Contents

Abstract

The work is dedicated to investigating a limiting procedure for extending "local" integral operator equalities to "global" ones in the sense explained below, and to applying it to obtaining generalizations of the Newton–Leibniz formula for operator-valued functions for a wide class of unbounded operators. The integral equalities considered have the form

$$g(R_F)\int f_x(R_F)\,d\mu(x) = h(R_F). \tag{1}$$

They involve functions of the kind

$$X \ni x \mapsto f_x(R_F) \in B(F),$$

where X is a general locally compact space, F runs over a suitable class of Banach subspaces of a fixed complex Banach space G, in particular $F = G$. The integrals are with respect to a general complex Radon measure on X and the $\sigma(B(F), \mathcal{N}_F)$-topology on $B(F)$, where $\mathcal{N}_F$ is a suitable subset of $B(F)^*$, the topological dual of $B(F)$. R_F is a possibly unbounded scalar type spectral operator in F such that $\sigma(R_F) \subseteq \sigma(R_G)$, and for all $x \in X$, f_x and g, h are complex-valued Borelian maps on the spectrum $\sigma(R_G)$ of R_G. If $F \neq G$ we call the integral equality (1) "local", while if $F = G$ we call it "global".

Acknowledgements. I am grateful to Professor Victor Burenkov for having supervised this work and to Professors Massimo Lanza de Cristoforis and Karl Michael Schmidt for their helpful comments.

Research supported by the Engineering and Physical Sciences Research Council (EPRSC).

2000 *Mathematics Subject Classification*: 46G10, 47B40, 47A60.

Key words and phrases: unbounded spectral operators in Banach spaces, functional calculus, integration of locally convex space valued maps.

Received 6.6.2008; revised version 9.2.2009.

Introduction

In this work we investigate a limiting procedure for extending "local" integral operator equalities to "global" ones in the sense explained below, and apply it to obtain generalizations of the Newton–Leibniz formula for operator-valued functions for a wide class of unbounded operators.

The integral equalities considered have the form

$$g(R_F)\int f_x(R_F)\,d\mu(x) = h(R_F). \tag{2}$$

They involve functions of the kind

$$X \ni x \mapsto f_x(R_F) \in B(F),$$

where X is a general locally compact space, F is a suitable Banach subspace of a fixed complex Banach space G, for example $F = G$. The integrals are with respect to a general complex Radon measure on X and the $\sigma(B(F), \mathcal{N}_F)$-topology ([1]) on $B(F)$. R_F is a possibly unbounded scalar type spectral operator in F such that $\sigma(R_F) \subseteq \sigma(R_G)$, and for all $x \in X$, f_x and g, h are complex-valued Borelian maps on the spectrum $\sigma(R_G)$ of R_G.

If $F \neq G$ we call the integral equalities (2) "local", while if $F = G$ we call them "global".

Let G be a complex Banach space and $B(G)$ the Banach algebra of all bounded linear operators on G. *Scalar type spectral operators* in G were defined in [DS, Definition 18.2.12] ([2]) (see Section 1.1), and were created for providing a general Banach space with a class of unbounded linear operators for which it is possible to establish a Borel functional calculus similar to the well-known one for unbounded self-adjoint operators in a Hilbert space.

We start with the following useful formula ([3]) for the resolvent of T:

$$(T - \lambda\mathbf{1})^{-1} = i\int_{-\infty}^{0} e^{-it\lambda} e^{itT}\,dt. \tag{3}$$

([1]) Here $\mathcal{N}_F$ is a suitable subset of $B(F)^*$, the topological dual of $B(F)$, associated with the resolution of the identity of R_F.

([2]) For the special case of bounded spectral operators on G see [Dow].

([3]) An important application of this formula is in proving the well-known Stone theorem for strongly continuous semigroups of unitary operators in Hilbert space (see Theorem 12.6.1 of [DS]). In [Dav] it has been used to show the equivalence of uniform convergence in strong operator topology of a one-parameter semigroup depending on a parameter and the convergence in strong operator topology of the resolvents of the corresponding generators (Theorem 3.17).

Here $\lambda \in \mathbb{C}$ with $\mathrm{Im}(\lambda) > 0$ and the integral is with respect to the Lebesgue measure and the strong operator topology on $B(G)$ ([4]). It is known that this formula holds for

- any bounded operator $T \in B(G)$ on a complex Banach space G with real spectrum $\sigma(T)$ (see for example [LN]);
- any infinitesimal generator T of a strongly continuous semigroup in a Banach space (see Corollary 8.1.16 of [DS]), in particular for any unbounded self-adjoint operator $T : \mathbf{D}(T) \subset H \to H$ in a complex Hilbert space H.

Next we consider a more general case. Let S be an entire function and $L > 0$. Then the Newton–Leibniz formula

$$R \int_{u_1}^{u_2} \frac{dS}{d\lambda}(tR)\, dt = S(u_2 R) - S(u_1 R) \tag{4}$$

for all $u_1, u_2 \in [-L, L]$ is known for any element R in a Banach algebra $\mathcal{A}$, where $S(tR)$ and $\frac{dS}{d\lambda}(tR)$ are understood in the standard framework of analytic functional calculus on Banach algebras, while the integral is with respect to the Lebesgue measure and the norm topology on $\mathcal{A}$ (see for example [Rud, Dieu, Schw]). If E is the resolution of the identity of R then for all $U \in \mathcal{B}(\mathbb{C})$ ([5]),

$$\mathfrak{L}_E^{\infty}(U) := \{f : \mathbb{C} \to \mathbb{C} \mid \|f\chi_U\|_{\infty}^{E} < \infty\}.$$

Here $\chi_U : \mathbb{C} \to \mathbb{C}$ is equal to 1 in U and 0 in $\complement U$, and for all maps $F : \mathbb{C} \to \mathbb{C}$,

$$\|F\|_{\infty}^{E} := E\text{-}\operatorname*{ess\,sup}_{\lambda \in \mathbb{C}} |F(\lambda)| := \inf_{\{\delta \in \mathcal{B}(\mathbb{C}) \mid E(\delta) = \mathbf{1}\}} \sup_{\lambda \in \delta} |F(\lambda)|.$$

See [DS].

We say (see Definition 2.11) that $\mathcal{N}$ is an *E-appropriate set* if

- $\mathcal{N} \subseteq B(G)^*$ is a linear subspace;
- $\mathcal{N}$ separates the points of $B(G)$, i.e.

$$(\forall T \in B(G) - \{\mathbf{0}\})(\exists \omega \in \mathcal{N})(\omega(T) \neq 0);$$

- $(\forall \omega \in \mathcal{N})(\forall \sigma \in \mathcal{B}(\mathbb{C}))$ we have

$$\omega \circ \mathcal{R}(E(\sigma)) \in \mathcal{N} \quad \text{and} \quad \omega \circ \mathcal{L}(E(\sigma)) \in \mathcal{N}. \tag{5}$$

Moreover, we say that $\mathcal{N}$ is an *E-appropriate set with the duality property* if in addition

$$\mathcal{N}^* \subseteq B(G). \tag{6}$$

Here for any Banach algebra $\mathcal{A}$, so in particular for $\mathcal{A} = B(G)$, we define $\mathcal{R} : \mathcal{A} \to \mathcal{A}^{\mathcal{A}}$ and $\mathcal{L} : \mathcal{A} \to \mathcal{A}^{\mathcal{A}}$ by

$$\mathcal{R}(T) : \mathcal{A} \ni h \mapsto Th \in \mathcal{A}, \qquad \mathcal{L}(T) : \mathcal{A} \ni h \mapsto hT \in \mathcal{A}, \tag{7}$$

([4]) Notice that if $\zeta := -i\lambda$ and $Q := iT$, then the equality (3) turns into

$$(Q + \zeta\mathbf{1})^{-1} = \int_0^{\infty} e^{-t\zeta} e^{-Qt}\, dt,$$

which is referred to in IX.1.3 of [Kat] as the fact that the resolvent of Q is the *Laplace* transform of the semigroup e^{-Qt}. Applications of this formula to perturbation theory are in IX.2 of [Kat].

([5]) $\mathcal{B}(\mathbb{C})$ is the class of all Borelian subsets of $\mathbb{C}$.

for all $T \in \mathcal{A}$. Notice that for all $T, h \in \mathcal{A}$ we have $\|\mathcal{R}(T)(h)\|_{\mathcal{A}} \leq \|T\|_{\mathcal{A}}\|h\|_{\mathcal{A}}$ and $\|\mathcal{L}(T)(h)\|_{\mathcal{A}} \leq \|T\|_{\mathcal{A}}\|h\|_{\mathcal{A}}$, so

$$\mathcal{R}(T), \mathcal{L}(T) \in B(\mathcal{A}) \tag{8}$$

with

$$\|\mathcal{R}(T)\|_{B(\mathcal{A})} \leq \|T\|_{\mathcal{A}}, \quad \|\mathcal{L}(T)\|_{B(\mathcal{A})} \leq \|T\|_{\mathcal{A}}. \tag{9}$$

Since $\mathcal{L}$ and $\mathcal{R}$ are linear mappings we can conclude that

$$\begin{cases} \mathcal{L}, \mathcal{R} \in B(\mathcal{A}, B(\mathcal{A})), \\ \|\mathcal{R}\|_{B(\mathcal{A},B(\mathcal{A}))}, \|\mathcal{L}\|_{B(\mathcal{A},B(\mathcal{A}))} \leq 1. \end{cases} \tag{10}$$

In (6) we mean

$$(\exists\, Y_0 \subseteq B(G))(\mathcal{N}^* = \{\hat{A} \restriction \mathcal{N} \mid A \in Y_0\}),$$

where $(\hat{\cdot}) : B(G) \to B(G)^{**}$ is the canonical isometric embedding of $B(G)$ into its bidual.

In this work the following generalizations of (4) are proved for the case when $R : \mathbf{D} \subset G \to G$ is an unbounded scalar type spectral operator in a complex Banach space G, in particular when $R : \mathbf{D} \subset H \to H$ is an unbounded self-adjoint operator in a complex Hilbert space H. We assume that $S : U \to \mathbb{C}$ is an analytic map on an open neighbourhood U of the spectrum $\sigma(R)$ of R such that there is $L > 0$ such that $]-L, L[\cdot U \subseteq U$ and

$$\widetilde{S}_t \in \mathfrak{L}_E^\infty(\sigma(R)), \quad \widetilde{\left(\frac{dS}{d\lambda}\right)_t} \in \mathfrak{L}_E^\infty(\sigma(R))$$

for all $t \in]-L, L[$, where $(S)_t(\lambda) := S(t\lambda)$ and $(\frac{dS}{d\lambda})_t(\lambda) := \frac{dS}{d\lambda}(t\lambda)$ for all $t \in]-L, L[$ and $\lambda \in U$, while for any map $F : U \to \mathbb{C}$ we let $\widetilde{F}$ be the $\mathbf{0}$-extension of F to $\mathbb{C}$. The following statements are proved:

- If
$$\int^* \left\| \widetilde{\left(\frac{dS}{d\lambda}\right)_t} \right\|_\infty^E dt < \infty \tag{11}$$
and for all $\omega \in \mathcal{N}$ the map $]-L, L[\ni t \mapsto \omega\big(\frac{dS}{d\lambda}(tR)\big) \in \mathbb{C}$ is Lebesgue measurable, then in Corollary 2.33 it is proved that formula (4) holds where the integral is the weak integral ([6]) with respect to the Lebesgue measure and the $\sigma(B(G), \mathcal{N})$-topology for any E-appropriate set $\mathcal{N}$ with the duality property. Moreover, in Corollary 2.34 it is proved that formula (4) also holds when $\widetilde{\big(\frac{dS}{d\lambda}\big)_t} \in \mathfrak{L}_E^\infty(\sigma(R))$ almost everywhere on $]-L, L[$ with respect to the Lebesgue measure.
- In particular, it is proved that formula (4) holds where the integral is the weak integral with respect to the Lebesgue measure and the sigma-weak operator topology, when G is a Hilbert space (Corollary 2.35).
- If in addition to (11), G is a reflexive complex Banach space then in Corollary 2.36 it is proved that formula (4) holds where the integral is the weak integral with respect to the Lebesgue measure and the weak operator topology.

([6]) See formula (21) below.

- If
$$\sup_{t\in]-L,L[}\left\|\widetilde{\left(\frac{dS}{d\lambda}\right)_t}\right\|_\infty^E<\infty, \tag{12}$$
then in Theorem 1.25 it is proved that formula (4) holds where the integral is with respect to the Lebesgue measure and the strong operator topology.
- In Theorem 1.23 it is proved that if in addition to (12),
$$\sup_{t\in]-L,L[}\|\widetilde{(S)_t}\|_\infty^E<\infty,$$
then for all $v\in\mathbf{D}$ the mapping $]-L,L[\ni t\mapsto S(tR)v\in G$ is differentiable, and $(\forall v\in\mathbf{D})(\forall t\in]-L,L[)$
$$\frac{dS(tR)v}{dt}=R\,\frac{dS}{d\lambda}\,(tR)v. \tag{13}$$
- In Corollary 1.27 formula (3) is deduced from formula (4) for any unbounded scalar type spectral operator $T:\mathbf{D}(T)\subset G\to G$ in a complex Banach space G with real spectrum.

In these statements $\frac{dS}{d\lambda}(tR)$ and $S(tR)$ are understood in the framework of the Borel functional calculus for unbounded scalar type spectral operators in G. See Definition 18.2.10 in Vol. 3 of the Dunford–Schwartz monograph [DS] (also see Section 1.1).

In order to prove equality (4) when R is an unbounded scalar type spectral operator in G, we proceed in two steps. First of all we consider the Banach spaces $G_{\sigma_n}:=E(\sigma_n)G$ where $\sigma_n:=B_n(\mathbf{0})\subset\mathbb{C}$, with $n\in\mathbb{N}$, the bounded operators $R_{\sigma_n}:=RE(\sigma_n)$, and their restrictions $R_{\sigma_n}\restriction G_{\sigma_n}$. Then by Key Lemma 1.7 the operators $R_{\sigma_n}\restriction G_{\sigma_n}$ are bounded *scalar type spectral* operators on G_{σ_n}, and for all $x\in G$,
$$S(R)x=\lim_{n\in\mathbb{N}}S(R_{\sigma_n}\restriction G_{\sigma_n})E(\sigma_n)x \tag{14}$$
and
$$(R_{\sigma_n}\restriction G_{\sigma_n})\int_{u_1}^{u_2}\frac{dS}{d\lambda}(t(R_{\sigma_n}\restriction G_{\sigma_n}))\,dt=S(u_2(R_{\sigma_n}\restriction G_{\sigma_n}))-S(u_1(R_{\sigma_n}\restriction G_{\sigma_n})). \tag{15}$$
The second and most important step is to set up a *limiting* procedure, which allows one, by using the convergence (14), to extend the "local" equality (15) to the "global" one (4).

As we shall see below, such a limiting procedure can be set up in a very general context. First we wish to point out that the following equalities for all $n\in\mathbb{N}$ and $t\in]-L,L[$, which follow from Key Lemma 1.7, are essential for making this limiting procedure possible:
$$\begin{cases}\dfrac{dS}{d\lambda}(tR)E(\sigma_n)=\dfrac{dS}{d\lambda}(t(R_{\sigma_n}\restriction G_{\sigma_n}))E(\sigma_n),\\ S(tR)E(\sigma_n)=S(t(R_{\sigma_n}\restriction G_{\sigma_n}))E(\sigma_n).\end{cases} \tag{16}$$

We note that in (15) one cannot replace $R_{\sigma_n}\restriction G_{\sigma_n}$ with the simpler operator R_{σ_n} for the following reason. Although R_{σ_n} is a bounded operator on G for $n\in\mathbb{N}$ and $Rx=\lim_{n\in\mathbb{N}}R_{\sigma_n}x$ in G, in general R_{σ_n} is not a scalar type spectral operator, hence the expression $\frac{dS}{d\lambda}(tR_{\sigma_n})$ is not defined in the Dunford–Schwartz functional calculus for scalar type spectral operators, which turns out to be mandatory when using general Borelian maps not necessarily analytic.

Next we formulate a rather general statement allowing one, by using a limiting procedure, to pass from "local" equalities similar to (15) to "global" ones similar to (4).

We generalize (4) in several directions. We replace

- the operator R to the left of the integral by a function $g(R)$, where g is a general Borelian map on $\sigma(R)$ (7),
- the compact set $[u_1, u_2]$ and the Lebesgue measure on it by a general locally compact space X and a complex Radon measure on it respectively,
- the map $[u_1, u_2] \ni t \mapsto \left(\frac{dS}{d\lambda}\right)_t \in \mathrm{Bor}(\sigma(R))$ by the map $X \ni x \mapsto f_x \in \mathrm{Bor}(\sigma(R))$ such that $\widetilde{f_x} \in \mathfrak{L}_E^\infty(\sigma(R))$ where $\widetilde{f_x}$ is the $\mathbf{0}$-extension of f_x to $\mathbb{C}$, and the map $X \ni x \mapsto f_x(R) \in B(G)$ is strongly integrable with respect to the measure μ (8);
- the map $S_{u_2} - S_{u_1}$ by a Borelian map h on $\sigma(R)$ such that $\widetilde{h} \in \mathfrak{L}_E^\infty(\sigma(R))$.

One of the main results of the work is Theorem 1.18 where we prove that if $\{\sigma_n\}_{n\in\mathbb{N}}$ is an E-sequence (9), and (10) for all $n \in \mathbb{N}$

$$R_{\sigma_n}\lceil G_{\sigma_n} := RE(\sigma_n)\lceil(G_{\sigma_n} \cap \mathrm{Dom}(R)), \tag{17}$$

and for all $n \in \mathbb{N}$ the following *local* inclusion

$$g(R_{\sigma_n}\lceil G_{\sigma_n}) \int f_x(R_{\sigma_n}\lceil G_{\sigma_n})\, d\mu(x) \subseteq h(R_{\sigma_n}\lceil G_{\sigma_n}) \tag{18}$$

holds, then $h(R) \in B(G)$ and the *global* equality holds, i.e.

$$g(R) \int f_x(R)\, d\mu(x) = h(R). \tag{19}$$

Here all the integrals are with respect to the strong operator topology.

Now we can describe the Extension Theorem and the Newton–Leibniz formula for the integration with respect to the $\sigma(B(G), \mathcal{N})$-topology, where $\mathcal{N}$ is a suitable subset of $B(G)^*$, which, roughly speaking, is the weakest among reasonable locally convex topologies on $B(G)$ for which the aforementioned limiting procedure can be performed.

In Section 2.2 we recall the definition of scalar essential μ-integrability and the weak integral of maps defined on X and with values in a Hausdorff locally convex space, where μ is a Radon measure on a locally compact space X.

Here we need just to apply these definitions to the case of $\sigma(B(G), \mathcal{N})$, i.e. the weak topology on $B(G)$ defined by the standard duality between $B(G)$ and $\mathcal{N}$ where $\mathcal{N}$ is a subset of the (topological) dual $B(G)^*$ of $B(G)$ that separates the points of $B(G)$.

---

(7) The most interesting case is when the operator $g(R)$ is unbounded.

(8) This means that $X \ni x \mapsto f_x(R)v \in G$ is integrable with respect to the measure μ for all $v \in G$, in the sense of Ch 4, §4 of Bourbaki [INT], and the map $G \ni v \mapsto \int f_x(R)v \in G$ is a (linear) bounded operator on G.

(9) By definition this means that for all $n \in \mathbb{N}$, $\sigma_n \in \mathcal{B}(\mathbb{C})$; for all $n, m \in \mathbb{N}$, $n > m \Rightarrow \sigma_n \supseteq \sigma_m$; $\mathrm{supp}(E) \subseteq \bigcup_{n\in\mathbb{N}} \sigma_n$; hence we have $\lim_{n\in\mathbb{N}} E(\sigma_n) = \mathbf{1}$ strongly.

(10) By Key Lemma 1.7, $R_{\sigma_n}\lceil G_{\sigma_n}$ is a scalar type spectral operator in the complex Banach space G_{σ_n}, but in contrast to the previous case where $\sigma_n := B_n(\mathbf{0})$ was bounded, here σ_n could be unbounded so it may happen that $G_{\sigma_n} \nsubseteq \mathrm{Dom}(R)$, hence the restriction $R_{\sigma_n}\lceil G_{\sigma_n}$ of R_{σ_n} to G_{σ_n} has to be defined on the set $G_{\sigma_n} \cap \mathrm{Dom}(R)$, and it could be an unbounded operator in G_{σ_n}.

