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Abstract

We study the C^*-algebras of Fell bundles. In particular, we prove the analogue of Renault's disintegration theorem for groupoids. As in the groupoid case, this result is the key step in proving a deep equivalence theorem for the C^*-algebras of Fell bundles.

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Introduction

Induced representations and the corresponding imprimitivity theorems constitute a substantial part of the representation theory of locally compact groups and play a critical role in harmonic analysis. These constructs extend naturally to the crossed products of C^*-algebras by locally compact groups as illustrated in [30]. One can push the envelope considerably further to include twisted crossed products of various flavors, and eventually arrive at the Banach $*$-algebraic bundles of Fell. (The latter are discussed in considerable detail in the two-volume treatise of Fell and Doran [5, 6].) There are also extensions of these concepts to locally compact groupoids, groupoid crossed products, and as we will describe here, to the groupoid analogue of Fell's Banach $*$-algebraic bundles over groups, which we call just *Fell bundles*. These bundles, their representation theory and their role in harmonic analysis will be our focus here.

In our approach, induction and imprimitivity are formalized using the Rieffel machinery as described in [24]. Therefore our results are stated in the form of Morita equivalences, and induction is defined algebraically as in [24, §2.4]. In this setting, the central result is Raeburn's Symmetric Imprimitivity Theorem [23; 30, Chap. 4].[1]

However, there are daunting technical difficulties to surmount. For example, the correspondence between representations of a groupoid C^*-algebra and "unitary" representations of the underlying groupoid is, unlike in the group case, a deep result which is known as Renault's Disintegration Theorem [27, Proposition 4.2] (see also, [22, Theorem 7.8] for another proof). The formulation of Renault's Disintegration Theorem is very general, and is designed to facilitate the proof of the Equivalence Theorem [20, Theorem 2.8] which states, in essence, that equivalent groupoids have Morita equivalent C^*-algebras. The equivalence theorem can be extended to groupoid crossed products [27, Corollaire 5.4]. (Groupoid crossed products and Renault's equivalence theorem are discussed at length in [22].) The equivalence theorem for groupoid crossed products is a very powerful imprimitivity type theorem with far reaching consequences. For example, it subsumes Raeburn's Symmetric Imprimitivity Theorem [22, Example 5.12].

In his 1987 preprint, Yamagami proposed a natural generalization of Fell's Banach $*$-algebraic bundles over a locally compact group to a Fell bundle over a locally compact groupoid. He also suggested that there should be a disintegration theorem [31, Theorem 2.1] and a corresponding equivalence theorem [31, Theorem 2.3]. However he gave

[1] The imprimitivity theorem for Fell's Banach $*$-algebraic bundles, [6, Theorem XI.14.17], is not, strictly speaking, a Morita equivalence result. Of course, such a result will be a consequence of our equivalence theorem: Theorem 6.4.

only bare outlines of how proofs might be constructed. The object of this paper is to work out carefully the details of these results in a slightly more general context (as indicated in the next paragraph). As will become clear, there are significant technical hurdles to clear. In the first part of this paper, we want to formalize the notion of a Fell bundle and its corresponding C^*-algebra, and then to prove the disintegration theorem. This will require considerable new technology which we develop here. The remainder of the paper is devoted to describing the appropriate notion of equivalence for Fell bundles, and then to stating and proving the equivalence theorem.

Since a groupoid can only act on objects which are fibered over its unit space, it has been clear from their inception that groupoid crossed products must involve C^*-bundles of some sort. Moving to equivalence theorems and Fell bundles means that we will have to widen the scope to include Banach bundles. Originally, Renault and Yamagami, for example, worked with continuous Banach bundles of the sort studied by Fell (cf. [5, 6]). However, more recently (for example, see [22]), it has become clear that it is unnecessarily restrictive to insist on continuous Banach bundles. Rather, the appropriate notion is what we call an *upper semicontinuous Banach bundle*. Fell called such bundles *loose Banach bundles* ([5, Remark C.1]) and they are called *(H) Banach bundles* by Dupré and Gillette ([4, p. 8]). For convenience, we have collated some of the relevant definitions and results in Appendix A. The case of upper semicontinuous C^*-bundles is treated in more detail in [30, Appendix C]. As an illustration of the appropriateness of upper semicontinuous bundles in the theory, we mention the fact that every $C_0(X)$-algebra A is the section algebra of an upper semicontinuous C^*-bundle over X [30, Theorem C.26]. Furthermore, in practice, most results for continuous bundles extend to upper semicontinuous bundles without significant change.

Extending the groupoid disintegration theorem to the setting of Fell bundles is a formidable task. The first obstacle one faces is the necessity of developing a useful theory of generalized Radon measures for linear functionals on the section algebras of upper semicontinuous Banach bundles. This requires a not-altogether-straightforward extension to our setting of a result of Dinculeanu [3, Theorem 28.32] for continuous Banach bundles. Some not-so-standard facts about complex Radon measures are also required.

We must also cope with the reality that Ramsay's selection theorems ([25, Theorem 5.1] and [26, Theorem 3.2])—which show, for example, that an almost everywhere homomorphism is equal almost everywhere to a homomorphism—are not available for Fell bundles. Instead, we must finesse this with new techniques. (These techniques are valuable in the scalar case as well; for example, they play a role in the proof of Renault's Disintegration Theorem given in Appendix B of [22].)

The "philosophy" of the proof in our setting is that the total space $\mathscr{B}$ of a Fell bundle $p : \mathscr{B} \to G$ over G may be thought of as a groupoid in the category whose objects are C^*-algebras and whose morphisms are (isomorphism classes) of imprimitivity bimodules—the composition being induced by balanced tensor products. The bundle map p may then be viewed as a homomorphism or functor from $\mathscr{B}$ to G. Thus, the "object" mapped to $u \in G^{(0)}$ is the C^*-algebra $B(u) = p^{-1}(u)$ and the "arrow" $x \in G$ is the image of the $B(r(p(x)))$-$B(s(p(x))$-imprimitivity bimodule $B(x) := p^{-1}(x)$. Then the idea is to ramp up the proof in the scalar case to achieve our result. Some support for this point of view is that we can recover the scalar disintegration theorem from our approach—see Theorem B.5. With the disintegration result in hand, the path to a proof of an equivalence theorem is clear—if a bit rocky.

We have made an effort to use standard notation and conventions throughout. Because of the use of direct integrals and measure theory, we want most of the objects with which we work to be separable. In particular, G will always denote a second countable locally compact Hausdorff groupoid with a Haar system $\{\lambda^u\}_{u\in G^{(0)}}$. Homomorphisms between C^*-algebras are always $*$-preserving, and representations of C^*-algebras are assumed to be nondegenerate. If A is a C^*-algebra, then $\widetilde{A}$ is the subalgebra of the multiplier algebra, $M(A)$, generated by A and 1_A; that is, $\widetilde{A}$ is just A if A has an identity, and it is A with an identity adjoined otherwise. The term "pre-compact" is used to describe a set contained in a compact set.

We start in Section 1 with the definition of a Fell bundle $p : \mathscr{B} \to G$ and its associated C^*-algebra $C^*(G, \mathscr{B})$. Although the definition might seem overly technical on a first reading, we hope that the examples in Section 2 show that the definition is in fact easy to apply in practice. In Section 3, we develop the tools necessary to associate measures to linear functionals, which we call *generalized Radon measures*, on the section algebra of an upper semicontinuous Banach bundle. In Section 4 we formalize various notions of representations of Fell bundles and formulate the disintegration result. We give the proof in Section 5. In Section 6 we formulate the equivalence theorem which is our ultimate goal. The proof is given in Section 7 except for some important, but technical details, about the existence of approximate identities of a special form. These details are dealt with in Section 8.

For convenience, we have included a brief appendix on upper semicontinuous Banach bundles (Appendix A). A more detailed treatment is available in the original papers [4, 10–12]. An elementary summary for upper semicontinuous C^*-bundles can be found in [30, Appendix C]. There is also a short appendix (Appendix B) showing how to derive the scalar version of the disintegration theorem from our results.

1. Preliminaries

There are a number of (equivalent) definitions of Fell bundles in the literature. For example, in addition to Yamagami's original definition in [31, Definition 1.1], there is Muhly's version from [19, Definition 6], and more recently Deaconu, Kumjian and Ramazan have advanced one [2, Definition 2.1]. Here we give another variant which is formulated primarily with the goal of being easy to check in examples rather than being the most succinct or most elegant. We hope that the formulation will be validated when we look at examples in Section 2.

Definition 1.1. Suppose that $p : \mathscr{B} \to G$ is a separable upper semicontinuous Banach bundle over a second countable locally compact Hausdorff groupoid G. Let

$$\mathscr{B}^{(2)} := \{(a, b) \in \mathscr{B} \times \mathscr{B} : (p(a), p(b)) \in G^{(2)}\}.$$

We say that $p : \mathscr{B} \to G$ is a *Fell bundle* if there is a continuous, bilinear, associative "multiplication" map $m : \mathscr{B}^{(2)} \to \mathscr{B}$ and a continuous involution $b \mapsto b^*$ such that

(a) $p(m(a,b)) = p(a)p(b)$,
(b) $p(b^*) = p(b)^{-1}$,
(c) $m(a,b)^* = m(b^*,a^*)$,
(d) for each $u \in G^{(0)}$, the Banach space $B(u)$ is a C^*-algebra with respect to the $*$-algebra structure induced by the involution and multiplication m,
(e) for each $x \in G$, $B(x)$ is a $B(r(x))$-$B(s(x))$-imprimitivity bimodule with the module actions determined by m and inner products

$$ {}_{B(r(x))}\langle a,b\rangle = m(a,b^*) \quad \text{and} \quad \langle a,b\rangle_{B(s(x))} = m(a^*,b). $$

We will normally suppress the map m, and simply write ab in place of $m(a,b)$. As pointed out in [19], the axioms imply that $B(x^{-1})$ is (isomorphic to) the dual module to $B(x)$, and that formulas like

$$ \|b^*b\|_{B(s(p(b)))} = \|b\|^2_{B(p(b))} \quad \text{and} \quad b^*b \geq 0 \quad \text{in } B(s(p(b))) $$

hold for all $b \in \mathscr{B}$. Note also that (a) implies that $B(x)B(y) \subset B(xy)$. In fact, we have the following (cf. [19, Definition 6(2)]):

Lemma 1.2. *Multiplication induces an imprimitivity bimodule isomorphism*

$$ B(x) \otimes_{B(s(x))} B(y) \cong B(xy). $$

Proof. Recall that we are suppressing the map m. Since m is bilinear, we obtain a map $\bar{m} : B(x) \odot B(y) \to B(xy)$, where $B(x) \odot B(y)$ denotes the algebraic tensor product of $B(x)$ and $B(y)$ balanced over $B(s(x)) = B(r(y))$, and then

$$ {}_{B(r(x))}\langle \bar{m}(a \otimes b), \bar{m}(c \otimes d)\rangle = (ab)(cd)^* = a(cdb^*)^* = {}_{B(r(x))}\langle a, c\, {}_{B(r(y))}\langle d,b\rangle\rangle. $$

It follows that $\bar{m}$ maps $B(x) \otimes_{B(s(x))} B(y)$ isometrically onto a closed $B(r(x))$-$B(s(x))$-sub-bimodule Y of $B(xy)$. Since ${}_{B(r(x))}\langle Y,Y\rangle$ contains ${}_{B(r(x))}\langle B(x) \odot B(y), B(x) \odot B(y)\rangle$, the ideal of $B(r(x))$ corresponding to Y in the Rieffel correspondence is all of $B(r(x))$. Hence Y must be all of $B(xy)$. ∎

Conventions. In fact, to make formulas like the above easier to read, if $b \in \mathscr{B}$, then we will usually write $s(b)$ in place of the more cumbersome $s(p(b))$ and similarly for $r(b)$.

Given a Fell bundle $p : \mathscr{B} \to G$, we want to make the section algebra $\Gamma_c(G;\mathscr{B})$ into a topological $*$-algebra in the inductive limit topology. The involution is not controversial. We define

$$ f^*(x) := f(x^{-1})^* \quad \text{for } f \in \Gamma_c(G;\mathscr{B}). \tag{1.1} $$

The product is to be given by the convolution formula

$$ f * g(x) := \int_G f(y)g(y^{-1}x)\, d\lambda^{r(x)}(y) \quad \text{for all } f,g \in \Gamma_c(G;\mathscr{B}). \tag{1.2} $$

There is no issue seeing that $f * g(x)$ is a well-defined element of $B(x)$. Clearly $y \mapsto f(y)g(y^{-1}x)$ is in $C_c(G^{r(x)}, B(x))$. Then [30, Lemma 1.91]) implies that the integral converges to an element in the Banach space $B(x)$. However, it is not so clear that $f * g$ is continuous. For this we need the following lemma.

LEMMA 1.3. *Let $G *_r G = \{(x,y) \in G \times G : r(x) = r(y)\}$ and define $q : G *_r G \to G$ by $q(x,y) = x$. Let $q^*\mathscr{B}$ be the pull-back. If $F \in \Gamma_c(G *_r G; q^*\mathscr{B})$, then*

$$f_F(x) := \int_G F(x,y)\, d\lambda^{r(x)}(y)$$

defines a section $f_F \in \Gamma_c(G;\mathscr{B})$.

Proof. A partition of unity argument (see Lemma A.4) implies that sections of the form

$$(x,y) \mapsto \psi(x,y)f(x)$$

for $\psi \in C_c(G *_r G)$ and $f \in \Gamma_c(G;\mathscr{B})$ span a dense subspace of $\Gamma_c(G *_r G; q^*\mathscr{B})$. Since we can approximate ψ in the inductive limit topology by sums of the form $(x,y) \mapsto \varphi_1(x)\varphi_2(y)$, and since $\varphi_1(x)\varphi_2(y)f(x) = \varphi_2(y)(\varphi_1 \cdot f)(x)$, it follows that sections of the form

$$(x,y) \mapsto \varphi(y)f(x)$$

with $\varphi \in C_c(G)$ and $f \in \Gamma_c(G;\mathscr{B})$ span a dense subspace $\mathcal{A}_0$ of $\Gamma_c(G *_r G; q^*\mathscr{B})$. Since $f_{F_i} \to f_F$ uniformly if $F_i \to F$ in the inductive limit topology in $\Gamma_c(G *_r G; q^*\mathscr{B})$, it suffices to show that f_F is continuous when $F \in \mathcal{A}_0$. But if $F(x,y) = \varphi(y)f(x)$, then

$$f_F(x) = \Big(\int_G \varphi(y)\, d\lambda^{r(x)}(y)\Big) f(x) = \lambda(\varphi)(r(x))f(x)$$

which is clearly in $\Gamma_c(G;\mathscr{B})$ (because $\lambda(\varphi)$ is in $C_c(G^{(0)})$ since $\{\lambda^u\}$ is a Haar system). ■

COROLLARY 1.4. *If f and g are in $\Gamma_c(G;\mathscr{B})$, then so is $f * g$.*

Proof. Note that $(x,y) \mapsto f(y)g(y^{-1}x)$ defines a section in $\Gamma_c(G *_r G; q^*\mathscr{B})$. ■

Just as for groupoid algebras, the *I-norm* on $\Gamma_c(G;\mathscr{B})$ is given by

$$\|f\|_I = \max\Big(\sup_{u\in G^{(0)}} \int_G \|f(x)\|\, d\lambda^u(x), \sup_{u\in G^{(0)}} \int_G \|f(x)\|\, d\lambda_u(x)\Big). \tag{1.3}$$

A $*$-homomorphism $L : \Gamma_c(G;\mathscr{B}) \to B(\mathcal{H}_L)$ is called an *I-norm decreasing representation* if $\|L(f)\| \le \|f\|_I$ for all $f \in \Gamma_c(G;\mathscr{B})$ and if $\operatorname{span}\{L(f)\xi : f \in \Gamma_c(G;\mathscr{B}) \text{ and } \xi \in \mathcal{H}_L\}$ is dense in $\mathcal{H}_L$. The *universal C^*-norm* on $\Gamma_c(G;\mathscr{B})$ is given by

$$\|f\| := \sup\{\|L(f)\| : L \text{ is an } I\text{-norm decreasing representation}\}.$$

The completion of $\Gamma_c(G;\mathscr{B})$ with respect to the universal norm is denoted by $C^*(G,\mathscr{B})$.

2. Examples

We want to review some of the examples of Fell bundles described by Muhly in Example 7 of [19, §3]. At the same time, we want to add a bit of detail, and make a few alterations.

EXAMPLE 2.1 (Groupoid crossed products). Let $\pi : \mathscr{A} \to G^{(0)}$ be an upper semicontinuous C^*-bundle over $G^{(0)}$. We assume that $(\mathscr{A}, G, \alpha)$ is a groupoid dynamical system. Unlike the treatment in [19, Example 7(1)] where the focus is on $s^*\mathscr{A}$, we want to work with the pull-back $\mathscr{B} := r^*\mathscr{A} = \{(a,x) : \pi(a) = r(x)\}$. The first step is to define a multiplication on $\mathscr{B}^{(2)} := \{((a,x),(b,y)) \in \mathscr{B}\times\mathscr{B} : (x,y) \in G^{(2)}\}$ as follows:

$$(a,x)(b,y) := (a\alpha_x(b), xy). \tag{2.1}$$

(This formula looks "even more natural" if we write $x\cdot b$ in place of $\alpha_x(b)$.) The involution is given by

$$(a,x)^* := (\alpha_x^{-1}(a^*), x^{-1}). \tag{2.2}$$

To verify that $B(x) = \{(a,x) : a \in A(r(x))\}$ is an $A(r(x))$-$A(s(x))$-imprimitivity bimodule, we proceed as follows. Keep in mind that the left $A(r(x))$- and right $A(s(x))$-actions are determined by (2.1). Since $\alpha_u = \mathrm{id}$ if $u \in G^{(0)}$, we have

$$b\cdot(a,x) = (b,r(x))(a,x) = (ba,x), \qquad (a,x)\cdot c = (a,x)(c,s(x)) = (a\alpha_x(c),x).$$

Again, by axiom, the inner products are supposed to be given by

$$\begin{aligned}\langle (a,x),(b,x)\rangle_{B(s(x))} &= (a,x)^*(b,x) = (\alpha_x^{-1}(a^*b), s(x)),\\ {}_{B(r(x))}\langle (a,x),(b,x)\rangle &= (a,x)(b,x)^* = (ab^*, r(x))\end{aligned}$$

Of course, these are the natural inner products and actions on $A(r(x))$ making it into an imprimitivity bimodule.[2]

Remark 2.2. Notice that a section $f \in \Gamma(G;\mathscr{B})$ is determined by a function $\check{f} : G \to \mathscr{A}$ such that $\check{f}(x) \in A(r(x))$. Then $f(x) = (\check{f}(x), x)$. We will often not distinguish between f and $\check{f}$.

Example 2.3 (Twists). The notion of a twist E over G, or a $\mathbb{T}$-groupoid over G, is due to Kumjian. Recall that E must be a principal circle bundle, say $j : E \to G$, over G and that E is also equipped with a groupoid structure such that

$$G^{(0)} \to G^{(0)} \times \mathbb{T} \xrightarrow{i} E \xrightarrow{j} G \to G^{(0)}$$

is a groupoid extension such that $t\cdot e = i(r(e),t)e$ and $(t\cdot e)(s\cdot f) = (ts)\cdot ef$ (see [21, p. 115]). In this case, we let $\mathscr{B}$ be the complex line bundle over G associated to E.[3] The multiplication on $\mathscr{B}^{(2)}$ goes as follows. We can view $\mathscr{B}$ as the quotient of $E\times\mathbb{C}$ by the action of $\mathbb{T}$ given by $z(e,\lambda) := (ze, \bar{z}\lambda)$. Then the product is just $[e,\lambda][f,\beta] := [ef, \lambda\beta]$.

Remark 2.4. Note that we can view $E \subset \mathscr{B}$ (in the model above, just send e to $[e,1]$). Furthermore, sections of $\mathscr{B}$ are identified in a natural way with continuous $\mathbb{C}$-valued functions $\check{f}$ on E which transform as follows:

$$\check{f}(ze) = \bar{z}\check{f}(e). \tag{2.3}$$

The section $f \in \Gamma(G;\mathscr{B})$ associated to $\check{f}$, transforming as in (2.3), is given by

$$f(j(e)) = \check{f}(e)e := [e, \check{f}(e)]. \tag{2.4}$$

[2] Recall that if $\theta : A \to B$ is an isomorphism, then there is a natural way to make A into an A-B-imprimitivity bimodule usually denoted A_θ.

[3] One obtains $\mathscr{B}$ from E by taking the $\mathbb{T}$-valued transition functions for E, and viewing them as $\mathrm{GL}_1(\mathbb{C})$-valued transition functions for a complex vector bundle. Thus $\mathscr{B}$ is the quotient of $\coprod U_i \times \mathbb{C}$ where $(i,x,\lambda) \sim (j,x,\sigma_{ij}(x)\lambda)$. Since the $\sigma_{ij}(x)$ act by multiplication, we get a vector bundle rather than a principal bundle with respect to the additive group action on the fibers.

EXAMPLE 2.5 (Green–Renault). As pointed out in [19, Example 7(3)], Examples 2.1 and 2.3 are subsumed by Renault's formalism from [27, 28]. In this case, we have a groupoid extension

$$G^{(0)} \to S \xrightarrow{i} \Sigma \xrightarrow{j} G \to G^{(0)}$$

of locally compact groupoids over $G^{(0)}$ where S is a group bundle of *abelian* groups with Haar system. We view S as a closed subgroupoid of Σ.

In the spirit of Green twisted dynamical systems, we assume that we have a groupoid dynamical system $(\mathscr{A}, \Sigma, \alpha)$ (so that $\pi : \mathscr{A} \to G^{(0)} = \Sigma^{(0)}$ is an upper semicontinuous C^*-bundle). We also need an element $\chi \in \prod_{s\in S} M(A(r(s)))$ such that

$$(s, a) \mapsto \chi(s)a \tag{2.5}$$

is continuous from $S * \mathscr{A} := \{(s, a) : r(s) = \pi(a)\}$ to $\mathscr{A}$, and such that

$$\alpha_s(a) = \chi(s)a\chi(s)^* \quad \text{for all } (s, a) \in S * \mathscr{A}, \tag{2.6}$$

$$\chi(\sigma s\sigma^{-1}) = \bar{\alpha}_\sigma(\chi(s)) \quad \text{for } (\sigma, s) \in \Sigma^{(2)}. \tag{2.7}$$

A little computation shows that $a\chi(s)^* = (\chi(s)a^*)^*$, and it follows that

$$(s, a) \mapsto a\chi(s)^* \tag{2.8}$$

is continuous. Therefore we can define an S-action on $r^*\mathscr{A} = \{(a, \sigma) : \pi(a) = r(\sigma)\}$:

$$(a, \sigma) \cdot s := (a\chi(s)^*, s\sigma). \tag{2.9}$$

We define $\mathscr{B}$ to be the quotient $r^*\mathscr{A}/S$, and define $p : \mathscr{B} \to G$ by $p([a, \sigma]) = j(\sigma)$.

LEMMA 2.6. *With the set-up above, $p : \mathscr{B} \to G$ is an upper semicontinuous Banach bundle over G.*

Proof. The proof is obtained by modifying the proof of [15, Proposition 2.15]. Specifically, we have to show that p is open, and that axioms B1–B4 of Definition A.1 are satisfied.[4] ∎

To get a Fell bundle, we will need a multiplication on

$$\mathscr{B}^{(2)} := \{([a, \sigma], [b, \tau]) : (j(\sigma), j(\tau)) \in G^{(2)}\}.$$

Since $(j(\sigma), j(\tau)) \in G^{(2)}$ exactly when $(\sigma, \tau) \in \Sigma^{(2)}$, we want to try

$$[a, \sigma][b, \tau] := [a\alpha_\sigma(b), \sigma\tau]. \tag{2.10}$$

To see that (2.10) is well-defined, consider

$$\begin{aligned}
[a\chi(s)^*, s\sigma][b\chi(t)^*, t\tau] &= [a\chi(s)^*\alpha_{s\sigma}(b\chi(t)^*), s\sigma t\tau] \\
&= [a\alpha_\sigma(b\chi(t)^*)\chi(s)^*, s\sigma t\sigma^{-1}\sigma\tau] && \text{(by (2.6))} \\
&= [a\sigma_\sigma(b)\chi(\sigma t\sigma^{-1})^*\chi(s)^*, (s\sigma t\sigma^{-1})\sigma\tau] && \text{(by (2.7))} \\
&= [a\alpha_\sigma(b)\chi(s\sigma t\sigma^{-1})^*, (s\sigma t\sigma^{-1})\sigma\tau] \\
&= [a\alpha_\sigma(b), \sigma\tau].
\end{aligned}$$

Thus (2.10) is well-defined and we can establish the following lemma without difficulty.

[4] In [15, Proposition 2.15], we are working with C^*-bundles. Here we have to adjust to upper semicontinuous bundles which are merely Banach bundles.

Lemma 2.7. *With respect to the multiplication defined above, $p : \mathscr{B} \to G$ is a Fell bundle over G.*

To get a section of $\mathscr{B}$, we need a continuous function $f : \Sigma \to \mathscr{A}$ such that

$$f(\sigma) \in A(r(\sigma)), \tag{2.11}$$

$$f(s\sigma) = f(\sigma)\chi(s)^* \quad \text{for } (s, \sigma) \in \Sigma^{(2)}. \tag{2.12}$$

The corresponding section is given by

$$\check{f}(j(\sigma)) := [f(\sigma), \sigma]. \tag{2.13}$$

Now recall that the set $\Gamma_c(G; \mathscr{B})$ of continuous compactly supported sections is endowed with a $*$-algebra structure as follows:

$$f * g(x) := \int_G f(y)g(y^{-1}x)\, d\lambda^{r(x)}(y), \tag{2.14}$$

$$f^*(x) := f(x^{-1})^*. \tag{2.15}$$

It is a worthwhile exercise to look a bit more closely at the $*$-algebra $\Gamma_c(G; \mathscr{B})$ in each of the basic examples above.

