigen Lösungen der Funktionalgleichung* $f(x+y) = f(x) + f(y)$, Math. Ann. 60, 459–462.

G. H. Hardy, J. E. Littlewood, G. Pólya

[1952] *Inequalities*, Cambridge Univ. Presss, Cambridge.

E. Hille

[1948] *Functional Analysis and Semigroups*, Amer. Math. Soc. Colloq. Publ. 31, Amer. Math. Soc., Providence, RI.

E. Hille, R. S. Phillips

[1957] *Functional Analysis and Semigroups*, Amer. Math. Soc. Colloq. Publ. 31, Amer. Math. Soc., Providence, RI.

H. Hudzik, L. Maligranda

[1994] *Some remarks on* s*-convex functions*, Aequationes Math. 48, 100–111.

J. L. W. V. Jensen

[1906] *Sur les fonctions convexes et les inégalités entre les valeurs moyennes*, Acta Math. 30, 175–193.

Z. Kominek

[1992] *On* (a,b)*-convex functions*, Arch. Math. (Basel) 58, 64–69.

Z. Kominek, M. Kuczma

[1989] *Theorems of Bernstein–Doetsch, Piccard and Mehdi and semilinear topology*, Arch. Math. (Basel) 52, 595–602.

S. Kryński

[1993] *Metrically convex functions in normed spaces*, Studia Math. 105, 1–11.

M. Kuczma

[1985] *An Introduction to the Theory of Functional Equations and Inequalities*, PWN–Polish Sci. Publ., Warszawa, and Uniwersytet Śląski, Katowice.

N. Kuhn

[1984] *A note on t-convex functions*, in: General Inequalities 4, Internat. Ser. Numer. Math. 71, Birkhäuser, Basel, 269–276.

[1987] *On the structure of (s,t)-convex functions*, in: General Inequalities 5, Internat. Ser. Numer. Math. 80, Birkhäuser, Basel, 161–174.

L. Losonczi

[1964] *Bestimmung aller nichtkonstanten Lösungen von linearen Funktionalgleichungen*, Acta Sci. Math. (Szeged) 25, 250–254.

J. Matkowski

[1990a] *The converse of the Minkowski's inequality theorem and its generalization*, Proc. Amer. Math. Soc. 109, 663–675.

[1990b] *A functional inequality characterizing convex functions, conjugacy and a generalization of Hölder's and Minkowski's inequalities*, Aequationes Math. 40, 168–180.

[1991] *A generalization of Hölder's and Minkowski's inequalities and conjugate functions*, in: Constantin Carathéodory: An International Tribute, Th. M. Rassias (ed.), World Sci., 819–827.

[1992a] *Functional inequality characterizing nonnegative concave functions in $(0,\infty)^k$*, Aequationes Math. 43, 219–224.

[1992b] *L^p-like paranorms*, in: Selected Topics in Functional Equations and Iteration Theory (Graz, 1991), Grazer Math. Ber. 316, Karl-Franzens-Univ. Graz, 103–138.

[1993] *Subadditive functions and a relaxation of the homogeneity condition of seminorms*, Proc. Amer. Math. Soc. 117, 991–1001.

[1994a] *The converse of the Hölder inequality and its generalizations*, Studia Math. 109, 171–182.

[1994b] *On (α,a)-subadditive functions*, unpublished.

[1996] *On a characterization of L^p-norm and converse of Minkowski's inequality*, Hiroshima Math. J. 26, 277–287.

J. Matkowski, M. Pycia

[1995a] *Convex-like inequality, homogeneity, subadditivity, and a characterization of L^p-norm*, Ann. Polon. Math. 60, 221–230.

[1995b] *On (α,a)-convex functions*, Arch. Math. (Basel) 64, 132–138.

[1996] *Positive homogeneous functionals related to L^p-norms*, J. Math. Anal. Appl. 200, 245–253.

J. Matkowski, T. Świątkowski

[1991] *Quasi-monotonicity, subadditive bijections of $\mathbb{R}_+$, and characterization of L^p-norm*, J. Math. Anal. Appl. 154, 493–506.

[1993a] *On subadditive functions*, Proc. Amer. Math. Soc. 119, 187–197.

[1993b] *Subadditive functions and partial converses of Minkowski's and Mulholland's inequalities*, Fund. Math. 143, 75–85.

[1994] *On (α,a)-subadditive functions*, unpublished.

M. R. Mehdi

[1964] *On convex functions*, J. London Math. Soc. 39, 321–326.

E. Mohr

[1952] *Beitrag zur Theorie der konvexen Funktionen*, Math. Nachr. 8, 133–148.

A. Ostrowski

[1929] *Zur Theorie der konvexen Funktionen*, Comment. Math. Helv. 1, 157–159.

M. Pycia

[1993] *Functional inequalities*, Delta 3/1993, 4–5 (in Polish).
[1997] *Random walks and Wiener–Hopf functional inequalities*, M.Sc. thesis, Warsaw Univ. (in Polish).
[1998] *Linear functional inequalities*, Ph.D. thesis, Inst. Math., Polish Acad. Sci., Warszawa.

S. Rolewicz

[1984] *Metric Linear Spaces*, PWN–Polish Sci. Publ., Warszawa.

R. A. Rosenbaum

[1950] *Subadditive functions*, Duke Math. J. 17, 227–242.

W. Sierpiński

[1920] *Sur les fonctions convexes mesurables*, Fund. Math. 1, 125–128.

W. Wnuk

[1984] *Orlicz spaces cannot be normed analogously to L^p-spaces*, Indag. Math. 46, 357–359.

A. C. Zaanen

[1982] *Some remarks about the definition of an Orlicz space*, in: Lecture Notes in Math. 945, Springer, Berlin, 263–268.