Thus $f : X \to \langle B(G), \sigma(B(G), \mathcal{N})\rangle$ is by definition scalarly essentially μ-integrable or equivalently $f : X \to B(G)$ is scalarly essentially μ-integrable with respect to the $\sigma(B(G), \mathcal{N})$-topology on $B(G)$ if for all $\omega \in \mathcal{N}$ the map $\omega \circ f : X \to \mathbb{C}$ is essentially μ-integrable ([11]), so we can define its *integral* as the linear operator

$$\mathcal{N} \ni \omega \mapsto \int \omega(f(x))\, d\mu(x) \in \mathbb{C}.$$

Let $f : X \to \langle B(G), \sigma(B(G), \mathcal{N})\rangle$ be scalarly essentially μ-integrable and assume that

$$(\exists B \in B(G))(\forall \omega \in \mathcal{N})\Big(\omega(B) = \int \omega(f(x))\, d\mu(x)\Big). \tag{20}$$

Notice that the operator B is uniquely defined by this condition. In this case, by definition $f : X \to \langle B(G), \sigma(B(G), \mathcal{N})\rangle$ is *scalarly essentially* $(\mu, B(G))$*-integrable* or $f : X \to B(G)$ is *scalarly essentially* $(\mu, B(G))$*-integrable with respect to the* $\sigma(B(G), \mathcal{N})$*-topology*, and its *weak integral with respect to the measure* μ *and the* $\sigma(B(G), \mathcal{N})$*-topology*, or simply its *weak integral*, is defined by

$$\int f(x)\, d\mu(x) := B. \tag{21}$$

Next we can state the main result of this work (see Theorem 2.25).

THEOREM 0.1 ($\sigma(B(G), \mathcal{N})$-Extension Theorem). *Let G be a complex Banach space, X a locally compact space, and μ a complex Radon measure on it. In addition let R be a possibly unbounded scalar type spectral operator in G, $\sigma(R)$ its spectrum, E its resolution of the identity, and $\mathcal{N}$ an E-appropriate set. Let $X \ni x \mapsto f_x \in \mathrm{Bor}(\sigma(R))$ be such that $\widetilde{f_x} \in \mathfrak{L}_E^\infty(\sigma(R))$ μ-locally almost everywhere on X and $X \ni x \mapsto f_x(R) \in \langle B(G), \sigma(B(G), \mathcal{N})\rangle$ is scalarly essentially $(\mu, B(G))$-integrable. Finally, let $g, h \in \mathrm{Bor}(\sigma(R))$ and $\widetilde{h} \in \mathfrak{L}_E^\infty(\sigma(R))$. If $\{\sigma_n\}_{n\in\mathbb{N}}$ is an E-sequence and for all $n \in \mathbb{N}$,*

$$g(R_{\sigma_n} \restriction G_{\sigma_n}) \int f_x(R_{\sigma_n} \restriction G_{\sigma_n})\, d\mu(x) \subseteq h(R_{\sigma_n} \restriction G_{\sigma_n}) \tag{22}$$

then $h(R) \in B(G)$ and

$$g(R) \int f_x(R)\, d\mu(x) = h(R). \tag{23}$$

In (22) *the weak integral is with respect to the measure μ and the $\sigma(B(G_{\sigma_n}), \mathcal{N}_{\sigma_n})$-topology* ([12])*, while in* (23) *it is with respect to μ and the $\sigma(B(G), \mathcal{N})$-topology.*

Notice that $g(R)$ is a possibly *unbounded* operator in G.

We list the most important results that allow us to prove Theorem 2.25:

- Key Lemma 1.7;
- "Commutation" property (Theorem 2.13):

$$(\forall \sigma \in \mathcal{B}(\mathbb{C}))\Big[\int f_x(R)\, d\mu(x),\ E(\sigma)\Big] = \mathbf{0}; \tag{24}$$

([11]) See for the definition Ch. 5, §1, n° 3 of [INT].

([12]) $\mathcal{N}_{\sigma_n}$ is, roughly speaking, the set of the restrictions to $B(G_{\sigma_n})$ of all the functionals belonging to $\mathcal{N}$. For the exact definition and properties see Definition 2.20 and Lemma 2.17.

- "Restriction" property (Theorem 2.22): for all $\sigma \in \mathcal{B}(\mathbb{C})$ we have $f_x(R_\sigma \restriction G_\sigma) \in B(G_\sigma)$ μ-locally almost everywhere on X, $X \ni x \mapsto f_x(R_\sigma \restriction G_\sigma) \in \langle B(G_\sigma), \sigma(B(G_\sigma), \mathcal{N}_\sigma)\rangle$ is scalarly essentially $(\mu, B(G_\sigma))$-integrable, and
$$\int f_x(R_\sigma \restriction G_\sigma)\, d\mu(x) = \int f_x(R)\, d\mu(x) \restriction G_\sigma; \tag{25}$$
- the fact that
$$\mathrm{Dom}\Big(g(R) \int f_x(R)\, d\mu(x)\Big) \text{ is dense in } G.$$

We remark that the reason for introducing the concept of an E-appropriate set is primarily to obtain the commutation and restriction properties.

Now we define
$$\mathcal{N}_{\mathrm{st}}(G) := \langle B(G), \tau_{\mathrm{st}}(G)\rangle^* = \mathfrak{L}_{\mathbb{C}}(\{\psi_{(\phi,v)} \mid (\phi, v) \in G^* \times G\}). \tag{26}$$
Here $\langle B(G), \tau_{\mathrm{st}}(G)\rangle^*$ is the topological dual of $B(G)$ with respect to the strong operator topology, $\psi_{(\phi,v)} : B(G) \ni T \mapsto \phi(Tv) \in \mathbb{C}$, while $\mathfrak{L}_{\mathbb{C}}(J)$ is the complex linear space generated by the set $J \subseteq B(G)^*$. Then $\sigma(B(G), \mathcal{N}_{\mathrm{st}}(G))$ is the weak operator topology on $B(G)$, and $\mathcal{N}_{\mathrm{st}}(G)$ is an E-appropriate set for any spectral measure E.

Moreover, we set, in the case where G is a complex Hilbert space,
$$\mathcal{N}_{\mathrm{pd}}(G) := \text{predual of } B(G),$$
which is by definition the linear space of all sigma-weakly continuous linear functionals on $B(G)$.

Note that
$$\mathcal{N}_{\mathrm{pd}}(G)^* = B(G). \tag{27}$$
(See Theorem 2.6(iii), Ch. 2 of [Tak], or Proposition 2.4.18 of [BR]). Here we mean that the normed subspace $\mathcal{N}_{\mathrm{pd}}(G)^*$ of the bidual $B(G)^{**}$ is isometric to $B(G)$, through the canonical embedding of $B(G)$ into $B(G)^{**}$.

Hence we can apply the Extension Theorem 2.25 to the case $\mathcal{N} := \mathcal{N}_{\mathrm{st}}(G)$ or $\mathcal{N} := \mathcal{N}_{\mathrm{pd}}(G)$ and use the following additional property which is proved in Proposition 2.23:
$$(\mathcal{N}_{\mathrm{st}}(G))_\sigma = \mathcal{N}_{\mathrm{st}}(G_\sigma) \quad \text{and} \quad (\mathcal{N}_{\mathrm{pd}}(G))_\sigma = \mathcal{N}_{\mathrm{pd}}(G_\sigma). \tag{28}$$

The reason for introducing the concept of duality property for E-appropriate sets is primarily to ensure that a map $f : X \to \langle B(G), \sigma(B(G), \mathcal{N})\rangle$ that is scalarly essentially μ-integrable is also $(\mu, B(G))$-integrable.

As an application of this fact and of the Extension Theorem we obtain the Newton–Leibniz formula in (4) by replacing $\mathcal{A}$ with $B(G)$, R with an unbounded scalar type spectral operator in a complex Banach space G, by considering S analytic in an open neighbourhood U of $\sigma(R)$ such that $]-L, L[\, \cdot U \subseteq U$, and the integral with respect to the $\sigma(B(G), \mathcal{N})$-topology, where $\mathcal{N}$ is an E-appropriate set with the duality property (Corollary 2.33).

Finally, in a similar way we obtain the corresponding results for the sigma-weak operator topology (Corollary 2.35), and weak operator topology (Corollary 2.36). The last result is a complement to Theorem 1.25.

Summary of the main results

Let G be a complex Banach space, R an unbounded scalar type spectral operator in G, for example an unbounded self-adjoint operator in a Hilbert space, $\sigma(R)$ its spectrum and E its resolution of identity. The *main results* of the work are the following:

- Extension procedure leading from local equality (22) to global equality (23) for integration with respect to the $\sigma(B(G),\mathcal{N})$-topology (Theorem 2.25 if $\mathcal{N}$ is an E-appropriate set and Corollary 2.26 if $\mathcal{N}$ is an E-appropriate set with the duality property).
- Extension procedure leading from local equality (22) to global equality (23) for integration with respect to the sigma-weak topology (Corollary 2.28 and Theorem 2.29) and for integration with respect to the weak operator topology (Corollary 2.27 and Theorem 2.30 or Theorem 1.18 and Corollary 1.19).
- Newton–Leibniz formula (4) for a suitable analytic map S for integration with respect to the $\sigma(B(G),\mathcal{N})$-topology, where $\mathcal{N}$ is an E-appropriate set with the duality property (Corollaries 2.33 and 2.34); for integration with respect to the sigma-weak topology (Corollary 2.35) and for integration with respect to the weak operator topology (Corollary 2.36 and Theorem 1.25).
- Differentiation formula (13) for a suitable analytic map S (Theorems 1.21 and 1.23).
- A new proof for the resolvent formula (3) via formula (4) (Corollary 1.27).

1. Extension theorem. The case of the strong operator topology

1.1. Key lemma

Preliminaries 1.1 (Integrals of bounded Borelian functions with respect to a vector valued measure). In the following, $G := \langle G, \|\cdot\|_G\rangle$ will be a complex Banach space. Denote by $\Pr(G)$ the class of all projectors on G, that is, the class of $P \in B(G)$ such that $P^2 = P$.

Consider a Boolean algebra $\mathcal{B}_X$ (see Sec. 1.12 of [DS]) of subsets of a set X, with respect to the order relation defined by $\sigma \geq \delta \Leftrightarrow \sigma \supseteq \delta$ and complemented by the operation $\sigma' := \complement\sigma$. In particular, $\mathcal{B}_X$ contains $\emptyset$ and X and is closed under finite intersections and finite unions.

The map $E : \mathcal{B}_X \to B(G)$ is called a *spectral measure* in G on $\mathcal{B}_X$, or simply on X if X is a topological space and $\mathcal{B}_X$ is the Boolean algebra of its Borelian subsets, if

(1) $E(\mathcal{B}_X) \subseteq \Pr(G)$;
(2) $(\forall\sigma_1,\sigma_2 \in \mathcal{B}_X)(E(\sigma_1 \cap \sigma_2) = E(\sigma_1)E(\sigma_2))$;
(3) $(\forall\sigma_1,\sigma_2 \in \mathcal{B}_X)(E(\sigma_1 \cup \sigma_2) = E(\sigma_1) + E(\sigma_2) - E(\sigma_1)E(\sigma_2))$;
(4) $E(X) = \mathbf{1}$;
(5) $E(\emptyset) = \mathbf{0}$.

(See Definition 15.2.1 of [DS].)

If condition (3) is replaced by

(3′) $(\forall\sigma_1,\sigma_2 \in \mathcal{B}_X \mid \sigma_1 \cap \sigma_2 = \emptyset)(E(\sigma_1 \cup \sigma_2) = E(\sigma_1) + E(\sigma_2))$,

we obtain an equivalent definition.

Notice that if E is a spectral measure in G on $\mathcal{B}_X$, then it is a Boolean homomorphism onto the Boolean algebra $E(\mathcal{B}_X)$ with respect to the order relation induced by that defined in $\mathrm{Pr}(G)$ by $P \geq Z \Leftrightarrow Z = ZP$, and complemented by the operation $P' := \mathbf{1} - P$. Indeed, for all $\sigma, \delta \in \mathcal{B}_X$ we have $\delta \subseteq \sigma \Rightarrow E(\delta) = E(\delta \cap \sigma) \doteq E(\delta)E(\sigma) \Leftrightarrow E(\delta) \leq E(\sigma)$, while $\mathbf{1} = E(\sigma \cup \complement\sigma) = E(\sigma) + E(\complement\sigma)$.

A spectral measure E is called (*weakly*) *countable additive* if for all sequences $\{\varepsilon_n\}_{n\in\mathbb{N}} \subset \mathcal{B}_X$ of disjoint sets, all $x \in G$ and all $\phi \in G^*$ we have

$$\phi\Big(E\Big(\bigcup_{n\in\mathbb{N}} \varepsilon_n\Big)x\Big) = \sum_{n=1}^{\infty} \phi(E(\varepsilon_n)x).$$

If $\mathcal{B}_X$ is a σ-field, i.e. a Boolean algebra closed under the operation of forming countable unions, Corollary 15.2.4 of [DS] shows that E is countably additive with respect to the strong operator topology, i.e. for all sequences $\{\varepsilon_n\}_{n\in\mathbb{N}} \subset \mathcal{B}(\mathbb{C})$ of disjoint sets and for all $x \in G$ we have (1)

$$E\Big(\bigcup_{n\in\mathbb{N}} \varepsilon_n\Big)x = \sum_{n=1}^{\infty} E(\varepsilon_n)x = \sum_{n\in\mathbb{N}} E(\varepsilon_n)x. \tag{1.1}$$

Since $E(\bigcup_{n\in\mathbb{N}} \varepsilon_n) = E(\bigcup_{n\in\mathbb{N}} \varepsilon_{\rho(n)})$ for any permutation ρ of $\mathbb{N}$, we have $\sum_{n=1}^{\infty} E(\varepsilon_n)x = \sum_{n=1}^{\infty} E(\varepsilon_{\rho(n)})x$ for all $x \in G$, therefore by Proposition 9, §5.7, Ch. 3 of [GT] we obtain the second equality in (1.1). By $\mathcal{B}(\mathbb{C})$ we denote the set of the Borelian subsets of $\mathbb{C}$, and by $\mathrm{Bor}(U)$ the complex linear space of all Borelian complex maps defined on a Borelian subset U of $\mathbb{C}$. We denote by $\mathbf{TM}$ the space of totally $\mathcal{B}(\mathbb{C})$-measurable maps (2), which is the closure in the Banach space $\langle \mathbf{B}(\mathbb{C}), \|\cdot\|_{\sup}\rangle$ of all bounded complex functions on $\mathbb{C}$ with respect to the norm $\|g\|_{\sup} := \sup_{\lambda\in\mathbb{C}} |g(\lambda)|$, of the linear space generated by the set $\{\chi_\sigma \mid \sigma \in \mathcal{B}(\mathbb{C})\}$, where χ_σ is the characteristic function of the set σ. $\langle \mathbf{TM}, \|\cdot\|_{\sup}\rangle$ is a Banach space, and the space of all bounded Borelian complex functions is contained in $\mathbf{TM}$, so is dense in it. Finally, $\langle \mathbf{TM}, \|\cdot\|_{\sup}\rangle$ is a C^*-subalgebra, in particular a Banach subalgebra, of $\langle \mathbf{B}(\mathbb{C}), \|\cdot\|_{\sup}\rangle$ if we define the pointwise operations of product and involution on $\mathbf{B}(\mathbb{C})$.

Let X be a complex Banach space and $F : \mathcal{B}(\mathbb{C}) \to X$ a weakly countably finite additive vector valued measure (see Section 4.10 of [DS]). Then we can define the integral with respect to F (see Section 10.1 of [DS]), which will be denoted by $\int_{\mathbb{C}} f\, dF$. The operator

$$\mathbf{I}_{\mathbb{C}}^F : \mathbf{TM} \ni f \mapsto \int_{\mathbb{C}} f\, dF \in X \tag{1.2}$$

is linear and norm-continuous (3). We have the following useful property: if Y is a $\mathbb{C}$-Banach space and $Q \in B(X, Y)$, then

$$Q \circ \mathbf{I}_{\mathbb{C}}^F = \mathbf{I}_{\mathbb{C}}^{Q\circ F} \tag{1.3}$$

(see Theorem 4.10.8(f) of [DS]).

(1) By definition (see Ch. 3 of [GT]), $v = \sum_{n\in\mathbb{N}} E(\varepsilon_n)x$ if $v = \lim_{J\in\mathcal{P}_\omega(\mathbb{N})} \sum_{n\in J} E(\varepsilon_n)x$, where $\mathcal{P}_\omega(\mathbb{N})$ is the directed ordered set of all finite subsets of $\mathbb{N}$ ordered by inclusion.

(2) In [DS] denoted by $B(\mathbb{C}, \mathcal{B}(\mathbb{C}))$, while by using the notations of [Din2] and considering $\mathbb{C}$ as a real Banach space we have $\mathbf{TM} = \mathbf{TM}_{\mathbb{R}}(\mathcal{B}(\mathbb{C}))$.

(3) Notice that if we identify $B(G)$ with $B(\mathbb{R}, B(G))$ and recall that $\mathbf{TM} = \mathbf{TM}_{\mathbb{R}}(\mathcal{B}(\mathbb{C}))$, then with the notations of Definition 24, §1, Ch. 1 of [Din2] we find that $\mathbf{I}_{\mathbb{C}}^E$ is the immediate integral with respect to the vector-valued measure $E : \mathcal{B}(\mathbb{C}) \to B(\mathbb{R}, B(G))$.

If $X := B(G)$, the case we are most interested in, as an immediate consequence of this property and the fact that the map $Q_x : B(G) \ni A \mapsto Ax \in G$ is linear and continuous for all $x \in G$, we have

$$(\forall x \in G)(\forall f \in \mathbf{TM})(\mathbf{I}_{\mathbb{C}}^{F}(f)x = \mathbf{I}_{\mathbb{C}}^{F^x}(f)). \tag{1.4}$$

Here $F^x : \mathcal{B}(\mathbb{C}) \ni \sigma \mapsto F(\sigma)x$. Finally, if E is a spectral measure on $\mathbb{C}$, then $\mathbf{I}_{\mathbb{C}}^{E}$ is a continuous unital homomorphism between the Banach algebras $\langle \mathbf{TM}, \|\cdot\|_{\sup}\rangle$, and $\langle B(G), \|\cdot\|_{B(G)}\rangle$ and $\mathbf{I}_{\mathbb{C}}^{E}(\chi_{\operatorname{supp} E}) = \mathbf{1}$ (see (1.5) and Section (2), Ch. 15 of [DS]).

Borel functional calculus for possibly unbounded scalar type spectral operators in G. If $T : \mathcal{D}(T) \subseteq G \to G$ is a possibly unbounded linear operator then we denote by $\sigma(T)$ its standard spectrum. A possibly unbounded linear operator $T : \mathcal{D}(T) \subseteq G \to G$ is called a *spectral operator in G* if it is closed and there exists a countably additive spectral measure $E : \mathcal{B}(\mathbb{C}) \to \Pr(G)$ such that

(i) for all bounded sets $\delta \in \mathcal{B}(\mathbb{C})$,
$$E(\delta)G \subseteq \mathcal{D}(T);$$
(ii) $(\forall \delta \in \mathcal{B}(\mathbb{C}))(\forall x \in \mathcal{D}(T))$ we have
- $E(\delta)\mathcal{D}(T) \subseteq \mathcal{D}(T)$,
- $TE(\delta)x = E(\delta)Tx$;

(iii) for all $\delta \in \mathcal{B}(\mathbb{C})$ we have
$$\sigma(T\restriction(\mathcal{D}(T) \cap E(\delta)G)) \subseteq \overline{\delta}.$$
Here $\sigma(T\restriction(\mathcal{D}(T) \cap E(\delta)G))$ is the spectrum of the restriction of T to $\mathcal{D}(T) \cap E(\delta)G$.

(See Definition 18.2.1 of [DS].) We call any E with the above properties a *resolution of the identity of T*. Theorem 18.2.5 of [DS] states that the resolution of the identity of a spectral operator is unique.

Finally, the *support* of a spectral measure E on $\mathcal{B}_X$ is the set

$$\operatorname{supp} E := \bigcap_{\{\sigma \in \mathcal{B}_X \mid E(\sigma) = \mathbf{1}\}} \overline{\sigma}.$$

It is easy to see (4) that

$$E(\operatorname{supp} E) = \mathbf{1}. \tag{1.5}$$

Notice that an unbounded spectral operator T is closed by definition. Now we will show that T is also densely defined. In fact, if E is the resolution of the identity of T and if $\{\sigma_n\}_{n \in \mathbb{N}} \subset \mathcal{B}(\mathbb{C})$ is a nondecreasing sequence of Borelian sets such that $\sigma(T) \subseteq \bigcup_{n \in \mathbb{N}} \sigma_n$, then by the strong countable additivity of E and the fact that $E(\sigma(T)) = \mathbf{1}$ we can deduce $\mathbf{1} = \lim_{n \in \mathbb{N}} E(\sigma_n)$ in the strong operator topology of $B(G)$ (see (1.18)). Now we

(4) Indeed, let $S := \operatorname{supp} E$. Then $\complement S = \bigcup_{\{\sigma \in \mathcal{B}_X \mid E(\sigma) = \mathbf{1}\}} \complement \overline{\sigma}$. Moreover, E is order-preserving so for all $\sigma \in \mathcal{B}_X$ such that $E(\sigma) = \mathbf{1}$ we have $E(\complement \overline{\sigma}) \leq E(\complement \sigma) = \mathbf{1} - E(\sigma) = \mathbf{0}$. Hence by the definition of the order $E(\complement \overline{\sigma}) = E(\complement \overline{\sigma})\mathbf{0} = \mathbf{0}$. Therefore by the principle of localization (Corollary, Ch. 3, §2, n° 1 of [INT]) which holds also for vector measures (footnote in Ch. 6, §2, n° 1 of [INT]) we deduce that $E(\complement S) = \mathbf{0}$. Finally, $E(S) = \mathbf{1} - E(\complement S) = \mathbf{1}$.

can choose $\{\sigma_n\}_{n\in\mathbb{N}}$ such that $\sigma_n := B_n(\mathbf{0}) := \{\lambda \in \mathbb{C} \mid |\lambda| < n\}$, or $\sigma_n := W(\mathbf{0}, 2n) := \{\lambda \in \mathbb{C} \mid |\mathrm{Re}(\lambda)| < n, |\mathrm{Im}(\lambda)| < n\}$. But by the property (i) of Definition 18.2.1 of [DS], we know that for all bounded sets $\sigma \in \mathcal{B}(\mathbb{C})$ we have $E(\sigma)G \subseteq \mathrm{Dom}(T)$. Therefore we conclude that for all $v \in G$, $v = \lim_{n\in\mathbb{N}} E(\sigma_n)v$, and for all $n \in \mathbb{N}$, $E(\sigma_n)v \in \mathrm{Dom}(T)$, so $\mathrm{Dom}(T)$ is dense in G.