Example 2.8 (The crossed product for Example 2.1). Let $p : \mathscr{B} \to G$ be the Fell bundle associated to a dynamical system $(\mathscr{A}, G, \alpha)$ as in Example 2.1. Let $f, g \in \Gamma_c(G; \mathscr{B})$, and let $\check{f}$ and $\check{g}$ be the corresponding $\mathscr{A}$-valued functions on G as in Remark 2.2. Then

$$f * g(x) = \int_G (\check{f}(y), y)(g(y^{-1}x), y^{-1}x)\, d\lambda^{r(x)}(y) = \int_G (\check{f}(y)\alpha_y(\check{g}(y^{-1}x)), x)\, d\lambda^{r(x)}(y).$$

Similarly,

$$f^*(x) = f(x^{-1})^* = (\check{f}(x^{-1}), x^{-1})^* = (\alpha_x(f(x^{-1}))^*, x)$$

Thus if we confound f and $\check{f}$, as is usually the case, we obtain the "usual" convolution formula:

$$f * g(x) := \int_G f(y)\alpha_y(g(y^{-1}x))\, d\lambda^{r(x)}(y), \tag{2.14$'$}$$

and the "usual" involution formula:

$$f^*(x) = \alpha_x(f(x^{-1})^*). \tag{2.15$'$}$$

In this case, after completing as in Section 4, we obtain the crossed product $\mathscr{A} \rtimes_\alpha G$ (or $A \rtimes_\alpha G$). (For more on groupoid crossed products, see [22].)

Example 2.9 (The crossed product for Example 2.3). In this case, we work with functions on E transforming as in (2.3). Then

$$f * g(e) = \int_G [\check{f}(d)\check{g}(d^{-1}e), e]\, d\lambda^{r(e)}(j(d)),$$

and $f * g$ is represented by the function on E given by

$$e \mapsto \int_G \check{f}(d)\check{g}(d^{-1}e)\, d\lambda^{r(j(e))}(j(d)).$$

Thus the completion is the algebra $C^*(G; E)$ as in [21].

Example 2.10 (The crossed product for Example 2.5). Here we replace f with $\check{f}$. Then the functions transform as in (2.12), and the operations on functions on Σ are given by

$$f * g(\sigma) = \int_G f(\tau)\alpha_\tau(g(\tau^{-1}\sigma)\,d\lambda^{r(j(\sigma))}(\tau), \qquad f^*(\sigma) = \alpha_\sigma(f(\sigma^{-1})^*).$$

The completion is Renault's $C^*(G, \Sigma, \mathscr{A}, \lambda)$ from [27, 28].

3. Generalized Radon measures on Fell bundles

For the proof of the disintegration theorem for representations of Fell bundles (Theorem 4.13), we will need a version of Yamagami's [31, Lemma 2.2] suitable for upper semicontinuous Banach bundles. Note that [31, Lemma 2.2] is intended to be a restatement of [3, Theorem 28.32] due to Dinculeanu. In fact, there is a bit of work to do just to coax out the result Yamagami claims in the (continuous) Banach bundle case from [3, Theorem 28.32]. Therefore it seems more than reasonable to work out the details of the more general result here.

First we need some terminology and notation. Let $p : \mathscr{B} \to G$ be an upper semicontinuous Banach bundle over a *second countable* Hausdorff groupoid G such that the corresponding Banach space $B := \Gamma_0(G; \mathscr{B})$ is *separable*. We call a linear functional

$$\nu : \Gamma_c(G; \mathscr{B}) \to \mathbb{C}$$

a *generalized Radon measure* provided that ν is continuous in the inductive limit topology. Of course, if $\mathscr{B}$ is the trivial (complex) line bundle over G, then a generalized Radon measure is just a complex Radon measure. Some useful comments on complex Radon measures can be found in [22, Appendix A.1].

Lemma 3.1. *Let $\nu : \Gamma_c(G; \mathscr{B}) \to \mathbb{C}$ be a generalized Radon measure on G. Then there is a Radon measure μ on G such that for all $\varphi \in C_c^+(G)$,*

$$\mu(\varphi) := \sup\{|\nu(f)| : \|f\| \le \varphi\}. \tag{3.1}$$

Proof. We will produce a function $\mu : C_c^+(G) \to \mathbb{R}^+$ satisfying (3.1) and such that for all $\alpha \ge 0$ and $\varphi_i \in C_c^+(G)$ we have

(a) $\mu(\alpha\varphi_1) = \alpha\mu(\varphi_1)$,
(b) $\mu(\varphi_1 + \varphi_2) = \mu(\varphi_1) + \mu(\varphi_2)$.

Then it is not hard to see that we can extend μ to all of $C_c(G)$ in the expected way (cf. [3, Proposition 2.20] or [9, Theorem B.38]).

Naturally, we define μ on $C_c^+(G)$ using (3.1). Then part (a) follows immediately from the definition of μ. Note that if $K \subset G$ is compact, then the continuity of ν implies that there is a constant $a_K \ge 0$ such that

$$|\nu(f)| \le a_K\|f\|_\infty \quad \text{for all } f \in \Gamma_c(G; \mathscr{B}) \text{ such that } \operatorname{supp} f \subset K.$$

(If not, then for each n we can find f_n with $\operatorname{supp} f_n \subset K$, $\|f_n\|_\infty \le 1$ and $|\nu(f_n)| \ge n^2$. Then $\frac{1}{n}f_n$ tends to 0 in the inductive limit topology, while $\nu(\frac{1}{n}f_n) \not\to 0$.) It follows that $\mu(\varphi) < \infty$ for all $\varphi \in C_c^+(G)$.

Fix $\varphi_1, \varphi_2 \in C_c^+(G)$ and $\epsilon > 0$. Choose $f_i \in \Gamma_c(G; \mathscr{B})$ such that $\|f_i\| \leq \varphi_i$ and

$$\mu(\varphi_i) \leq |\nu(f_i)| + \epsilon/2.$$

Let τ_i be a unimodular scalar such that $|\nu(f_i)| = \nu(\tau_i f)$. Then

$$\|\tau_1 f_1 + \tau_2 f_2\| \leq \|f_1\| + \|f_2\| \leq \varphi_1 + \varphi_2.$$

Thus

$$\mu(\varphi_1) + \mu(\varphi_2) \leq |\nu(f_1)| + |\nu(f_2)| + \epsilon = \nu(\tau_1 f_1 + \tau_2 f_2) + \epsilon = \mu(\varphi_1 + \varphi_2) + \epsilon.$$

Since ϵ was arbitrary, $\mu(\varphi_1) + \mu(\varphi_2) \leq \mu(\varphi_1 + \varphi_2)$.

Now suppose that $h \in \Gamma_c(G; \mathscr{B})$ is such that $\|h\| \leq \varphi_1 + \varphi_2$. Define

$$\sigma_1(x) := \begin{cases} \dfrac{\varphi_1(x)}{\varphi_1(x) + \varphi_2(x)} & \text{if } \varphi_1(x) + \varphi_2(x) > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Let $h_i := \sigma_i \cdot h$. We want to see that each $h_i \in \Gamma_c(G; \mathscr{B})$. For this, we just need to see that $x \mapsto h_i(x)$ is continuous from G to $\mathscr{B}$. Fix $x_0 \in G$. If $h(x_0) \neq 0$, then $\varphi_1(x_0) + \varphi_2(x_0) > 0$ and $\varphi_1(x) + \varphi_2(x) > 0$ near x_0. Consequently, σ_i is continuous near x_0. Therefore h_i is continuous at x_0 (see [30, Proposition C.17]). On the other hand, suppose that $h(x_0) = 0$ and $\epsilon > 0$. Since $x \mapsto \|h(x)\|$ is upper semicontinuous, there is a neighborhood V of x_0 such that $\|h(x)\| < \epsilon$ for all $x \in V$. However, $|\sigma_i(x)| \leq 1$ for all x. Hence

$$\|h_i(x)\| = |\sigma_i(x)| \, \|h(x)\| \leq \|h(x)\| < \epsilon \quad \text{provided } x \in V.$$

It follows that h_i is continuous at x_0 (for example, see axiom B4 of Definition A.1).

Furthermore, if $h_i(x) \neq 0$, then $\varphi_1(x) + \varphi_2(x) > 0$ and

$$\|h_i(x)\| = \varphi_i(x) \frac{1}{\varphi_1(x) + \varphi_2(x)} \|h(x)\| \leq \varphi_i(x),$$

since $\|h(x)\| \leq \varphi_i(x) + \varphi_2(x)$. Thus if $\|h\| \leq \varphi_1 + \varphi_2$, then there are h_i such that $h = h_1 + h_2$ and $\|h_i\| \leq \varphi_i$. Therefore we can compute as follows:

$$\begin{aligned} \mu(\varphi_1 + \varphi_2) &= \sup\{|\nu(h)| : \|h\| \leq \varphi_1 + \varphi_2\} = \sup\{|\nu(h_1 + h_2)| : \|h_i\| \leq \varphi_i\} \\ &\leq \sup\{|\nu(h_1)| + |\nu(h_2)| : \|h_i\| \leq \varphi_i\} \leq \mu(\varphi_1) + \mu(\varphi_2). \end{aligned}$$

This establishes (b). Since G is second countable and since we established that μ was finite on $C_c(G)$, μ is a Radon measure (by, for example, [29, Theorem 2.18]). ■

EXAMPLE 3.2. Suppose that $\nu : C_c(G) \to \mathbb{C}$ is a *complex* Radon measure on G. Then the measure μ associated to ν via (3.1) in Lemma 3.1 is the total variation $|\nu|$ of ν.

Proof. It suffices to see that for all $\varphi \in C_c^+(G)$ we have

$$|\nu|(\varphi) = \sup\{|\nu(f)| : \|f\| \leq \varphi\}.$$

Since $\nu = \tau|\nu|$ for a unimodular function τ (see [7, Proposition 3.13] in the bounded case and [22, Appendix A.1] in general) and since we can let $f = \bar{\tau}\varphi$, we clearly get equality. ■

In view of Example 3.2, we call the measure μ appearing in Lemma 3.1 the *total variation* of the generalized Radon measure ν and write $|\nu|$ in place of μ. Then what we need to prove is the following.

PROPOSITION 3.3. *Suppose that $p : \mathscr{B} \to G$ is an upper semicontinuous Banach bundle over a second countable locally compact Hausdorff space G such that the section algebra $B := \Gamma_0(G; \mathscr{B})$ is separable.*[5] *If $\nu : \Gamma_c(G; \mathscr{B}) \to \mathbb{C}$ is a generalized Radon measure on $\mathscr{B}$ with total variation $|\nu|$, then for all $x \in G$ there are linear functionals $\epsilon_x \in B(x)^*$ of norm at most one such that*

(a) *for each $f \in \Gamma_c(G; \mathscr{B})$, $x \mapsto \epsilon_x(f(x))$ is in $\mathcal{B}_c^b(G)$,*[6]
(b) *$\nu(f) = \int_G \epsilon_x(f(x))\, d|\nu|(x)$ for all $f \in \Gamma_c(G; \mathscr{B})$.*

Before proceeding with the proof, we need to deal with the reality that—unlike the case in [3] where continuous Banach bundles are used—$\|f\|$ need not be in $C_c(G)$ if $f \in \Gamma_c(G; \mathscr{B})$. But it is at least upper semicontinuous (by axiom B1 of Definition A.1). Therefore $x \mapsto \|f(x)\|$ is in $\mathcal{B}_c^b(G)$. In particular, it is integrable with respect to any Radon measure on G.

We need the following observations.

LEMMA 3.4. *Suppose that f is a bounded nonnegative upper semicontinuous function with compact support on G. Then if μ is a Radon measure on G,*

$$\int_G f(x)\, d\mu = \inf\Big\{ \int_G g(x)\, d\mu(x) : g \in C_c^+(G) \text{ and } f \le g \Big\}. \tag{3.2}$$

Proof. Fix $\varphi \in C_c^+(G)$ such that $f \le \varphi$. Then $\varphi - f$ is a nonnegative *lower* semicontinuous function on G. By [7, Corollary 7.13], given $\epsilon > 0$, there is a $g \in C_c(G)$ such that $0 \le g \le \varphi - f$ and

$$\int_G (\varphi(x) - f(x))\, d\mu(x) \le \int_G g(x)\, d\mu(x) + \epsilon.$$

But then

$$\int_G (\varphi(x) - g(x)) \le \int_G f(x)\, d\mu(x) + \epsilon.$$

Since $f \le \varphi - g \le \varphi$, we have $\varphi - g \in C_c(G)$ and dominates f. Since $\epsilon > 0$ was arbitrary, the right-hand side of (3.2) is at least the left-hand side. The other inequality is clear, so the result is proved. ∎

COROLLARY 3.5. *Suppose that $\nu : \Gamma_c(G; \mathscr{B}) \to \mathbb{C}$ is a generalized Radon measure on G. Then for all $f \in \Gamma_c(G; \mathscr{B})$,*

$$|\nu(f)| \le |\nu|(\|f\|).$$

In particular, if μ is any Radon measure on G such that $|\nu(f)| \le \mu(\|f\|)$ for all $f \in \Gamma_c(G; \mathscr{B})$, then $|\nu|(\varphi) \le \mu(\varphi)$ for all $\varphi \in C_c^+(G)$.

Proof. Suppose that $\varphi \in C_c^+(G)$ and $\|f\| \le \varphi$. Then by definition, $|\nu|(\varphi) \ge |\nu(f)|$. By the previous lemma,

$$|\nu|(\|f\|) = \inf\{ |\nu|(\varphi) : \varphi \in C_c^+(G) \text{ and } \|f\| \le \varphi \} \ge |\nu(f)|.$$

---

[5] This hypothesis of separability seems to be crucial. In the proof we will have to collect null sets for a countable dense subset reminiscent of direct integral type arguments.

[6] We use $\mathcal{B}_c^b(G)$ to denote the bounded Borel functions on G which vanish off a compact set.

Now suppose that μ is as above. Then for all $\varphi \in C_c^+(G)$,

$$|\nu|(\varphi) = \sup\{|\nu(f)| : \|f\| \le \varphi\} \le \sup\{\mu(\|f\|) : \|f\| \le \varphi\} \le \mu(\varphi). \ \blacksquare$$

Proof of Proposition 3.3. Suppose $\nu : \Gamma_c(G;\mathscr{B}) \to \mathbb{C}$ is a generalized Radon measure as in the statement of the proposition. For each $f \in \Gamma_c(G;\mathscr{B})$, define a (scalar-valued) complex Radon measure on G by $\nu_f(\varphi) := \nu(\varphi \cdot f)$. Then by Corollary 3.5,

$$|\nu_f(\varphi)| = |\nu(\varphi \cdot f)| \le |\nu|(|\varphi|\,\|f\|). \tag{3.3}$$

Therefore

$$|\nu_f(\varphi)| \le \|f\|_\infty \|\varphi\|_1, \tag{3.4}$$

where $\|\varphi\|_1 = |\nu|(|\varphi|)$ is the norm of φ in $L^1(G,|\nu|)$.

It follows that $f \mapsto \nu_f$ is a bounded linear map Φ of $\Gamma_c(G;\mathscr{B}) \subset B = \Gamma_0(G;\mathscr{B})$ into $L^1(|\nu|)^*$. Of course, we can identify $L^1(|\nu|)^*$ with $L^\infty(|\nu|)$. Then, since B is separable and Φ is bounded, we can identify $S := \Phi(\Gamma_c(G;\mathscr{B}))$ with a separable subspace of $L^\infty(|\nu|)$. By [30, Lemma I.8], there is a linear map $\rho : S \to \mathcal{B}^b(G)$ such that the $|\nu|$-almost everywhere equivalence class of $\rho(b)$ is b.[7]

Thus if we let $b_f := \rho(\nu_f)$, then b_f is a bounded Borel function such that

$$\nu_f(\varphi) = \int_G \varphi(x) b_f(x)\, d|\nu|(x).$$

Furthermore, the linearity of ρ implies that

$$b_{\alpha f + \beta g} = \alpha b_f + \beta b_g \quad \text{for all } f, g \in \Gamma_c(G;\mathscr{B}) \text{ and } \alpha, \beta \in \mathbb{C}. \tag{3.5}$$

It follows from Corollary 3.5 (as well as Example 3.2) and (3.3) that for all $\varphi \in C_c^+(G)$,

$$|\nu_f|(\varphi) \le |\nu|(\varphi\|f\|). \tag{3.6}$$

Since $|\nu_f| = |b_f|\,|\nu|$, (3.6) amounts to

$$\int_G \varphi(x)|b_f(x)|\, d|\nu|(x) \le \int_G \varphi(x)\|f(x)\|\, d|\nu|(x) \quad \text{for all } \varphi \in C_c^+(G).$$

Therefore,

$$|b_f(x)| \le \|f(x)\| \quad \text{for } |\nu|\text{-almost all } x.$$

Using (3.5), this means that

$$|b_f(x) - b_g(x)| \le \|f(x) - g(x)\| \quad \text{for } |\nu|\text{-almost all } x.$$

Since B is separable, there is a sequence $\{f_n\} \subset \Gamma_c(G;\mathscr{B})$ such that $\{f_n(x)\}$ is dense in $B(x)$ for all x. Let B_0 be the rational span of the f_n. Then

$$B_0(x) := \{f(x) : f \in B_0\}$$

is a vector space over $\mathbb{Q}$ which is dense in $B(x)$. Since B_0 is countable, there is a $|\nu|$-null set N such that for all $x \notin N$ and all $f, g \in B_0$,

$$|b_f(x)| \le \|f(x)\|, \tag{3.7}$$

$$|b_f(x) - b_g(x)| \le \|f(x) - g(x)\|. \tag{3.8}$$

---

[7]The function ρ is called a "lift" for $S \subset L^\infty(|\nu|)$. More comments and references about lifts can be found in the paragraph preceding [30, Lemma I.8].

We can simply alter each b_f so that $b_f(x) = 0$ if $x \in N$. Then, for all $f, g \in B_0$, equations (3.5) (for α and β rational), (3.7) and (3.8) are valid for *all* $x \in G$.

In particular, (3.8) implies that if $f, g \in B_0$ and $f(x) = g(x)$, then $b_f(x) = b_g(x)$. Therefore we get a well-defined $\mathbb{Q}$-linear map

$$\epsilon_x : B_0(x) \to \mathbb{C}$$

by letting $\epsilon_x(a) = b_f(x)$, where f is any section in B_0 such that $f(x) = a$. In view of (3.7), each ϵ_x has norm at most one. It follows that ϵ_x extends uniquely to an element in $B(x)^*$ of norm at most one which we continue to denote by ϵ_x.

If $f \in B_0$, then $b_f(x) = \epsilon_x(f(x))$. Consequently, $x \mapsto \epsilon_x(f(x))$ is in $\mathcal{B}^b_c(G)$. Furthermore, if $f \in B_0$ and $\varphi \in C_c(G)$, then

$$\begin{aligned}\nu(\varphi \cdot f) = \nu_f(\varphi) &= \int_G \varphi(x) b_f(x)\, d|\nu|(x) = \int_G \varphi(x)\epsilon_x(f(x))\, d|\nu|(x) \\ &= \int_G \epsilon_x(\varphi \cdot f(x))\, d|\nu|(x).\end{aligned}$$

But, if $f \in \Gamma_c(G; \mathscr{B})$, then there is a sequence $\{g_k\} \subset \Gamma_c(G; \mathscr{B})$ converging to f in the inductive limit topology such that each g_k is a finite sum of the form $\sum \varphi_j \cdot f_j$ with each $\varphi_j \in C_c(G)$ and each $f_j \in B_0$. By the above, we have

$$\nu(g_k) = \int_G \epsilon_x(g_k(x))\, d|\nu|(x),$$

and since $g_k \to f$ in the inductive limit topology and since each ϵ_x has norm at most one, $\epsilon_x(g_k(x)) \to \epsilon_x(f(x))$ uniformly and the entire sequence vanishes off a compact set. Hence $x \mapsto \epsilon_x(f(x))$ is in $\mathcal{B}^b_c(G)$ as claimed in part (a) of the proposition, and

$$\nu(f) = \lim_k \nu(g_k) = \lim_k \int_G \epsilon_x(g_k(x))\, d|\nu|(x) = \int_G \epsilon_x(f(x))\, d|\nu|(x).$$

This completes the proof of the proposition. ∎

We close this section with some technicalities that will be needed later. In particular, we will need to deal with some not necessarily continuous sections.[8]

Definition 3.6. We let $\Sigma^1_c(G; \mathscr{B})$ be the set of bounded sections f of $p : \mathscr{B} \to G$ such that there is a uniformly bounded sequence $\{f_n\} \subset \Gamma_c(G; \mathscr{B})$ and a compact set $K \subset G$ such that $\operatorname{supp} f_n \subset K$ for all n and $f_n(x) \to f(x)$ for all $x \in G$.[9] Analogously, we let $\mathcal{B}^1_c(G)$ be the family of Borel functions φ on G such that there is a uniformly bounded sequence $\{\varphi_n\} \subset C_c(G)$ and a compact set $K \subset G$ such that $\operatorname{supp} \varphi_n \subset K$ for all n and $\varphi_n(x) \to \varphi(x)$ for all $x \in G$.

Example 3.7. Suppose that $f \in \Gamma_c(G; \mathscr{B})$ and $\varphi \in \mathcal{B}^1_c(G)$. Then by considering $\{\varphi_n \cdot f\}$ for appropriate φ_n, we see that $\varphi \cdot f \in \Sigma^1_c(G; \mathscr{B})$.

[8]Note that we are trying to avoid dealing with measurability issues for sections of $\mathscr{B}$. Consequently, we are going to introduce only those potentially discontinuous sections that we absolutely have to.

[9]Recall that we take the point of view that a section of $p : \mathscr{B} \to G$ is simply a function f from G to $\mathscr{B}$ such that $p(f(x)) = x$.

LEMMA 3.8. *Suppose that σ is a generalized Radon measure on $\mathscr{B}$ and that $\varphi \in \mathcal{B}_c^1(G)$ is such that there are $\{\varphi_n\} \subset C_c^+(G)$ with $\varphi_n(x) \searrow \varphi(x)$ for all $x \in G$. Then*

$$|\sigma|(\varphi) = \sup\{|\sigma(f)| : f \in \Sigma_c^1(G;\mathscr{B}) \text{ and } \|f\| \leq \sigma\}.$$

First, an even more specialized result.

LEMMA 3.9. *Suppose that $\varphi \in \mathcal{B}_c^1(G)$ is such that there is a sequence $\{\varphi_n\}$ in $C_c^+(G)$ such that $\varphi_n \searrow \varphi$. If $f \in \Sigma_c^1(G;\mathscr{B})$ is such that $\|f\| \leq \varphi_n$, then there are $f_1, f_2 \in \Sigma_c^1(G;\mathscr{B})$ such that $f = f_1 + f_2$, $\|f_i\| \leq \varphi$ and $\|f_2\| \leq \varphi_n - \varphi$.*

Proof. Define

$$\sigma(x) := \begin{cases} \dfrac{\varphi(x)}{\varphi_n(x)} & \text{if } \varphi_n(x) > 0, \\ 0 & \text{otherwise,} \end{cases} \qquad \tau(x) := \begin{cases} \dfrac{\varphi_n(x) - \varphi(x)}{\varphi_n(x)} & \text{if } \varphi_n(x) > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then we clearly have $f = \sigma \cdot f + \tau \cdot f$, $\|\sigma \cdot f\| \leq \varphi$ and $\|\tau \cdot f\| \leq \varphi_n - \varphi$. Therefore we just need to see that $\sigma \cdot f$ and $\tau \cdot f$ are in $\Sigma_c^1(G;\mathscr{B})$. But $\sigma(x) = \lim_{m \geq n} \sigma_m(x)$, where

$$\sigma_m(x) := \begin{cases} \varphi_m/\varphi_n & \text{if } \varphi_n(x) > 0, \\ 0 & \text{otherwise.} \end{cases}$$

As in the proof of Lemma 3.1, we have $\sigma_m \cdot f \in \Gamma_c(G;\mathscr{B})$ and $\sigma_m \cdot f \to \sigma \cdot f$ pointwise. It follows, that $\sigma \cdot f \in \Sigma_c^1(G;\mathscr{B})$. A similar argument shows that $\tau \cdot f \in \Sigma_c^1(G;\mathscr{B})$. This is what we needed to show. ∎

Proof of Lemma 3.8. As in Proposition 3.3, we have

$$\sigma(f) = \int_G \epsilon_x(f(x))\, d|\sigma|(x)$$

for linear functionals ϵ_x in the unit ball of $B(x)^*$. Thus if $\|f\| \leq \varphi$, then

$$|\sigma(f)| \leq \int_G \|f(x)\|\, d|\sigma|(x) \leq |\sigma|(\varphi).$$

By the dominated convergence theorem, $|\sigma|(\varphi_n) \to |\sigma|(\varphi)$. Therefore, there are $f_n \in \Gamma_c(G;\mathscr{B})$ such that $\|f_n\| \leq \varphi_n$ and $|\sigma(f_n)| \to |\sigma|(\varphi)$. By Lemma 3.9, we can decompose $f_n = f_n' + f_n''$ in $\Sigma_c^1(G;\mathscr{B})$ so that $\|f_n'\| \leq \varphi$ and $\|f_n''\| \leq \varphi_n - \varphi$. By the above,

$$|\sigma(f_n'')| \leq |\sigma|(\varphi_n - \varphi),$$

and $\sigma(f_n'') \to 0$ (as $n \to \infty$). It then follows that $|\sigma(f_n')| \to |\sigma|(\varphi)$. The result follows. ∎

Part (a) of Proposition 3.3 gives us "just enough" measurability of the ϵ_x to get by. We will need to amplify this a bit for the proof of the disintegration theorem in the next section. What we need is provided below.

Let $m : G^{(2)} \to G$ be the multiplication map and let $m^*\mathscr{B}$ be the pull back.

LEMMA 3.10. *Suppose that $\nu : \Gamma_c(G;\mathscr{B}) \to \mathbb{C}$ is a generalized Radon measure given by*

$$\nu(f) = \int_G \epsilon_x(f)\, d|\nu|(x) \tag{3.9}$$

as in Proposition 3.3. If $F \in \Gamma_c(G^{(2)}; m^*\mathscr{B})$, then

$$(x,y) \mapsto \epsilon_{xy}(F(x,y))$$

is a Borel function on $G^{(2)}$.

Proof. Let $\mathcal{A}_0$ be the subalgebra of $\Gamma_c(G^{(2)}; m^*\mathscr{B})$ spanned by sections of the form

$$(x, y) \mapsto \varphi(x, y) f(xy)$$

with $\varphi \in C_c(G^{(2)})$ and $f \in \Gamma_c(G; \mathscr{B})$. Clearly, $\mathcal{A}_0$ is closed under multiplication by functions from $C_c(G^{(2)})$, and $\{F(x, y) : F \in \mathcal{A}_0\}$ is dense in $m^*\mathscr{B}_{(x,y)} = \mathscr{B}_{xy}$ for all (x, y). Then a partition of unity argument (see Lemma A.4) implies that $\mathcal{A}_0$ is dense in $\Gamma_c(G^{(2)}; m^*\mathscr{B})$ in the inductive limit topology. Proposition 3.3 implies that for each $f \in \Gamma_c(G; \mathscr{B})$, $x \mapsto \epsilon_x(f(x))$ is a bounded Borel function with compact support. Since

$$(x, y) \mapsto \epsilon_{xy}(f(xy))$$

is the composition of two Borel functions, it too is Borel. Thus

$$(x, y) \mapsto \varphi(x, y)\epsilon_{xy}(f(xy))$$

is Borel for all $\varphi \in C_c(G^{(2)}))$ and $f \in \Gamma_c(G; \mathscr{B})$. Consequently, $(x, y) \mapsto \epsilon_{xy}(F(x, y))$ is Borel for all $F \in \mathcal{A}_0$. But if $F_i \to F$ in the inductive limit topology on $\Gamma_c(G; \mathscr{B})$, then the functions $(x, y) \mapsto \epsilon_{xy}(F_i(x, y))$ converge uniformly to $(x, y) \mapsto \epsilon_{xy}(F(x, y))$. Therefore the result follows, as $\mathcal{A}_0$ is dense in $\Gamma_c(G^{(2)}; m^*\mathscr{B})$. ∎

Lemma 3.11. *Suppose that ν is a generalized Radon measure given by* (3.9) *as in the statement of Lemma* 3.10. *Let $G^{(3)} = \{(x, y, z) \in G \times G \times G : s(x) = r(y)$ and $s(y) = r(z)\}$, let $\kappa : G^{(3)} \to B$ be given by $\kappa(x, y, z) := y^{-1}x$ and let $\kappa^*\mathscr{B}$ be the pull-back. If $F \in \Gamma_c(G^{(3)}; \kappa^*\mathscr{B})$, then*

$$(x, y, z) \mapsto \epsilon_{y^{-1}x}(F(x, y, z))$$

is Borel.

Proof. Sections of the form

$$(x, y, z) \mapsto \varphi(x, y, z) f(y^{-1}x)$$

with $\varphi \in C_c(G^{(3)})$ and $f \in \Gamma_c(G; \mathscr{B})$ are dense in $\Gamma_c(G^{(3)}; \kappa^*\mathscr{B})$. Since

$$(x, y, z) \mapsto \varphi(x, y, z)\epsilon_{y^{-1}x}(f(y^{-1}x))$$

is clearly Borel (it is even continuous in the z variable), the result follows as in the proof of Lemma 3.10. ∎

Example 3.12. We assume that we have the same set-up as in the previous two lemmas. If $f, g, h \in \Gamma_c(G; \mathscr{B})$, then $F(x, y, z) := g(y)^* f(z) h(z^{-1}x)$ defines a section in $\Gamma_c(G^{(3)}; \kappa^*\mathscr{B})$. Then

$$(x, y, z) \mapsto \epsilon_{y^{-1}x}(g(y^*) f(z) h(z^{-1}x))$$

is Borel.

4. Representations of Fell bundles

In order to prove an equivalence theorem generalizing that for groupoids (cf. [20, 27]), we will need to work with the sort of "weak representations" introduced by Renault in [27].

DEFINITION 4.1. Let $\mathcal{H}_0$ be a dense subspace of a Hilbert space $\mathcal{H}$, and denote by $\operatorname{Lin}(\mathcal{H}_0)$ the collection of *all* linear operators on the vector space $\mathcal{H}_0$. A *pre-representation* of $\mathscr{B}$ on $\mathcal{H}_0 \subset \mathcal{H}$ is a homomorphism $L : \Gamma_c(G;\mathscr{B}) \to \operatorname{Lin}(\mathcal{H}_0)$ such that for all $\xi, \eta \in \mathcal{H}_0$,

(a) $f \mapsto (L(f)\xi \mid \eta)$ is continuous in the inductive limit topology on $\Gamma_c(G;\mathscr{B})$,
(b) $(L(f)\xi \mid \eta) = (\xi \mid L(f^*)\eta)$,
(c) $\mathcal{H}_{00} := \operatorname{span}\{L(f)\zeta : f \in \Gamma_c(G;\mathscr{B}) \text{ and } \zeta \in \mathcal{H}_0\}$ is dense in $\mathcal{H}$.

Of course, two pre-representations $(L, \mathcal{H}_0, \mathcal{H})$ and $(L', \mathcal{H}_0', \mathcal{H}')$ are equivalent if there is a unitary $U : \mathcal{H} \to \mathcal{H}'$ intertwining L and L' on $\mathcal{H}_0$ and $\mathcal{H}_0'$, respectively.

The next result implies that each pre-representation has associated to it a unique measure class on $G^{(0)}$. This will be the measure class that appears in the disintegration.[10]

PROPOSITION 4.2. *Suppose that $L : \Gamma_c(G;\mathscr{B}) \to \operatorname{Lin}(\mathcal{H}_0)$ is a pre-representation of $\mathscr{B}$ on $\mathcal{H}_0 \subset \mathcal{H}$. Then there is a representation $M : C_0(G^{(0)}) \to B(\mathcal{H})$ such that for all $h \in C_0(G^{(0)})$, $f \in \Gamma_c(G;\mathscr{B})$ and $\xi \in \mathcal{H}_0$ we have*

$$M(h)L(f)\xi = L((h \circ r) \cdot f)\xi. \tag{4.1}$$

In particular, after replacing L by an equivalent representation, we may assume that $\mathcal{H} = L^2(G^{(0)} * \mathscr{V}, \mu)$ for a Borel Hilbert bundle $G^{(0)} * \mathscr{V}$ and a finite Radon measure μ on $G^{(0)}$ such that

$$M(h)\xi(u) = h(u)\xi(u) \quad \text{for all } h \in C_0(G^{(0)}) \text{ and } \xi \in L^2(G^{(0)} * \mathscr{V}, \mu).$$

Proof. We can easily make sense of $(h \circ r) \cdot f$ for $h \in C_0(G^{(0)})^{\sim}$. Furthermore, we can compute that

$$(L((h \circ r) \cdot f)\xi \mid L(g)\eta) = (L(f)\xi \mid L((\bar{h} \circ r) \cdot g)\eta).$$

Then, if $k \in C_0(G^{(0)})^{\sim}$ is such that

$$\|h\|_\infty^2 1 - |h|^2 = |k|^2,$$

we can compute that

$$\begin{aligned} \|h\|_\infty^2 \Big\| \sum_{i=1}^n L(f_i)\xi_i \Big\|^2 - \Big\| \sum_{i=1}^n L((h \circ r) \cdot f_i)\xi_i \Big\|^2 \\ &= \sum_{ij} (L(((\|h\|_\infty^2 1 - \mid h|^2) \circ r) \cdot f_i)\xi_i | L(f_j)\xi_j) \\ &= \Big\| \sum_{i=1}^n L((k \circ r) \cdot f_i)\xi_i \Big\|^2 \geq 0. \end{aligned}$$

Since $\mathcal{H}_{00}$ is dense in $\mathcal{H}$, it follows that there is a well-defined bounded operator $M(h)$ on all of $\mathcal{H}$ satisfying (4.1). It is not hard to see that M is a $*$-homomorphism. To see that M is a representation, by convention, we must also see that M is nondegenerate. If $f \in \Gamma_c(G;\mathscr{B})$, then $r(\operatorname{supp} f)$ is compact. Hence there is an $h \in C_0(G^{(0)})$ such that $M(h)f = f$. From this, it is straightforward to see that M is nondegenerate and therefore a representation.

[10] For the basics on Borel Hilbert bundles and direct integrals, see [30, Appendix F].

Since M is a representation of $C_0(G^{(0)})$, it is equivalent to a multiplication representation on $L^2(G^{(0)} * \mathscr{V}, \mu)$ for an appropriate Borel Hilbert bundle $G^{(0)} * \mathscr{V}$ and finite Radon measure μ by, for example, [30, Example F.25]. The second assertion follows, and this completes the proof. ∎

Definition 4.3. Suppose that $G^{(0)} * \mathscr{H}$ is a Borel Hilbert bundle over $G^{(0)}$. We let

$$\operatorname{End}(G^{(0)} * \mathscr{H}) = \{(u, T, v) : T \in B(\mathcal{H}(v), \mathcal{H}(u))\}.$$

We give $\operatorname{End}(G^{(0)} * \mathscr{H})$ the smallest Borel structure such that

$$\psi_{f,g}(u, T, v) := (Tf(u) \mid g(v))$$

is Borel for all Borel sections $f, g \in B(G^{(0)}, \mathscr{H})$ (see [30, Definitions F.1 and F.6]).

Remark 4.4. The Borel structure on $\operatorname{End}(G^{(0)} * \mathscr{H})$ is standard. To see this, first note that $B_k(\mathcal{H}, \mathcal{K}) := \{T \in B(\mathcal{H}, \mathcal{K}) : \|T\| \le k\}$ can be viewed as a closed subset of $B_k(\mathcal{H} \oplus \mathcal{K})$ in the weak operator topology. Since the latter is a compact Polish space (see [30, Lemma D.37]), so is $B_k(\mathcal{H}, \mathcal{K})$. On the other hand, $G^{(0)} * \mathscr{H}$ is isomorphic to $\coprod_n X_n \times \mathcal{H}_n$ for Hilbert spaces $\mathcal{H}_n$ and a Borel partition $\{X_n\}$ of X [30, Corollary F.12]. Then it is not hard to convince yourself that $\operatorname{End}(G^{(0)} * \mathscr{H})$ is Borel isomorphic to $\bigcup_k \coprod_{m,n} X_n \times B_k(\mathcal{H}_m, \mathcal{H}_n) \times X_m$.

Definition 4.5. We say that a map $\hat{\pi} : \mathscr{B} \to \operatorname{End}(G^{(0)} * \mathscr{H})$ is a *$*$-functor* if $\hat{\pi}(b) = (r(b), \pi(b), s(b))$ for an operator $\pi(b) : \mathcal{H}(s(b)) \to \mathcal{H}(r(b))$[11] such that

(a) $\pi(\lambda a + b) = \lambda \pi(a) + \pi(b)$ if $p(a) = p(b)$,
(b) $\pi(ab) = \pi(a)\pi(b)$ if $(a, b) \in \mathscr{B}^{(2)}$,
(c) $\pi(b^*) = \pi(b)^*$.

We say that a $*$-functor is *Borel* if $x \mapsto \hat{\pi}(f(x))$ is Borel from G to $\operatorname{End}(G^{(0)} * \mathscr{H})$ for all $f \in \Gamma_c(G; \mathscr{B})$.

Remark 4.6. Notice that if π is a $*$-functor, then for each $u \in G^{(0)}$, $\pi|_{B(u)}$ is a $*$-homomorphism and therefore bounded (since $B(u)$ is a C^*-algebra). But then

$$\|\pi(b)\|^2 = \|\pi(b)^*\pi(b)\| = \|\pi(b^*b)\| \le \|b^*b\| = \|b\|^2.$$

Hence $*$-functors are “naturally” norm decreasing.

Definition 4.7. Suppose that $p : \mathscr{B} \to G$ is a Fell bundle. Then a $*$-homomorphism L of $\Gamma_c(G; \mathscr{B})$ into $B(\mathcal{H})$ is a called a *representation* if

(a) it is continuous from $\Gamma_c(G; \mathscr{B})$ equipped with the inductive limit topology into $B(\mathcal{H})$ with the weak operator topology,
(b) it is nondegenerate in the sense that $\operatorname{span}\{L(f)\xi : f \in \Gamma_c(G; \mathscr{B}) \text{ and } \xi \in \mathcal{H}\}$ is dense in $\mathcal{H}$.

[11] Here we have adopted the notation $r(b)$ in place of the cumbersome $r(p(b))$—and similarly for $s(b)$.

Example 4.8. If $L : \Gamma_c(G;\mathscr{B}) \to B(\mathcal{H})$ is a nondegenerate $*$-homomorphism which is bounded with respect to the I-norm—that is, if $\|L(f)\| \le \|f\|_I$ for all $f \in \Gamma_c(G;\mathscr{B})$—then it is easy to see that L is a representation as defined in Definition 4.7.

Recall that a measure μ on $G^{(0)}$ is called *quasi-invariant* if the Radon measure $\nu = \mu\circ\lambda$ on G defined by

$$\nu(f) := \int_{G^{(0)}} \int_G f(x)\, d\lambda^u(x)\, d\mu(u) \tag{4.2}$$

is equivalent to the measure ν^{-1} defined by $\nu^{-1}(f) = \nu(\tilde{f})$, where $\tilde{f}(x) := f(x^{-1})$. (Alternatively, ν^{-1} is the push-forward of ν by the inversion map on G: $\nu^{-1}(E) = \nu(E^{-1})$ for Borel sets $E \subset G$.) Note that

$$\nu^{-1}(f) = \int_{G^{(0)}} \int_G f(x^{-1})\, d\lambda^u(x)\, d\mu(u) = \int_{G^{(0)}} \int_G f(x)\, d\lambda_u(x)\, d\mu(u). \tag{4.3}$$

If μ is quasi-invariant, so that ν is equivalent to ν^{-1}, then we can let $\Delta = d\nu/d\nu^{-1}$ be the Radon–Nikodym derivative for ν with respect to ν^{-1}. It will be important for the calculations below to note that can choose $\Delta : G \to \mathbb{R}^+$ to be a *bona fide* homomorphism (see [18, Theorem 3.15] or [8, Corollary 3.14]). As is standard, we will write ν_0 for the symmetrized measure $\Delta^{-1/2}\nu$.[12]

Definition 4.9. Suppose that $p : \mathscr{B} \to G$ is a Fell bundle over G. Then a *strict representation* of $\mathscr{B}$ is a triple $(\mu, G^{(0)} * \mathscr{H}, \pi)$ consisting of a quasi-invariant measure μ on $G^{(0)}$, a Borel Hilbert bundle $G^{(0)} * \mathscr{H}$ and a $*$-functor $\pi : \mathscr{B} \to \operatorname{End}(G^{(0)} * \mathscr{H})$.

Proposition 4.10. *Suppose that $p : \mathscr{B} \to G$ is a Fell bundle and that $(\mu, G^{(0)} * \mathscr{H}, \pi)$ is a strict representation of $\mathscr{B}$. Then there is an associated I-norm bounded $*$-homomorphism L, called the* integrated form of π, *on $L^2(G^{(0)} * \mathscr{H}, \mu)$ given by*

$$(L(f)\xi \mid \eta) = \int_G (\pi(f(x))\xi(s(x)) \mid \eta(r(x)))\Delta(x)^{-1/2}\, d\nu(x), \tag{4.4}$$

where Δ is the Radon–Nikodym derivative of ν^{-1} with respect to ν and $\nu = \mu \circ \lambda$.

Remark 4.11. Using vector-valued integrals, we can also write

$$L(f)\xi(u) = \int_G \pi(f(x))\xi(s(x))\Delta(x)^{-1/2}\, d\lambda^u(x). \tag{4.5}$$

Remark 4.12. Note that there is no reason to suspect that L is nondegenerate, and hence a representation, without some additional hypotheses on π.

Proof. The proof that L is bounded is standard and uses the quasi-invariance of μ. The "trick" (due to Renault) is to apply the Cauchy–Schwarz inequality in $L^2(\nu)$ (and to

[12] We say the ν_0 is "symmetrized" because it is invariant under the inverse map:

$$\begin{aligned}\textstyle\int_G f(x)\, d\nu_0(x) &= \textstyle\int_G f(x)\Delta(x)^{-1/2}\, d\nu(x) = \int_G f(x)\Delta(x)^{-1/2}\Delta(x)\, d\nu^{-1}(x)\\ &= \textstyle\int_G f(x^{-1})\Delta(x^{-1})^{-1/2}\Delta(x^{-1})\, d\nu(x) = \int_G f(x^{-1})\Delta(x)^{-1/2}\nu(x)\\ &= \textstyle\int_G f(x^{-1})\, d\nu_0(x).\end{aligned}$$

recall that π must be norm decreasing by Remark 4.6):

$$\begin{aligned}|(L(f)\xi \mid \eta)| &\le \int_G \|f(x)\|\,\|\xi(s(x))\|\,\|\eta(r(x))\|\Delta(x)^{-1/2}(x)\,d\nu(x)\\ &\le \Big(\int_G \|f(x)\|\,\|\xi(s(x))\|^2\Delta(x)^{-1}\,d\nu(x)\Big)^{1/2}\\ &\quad\times\Big(\int_G \|f(x)\|\,\|\eta(r(x))\|^2\,d\nu(x)\Big)^{1/2}\\ &\le (\|f\|_I\|\xi\|_2^2)^{1/2}(\|f\|_I\|\eta\|_2^2)^{1/2}\\ &\le \|f\|_I\|\xi\|_2\|\eta\|_2.\end{aligned}$$

To show that L is multiplicative, we use (4.5) and compute as follows:

$$L(f*g)\xi(u) = \int_G \pi(f*g(x))\xi(s(x))\Delta(x)^{-1/2}\,d\lambda^u(x),$$

which, since $\pi|_{B(x)}$ is a bounded linear map of $B(x)$ into $B(\mathcal{H}(s(x)),\mathcal{H}(r(x))$, is

$$= \int_G\int_G \pi(f(y)g(y^{-1}x))\xi(s(x))\Delta(x)^{-1/2}\,d\lambda^u(y)\,d\lambda^u(x)$$

which, after using Fubini and sending $x \mapsto yx$, is

$$\begin{aligned}&= \int_G\int_G \pi(f(y)g(x))\xi(s(x))\Delta(yx)^{-1/2}\,d\lambda^{s(y)}(x)\,d\lambda^u(y)\\ &= \int_G \pi(f(y))\int_G \pi(g(x))\xi(s(x))\Delta(x)^{-1/2}\,d\lambda^{s(y)}(x)\Delta(y)^{-1/2}\,d\lambda^u(y)\\ &= \int_G \pi(f(y))L(g)\xi(s(y))\Delta(y)^{-1/2}\,d\lambda^u(y)\\ &= L(f)L(g)\xi(u).\end{aligned}$$

To see that L is $*$-preserving, we will need to use the quasi-invariance of μ in the form of the invariance of $\Delta^{-1/2}\,d\nu$ under the inversion map. We compute that

$$\begin{aligned}(L(f^*\xi \mid \eta) &= \int_G (\pi(f^*(x))\xi(s(x)) \mid \eta(r(x)))\Delta(x)^{-1/2}\,d\nu(x)\\ &= \int_G (\pi(f(x^{-1})^*)\xi(s(x)) \mid \eta(r(x)))\Delta(x)^{-1/2}\,d\nu(x)\end{aligned}$$

which, after sending $x \mapsto x^{-1}$, is

$$= \int_G (\xi(r(x)) \mid \pi(f(x))\eta(s(x)))\Delta(x)^{-1/2}\,d\nu(x). \quad\blacksquare$$

The next step is to prove a very strong converse of Proposition 4.10 modeled after Renault's [27, Proposition 4.2]. The extra generality is needed to prove the equivalence theorem—which is our eventual goal. The proof given in the next section follows Yamagami's suggestion that Proposition 3.3 ought to “replace” the Radon–Nikodym theorem in Renault's proof in the presence of suitable approximate identities (see Proposition 5.1).

The argument here follows Muhly's version of Renault's argument (see [18, Theorem 3.32] or [22, Theorem 7.8]) with a couple of "vector upgrades".

THEOREM 4.13 (Disintegration Theorem). *Suppose that $L : \Gamma_c(G;\mathscr{B}) \to \mathrm{Lin}(\mathcal{H}_0)$ is a pre-representation of $\mathscr{B}$ on $\mathcal{H}_0 \subset \mathcal{H}$. Then L is bounded in the sense that $\|L(f)\| \le \|f\|_I$ for all $f \in \Gamma_c(G;\mathscr{B})$. Therefore L extends to a* bona fide *representation of $\Gamma_c(G;\mathscr{B})$ on $\mathcal{H}$ which is equivalent to the integrated form of a strict representation $(\mu, G^{(0)} * \mathscr{H}, \pi)$ of $\mathscr{B}$ where μ is the measure defined in Proposition 4.2.*[13] *In particular, L is bounded with respect to the universal C^*-norm on $\Gamma_c(G;\mathscr{B})$.*

We will take up the proof of the Disintegration Theorem in the next section. However, once we have Theorem 4.13 in hand, we can "adjust" our definition of the universal norm as follows.

REMARK 4.14. Since any pre-representation, and *a priori* any representation, L, of $\mathscr{B}$ is equivalent to the integrated form of a strict representation, it follows that L is I-norm bounded by Proposition 4.10. Conversely, I-norm bounded representations are clearly representations. Therefore, we could have defined the universal norm on $\Gamma_c(G;\mathscr{B})$ via

$$\|f\| := \sup\{\|L(f)\| : L \text{ is a representation of } \mathscr{B}\}.$$

5. Proof of the Disintegration Theorem

Naturally, we will break the proof up into a number of steps. The first is, as suggested by Yamagami, to produce a two-sided approximate identity for $\Gamma_c(G;\mathscr{B})$ in the inductive limit topology. This is a highly nontrivial result. However, it is an immediate consequence of the rather special approximate identities that we need for our proof of the equivalence theorem. Thus the next result is an immediate consequence of Proposition 6.10 (see the comments immediately following that proposition).

PROPOSITION 5.1. *There is a self-adjoint approximate identity for $\Gamma_c(G;\mathscr{B})$ in the inductive limit topology.*

COROLLARY 5.2. *Suppose that L is a pre-representation of $\mathscr{B}$ on $\mathcal{H}_0 \subset \mathcal{H}$. Let $\mathcal{H}_{00}$ be the necessarily dense subspace* $\mathrm{span}\{L(f)\xi : f \in \Gamma_c(G;\mathscr{B})$ *and* $\xi \in \mathcal{H}_0\}$. *If $\mathcal{H}_{00}'$ is a dense subspace of $\mathcal{H}_{00}$, then*

$$\mathrm{span}\{L(f)\xi : f \in \Gamma_c(G;\mathscr{B}) \text{ and } \xi \in \mathcal{H}_{00}'\}$$

is dense in $\mathcal{H}$.

Proof. Let $\{e_i\}$ be a self-adjoint approximate identity for $\Gamma_c(G;\mathscr{B})$ in the inductive limit topology. Then if $L(f)\xi \in \mathcal{H}_{00}$, we see that

$$\begin{aligned}\|L(e_i)L(f)\xi - L(f)\xi\|^2 \\ &= (L(f^* * e_i * e_i * f)\xi \mid \xi) - 2\,\mathrm{Re}(L(f^* * e_i * f)\xi \mid \xi) + (L(f^* * f)\xi \mid \xi),\end{aligned}$$

[13] Notice that we are asserting that μ is necessarily quasi-invariant.

which tends to zero by part (a) of Definition 4.1. It follows that $\mathcal{H}_{00}' \subset \overline{\operatorname{span}}\{L(f)\xi : \xi \in \mathcal{H}_{00}' \text{ and } f \in \Gamma_c(G;\mathscr{B})\}$. Since $\mathcal{H}_{00}'$ is dense, the result follows. ∎

LEMMA 5.3. *Suppose that L is a pre-representation of $\Gamma_c(G;\mathscr{B})$ on $\mathcal{H}_0 \subset \mathcal{H}$. Then there is a positive sesquilinear form $\langle \cdot\,, \cdot\rangle$ on $\Gamma_c(G;\mathscr{B}) \odot \mathcal{H}_0$ such that*

$$\langle f \otimes \xi, g \otimes \eta\rangle = (L(g^* * f)\xi \mid \eta). \tag{5.1}$$

Furthermore, the Hilbert space completion $\mathcal{K}$ of $\Gamma_c(G;\mathscr{B}) \odot \mathcal{H}_0$ is isomorphic to $\mathcal{H}$. In fact, if $[f \otimes \xi]$ is the class of $f \otimes \xi$ in $\mathcal{K}$, then $[f \otimes \xi] \mapsto L(f)\xi$ is well-defined and induces an isomorphism of $\mathcal{K}$ with $\mathcal{H}$ which maps the quotient $\Gamma_c(G;\mathscr{B}) \odot \mathcal{H}_0/\mathscr{N}$, where $\mathscr{N}$ is the subspace $\mathscr{N} = \{\sum_i f_i \otimes \xi_i : \sum_i L(f_i)\xi_i = 0\}$ of vectors in $\Gamma_c(G;\mathscr{B}) \odot \mathcal{H}_0$ of length zero, onto $\mathcal{H}_{00}$ (as defined in part (c) of Definition 4.1).