We remark that for each possibly unbounded spectral operator T in G, denoting by $\sigma(T)$ its spectrum and by $E : \mathcal{B}(\mathbb{C}) \to \mathrm{Pr}(G)$ its resolution of the identity, we deduce by Lemma 18.2.25 of [DS] that $\sigma(T)$ is closed and $\operatorname{supp} E = \sigma(T)$, so by (1.5),

$$E(\sigma(T)) = \mathbf{1}.$$

Now we will give the definition of the Borel functional calculus for unbounded spectral operators in a complex Banach space G, essentially the same as in Definition 18.2.10 of [DS].

Definition 1.2. Let X be a set, $S \subset X$, V a vector space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, and $f : S \to V$. Then we define $\widetilde{f}^X$, or simply $\widetilde{f}$ when it causes no confusion, to be the $\mathbf{0}$-extension of f to X, i.e. $\widetilde{f} : X \to V$ with $\widetilde{f}\restriction S = f$ and $\widetilde{f}(x) = \mathbf{0}$ for all $x \in X - S$, where $\mathbf{0}$ is the zero vector of V.

Definition 1.3. Assume that

- $E : \mathcal{B}(\mathbb{C}) \to \mathrm{Pr}(G)$ is a countably additive spectral measure and S its support;
- $f \in \mathrm{Bor}(S)$;
- for all $\sigma \subseteq \mathbb{C}$ we define $f_\sigma : \mathbb{C} \to \mathbb{C}$ by $f_\sigma := \widetilde{f} \cdot \chi_\sigma$;
- $\delta_n := [-n, +n]$ and

$$f_n := f_{|f|^{-1}(\delta_n)}.$$

Here $(\forall \sigma \subseteq \mathbb{C})(\forall g : \mathcal{D} \to \mathbb{C})(g^{-1}(\sigma) := \{\lambda \in \mathcal{D} \mid g(\lambda) \in \sigma\})$.

Of course $f_n \in \mathbf{TM}$ for all $n \in \mathbb{N}$ so we can define the following operator in G:

$$\begin{cases} \mathrm{Dom}(f(E)) := \{x \in G \mid \lim_{n\in\mathbb{N}} \mathbf{I}_{\mathbb{C}}^{E}(f_n)x \text{ exists}\}, \\ (\forall x \in \mathrm{Dom}(f(E)))(f(E)x := \lim_{n\in\mathbb{N}} \mathbf{I}_{\mathbb{C}}^{E}(f_n)x). \end{cases} \tag{1.6}$$

Here all limits are considered in the space G. We call the map $f \mapsto f(E)$ the *Borel functional calculus of the spectral measure* E.

In the case where E is the resolution of the identity of a possibly unbounded spectral operator T, recalling Lemma 18.2.25 of [DS] stating that $\sigma(T)$ is the support of E, we can define $f(T) := f(E)$ for any map $f \in \mathrm{Bor}(\sigma(T))$ and call the map

$$\mathrm{Bor}(\sigma(T)) \ni f \mapsto f(T)$$

the *Borel functional calculus of the operator* T.

Definition 1.4 (18.2.12 of [DS]). A *spectral operator of scalar type in* G or a *scalar type spectral operator in* G is a possibly unbounded linear operator R in G such that there exists a countably additive spectral measure $E : \mathcal{B}(\mathbb{C}) \to \mathrm{Pr}(G)$ with support S and the property

$$R = \imath(E).$$

Here $\imath : S \ni \lambda \mapsto \lambda \in \mathbb{C}$, and $\imath(E)$ is relative to the Borel functional calculus of the spectral measure E. We call E a *resolution of the identity of* R.

Let R be a scalar type spectral operator in G, and E a resolution of the identity of R. Then we have the following statements by [DS]:

- T is a spectral operator in G;
- E is the resolution of the identity of T as spectral operator;
- E is unique.

DEFINITION 1.5 ([DS]). Let $E : \mathcal{B}(\mathbb{C}) \to \mathrm{Pr}(G)$ be a countably additive spectral measure and $U \in \mathcal{B}(\mathbb{C})$. Then the space of all *E-essentially bounded* maps is

$$\mathfrak{L}_E^{\infty}(U) := \{f : \mathbb{C} \to \mathbb{C} \mid \|f\chi_U\|_{\infty}^E < \infty\}.$$

Here $\chi_U : \mathbb{C} \to \mathbb{C}$ is the characteristic map of U, and for each map $F : \mathbb{C} \to \mathbb{C}$,

$$\|F\|_{\infty}^E := E\text{-}\operatorname*{ess\,sup}_{\lambda \in \mathbb{C}} |F(\lambda)| := \inf_{\{\delta \in \mathcal{B}(\mathbb{C}) \mid E(\delta) = \mathbf{1}\}} \sup_{\lambda \in \delta} |F(\lambda)|.$$

For a Borelian map $f : U \supset \sigma(R) \to \mathbb{C}$ with $U \in \mathcal{B}(\mathbb{C})$, we define $f(R)$ to be the operator $(f \restriction \sigma(R))(R)$. Let $g : U \subseteq \mathbb{C} \to \mathbb{C}$ be a Borelian map. Then g is E-essentially bounded if

$$E\text{-}\operatorname*{ess\,sup}_{\lambda \in U} |g(\lambda)| := \|\widetilde{g}\|_{\infty}^E < \infty.$$

See Definition 17.2.6 of [DS]. One formula arising by Spectral Theorem 18.2.11(i) of [DS], which will be used many times in this work, is the following: for all Borelian complex functions $f : \sigma(R) \to \mathbb{C}$ and for all $\phi \in G^*$ and $y \in \mathrm{Dom}(f(R))$,

$$\phi(f(R)y) = \int_{\mathbb{C}} \widetilde{f}\, dE_{(\phi,y)}. \tag{1.7}$$

Here G^* is the topological dual of G, that is, the normed space of all $\mathbb{C}$-linear and continuous functionals on G with the sup norm, and for all $\phi \in G^*$ and $y \in G$ we define $E_{(\phi,y)} : \mathcal{B}(\mathbb{C}) \ni \sigma \mapsto \phi(E(\sigma)y) \in \mathbb{C}$. Finally, if $P \in \mathrm{Pr}(G)$ then $\langle P(G), \|\cdot\|_{P(G)}\rangle$ with $\|\cdot\|_{P(G)} := \|\cdot\|_G \restriction P(G)$, is a Banach space. In fact, let $\{v_n\}_{n\in\mathbb{N}} \subset G$ be such that $v = \lim_{n\in\mathbb{N}} Pv_n$ in $\|\cdot\|_G$. Since $P = P^2$ is continuous we have $Pv = \lim_{n\in\mathbb{N}} P^2 v_n = \lim_{n\in\mathbb{N}} Pv_n = v$, so $v \in P(G)$. Thus $P(G)$ is closed in $\langle G, \|\cdot\|_G\rangle$, hence $\langle P(G), \|\cdot\|_{P(G)}\rangle$ is a Banach space. If $E : \mathcal{B}_Y \to \mathrm{Pr}(G)$ is a spectral measure in G on $\mathcal{B}_Y$ and $\sigma \in \mathcal{B}_Y$, then we shall denote by G_σ^E or simply G_σ the complex Banach space $E(\sigma)G$, without indicating its dependence on E whenever it causes no confusion. In addition for any possibly unbounded operator Q in G we define, for all $\sigma \in \mathcal{B}_Y$, the following possibly unbounded operator in G:

$$Q_\sigma := QE(\sigma).$$

Finally, we shall denote by $\mathcal{B}_b(\mathbb{C})$ the subclass of all bounded subsets of $\mathcal{B}(\mathbb{C})$.

DEFINITION 1.6. Let F be a $\mathbb{C}$-Banach space, $P \in \mathrm{Pr}(F)$ and $S : \mathrm{Dom}(S) \subseteq F \to F$. Then we define

$$SP \restriction P(F) := SP \restriction (P(F) \cap \mathrm{Dom}(SP)). \tag{1.8}$$

Notice that as $P^2 = P$ we have $P(F) \cap \mathrm{Dom}(S) = P(F) \cap \mathrm{Dom}(SP)$, and that

$$SP{\restriction}P(F) = S{\restriction}(P(F) \cap \mathrm{Dom}(S)).$$

Moreover, if $PS \subseteq SP$ then

$$SP{\restriction}P(F) : P(F) \cap \mathrm{Dom}(S) \to P(F).$$

That is, $SP{\restriction}P(F)$ is a linear operator in the Banach space $P(F)$.

Let $E : \mathcal{B}_Y \to \mathrm{Pr}(G)$ be a spectral measure in G on $\mathcal{B}_Y$, $\sigma \in \mathcal{B}_Y$, and Q a possibly unbounded operator in G such that $E(\sigma)Q \subseteq QE(\sigma)$. Then

$$Q_\sigma{\restriction}G_\sigma : G_\sigma \cap \mathrm{Dom}(Q) \to G_\sigma.$$

In particular, if R is a possibly unbounded scalar type spectral operator in G, E its resolution of the identity, and $f \in \mathrm{Bor}(\sigma(R))$, then by Theorem 18.2.11(g) of [DS], for all $\sigma \in \mathcal{B}(\mathbb{C})$ we have

$$E(\sigma)f(R) \subseteq f(R)E(\sigma).$$

Hence for all $\sigma \in \mathcal{B}(\mathbb{C})$,

$$\begin{cases} R_\sigma{\restriction}G_\sigma = R_\sigma{\restriction}(G_\sigma \cap \mathrm{Dom}(R)) = R{\restriction}(G_\sigma \cap \mathrm{Dom}(R)), \\ f(R)_\sigma{\restriction}G_\sigma = f(R)_\sigma{\restriction}(G_\sigma \cap \mathrm{Dom}(f(R))) = f(R){\restriction}(G_\sigma \cap \mathrm{Dom}(f(R))) \end{cases} \tag{1.9}$$

are linear operators in G_σ. Finally, $E(\sigma(R)) = \mathbf{1}$ implies $E(\sigma) = E(\sigma \cap \sigma(R))$ for all $\sigma \in \mathcal{B}(\mathbb{C})$, so by (1.9),

$$\begin{cases} R_\sigma{\restriction}G_\sigma = R_{\sigma\cap\sigma(R)}{\restriction}G_{\sigma\cap\sigma(R)}, \\ f(R)_\sigma{\restriction}G_\sigma = f(R)_{\sigma\cap\sigma(R)}{\restriction}G_{\sigma\cap\sigma(R)}. \end{cases} \tag{1.10}$$

Lemma 1.7 (Key Lemma). *Let R be a possibly unbounded scalar type spectral operator in G, E its resolution of the identity, $\sigma(R)$ its spectrum, and $f \in \mathrm{Bor}(\sigma(R))$. Then for all $\sigma \in \mathcal{B}(\mathbb{C})$:*

(1) *$R_\sigma{\restriction}G_\sigma$ is a scalar type spectral operator in G_σ whose resolution of the identity $\widetilde{E}_\sigma$ is such that for all $\delta \in \mathcal{B}(\mathbb{C})$,*

$$\widetilde{E}_\sigma(\delta) = E(\delta){\restriction}G_\sigma \in B(G_\sigma),$$

(2) *$f(R)_\sigma{\restriction}G_\sigma = f(R_\sigma{\restriction}G_\sigma)$,*
(3) *for all $g \in \mathrm{Bor}(\sigma(R))$ such that $g(\sigma \cap \sigma(R))$ is bounded, we have*

$$g(R)E(\sigma) = \mathbf{I}_{\mathbb{C}}^{E}(\widetilde{g} \cdot \chi_\sigma) \in B(G).$$

Proof. Let $\sigma \in \mathcal{B}(\mathbb{C})$. Since $E(\sigma \cap \delta) = E(\delta)E(\sigma) = E(\sigma)E(\delta)$ for all $\delta \in \mathcal{B}(\mathbb{C})$, and $E(\sigma){\restriction}G_\sigma = \mathbf{1}_\sigma$, the unity operator on G_σ, for all $\delta \in \mathcal{B}(\mathbb{C})$ we have

$$\widetilde{E}_\sigma(\delta) = E(\sigma \cap \delta){\restriction}G_\sigma \in B(G_\sigma). \tag{1.11}$$

In particular, $\widetilde{E}_\sigma : \mathcal{B}(\mathbb{C}) \to B(G_\sigma)$, and E is a countably additive spectral measure in G, so

$$\widetilde{E}_\sigma \text{ is a countably additive spectral measure in } G_\sigma. \tag{1.12}$$

By Lemma 18.2.2 of [DS], $\widetilde{E}_\sigma$ is the resolution of identity of the spectral operator $R_\sigma \upharpoonright G_\sigma$ so by Lemma 18.2.25 of [DS] applied to $R_\sigma \upharpoonright G_\sigma$,

$$\operatorname{supp} \widetilde{E}_\sigma = \sigma(R_\sigma \upharpoonright G_\sigma). \tag{1.13}$$

Furthermore, by (1.10) and Definition 18.2.1(iii) of [DS] we have $\sigma(R_\sigma \upharpoonright G_\sigma) \subseteq \overline{\sigma \cap \sigma(R)}$, and since $\overline{\overline{\sigma} \cap \sigma(R)} = \overline{\sigma} \cap \sigma(R)$, we deduce

$$\sigma(R_\sigma \upharpoonright G_\sigma) \subseteq \overline{\sigma} \cap \sigma(R) \subseteq \sigma(R). \tag{1.14}$$

Hence (1.13) and (1.14) imply that the operator function $f(\widetilde{E}_\sigma)$ is well defined. For all $x \in \operatorname{Dom}(f(R)_\sigma \upharpoonright G_\sigma)$,

$$\begin{aligned}
(f(R)_\sigma \upharpoonright G_\sigma)x &= f(R)x && \text{by (1.9)} \\
&= \lim_{n\in\mathbb{N}} \mathbf{I}_{\mathbb{C}}^{E^x}(\widetilde{f} \cdot \chi_{|f|^{-1}(\delta_n)}) && \text{by (1.6), (1.4)} \\
&= \lim_{n\in\mathbb{N}} \mathbf{I}_{\mathbb{C}}^{\widetilde{E}_\sigma^x}(\widetilde{f} \cdot \chi_{|f|^{-1}(\delta_n)}) && \text{by } x \in G_\sigma, \text{ (1.12)} \\
&= \lim_{n\in\mathbb{N}} \mathbf{I}_{\mathbb{C}}^{\widetilde{E}_\sigma}(\widetilde{f} \cdot \chi_{|f|^{-1}(\delta_n)})x && \text{by (1.4)} \\
&= f(\widetilde{E}_\sigma)x && \text{by (1.6),}
\end{aligned}$$

So $f(R)_\sigma \upharpoonright G_\sigma \subseteq f(\widetilde{E}_\sigma)$. For all $x \in \operatorname{Dom}(f(\widetilde{E}_\sigma))$,

$$\begin{aligned}
f(\widetilde{E}_\sigma)x &= \lim_{n\in\mathbb{N}} \mathbf{I}_{\mathbb{C}}^{\widetilde{E}_\sigma^x}(\widetilde{f} \cdot \chi_{|f|^{-1}(\delta_n)}) && \text{by (1.6), (1.4)} \\
&= \lim_{n\in\mathbb{N}} \mathbf{I}_{\mathbb{C}}^{E^x}(\widetilde{f} \cdot \chi_{|f|^{-1}(\delta_n)}) \\
&= \lim_{n\in\mathbb{N}} \mathbf{I}_{\mathbb{C}}^{E}(\widetilde{f} \cdot \chi_{|f|^{-1}(\delta_n)})x && \text{by (1.4)} \\
&= (f(R)_\sigma \upharpoonright G_\sigma)x && \text{by (1.6), (1.9).}
\end{aligned}$$

So $f(\widetilde{E}_\sigma) \subseteq f(R)_\sigma \upharpoonright G_\sigma$, hence

$$f(R)_\sigma \upharpoonright G_\sigma = f(\widetilde{E}_\sigma). \tag{1.15}$$

Therefore statement (1) follows by setting $f = \imath$, while (2) follows from (1) and (1.15). Let $g \in \operatorname{Bor}(\sigma(R))$, such that $g(\sigma \cap \sigma(R))$ is bounded. Then

$$(\exists n \in \mathbb{N})(\forall m > n)(\sigma \cap |g|^{-1}(\delta_m) = \sigma \cap \sigma(R)).$$

Next $E(\sigma(R)) = \mathbf{1}$, so $E(\sigma) = E(\sigma)E(\sigma(R)) = E(\sigma(R) \cap \sigma)$. Since $\mathbf{I}_{\mathbb{C}}^E$ is an algebra homomorphism, for all $m \in \mathbb{N}$ we have

$$\begin{aligned}
\mathbf{I}_{\mathbb{C}}^E(\widetilde{g} \cdot \chi_{|g|^{-1}(\delta_m)})E(\sigma) &= \mathbf{I}_{\mathbb{C}}^E(\widetilde{g} \cdot \chi_{|g|^{-1}(\delta_m)})E(\sigma \cap \sigma(R)) = \mathbf{I}_{\mathbb{C}}^E(\widetilde{g} \cdot \chi_{|g|^{-1}(\delta_m)})\mathbf{I}_{\mathbb{C}}^E(\chi_{\sigma\cap\sigma(R)}) \\
&= \mathbf{I}_{\mathbb{C}}^E(\widetilde{g} \cdot \chi_{|g|^{-1}(\delta_m)} \cdot \chi_{\sigma\cap\sigma(R)}) = \mathbf{I}_{\mathbb{C}}^E(\widetilde{g} \cdot \chi_{|g|^{-1}(\delta_m)\cap\sigma\cap\sigma(R)}) \\
&= \mathbf{I}_{\mathbb{C}}^E(\widetilde{g} \cdot \chi_{|g|^{-1}(\delta_m)\cap\sigma}).
\end{aligned}$$

This equality implies that

$$(\exists n \in \mathbb{N})(\forall m > n)(\mathbf{I}_{\mathbb{C}}^E(\widetilde{g} \cdot \chi_{|g|^{-1}(\delta_m)})E(\sigma) = \mathbf{I}_{\mathbb{C}}^E(\widetilde{g} \cdot \chi_{\sigma\cap\sigma(R)})). \tag{1.16}$$

Furthermore,

$$\mathbf{I}_{\mathbb{C}}^E(\widetilde{g} \cdot \chi_{\sigma\cap\sigma(R)}) = \mathbf{I}_{\mathbb{C}}^E(\widetilde{g}\chi_\sigma\chi_{\sigma(R)}) = \mathbf{I}_{\mathbb{C}}^E(\widetilde{g}\chi_\sigma)\mathbf{I}_{\mathbb{C}}^E(\chi_{\sigma(R)}) = \mathbf{I}_{\mathbb{C}}^E(\widetilde{g}\chi_\sigma)E(\sigma(R)) = \mathbf{I}_{\mathbb{C}}^E(\widetilde{g}\chi_\sigma).$$

Therefore by (1.16),

$$(\exists n \in \mathbb{N})(\forall m > n)(\mathbf{I}_{\mathbb{C}}^{E}(\widetilde{g} \cdot \chi_{|g|^{-1}(\delta_m)})E(\sigma) = \mathbf{I}_{\mathbb{C}}^{E}(\widetilde{g} \cdot \chi_{\sigma})). \tag{1.17}$$

Moreover, by definition in (1.6) we have, for all $x \in \mathrm{Dom}(g(R))$,

$$g(R)x := \lim_{n\to\infty} \mathbf{I}_{\mathbb{C}}^{E}(\widetilde{g} \cdot \chi_{|g|^{-1}(\delta_n)})x,$$

and $\mathrm{Dom}(g(R))$ is the set of $x \in G$ such that the limit exists; thus by (1.17) we conclude that $E(\sigma)G \subseteq \mathrm{Dom}(g(R))$ and $g(R)E(\sigma) = \mathbf{I}_{\mathbb{C}}^{E}(\widetilde{g} \cdot \chi_{\sigma}) \in B(G)$, which is statement (3). ∎