Proof. Using the universal properties of the *algebraic* tensor product, as in the proof of [24, Proposition 2.64] for example, it is not hard to see that there is a unique sesquilinear form on $\Gamma_c(G;\mathscr{B}) \odot \mathcal{H}_0$ satisfying (5.1).[14] Thus to see that $\langle \cdot\,, \cdot\rangle$ is a pre-inner product, we just have to see that it is positive. But

$$\begin{aligned} \Big\langle \sum_i f_i \otimes \xi_i, \sum_i f_i \otimes \xi_i\Big\rangle &= \sum_{ij} (L(f_j^* * f_i)\xi_i \mid \xi_j) \\ &= \sum_{ij} (L(f_i)\xi_i \mid L(f_j)\xi_i) = \Big\| \sum_i L(f_i)\xi_i \Big\|^2. \end{aligned} \tag{5.2}$$

As in [24, Lemma 2.16], $\langle \cdot\,, \cdot\rangle$ defines an inner product on $\Gamma_c(G;\mathscr{B}) \odot \mathcal{H}_0/\mathscr{N}$, and $[f_i \otimes \xi] \mapsto L(f_i)\xi$ is well-defined in view of (5.2). Since this map has range $\mathcal{H}_{00}$ and since $\mathcal{H}_{00}$ is dense in $\mathcal{H}$ by definition, the map extends to an isomorphism of $\mathcal{K}$ onto $\mathcal{H}$ as claimed. ∎

CONVENTIONS. Using Lemma 5.3, we will identify $\mathcal{H}$ with $\mathcal{K}$, and $\mathcal{H}_{00}$ with $\Gamma_c(G;\mathscr{B}) \odot \mathcal{H}_0/\mathscr{N}$. Thus we will interpret $[f \otimes \xi]$ as a vector in $\mathcal{H}_{00} \subset \mathcal{H}_0 \subset \mathcal{H}$. Then we have

$$L(g)[f \otimes \xi] = [g * f \otimes \xi], \tag{5.3}$$

$$M(h)[f \otimes \xi] = [(h \circ r) \cdot f \otimes \xi], \tag{5.4}$$

where M is the representation of $C_0(G^{(0)})$ defined in Proposition 4.2, $g \in \Gamma_c(G;\mathscr{B})$ and $h \in C_0(G^{(0)})$.

REMARK 5.4. In view of Proposition 4.2, M extends to a $*$-homomorphism of $\mathcal{B}_c^b(G)$ into $B(\mathcal{H})$ such that $M(h) = 0$ if $h(u) = 0$ for μ-almost all u (where μ is the measure defined in that proposition). However, at this point, we cannot assert that (5.4) holds for any $h \notin C_0(G^{(0)})$.

A critical step in producing a strict representation is producing a quasi-invariant measure class. While we have the measure μ courtesy of Proposition 4.2, showing that

[14] For fixed g and η, the left-hand side of (5.1) is bilinear in f and ξ. Therefore, by the universal properties of the algebraic tensor product, (5.1) defines a linear map $m(g,\eta) : \Gamma_c(G;\mathscr{B}) \odot \mathcal{H}_0 \to \mathbb{C}$. Then $(g,\eta) \mapsto \overline{m(g,\eta)}$ is a bilinear map into the space $\mathrm{CL}(\Gamma_c(G;\mathscr{B}) \odot \mathcal{H}_0)$ of conjugate linear functionals on $\Gamma_c(G;\mathscr{B}) \odot \mathcal{H}_0$. Then we get a linear map $N : \Gamma_c(G;\mathscr{B}) \odot \mathcal{H}_0 \to \mathrm{CL}(\Gamma_c(G;\mathscr{B}) \odot \mathcal{H}_0)$. We can then define $\langle \alpha, \beta\rangle := \overline{N(\beta)(\alpha)}$. Clearly $\alpha \mapsto \langle \alpha, \beta\rangle$ is linear and it is not hard to check that $\overline{\langle \alpha, \beta\rangle} = \langle \beta, \alpha\rangle$.

μ is quasi-invariant requires that we extend equations (5.3) and (5.4) to a larger class of functions. This cannot be done without also enlarging the domain of definition of L. This is problematic as we do not as yet know that each $L(f)$ is bounded in any sense, nor have we assumed that $\mathcal{H}_0$ is complete. Motivated by Muhly's proof in [18], we have introduced $\Sigma_c^1(G;\mathscr{B})$ and $\mathcal{B}_c^1(G)$ in Definition 3.6 in order to deal with only those additional functions that we absolutely need.

"Not to put too fine a point on it," $\Sigma_c^1(G;\mathscr{B})$ is not a well-behaved class of functions on G. For example, there is no reason to suspect that it is closed under the sort of uniformly bounded pointwise convergence used in its definition. Nevertheless, we have the following useful observation.

LEMMA 5.5. *Suppose that σ is a generalized Radon measure on $\mathscr{B}$ given by*

$$\sigma(f) = \int_G \epsilon_x(f(x))\,d|\sigma|(x) \tag{5.5}$$

as in Proposition 3.3. If $f \in \Sigma_c^1(G;\mathscr{B})$, then $x \mapsto \epsilon_x(f(x))$ is in $\mathcal{B}_c^b(G)$ and we can extend σ to a linear functional on $\Sigma_c^1(G;\mathscr{B})$. In particular, if $\{f_n\} \subset \Sigma_c^1(G;\mathscr{B})$ is a uniformly bounded sequence whose supports are contained in a fixed compact set and which converges pointwise to $f \in \Sigma_c^1(G;\mathscr{B})$, then $\sigma(f_n) \to \sigma(f)$.

Proof. Let $\{f_n\} \subset \Gamma_c(G;\mathscr{B})$ be as in the second part of the lemma. Let $\tau_n(x) := \epsilon_x(f_n(x))$. Proposition 3.3 implies that $\tau_n \in \mathcal{B}_c^b(G)$ and clearly $\tau_n(x) \to \tau(x) := \epsilon_x(f(x))$ for each $x \in G$. Moreover the sequence $\{\tau_n\}$ is uniformly bounded and vanishes off some compact set. Therefore $\tau \in \mathcal{B}_c^b(G)$ and we can extend σ using (5.5). Furthermore, using the dominated convergence theorem we have

$$\sigma(f) := \int_G \epsilon_x(f(x))\,d|\sigma|(x) = \lim_n \int_G \epsilon_x(f_n(x))\,d|\sigma|(x) = \lim_n \sigma(f_n).$$

Using this, it is not hard to see that the extension of σ is a linear functional.

If $\{f_n\} \subset \Sigma_c^1(G;\mathscr{B})$ converges to $f \in \Sigma_c^1(G;\mathscr{B})$ as in the second part of the lemma, then we can define τ_n and τ as above. Thus just as above, the dominated convergence theorem implies the final assertion in the lemma. ∎

LEMMA 5.6. *Suppose that $f, g \in \Sigma_c^1(G;\mathscr{B})$. Then*

$$f * g(x) := \int_G f(y)g(y^{-1}x)\,d\lambda^{r(x)}(y)$$

is a well-defined element of $B(x)$, and $f * g$ defines a section in $\Sigma_c^1(G;\mathscr{B})$.

Proof. Let $\{f_n\}$ and $\{g_n\}$ be uniformly bounded sequences in $\Gamma_c(G;\mathscr{B})$ with supports all in the same compact set such that $f_n \to f$ and $g_n \to g$ pointwise. Then for each x, $y \mapsto f_n(y)g_n(y^{-1}x)$ converges pointwise to $y \mapsto f(y)g(y^{-1}x)$. Therefore the latter is a bounded Borel function from $G^{r(x)}$ to $B(x)$ vanishing off a compact set. Thus $f * g(x)$ is a well-defined element of $B(x)$ (for example, by [30, Lemma 1.91]).

Furthermore, since

$$\|f_n * g_n(x)\| \le \|f_n\|_\infty \|g_n\|_\infty \sup_u \lambda^u((\operatorname{supp} f_n)(\operatorname{supp} g_n)),$$

$\{f_n * g_n\}$ is uniformly bounded sequence in $\Gamma_c(G;\mathscr{B})$ (by Corollary 1.4) whose supports are all contained in a fixed compact set and which converges pointwise to $f * g$. Hence $f * g \in \Sigma_c^1(G;\mathscr{B})$ as claimed. ∎

For each ξ and η in $\mathcal{H}_0$, part (b) of Definition 4.1 implies that

$$L_{\xi,\eta}(f) := (L(f)\xi \mid \eta) \tag{5.6}$$

defines a generalized Radon measure on $\mathscr{B}$. We will use Lemma 5.5 to extend $L_{\xi,\eta}$ to $\Sigma_c^1(G;\mathscr{B})$.

Lemma 5.7. *Suppose that L is a pre-representation of $\mathscr{B}$ on $\mathcal{H}_0 \subset \mathcal{H}$. Then there is a positive sesquilinear form on $\Sigma_c^1(G;\mathscr{B}) \odot \mathcal{H}_0$, extending that on $\Gamma_c(G;\mathscr{B}) \odot \mathcal{H}_0$, such that*

$$\langle f\otimes\xi, g\otimes\eta\rangle = L_{\xi,\eta}(g^* * f) \quad \text{for all } f,g\in\Sigma_c^1(G;\mathscr{B}) \text{ and } \xi,\eta\in\mathcal{H}_0.$$

In particular, if

$$\mathscr{N}_b := \Big\{\sum_i f_i\otimes\xi \in \Sigma_c^1(G;\mathscr{B})\odot\mathcal{H}_0 : \Big\langle \sum_i f_i\otimes\xi, \sum_i f_i\otimes\xi_i\Big\rangle = 0\Big\}$$

is the subspace of vectors of zero length, then the quotient $\Sigma_c^1(G;\mathscr{B})\odot\mathcal{H}_0/\mathscr{N}_b$ can be identified with a subspace of $\mathcal{H}$ containing $\mathcal{H}_{00} := \Gamma_c(G;\mathscr{B})\odot\mathcal{H}_0/\mathscr{N}$.

Remark 5.8. As before, we will write $[f\otimes\xi]$ for the class of $f\otimes\xi$ in the quotient $\Sigma_c^1(G;\mathscr{B})\odot\mathcal{H}_0/\mathscr{N}_b \subset \mathcal{H}$.

Proof. Just as in Lemma 5.3, there is a well-defined sesquilinear form on $\Sigma_c^1(G;\mathscr{B})\odot\mathcal{H}_0$ extending that on $\Gamma_c(G;\mathscr{B})\odot\mathcal{H}_0$. (Note that the right-hand side of (5.1) can be rewritten as $L_{\xi,\eta}(g^* * f)$.) In particular, we have

$$\Big\langle \sum_i f_i\otimes\xi_i, \sum_j g_j\otimes\eta_j\Big\rangle = \sum_{ij} L_{\xi_i,\eta_j}(g_j^* * f_i).$$

We need to see that the form is positive. Let $\alpha := \sum_i f_i\otimes\xi_i$, and let $\{f_{i,n}\}$ be a uniformly bounded sequence in $\Gamma_c(G;\mathscr{B})$ converging pointwise to f_i with all the supports contained in a fixed compact set. Then for each i and j, $f_{j,n}^* * f_{i,n} \to f_j^* * f_i$ in the appropriate sense. In particular, Lemma 5.5 implies that

$$\langle\alpha,\alpha\rangle = \sum_{ij} L_{\xi_i,\xi_j}(f_j^* * f_i) = \lim_n \sum_{ij} L_{\xi_i,\xi_j}(f_{j,n}^* * f_{i,n}) = \lim_n \langle\alpha_n,\alpha_n\rangle,$$

where $\alpha_n := \sum_i f_{i,n}\otimes\xi_i$. Since $\langle\cdot\,,\cdot\rangle$ is positive on $\Gamma_c(G;\mathscr{B})\odot\mathcal{H}_0$ by Lemma 5.3, we have $\langle\alpha_n,\alpha_n\rangle \geq 0$, and we have shown that $\langle\cdot\,,\cdot\rangle$ is still positive on $\Sigma_c^1(G;\mathscr{B})\odot\mathcal{H}_0$.

Clearly the map sending the class $f\otimes\xi+\mathscr{N}$ to $f\otimes\xi+\mathscr{N}_b$ is isometric and therefore extends to an isometric embedding of $\mathcal{H}$ into the Hilbert space completion $\mathcal{H}_b$ of $\Sigma_c^1(G;\mathscr{B})\odot\mathcal{H}_0$ with respect to $\langle\cdot\,,\cdot\rangle$. However, if $g\otimes\xi\in\Sigma_c^1(G;\mathscr{B})\odot\mathcal{H}_0$ and if $\{g_n\}$ is a sequence in $\Gamma_c(G;\mathscr{B})$ such that $g_n\to g$ in the usual way, then

$$\|(g_n\otimes\xi+\mathscr{N}_b)-(g\otimes\xi+\mathscr{N}_b)\|^2 = L_{\xi,\xi}(g_n^* * g_n - g_n^* * g - g * g_n^* + g^* * g),$$

and this tends to zero by Lemma 5.5. Thus the image of $\mathcal{H}$ in $\mathcal{H}_b$ is all of $\mathcal{H}_b$. Consequently, we can identify the completion of $\Sigma_c^1(G;\mathscr{B})\odot\mathcal{H}_0$ with $\mathcal{H}$, and $\Sigma_c^1(G;\mathscr{B})\odot\mathcal{H}_0/\mathscr{N}_b$ with a subspace of $\mathcal{H}$ containing $\mathcal{H}_{00}$. ∎

The "extra" vectors provided by $\Sigma_c^1(G;\mathscr{B})\odot\mathcal{H}_0/\mathscr{N}_b$ are just enough to allow us to use a bit of general nonsense about unbounded operators to extend the domain of each $L(f)$. More precisely, for $f\in\Gamma_c(G;\mathscr{B})$, we can view $L(f)$ as an operator in $\mathcal{H}$ with domain $\mathscr{D}(L(f))=\mathcal{H}_{00}$. Then using part (b) of Definition 4.1, we see that

$$L(f^*)\subset L(f)^*.$$

This implies that $L(f)^*$ is a densely defined operator. Hence $L(f)$ is closable [1, Proposition X.1.6]. Consequently, the closure of the graph of $L(f)$ in $\mathcal{H}\times\mathcal{H}$ is the graph of the closure $\overline{L(f)}$ of $L(f)$ [1, Proposition X.1.4].

Suppose that $g\in\Sigma_c^1(G;\mathscr{B})$. Let $\{g_n\}$ be a uniformly bounded sequence in $\Gamma_c(G;\mathscr{B})$ all supported in a fixed compact set such that $g_n\to g$ pointwise. Then

$$\|[g_n\otimes\xi]-[g\otimes\xi]\|^2=L_{\xi,\xi}(g_n^**g_n-g^**g_n-g_n^**g+g*g). \tag{5.7}$$

However, $\{g_n^**g_n-g^**g_n-g_n^**g+g*g\}$ is uniformly bounded and converges pointwise to zero. Since the supports are all contained in a fixed compact set, the left-hand side of (5.7) tends to zero by Lemma 5.5. Similarly,

$$\|[f*g_n\otimes\xi]-[f*g\otimes\xi]\|^2\to 0.$$

It follows that

$$([g_n\otimes\xi,L(f)[g_n\otimes\xi])\to([g\otimes\xi],[f*g\otimes\xi])$$

in $(\Sigma_c^1(G;\mathscr{B})\odot\mathcal{H}_0/\mathscr{N}_b)\times(\Sigma_c^1(G;\mathscr{B})\odot\mathcal{H}_0/\mathscr{N}_b)\subset\mathcal{H}\times\mathcal{H}$. Therefore $[g\otimes\xi]\in\mathscr{D}(\overline{L(f)})$ and $\overline{L(f)}[g\otimes\xi]=[f*g\otimes\xi]$. We have proved the following.

Lemma 5.9. *For each $f\in\Gamma_c(G;\mathscr{B})$, $L(f)$ is a closable operator in $\mathcal{H}$ with domain $\mathscr{D}(L(f))=\mathcal{H}_{00}=\Gamma_c(G;\mathscr{B})\odot\mathcal{H}_0/\mathscr{N}$. Furthermore $\Sigma_c^1(G;\mathscr{B})\odot\mathcal{H}_0/\mathscr{N}_b$ belongs to $\mathscr{D}(\overline{L(f)})$, and*

$$\overline{L(f)}[g\otimes\xi]=[f*g\otimes\xi]\quad\text{for all } f\in\Gamma_c(G;\mathscr{B}),\ g\in\Sigma_c^1(G;\mathscr{B}) \text{ and } \xi\in\mathcal{H}_0.$$

Lemma 5.10. *For each $f\in\Sigma_c^1(G;\mathscr{B})$, there is a well-defined operator $L_b(f)\in\operatorname{Lin}(\Sigma_c^1(G;\mathscr{B})\odot\mathcal{H}_0)/N_b)$ such that*

$$L_b(f)[g\otimes\xi]=[f*g\otimes\xi]. \tag{5.8}$$

If $f\in\Gamma_c(G;\mathscr{B})$, then $L_b(f)\subset\overline{L(f)}$.

Proof. To see that (5.8) determines a well-defined operator, we need to see that

$$\sum_i[g_i\otimes\xi_i]=0\quad\text{implies}\quad\sum_i[f*g_i\otimes\xi_i]=0. \tag{5.9}$$

However,

$$\Big\|\sum_i[f*g_i\otimes\xi_i]\Big\|^2=\sum_{i,j}L_{\xi_i,\xi_j}(g_j^**f^**f*g_i). \tag{5.10}$$

Since $f\in\Sigma_c^1(G;\mathscr{B})$, we can approximate the right-hand side of (5.10) by sums of the form

$$\sum_{i,j}L_{\xi_i,\xi_j}(g_j^**h^**h*g_i), \tag{5.11}$$

where $h \in \Gamma_c(G;\mathscr{B})$. But (5.11) equals

$$\Big\|\overline{L(h)}\sum_i [g_i \otimes \xi_i]\Big\|^2,$$

which is zero if the left-hand side of (5.9) is zero. Hence the right-hand side of (5.9) is also zero and $L_b(f)$ is well-defined.

If $f \in \Gamma_c(G;\mathscr{B})$, then $L_b(f) \subset \overline{L(f)}$ by Lemma 5.9. ■

Now we prove the analogue of Muhly's technical lemma ([18, Lemma 3.33] or [22, Lemma B.12]) which will allow us to compute with Borel functions as we would expect.

LEMMA 5.11. *Suppose that $f \in \Sigma_c^1(G;\mathscr{B})$ and that k is a bounded Borel function on $G^{(0)}$ which is the pointwise limit of a uniformly bounded sequence from $C_0(G^{(0)})$. Then for all $g, h \in \Gamma_c(G;\mathscr{B})$ and $\xi, \eta \in \mathcal{H}_0$, we have the following.*

(a)
$$\begin{aligned}(L_b(f)[g\otimes\xi] \mid [h\otimes\eta]) &= ([f*g\otimes\xi] \mid [h\otimes\eta])\\ &= L_{\xi,\eta}(g^* * f * h)\\ &= L_{[g\otimes\xi],[h\otimes\eta]}(f),\end{aligned}$$

(b)
$$\begin{aligned}\big(M(k)[g\otimes\xi] | [h\otimes\eta]\big) &= L_{\xi,\eta}(h^* * ((k\circ r)\cdot g))\\ &= ([(k\circ r)\cdot g\otimes\xi] \mid [h\otimes\eta])\\ &= (M(k)L(g)\xi \mid L(h)\eta),\end{aligned}$$

(c)
$$(M(k)L_b(f)[g\otimes\xi] \mid [h\otimes\eta]) = (L_b((k\circ r)\cdot f)[g\otimes\xi] \mid [h\otimes\eta]).$$

Proof. We start with (a). The first equality is just the definition of $L_b(f)$. The second follows from the definition of the inner product on $\Sigma_c^1(G;\mathscr{B})\odot\mathcal{H}_0/\mathscr{N}_b$. If f is in $\Gamma_c(G;\mathscr{B})$, then the third equation holds just by untangling the definition of the generalized Radon measures $L_{\xi,\eta}$ and using property (b) of Definition 4.1. Therefore the third equality holds for $f \in \Sigma_c^1(G;\mathscr{B})$ by applying the continuity assertion in Lemma 5.5.

Part (b) is proved similarly. The first equation holds if $k \in C_0(G^{(0)})$ by definition of $M(k)$ and $L_{\xi,\eta}$. If $\{k_n\} \subset C_0(G^{(0)})$ is a bounded sequence converging pointwise to k, then $M(k_k) \to M(k)$ in the weak operator topology by the dominated convergence theorem. On the other hand, $g^* * (k_n \circ r)\cdot h \to g^* * (k\circ r)h$ in the required way. Thus $L_{\xi,\eta}(g^* * (k_n\circ r)\cdot h) \to L_{\xi,\eta}(g^* * (k\circ r)h)$ by Lemma 5.5. Thus the first equality is valid. The second equality is clear if $k \in C_0(G^{(0)})$ and passes to the limit as above. The third equality is simply our identification of $[g\otimes\xi]$ with $L(g)\xi$ as in Lemma 5.3.

For part (c), first note that if $f_n \to f$ and $k_n \to k$ are uniformly bounded sequences converging pointwise with supports in fixed compact sets independent of n, then $(k\circ r)\cdot f = \lim_n (k_n \circ r)\cdot f_n$. It follows that $(k\circ r)\cdot f \in \Sigma_c^1(G;\mathscr{B})$. Also, $[f\otimes\xi] = \lim[f_n\otimes\xi]$, and since $M(k)$ is bounded, part (b) implies that

$$M(k)[f\otimes\xi] = \lim_n M(k)[f_n\otimes\xi] = \lim_n[(k\circ r)\cdot f_n\otimes\xi] = [(k\circ r)\cdot f\otimes\xi].$$

Since it is not hard to verify that $M(k)^*[f\otimes\xi]=(\bar{k}\circ r)\cdot f\otimes\xi]$, we can compute that

$$\begin{aligned}(M(k)L_b(f)[g\otimes\xi]\mid[h\otimes\eta])&=([f*g\otimes\xi]\mid(\bar{k}\circ r)\cdot h\otimes\eta])\\&=([k\circ r)\cdot(f*g)\otimes\xi]\mid[h\otimes\eta])\\&=([((k\circ r)\cdot f)*g\otimes\xi]\mid[h\otimes\eta])\\&=(L_b((k\circ r)\cdot f)[g\otimes\xi]\mid[h\otimes\eta]).\ \blacksquare\end{aligned}$$

PROPOSITION 5.12. *Let μ be the Radon measure on $G^{(0)}$ associated to the pre-representation L by Proposition 4.2. Then μ is quasi-invariant.*

Proof. We need to show that the measures ν and ν^{-1} (defined in (4.2) and (4.3), respectively) are equivalent. Therefore, we have to show that if A is pre-compact in G, then $\nu(A)=0$ if and only if $\nu(A^{-1})=0$. Since $(A^{-1})^{-1}=A$, it is enough to show that A ν-null implies that A^{-1} is too. Since ν is regular, we may as well assume that A is a G_δ-set so that $\varphi:=\mathbb{1}_A$ is in $\mathcal{B}^1_c(G)$. Let $\tilde{\varphi}(x)=\varphi(x^{-1})$. We need to show that $\tilde{\varphi}(x)=0$ for ν-almost all x. Since A is a G_δ, we can find a sequence $\{\varphi_n\}\subset C_c^+(G)$ such that $\varphi_n\searrow\varphi$.

If ψ is any function in $C_c(G)$, then $\psi\varphi\bar{\psi}=|\psi|^2\varphi\in\mathcal{B}^1_c(G)$ and vanishes ν-almost everywhere. By the monotone convergence theorem,

$$\lambda(\psi\varphi\bar{\psi})(u):=\int_G|\psi(x)|^2\varphi(x)\,d\lambda^u(x)$$

defines a function in $\mathcal{B}^1_c(G^{(0)})$ which is equal to 0 for μ-almost all u. In particular, $M(\lambda(\psi\varphi\bar{\psi}))=0$.

Therefore

$$0=(M(\lambda(\psi\varphi\bar{\psi}))L(g)\xi\mid L(g)\xi)=L_{\xi,\xi}(g^**(\lambda(\psi\varphi\bar{\psi})\circ r)\cdot g)\tag{5.12}$$

for all $g\in\Gamma_c(G;\mathscr{B})$ and $\xi\in\mathcal{H}_0$. On the other hand, if (5.12) holds for all $g\in\Gamma_c(G;\mathscr{B})$, $\xi\in\mathcal{H}_0$ and $\psi\in C_c(G)$, then we must have $M(\lambda(\psi\varphi\bar{\psi}))=0$ for all $\psi\in C_c(G)$. Since $\varphi(x)\geq 0$ everywhere, this forces $|\psi(x)|^2\varphi(x)=0$ for ν-almost all x. Since ψ is arbitrary, we conclude that $\varphi(x)=0$ for ν-almost all x. Therefore it will suffice to show that

$$L_{\xi,\xi}(g^**(\lambda(\psi\tilde{\varphi}\bar{\psi})\circ r)\cdot g)=0\text{ for all }g\in\Gamma_c(G;\mathscr{B}),\ \xi\in\mathcal{H}_0\text{ and }\psi\in C_c(G),\tag{5.13}$$

where we have replaced φ with $\tilde{\varphi}$ in the right-hand side of (5.12). First, we compute that with φ in (5.12) we have

$$\begin{aligned}g^**(\lambda(\psi\varphi\bar{\psi})\circ r)\cdot g(z)&=\int_G g(x^{-1})^*(\lambda(\psi\varphi\bar{\psi})\circ r)\cdot g(x^{-1}z)\,d\lambda^{r(z)}(x)\\&=\int_G g(x^{-1})^*\lambda(\psi\varphi\bar{\psi})(s(x))g(x^{-1}z)\,d\lambda^{r(z)}(x)\\&=\int_G\int_G g(x^{-1})^*\overline{\psi(y)}\varphi(y)\psi(y)g(x^{-1}z)\,d\lambda^{s(x)}(y)\,d\lambda^{r(z)}(x),\end{aligned}$$

which, after sending $y\mapsto x^{-1}y$ and using left-invariance of the Haar system, is

$$=\int_G\int_G g(x^{-1})^*\overline{\psi(x^{-1}y)}\varphi(x^{-1}y)\psi(x^{-1}y)g(x^{-1}z)\,d\lambda^{r(z)}(y)\,d\lambda^{r(z)}(x)$$

which, after defining $F(x,y) := \psi(x^{-1}y)g(x^{-1})$ and $\varphi\cdot F(x,y) := \varphi(x^{-1}y)F(x,y)$ for $(x,y)\in G *_r G$, is

$$= \int_G\int_G F(x,y)^*\varphi\cdot F(z^{-1}x,z^{-1}y)\,d\lambda^{r(z)}(y)\,d\lambda^{r(z)}(x). \tag{5.14}$$

We will have to look at vector-valued integrals of the form (5.14) in some detail. Define $\iota : G *_r G \to G$ by $\iota(x,y) = x^{-1}$, and let $\iota^*\mathscr{B} = \{(x,y,b) : p(b) = x^{-1}\}$ be the pull-back bundle. If $\psi \in C_c(G)$ and $g\in\Gamma_c(G;\mathscr{B})$, then

$$(x,y)\mapsto \psi(y)g(x^{-1})$$

is a section in $\Gamma_c(G *_r G;\iota^*\mathscr{B})$, and it is not hard to see that such sections span a subspace dense in the inductive limit topology.[15]

LEMMA 5.13. *Suppose that $F_1,F_2\in\Gamma_c(G *_r G;\iota^*\mathscr{B})$. Then*

$$z\mapsto \int_G\int_G F_1(x,y)^*F_2(z^{-1}x,z^{-1}y)\,d\lambda^{r(z)}(y)\,d\lambda^{r(z)}(x)$$

defines a section in $\Gamma_c(G;\mathscr{B})$ which we denote by $F_1^* *_{\lambda*\lambda} F_2$.

Proof. Let $K = K^{-1}$ be a compact set in G such that $\operatorname{supp} F_i\subset K\times K$. Then $\operatorname{supp} F_1 *_{\lambda*\lambda} F_2\subset K^2$, and

$$\|F_1^* *_{\lambda*\lambda} F_2\|_\infty \le \|F_1\|_\infty\|F_2\|_\infty\lambda^u(K)^2.$$

Thus it suffices to show the result for F_i which span a dense subspace in the inductive limit topology. Therefore we may as well assume that $F_i(x,y) = \psi_i(y)g_i(x^{-1})$ as above. Then

$$\begin{aligned}F_1^* *_{\lambda*\lambda} F_2(z) &= \int_G\int_G \overline{\psi_1(y)}\psi_2(z^{-1}y)g_1(x^{-1})^*g_2(x^{-1}z)\,d\lambda^{r(z)}(y)\,d\lambda^{r(z)}(x)\\ &= \bar{\psi}_1 * \tilde{\psi}_2(z)g_1^* * g_2(z),\end{aligned}$$

and the result follows by Corollary 1.4. ■

LEMMA 5.14. *If $\psi\in C_c(G)$ and $g\in\Gamma_c(G;\mathscr{B})$, then*

$$(x,y)\mapsto\psi(x^{-1}y)g(x^{-1})$$

is a section in $\Gamma_c(G *_r G;\iota^*\mathscr{B})$ and the sections of this form span a dense subspace in the inductive limit topology.

Proof. It suffices to see that we can approximate sections of the form

$$(x,y)\mapsto \theta(x,y)f(x^{-1}) \tag{5.15}$$

with $\theta\in C_c(G *_r G)$ and $f\in\Gamma_c(G;\mathscr{B})$. A Stone–Weierstrass argument shows that we can approximate θ with sums of functions of the form $(x,y)\mapsto\psi(x^{-1}y)\psi'(x^{-1})$. Then we can approximate (5.15) by sums of sections of the required form: $(x,y)\mapsto\psi(x^{-1}y)g(x^{-1})$ where $g(x) := \psi'(x)f(x)$. This completes the proof. ■

[15] Our by now standard partition of unity argument (Lemma A.4) shows that sections of the form $(x,y)\mapsto\theta(x,y)g(x^{-1})$ for $\theta\in C_c(G *_r G)$ and $g\in\Gamma_c(G;\mathscr{B})$ span a dense subspace. We can approximate θ by sums of the form $\psi_1(x)\psi_2(y)$. But $\psi_1(x)\psi_2(y)g(x^{-1}) = \psi_2(y)\tilde{\psi}_1\cdot g(x^{-1})$.

Let $\mathcal{A}_0 \subset \Gamma_c(G *_r G; \iota^*\mathscr{B})$ be the dense subspace of sections of the form considered in Lemma 5.14. We continue to write φ for the characteristic function of our fixed pre-compact, ν-null set. Then we know from (5.12) that

$$L_{\xi,\xi}(F^* *_{\lambda*\lambda} (\varphi \cdot F)) = 0 \quad \text{for all } F \in \mathcal{A}_0. \tag{5.16}$$

It is not hard to check that if $\varphi' \in \mathcal{B}^1_c(G)$, then $F^* *_{\lambda*\lambda} (\varphi' \cdot F) \in \Sigma^1_c(G;\mathscr{B})$, and that if $F_n \to F$ in the inductive limit topology in $\Gamma_c(G *_r G; \iota^*\mathscr{B})$, then $\{F^*_n *_{\lambda*\lambda} (\varphi' \cdot F)\}$ is uniformly bounded and converges pointwise to $F *_{\lambda*\lambda} (\varphi' \cdot F)$. In particular, the continuity of the $L_{\xi,\xi}$ (see Lemma 5.5) implies that (5.16) holds for all $F \in \Gamma_c(G *_r G; \iota^*\mathscr{B})$. But if we define $\tilde{F}(x,y) := F(y,x)$, then we see from the definition that

$$\tilde{F}^* *_{\lambda*\lambda} (\varphi \cdot \tilde{F}) = F^* *_{\lambda*\lambda} (\tilde{\varphi} \cdot F)),$$

where we recall that $\tilde{\varphi}(x) := \varphi(x^{-1})$. Thus

$$L_{\xi,\xi}(F^* *_{\lambda*\lambda} (\tilde{\varphi} \cdot F))) = 0 \quad \text{for all } F \in \Gamma_c(G *_r G; \iota^*\mathscr{B}).$$

Since the above holds in particular for $F \in \mathcal{A}_0$, this implies (5.13) and completes the proof. ∎

We can now turn our attention to creating the Borel Hilbert bundle. We still need some "Borelogy" in the form of Lemma 5.11.

Lemma 5.15. *Let a and b be vectors in $\mathcal{H}_{00}$ (identified with $\Gamma_c(G;\mathscr{B}) \odot \mathcal{H}_0/\mathscr{N}$). Let $L_{a,b}$ be the generalized Radon measure given by*

$$L_{a,b}(f) := (L(f)a \mid b). \tag{5.17}$$

Then $|L_{a,b}| \ll \nu$, where ν is the measure on G given by (4.2), *and $|L_{a,b}|$ is the total variation of $L_{a,b}$ as defined in the paragraph preceding Proposition* 3.3.

Remark 5.16. Although the generalized Radon measure defined in (5.17) makes perfectly good sense for any $a, b \in \mathcal{H}_0$, notice that we are only claiming the result of the proposition when $a, b \in \mathcal{H}_{00}$ (because that is all we are able to prove).

Proof. It is enough to show that if M is a pre-compact ν-null set, then $|L_{a,b}|(M) = 0$. Since ν is a Radon measure, and therefore regular, we may as well assume that M is a G_δ-set. Thus if $\varphi := \mathbb{1}_M$, then there are $\varphi_n \in C^+_c(G)$ such that $\varphi_n(x) \searrow \varphi(x)$ as in Lemma 3.8 (which has been cooked up for just this purpose). Then, using Lemma 3.8, it will suffice to see that $L_{a,b}(f) = 0$ whenever $f \in \Sigma^1_c(G;\mathscr{B})$ and $\|f\| \le \varphi$.

On the other hand,

$$0 = \int_{G^{(0)}} \int_G \varphi(x)\, d\lambda^u(x)\, d\mu(u),$$

so there is a μ-null set $N \subset G^{(0)}$ such that $\lambda^u(M \cap G^u) = 0$ if $u \notin N$. As above, we can assume that N is a G_δ set. Thus if $f \in \Sigma^1_c(G;\mathscr{B})$ is such that $\|f\| \le \varphi$ and if $g \in \Gamma_c(G;\mathscr{B})$, then

$$f * g(x) = \int_G f(y)g(y^{-1}x)\, d\lambda^{r(x)}(y) = 0$$

whenever $r(x) \notin N$. Since $\operatorname{supp} \lambda^{r(x)} = G^{r(x)}$, it follows that for all $x \in G$ (without exception),

$$f * g(x) = \mathbb{1}_N(r(x)) f * g(x) = ((\mathbb{1}_N \circ r) \cdot f) * g(x). \tag{5.18}$$

Since $a, b \in \mathcal{H}_{00}$, it suffices to consider $a = [g \otimes \xi]$ and $b = [h \otimes \eta]$ (with $g, h \in \Gamma_c(G; \mathscr{B})$ and $\xi, \eta \in \mathcal{H}_0$). Note that f and $\mathbb{1}_N$ satisfy the hypotheses of Lemma 5.11. Therefore, by part (a) of that lemma,

$$\begin{aligned} L_{[g\otimes\xi, h\otimes\eta]}(f) &= ([f * g \otimes \xi] \mid [h \otimes \eta]) \\ &= ([((\mathbb{1}_N \circ r) \cdot f) * g \otimes \xi] \mid [h \otimes \eta]) \qquad \text{(by (5.18))} \\ &= (L_b((\mathbb{1}_N \circ r) \cdot f)[g \otimes \xi] \mid [h \otimes \eta]) \qquad \text{(by Lemma 5.11(a))} \\ &= (M(\mathbb{1}_N) L_b(f)[g \otimes \xi] \mid [h \otimes \eta]) \qquad \text{(by Lemma 5.11(c))} \\ &= 0 \quad \text{(since } M(\mathbb{1}_N) = 0\text{)}, \end{aligned}$$

as desired. This completes the proof. ∎

Since the measures ν and ν_0 are equivalent, for each $\xi, \eta \in \mathcal{H}_{00}$, we can, in view of Lemma 5.15, let $\rho_{\xi,\eta}$ be the Radon–Nikodym derivative of $|L_{\xi,\eta}|$ with respect to ν_0. Then for each $\xi, \eta \in \mathcal{H}_{00}$, we have

$$\begin{aligned} (L(f)\xi \mid \eta) = L_{\xi,\eta}(f) &= \int_G \epsilon(\xi,\eta)_x(f(x))\, d|L_{\xi,\eta}|(x) \\ &= \int_G \epsilon(\xi,\eta)_x(f(x)) \rho_{\xi,\eta}(x) \Delta(x)^{-1/2}\, d\nu(x) \\ &= \int_{G^{(0)}} \int_G \epsilon(\xi,\eta)_x(f(x)) \rho_{\xi,\eta}(x) \Delta(x)^{-1/2}\, d\lambda^u(x)\, d\mu(u), \end{aligned}$$

where, of course, $\epsilon(\xi,\eta)_x$ denotes a linear functional in the unit ball of $B(x)^*$ which depends on the choice of ξ and η in $\mathcal{H}_{00}$.

Remark 5.17. It is surely the case that there are interesting uniqueness conditions satisfied by the $\rho_{\xi,\eta}$ and the $\epsilon(\xi,\eta)_x$ that would make our subsequent calculations a bit tidier. That is, we would expect that, in some "almost everywhere" sense, $\epsilon(\xi,\eta)$ is linear in ξ and conjugate linear in η. A similar statement should hold for $\rho_{\xi,\eta}$. If true, this would make defining an inner product in Lemma 5.18 more straightforward. We will finesse these issues below by restricting to a countable set of ξ's and η's.

Our next computation serves to motivate the construction in Lemma 5.18. We need $\xi, \eta \in \mathcal{H}_{00}$ to be able to apply Lemma 5.15. To simplify the notation, we write ϵ in place of $\epsilon(\xi,\eta)$ and ρ in place of $\rho_{\xi,\eta}$. Then

$$\begin{aligned} (L(f)\xi \mid L(g)\eta) = (L(g^* * f)\xi \mid \eta) &= L_{\xi,\eta}(g^* * f) \\ &= \int_{G^{(0)}} \int_G \epsilon_x(g^* * f(x)) \rho(x) \Delta(x)^{-1/2}\, d\lambda^u(x)\, d\mu(u) \\ &= \int_{G^{(0)}} \int_G \int_G \epsilon_x(g(y^{-1})^* f(y^{-1}x)) \rho(x) \Delta(x)^{-1/2}\, d\lambda^u(u)\, d\lambda^u(x)\, d\mu(u) \end{aligned}$$

which, by Fubini[16] and sending $x \mapsto yx$, is

$$= \int_{G^{(0)}} \int_G \int_G \epsilon_{yx}(g(y^{-1})^* f(x))\rho(yx)\Delta(yx)^{-1/2}\, d\lambda^{s(y)}(x)\, d\lambda^u(y)\, d\mu(u)$$

which, after sending $y \mapsto y^{-1}$, and using the symmetry of ν_0, is

$$= \int_{G^{(0)}} \int_G \int_G \epsilon_{y^{-1}x}(g(y)^* f(x))\rho(y^{-1}x)\Delta(y)^{-1/2}\Delta(x)^{-1/2} \\ d\lambda^u(x)\, d\lambda^u(y)\, d\mu(u).$$

It will be convenient to establish some additional notation. We fix once and for all a countable orthonormal basis $\{\zeta_i\}$ for $\mathcal{H}_{00}$. (Actually, any countable linearly independent set whose span is dense in $\mathcal{H}_{00}$ will do.) We let

$$\mathcal{H}'_{00} := \operatorname{span}\{\zeta_i\}.$$

To make the subsequent formulas a bit easier to read, we will write ρ_{ij} in place of the Radon–Nikodym derivative ρ_{ζ_i,ζ_j} and ϵ^{ij}_x in place of the linear functional $\epsilon(\zeta_i,\zeta_j)_x$. The linear independence of the ζ_i guarantees that each $\alpha \in \Gamma_c(G;\mathscr{B}) \odot \mathcal{H}'_{00}$ can be written *uniquely* as

$$\alpha = \sum_i f_i \otimes \zeta_i$$

where all by finitely many f_i are zero.

LEMMA 5.18. *For each $u \in G^{(0)}$, there is a sesquilinear form $\langle \cdot\,,\cdot\rangle_u$ on $\Gamma_c(G;\mathscr{B}) \odot \mathcal{H}'_{00}$ such that*

$$\langle f \otimes \zeta_i, g \otimes \zeta_j\rangle_u = \int_G \int_G \epsilon^{ij}_{y^{-1}x}(g(y)^* f(x))\rho_{ij}(y^{-1}x)\Delta(yx)^{-1/2}\, d\lambda^u(x)\, d\lambda^u(y). \tag{5.19}$$

Furthermore, there is a μ-conull set $F \subset G^{(0)}$ such that $\langle \cdot\,,\cdot\rangle_u$ is a pre-inner product for all $u \in F$.

REMARK 5.19. As we noted in Remark 5.17, we fixed the ζ_i because it is not clear that the right-hand side of (5.19) is linear in ζ_i or conjugate linear in ζ_j.

Proof. The integrand in (5.19) has compact support and is Borel by Lemma 3.10. Therefore the integrals there and below are well-defined. Given $\alpha = \sum_i f_i \otimes \zeta_i$ and $\beta = \sum_j g_j \otimes \zeta_j$, we get a well-defined form via the definition

$$\langle \alpha, \beta\rangle_u = \sum_{ij} \int_G \int_G \epsilon^{ij}_{y^{-1}x}(g_j(y)^* f_i(x))\rho_{ij}(y^{-1}x)\Delta(yx)^{-1/2}\, d\lambda^u(x)\, d\lambda^u(y).$$

This clearly satisfies (5.19), and the linearity of the ϵ^{ij}_z implies that $(\alpha,\beta) \mapsto \langle \alpha,\beta\rangle_u$ is linear in α and conjugate linear in β. It only remains to provide a conull Borel set F such that $\langle \cdot\,,\cdot\rangle_u$ is positive for all $u \in F$.

[16] The integrand is Borel by Lemma 3.10, so there is no problem applying Fubini's Theorem here.

However, (5.19) was inspired by the calculation preceding the lemma. Hence if $\alpha := \sum_i f_i \otimes \zeta_i$, then

$$\begin{aligned}
(5.20)\qquad \Big\|\sum L(f_i)\zeta_i\Big\|^2 &= \sum_{ij} (L(f_i)\zeta_i \mid L(f_j)\zeta_j) \\
&= \sum_{ij} (L(f_j^* * f_i)\zeta_i \mid \zeta_j) \\
&= \sum_{ij} \int_{G^{(0)}} \int_G \int_G \epsilon^{ij}_{y^{-1}x}(f_j(y)^* f_i(x))\rho_{ij}(y^{-1}x)\Delta(xy)^{-1/2} \\
&\qquad\qquad d\lambda^u(x)\, d\lambda^u(y)\, d\mu(u) \\
&= \sum_{ij} \int_{G^{(0)}} \langle f_i \otimes \zeta_i, f_j \otimes \zeta_j\rangle_u \, d\mu(u) \\
&= \int_{G^{(0)}} \langle \alpha, \alpha\rangle_u \, d\mu(u).
\end{aligned}$$

Thus, for μ-almost all u, we have $\langle \alpha, \alpha\rangle_u \geq 0$. The difficulty is that the exceptional null set depends on α. However, since we have assumed $B := \Gamma_0(G; \mathscr{B})$ is separable, there is a sequence $\{f_i\} \subset \Gamma_c(G; \mathscr{B})$ which is dense in $\Gamma_c(G; \mathscr{B})$ in the inductive limit topology. Let $\mathcal{A}_0$ be the rational vector space spanned by the countable set $\{f_i \otimes \zeta_j\}_{i,j}$. Since $\mathcal{A}_0$ is countable, there is a μ-conull set F such that $\langle \cdot\,, \cdot\rangle_u$ is a positive $\mathbb{Q}$-sesquilinear form on $\mathcal{A}_0$. However, if $g_i \to g$ and $h_i \to h$ in the inductive limit topology in $\Gamma_c(G; \mathscr{B})$, then, since $\lambda^u \times \lambda^u$ is a Radon measure on $G \times G$, we have $\langle g_i \otimes \zeta_j, h_i \otimes \zeta_k\rangle_u \to \langle g \otimes \zeta_j, h \otimes \zeta_k\rangle_u$. It follows that for all $u \in F$, $\langle \cdot\,, \cdot\rangle_u$ is a positive sesquilinear form (over $\mathbb{C}$) on the complex vector space generated by

$$\{f \otimes \zeta_i : f \in \Gamma_c(G; \mathscr{B})\}.$$

However, as that is all of $\Gamma_c(G; \mathscr{B}) \odot \mathcal{H}'_{00}$, the proof is complete. ∎

Note that for *any* $u \in G^{(0)}$, the value of $\langle f \otimes \zeta_i, g \otimes \zeta_j\rangle_u$ depends only on $f|_{G^u}$ and $g|_{G^u}$. Furthermore, using a suitable vector-valued Tietze Extension Theorem (see Proposition A.5), we can view $\langle \cdot\,, \cdot\rangle_u$ as a sesquilinear form on $\Gamma_c(G^u; \mathscr{B}|_{G^u})$. (Clearly, each $f \in \Gamma_c(G; \mathscr{B})$ determines a section of $\Gamma_c(G^u; \mathscr{B}|_{G^u})$. We need the extension theorem to know that every section in $\Gamma_c(G^u; \mathscr{B}|_{G^u})$ arises in this fashion.)

Using our Tietze Extension Theorem as above, given $f \in \Gamma_c(G; \mathscr{B})$ and $b \in \mathscr{B}$ there is a section, denoted by $\check{\pi}(b)f$, such that[17]

$$(\check{\pi}(b)f)(x) = \Delta(z)^{1/2} b f(z^{-1}x) \qquad \text{for all } x \in G^{r(b)}.$$

Of course, $\check{\pi}(b)f$ is only well-defined on $G^{r(b)}$. Then, if $b \in \mathscr{B}$, we can compute that

$$\langle \check{\pi}(b)f \otimes \zeta_i, g \otimes \zeta_j\rangle_{r(b)} = \int_G \int_G \epsilon^{ij}_{y^{-1}x}(g(y)^* b f(z^{-1}x))\rho_{ij}(x^{-1}y)\Delta(z^{-1}xy)^{-1/2} \\ d\lambda^{r(b)}(y)\, d\lambda^{r(b)}(x)$$

[17] Recall that $r(b)$ is a shorthand for $r(p(b))$, and similarly for $s(b)$.

which, after sending $x \mapsto zx$, is

$$= \int_G \int_G \epsilon^{ij}_{y^{-1}zx}(g(y)^* b f(x))\rho_{ij}(x^{-1}z^{-1}y)\Delta(xy)^{-1/2} \, d\lambda^{r(b)}(y)\, d\lambda^{s(b)}(x)$$

which, after sending $y \mapsto zy$, is

$$\begin{aligned} &= \int_G \int_G \epsilon^{ij}_{y^{-1}x}(g(zy)^* b f(x))\rho_{ij}(x^{-1}y)\Delta(z)^{-1/2}\Delta(xy)^{-1/2} \, d\lambda^{s(b)}(y)\, d\lambda^{s(b)}(x) \\ &= \int_G \int_G \epsilon^{ij}_{y^{-1}x}((\check\pi(b^*)g)(y) f(x))\Delta(xy)^{-1/2}\, d\lambda^{s(b)}(y)\, d\lambda^{s(b)}(x) \\ &= \langle f\otimes \zeta_i, \check\pi(b^*)g\otimes\eta\rangle_{s(b)}. \end{aligned}$$

In particular, if $a \in B(u)$, then $(\check\pi(a)f)(x) = af(x)$ and

$$\langle \check\pi(a)f\otimes\zeta_i, g\otimes\zeta_j\rangle_u = \langle f\otimes\zeta_i, \check\pi(a^*)g\otimes\zeta_j\rangle_u. \tag{5.21}$$

Recall that G acts continuously on the left of $G^{(0)}$: $x\cdot s(x) = r(x)$. In particular, if C is compact in G and if K is compact in $G^{(0)}$, then

$$C\cdot K = \{x\cdot u : (x,u)\in G^{(2)}\cap (C\times K)\}$$

is compact. If $U \subset G^{(0)}$, then we say that U is *saturated* if U is G-invariant. More simply, U is saturated if $s(x)\in U$ implies $r(x)$ is in U. If $V\subset G^{(0)}$, then its *saturation* is the set $[V] = G\cdot V$, which is the smallest saturated set containing V.

The next result is a key technical step in our proof and takes the place of the Ramsay selection theorems ([25, Theorem 5.1; 26, Theorem 3.2]) used in Muhly's and Renault's proofs.