Corollary 1.8. *Let R be a possibly unbounded scalar type spectral operator in G, and $f \in \mathrm{Bor}(\sigma(R))$. Then for all $\sigma \in \mathcal{B}(\mathbb{C})$,*

$$f(R)E(\sigma) = f(R_{\sigma}\restriction G_{\sigma})E(\sigma).$$

Moreover, if $f(\sigma \cap \sigma(R))$ is bounded then

$$f(R_{\sigma}\restriction G_{\sigma})E(\sigma) \in B(G).$$

Proof. Let $y \in \mathrm{Dom}(f(R)E(\sigma))$. Then $E(\sigma)y \in G_{\sigma} \cap \mathrm{Dom}(f(R))$, hence by (1.9) and Lemma 1.7,

$$f(R)E(\sigma)y = (f(R)_{\sigma}\restriction G_{\sigma})E(\sigma)y = f(R_{\sigma}\restriction G_{\sigma})E(\sigma)y.$$

So $f(R)E(\sigma) \subseteq f(R_{\sigma}\restriction G_{\sigma})E(\sigma)$. Next let $y \in \mathrm{Dom}(f(R_{\sigma}\restriction G_{\sigma})E(\sigma))$. Then $E(\sigma)y \in \mathrm{Dom}(f(R_{\sigma}\restriction G_{\sigma}))$, hence by Lemma 1.7 and (1.9),

$$f(R_{\sigma}\restriction G_{\sigma})(E(\sigma)y) = f(R)E(\sigma)E(\sigma)y = f(R)E(\sigma)y.$$

So $f(R_{\sigma}\restriction G_{\sigma})E(\sigma) \subseteq f(R)E(\sigma)$. Thus we obtain the first statement. The second follows from the first and Lemma 1.7(3). ∎

1.2. Extension theorem for strong operator integral equalities

Notations 1.9. Let X be a locally compact space and μ a measure on X in the sense of [INT, III.7, Definition 2], that is, a continuous linear $\mathbb{C}$-functional on the $\mathbb{C}$-locally convex space $H(X)$ of all compactly supported continuous complex functions on X, with the topology of the direct limit (or inductive limit) of the spaces $H(X;K)$ with K running over the class of all compact subsets of X, where $H(X;K)$ is the space of all continuous functions $f : X \to \mathbb{C}$ such that $\mathrm{supp}(f) := \overline{\{x \in X \mid f(x) \neq 0\}} \subseteq K$ with the norm topology of uniform convergence ([5]). In this work any measure μ on X in the sense of [INT] will be called a complex Radon measure on X. For the definition of *μ-integrable functions defined on X and with values in a $\mathbb{C}$-Banach space G* see IV.23, Definition 2 of [INT], while the *integral with respect to μ of a μ-integrable function $f : X \to G$* , which will be denoted by $\int f(x)\, d\mu(x) \in G$, is defined in Definition 1, III.33 and Definition 1, IV.33 of [INT]. For the definition of the total variation $|\mu|$, and definition and properties of the upper integral $\int^{*} g\, d|\mu|(x)$ see Ch. 3–4 of [INT]. We denote by $\mathrm{Comp}(X)$ the class of all compact subsets of X and by $\mathfrak{F}_1(X;\mu)$ the seminormed space, with seminorm

([5]) $H(X;K)$ is isometric to the Banach space of all continuous maps $g : K \to \mathbb{C}$ equal to 0 on ∂K, with the norm topology of uniform convergence.

$\|\cdot\|_{\mathfrak{F}_1(X;\mu)}$, of all maps $F : X \to \mathbb{C}$ such that

$$\|F\|_{\mathfrak{F}_1(X;\mu)} := \int^* |F(x)|\, d|\mu|(x) < \infty.$$

In this section it will be assumed, unless otherwise stated, that X is a locally compact space and μ is a complex Radon measure over X. Let $B \subseteq X$ be a μ-measurable set. Then by writing μ-a.e.(B) we mean "almost everywhere in B with respect to the measure μ". Let $f : X \to B(G)$ be a μ-integrable map into the normed space $B(G)$ (Definition 2, Ch. IV, §3, n° 4 of [INT]). Then we denote by

$$\oint f(x)\, d\mu(x) \in B(G)$$

its integral in $B(G)$ (Definition 1, Ch. IV, §4, n° 1 of [INT]), which is uniquely determined by the following property: for all $\phi \in B(G)^*$,

$$\phi\Big(\oint f(x)\, d\mu(x)\Big) = \int \phi(f(x))\, d\mu(x).$$

For any scalar type spectral operator S in a complex Banach space G and for any Borelian map $f : U \supseteq \sigma(S) \to \mathbb{C}$ we assume that $f(S)$ is the closed operator defined in (1.6) and recall that we denote by $\widetilde{f}$ the $\mathbf{0}$-extension of f to $\mathbb{C}$ (see Definition 1.2).

Definition 1.10. Let $E : \mathcal{B}_Y \to \Pr(G)$ be a spectral measure in G on $\mathcal{B}_Y$. Then we say that $\{\sigma_n\}_{n\in\mathbb{N}}$ is an *E-sequence* if there exists an $S \in \mathcal{B}_Y$ such that $E(S) = \mathbf{1}$ and

- $(\forall n \in \mathbb{N})(\sigma_n \in \mathcal{B}_Y)$;
- $(\forall n, m \in \mathbb{N})(n > m \Rightarrow \sigma_n \supseteq \sigma_m)$;
- $S \subseteq \bigcup_{n\in\mathbb{N}} \sigma_n$.

Proposition 1.11. *Let $E : \mathcal{B}_Y \to \Pr(G)$ be a countably additive spectral measure in G on a σ-field $\mathcal{B}_Y$, and $\{\sigma_n\}_{n\in\mathbb{N}}$ an E-sequence. Then*

$$\lim_{n\to\infty} E(\sigma_n) = \mathbf{1} \quad \text{in the strong operator topology.} \tag{1.18}$$

Proof. Let $S \in \mathcal{B}_Y$ be associated to the E-sequence $\{\sigma_n\}_{n\in\mathbb{N}}$ as in Definition 1.10. So $E(S) = \mathbf{1}$ and since E is an order-preserving map, we have $E(\bigcup_{n\in\mathbb{N}} \sigma_n) \geq E(S) = \mathbf{1}$. Since $\mathbf{1}$ is a maximal element in $\langle E(\mathcal{B}_Y), \geq\rangle$,

$$E\Big(\bigcup_{n\in\mathbb{N}} \sigma_n\Big) = \mathbf{1}.$$

Set $\eta_1 := \sigma_1$, and for all $n \geq 2$, $\eta_n := \sigma_n \cap \complement\sigma_{n-1}$, so for all $n \in \mathbb{N}$, $\sigma_n = \bigcup_{k=1}^n \eta_k$, and for all $n \neq m \in \mathbb{N}$, $\eta_n \cap \eta_m = \emptyset$, and finally $\bigcup_{n\in\mathbb{N}} \eta_n = \bigcup_{n\in\mathbb{N}}(\bigcup_{k=1}^n \eta_k) = \bigcup_{n\in\mathbb{N}} \sigma_n$. Therefore by the countable additivity of E with respect to the strong operator topology,

$$E\Big(\bigcup_{n\in\mathbb{N}} \sigma_n\Big) = E\Big(\bigcup_{n\in\mathbb{N}} \eta_n\Big) = \sum_{n=1}^{\infty} E(\eta_n) = \lim_{n\to\infty} \sum_{k=1}^{n} E(\eta_k) = \lim_{n\to\infty} E\Big(\bigcup_{k=1}^{n} \eta_k\Big) = \lim_{n\to\infty} E(\sigma_n).$$

Here all limits are with respect to the strong operator topology, yielding the statement. ∎

Definition 1.12. Let G_1, G_2 be two complex Banach spaces, and $f : X \to B(G_1, G_2)$. Then we say that f *is μ-integrable with respect to the strong operator topology* if

- for all $v \in G_1$ the map $X \ni x \mapsto f(x)v \in G_2$ is μ-integrable;
- if we set

$$F : G_1 \ni v \mapsto \int f(x)(v)\, d\mu(x) \in G_2$$

then $F \in B(G_1, G_2)$.

In that case we set $\int f(x)\, d\mu(x) := F$, in other words the integral $\int f(x)\, d\mu(x)$ of f with respect to the measure μ and the strong operator topology is a bounded linear operator from G_1 to G_2 such that for all $v \in G_1$,

$$\Big(\int f(x)\, d\mu(x)\Big)(v) = \int f(x)(v)\, d\mu(x).$$

We shall need the following version of the Minkowski inequality:

Proposition 1.13. *Let G_1, G_2 be two complex Banach spaces, and let $f : X \to B(G_1, G_2)$ be such that*

(1) $(\forall v \in G_1)(\forall \phi \in G_2^*)$ *the complex map $X \ni x \mapsto \phi(f(x)v) \in \mathbb{C}$ is μ-measurable;*
(2) *for all $v \in G_1$ and $K \in \mathrm{Comp}(X)$ there is $H \subset G_2$ such that H is countable and $f(x)v \in \overline{H}$ μ-a.e.(K);*
(3) $(X \ni x \mapsto \|f(x)\|_{B(G_1,G_2)}) \in \mathfrak{F}_1(X; \mu)$,

Then f is μ-integrable with respect to the strong operator topology, and

$$\Big\|\int f(x)\, d\mu(x)\Big\|_{B(G_1,G_2)} \leq \int^* \|f(x)\|_{B(G_1,G_2)}\, d|\mu|(x).$$

Proof. By hypothesis (3) we have, for all $v \in G_1$,

$$\int^* \|f(x)v\|_{G_2}\, d|\mu|(x) \leq \|v\|_{G_1} \int^* \|f(x)\|_{B(G_1,G_2)}\, d|\mu|(x) < \infty. \tag{1.19}$$

By hypotheses (1–2) and Corollary 1, IV.70 of [INT], for all $v \in G_1$ the map $X \mapsto f(x)v \in G_2$ is μ-measurable. Therefore by (1.19) and by Theorem 5, IV.71 of [INT] we deduce that $X \mapsto f(x)v \in G_2$ is μ-integrable for all $v \in G_1$. So in particular by Definition 1, IV.33 of [INT] for all $v \in G_1$ we have $\int f(x)v\, d\mu(x) \in G_2$ while by Proposition 2, IV.35 of [INT] and (1.19),

$$\Big\|\int f(x)v\, d\mu(x)\Big\|_{G_2} \leq \|v\|_{G_1} \int^* \|f(x)\|_{B(G_1,G_2)}\, d|\mu|(x).$$

Hence the statement follows. ■

Remark 1.14. By the above proof, Proposition 1.13 is also valid if we replace hypotheses (1–2) with the following one:

$$\forall v \in G_1 \quad \text{the map } X \ni x \mapsto f(x)v \in G_2 \text{ is } \mu\text{-measurable.} \tag{1'}$$

Lemma 1.15. *Let X, Y, Z be normed spaces over the same field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, $R : \mathrm{Dom}(R) \subseteq Y \to Z$ a possibly unbounded closed linear operator, and $A \in B(X, Y)$. Then $R \circ A : \mathsf{D} \to Z$ is a closed operator, where $\mathsf{D} := \mathrm{Dom}(R \circ A)$.*

Proof. Let $\{x_n\}_{n \in \mathbb{N}} \subset \mathsf{D} := \{x \in X \mid A(x) \in \mathrm{Dom}(R)\}$, and $(x, z) \in X \times Z$ such that $x = \lim_{n\to\infty} x_n$ and $z = \lim_{n\to\infty} R \circ A(x_n)$. Since A is continuous we have $A(x) =$

$\lim_{n\to\infty} A(x_n)$; but $z = \lim_{n\to\infty} R(Ax_n)$, and R is closed, so $z = R(A(x)) := R \circ A(x)$, hence $(x,z) \in \mathrm{Graph}(R \circ A)$, which is just the statement. ∎

Lemma 1.16. *Let X be a normed space, Y a Banach space over the same field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, and $U : \mathsf{D} \subseteq X \to Y$ a linear operator. If U is continuous and closed then D is closed.*

Proof. Let $\{x_n\}_{n\in\mathbb{N}} \subset \mathsf{D}$ and $x \in X$ be such that $x = \lim_{n\to\infty} x_n$. So by the continuity of U, for all $n, m \in \mathbb{N}$ we have $\|U(x_n) - U(x_m)\| = \|U(x_n - x_m)\| \leq \|U\| \, \|x_n - x_m\|$, hence $\lim_{(n,m)\in\mathbb{N}^2} \|U(x_n) - U(x_m)\| = 0$, thus ($Y$ being a Banach space) there is $y \in Y$ such that $y = \lim_{n\to\infty} U(x_n)$. But U is closed, therefore $y = U(x)$, so $x \in \mathsf{D}$, which is the statement. ∎

Theorem 1.17. *Let R be a possibly unbounded scalar type spectral operator in G, $\sigma(R)$ its spectrum, and E its resolution of the identity. Let $X \ni x \mapsto f_x \in \mathrm{Bor}(\sigma(R))$ be such that for all $x \in X$, $\widetilde{f}_x \in \mathfrak{L}_E^\infty(\sigma(R))$ where $X \ni x \mapsto f_x(R) \in B(G)$ is μ-integrable with respect to the strong operator topology. Then:*

(1) *For all $\sigma \in \mathcal{B}(\mathbb{C})$ the map $X \ni x \mapsto f_x(R_\sigma \restriction G_\sigma) \in B(G_\sigma)$ is μ-integrable with respect to the strong operator topology and*

$$\left\| \int f_x(R_\sigma \restriction G_\sigma)\, d\mu(x) \right\|_{B(G_\sigma)} \leq \left\| \int f_x(R)\, d\mu(x) \right\|_{B(G)}.$$

(2) *If $g, h \in \mathrm{Bor}(\sigma(R))$, $\{\sigma_n\}_{n\in\mathbb{N}}$ is an E-sequence, and for all $n \in \mathbb{N}$,*

$$g(R_{\sigma_n} \restriction G_{\sigma_n}) \int f_x(R_{\sigma_n} \restriction G_{\sigma_n})\, d\mu(x) \subseteq h(R_{\sigma_n} \restriction G_{\sigma_n}). \tag{1.20}$$

then

$$g(R) \int f_x(R)\, d\mu(x) \restriction \Theta = h(R) \restriction \Theta, \tag{1.21}$$

where $\Theta := \mathrm{Dom}(g(R) \int f_x(R)\, d\mu(x)) \cap \mathrm{Dom}(h(R))$ and all the integrals are with respect to the strong operator topologies.

Notice that $g(R)$ is possibly an *unbounded* operator in G.

Proof. Let $\sigma \in \mathcal{B}(\mathbb{C})$. Then by (1.14),

$$(\forall \sigma \in \mathcal{B}(\mathbb{C}))(\sigma(R_\sigma \restriction G_\sigma) \subseteq \overline{\sigma} \cap \sigma(R) \subseteq \sigma(R)),$$

which implies that the operator functions $g(R_\sigma \restriction G_\sigma)$, $h(R_\sigma \restriction G_\sigma)$ and $f_x(R_\sigma \restriction G_\sigma)$, for all $x \in X$, are all well defined.

Since $\{\delta \in \mathcal{B}(\mathbb{C}) \mid E(\delta) = \mathbf{1}\} \subseteq \{\delta \in \mathcal{B}(\mathbb{C}) \mid \widetilde{E}_\sigma(\delta) = \mathbf{1}_\sigma\}$ (by Lemma 1.7(1)), we deduce, for all $x \in X$,

$$\|\widetilde{f}_x\|_\infty^{\widetilde{E}_\sigma} \leq \|\widetilde{f}_x\|_\infty^E = \|\widetilde{f}_x \chi_{\sigma(R)}\|_\infty^E < \infty,$$

where the last equality comes from $\widetilde{f}_x \chi_{\sigma(R)} = \widetilde{f}_x$, while the boundedness from the hypothesis $\widetilde{f}_x \in \mathfrak{L}_E^\infty(\sigma(R))$. Thus $\widetilde{f}_x \in \mathfrak{L}_{\widetilde{E}_\sigma}^\infty(\mathbb{C})$, hence by Theorem 18.2.11(c) of [DS] applied to the scalar type spectral operator $R_\sigma \restriction G_\sigma$,

$$(\forall \sigma \in \mathcal{B}(\mathbb{C}))(f_x(R_\sigma \restriction G_\sigma) \in B(G_\sigma)). \tag{1.22}$$

A more direct way of obtaining (1.22) is to use Lemma 1.7(2) and the fact that $\widetilde{f}_x \in \mathfrak{L}_E^\infty(\sigma(R))$ implies $f_x(R) \in B(G)$. For all $\sigma \in \mathcal{B}(\mathbb{C})$ we claim that $X \ni x \mapsto f_x(R_\sigma \restriction G_\sigma) \in$

$B(G_\sigma)$ is μ-integrable with respect to the strong operator topology. By Lemma 1.7 we have, for all $\sigma \in \mathcal{B}(\mathbb{C})$ and all $v \in G_\sigma$,

$$\int^* \|f_x(R_\sigma \restriction G_\sigma)v\|_{G_\sigma} \, d|\mu|(x) = \int^* \|f_x(R)v\|_G \, d|\mu|(x) < \infty. \tag{1.23}$$

Here the boundedness comes from Theorem 5, IV.71 of [INT] applied to the μ-integrable map $X \ni x \mapsto f_x(R)v \in G$. By Corollary 1, IV.70 and Theorem 5, IV.71 of [INT] applied, for any $v \in G$, to the μ-integrable map $X \ni x \mapsto f_x(R)v \in G$, we find that for all $v \in G$ and $K \in \mathrm{Comp}(X)$ there is a countable $H^v \subseteq G$ such that $f_x(R)v \in \overline{H^v}$, μ-a.e.(K). But by Theorem 18.2.11(g) of [DS] and since $f_x(R) \in B(G)$, we have for all $\sigma \in \mathcal{B}(\mathbb{C})$, $[f_x(R), E(\sigma)] = \mathbf{0}$, hence by the previous equation and by the fact that $E(\sigma)$ is in $B(G)$, so it is continuous, we obtain, for all $\sigma \in \mathcal{B}(\mathbb{C})$, $v \in G$, and $K \in \mathrm{Comp}(X)$,

$$(\exists H^v \subseteq G \text{ countable})(f_x(R)E(\sigma)v = E(\sigma)f_x(R)v \in \overline{H^v_\sigma},\ \mu\text{-a.e.}(K)).$$

Here $H^v_\sigma := E(\sigma)H^v$. Therefore by Lemma 1.7, for all $\sigma \in \mathcal{B}(\mathbb{C})$, $v \in G_\sigma$, and $K \in \mathrm{Comp}(X)$,

$$(\exists H^v_\sigma \subset G_\sigma \text{ countable})(f_x(R_\sigma \restriction G_\sigma)v \in \overline{H^v_\sigma} \subseteq G_\sigma,\ \mu\text{-a.e.}(K)). \tag{1.24}$$

That $\overline{H^v_\sigma} \subseteq G_\sigma$ follows from the fact that G_σ is closed in G. Therefore we can consider $\overline{H^v_\sigma}$ as the closure in the Banach space G_σ. By the Hahn–Banach theorem (see Corollary 3, II.23 of [TVS]), for all $\sigma \in \mathcal{B}(\mathbb{C})$,

$$\{\phi \restriction G_\sigma \mid \phi \in G^*\} = (G_\sigma)^*. \tag{1.25}$$

Moreover, by Corollary 1, IV.70 and Theorem 5, IV.71 of [INT] applied, for any $v \in G$, to the μ-integrable map $X \ni x \mapsto f_x(R)E(\sigma)v \in G$, we see that for all $\phi \in G^*$,

$$X \ni x \mapsto \phi(f_x(R)E(\sigma)v) \in \mathbb{C} \text{ is } \mu\text{-measurable}.$$

Thus by Lemma 1.7, for all $\sigma \in \mathcal{B}(\mathbb{C})$, $v \in G_\sigma$, and $\phi \in G^*$,

$$X \ni x \mapsto \phi(f_x(R_\sigma \restriction G_\sigma)v) \in \mathbb{C} \text{ is } \mu\text{-measurable}.$$

Hence by (1.25), for all $\sigma \in \mathcal{B}(\mathbb{C})$ and $v \in G_\sigma$,

$$(\forall \phi_\sigma \in (G_\sigma)^*)(X \ni x \mapsto \phi_\sigma(f_x(R_\sigma \restriction G_\sigma)v) \in \mathbb{C} \text{ is } \mu\text{-measurable}). \tag{1.26}$$