LEMMA 5.20. *We can choose the μ-conull Borel set $F\subset G^{(0)}$ in Lemma 5.18 to be saturated for the G-action on $G^{(0)}$.*

Proof. Let F be the Borel set from Lemma 5.18. We want to see that $\langle\cdot\,,\cdot\rangle_v$ is positive for all v in the saturation of F. To this end, suppose that $u\in F$ and that $z\in G$ is such that $s(z) = u$ and $r(z) = v$. If $b\in B(z)$, then

$$x\mapsto \Delta(z)^{1/2} b f(z^{-1}x)$$

is a section in $\Gamma_c(G^u;\mathscr{B}|_{G^u})$, and an application of Lemma A.4 shows that such sections span a dense subspace of $\Gamma_c(G^u;\mathscr{B}|_{G^u})$ in the inductive limit topology. Moreover, as we observed at the end of the proof of Lemma 5.18,

$$\langle f_i\otimes\zeta_j, g_i\otimes\zeta_k\rangle_v \to \langle f\otimes\zeta_j, g\otimes\zeta_k\rangle_v$$

provided $f_i\to f$ and $g_i\to g$ in the inductive limit topology in $\Gamma_c(G^v;\mathscr{B}|_{G^v})$. Therefore, to show that $\langle\cdot\,,\cdot\rangle_v$ is positive, it will suffice to check on vectors of the form $\alpha := \sum_i \check\pi(b_i)(f_i)\otimes\zeta_i$. Then using the calculation preceding (5.21), we have

$$\langle\alpha,\alpha\rangle_v = \sum_{ij}\langle\check\pi(b_j^* b_i)f_i\otimes\zeta_i, f_j\otimes\zeta_j\rangle_u. \tag{5.22}$$

However, since $B(z)$ is, in particular, a right Hilbert $B(u)$-module with inner product $\langle b_j, b_i\rangle_{B(u)} = b_j^* b_i$ (by part (e) of Definition 1.1), the matrix $(b_j^* b_i)$ is positive in $M_n(B(u))$ by [24, Lemma 2.65]. Therefore there are $d_{rs} \in B(u)$ such that $b_j^* b_i = \sum_k d_{kj}^* d_{ki}$. Then, using (5.22), the right-hand side of (5.22) is

$$\begin{aligned}\sum_{ijk} \langle \check\pi(d_{kj}^* d_{ki}) f_i \otimes \zeta_i, f_j \otimes \zeta_j\rangle_u &= \sum_{ijk} \langle \check\pi(d_{ki}) f_i \otimes \zeta_i, \check\pi(d_{kj}) f_j \otimes \zeta_j\rangle_u \\ &= \sum_k \Big\langle \sum_i \check\pi(d_{ki}) f_i \otimes \zeta_i, \sum_i \check\pi(d_{ki}) f_i \otimes \zeta_i \Big\rangle_u,\end{aligned}$$

which is positive since $u \in F$.

It only remains to verify that the saturation of F is Borel. Since μ is a Radon measure—and therefore regular—we can shrink F a bit if necessary, and assume it is σ-compact. Say $F = \bigcup K_n$. On the other hand, G is second countable and therefore σ-compact. If $G = \bigcup C_m$, then $[F] = \bigcup C_m \cdot K_n$. Since each $C_m \cdot K_n$ is compact, $[F]$ is σ-compact and therefore Borel. This completes the proof. ∎

From here on, we will assume that F is saturated. In view of Lemma 5.18, for each $u \in F$ we can define $\mathcal{H}(u)$ to be the Hilbert space completion of $\Gamma_c(G; \mathscr{B}) \odot \mathcal{H}_{00}'$ with respect to $\langle \cdot\,, \cdot\rangle_u$. We will denote the image of $f \otimes \zeta_i$ in $\mathcal{H}(u)$ by $f \otimes_u \zeta_i$. Since the complement of F is μ-null and also saturated, what we do off F has little consequence. In particular, G is the disjoint union of $G|_F$ and the ν-null set $G|_{G^{(0)}\setminus F}$.[18] Nevertheless, for niceties sake, we let $\mathcal{V}$ be a Hilbert space with an orthonormal basis $\{e_{ij}\}$ doubly indexed by the same index sets as for $\{f_i\}$ and $\{\zeta_j\}$, and set $\mathcal{H}(u) := \mathcal{V}$ for all $u \in G^{(0)} \setminus F$. We then let

$$G^{(0)} * \mathscr{H} = \{(u, h) : u \in F \text{ and } h \in \mathcal{H}(u)\},$$

and define $\Phi_{ij} : F \to F * \mathscr{H}$ by

$$\Phi_{ij}(u) := \begin{cases} f_i \otimes_u \zeta_j & \text{if } u \in F, \\ e_{ij} & \text{if } u \notin F. \end{cases}$$

(Technically, $\Phi_{ij}(u) = (u, f_i \otimes_u \zeta_j)$—at least for $u \in F$—but we have agreed to obscure this subtlety.) Then [30, Proposition F.8] implies that we can make $G^{(0)} * \mathscr{H}$ into a Borel Hilbert bundle over $G^{(0)}$ in such a way that the $\{\Phi_{ij}\}$ form a fundamental sequence (see [30, Definition F.1]). Note that if $f \otimes \zeta_i \in \Gamma_c(G; \mathscr{B}) \odot \mathcal{H}_{00}'$ and if $\Phi(u) := f \otimes_u \zeta_i$, then

$$u \mapsto \langle \Phi(u), \Phi_{ij}(u)\rangle_u$$

is Borel on F.[19] It follows that Φ is a Borel section of $G^{(0)} * \mathscr{H}$ and defines a class in $L^2(G^{(0)} * \mathscr{H}, \mu)$.

We can extend $\check\pi$ so that (5.21) holds for all $a \in B(u)^\sim$. If $p(b) \in G_u^v$, then $\|b\|^2 1_{B(u)} - b^* b$ is a positive element in the C^*-algebra $B(u)$. Therefore, there is a $k \in B(u)^\sim$ such

[18] The saturation of F is critical to what follows. If F is not saturated, then in general G is not the union of $G|_F$ and $G|_{G^{(0)}\setminus F}$. But as F *is* saturated, note that a homomorphism $\varphi : G|_F \to H$ can be trivially extended to a homomorphism on all of G by letting φ be suitably trivial on $G|_{G^{(0)}\setminus F}$. This is certainly not the case unless F is saturated.

[19] We can define $\Phi(u)$ to be zero off F. We are going to continue to pay as little attention as possible to the null complement of F in the following.

that

$$\|b\|^2 1_{B(u)} - b^*b = k^*k.$$

Then, using (5.21) and the computation that preceded it, we have

$$\begin{aligned}\|b\|^2\Big\langle \sum_i f_i \otimes \zeta_i, \sum_j f_j \otimes \zeta_j\Big\rangle_u - \Big\langle \sum_i \check\pi(b) f_i \otimes \zeta_i, \sum_j \check\pi(b) f_j \otimes \zeta_j\Big\rangle_v \\ = \sum_{ij} \langle \check\pi(\|b\|^2 1 - b * b) f_i \otimes \zeta_i, f_j \otimes \zeta_j\rangle_u \\ = \sum_{ij} \langle \check\pi(k) f_i \otimes \zeta_i, \check\pi(k) f_j \otimes \zeta_j\rangle_u \\ = \Big\langle \sum_i \check\pi(k) f_i \otimes \zeta_i, \sum_j \check\pi(k) f_j \otimes \zeta_j\Big\rangle_u \geq 0.\end{aligned}$$

In other words, if we define $\pi(b)$ by

$$\pi(b)\Big(\sum_i f_i \otimes \zeta_i\Big) = \sum_i \check\pi(b) f_i \otimes \zeta_i,$$

then

$$\langle \pi(b)(\alpha), \pi(b)(\alpha)\rangle_{r(b)} \leq \|b\|^2 \langle \alpha, \alpha\rangle_{s(b)} \quad \text{for } b \in p^{-1}(G|_F). \tag{5.23}$$

Therefore we get a bounded operator, also denoted by $\pi(b)$, from $\mathcal{H}(u)$ to $\mathcal{H}(v)$ with $\|\pi(b)\| \leq \|b\|$. Since (5.23) implies that $\pi(b)$ takes vectors of length zero to vectors of length zero, we have

$$\pi(b)(f \otimes_{s(b)} \zeta_i) = \check\pi(b) f \otimes_{r(b)} \zeta_i \quad \text{for all } b \in p^{-1}(G|_F).$$

If $b \notin p^{-1}(G|_F)$, then $\mathcal{H}(s(b)) = \mathcal{H}(r(b)) = \mathcal{V}$, and we can let $\pi(b)$ be the identity operator.

LEMMA 5.21. *The map $\hat\pi$ from $\mathscr{B}$ to $\operatorname{End}(G^{(0)} * \mathscr{H})$ defined by $\hat\pi(b) := (r(b), \pi(b), s(b))$ is a Borel $*$-functor. Consequently, $(\mu, G^{(0)} * \mathscr{H}, \hat\pi)$ is a strict representation of $\mathscr{B}$ on $L^2(G^{(0)} * \mathscr{H}, \mu)$.*

Proof. If $f \in \Gamma_c(G; \mathscr{B})$ and if $z \in G|_F$, then

$$\begin{aligned}&(\pi(f(z))\Phi_{ij}(s(z)) \mid \Phi_{kl}(r(z))) \\ &\quad = \int_G\int_G \epsilon^{jl}_{y^{-1}x}(f_k^*(y) f(z) f_i(z^{-1}x))\rho_{jl}(y^{-1}x)\Delta(z^{-1}xy)^{-1/2}\, d\lambda^{r(z)}(y)\, d\lambda^{r(z)}(x).\end{aligned}$$

Thus $z \mapsto (\pi(f(z))\Phi_{ij}(s(z)) \mid \Phi_{kl}(r(z)))$ is Borel on F by Lemma 3.11 (in the form of Example 3.12) and Fubini's Theorem. Since it is clearly Borel on the complement of F, $\hat\pi$ satisfies the Borel condition in Definition 4.5.[20] It is straightforward to verify the algebraic properties (that is, properties (a), (b) and (c) of Definition 4.5). For example, assuming that $x \in G^{r(a)}$, we have on the one hand,

$$(\check\pi(ab)f)(x) = \Delta(p(ab))^{1/2} abf(p(ab)^{-1}x),$$

[20]It suffices to check on a fundamental sequence.

while

$$\begin{aligned}(\check\pi(a)\check\pi(b)f)(x) &= \Delta(p(a))^{1/2}a(\check\pi(b)f)(p(a)^{-1}x)\\ &= \Delta(p(a)p(b))^{1/2}abf(p(b)^{-1}p(a)^{-1}x).\end{aligned}$$

Since $p(ab) = p(a)p(b)$, it follows that $\hat\pi$ is multiplicative on $p^{-1}(G|_F)$. Of course, it is clearly multiplicative on the complement (which is $p^{-1}(G|_{G^{(0)}\setminus F})$ since F is saturated). The other properties follow similarly. ∎

Lemma 5.22. *Each $f \otimes \zeta_i \in \Gamma_c(G;\mathscr{B}) \odot \mathcal{H}'_{00}$ determines a Borel section $\Phi(u) := f \otimes_u \zeta_i$ whose class in $L^2(G^{(0)} * \mathscr{H}, \mu)$ depends only on the class of $[f\otimes\zeta_i] \in \Gamma_c(G;\mathscr{B})\odot\mathcal{H}'_{00}/\mathscr{N} \subset \Gamma_c(G;\mathscr{B}) \odot \mathcal{H}_0/\mathscr{N} = \mathcal{H}_{00}$. Furthermore, there is a unitary isomorphism V of $\mathcal{H}$ onto $L^2(G^{(0)} * \mathscr{H}, \mu)$ such that $V(L(f)\zeta_i) = [\Phi]$.*

Proof. We have already seen that Φ is in $L^2(F * \mathscr{H}, \mu) \cong L^2(G^{(0)} * \mathscr{H}, \mu)$. More generally, the computation (5.20) in the proof of Lemma 5.18 shows that if $\alpha = \sum_i f_i \otimes \zeta_i$ and $\Psi(u) := \sum_i f_i \otimes_u \zeta_i$, then

$$\|\Psi\|_2^2 = \Big\|\sum_i L(f_i)\zeta_i\Big\|^2.$$

Thus there is a well defined isometric map V as in the statement of lemma mapping $\operatorname{span}\{L(f)\zeta_i : f \in \Gamma_c(G;\mathscr{B})\}$ onto a dense subspace of $L^2(F * \mathscr{H}, \mu)$. Since $\mathcal{H}'_{00}$ is dense in $\mathcal{H}_{00}$, and therefore in $\mathcal{H}$, the result follows by Corollary 5.2. ∎

The proof of Theorem 4.13 now follows almost immediately from the next proposition.

Proposition 5.23. *The unitary V defined in Lemma 5.22 intertwines L with a representation L' which is the integrated form of the strict representation $(\mu, G^{(0)} * \mathscr{H}, \hat\pi)$ from Lemma 5.21.*

Proof. We have $L'(f_1) = VL(f_1)V^*$. On the one hand,

$$(L(f_1)[f\otimes\zeta_i] \mid [g\otimes\zeta_j])_{\mathcal{H}} = (VL(f_1)[f\otimes\zeta_i] \mid V[g\otimes\zeta_j]) = (L'(f_1)V[f\otimes\zeta_1] \mid V[g\otimes\zeta_j]).$$

But the left-hand side is

$$\begin{aligned}(L(f_1*f)\zeta_i \mid L(g)\zeta_j) &= L_{\zeta_i,\zeta_j}(g^* * f_1 * f)\\ &= \int_{G^{(0)}}\int_G\int_G \epsilon^{ij}_{y^{-1}x}(g(y)^* f_1 * f(x))\rho_{ij}(y^{-1}x)\Delta(yx)^{-1/2}\,d\lambda^u(x)\,d\lambda^u(y)\,d\mu(u)\\ &= \int_{G^{(0)}}\int_G\int_G\int_G \epsilon^{ij}_{y^{-1}x}(g(y)^* f_1(z)f(z^{-1}x))\rho_{ij}(y^{-1}x)\Delta(yx)^{-1/2}\\ &\qquad\qquad d\lambda^u(z)\,d\lambda^u(x)\,d\lambda^u(y)\,d\mu(u)\\ &= \int_{G^{(0)}}\int_G\int_G\int_G \epsilon^{ij}_{y^{-1}x}(g(y)^*\check\pi(f_1(z))(f)(x))\rho_{ij}(y^{-1}x)\Delta(yx)^{-1/2}\Delta(z)^{-1/2}\\ &\qquad\qquad d\lambda^u(z)\,d\lambda^u(x)\,d\lambda^u(y)\,d\mu(u)\end{aligned}$$

$$
\begin{aligned}
&= \int_F \int_G \langle \check{\pi}(f_1(z)) f \otimes \zeta_i, g \otimes \zeta_j \rangle_u \Delta(z)^{-1/2} \, d\lambda^u(z) \, d\mu(u) \\
&= \int_F \int_G \langle \pi(f_1(z))(f \otimes_{s(z)} \zeta_i), (g \otimes_u \zeta_j \rangle_u \Delta(z)^{-1/2} \, d\lambda^u(z) \, d\mu(u) \\
&= \int_G \langle \pi(f_1(z)) V[f \otimes \zeta_i](s(z)), V[g \otimes \zeta_j](r(z)) \rangle_{r(z)} \Delta(z)^{-1/2} \, d\nu(z).
\end{aligned}
$$

Thus L' is the integrated form as claimed. ∎

6. Equivalence of Fell bundles

We want to formalize the notion of the equivalence of two Fell bundles. As with the definition of a Fell bundle presented in Definition 1.1, our formulation will be a modification of the existing definitions (cf. [31, Definition 1.5] and [19, Definition 10]). We are aiming to give a version which will be readily "check-able" even at the expense of length or elegance. First, however, we need to establish some notation and terminology.

Suppose that $p : \mathscr{B} \to G$ is a Fell bundle and that $q : \mathscr{E} \to T$ is an *upper semicontinuous* Banach bundle over a left G-space T. Then we say that $\mathscr{B}$ *acts* on (the left) of $\mathscr{E}$ if there is a continuous map $(b, e) \mapsto b \cdot e$ from

$$\mathscr{B} * \mathscr{E} := \{(b, e) \in \mathscr{B} \times \mathscr{E} : s(p(b)) = r(q(e))\}$$

to $\mathscr{E}$ such that

(a) $q(b \cdot e) = p(b) \cdot q(e)$,
(b) $a \cdot (b \cdot e) = (ab) \cdot e$ for appropriate $a, b \in \mathscr{B}$ and $e \in \mathscr{E}$,
(c) $\|b \cdot e\| = \|b\| \, \|e\|$.

Of course, there is an analogous notion of a right action of a Fell bundle.

If T is a (G, H)-*equivalence*, it is important to keep in mind that $[x \cdot t, t] \mapsto x$ is an isomorphism of the orbit space $(T *_s T)/H$ onto G and that $[t, t \cdot h] \mapsto h$ is an isomorphism of $G \backslash (T *_r T)$ onto H (see [20, §2]). We will write $[t, s]_G$ for the image of the orbit of (t, s) in G. Similarly, we will write $[t, s]_H$ for the image of the corresponding orbit in H.

If $q : \mathscr{E} \to T$ is a Banach bundle over a (G, H)-equivalence, then we will write $\mathscr{E} *_s \mathscr{E}$ for $\{(e, f) \in \mathscr{E} \times \mathscr{E} : s(q(e)) = s(q(f))\}$, and similarly for $\mathscr{E} *_r \mathscr{E}$.

Definition 6.1. Suppose that T is a (G, H)-equivalence, that $p_G : \mathscr{B} \to G$ and $p_H : \mathscr{C} \to H$ are Fell bundles, and that $q : \mathscr{E} \to T$ is an upper semicontinuous Banach bundle. We say that $q : \mathscr{E} \to T$ is a $\mathscr{B}$-$\mathscr{C}$-*equivalence* if the following conditions hold.

(a) There is a left $\mathscr{B}$-action and a right $\mathscr{C}$-action on $\mathscr{E}$ such that $b \cdot (e \cdot c) = (b \cdot e) \cdot c$ for all $b \in \mathscr{B}$, $e \in \mathscr{E}$ and $c \in \mathscr{C}$.
(b) There are sesquilinear maps $(e, f) \mapsto {}_{\mathscr{B}}\langle e, f \rangle$ from $\mathscr{E} *_s \mathscr{E}$ to $\mathscr{B}$ and $(e, f) \mapsto \langle e, f \rangle_{\mathscr{C}}$ from $\mathscr{E} *_r \mathscr{E}$ to $\mathscr{C}$ such that
   (i) $p_G({}_{\mathscr{B}}\langle e, f \rangle) = [q(e), q(f)]_G$ and $p_H(\langle e, f \rangle_{\mathscr{C}}) = [q(e), q(f)]_H$,
   (ii) ${}_{\mathscr{B}}\langle e, f \rangle^* = {}_{\mathscr{B}}\langle f, e \rangle$ and $\langle e, f \rangle_{\mathscr{C}}^* = \langle f, e \rangle_{\mathscr{C}}$,
   (iii) ${}_{\mathscr{B}}\langle b \cdot e, f \rangle = b \, {}_{\mathscr{B}}\langle e, f \rangle$ and $\langle e, f \cdot c \rangle_{\mathscr{C}} = \langle e, f \rangle_{\mathscr{C}} \, c$,
   (iv) ${}_{\mathscr{B}}\langle e, f \rangle \cdot g = e \cdot \langle f, g \rangle_{\mathscr{C}}$.

(c) With the actions coming from (a) and the inner products coming from (b), each $E(t)$ is a $B(r(t))$-$B(s(t))$-imprimitivity bimodule.

Lemma 6.2. *The map* $(b,e)\mapsto b\cdot e$ *induces an imprimitivity bimodule isomorphism of* $B(p(b))\otimes_{B(s(p(b)))} E(t)$ *onto* $E(p(b)\cdot t)$.

Proof. The proof is similar to that for Lemma 1.2. ∎

Our next observation is straightforward, but it will be helpful to keep it in mind in what follows.

Example 6.3. Suppose that $p:\mathscr{B}\to G$ is a Fell bundle over G. Since G is naturally a (G,G)-equivalence, we see immediately that $\mathscr{B}$ acts on the right and the left of itself with $\mathscr{B}*\mathscr{B}=\mathscr{B}^{(2)}$. If we define

$$_{\mathscr{B}}\langle a,b\rangle := ab^* \quad\text{and}\quad \langle a,b\rangle_{\mathscr{B}} := a^*b,$$

then it is a simple matter to check that $p:\mathscr{B}\to G$ is a $\mathscr{B}$-$\mathscr{B}$-equivalence.

We can now state our main result.

Theorem 6.4 (The equivalence theorem for Fell bundles). *Suppose that G and H are second countable groupoids with Haar systems $\{\lambda_G^u\}_{u\in G^{(0)}}$ and $\{\lambda_H^v\}_{v\in H^{(0)}}$, respectively. Suppose also that $p_{\mathscr{B}}:\mathscr{B}\to G$ and $p_H:\mathscr{C}\to H$ are Fell bundles and that $q:\mathscr{E}\to T$ is a $\mathscr{B}$-$\mathscr{C}$-equivalence. Then $\mathsf{X}_0:=\Gamma_c(T;\mathscr{E})$ becomes a $C^*(G,\mathscr{B})$-$C^*(H,\mathscr{C})$-pre-imprimitivity bimodule with respect to the actions and inner products given by*

$$f\cdot\xi(t) := \int_G f(x)\xi(x^{-1}\cdot t)\,d\lambda_G^{r(t)}(x), \tag{6.1}$$

$$\xi\cdot g(t) := \int_H \xi(t\cdot h)g(h^{-1})\,d\lambda_H^{s(t)}(h), \tag{6.2}$$

$$_{\star}\langle\!\langle \xi,\eta\rangle\!\rangle(x) := \int_H {}_{\mathscr{B}}\langle \xi(xth),\eta(th)\rangle\,d\lambda_H^{s(t)}(h), \tag{6.3}$$

$$\langle\!\langle \xi,\eta\rangle\!\rangle_{\star}(h) = \int_G \langle \xi(x^{-1}t),\eta(x^{-1}th)\rangle_{\mathscr{C}}\,d\lambda_G^{r(t)}(x). \tag{6.4}$$

Consequently, $C^*(G,\mathscr{B})$ and $C^*(H,\mathscr{C})$ are Morita equivalent.

Remark 6.5. Since T is a (G,H)-equivalence, $r_T(t)=r_T(s)$ implies that $s=t\cdot h$ for some $h\in H$. Thus, we are free to choose any t in (6.3) satisfying $r_T(t)=s_G(x)$. Similarly, in (6.4), any t satisfying $s_T(t)=r_H(h)$ will do.

The proof of Theorem 6.4 is a bit involved—for example, it is not even obvious that (6.1)–(6.4) define continuous sections of the appropriate bundles. In any event, we require some preliminary comments and set-up before launching into the proof in the next section. However, the next example is fundamental and should help to motivate some of what follows.

Example 6.6. If $p:\mathscr{B}\to G$ is a Fell bundle, then we can also view it as a $\mathscr{B}$-$\mathscr{B}$-equivalence as described in Example 6.3. Then the pre-imprimitivity bimodule structure imposed on $\Gamma_c(G;\mathscr{B})$ by Theorem 6.4 is the expected one. The left and right actions are given by convolution, as are the inner products

$$_\star\langle\langle f,g\rangle\rangle = f * g^* \quad\text{and}\quad \langle\langle f,b\rangle\rangle_\star = f^* * g.$$

Another example that will be useful illustrates the symmetry inherent in the definition of equivalence. Sadly, the notation obscures what is actually quite straightforward.

EXAMPLE 6.7. If T is a (G,H)-equivalence, then we can make the same space into a (H,G)-equivalence T^{op} as follows. Let $\iota : T \to T^{\mathrm{op}}$ be the identity map and define a left H-action and a left G-action by

$$\begin{aligned} r(\iota(t)) &:= s(t), & s(\iota(t)) &:= r(t),\\ h\cdot\iota(t) &:= \iota(t\cdot h^{-1}), & \iota(t)\cdot x &:= \iota(x^{-1}c\dot{t}). \end{aligned}$$

Similarly, if $p_{\mathscr{B}} : \mathscr{B} \to G$ and $p_{\mathscr{C}} : \mathscr{C} \to H$ are Fell bundles and if $q : \mathscr{E} \to T$ is a $\mathscr{B}$-$\mathscr{C}$-equivalence, then we can form a $\mathscr{C}$-$\mathscr{B}$-equivalence $\bar{q} : \overline{\mathscr{E}} \to T^{\mathrm{op}}$. Let $\overline{\mathscr{E}}$ be the conjugate vector space to $\mathscr{E}$. Thus if $\flat : \mathscr{E} \to \overline{\mathscr{E}}$ is the identity map, then scalar multiplication is given by $z\cdot\flat(e) := \flat(\bar{z}\cdot e)$ for all $z \in \mathbb{C}$. Let $\bar{q}(\flat(e)) := \iota(q(e))$. We then define $\mathscr{C}$- and $\mathscr{B}$-actions by

$$c\cdot\flat(e) := \flat(e\cdot c^*) \quad\text{and}\quad \flat(e)\cdot b := \flat(b^*\cdot e),$$

and sesquilinear forms by

$$_{\mathscr{C}}\langle\flat(e),\flat(f)\rangle := \langle e,f\rangle_{\mathscr{C}} \quad\text{and}\quad \langle\flat(e),\flat(f)\rangle_{\mathscr{B}} := {}_{\mathscr{B}}\langle e,f\rangle.$$

Now it is simply a matter of verifying the axioms to see that $\bar{q} : \overline{\mathscr{E}} \to T^{\mathrm{op}}$ is a $\mathscr{C}$-$\mathscr{B}$-equivalence such that $\overline{\mathscr{E}}(t) = E(t)^\sim$, where $E(t)^\sim$ is the dual imprimitivity bimodule to $E(t)$ (see [24, pp. 49–50]).

Of course, we obtain actions and inner products on $\Gamma_c(T^{\mathrm{op}};\overline{\mathscr{E}})$ using (6.1)–(6.4). When using these side-by-side with those on $\Gamma_c(T;\mathscr{E})$ it will usually be clear from context which we are applying. Nevertheless, to make matters a bit easier to decode, the inner products on $\Gamma_c(T^{\mathrm{op}};\overline{\mathscr{E}})$ will be denoted by $_{\bar\star}\langle\langle\cdot\,,\cdot\rangle\rangle$ and $\langle\langle\cdot\,,\cdot\rangle\rangle_{\bar\star}$.

The following technical lemma will be useful in exploiting the symmetry in Theorem 6.4.

LEMMA 6.8. *Define* $\Phi : \Gamma_c(T;\mathscr{E}) \to \Gamma_c(T^{\mathrm{op}};\overline{\mathscr{E}})$ *by* $\Phi(\xi)(\iota(t)) := \flat(\xi(t))$. *Then*

$$\begin{aligned} \Phi(\xi\cdot g) &= g^*\cdot\Phi(\xi), & \Phi(f\cdot\xi) &= \Phi(\xi)\cdot f^*,\\ _{\bar\star}\langle\langle\Phi(\xi),\Phi(\eta)\rangle\rangle &:= \langle\langle\xi,\eta\rangle\rangle_\star, & \langle\langle\Phi(\xi),\Phi(\eta)\rangle\rangle_{\bar\star} &= {}_\star\langle\langle\xi,\eta\rangle\rangle. \end{aligned}$$

Proof. The lemma follows from routine computations. For example,

$$\begin{aligned} g^*\cdot\Phi(\xi)(\iota(t)) &= \int_H g(h^{-1})^*\Phi(\xi)(h^{-1}\cdot\iota(t))\,d\lambda_H^{s(t)}(h)\\ &= \int_H \flat(\xi(t\cdot h)g(h^{-1}))\,d\lambda_H^{s(t)}(h) = \Phi(\xi\cdot g)(\iota(t)), \end{aligned}$$

while

$$\begin{aligned} _{\bar\star}\langle\langle\Phi(\xi),\Phi(\eta)\rangle\rangle(h) &= \int_G {}_{\mathscr{C}}\langle\Phi(\xi)(h\cdot\iota(t')\cdot x),\Phi(\eta)(\iota(t')\cdot x)\rangle\,d\lambda_G^{r(t')}(x)\\ &= \int_G {}_{\mathscr{C}}\langle\Phi(\iota(x^{-1}\cdot t'\cdot h^{-1})),\Phi(\eta)(\iota(x^{-1}\cdot t'))\rangle\,d\lambda_G^{r(t')}(x) \end{aligned}$$

which, after letting $t = t' \cdot h^{-1}$, is

$$= \int_G {}_{\mathscr{C}}\langle \flat(\xi(x^{-1} \cdot t)), \flat(\eta(x^{-1} \cdot t \cdot h))\rangle \, d\lambda_G^{r(t)}(x)$$
$$= \int_G \langle \xi(x^{-1} \cdot t), \eta(x^{-1} \cdot t \cdot h)\rangle_{\mathscr{C}} \, d\lambda_G^{r(t)}(x) = \langle\!\langle \xi, \eta\rangle\!\rangle_\star(h).$$

The other formulas are established similarly. ∎

REMARK 6.9. Lemma 6.8 will be very useful in the proof of Theorem 6.4. For example, once we establish that $\langle\!\langle \cdot\,, \cdot\rangle\!\rangle_\star$ is positive, it follows immediately that $\langle\!\langle \cdot\,, \cdot\rangle\!\rangle_{\bar\star}$ is also positive. Then, since

$${}_\star\langle\!\langle \xi, \xi\rangle\!\rangle = \langle\!\langle \Phi(\xi), \Phi(\xi)\rangle\!\rangle_{\bar\star},$$

we can say that the positivity of ${}_\star\langle\!\langle \cdot\,, \cdot\rangle\!\rangle$ “follows by symmetry”.

As is now standard, the key result needed for the proof of Theorem 6.4 is that we have approximate identities for $\Gamma_c(G; \mathscr{B})$ in the inductive limit topology of a special form. In the case of Fell bundles, even the existence of a one-sided approximate identity for $\Gamma_c(G; \mathscr{B})$ of any form is not so obvious.[21] Nevertheless, the result we want is the following.

PROPOSITION 6.10. *Suppose that $q : \mathscr{E} \to T$ is a $\mathscr{B}$-$\mathscr{C}$-equivalence. Then there is a net $\{e_\lambda\}$ in $\Gamma_c(G; \mathscr{B})$ consisting of elements of the form*

$$e_\lambda = \sum_{i=1}^{n_\lambda} {}_\star\langle\!\langle \xi_i^\lambda, \xi_i^\lambda\rangle\!\rangle,$$

with each ξ_i^λ in $\Gamma_c(T; \mathscr{E})$, which is an approximate identity in the inductive limit topology for the left action of $\Gamma_c(G; \mathscr{B})$ on itself and on $\Gamma_c(T; \mathscr{E})$.

Since the proof of Proposition 6.10 is rather technical, we will postpone it to Section 8. However, if $p : \mathscr{B} \to G$ is a Fell bundle, then, since $\mathscr{B}$ is naturally a $\mathscr{B}$-$\mathscr{B}$-equivalence and since each e_λ in Proposition 6.10 is self-adjoint by construction, an immediate corollary is that $\Gamma_c(G; \mathscr{B})$ itself has a two-sided approximate identity in the inductive limit topology. (As promised, this proves Proposition 5.1.)

7. Proof of the main theorem

It is high time to see that (6.1)–(6.4) define continuous sections as claimed. To see that (6.1) defines an element of $\Gamma_c(T; \mathscr{E})$, we note that our assumptions on $\mathscr{E}$ imply that $(x, t) \mapsto f(x)\xi(x^{-1} \cdot t)$ defines a section in $\Gamma_c(G * T; \tau^*\mathscr{E})$, where $\tau : G * T \to T$ is the projection map. Therefore, we will get what we need from the following lemma:

LEMMA 7.1. *If $f \in \Gamma_c(G * T; \tau^*\mathscr{E})$ and if*

$$\theta(f)(t) := \int_G f(x, t) \, d\lambda_G^{r(t)}(x),$$

then $\theta(f) \in \Gamma_c(T; \mathscr{E})$.

[21] The “standard” technique of using functions with small support and unit integral fail as integrals of form $\int_G f(x) \, d\lambda_G^u(x)$ are meaningless for $f \in \Gamma_c(G; \mathscr{B})$.

Proof. Suppose that $\operatorname{supp} f \subset K_G \times K_T$ with each factor compact. Then $\operatorname{supp} \theta(f) \subset K_T$ and

$$\|\theta(f)\|_\infty \leq M \|f\|_\infty$$

where M is an upper bound for $\lambda_G^u(K_G)$ (for $u \in G^{(0)}$). It follows that the collection of f for which $\theta(f) \in \Gamma_c(T; \mathscr{E})$ is closed in the inductive limit topology. Since sections of the form $(x,t) \mapsto \varphi(x)\psi(t)e(t)$ for $\varphi \in C_c(G)$, $\psi \in C_c(T)$ and $e \in \Gamma_c(T; \mathscr{E})$ span a dense subspace of $\Gamma_c(G * T; \tau^*\mathscr{E})$ by Lemma A.4, it suffices to consider f of the form $f(x,t) = \varphi(x)\psi(t)e(t)$. But then $\theta(f) = \psi(t)\lambda_G(\varphi)(r(t))e(t)$, which defines a continuous section on T since $\{\lambda_G^u\}_{u \in G^{(0)}}$ is a Haar system. ■

Of course, the argument for (6.2) is similar, as are the arguments for (6.3) and (6.4). For example, to establish (6.3), let $\sigma : X *_s X \to G$ be given by $\sigma(t,t') := [t,t']_G$. Then we will need the following analogue of Lemma 7.1.

Lemma 7.2. *Suppose that $F \in \Gamma_c(X *_s X; \sigma^*\mathscr{B})$. Then*

$$\lambda(F)([t,t']_G) := \int_H F(t \cdot h, t' \cdot h)\, d\lambda_H^{s(t')}(h)$$

defines a section $\lambda(F) \in \Gamma_c(G; \mathscr{B})$.

Sketch of the proof. Using Lemma A.4, we see that sections of the form $F(t,t') = \varphi(t,t')b([t,t']_G)$ with $\varphi \in C_c(X *_s X)$ and $b \in \Gamma_c(G; \mathscr{B})$ are dense in $\Gamma_c(X *_s X; \sigma^*\mathscr{B})$ in the inductive limit topology. But if F has this form, then $\lambda(F) = \lambda(\varphi) \cdot b$, where

$$\lambda(\varphi)([t,t']_G) := \int_H \varphi(t \cdot h, t' \cdot h)\, d\lambda_H^{s(t')}(h).$$

Since $\lambda(\varphi) \in C_c(X *_s X)$ by [20, Lemma 2.9(b)], $\lambda(F) \in \Gamma_c(G; \mathscr{B})$. We now proceed as in Lemma 7.1. ■

We apply this to (6.3) as follows. Notice that $F(t,t') := {}_{\mathscr{B}}\langle \xi(t), \eta(t') \rangle$ defines a section in $\Gamma_c(X *_s X; \sigma^*\mathscr{B})$, and then ${}_\star\langle\!\langle \xi, \eta \rangle\!\rangle(x) = \lambda(F)([xt,t]_G)$. Of course, (6.4) is dealt with in a similar fashion.

To complete the proof, we are going to appeal to [24, Proposition 3.12]. Thus we must to do the following.

IB1: Show that $\Gamma_c(T; \mathscr{E})$ is both a left $\Gamma_c(G; \mathscr{B})$- and a right $\Gamma_c(H; \mathscr{C})$-pre-inner product module (as in [24, Lemma 2.16]).

IB2: Show that the inner products span dense ideals.

IB3: Show that the actions are bounded in that

$$\langle\!\langle f \cdot \xi, f \cdot \xi \rangle\!\rangle_\star \leq \|f\|^2_{C^*(G,\mathscr{B})} \langle\!\langle \xi, \xi \rangle\!\rangle_\star \quad \text{and} \quad {}_\star\langle\!\langle \xi \cdot g, \xi \cdot g \rangle\!\rangle \leq \|g\|^2_{C^*(H,\mathscr{C})}\, {}_\star\langle\!\langle \xi, \xi \rangle\!\rangle.$$

IB4: Show that ${}_\star\langle\!\langle \xi, \eta \rangle\!\rangle \cdot \zeta = \xi \cdot \langle\!\langle \eta, \zeta \rangle\!\rangle_\star$.

Verifying IB4 is a nasty little computation: recall that

$${}_\star\langle\!\langle \xi, \eta \rangle\!\rangle(x) = \int_H {}_{\mathscr{B}}\langle \xi(x \cdot t' \cdot h), \eta(t' \cdot h) \rangle\, d\lambda_H^{s(t')},$$

where we are free to choose any t' such that $r_T(t') = r_G(x)$. Thus if $r(x) = r(t)$, then we can let $t' = x^{-1} \cdot t$. Therefore

$$\begin{aligned}{}_\star\langle\!\langle \xi, \eta\rangle\!\rangle \cdot \zeta(t) &= \int_G \langle\!\langle \xi, \eta\rangle\!\rangle x \cdot \zeta(x^{-1} \cdot t)\, d\lambda_G^{r(t)}(x) \\ &= \int_G \Big(\int_H {}_{\mathscr{B}}\langle \xi(t \cdot h), \eta(x^{-1} \cdot t \cdot h)\rangle \, d\lambda_H^{s(t)}(h) \Big) \cdot \zeta(x^{-1} \cdot t)\, d\lambda_G^{r(t)}(x).\end{aligned}$$

But if $e \in E(x^{-1} \cdot t)$, then $b \mapsto b \cdot e$ is a bounded linear map from $B(x)$ into $E(t)$. Thus the properties of vector-valued integrals (cf. [30, Lemma 1.91]) imply that

$${}_\star\langle\!\langle \xi, \eta\rangle\!\rangle \cdot \zeta(t) = \int_G \int_H {}_{\mathscr{B}}\langle \xi(t \cdot h), \eta(x^{-1} \cdot t \cdot h)\rangle \cdot \zeta(x^{-1} \cdot t)\, d\lambda_H^{s(t)}(h)\, d\lambda_G^{r(t)}(x)$$

which, since $E(t)$ is an imprimitivity bimodule, is

$$= \int_G \int_H \xi(t \cdot h) \cdot \langle \eta(x^{-1} \cdot t \cdot h), \zeta(x^{-1} \cdot t)\rangle_{\mathscr{C}}\, d\lambda_H^{s(t)}(h)\, d\lambda_G^{r(t)}(x)$$

which, after using Fubini and the properties of vector-valued integrals, is

$$= \int_H \xi(t \cdot h) \cdot \Big(\int_G \langle \eta(x^{-1} \cdot t \cdot h), \zeta(x^{-1} \cdot t)\rangle_{\mathscr{C}}\, d\lambda_G^{r(t)}(x) \Big)\, d\lambda_H^{s(t)}(h)$$

which, after replacing $t \cdot h$ by t' in the inner integral, is

$$\begin{aligned}&= \int_H \xi(t \cdot h) \cdot \Big(\int_G \langle \eta(x^{-1} \cdot t'), \zeta(x^{-1} \cdot t' \cdot h^{-1})\rangle_{\mathscr{C}}\, d\lambda_G^{r(t')}(x) \Big)\, d\lambda_H^{s(t)}(h) \\ &= \int_H \xi(t \cdot h) \cdot \langle\!\langle \eta, \zeta\rangle\!\rangle_\star (h^{-1})\, d\lambda_H^{s(t)}(h) = \xi \cdot \langle\!\langle \eta, \zeta\rangle\!\rangle_\star (t).\end{aligned}$$

This proves IB4.

To verify IB1, it suffices, by symmetry, to consider only ${}_\star\langle\!\langle \cdot\,, \cdot\rangle\!\rangle$. The algebraic properties are routine. For example, the axioms of an equivalence guarantee that $f \mapsto {}_{\mathscr{B}}\langle f, e\rangle$ is a bounded linear map of $E(x \cdot t)$ into $B(x)$. Thus the usual properties of vector-valued integration (cf. [30, Lemma 1.91]) imply that

$${}_\star\langle\!\langle f \cdot \xi(t), \eta(t)\rangle\!\rangle = \int_G {}_{\mathscr{B}}\langle f(x) \cdot \xi(x^{-1} \cdot t), \eta(t)\rangle \, d\lambda_G^{r(t)}(x).$$

Thus we can compute as follows:

$$\begin{aligned}{}_\star\langle\!\langle f \cdot \xi, \eta\rangle\!\rangle(x) &= \int_H {}_{\mathscr{B}}\langle f \cdot \xi(x \cdot t \cdot h), \eta(t \cdot h)\rangle \, d\lambda_H^{s(t)}(h) \\ &= \int_H \int_G {}_{\mathscr{B}}\langle f(y) \cdot \xi(y^{-1}x \cdot t \cdot h), \eta(t \cdot h)\rangle \, d\lambda_G^{r(x)}(y)\, d\lambda_H^{s(t)}(h)\end{aligned}$$

which, in view of part (b)(iii) of Definition 6.1, is

$$\begin{aligned}&= \int_H \int_G f(y)\, {}_{\mathscr{B}}\langle \xi(y^{-1}x \cdot t \cdot h), \eta(t \cdot h)\rangle \, d\lambda_G^{r(x)}(y)\, d\lambda_H^{s(t)}(h) \\ &= \int_G f(y) \cdot {}_\star\langle\!\langle \xi, \eta\rangle\!\rangle(y^{-1}x)\, d\lambda_G^{r(x)}(y) \\ &= f * {}_\star\langle\!\langle \xi, \eta\rangle\!\rangle(x).\end{aligned}$$

To complete the verification of IB1, we only need to see that the pre-inner products are positive. We will show this (as well as IB2) using the approximate identity developed in Proposition 6.10.

It is not hard to see that the inner products respect the inductive limit topology, and convergence in the inductive limit topology in $\Gamma_c(G;\mathscr{B})$ certainly implies convergence in the C^*-norm. Thus we have

$$\begin{aligned}\langle\!\langle \xi,\xi\rangle\!\rangle_\star &= \lim_\lambda \langle\!\langle e_\lambda * \xi,\xi\rangle\!\rangle_\star = \lim_\lambda \sum_{i=1}^{n_\lambda} \langle\!\langle {}_\star\langle\!\langle \xi_i^\lambda,\xi_i^\lambda\rangle\!\rangle \cdot \xi,\xi\rangle\!\rangle_\star \\ &= \lim_\lambda \sum_i \langle\!\langle \xi_i^\lambda \cdot \langle\!\langle \xi_i^\lambda,\xi\rangle\!\rangle_\star,\xi\rangle\!\rangle_\star = \lim_\lambda \sum_i \langle\!\langle \xi_i^\lambda,\xi\rangle\!\rangle_\star^* \langle\!\langle \xi_i^\lambda,\xi\rangle\!\rangle_\star,\end{aligned}$$

which shows positivity of $\langle\!\langle \cdot\,,\cdot\rangle\!\rangle_\star$. The positivity of ${}_\star\langle\!\langle \cdot\,,\cdot\rangle\!\rangle$ follows by symmetry.

Since $\xi = \lim_\lambda e_\lambda * \xi$, similar considerations show the span of the inner products are dense as required in IB2.

This leaves only IB3 to establish, and by symmetry, it is enough to show that

$$\langle\!\langle f\cdot\xi, f\cdot\xi\rangle\!\rangle_\star \le \|f\|^2 \langle\!\langle \xi,\xi\rangle\!\rangle_\star. \tag{7.1}$$

If ρ is a state on $C^*(G,\mathscr{B})$, then the pre-inner product

$$(\cdot \mid \cdot)_\rho := \rho(\langle\!\langle \cdot\,,\cdot\rangle\!\rangle_\star)$$

makes $\Gamma_c(T;\mathscr{E})$ into a pre-Hilbert space. Let $\mathcal{H}_0$ be the dense image of $\Gamma_c(T;\mathscr{E})$ in the Hilbert space completion. It is not hard to check that the left action of $\Gamma_c(G;\mathscr{B})$ on $\Gamma_c(T;\mathscr{E})$ defines a pre-representation L of $\mathscr{B}$ on $\mathcal{H}_0$. Now we employ the full power of the Disintegration Theorem (Theorem 4.13) to conclude that L extends to a bona fide representation of $\Gamma_c(G;\mathscr{B})$ (which is therefore norm-reducing for the universal norm). Therefore

$$\rho(\langle\!\langle f\cdot\xi, f\cdot\xi\rangle\!\rangle_\star) \le \|f\|^2_{C^*(G,\mathscr{B})}\rho(\langle\!\langle \xi,\xi\rangle\!\rangle_\star).$$

Since this holds for all ρ, we have established (7.1). This completes the proof of Theorem 6.4 with the exception of the proof of the existence of approximate identities of the required type.

8. Proof of Proposition 6.10

Recall that we write A for the C^**-algebra* $\Gamma_0(G^{(0)};\mathscr{B})$. Abusing notation a bit, we will let $\Gamma_c(G^{(0)};\mathscr{B})^+$ denote the positive elements in A with compact support.

LEMMA 8.1. *Let* $\Lambda_c := \{a \in \Gamma_c(G^{(0)};\mathscr{B})^+ : \|a\| \le 1\}$. *Then* Λ_c *is a net directed by itself such that for all* $\xi \in \Gamma_c(T;\mathscr{E})$, *we have* $a\cdot\xi \to \xi$ *uniformly, where* $a\cdot\xi(t) := a(r(t))\cdot\xi(t)$.

Proof. Note that if $a \ge b \ge 0$ in A^+, then for each $u \in G^{(0)}$, we have $a(u) \ge b(u) \ge 0$. Also, given $b \in A(u)^+$, there is an $a \in \Lambda_c$ such that $a(u) = b$. It follows that, for any $t \in T$, $a(r(t))\cdot\xi(t) \to \xi(t)$ as $a \nearrow 1$ (since $\xi(t) \in E(t)$, and $E(t)$ is a left Hilbert $A(r(t))$-module).

If $a \cdot g$ does not converge to g uniformly, then there is an $\epsilon > 0$, a subnet $\{a_i\}$ and $t_i \in \operatorname{supp} \xi$ such that

$$\|a(r(t_i)) \cdot \xi(t_i) - \xi(t_i)\| \geq \epsilon. \tag{8.1}$$

Since $\operatorname{supp} \xi$ is compact, we can pass to a subnet, relabel, and assume that $t_i \to t$. Then

$$\begin{aligned}\|a(r(t_i)) \cdot \xi(t_i) - \xi(t_i)\| \leq \|a(r(t_i)) \cdot \xi(t_i) - a(r(t)) \cdot \xi(t)\| \\ + \|a(r(t)) \cdot \xi(t) - \xi(t)\| + \|\xi(t) - \xi(t_i)\|.\end{aligned}$$

Since the terms on the right-hand side all tend to zero with i, we eventually contradict (8.1). This completes the proof. ■

LEMMA 8.2. *Suppose that $a \in \Lambda_c := \Gamma_c(G^{(0)}; \mathscr{B})^+$, that $\epsilon > 0$ and that $K \subset T$ is compact. Then there are $\xi_1, \ldots, \xi_n \in \Gamma_c(T; \mathscr{E})$ such that*

$$\Big\|a(r(t)) - \sum_{i=1}^{n} {}_{\mathscr{B}}\langle \xi_i(t), \xi_i(t)\rangle\Big\| < \epsilon \quad \textit{for all } t \in K. \tag{8.2}$$

Proof. Since $\mathscr{E}$ is an equivalence, $E(t)$ is a full left Hilbert $B(r(t))$-module. It follows from [22, Lemma 6.3] that for each $t \in T$, there are $\zeta_1^t, \ldots, \zeta_{n_t}^t \in \Gamma_c(T; \mathscr{E})$ such that (8.2) holds at t. Since K is compact and since $b \mapsto \|b\|$ is upper semicontinuous, there is a finite cover $\{V_1, \ldots, V_m\}$ of K and sections $\zeta_1^j, \ldots, \zeta_{n_j}^j$ such that

$$\Big\|a(r(t)) - \sum_{i=1}^{n_j} {}_{\mathscr{B}}\langle \zeta_i^j(t), \zeta_i^j(t)\rangle\Big\| < \epsilon \quad \text{for all } t \in V_j.$$

Let $\{\varphi_j\}$ be a partition of unity subordinate to the $\{V_j\}$ so that each $\varphi_j \in C_c^+(T)$ with $\operatorname{supp} \varphi_j \subset V_j$, $\sum_j \varphi_j(t) = 1$ if $t \in K$ and the sum is less than or equal to 1 otherwise. Then

$$\|a(r(t)) - \sum_{j=1}^{m} \varphi_j(t) \sum_{n=1}^{n_j} {}_{\mathscr{B}}\langle \zeta_i^j(t), \zeta_i^j(t)\rangle\| < \epsilon \quad \text{for all } t \in K.$$

Now we can let $\xi_{ij}(t) := \varphi_j(t)^{\frac{1}{2}} \zeta_i^j(t)$. This suffices. ■

REMARK 8.3. To ease the notation a bit, we are going to let

$$\Upsilon(x, t) = \sum_{i=1}^{n} {}_{\mathscr{B}}\langle \xi_i(x \cdot t), \xi_i(t)\rangle \tag{8.3}$$

so that $\Upsilon(r(t), t))$ is the sum appearing in (8.2). Notice that $\Upsilon(x, t) \in B(x)$.

Proof of Proposition 6.10. In view of Examples 6.3 and 6.6, it suffices to treat only the case of $\Gamma_c(G; \mathscr{B})$ acting on $\Gamma_c(T; \mathscr{E})$.

As a first step, for each $a \in \Lambda_c$ and finite set $F \subset \Gamma_c(T; \mathscr{E})$, we will produce a net $\{e_j^{a,F}\}_{j \in J}$ of the required form such that, as j increases, $e_j^{a,F} * \xi \to a \cdot \xi$ in the inductive limit topology for each $\xi \in F$. These nets are to be indexed by pairs $j = (V, \epsilon)$ where $\epsilon > 0$ and V is a conditionally compact neighborhood of $G^{(0)}$ all contained in a fixed conditionally compact neighborhood V_0. Since we will arrange that $\operatorname{supp} e_{V,\epsilon}^{a,F} \subset V$, it will follow that

$$\operatorname{supp} e_{V,\epsilon}^{a,F} * \xi \subset V \cdot \operatorname{supp} \xi \subset V_0 \cdot \operatorname{supp} \xi.$$

Since V_0 is conditionally compact, there is a compact set K_F, depending only on F, such that

$$\operatorname{supp} e_{V,\epsilon}^{a,F} * \xi \subset V_0 \cdot \operatorname{supp} \xi \subset V_0 \cap r_G^{-1}(r_T(\operatorname{supp} \xi)) \cdot \operatorname{supp} \xi \subset K_F.$$

Consequently, we just have to show that $e_j^{a,F} * \xi \to a \cdot \xi$ uniformly.

Therefore, we fix $a \in \Lambda_c$ and $F \subset \Gamma_c(T; \mathscr{E})$. We also fix $(V, \epsilon) \in J$. Let $D \subset T$ be a compact set such that $V_0 \cdot \operatorname{supp} \xi \subset D$ for all $\xi \in F$. Given $\epsilon > 0$, Lemma 8.2 implies that there are $\xi_1, \dots, \xi_n \in \Gamma_c(T; \mathscr{E})$ such that

$$\|a(r(t)) - \Upsilon(r(t), t)\| < \epsilon \quad \text{for all } t \in D, \tag{8.4}$$

where Υ is defined as in (8.3).

Remark 8.4. Since $\|a(u)\| \leq 1$ for all u, we can assume that $\|\Upsilon(r(t), t)\| \leq 2$ for all $t \in D$.