Now by combining (1.26), (1.23) and (1.24), where $\overline{H^v_\sigma}$ is to be understood as the closure in the Banach space G_σ, we can apply Corollary 1, IV.70 and Theorem 5, IV.71 of [INT] to the map $X \ni x \mapsto f_x(R_\sigma \restriction G_\sigma)v \in G_\sigma$, to deduce that

$$(\forall \sigma \in \mathcal{B}(\mathbb{C}))(\forall v \in G_\sigma)(X \ni x \mapsto f_x(R_\sigma \restriction G_\sigma)v \in G_\sigma \text{ is } \mu\text{-integrable}). \tag{1.27}$$

This means in particular that its integral exists, so for all $\sigma \in \mathcal{B}(\mathbb{C})$ and $v \in G_\sigma$,

$$\begin{aligned} \left\| \int f_x(R_\sigma \restriction G_\sigma)v \, d\mu(x) \right\|_{G_\sigma} &= \left\| \int f_x(R)v \, d\mu(x) \right\|_G \qquad \text{by Lemma 1.7} \\ &\le \left\| \int f_x(R) \, d\mu(x) \right\|_{B(G)} \|v\|_{G_\sigma}. \end{aligned} \tag{1.28}$$

Here the inequality follows by the hypothesis that $X \ni x \mapsto f_x(R) \in B(G)$ is μ-integrable in the strong operator topology. Therefore by Definition 1.12 and (1.22), (1.27), (1.28) we conclude that

$$\begin{cases}(\forall\sigma\in\mathcal{B}(\mathbb{C}))(X\ni x\mapsto f_x(R_\sigma\lceil G_\sigma)\in B(G_\sigma)\text{ is }\mu\text{-integrable}\\ \qquad\qquad\qquad\qquad\qquad\qquad\text{in the strong operator topology}),\\ \|\int f_x(R_\sigma\lceil G_\sigma)\,d\mu(x)\|_{B(G_\sigma)}\le\|\int f_x(R)\,d\mu(x)\|_{B(G)},\end{cases}\tag{1.29}$$

so statement (1) follows. It proves that the assumption (1.20) is meaningful, so we are able to start the proof of (2). For all $y\in\Theta$,

$$\begin{aligned}
&g(R)\int f_x(R)\,d\mu(x)y\\
&=\lim_{n\in\mathbb{N}}E(\sigma_n)g(R)\int f_x(R)\,d\mu(x)y && \text{by (1.18)}\\
&=\lim_{n\in\mathbb{N}}g(R)E(\sigma_n)\int f_x(R)\,d\mu(x)y && \text{by Theorem 18.2.11(g) of [DS]}\\
&=\lim_{n\in\mathbb{N}}g(R)E(\sigma_n)\int f_x(R)y\,d\mu(x) && \text{by Definition 1.12}\\
&=\lim_{n\in\mathbb{N}}g(R)E(\sigma_n)\int E(\sigma_n)f_x(R)y\,d\mu(x) && \text{by Theorem 1, IV.35 of [INT]}\\
&=\lim_{n\in\mathbb{N}}g(R)E(\sigma_n)\int f_x(R)E(\sigma_n)y\,d\mu(x) && \text{by Theorem 18.2.11(g) of [DS]}\\
&=\lim_{n\in\mathbb{N}}g(R_{\sigma_n}\lceil G_{\sigma_n})\int f_x(R_{\sigma_n}\lceil G_{\sigma_n})E(\sigma_n)y\,d\mu(x) && \text{by Lemma 1.7}\\
&=\lim_{n\in\mathbb{N}}g(R_{\sigma_n}\lceil G_{\sigma_n})\int f_x(R_{\sigma_n}\lceil G_{\sigma_n})\,d\mu(x)E(\sigma_n)y && \text{by (1) and Definition 1.12}\\
&=\lim_{n\in\mathbb{N}}h(R_{\sigma_n}\lceil G_{\sigma_n})E(\sigma_n)y && \text{by (1.20)}\\
&=\lim_{n\in\mathbb{N}}h(R)E(\sigma_n)y && \text{by Lemma 1.7}\\
&=\lim_{n\in\mathbb{N}}E(\sigma_n)h(R)y && \text{by Theorem 18.2.11(g) of [DS]}\\
&=h(R)y && \text{by (1.18).}
\end{aligned}$$

Therefore

$$g(R)\int f_x(R)\,d\mu(x)\lceil\Theta=h(R)\lceil\Theta.\ \blacksquare$$

THEOREM 1.18 (Strong Extension Theorem). *Let X be a locally compact space, μ a complex Radon measure on X, R a possibly unbounded scalar type spectral operator in G, $\sigma(R)$ its spectrum, and E its resolution of the identity. Let $X\ni x\mapsto f_x\in\mathrm{Bor}(\sigma(R))$ be such that for all $x\in X$, $\widetilde{f_x}\in\mathfrak{L}_E^\infty(\sigma(R))$, where $X\ni x\mapsto f_x(R)\in B(G)$ is μ-integrable with respect to the strong operator topology. Finally, let $g,h\in\mathrm{Bor}(\sigma(R))$ and $\widetilde{h}\in\mathfrak{L}_E^\infty(\sigma(R))$. If $\{\sigma_n\}_{n\in\mathbb{N}}$ is an E-sequence and for all $n\in\mathbb{N}$,*

$$g(R_{\sigma_n}\lceil G_{\sigma_n})\int f_x(R_{\sigma_n}\lceil G_{\sigma_n})\,d\mu(x)\subseteq h(R_{\sigma_n}\lceil G_{\sigma_n})\tag{1.30}$$

then $h(R) \in B(G)$ *and*

$$g(R)\int f_x(R)\,d\mu(x) = h(R).$$

Here all the integrals are with respect to the strong operator topologies.

Notice that $g(R)$ is possibly an *unbounded* operator on G.

Proof. $h(R) \in B(G)$ by Theorem 18.2.11. of [DS] and the hypothesis $\widetilde{h} \in \mathfrak{L}_E^\infty(\sigma(R))$, so by (1.21),

$$g(R)\int f_x(R)\,d\mu(x) \subseteq h(R). \tag{1.31}$$

Let us set

$$(\forall n \in \mathbb{N})(\delta_n := |g|^{-1}([0,n])). \tag{1.32}$$

We claim that

$$\begin{cases} \bigcup_{n\in\mathbb{N}} \delta_n = \sigma(R), \\ n \geq m \Rightarrow \delta_n \supseteq \delta_m, \\ (\forall n \in \mathbb{N})(g(\delta_n) \text{ is bounded}). \end{cases} \tag{1.33}$$

In addition, since $|g| \in \mathrm{Bor}(\sigma(R))$ we have $\delta_n \in \mathcal{B}(\mathbb{C})$ for all $n \in \mathbb{N}$, so $\{\delta_n\}_{n\in\mathbb{N}}$ is an E-sequence, hence by (1.18),

$$\lim_{n\in\mathbb{N}} E(\delta_n) = \mathbf{1} \tag{1.34}$$

with respect to the strong operator topology on $B(G)$. Indeed, the first equality of (1.33) follows since $\bigcup_{n\in\mathbb{N}} \delta_n \doteq \bigcup_{n\in\mathbb{N}} |g|^{-1}([0,n]) = |g|^{-1}(\bigcup_{n\in\mathbb{N}}[0,n]) = |g|^{-1}(\mathbb{R}^+) = \mathrm{Dom}(g)$ $:= \sigma(R)$, the second by the fact that $|g|^{-1}$ preserves the inclusion, the third since $|g|(\delta_n) \subseteq [0,n]$. This yields our claim. By the third statement of (1.33), $\delta_n \in \mathcal{B}(\mathbb{C})$ and Lemma 1.7(3),

$$(\forall n \in \mathbb{N})(E(\delta_n)G \subseteq \mathrm{Dom}(g(R))). \tag{1.35}$$

As $f_x(R)E(\delta_n) = E(\delta_n)f_x(R)$, by Theorem 18.2.11(g) of [DS], for all $v \in G$ we have

$$\begin{aligned} \int f_x(R)\,d\mu(x)E(\delta_n)v &\doteq \int f_x(R)E(\delta_n)v\,d\mu(x) \\ &= \int E(\delta_n)f_x(R)v\,d\mu(x) = E(\delta_n)\int f_x(R)v\,d\mu(x), \end{aligned}$$

where the last equality follows by applying Theorem 1, IV.35 of [INT]. Hence for all $n \in \mathbb{N}$,

$$\int f_x(R)\,d\mu(x)E(\delta_n)G \subseteq E(\delta_n)G \subseteq \mathrm{Dom}(g(R)),$$

where the last inclusion is by (1.35). Therefore

$$(\forall n \in \mathbb{N})(\forall v \in G)\left(E(\delta_n)v \in \mathrm{Dom}\left(g(R)\int f_x(R)\,d\mu(x)\right)\right).$$

Hence by (1.34),

$$\mathbf{D} := \mathrm{Dom}\left(g(R)\int f_x(R)\,d\mu(x)\right) \text{ is dense in } G. \tag{1.36}$$

Next $\int f_x(R)\,d\mu(x) \in B(G)$ and $g(R)$ is closed by Theorem 18.2.11 of [DS], so by Lemma 1.15,

$$g(R) \int f_x(R)\,d\mu(x) \text{ is closed.} \tag{1.37}$$

Moreover, $h(R) \in B(G)$, hence by (1.31),

$$g(R) \int f_x(R)\,d\mu(x) \in B(\mathbf{D}, G). \tag{1.38}$$

Since (1.37), (1.38) and Lemma 1.16 imply that $\mathbf{D}$ is closed in G, by (1.36) we have $\mathbf{D} = G$. Therefore by (1.31) the statement follows. ∎

Now we give conditions ensuring the strong operator integrability of the map $f_x(R)$.

Corollary 1.19. *Let R be a possibly unbounded scalar type spectral operator in G. Let $\{\sigma_n\}_{n\in\mathbb{N}}$ be an E-sequence and for all $x \in X$, let $f_x \in \mathrm{Bor}(\sigma(R))$ be such that*

$$(X \ni x \mapsto \|\widetilde{f_x}\|_\infty^E) \in \mathfrak{F}_1(X;\mu)$$

and $X \ni x \mapsto f_x(R) \in B(G)$ satisfies the conditions (1–2) *of Proposition* 1.13 *(respectively for all $v \in G$ the map $X \ni x \mapsto f_x(R)v \in G$ is μ-measurable). Finally, let $g, h \in$* $\mathrm{Bor}(\sigma(R))$*. If we assume that* (1.30) *holds for all $n \in \mathbb{N}$ and that $\widetilde{h} \in \mathfrak{L}_E^\infty(\sigma(R))$, then the conclusions of Theorem* 1.18 *hold.*

Proof. By Theorem 18.2.11(c) of [DS] and Proposition 1.13 (respectively Remark 1.14) the map $X \ni x \mapsto f_x(R) \in B(G)$ is μ-integrable with respect to the strong operator topology and

$$\left\|\int f_x(R)\,d\mu(x)\right\|_{B(G)} \le 4M \int^* \|\widetilde{f_x}\|_\infty^E\,d|\mu|(x).$$

Here $M := \sup_{\sigma\in\mathcal{B}(\mathbb{C})} \|E(\sigma)\|_{B(G)}$. Therefore the statement follows by Theorem 1.18. ∎

1.3. Generalization of the Newton–Leibniz formula. The main result of this section is Theorem 1.25 which generalizes the Newton–Leibniz formula to the case of unbounded scalar type spectral operators in G. To prove it we need two preliminary results; the first is Theorem 1.21, concerning the Newton–Leibniz formula for any bounded scalar type spectral operator on G and any analytic map on an open neighbourhood of its spectrum. The second, Theorem 1.23, concerns strong operator continuity, and under additional conditions also differentiability, for operator maps of the type $K \ni t \mapsto S(tR) \in B(G)$, where K is an open interval of $\mathbb{R}$, $S(tR)$ arises by the Borel functional calculus for the unbounded scalar type spectral operator R in G, and S is any analytic map on an open neighbourhood U of $\sigma(R)$ such that $K \cdot U \subseteq U$. Let Z be a nonempty set, Y a $\mathbb{K}$-linear space ($\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$), $U \subseteq Y$, $K \subseteq \mathbb{K}$ such that $K \cdot U \subseteq U$ and $F : U \to Z$. Then we define $F_t : U \to Z$ by $F_t(\lambda) := F(t\lambda)$ for all $t \in K$ and $\lambda \in U$.

If H, G are two $\mathbb{C}$-Banach spaces, $A \subseteq H$ open and $f : A \subseteq H \to G$ a map, we denote the real Banach spaces $H_{\mathbb{R}}$ and $G_{\mathbb{R}}$ associated to H and G again by H and G respectively, and by f the map $f^{\mathbb{R}} : A \subseteq H_{\mathbb{R}} \to G_{\mathbb{R}}$.

Lemma 1.20. *Let $\langle Y, d\rangle$ be a metric space, U an open subset of Y, and σ a compact set such that $\sigma \subseteq U$. Then there is $Q > 0$ such that*

$$K := \overline{\bigcup_{y\in\sigma} \overline{B}_Q(y)} \subseteq U, \tag{1.39}$$

and if σ is of finite diameter then K is of finite diameter.

Proof. By Remark, §2.2, Ch. 9 of [GT] we deduce

$$P := \operatorname{dist}(\sigma, \complement U) \neq 0,$$

where $\operatorname{dist}(A, B) := \inf_{\{x\in A, y\in B\}} d(x, y)$ for all $A, B \subseteq Y$. Set

$$Q := P/2.$$

Then for all $y \in \sigma$, $x \in \overline{B}_Q(y)$, $z \in \complement U$ we have

$$d(x, z) \geq d(z, y) - d(y, x) \geq P/2 \neq 0. \tag{1.40}$$

Thus by applying Proposition 2, §2.2, Ch. 9 of [GT], $\overline{B}_Q(y) \cap \complement U = \emptyset$, i.e. $\overline{B}_Q(y) \subseteq U$, so

$$A := \bigcup_{y\in\sigma} \overline{B}_Q(y) \subseteq U.$$

Moreover, by Proposition 3, §2.2, Ch. 9 of [GT] the map $x \mapsto d(x, \complement U)$ is continuous on Y, hence by (1.40) for all $x \in \overline{A}$,

$$d(x, \complement U) = \lim_{n\in\mathbb{N}} d(x_n, \complement U) \geq P/2 \neq 0$$

for all $\{x_n\}_{n\in\mathbb{N}} \subset A$ such that $x = \lim_{n\in\mathbb{N}} x_n$. Therefore by Proposition 2, §2.2, Ch. 9 of [GT], (1.39) follows. Let $B \subset Y$ be of finite diameter. Then by the continuity of the map $d : Y \times Y \to \mathbb{R}^+$ also $\overline{B}$ is of finite diameter. Indeed, let $\operatorname{diam}(B) := \sup_{x,y\in B} d(x, y)$. If $\sup_{x,y\in\overline{B}} d(x, y) = \infty$ then

$$(\exists x_0, y_0 \in \overline{B})(d(x_0, y_0) > \operatorname{diam}(B) + 1). \tag{1.41}$$

Let $\{(x_\alpha, y_\alpha)\}_{\alpha\in D} \subset B \times B$ be a net such that $\lim_{\alpha\in D}(x_\alpha, y_\alpha) = (x_0, y_0)$ (limit in $\langle Y, d\rangle \times \langle Y, d\rangle$). Thus by the continuity of d,

$$d(x_0, y_0) = \lim_{\alpha\in D} d(x_\alpha, y_\alpha) \leq \operatorname{diam}(B),$$

which contradicts (1.41), so $\sup_{x,y\in\overline{B}} d(x, y) < \infty$. Therefore if A is of finite diameter, so is K. Let $z_1, z_2 \in A$. Then there exist $y_1, y_2 \in \sigma$ such that $z_k \in \overline{B}_Q(y_k)$ for $k \in \{1, 2\}$. Then

$$d(z_1, z_2) \leq d(z_1, y_1) + d(y_1, y_2) + d(y_2, z_2) \leq 2Q + \operatorname{diam}(\sigma) < \infty.$$

Hence A is of finite diameter. ∎

Theorem 1.21. *Let $T \in B(G)$ be a scalar type spectral operator, and $\sigma(T)$ its spectrum. Assume that $0 < L \leq \infty$, U is an open neighbourhood of $\sigma(T)$ such that $]-L, L[\cdot U \subseteq U$, and $F : U \to \mathbb{C}$ an analytic map. Then for all $t \in]-L, L[$,*

(1) *we have*

$$F(tT) = F_t(T); \tag{1.42}$$

(2) *we have*

$$\frac{dF(tT)}{dt} = T\,\frac{dF}{d\lambda}(tT); \tag{1.43}$$

(3) *for all* $u_1, u_2 \in \,]-L, L[$,

$$T \oint_{u_1}^{u_2} \frac{dF}{d\lambda}(tT)\,dt = F(u_2 T) - F(u_1 T). \tag{1.44}$$

Here $F_t(T)$ *(respectively* $\frac{dF}{d\lambda}(tT)$ *and* $F(tT)$*) are the operators arising by the Borel functional calculus of the operator* T *(respectively* tT*) for all* $t \in \,]-L, L[$.

Proof. T is a bounded operator on G so $\sigma(T)$ is compact. Let $\langle \mathcal{C}(\sigma(T)), \|\cdot\|_{\sup}\rangle$ be the Banach algebra of all continuous complex valued maps defined on $\sigma(T)$ with the sup norm. Set

$$\begin{cases} \widetilde{\mathcal{C}}(\sigma(T)) := \{f : \mathbb{C} \to \mathbb{C} \mid f{\restriction}\sigma(T) \in \mathcal{C}(\sigma(T)),\ f{\restriction}\complement\sigma(T) = \mathbf{0}\}, \\ J : \mathcal{C}(\sigma(T)) \ni g \mapsto \widetilde{g} \in \widetilde{\mathcal{C}}(\sigma(T)). \end{cases} \tag{1.45}$$

Notice that $\widetilde{\mathcal{C}}(\sigma(T))$ is an algebra, J is a surjective morphism of algebras, and we have $\sup_{\lambda\in\mathbb{C}} |J(g)(\lambda)| = \|g\|_{\sup}$ for all $g \in \mathcal{C}(\sigma(T))$. Furthermore, $J(g) \in \mathrm{Bor}(\mathbb{C})$ since $g \in \mathrm{Bor}(\sigma(T))$ and $\sigma(T) \in \mathcal{B}(\mathbb{C})$. Hence $\widetilde{\mathcal{C}}(\sigma(T))$ is a subalgebra of $\mathbf{TM}$, and J is an isometry between $\langle \mathcal{C}(\sigma(T)), \|\cdot\|_{\sup}\rangle$ and $\langle \widetilde{\mathcal{C}}(\sigma(T)), \|\cdot\|_{\sup}\rangle$. Thus $\langle \widetilde{\mathcal{C}}(\sigma(T)), \|\cdot\|_{\sup}\rangle$ is a Banach subalgebra of the Banach algebra $\langle \mathbf{TM}, \|\cdot\|_{\sup}\rangle$ and J is an isometric isomorphism of algebras. Therefore if E is the resolution of the identity of T, (1.2) implies that $\mathbf{I}_{\mathbb{C}}^{E} \circ J$ is a unital (6) morphism of algebras such that $\mathbf{I}_{\mathbb{C}}^{E} \circ J \in B(\langle \mathcal{C}(\sigma(T)), \|\cdot\|_{\sup}\rangle, B(G))$. For brevity we will write $\mathbf{I}_{\mathbb{C}}^{E}$ for $\mathbf{I}_{\mathbb{C}}^{E} \circ J$ so

$$\begin{cases} \mathbf{I}_{\mathbb{C}}^{E} \in B(\langle \mathcal{C}(\sigma(T)), \|\cdot\|_{\sup}\rangle, B(G)), \\ \mathbf{I}_{\mathbb{C}}^{E} \text{ is a unital morphism of algebras}, \\ (\forall g \in \mathcal{C}(\sigma(T)))(g(T) = \mathbf{I}_{\mathbb{C}}^{E}(g)). \end{cases} \tag{1.46}$$

In particular, $\mathbf{I}_{\mathbb{C}}^{E}$ is Fréchet differentiable with constant differential map equal to $\mathbf{I}_{\mathbb{C}}^{E}$. Let $\mathbf{0}$ be the zero element of $\langle \mathcal{C}(\sigma(T)), \|\cdot\|_{\sup}\rangle$. Let $t \in \,]-L, L[- \{0\}$, and $\imath_t := t \cdot \imath$, where $\imath : \sigma(T) \ni \lambda \mapsto \lambda$. So $\imath_t(T) = \mathbf{I}_{\mathbb{C}}^{E}(t \cdot \imath) = t\mathbf{I}_{\mathbb{C}}^{E}(\imath) = tT$. Hence by the general spectral mapping theorem 18.2.21 of [DS] applied to the map $\imath_t$, in view of the fact that $\sigma(T)$ is closed and multiplication by a nonzero scalar in $\mathbb{C}$ is a homeomorphism, we deduce that tT is a scalar type spectral operator and $E_t : \mathcal{B}(\mathbb{C}) \ni \delta \mapsto E(t^{-1}\delta)$ is its resolution of the identity. Finally,

$$(\forall t \in \,]-L, L[)(\sigma(tT) = t\sigma(T) \subseteq U);$$

the inclusion is by hypothesis. So $F(tT)$ arising by the Borel functional calculus of the operator tT is well defined, and by (1.46),

$$\begin{aligned} F(tT) &= \mathbf{I}_{\mathbb{C}}^{E_t}(F{\restriction}\sigma(tT)) \doteq \mathbf{I}_{\mathbb{C}}^{E\circ\imath_{t^{-1}}}(F{\restriction}\sigma(tT)) \\ &= \mathbf{I}_{\mathbb{C}}^{E}(F \circ \imath_t) = \mathbf{I}_{\mathbb{C}}^{E}(F_t{\restriction}\sigma(T)) = F_t(T), \end{aligned} \tag{1.47}$$

(6) Indeed, the unit element in $\mathbf{TM}$ is $\mathbf{1} : \mathbb{C} \ni \lambda \mapsto 1 \in \mathbb{C}$ and $\mathbf{I}_{\mathbb{C}}^{E} \circ J(\mathbf{1}{\restriction}\sigma(T)) = \mathbf{I}_{\mathbb{C}}^{E}(\mathbf{1}\cdot\chi_{\sigma(T)}) = \mathbf{I}_{\mathbb{C}}^{E}(\mathbf{1})\mathbf{I}_{\mathbb{C}}^{E}(\chi_{\sigma(T)}) = \mathbf{1}$.

proving (1.42). Set

$$\Delta :]-L, L[\ni t \mapsto F \circ \imath_t \in \langle \mathcal{C}(\sigma(T)), \|\cdot\|_{\sup} \rangle.$$

By the third equality in (1.47),

$$(\forall t \in]-L, L[)(F(tT) = \mathbf{I}_{\mathbb{C}}^{E} \circ \Delta(t)). \tag{1.48}$$

We claim that Δ is derivable (i.e. Fréchet differentiable) and for all $t \in]-L, L[$,

$$\frac{d\Delta}{dt}(t) = \imath \cdot \left(\frac{dF}{d\lambda}\right)_t \upharpoonright \sigma(T). \tag{1.49}$$

Set

$$\begin{cases} \mathcal{C}_U(\sigma(T)) := \{f \in \mathcal{C}(\sigma(T)) \mid f(\sigma(T)) \subseteq U\}, \\ \zeta :]-L, L[\ni t \mapsto \imath_t \in \mathcal{C}_U(\sigma(T)) \subset \langle \mathcal{C}(\sigma(T)), \|\cdot\|_{\sup} \rangle, \\ \Upsilon : \mathcal{C}_U(\sigma(T)) \ni f \mapsto F \circ f \in \langle \mathcal{C}(\sigma(T)), \|\cdot\|_{\sup} \rangle. \end{cases}$$

Notice

$$\Delta = \Upsilon \circ \zeta, \tag{1.50}$$

moreover ζ is Fréchet differentiable and for all $t \in]-L, L[$,

$$\frac{d\zeta}{dt}(t) = \imath. \tag{1.51}$$

Next for all $f \in \mathcal{C}_U(\sigma(T))$ by Lemma 1.20 applied to the compact set $f(\sigma(T))$, there is $Q_f > 0$ such that

$$K_f := \overline{\bigcup_{\lambda \in \sigma(T)} \overline{B}_{Q_f}(f(\lambda))} \subseteq U, \tag{1.52}$$

in particular

$$\overline{B}_{Q_f}(f) \subseteq \mathcal{C}_U(\sigma(T)). \tag{1.53}$$

Thus $\mathcal{C}_U(\sigma(T))$ is an open set in $\langle \mathcal{C}(\sigma(T)), \|\cdot\|_{\sup} \rangle$, so Υ is Fréchet differentiable and its differential map $\Upsilon^{[1]} : \mathcal{C}_U(\sigma(T)) \to B(\mathcal{C}(\sigma(T)))$ is such that for all $f \in \mathcal{C}_U(\sigma(T))$, $h \in \mathcal{C}(\sigma(T))$, and $\lambda \in \sigma(T)$,

$$\begin{cases} \Upsilon^{[1]}(f)(h)(\lambda) = \dfrac{dF}{d\lambda}(f(\lambda))h(\lambda), \\ \|\Upsilon^{[1]}(f)\|_{B(\mathcal{C}(\sigma(T)))} \leq \left\| \dfrac{dF}{d\lambda} \circ f \right\|_{\sup}. \end{cases} \tag{1.54}$$

Fix $f \in \mathcal{C}_U(\sigma(T))$ and K_f as in (1.52). By the boundedness of $f(\sigma(T))$ and Lemma 1.20, K_f is compact. Morever, $dF/d\lambda$ is continuous on U, hence uniformly continuous on the compact K_f, hence $(\forall \varepsilon > 0)(\exists \delta > 0)(\forall h \in \overline{B}_{Q_f}(\mathbf{0}) \cap \overline{B}_{\delta}(\mathbf{0}))$

$$\sup_{t \in [0,1]} \sup_{\lambda \in \sigma(T)} \left| \frac{dF}{d\lambda}(f(\lambda) + th(\lambda)) - \frac{dF}{d\lambda}(f(\lambda)) \right| \leq \varepsilon; \tag{1.55}$$

indeed, $f(\lambda) + th(\lambda) \in K_f$ and $|f(\lambda) + th(\lambda) - f(\lambda)| \leq |h(\lambda)| \leq \delta$ for all $\lambda \in \sigma(T)$ and $t \in [0, 1]$. Let us identify for the moment $\mathbb{C}$ as the $\mathbb{R}$-normed space $\mathbb{R}^2$. Then the usual product $(\cdot) : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ is $\mathbb{R}$-bilinear, so $F : U \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ is Fréchet differentiable and for all $x \in U$ and $h \in \mathbb{R}^2$,

$$F^{[1]}(x)(h) = \frac{dF}{d\lambda}(x) \cdot h. \tag{1.56}$$

For all $h \in \overline{B}_{Q_f}(\mathbf{0})$,

$$\begin{aligned}
\sup_{\lambda\in\sigma(T)}&\left|(F(f(\lambda)+h(\lambda))-F(f(\lambda))-\frac{dF}{d\lambda}(f(\lambda))h(\lambda)\right| \\
&= \sup_{\lambda\in\sigma(T)}|(F(f(\lambda)+h(\lambda))-F(f(\lambda))-F^{[1]}(f(\lambda))(h(\lambda)))| \\
&\leq \sup_{t\in[0,1]}\sup_{\lambda\in\sigma(T)}\|F^{[1]}(f(\lambda)+th(\lambda))-F^{[1]}(f(\lambda))\|_{B(\mathbb{R}^2)}\sup_{\lambda\in\sigma(T)}|h(\lambda)| \\
&= \sup_{t\in[0,1]}\sup_{\lambda\in\sigma(T)}\left|\frac{dF}{d\lambda}(f(\lambda)+th(\lambda))-\frac{dF}{d\lambda}(f(\lambda))\right|\|h\|_{\sup}. \qquad (1.57)
\end{aligned}$$

Here in the first equality we use (1.56), in the first inequality we apply the mean value theorem to the Fréchet differentiable map $F : U \subset \mathbb{R}^2 \to \mathbb{R}^2$, and in the second equality we use a corollary of (1.56). Finally, by (1.57) and (1.55), $(\forall\varepsilon > 0)(\exists\delta > 0)(\forall h \in \overline{B}_{Q_f}(\mathbf{0}) \cap \overline{B}_\delta(\mathbf{0}) - \{\mathbf{0}\})$

$$\frac{\sup_{\lambda\in\sigma(T)}|(F(f(\lambda)+h(\lambda))-F(f(\lambda))-\frac{dF}{d\lambda}(f(\lambda))h(\lambda)|}{\|h\|_{\sup}} \leq \varepsilon.$$

Equivalently,

$$\lim_{\substack{h\to\mathbf{0}\\ h\neq\mathbf{0}}}\frac{\|\Upsilon(f+h)-\Upsilon(f)-\Upsilon^{[1]}(f)(h)\|_{\sup}}{\|h\|_{\sup}} = 0. \qquad (1.58)$$

Moreover,

$$\|\Upsilon^{[1]}(f)(h)\|_{\sup} \leq \left\|\frac{dF}{d\lambda}\circ f\right\|_{\sup}\|h\|_{\sup}.$$

Thus by (1.58) we have proved (1.54). By (1.50), (1.51) and (1.54) we deduce that Δ is derivable; in addition, for all $t \in \,]-L, L[$ and $\lambda \in \sigma(T)$,

$$\frac{d\Delta}{dt}(t)(\lambda) = \Upsilon^{[1]}(\zeta(t))(\imath)(\lambda) = \frac{dF}{d\lambda}(\zeta_t(\lambda))\imath(\lambda) = \imath\left(\frac{dF}{d\lambda}\right)_t(\lambda),$$

proving (1.49). In conclusion, from the fact that $\mathbf{I}_{\mathbb{C}}^E$ is a morphism of algebras, (1.48), (1.46) and (1.49), for all $t \in \,]-L, L[$,

$$\begin{aligned}
\frac{dF(tT)}{dt} &= \frac{d}{dt}(\mathbf{I}_{\mathbb{C}}^E\circ\Delta)(t) = \mathbf{I}_{\mathbb{C}}^E\left(\frac{d\Delta}{dt}(t)\right) = \mathbf{I}_{\mathbb{C}}^E\left(\imath\cdot\left(\frac{dF}{d\lambda}\right)_t\upharpoonright\sigma(T)\right) \\
&= \mathbf{I}_{\mathbb{C}}^E(\imath)\mathbf{I}_{\mathbb{C}}^E\left(\left(\frac{dF}{d\lambda}\right)_t\upharpoonright\sigma(T)\right) = T\left(\frac{dF}{d\lambda}\right)_t(T).
\end{aligned}$$

Therefore statement (2) follows by applying (1) to $dF/d\lambda$. The map $]-L, L[\, \ni t \mapsto \frac{dF}{d\lambda}(tT) \in B(G)$ is continuous by (1.43) (replacing F with $dF/d\lambda$), hence it is Lebesgue measurable in $B(G)$. Let $u_1, u_2 \in \,]-L, L[$. By statement (1) and Theorem 18.2.11 of [DS],

$$\begin{aligned}
\int_{[u_1,u_2]}^{*}\left\|\frac{dF}{d\lambda}(tT)\right\|dt &= \int_{[u_1,u_2]}^{*}\left\|\left(\frac{dF}{d\lambda}\right)_t(T)\right\|dt \leq 4M\int_{[u_1,u_2]}^{*}\left\|\left(\frac{dF}{d\lambda}\right)_t\upharpoonright\sigma(T)\right\|_{\sup}dt \\
&\leq 4MD|u_2-u_1| < \infty,
\end{aligned}$$

where $M := \sup_{\delta\in\mathcal{B}(\mathbb{C})} \|E(\delta)\|$, and

$$D := \sup_{t\in[u_1,u_2]} \left\| \left(\frac{dF}{d\lambda}\right)_t \restriction\sigma(T) \right\|_{\sup} = \sup_{(t,\lambda)\in[u_1,u_2]\times\sigma(T)} \left|\frac{dF}{d\lambda}(t\lambda)\right| < \infty,$$

indeed, $[u_1,u_2]\times\sigma(T)$ is compact and the map $(t,\lambda)\mapsto \frac{dF}{d\lambda}(t\lambda)$ is continuous on $]-L,L[\times U$. Therefore by Theorem 5, IV.71 of [INT] , $]-L,L[\ni t\mapsto \frac{dF}{d\lambda}(tT)$ is Lebesgue integrable with respect to the norm topology on $B(G)$, so in particular by Definition 1, IV.33 of [INT], the integral

$$\oint_{u_1}^{u_2} \frac{dF}{d\lambda}(tT)dt \in B(G) \text{ exists.} \tag{1.59}$$

Therefore by (8), (1.59), Theorem 1, IV.35 of [INT] and (1.43),

$$T\oint_{u_1}^{u_2} \frac{dF}{d\lambda}(tT)dt = \oint_{u_1}^{u_2} T\frac{dF}{d\lambda}(tT)\,dt = \oint_{u_1}^{u_2} \frac{dF(tT)}{dt}\,dt. \tag{1.60}$$

By (1.43) the map $]-L,L[\ni t\mapsto F(tT)$ is derivable, and its derivative $]-L,L[\ni t\mapsto dF(tT)/dt$ is continuous in $B(G)$ by (1.43) and the continuity of the map $]-L,L[\ni t\mapsto dF/d\lambda(tT)$ in $B(G)$. Therefore $[u_1,u_2]\ni t\mapsto dF(tT)/dt$ is Lebesgue integrable in $B(G)$, where the integral has to be understood as defined in Ch. II of [FVR] (see Proposition 3, n° 3, §1, Ch. II of [FVR]).

Finally, the Lebesgue integral for functions with values in a Banach space as defined in Ch. II of [FVR] turns out to be the integral with respect to the Lebesgue measure as defined in Ch. IV, §4, n° 1 of [INT] (see Ch. III, §3, n° 3 and example in Ch. IV, §4, n° 4 of [INT]). Thus statement (3) follows by (1.60). ∎

LEMMA 1.22. *Let R be a possibly unbounded scalar type spectral operator in G, $\sigma(R)$ its spectrum, E its resolution of the identity, $K\neq\emptyset$, and for all $t\in K$ let $f_t\in\mathrm{Bor}(\sigma(R))$ be such that*

$$N := \sup_{t\in K}\|\widetilde{f_t}\|_\infty^E < \infty. \tag{1.61}$$

If $g\in\mathrm{Bor}(\sigma(R))$ and $\{\sigma_n\}_{n\in\mathbb{N}}$ is an E-sequence then for all $v\in\mathrm{Dom}(g(R))$,

$$\lim_{n\in\mathbb{N}}\sup_{t\in K}\|f_t(R)g(R)v - f_t(R)g(R)E(\sigma_n)v\| = 0.$$

Proof. By Theorem 18.2.11(g) of [DS] the statement is meaningful. We define $M := \sup_{\sigma\in\mathcal{B}(\mathbb{C})}\|E(\sigma)\|_{B(G)}$. Then $M<\infty$ by Corollary 15.2.4 of [DS]. Hypothesis (1.61) together with Theorem 18.2.11(c) of [DS] implies that for all $t\in K$, $f_t(R)\in B(G)$ and

$$\sup_{t\in K}\|f_t(R)\|_{B(G)} \leq 4MN.$$

Therefore for all $v\in\mathrm{Dom}(g(R))$ we have

$$\begin{aligned}
&\lim_{n\in\mathbb{N}}\sup_{t\in K}\|f_t(R)g(R)v - f_t(R)g(R)E(\sigma_n)v\| \\
&\leq \lim_{n\in\mathbb{N}}\sup_{t\in K}\|f_t(R)\|\cdot\|g(R)v - g(R)E(\sigma_n)v\| \leq 4MN\lim_{n\in\mathbb{N}}\|g(R)v - g(R)E(\sigma_n)v\| \\
&= 4MN\lim_{n\in\mathbb{N}}\|g(R)v - E(\sigma_n)g(R)v\| \qquad \text{by Theorem 18.2.11(g) of [DS]} \\
&= 0 \qquad \text{by (1.18).}
\end{aligned}$$
∎

THEOREM 1.23 (Strong operator derivability). *Let R be a possibly unbounded scalar type spectral operator in G, $K \subseteq \mathbb{R}$ an open interval of $\mathbb{R}$, and U an open neighbourhood of $\sigma(R)$ such that $K \cdot U \subseteq U$. Assume that $f : U \to \mathbb{C}$ is an analytic map and*

$$\sup_{t \in K} \|\widetilde{f}_t\|_{\infty}^{E} < \infty.$$

Then

(1) *the map $K \ni t \mapsto f(tR) \in B(G)$ is continuous in the strong operator topology,*
(2) *if*

$$\sup_{t \in K} \left\| \widetilde{\left(\frac{df}{d\lambda}\right)_t} \right\|_{\infty}^{E} < \infty, \tag{1.62}$$

then for all $v \in \mathrm{Dom}(R)$ and $t \in K$,

$$\frac{df(tR)v}{dt} = R\,\frac{df}{d\lambda}(tR)v \in G.$$

Proof. Let $\{\sigma_n\}_{n \in \mathbb{N}}$ be an E-sequence of compact sets. Then by Lemma 1.22 applied to the Borelian map $g : \sigma(R) \ni \lambda \mapsto 1 \in \mathbb{C}$, so $g(R) = \mathbf{1}$, and by (1.42) we have, for all $v \in G$,

$$\lim_{n \in \mathbb{N}} \sup_{t \in K} \|f(tR)v - f(tR)E(\sigma_n)v\| = 0. \tag{1.63}$$

By (1.42) and Key Lemma 1.7, for all $n \in \mathbb{N}$,

$$\begin{aligned} f(tR)E(\sigma_n) &= f_t(R)E(\sigma_n) = f_t(R_{\sigma_n}\restriction G_{\sigma_n})E(\sigma_n) \\ &= f(t(R_{\sigma_n}\restriction G_{\sigma_n}))E(\sigma_n). \end{aligned} \tag{1.64}$$

As σ_n is bounded, Key Lemma 1.7 shows that $R_{\sigma_n}\restriction G_{\sigma_n}$ is a scalar type spectral operator such that $R_{\sigma_n}\restriction G_{\sigma_n} \in B(G_{\sigma_n})$. Moreover, by (1.14), U is an open neighbourhood of $\sigma(R_{\sigma_n}\restriction G_{\sigma_n})$. Thus by Theorem 1.21(2) the map

$$K \ni t \mapsto f(t(R_{\sigma_n}\restriction G_{\sigma_n})) \in B(G_{\sigma_n})$$

is derivable, so in particular $\|\cdot\|_{B(G_{\sigma_n})}$-continuous. Now for all $n \in \mathbb{N}$ and $v_n \in G_{\sigma_n}$ define $\xi_{v_n} : B(G_{\sigma_n}) \ni A \mapsto Av_n \in G$. Then $\xi_{v_n} \in B(B(G_{\sigma_n}), G)$. Thus as a composition of two continuous maps, also the map

$$K \ni t \mapsto \xi_{E(\sigma_n)v}(f(t(R_{\sigma_n}\restriction G_{\sigma_n}))) \in G \tag{1.65}$$

is $\|\cdot\|_G$-continuous, for all $n \in \mathbb{N}$ and $v \in G$. Hence by (1.64) for all $n \in \mathbb{N}$,

$$K \ni t \mapsto f(tR)E(\sigma_n) \in B(G) \text{ is strongly continuous.} \tag{1.66}$$

Finally, by (1.66) and (1.63) we can apply Theorem 2, §1.6, Ch. 10 of [GT] to the uniform space $B(G)_{\mathrm{st}}$ whose uniformity is generated by the set of seminorms defining the strong operator topology on $B(G)$. Thus we conclude that $K \ni t \mapsto f(tR) \in B(G)$ is strongly continuous, and statement (1) follows.