As in [22, §6], we can find functions $\varphi_1, \dots, \varphi_m \in C_c^+(T)$ such that if

$$F(x, t) := \sum_{j=1}^{m} \varphi_j(t) \varphi_j(x^{-1} \cdot t),$$

then

$$F(x, t) = 0 \quad \text{if } x \notin V \text{ or } t \notin D, \tag{8.5}$$

$$\left| \int_G \int_H F(x, t \cdot h) \, d\lambda_H^{s(t)}(h) \, d\lambda_G^{r(t)}(x) - 1 \right| < \epsilon \quad \text{provided } t \in D, \tag{8.6}$$

$$\int_G \int_H F(x, t \cdot h) \, d\lambda_H^{s(t)}(h) \, d\lambda_G^{r(t)}(x) \leq 2 \quad \text{for all } t. \tag{8.7}$$

Then we define $e = e_{V,\epsilon}^{a,F}$ by

$$e(x) = \sum_{ij} {}_\star\langle\!\langle \omega_{ij}, \omega_{ij} \rangle\!\rangle,$$

where $\omega_{ij}(t) := \varphi_i(t) \xi_j(t)$. Thus

$$e(x) = \int_H F(x, t \cdot h) \Upsilon(x, t \cdot h) \, \lambda_H^{s(t)}(h).$$

Claim. *There is a conditionally compact neighborhood V of $G^{(0)}$ such that $y \in V$ and $(s, t) \in D *_r D$ implies that*

$$\|\Upsilon(y, t) \xi(y^{-1} \cdot s) - \Upsilon(r(t), t) \xi(s)\| < \epsilon \quad \textit{for all } \xi \in F. \tag{8.8}$$

Proof of Claim. It suffices to produce V such that (8.8) holds for a fixed $\xi \in F$. If the claim were false, then for some $\epsilon_0 > 0$ and every neighborhood V of $G^{(0)}$ inside some fixed conditionally compact neighborhood V_0, we could find $(s_V, t_V) \in D *_r D$ and $y_V \in V \cap r_G^{-1}(r_X(D))$ such that

$$\|\Upsilon(y_V, t_V) \xi(y_V^{-1} \cdot s_V) - \Upsilon(r(t_V), t_V) \xi(s_V)\| \geq \epsilon_0. \tag{8.9}$$

Since V_0 is conditionally compact, $V_0 \cap r_G^{-1}(r_X(D))$ has compact closure. Therefore, we can pass to a subnet, relabel, and assume that $(s_V, t_V) \to (s, t)$ while $y_V \to r(t) = r(s)$. Since $b \mapsto \|b\|$ is upper semicontinuous on $\mathscr{B}$, this eventually contradicts (8.9). This completes the proof of the claim. ∎

Now we compute as follows:

$$
\begin{aligned}
&\|e * \xi(t) - a \cdot \xi(t)\| \\
&\quad = \Big\| \int_G \int_H F(y, t\cdot h)\Upsilon(y, t\cdot h)\xi(y^{-1}\cdot t)\, \lambda_H^{s(t)}(h)\, \lambda_G^{r(t)}(y) - a(r(t))\cdot g(t) \Big\| \\
&\quad = \int_G \int_H F(y, t\cdot h) \|\Upsilon(y, t\cdot h)\xi(y^{-1}\cdot t) - \Upsilon(r(t), t\cdot h)\cdot \xi(t)\|\, d\lambda_H^{s(t)}(h)\, d\lambda_G^{r(t)}(y) \qquad (8.10)\\
&\qquad + \int_G \int_H F(y, t\cdot h) \|(\Upsilon(r(t), t\cdot h) - a(r(t)))\cdot \xi(t))\|\, d\lambda_H^{s(t)}(h)\, d\lambda_G^{r(t)}(y) \qquad (8.11)\\
&\qquad + \Big| \int_G \int_H F(y, t\cdot h)\, d\lambda_H^{s(t)}(h)\, d\lambda_G^{r(t)}(y) - 1 \Big| \, \|a(r(t))\cdot \xi(t)\|. \qquad (8.12)
\end{aligned}
$$

Since $\operatorname{supp} e * \xi \subset D$, $\|e * \xi(t) - a \cdot \xi(t)\| = 0$ if $t \notin D$. Therefore the integrand in (8.10) is nonzero only if $y \in V$ and $(t, t \cdot h) \in D *_r D$. Thus (8.10) is bounded by 2ϵ by our choice of V and by (8.7). Since (8.11) also vanishes if $t \cdot h \notin D$, (8.11) is bounded by $2\epsilon\|\xi\|_\infty$ by (8.4) and (8.7). Equation (8.12) is bounded by $\epsilon\|\xi\|_\infty$ by (8.6) (since $t \in D$) and since $\|a(u)\| \leq 1$ for all u. Thus $\|e * \xi(t) - a \cdot \xi(t)\| \leq 2\epsilon + 2\epsilon\|\xi\|_\infty + \epsilon\|\xi\|_\infty$. It follows that $e_{V,\epsilon}^{a,F} * \xi \to a \cdot \xi$ uniformly with (V, ϵ) for all $\xi \in F$. This completes the "first step".

For the next step, view $\{e_{V,\epsilon}^{a,F}\}$ as net indexed by increasing a and F and decreasing V and ϵ. Since we have already dealt with the supports, we will complete the proof by showing that $\{e_{V,\epsilon}^{a,F}\}$ has a subnet $\{d_\lambda\}_{\lambda\in\Lambda}$ such that $d_\lambda * \xi \to \xi$ uniformly for all $\xi \in \Gamma_c(T; \mathscr{E})$.

Let Λ be the collection of 5-tuples (a, F, V, ϵ, n) where $a \in \Lambda_c$, F is a finite subset of $\Gamma_c(T; \mathscr{E})$, V is a conditionally compact neighborhood of $G^{(0)}$ contained in V_0, $\epsilon > 0$ and $n \in \mathbf{Z}^+$ is such that

$$\|e_{V,\epsilon}^{a,F} * \xi - \xi\|_\infty < 1/n \quad \text{for all } \xi \in F.$$

To see that Λ is directed, suppose that $(a_i, F_i, V_i, \epsilon_i, n_i)$ is an element of Λ for $i = 1, 2$. Let $F = F_1 \cup F_2$. Using Lemma 8.1, there is a $b \in \Lambda_c$ such that

$$\|b \cdot \xi - \xi\|_\infty < \frac{1}{2(n_1 + n_2)} \quad \text{for all } \xi \in F.$$

By the first part of this proof, we can find (V, ϵ) dominating (V_i, ϵ_i) for $i = 1, 2$ such that

$$\|e_{V,\epsilon}^{b,F} * \xi - b \cdot \xi\|_\infty < \frac{1}{2(n_1 + n_2)} \quad \text{for all } \xi \in F.$$

Then $(b, F, V, \epsilon, n_1 + n_2) \in \Lambda$ and Λ is directed.

We then get a subnet $\{d_\lambda\}_{\lambda\in\Lambda}$ of $\{e_{V,\epsilon}^{a,F}\}$ by letting $d_{a,F,V,\epsilon,n} = e_{V,\epsilon}^{a,F}$. To see that $d_\lambda * \xi \to \xi$ in the inductive limit topology, it suffices, since $V \subset V_0$ gives control of the supports, to show that $d_\lambda * \xi \to \xi$ uniformly. If $\delta > 0$, then there is an n_0 such that $1/n_0 < \delta$, and we can find a_0 such that

$$\|a_0 \cdot \xi - \xi\|_\infty < \frac{1}{2n_0}.$$

If $F := \{\xi\}$, then we can find (V_0, ϵ_0) such that

$$\|e_{V_0,\epsilon_0}^{a_0,F_0} * \xi - a_0 \cdot \xi\|_\infty < \frac{1}{2n_0}.$$

Therefore $(a_0, F_0, V_0, \epsilon_0, n_0) \in \Lambda$ and if $(a, F, V, \epsilon, n) \geq (a_0, F_0, V_0, \epsilon_0, n_0)$, then we have

$$\|e^{a,F}_{V,\epsilon} * \xi - \xi\|_\infty < \delta.$$

This completes the proof. ∎

Appendix A. Upper semicontinuous Banach bundles

We are interested in fibered C^*-algebras as a groupoid G must act on the sections of a bundle that is fibered over the unit space (or over some G-space). In [27] and in [15], it was assumed that the algebra A was the section algebra of a C^*-bundle as defined, for example, by Fell in [5]. However, recent work has made it clear that the notion of a C^*-bundle, or for that matter a Banach bundle, as defined in this way is unnecessarily restrictive, and that it is sufficient to assume only that A is a $C_0(G^{(0)})$-algebra [13, 14, 16, 17]. However, our approach here, as in [15] (and in [27]), makes substantial use of the total space of the underlying bundle. Although it predates the term "$C_0(X)$-algebra", the existence of a Banach bundle whose section algebra is a given Banach $C_0(X)$-module goes back to [4, 10–12]. We give some of the basic definitions and properties here for the sake of completeness.

The basics on $C_0(X)$-algebras is available from numerous sources. A summary can be found in [30, Appendix C.1]. A discussion of upper semicontinuous C^*-bundles and their connection to $C_0(X)$-algebras is laid out in [30, Appendix C.2]. Here we need a bit more as the definition of Fell bundles requires upper semicontinuous Banach bundles (as opposed to upper semicontinuous C^*-bundles).

This definition is a minor variation on [4, Definition 1.1] and should be compared with [30, Definition C.16].

Definition A.1. An *upper semicontinuous Banach bundle* over a topological space X is a topological space $\mathscr{A}$ together with a continuous, open surjection $p = p_{\mathscr{A}} : \mathscr{A} \to X$ and complex Banach space structures on each fiber $\mathscr{A}_x := p^{-1}(\{x\})$ satisfying the following axioms.

- B1: The map $a \mapsto \|a\|$ is upper semicontinuous from $\mathscr{A}$ to $\mathbb{R}^+$. (That is, for all $\epsilon > 0$, $\{a \in \mathscr{A} : \|a\| \geq \epsilon\}$ is closed.)
- B2: If $\mathscr{A} * \mathscr{A} := \{(a, b) \in \mathscr{A} \times \mathscr{A} : p(a) = p(b)\}$, then $(a, b) \mapsto a + b$ is continuous from $\mathscr{A} * \mathscr{A}$ to $\mathscr{A}$.
- B3: For each $\lambda \in \mathbb{C}$, $a \mapsto \lambda a$ is continuous from $\mathscr{A}$ to $\mathscr{A}$.
- B4: If $\{a_i\}$ is a net in $\mathscr{A}$ such that $p(a_i) \to x$ and $\|a_i\| \to 0$, then $a_i \to 0_x$ (where 0_x is the zero element in $\mathscr{A}_x$).

Since $\{a \in \mathscr{A} : \|a\| < \epsilon\}$ is open for all $\epsilon > 0$, it follows that whenever $a_i \to 0_x$ in $\mathscr{A}$, then $\|a_i\| \to 0$. Therefore the proof of [5, Proposition II.13.10] implies that

B3′: The map $(\lambda, a) \to \lambda a$ is continuous from $\mathbb{C} \times \mathscr{A}$ to $\mathscr{A}$.

Definition A.2. An *upper semicontinuous C^*-bundle* is an upper semicontinuous Banach bundle : $\mathscr{A} \to X$ such that each fiber is a C^*-algebra such that

B5: The map $(a, b) \mapsto ab$ is continuous from $\mathscr{A} * \mathscr{A}$ to $\mathscr{A}$.
B6: The map $a \mapsto a^*$ is continuous from $\mathscr{A}$ to $\mathscr{A}$.

If axiom B1 is replaced by

B1′: The map $a \mapsto \|a\|$ is continuous,

then $p : \mathscr{A} \to X$ is called a Banach bundle (or a C^*-bundle). Banach bundles are studied in considerable detail in §§13–14 of Chapter II of [5].

If $p : \mathscr{A} \to X$ is an upper semicontinuous Banach bundle, then a continuous function $f : X \to \mathscr{A}$ such that $p \circ f = \mathrm{id}_X$ is called a *section*. The set of sections is denoted by $\Gamma(X; \mathscr{A})$. We say that $f \in \Gamma(X; \mathscr{A})$ *vanishes at infinity* if the the closed set $\{x \in X : |f(x)| \geq \epsilon\}$ is compact for all $\epsilon > 0$. The set of sections which vanish at infinity is denoted by $\Gamma_0(X; \mathscr{A})$, and the latter is a Banach space with respect to the supremum norm: $\|f\| = \sup_{x \in X} \|f(x)\|$ (cf. [4, p. 10] or [30, Proposition C.23]); in fact, $\Gamma_0(X; \mathscr{A})$ is a Banach $C_0(X)$-module for the natural $C_0(X)$-action on sections. (In particular, the uniform limit of sections is a section.) We also use $\Gamma_c(X; \mathscr{A})$ for the vector space of sections with compact support (i.e., $\{x \in X : f(x) \neq 0_x\}$ has compact closure). Moreover, if $p : \mathscr{A} \to X$ is an upper semicontinuous C^*-bundle, then the set of sections is clearly a $*$-algebra with respect to the usual pointwise operations, and $\Gamma_0(X; \mathscr{A})$ becomes a $C_0(X)$-algebra with the obvious $C_0(X)$-action. However, for arbitrary X, there is no reason to expect that there are any nonzero sections—let alone nonzero sections vanishing at infinity or which are compactly supported. An upper semicontinuous Banach bundle is said to have *enough sections* if given $x \in X$ and $a \in \mathscr{A}_x$ there is a section f such that $f(x) = a$. If X is a Hausdorff locally compact space and if $p : \mathscr{A} \to X$ is a Banach bundle, then a result of Douady and Soglio-Hérault implies there are enough sections [5, Appendix C]. Hofmann has noted that the same is true for upper semicontinuous Banach bundles over Hausdorff locally compact spaces [11] (although the details remain unpublished [10]). In this article, we are assuming all our upper semicontinuous Banach bundles have enough sections.

The following lemma is useful as it shows the topology on $\mathscr{A}$ is tied to the continuous sections.

Lemma A.3. *Suppose that $p : \mathscr{A} \to X$ is an upper semicontinuous Banach bundle. Suppose that $\{a_i\}$ is a net in $\mathscr{A}$, that $a \in A$ and that $f \in \Gamma_0(X; \mathscr{A})$ is such that $f(p(a)) = a$. If $p(a_i) \to p(a)$ and $\|a_i - f(p(a_i))\| \to 0$, then $a_i \to a$ in $\mathscr{A}$.*

Proof. We have $a_i - f(p(a_i)) \to 0_{p(a)}$ by axiom B4. Hence

$$a_i = (a_i - f(p(a_i)) + f(p(a_i)) \to 0_{p(a)} + a = a. \blacksquare$$

A slightly more general result is [30, Proposition C.20]. Results such as these can be used to show that the section algebra $\Gamma_0(X; \mathscr{A})$ is complete—see for example the proof of [30, Proposition C.23].

We will want to make repeated use of the following. It has a straightforward proof similar to that given in [30, Proposition C.24].

Lemma A.4. *Suppose that $p : \mathscr{A} \to X$ is an upper semicontinuous Banach bundle over a locally compact Hausdorff space X, and that B is a subspace of $A = \Gamma_0(X; \mathscr{A})$ which is* *closed under multiplication by functions in* $C_0(X)$ *and such that* $\{f(x) : f \in B\}$ *is dense in* $A(x)$ *for all* $x \in X$*. Then* B *is dense in* A*.*

Now we come to our vector-valued Tietze Extension Theorem. The proof is lifted from [5, Theorem II.14.8]. However, we have to make accommodations for the lack of continuity of the norm function on an upper semicontinuous Banach bundle.

PROPOSITION A.5. *Suppose that* $p : G \to \mathscr{B}$ *is an upper semicontinuous Banach bundle and that* Y *is a closed subset of* G*. If* $g \in \Gamma_c(Y; \mathscr{B}|_Y)$*, then there is an* $f \in \Gamma_c(G; \mathscr{B})$ *such that* $f(y) = g(y)$ *for all* $y \in Y$*.*

Proof. Let $C := \operatorname{supp} g$ and let U be a pre-compact open neighborhood of C in G. Let

$$\mathcal{A}_0 := \{f|_Y : f \in \Gamma_c(G; \mathscr{B})\} \subset \Gamma_c(Y; \mathscr{B}|_Y).$$

Since every $\psi \in C_c(Y)$ is the restriction of some $\varphi \in C_c(G)$ (by the scalar-valued Tietze theorem [30, Lemma 1.42]), $\mathcal{A}_0$ is dense in $\Gamma_c(Y; \mathscr{B}|_Y)$ in the inductive limit topology by Lemma A.4. Hence there is $\{f_n\} \subset \Gamma_c(G; \mathscr{B})$ such that

$$f_h|_Y \to g \quad \text{uniformly on } \overline{U} \cap Y.$$

By multiplying by a function which is 1 on C and vanishes off U, we can assume that each f_n vanishes off U. Passing to a subsequence and relabeling, we can assume that

$$\sup\{\|f_n(y) - f_{n-1}(y)\| : y \in \overline{U} \cap Y\} < 1/2^n \quad (n \geq 2).$$

Let $h_n' := f_n - f_{n-1}$. Since $x \mapsto \|h_n'(x)\|$ is upper semicontinuous,

$$A_n := \{x \in G : \|h_n'(x)\| \geq 1/2^n\}$$

is closed and disjoint from the closed set $\overline{U} \cap Y$. Therefore there is a $\varphi_n \in C_c^+(G)$ such that $0 \leq \varphi_n(x) \leq 1$ for all x, $\varphi_n(y) = 1$ if $y \in \overline{U} \cap Y$, and $\varphi_n(x) = 0$ if $x \in A_n$. Then if we let $h_n = \varphi_n \cdot h_n'$, we have arranged that $\|h_n(x)\| \leq 2^{-n}$ for all $x \in G$, and that $h_n(y) = h_n'(y)$ if $y \in \overline{U} \cap Y$. Since $B(x)$ is complete, we can define a section $f : G \to \mathscr{B}$ by

$$f(x) := f_1(x) + \sum_{n=2}^{\infty} h_n(x).$$

Clearly f vanishes off U and as it is the uniform limit of elements of $\Gamma_c(G; \mathscr{B})$, it too is in $\Gamma_c(G; \mathscr{B})$.

On one hand, if $y \in \overline{U} \cap Y$, then

$$f(y) = f_1(y) + \sum_{n=2}^{\infty} h_n'(y) = \lim_n f_n(y) = g(y).$$

On the other hand, if $y \notin \overline{U} \cap Y$, then both $g(y)$ and $f(y)$ are zero. Thus $g = f|_Y$ as required. ∎

Appendix B. An example: the scalar case

The most basic example of a Fell bundle over G is the trivial bundle $\mathscr{B} := G \times \mathbb{C}$. Then we can identify $\Gamma_c(G; \mathscr{B})$ with $C_c(G)$. Hence, we can talk about representations and pre-representations of $C_c(G)$ (see Definitions 4.1 and 4.7). In this section, we want to review the disintegration theorem in the scalar case as it appears in the literature. Then we want to obtain that formulation as a corollary to our Theorem 4.13. Let us take our time and recall the basic definitions.

DEFINITION B.1. If $X * \mathscr{H}$ is a Borel Hilbert bundle, then its *isomorphism groupoid* is the groupoid

$$\mathrm{Iso}(X * \mathscr{H}) := \{(u, V, v) : V : \mathcal{H}(v) \to \mathcal{H}(u) \text{ is a unitary}\}$$

with the weakest Borel structure such that

$$(u, V, v) \mapsto (V f_n(v) \mid f_m(u))$$

is Borel for each n and m with $\{f_n\}$ a fundamental system for $X * \mathscr{H}$.

DEFINITION B.2. A *unitary representation* of a groupoid G with Haar system $\{\lambda^u\}_{u\in G^{(0)}}$ is a triple $(\mu, G^{(0)} * \mathscr{H}, L)$ consisting of a quasi-invariant measure μ on $G^{(0)}$, a Borel Hilbert bundle $G^{(0)} * \mathscr{H}$ over $G^{(0)}$ and a Borel homomorphism[22] $\hat{L} : G \to \mathrm{Iso}(G^{(0)} * \mathscr{H})$ such that

$$\hat{L}(x) = (r(x), L_x, s(x)). \tag{B.1}$$

REMARK B.3. In Definition B.2, it is important to note that the groupoid homomorphism $\hat{L} : G \to \mathrm{Iso}(G^{(0)} * \mathscr{H})$ has the form specified in (B.1). In general, a groupoid homomorphism $L' : G \to \mathrm{Iso}(G^{(0)} * \mathscr{H})$ need only satisfy $L'(x) = (\rho(x), L'_x, \sigma(x))$ for appropriate maps ρ, σ of G into $G^{(0)}$. This makes passing from "almost everywhere" homomorphisms to everywhere homomorphisms via Ramsay's results a bit more problematic than indicated in the literature. We will pay attention to this detail below.

Then, in analogy with Proposition 4.10, we have the following.

PROPOSITION B.4. *If $(\mu, G^{(0)} * \mathscr{H}, L)$ is a unitary representation of a groupoid G, then we obtain a $\|\cdot\|_I$-norm bounded representation of $C_c(G)$ on*

$$\mathcal{H} := L^2(G^{(0)} * \mathscr{H}, \mu),$$

called the integrated form of $(\mu, G^{(0)} * \mathscr{H}, L)$, determined by

$$(L(f)h \mid k) = \int_G f(x)(L_x(h(s(x)) \mid k(r(x)))\Delta(x)^{-1/2}\, d\nu(x).$$

Then the classical form of the disintegration in the scalar case is given as follows.

THEOREM B.5 (Renault's Proposition 4.2). *Suppose that $L : C_c(G) \to \mathrm{Lin}(\mathcal{H}_0)$ is a pre-representation of $G \times \mathbb{C}$ on $\mathcal{H}_0 \subset \mathcal{H}$. Then L is bounded for the $\|\cdot\|_I$-norm and extends to a* bona fide *representation of $C_c(G)$ on $\mathcal{H}$ which is equivalent to the integrated form of a unitary representation $(\mu, G^{(0)} * \mathscr{H}, \hat{\sigma})$ of G.*

Proof. Notice that Theorem 4.13 implies that L is bounded and is equivalent to a *bona fide* representation, still called L, on $L^2(G^{(0)} * \mathscr{H}, \mu)$ which is the integrated form of a

[22] The natural thing is really an "almost everywhere representation" (in the sense of [18, Remark 3.23]). But we can produce a strict homomorphism in the disintegration theorem, so that is what we work with now.

strict representation $(\mu, G^{(0)} * \mathscr{H}, \pi)$ for a Borel $*$-functor $\pi : G \times \mathbb{C} \to \operatorname{End}(G^{(0)} * \mathscr{H})$ (as in Proposition 4.10).

For each $x \in G$, let $\sigma(x) := \pi(x, 1)$. Note that for all $\tau \in C$, we have $\pi(x, \tau) = \tau\sigma(x)$. Then

$$x \mapsto (\pi(f(x))\xi(s(x)) \mid \eta(r(x))) = f(x)(\sigma(x)\xi(s(x)) \mid \eta(r(x)))$$

is Borel for all $f \in C_c(G)$. Since we can find $f_n \in C_c(G)$ such that $f_n(x) \nearrow 1$ for all $x \in G$, it follows that

$$x \mapsto (\sigma(x)\xi(s(x)) \mid \eta(r(x))$$

is Borel. In particular, we can define a decomposable operator P in $B(L^2(G^{(0)} * \mathscr{H}, \mu))$ by

$$(P\xi)(u) = \sigma(u)\xi(u) \quad \text{ for all } u \in G^{(0)}.$$

Since $u \in G^{(0)}$ implies that $\sigma(u) = \sigma(u)^2 = \sigma(u)^*$, $\sigma(u)$ is a projection for all u. Hence P is a projection as well. Furthermore,

$$\begin{aligned} L(f)\xi(u) &= \int_G \pi(f(x))\xi(s(x))\Delta(x)^{-1/2}\, d\lambda^u(x) \\ &= \int_G f(x)\sigma(x)\xi(s(x))\Delta(x)^{-1/2}\, d\lambda^u(x) = \sigma(u)L(f)\xi(u). \end{aligned} \tag{B.2}$$

Therefore $L(f) = PL(f)$. Since L is nondegenerate, we must have $P = I$, and $\sigma(u) = I_{\mathcal{H}(u)}$ for μ-almost all $u \in G^{(0)}$. Therefore we can replace $\mathcal{H}(u)$ by $\mathcal{H}'(u) := \sigma(u)\mathcal{H}(u)$ and assume that $\sigma(u) = I_{\mathcal{H}(u)}$.[23] Having done so, we note that each $\sigma(x)$ is unitary:

$$\sigma(x)^*\sigma(x) = \sigma(s(x)) = I_{\mathcal{H}(s(x))} \quad \text{and} \quad \sigma(x)\sigma(x)^* = \sigma(r(x)) = I_{\mathcal{H}(r(x))}.$$

Now the result follows from (B.2) after defining $\hat{\sigma}(x) := (r(x), \sigma(x), s(x))$.[24] ∎

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[23]Technically, $L^2(G^{(0)} * \mathscr{H}, \mu)$ is unitarily isomorphic to $L^2(G^{(0)} * \mathscr{H}', \mu)$ and we are replacing L by its counterpart on the latter space.

[24]It is possible that some of the $\sigma(u)$, for $u \in G^{(0)}$, are zero. Since it may seem inappropriate to call the zero operator on the zero space "unitary", we can proceed as follows. The set $Q \subset G^{(0)}$ such that $\sigma(u) = 0$ is Borel and clearly saturated. If $F := G^{(0)} \setminus Q$, then $G = G|_F \cup G|_Q$. Since $P = I$, Q is μ-null and $G|_Q$ is ν-null. For $u \in Q$, we may simply redefine $\mathcal{H}(u)$ to be $\mathbb{C}$ and for $x \in G|_Q$ let $\hat{\sigma}(x) := (r(x), 1, s(x))$.

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