Let $n \in \mathbb{N}$ and $v_n \in G_{\sigma_n}$, so $\xi_{v_n} \in B(B(G_{\sigma_n}), G)$, thus ξ_{v_n} is Fréchet differentiable with constant differential map $\xi_{v_n}^{[1]} : B(G_{\sigma_n}) \ni A \mapsto \xi_{v_n} \in B(B(G_{\sigma_n}), G)$. Therefore by Theorem 1.21(2) for all $n \in \mathbb{N}$ and $v \in G$ the map in (1.65) is Fréchet differentiable as composition of two Fréchet differentiable maps, and its derivative is, for all $t \in K$,

$$
\begin{aligned}
\frac{d}{dt}(f(t(R_{\sigma_n}\restriction G_{\sigma_n}))E(\sigma_n)v) &= \xi_{E(\sigma_n)v}\left(\frac{d}{dt}(f(t(R_{\sigma_n}\restriction G_{\sigma_n})))\right)\\
&= \frac{d}{dt}(f(t(R_{\sigma_n}\restriction G_{\sigma_n})))E(\sigma_n)v\\
&= (R_{\sigma_n}\restriction G_{\sigma_n})\frac{df}{d\lambda}(t(R_{\sigma_n}\restriction G_{\sigma_n}))E(\sigma_n)v \qquad \text{by (1.43)}\\
&= \frac{df}{d\lambda}(t(R_{\sigma_n}\restriction G_{\sigma_n}))(R_{\sigma_n}\restriction G_{\sigma_n})E(\sigma_n)v \qquad \text{by [DS, 18.2.11]}\\
&= \left(\frac{df}{d\lambda}\right)_t(R_{\sigma_n}\restriction G_{\sigma_n})(R_{\sigma_n}\restriction G_{\sigma_n})E(\sigma_n)v \qquad \text{by (1.42)}\\
&= \left(\frac{df}{d\lambda}\right)_t(R)(R_{\sigma_n}\restriction G_{\sigma_n})E(\sigma_n)v \qquad \text{by Lemma 1.7}\\
&= \frac{df}{d\lambda}(tR)(R_{\sigma_n}\restriction G_{\sigma_n})E(\sigma_n)v \qquad \text{by (1.42).}
\end{aligned} \tag{1.67}
$$

Thus by (1.64), for all $n \in \mathbb{N}$ and $v \in G$,

$$
\begin{cases}
K \ni t \mapsto f(tR)E(\sigma_n)v \in G \text{ is differentiable and}\\
K \ni t \mapsto \dfrac{df}{d\lambda}(tR)(R_{\sigma_n}\restriction G_{\sigma_n})E(\sigma_n)v \in G \text{ is its derivative.}
\end{cases} \tag{1.68}
$$

By (1.62) we can apply Lemma 1.22 to the maps $\left(\frac{df}{d\lambda}\right)_t\restriction\sigma(R)$ and $g = \imath : \sigma(R) \ni \lambda \mapsto \lambda \in \mathbb{C}$, so $g(R) = R$, hence by (1.42), for all $v \in \mathrm{Dom}(R)$,

$$
\lim_{n\in\mathbb{N}} \sup_{t\in K}\left\|\frac{df}{d\lambda}(tR)Rv - \frac{df}{d\lambda}(tR)(R_{\sigma_n}\restriction G_{\sigma_n})E(\sigma_n)v\right\| = 0. \tag{1.69}
$$

Moreover, for all $a \in K$, let $r_a \in \mathbb{R}^+$ be such that $B_{r_a}(a) \subset K$ (it exists as K is open). Then (1.69), (1.68) and (1.63) hold again if we replace K by $B_{r_a}(a)$. Hence we can apply Theorem 8.6.3 of [Dieu] and deduce for all $v \in \mathrm{Dom}(R)$ that the map $K \ni t \mapsto f(tR)v \in G$ is derivable, and its derivative map is

$$
K \ni t \mapsto \frac{df}{d\lambda}(tR)Rv \in G.
$$

Finally, for all $v \in \mathrm{Dom}(R)$, we have $R\frac{df}{d\lambda}(tR)v = \frac{df}{d\lambda}(tR)Rv$, from $\mathrm{Dom}\left(\frac{df}{d\lambda}(tR)\right) = G$ and the commutativity property of the Borel functional calculus expressed in Theorem 18.2.11(f) of [DS]. Hence the statement follows. ∎

Corollary 1.24. *Let R be a possibly unbounded scalar type spectral operator in G, U an open neighbourhood of $\sigma(R)$, and $S : U \to \mathbb{C}$ an analytic map. Assume that there is $L > 0$ such that $]-L, L[\cdot U \subseteq U$ and*

(1) $\widetilde{S}_t \in \mathfrak{L}_E^\infty(\sigma(R))$ *and* $\widetilde{\left(\frac{dS}{d\lambda}\right)_t} \in \mathfrak{L}_E^\infty(\sigma(R))$ *for all* $t \in]-L, L[$;

(2) $\int^* \left\|\widetilde{\left(\frac{dS}{d\lambda}\right)_t}\right\|_\infty^E dt < \infty$ *(here the upper integral is with respect to the Lebesgue measure on* $]-L, L[$*);*

(3) *for all $v \in G$ the map $]-L, L[\ni t \mapsto \frac{dS}{d\lambda}(tR)v \in G$ is Lebesgue measurable.*

Then for all $u_1, u_2 \in]-L, L[$,

$$R\int_{u_1}^{u_2} \frac{dS}{d\lambda}(tR)\,dt = S(u_2R) - S(u_1R) \in B(G).$$

Here the integral is with respect to the Lebesgue measure on $[u_1, u_2]$ *and with respect to the strong operator topology on* $B(G)$ (*see Definition* 1.12).

Proof. Let $M := \sup_{\sigma\in\mathcal{B}(\mathbb{C})} \|E(\sigma)\|_G$ and μ the Lebesgue measure on $[u_1, u_2]$. Then by (1.42), the hypotheses, and Theorem 18.2.11(c) of [DS] we have

(a) for all $t \in [u_1, u_2]$, $S(tR) \in B(G)$;
(b) for all $t \in [u_1, u_2]$, $\frac{dS}{d\lambda}(tR) \in B(G)$;
(c) $([u_1, u_2] \ni t \mapsto \|\frac{dS}{d\lambda}(tR)\|_{B(G)}) \in \mathfrak{F}_1([u_1, u_2]; \mu)$.

So by hypothesis (3), (c) and Remark 1.14 the map

$$[u_1, u_2] \ni t \mapsto \frac{dS}{d\lambda}(tR) \in B(G)$$

is Lebesgue integrable with respect to the strong operator topology. This means that, except for (1.30), the hypotheses of Theorem 1.18 hold for $X := [u_1, u_2]$, $h := (S_{u_2} - S_{u_1})\restriction \sigma(R)$, $g : \sigma(R) \ni \lambda \mapsto \lambda \in \mathbb{C}$ and the maps $f_t := \left(\frac{dS}{d\lambda}\right)_t\restriction\sigma(R)$, for all $t \in [u_1, u_2]$. Next let $\sigma \in \mathcal{B}(\mathbb{C})$ be bounded. By Key Lemma 1.7, $R_\sigma\restriction G_\sigma$ is a scalar type spectral operator such that $R_\sigma\restriction G_\sigma \in B(G_\sigma)$, and by (1.14), U is an open neighbourhood of $\sigma(R_\sigma\restriction G_\sigma)$. Thus we can apply Theorem 1.21(3) to the Banach space G_σ, the analytic map S and the operator $R_\sigma\restriction G_\sigma$. In particular, the map $[u_1, u_2] \ni t \mapsto \frac{dS}{d\lambda}(t(R_\sigma\restriction G_\sigma)) \in B(G_\sigma)$ is Lebesgue integrable in the $\|\cdot\|_{B(G_\sigma)}$-topology, that is, in the sense of Definition 2, IV.23 of [INT]. Next we consider, for all $v \in G_\sigma$, the map

$$T \in B(G_\sigma) \mapsto Tv \in G_\sigma$$

which is linear and continuous in the norm topologies. Thus by Theorem 1, IV.35 of [INT], $[u_1, u_2] \ni t \mapsto \frac{dS}{d\lambda}(t(R_\sigma\restriction G_\sigma))v \in G_\sigma$ is Lebesgue integrable for all $v \in G_\sigma$ and

$$\int_{u_1}^{u_2} \frac{dS}{d\lambda}(t(R_\sigma\restriction G_\sigma))v\,dt = \left(\oint_{u_1}^{u_2} \frac{dS}{d\lambda}(t(R_\sigma\restriction G_\sigma))\,dt\right)v.$$

Therefore by Definition 1.12, $[u_1, u_2] \ni t \mapsto \frac{dS}{d\lambda}(t(R_\sigma\restriction G_\sigma)) \in B(G_\sigma)$ is Lebesgue integrable with respect to the strong operator topology on $B(G_\sigma)$ and

$$\int_{u_1}^{u_2} \frac{dS}{d\lambda}(t(R_\sigma\restriction G_\sigma))\,dt = \oint_{u_1}^{u_2} \frac{dS}{d\lambda}(t(R_\sigma\restriction G_\sigma))\,dt. \tag{1.70}$$

Here $\int_{u_1}^{u_2} \frac{dS}{d\lambda}(t(R_\sigma\restriction G_\sigma))\,dt$ is the integral of $\frac{dS}{d\lambda}(t(R_\sigma\restriction G_\sigma))$ with respect to the Lebesgue measure on $[u_1, u_2]$ and the strong operator topology on $B(G_\sigma)$. Furthermore, by Theorem 1.21(3),

$$(R_\sigma\restriction G_\sigma)\oint_{u_1}^{u_2} \frac{dS}{d\lambda}(t(R_\sigma\restriction G_\sigma))\,dt = S(u_2(R_\sigma\restriction G_\sigma)) - S(u_1(R_\sigma\restriction G_\sigma)).$$

Thus by (1.70),

$$(R_\sigma\restriction G_\sigma)\int_{u_1}^{u_2} \frac{dS}{d\lambda}(t(R_\sigma\restriction G_\sigma))\,dt = S(u_2(R_\sigma\restriction G_\sigma)) - S(u_1(R_\sigma\restriction G_\sigma)). \tag{1.71}$$

This implies (1.30), by choosing for example $\sigma_n := B_n(\mathbf{0})$ for all $n \in \mathbb{N}$. Therefore by Theorem 1.18 we obtain the statement. ■

THEOREM 1.25 (Strong operator Newton–Leibniz formula). *Let R be a possibly unbounded scalar type spectral operator in G, U an open neighbourhood of $\sigma(R)$ and $S : U \to \mathbb{C}$ an analytic map. Assume that there is $L > 0$ such that $]-L, L[\cdot U \subseteq U$ and*

(a) $\widetilde{S}_t \in \mathfrak{L}_E^\infty(\sigma(R))$ *for all* $t \in]-L, L[$;
(b) $\sup_{t\in]-L,L[} \left\| \widetilde{\left(\frac{dS}{d\lambda}\right)_t} \right\|_\infty^E < \infty$.

Then

(1) *For all* $u_1, u_2 \in]-L, L[$,
$$R \int_{u_1}^{u_2} \frac{dS}{d\lambda}(tR)\, dt = S(u_2 R) - S(u_1 R) \in B(G).$$
Here the integral is with respect to the Lebesgue measure on $[u_1, u_2]$ and the strong operator topology on $B(G)$.
(2) *If also* $\sup_{t\in]-L,L[} \|\widetilde{S}_t\|_\infty^E < \infty$, *then for all* $v \in \mathrm{Dom}(R), t \in]-L, L[$
$$\frac{dS(tR)v}{dt} = R\, \frac{dS}{d\lambda}(tR)v.$$

Proof. By hypothesis (b) and Theorem 1.23(1) for all $v \in G$ the map $]-L, L[\ni t \mapsto \frac{dS}{d\lambda}(tR)v \in G$ is continuous. Thus statement (1) follows from Corollary 1.24 and the fact that continuity implies measurability. Statement (2) follows from Theorem 1.23(2). ■

REMARK 1.26. We end this section by remarking that $f : X \to B(G)$ is μ-integrable with respect to the strong operator topology as defined in Definition 1.12 if and only if $f : X \to B(G)$ is scalarly $(\mu, B(G))$-integrable with respect to the weak operator topology in the sense explained in Notations 2.1. In Chapter 2 we shall extend the results of Chapter 1 to the case of integration with respect to the measure μ and the $\sigma(B(G), \mathcal{N})$-topology, where $\mathcal{N} \subset B(G)^*$ is a suitable linear subspace of the topological dual of $B(G)$.

1.4. Application to resolvents of unbounded scalar type spectral operators in a Banach space G

COROLLARY 1.27. *Let T be a possibly unbounded scalar type spectral operator in G with real spectrum $\sigma(T)$. Then*

(1) *For all* $\lambda \in \mathbb{C}$ *with* $\mathrm{Im}(\lambda) > 0$,
$$(T - \lambda\mathbf{1})^{-1} = i \int_{-\infty}^{0} e^{-it\lambda} e^{itT}\, dt \in B(G). \tag{1.72}$$
(2) *For all* $v \in \mathrm{Dom}(T)$ *and* $t \in \mathbb{R}$
$$\frac{de^{it(T-\lambda\mathbf{1})}v}{dt} = i(T - \lambda\mathbf{1})e^{i(T-\lambda\mathbf{1})t}v.$$

REMARK 1.28. If we define $S(\lambda) := \exp(i\lambda)$ for all $\lambda \in \mathbb{C}$ then the operator functions in Corollary 1.27 are $e^{itT} := S_t(T)$ and $e^{it(T-\lambda\mathbf{1})} := S_t(T - \lambda\mathbf{1})$, in the sense of the Borelian functional calculus for the scalar type spectral operators T and $T-\lambda\mathbf{1}$, respectively, as defined in Definition 1.3.

The integral in Corollary 1.27 is with respect to the Lebesgue measure and the strong operator topology on $B(G)$, meaning by definition that

$$\int_{-\infty}^{0} e^{-it\lambda}e^{itT}\,dt \in B(G)$$

and for all $v\in G$,

$$\left(\int_{-\infty}^{0} e^{-it\lambda}e^{itT}\,dt\right)v := \lim_{u\to-\infty}\left(\int_{u}^{0} e^{-it\lambda}e^{itT}\,dt\right)v = \lim_{u\to-\infty}\int_{u}^{0} e^{-it\lambda}e^{itT}v\,dt.$$

Here the integral on the right side of the first equality is with respect to the Lebesgue measure on $[u,0]$ and the strong operator topology on $B(G)$.

Proof. Let $\lambda\in\mathbb{C}$ and set $R := T-\mathbf{1}\lambda$. Then R is a scalar type spectral operator (see Theorem 18.2.17 of [DS]). Let $\lambda\in\mathbb{C}$ with $\operatorname{Im}(\lambda)\neq 0$ and E be the resolution of the identity of R. Then $\sigma(R)=\sigma(T)-\lambda$, as a corollary of the well-known spectral mapping theorem. Then for all $t\in\mathbb{R}$,

$$\begin{aligned} E\text{-}\operatorname*{ess\,sup}_{\nu\in\sigma(R)}\left|\frac{dS}{d\lambda}(t\nu)\right| &= E\text{-}\operatorname*{ess\,sup}_{\nu\in\sigma(R)}|S(t\nu)| \\ &\le \sup_{\nu\in\sigma(R)}|S(t\nu)| = \sup_{\mu\in\sigma(T)}|e^{i(\mu-\lambda)t}| = e^{\operatorname{Im}(\lambda)t}. \end{aligned}$$

Therefore the hypotheses of Corollary 1.25 hold with $R := T-\lambda\mathbf{1}$. Thus for all $v\in G$ and $u\in\mathbb{R}$,

$$i(T-\lambda\mathbf{1})\int_{u}^{0} e^{it(T-\lambda\mathbf{1})}v\,dt = v - e^{iu(T-\lambda\mathbf{1})}v. \tag{1.73}$$

Here $e^{it(T-\lambda\mathbf{1})} := S_t(R)$. One should note an apparent ambiguity about the symbol $e^{it(T-\lambda\mathbf{1})}$, standing here for the operator $S_t(R)=S(tR)$, which could also be seen as a Borelian function of the operator T. By setting $g^{[\lambda]}(\mu):=\mu-\lambda$, so $g^{[\lambda]}=\imath-\lambda\cdot\mathbf{1}$ with $\mathbf{1}:\mathbb{C}\ni\lambda\mapsto 1$, considering that by the composition rule (see Theorem 18.2.24 of [DS]), we have $S_t\circ g^{[\lambda]}(T)=S_t(g^{[\lambda]}(T))$, and finally $R=\imath(T)-\lambda\mathbf{1}(T)=(\imath-\lambda\cdot\mathbf{1})(T)=g^{[\lambda]}(T)$, we can assert

$$\begin{cases} T-\lambda\mathbf{1}=g^{[\lambda]}(T):=T-\lambda, \\ e^{it(T-\lambda\mathbf{1})}:=S_t(T-\lambda\mathbf{1})=S_t\circ g^{[\lambda]}(T)=e^{it(T-\lambda)}. \end{cases} \tag{1.74}$$

Therefore we can consider the operator $e^{it(T-\lambda\mathbf{1})}$ as an operator function of R or of T. Now for all $t\in\mathbb{R}$, $\sup_{\mu\in\sigma(T)}|\exp(i\mu t)|=1$, so by Theorem 18.2.11(c) of [DS],

$$\sup_{t\in\mathbb{R}}\|\exp(iTt)\|_{B(G)}\le 4M. \tag{1.75}$$

Here $M:=\sup_{\sigma\in\mathcal{B}(\mathbb{C})}\|E(\sigma)\|_G$. But with the notations adopted before we have, for all $\mu\in\mathbb{C}$, $S_t\circ g^{[\lambda]}(\mu)=\exp(it(\mu-\lambda))=\exp(-it\lambda)S_t(\mu)$, so in view of $S_t(T)=S(tT)$ (see (1.42)), we have $S_t\circ g^{[\lambda]}(T)=\exp(-it\lambda)S_t(T)=\exp(-it\lambda)S(tT)$. Thus by (1.74) we have, for all $t\in\mathbb{R}$ and $\lambda\in\mathbb{C}$ with $\operatorname{Im}(\lambda)>0$,

$$e^{it(T-\lambda\mathbf{1})}=\exp(-it\lambda)S(tT)\doteq\exp(-it\lambda)e^{itT}. \tag{1.76}$$

By (1.76) and (1.75) we have

$$\lim_{u\to-\infty}\|e^{iu(T-\lambda\mathbf{1})}\|_{B(G)} \leq 4M \lim_{u\to-\infty}\exp(\mathrm{Im}(\lambda)u) = 0$$

or equivalently $\lim_{u\to-\infty} e^{iu(T-\lambda\mathbf{1})} = \mathbf{0}$ in the $\|\cdot\|_{B(G)}$-topology. Hence by (1.73), for all $v \in G$,

$$v = i \lim_{u\to-\infty}(T-\lambda\mathbf{1})\int_u^0 e^{it(T-\lambda\mathbf{1})}v\,dt \quad \text{in } \|\cdot\|_G. \tag{1.77}$$

As $\mathrm{Im}(\lambda) \neq 0$ we have $\{\mu \in \mathbb{C} \mid g^{[\lambda]}(\mu) = 0\} \cap \sigma(T) = \emptyset$, so if we denote by F the resolution of the identity of T, we have $F(\sigma(T)) = \mathbf{1}$ so $F(\{\mu \in \mathbb{C} \mid g^{[\lambda]}(\mu) = 0\}) = F(\{\mu \in \mathbb{C} \mid g^{[\lambda]}(\mu) = 0\} \cap \sigma(T)) = F(\emptyset) := \mathbf{0}$. Thus by Theorem 18.2.11(h) of [DS],

$$\exists (T-\lambda)^{-1} = \frac{1}{g^{[\lambda]}}(T) \doteq \frac{1}{T-\lambda}.$$

Finally,

$$\begin{aligned} F\text{-}\operatorname*{ess\,sup}_{\mu\in\sigma(T)}\left|\frac{1}{g^{[\lambda]}(\mu)}\right| &\leq \sup_{\mu\in\sigma(T)}\left|\frac{1}{g^{[\lambda]}(\mu)}\right| = \sup_{\mu\in\sigma(T)}\left|\frac{1}{\mu-\lambda}\right| = \frac{1}{\inf_{\mu\in\sigma(T)}|(\mu-\lambda)|} \\ &\leq \frac{1}{|\mathrm{Im}(\lambda)|} < \infty, \end{aligned}$$

so

$$\frac{1}{g^{[\lambda]}}(T) \in B(G).$$

Hence by the previous equation and the fact $T - \lambda = T - \lambda\mathbf{1}$ (see (1.74)), we obtain

$$(T-\lambda\mathbf{1})^{-1} \in B(G).$$

Finally, by a standard argument (see for example [LN]) and (1.77) we deduce for all $v \in G$ that

$$(T-\lambda\mathbf{1})^{-1}v = i \lim_{u\to-\infty}(T-\lambda\mathbf{1})^{-1}(T-\lambda\mathbf{1})\int_u^0 e^{it(T-\lambda\mathbf{1})}v\,dt = i \lim_{u\to-\infty}\int_u^0 e^{it(T-\lambda\mathbf{1})}v\,dt.$$

So (1) follows by (1.76). By (1.76), the fact that $S_t(T) = S(tT)$ and Theorem 1.23(2) applied to the operator T and to the map $S : \mathbb{C} \ni \mu \mapsto e^{i\mu}$, we obtain statement (2). ∎

Remark 1.29. An important application of (1.72) is to prove the well-known Stone theorem for strongly continuous semigroups of unitary operators in Hilbert space (see Theorem 12.6.1 of [DS]). In [Dav] it has been used to show the equivalence of uniform convergence in the strong operator topology of a one-parameter semigroup depending on a parameter and the convergence in strong operator topology of the resolvents of the corresponding generators, Theorem 3.17.

Notice that if $\zeta := -i\lambda$ and $Q := iT$, then the equality (1.72) turns into

$$(Q+\zeta\mathbf{1})^{-1} = \int_0^\infty e^{-t\zeta}e^{-Qt}\,dt,$$

which is referred in IX.1.3 of [Kat] by saying that the resolvent of Q is the *Laplace* transform of the semigroup e^{-Qt}. Applications of this formula to perturbation theory are in IX.2 of [Kat].

2. Extension theorem. The case of the topology $\sigma(B(G),\mathcal{N})$

2.1. Introduction. Let R be an unbounded scalar type spectral operator R in a complex Banach space G, and E its resolution of identity. The main results of this chapter and of the entire work are of two types.

The results of the first type are Extension Theorems for integration with respect to the $\sigma(B(G),\mathcal{N})$-topology, when $\mathcal{N}$ is an E-appropriate set (Theorems 2.25) and when $\mathcal{N}$ is an E-appropriate set with the duality property (Corollary 2.26).

As an application we will prove, by using (2.37), the Extension Theorems for integration with respect to the sigma-weak topology (Corollaries 2.28 and 2.29), and for integration with respect to the weak operator topology (Corollaries 2.27 and 2.30).

The results of the second type are Newton–Leibniz formulas for integration with respect to the $\sigma(B(G),\mathcal{N})$-topology, when $\mathcal{N}$ is an E-appropriate set with the duality property (Corollaries 2.33 and 2.34); for integration with respect to the sigma-weak topology (Corollary 2.35); for integration with respect to the weak operator topology (Corollary 2.36).

To obtain the Extension Theorem 2.25 we need to introduce the concept of E-appropriate set (Definition 2.11), which allows us to establish two properties important for the proof of Theorem 2.25, namely the "commutation" property (Theorem 2.13), and the "restriction" property (Theorem 2.22).

Finally, to obtain Corollary 2.26 and the Newton–Leibniz formula in Corollary 2.33 we have to introduce the concept of an E-appropriate set $\mathcal{N}$ with the duality property (Definition 2.11), which allows us to establish conditions ensuring that a map is scalarly essentially $(\mu, B(G))$-integrable with respect to the $\sigma(B(G),\mathcal{N})$-topology (Theorem 2.2). Similar results for the weak operator topology are contained in Theorem 2.5 and Corollary 2.6.

2.2. Existence of the weak integral with respect to the $\sigma(B(G),\mathcal{N})$-topology. In this section we shall obtain a general result (Theorem 2.2) about conditions ensuring that a map is scalarly essentially $(\mu, B(G))$-integrable with respect to the $\sigma(B(G),\mathcal{N})$-topology, where $\mathcal{N}$ is a suitable subset of $B(G)^*$.

Notations with Comments 2.1. Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, Z a linear space over $\mathbb{K}$, and τ a locally convex topology on Z; let $\langle Z, \tau\rangle$ denote the associated locally convex space over $\mathbb{K}$. We denote by $\mathrm{LCS}(\mathbb{K})$ the class of all locally convex spaces over $\mathbb{K}$ and for any $\langle Z, \tau\rangle \in \mathrm{LCS}(\mathbb{K})$ we write $\langle Z, \tau\rangle^*$ for its topological dual, that is, the $\mathbb{K}$-linear space of all $\mathbb{K}$-linear continuous functionals on Z.

Let Y be a linear space over $\mathbb{K}$, and U a subspace of $\mathrm{Hom}(Y, \mathbb{K})$. Then $\sigma(Y, U)$ denotes the weakest (locally convex) topology on Y such that $U \subseteq \langle Y, \sigma(Y, U)\rangle^*$ (Def. 2, II.42 of [TVS]), which coincides with the locally convex topology on Y generated by the set of seminorms $\Gamma(U) := \{q_\phi : Y \ni y \mapsto |\phi(y)| \mid \phi \in U\}$.

It is not hard to see that $\sigma(Y, U)$ is the topology generated by the set of seminorms $\Gamma(S)$ for any S such that $U = \mathfrak{L}_{\mathbb{K}}(S)$, where $\mathfrak{L}_{\mathbb{K}}(S)$ is the $\mathbb{K}$-linear space generated by S.

By Proposition 2, II.43 of [TVS], $\sigma(Y, U)$ is a Hausdorff topology if and only if U separates the points of Y, i.e.

$$(\forall T \in Y)(T \neq \mathbf{0} \Rightarrow (\exists \phi \in U)(\phi(T) \neq 0)). \tag{2.1}$$

Also by Proposition 3, II.43 of [TVS],

$$\langle Y, \sigma(Y, U)\rangle^* = U. \tag{2.2}$$

Let X be a locally compact space and μ a $\mathbb{K}$-Radon measure on X (Definition 2, §1, n° 3, Ch. 3 of [INT] where it is called just a measure). We denote by $|\mu|$ the total variation of μ (§1, n° 6, Ch. 3 of [INT]), and by $\int^*$ the upper integral with respect to a positive measure, for example $|\mu|$ (Definition 1, §1, n° 1, Ch. 4 of [INT]). By $\int^\bullet$ we denote the essential upper integral with respect to a positive measure (Definition 1, §1, n° 1, Ch. 5 of [INT]) ([1]). For the definition of essentially μ-integrable map $f : X \to \mathbb{K}$ we refer to Ch. 5, §1, n° 3 of [INT].

Let $\langle Y, \tau\rangle \in \mathrm{LCS}(\mathbb{K})$ be Hausdorff. Then $f : X \to \langle Y, \tau\rangle$ is *scalarly essentially μ-integrable* or equivalently $f : X \to Y$ is *scalarly essentially μ-integrable with respect to the measure μ and the τ-topology on Y* if for all $\omega \in \langle Y, \tau\rangle^*$ the map $\omega \circ f : X \to \mathbb{K}$ is essentially μ-integrable, so we can define its *integral* as the following linear operator:

$$\langle Y, \tau\rangle^* \ni \omega \mapsto \int \omega(f(x))\, d\mu(x) \in \mathbb{K}.$$

See Ch. 6, §1, n° 1 of [INT] for $\mathbb{K} = \mathbb{R}$, and for the extension to the case $\mathbb{K} = \mathbb{C}$ see the end of §2, n° 10 of [INT].

Notice that the previous definitions depend only on the dual space $\langle Y, \tau\rangle^*$, hence both the concepts of scalar essential μ-integrability and integral will be invariant if we replace τ with any other Hausdorff locally convex topology τ_2 on Y compatible with the duality $(Y, \langle Y, \tau\rangle^*)$, i.e. such that $\langle Y, \tau\rangle^* = \langle Y, \tau_2\rangle^*$.

Therefore as a corollary of the well-known Mackey–Arens theorem (see Theorem 1, IV.2 of [TVS] or Theorem 5 §8.5 of [Jar]), given a locally convex space $\langle Y, \tau\rangle$ and denoting by $\mathcal{N} := \langle Y, \tau\rangle^*$ its topological dual, we see that scalar essential μ-integrability (respectively integral) is an invariant property (respectively functional) under the variation of any Hausdorff locally convex topology τ_1 on Y such that

$$\sigma(Y, \mathcal{N}) \leq \tau_1 \leq \tau(Y, \mathcal{N}).$$

Here $a \leq b$ means a is weaker than b, and $\tau(Y, \mathcal{N})$ is the Mackey topology associated to the canonical duality $(Y, \mathcal{N})$.

Let $f : X \to \langle Y, \tau\rangle$ be scalarly essentially μ-integrable and assume that

$$(\exists B \in Y)(\forall \omega \in \langle Y, \tau\rangle^*)\Big(\omega(B) = \int \omega(f(x))\, d\mu(x)\Big). \tag{2.3}$$

Notice that by the Hahn–Banach theorem $\langle Y, \tau\rangle^*$ separates the points of Y, so the element B is uniquely defined by this condition. In this case, by definition $f : X \to \langle Y, \tau\rangle$ *is scalarly essentially (μ, Y)-integrable* (or $f : X \to Y$ *is scalarly essentially (μ, Y)-integrable*

([1]) In general $\int^\bullet \leq \int^*$, but if X is σ-compact, in particular compact, then $\int^\bullet = \int^*$.

with respect to the τ*-topology on* Y), and its *weak integral with respect to the measure* μ *and the* τ*-topology*, or briefly its *weak integral*, is defined by

$$\int f(x)\,d\mu(x) := B. \tag{2.4}$$

We shall use this integral for $\langle Y, \tau\rangle := \langle B(G), \sigma(B(G), \mathcal{N})\rangle$, where $\mathcal{N}$ is a linear subspace of $B(G)^*$ which separates the points of $B(G)$. Notice that by (2.2), $\langle B(G), \sigma(B(G), \mathcal{N})\rangle^* = \mathcal{N}$.

Let G be a $\mathbb{K}$-normed space. Then the *strong operator topology* $\tau_{\mathrm{st}}(G)$ on $B(G)$ is defined to be the locally convex topology generated by the set of seminorms $\{q_v : B(G) \ni A \mapsto \|Av\|_G \mid v \in G\}$. Hence $\tau_{\mathrm{st}}(G)$ is a Hausdorff topology; a base of neighbourhoods of $A \in B(G)$ consists of the sets $U_{\overline{v},\varepsilon}(A) := \{B \in B(G) \mid \sup_{k=1,\dots,n} \|(A-B)\overline{v}_k\|_G < \varepsilon\}$ with $\overline{v}$ running over $\bigcup_{n\in\mathbb{N}} G^n$ and ε over $\mathbb{R}^+ - \{0\}$. So $B \in \overline{\{\mathbf{0}\}}$, the closure of $\{\mathbf{0}\}$ in the strong operator topology, if and only if $\|Bv\|_G < \varepsilon$ for all $\varepsilon \in \mathbb{R}^+ - \{0\}$ with $v \in G$, that is, $B = \mathbf{0}$. Hence $\overline{\{\mathbf{0}\}} = \{\mathbf{0}\}$ and so $\tau_{\mathrm{st}}(G)$ is Hausdorff. By Ch. 6, §1, n° 3 of [INT],

$$\mathcal{N}_{\mathrm{st}}(G) := \langle B(G), \tau_{\mathrm{st}}(G)\rangle^* = \mathfrak{L}_{\mathbb{K}}(\{\psi_{(\phi,v)} \mid (\phi, v) \in G^* \times G\}). \tag{2.5}$$

Here

$$\psi_{(\phi,v)} : B(G) \ni T \mapsto \phi(Tv) \in \mathbb{K}.$$

Here if Z is a $\mathbb{K}$-linear space and $S \subseteq Z$ then $\mathfrak{L}_{\mathbb{K}}(S)$ is the space of all $\mathbb{K}$-linear combinations of elements in S.

The first locally convex space we are mainly interested in is $\langle B(G), \sigma(B(G), \mathcal{N}_{\mathrm{st}}(G))\rangle$, for which by (2.2) we have

$$\langle B(G), \sigma(B(G), \mathcal{N}_{\mathrm{st}}(G))\rangle^* = \mathcal{N}_{\mathrm{st}}(G). \tag{2.6}$$

Notice that by what was said above, $\sigma(B(G), \mathcal{N}_{\mathrm{st}}(G))$ is the topology on $B(G)$ generated by the set of seminorms associated to the set $\{\psi_{(\phi,v)} \mid (\phi, v) \in G^* \times G\}$, hence $\sigma(B(G), \mathcal{N}_{\mathrm{st}}(G))$ is just the usual weak operator topology on $B(G)$.

Notice that by (2.1), and the Hahn–Banach theorem applied to G, we see that $\sigma(B(G), \mathcal{N}_{\mathrm{st}}(G))$ is a Hausdorff topology.

Let G be a complex Hilbert space. We define

$$\mathcal{N}_{\mathrm{pd}}(G) := B(G)_*.$$

Here $B(G)_*$ is the "predual" of the von Neumann algebra $B(G)$ (see for example Definition 2.4.17 of [BR], or Definition 2.13, Ch. 2 of [Tak]). So every $\omega \in \mathcal{N}_{\mathrm{pd}}(G)$ has the following form (see Proposition 2.4.6 of [BR] or Theorem 2.6(ii.4), Ch. 2 of [Tak]):

$$\omega : B(G) \ni B \mapsto \sum_{n=0}^{\infty} \langle u_n, Bw_n\rangle \in \mathbb{C}. \tag{2.7}$$

Here $\{u_n\}_{n\in\mathbb{N}}, \{w_n\}_{n\in\mathbb{N}} \subset G$ are such that $\sum_{n=0}^{\infty} \|u_n\|^2 < \infty$ and $\sum_{n=0}^{\infty} \|w_n\|^2 < \infty$.

We say that ω *is determined by* $\{u_n\}_{n\in\mathbb{N}}$, $\{w_n\}_{n\in\mathbb{N}}$ if (2.7) holds. Notice that ω is well-defined: indeed, for all $B \in B(G)$ we have

$$\sum_{n=0}^{\infty} |\langle u_n, Bw_n\rangle|^2 \leq \|B\|^2 \Big(\sum_{n=0}^{\infty} \|u_n\|^2\Big)\Big(\sum_{n=0}^{\infty} \|w_n\|^2\Big) < \infty,$$

hence $\omega(B)$ exists and $\omega \in B(G)^*$, so

$$\mathcal{N}_{\mathrm{pd}}(G) \subseteq B(G)^*. \tag{2.8}$$

The second locally convex space we are mainly interested in is $\langle B(G), \sigma(B(G), \mathcal{N}_{\mathrm{pd}}(G))\rangle$, for which by (2.2) we have

$$\langle B(G), \sigma(B(G), \mathcal{N}_{\mathrm{pd}}(G))\rangle^* = \mathcal{N}_{\mathrm{pd}}(G). \tag{2.9}$$

As every $\omega \in \mathcal{N}_{\mathrm{st}}(G)$ is determined by $\{u_n\}_{n=1}^N$, $\{w_n\}_{n=1}^N$ for some $N \in \mathbb{N}$, we have $\mathcal{N}_{\mathrm{st}}(G) \subset \mathcal{N}_{\mathrm{pd}}(G)$. Since $\sigma(B(G), \mathcal{N}_{\mathrm{st}}(G))$ is Hausdorff we conclude by (2.1) that so also is $\sigma(B(G), \mathcal{N}_{\mathrm{pd}}(G))$.

Notice that by the above, $\sigma(B(G), \mathcal{N}_{\mathrm{pd}}(G))$ is the topology on $B(G)$ generated by the set of seminorms associated to the set $\mathcal{N}_{\mathrm{pd}}(G)$, hence it is just the usual sigma-weak operator topology on $B(G)$ (see for example Section 2.4.1 of [BR]), so we shall often refer to it just as the sigma-weak operator topology on $B(G)$.

We remark that as a corollary of the aforementioned invariance property of weak integration, when we change the topology τ on Y to any other Hausdorff topology compatible with it, we deduce by (2.5) that $f : X \to B(G)$ is scalarly essentially $(\mu, B(G))$-integrable with respect to the measure μ and the $\sigma(B(G), \mathcal{N}_{\mathrm{st}}(G))$ topology on $B(G)$ if and only if it is so with respect to the strong topology $\tau_{\mathrm{st}}(G)$ on $B(G)$, and in this case their weak integrals coincide.

Let $\mathcal{A}$ be a $\mathbb{K}$-Banach algebra. Then for all $A, B \in \mathcal{A}$ set $[A, B] := AB - BA$. The map $\mathcal{R} : \mathcal{A} \to B(\mathcal{A})$ and $\mathcal{L} : \mathcal{A} \to B(\mathcal{A})$ have been defined in (7). Let G be a $\mathbb{K}$-Banach space and $\mathcal{N} \subseteq B(G)^*$ a linear subspace of the normed space $B(G)^*$. Then we introduce the following notations:

$$\begin{cases} \mathcal{N}^* \subseteq B(G) \Leftrightarrow (\exists Y_0 \subseteq B(G))(\mathcal{N}^* = \{\hat{A}\restriction\mathcal{N} \mid A \in Y_0\}); \\ \mathcal{N}^* \overset{\|\cdot\|}{\subseteq} B(G) \Leftrightarrow \\ (\exists Y_0 \subseteq B(G))(\forall \phi \in \mathcal{N}^*)(\exists A \in Y_0)((\phi = \hat{A}\restriction\mathcal{N}) \wedge (\|\phi\|_{\mathcal{N}^*} = \|A\|_{B(G)})). \end{cases}$$

Here $\hat{(\cdot)} : B(G) \to B(G)^{**}$ is the canonical isometric embedding of $B(G)$ into its bidual.

By Theorem 2.6(iii), Ch. 2 of [Tak], or Proposition 2.4.18 of [BR],

$$\mathcal{N}_{\mathrm{pd}}(G)^* \overset{\|\cdot\|}{\subseteq} B(G). \tag{2.10}$$

Let $\mathbf{H} : \mathcal{B}_Y \to \mathrm{Pr}(G)$ be a spectral measure in G on $\mathcal{B}_Y$. We write

$$(\forall \sigma \in \mathcal{B}(\mathbb{C}))(G_\sigma := \mathbf{H}(\sigma)G),$$

without expressing the dependence on $\mathbf{H}$ whenever it causes no confusion.

In this chapter we fix a complex Banach space G, a locally compact space X, a complex Radon measure μ on X, a possibly unbounded scalar type spectral operator R with spectrum $\sigma(R)$ and resolution of the identity E.

For each map $f : U \subset \mathbb{C} \to \mathbb{C}$ we denote by $\widetilde{f}$ the $\mathbf{0}$-extension of f to $\mathbb{C}$.

Finally, we denote by $\mathfrak{F}_{\mathrm{ess}}(X;\mu)$ the seminormed space of all maps $H : X \to \mathbb{C}$ such that

$$\|H\|_{\mathfrak{F}_{\mathrm{ess}}(X;\mu)} := \int^{\bullet} |H(x)|\, d|\mu|(x) < \infty.$$

By writing μ-l.a.e.(X) we shall mean "locally almost everywhere on X with respect to μ". Moreover, if $f : X_0 \to \mathbb{C}$ is defined μ-l.a.e.(X), then we say that $f \in \mathfrak{F}_{\mathrm{ess}}(X;\mu)$ if there exists a map $F : X \to \mathbb{C}$ such that $F{\restriction}X_0 = f$ and $F \in \mathfrak{F}_{\mathrm{ess}}(X;\mu)$. In that case we set

$$\|f\|_{\mathfrak{F}_{\mathrm{ess}}(X;\mu)} := \|F\|_{\mathfrak{F}_{\mathrm{ess}}(X;\mu)}. \tag{2.11}$$

The definition is independent of the choice of F by Proposition 1(a), n° 1, §1, Ch. V of [INT]. Moreover, let $\langle Y, \tau\rangle$ be a locally convex space and $f : X_0 \to Y$ a map defined μ-l.a.e.(X). Then for brevity we say that the map $f : X \to \langle Y, \tau\rangle$ is scalarly essentially (μ, Y)-integrable if there exists a map $F : X \to Y$ such that $F{\restriction}X_0 = f$ and $F : X \to \langle Y, \tau\rangle$ is scalarly essentially (μ, Y)-integrable. In this case we define

$$\int f(x)\, d\mu(x) := \int F(x)\, d\mu(x). \tag{2.12}$$

This does not depend on the choice of F. Indeed, for $k \in \{1, 2\}$ let $F_k : X \to Y$ be such that $F_k{\restriction}X_0 = f$ and $F_k : X \to \langle Y, \tau\rangle$ is scalarly essentially (μ, Y)-integrable. Then for all $\omega \in \langle Y, \tau\rangle^*$ and $k \in \{1, 2\}$,

$$\omega\Big(\int F_k(x)\, d\mu(x)\Big) = \int \omega(F_k(x))\, d\mu(x) = \int \chi_{X_0}(x)\omega(F_k(x))\, d\mu(x).$$

Next for all $x \in X$, $\chi_{X_0}(x)\omega(F_1(x)) = \chi_{X_0}(x)\omega(F_2(x))$, so for all $\omega \in \langle Y, \tau\rangle^*$,

$$\omega\Big(\int F_1(x)\, d\mu(x)\Big) = \omega\Big(\int F_2(x)\, d\mu(x)\Big).$$

Then (2.1) yields $\int F_1(x)\, d\mu(x) = \int F_2(x)\, d\mu(x)$.

Now we show some results about functions which are scalarly essentially $(\mu, B(G))$-integrable with respect to the $\sigma(B(G), \mathcal{N})$-topology. Here $\mathcal{N} \subseteq B(G)^*$ separates the points of $B(G)$ and $\mathcal{N}^* \subseteq B(G)$. Then we apply these results to the case when G is a Hilbert space and $\mathcal{N} = \mathcal{N}_{\mathrm{pd}}(G)$.

Theorem 2.2. *Let G be a complex Banach space, and $\mathcal{N} \subseteq B(G)^*$ be a subspace such that $\mathcal{N}$ separates the points of $B(G)$ and*

$$\mathcal{N}^* \subseteq B(G).$$

Let $F : X \to B(G)$ be such that for all $\omega \in \mathcal{N}$ the map $\omega \circ F : X \to \mathbb{C}$ is μ-measurable and

$$(X \ni x \mapsto \|F(x)\|_{B(G)}) \in \mathfrak{F}_{\mathrm{ess}}(X;\mu). \tag{2.13}$$

Then the map $F : X \to \langle B(G), \sigma(B(G), \mathcal{N})\rangle$ is scalarly essentially $(\mu, B(G))$-integrable. If in addition $\mathcal{N}^* \overset{\|\cdot\|}{\subseteq} B(G)$ then the weak integral of F satisfies

$$\Big\|\int F(x)\, d\mu(x)\Big\|_{B(G)} \le \int^{\bullet} \|F(x)\|_{B(G)}\, d|\mu|(x). \tag{2.14}$$

Proof. For all $\omega \in \mathcal{N}$ we have $|\omega(F(x))| \leq \|\omega\| \, \|F(x)\|_{B(G)}$, hence for all $\omega \in \mathcal{N}$,

$$\int^{\bullet} |\omega(F(x))| \, d|\mu|(x) \leq \|\omega\| \int^{\bullet} \|F(x)\|_{B(G)} \, d|\mu|(x). \tag{2.15}$$

Moreover, the map $\omega \circ F$ is μ-measurable by hypothesis, therefore by (2.15) and Proposition 9, §1, n° 3, Ch. 5 of [INT] we see that $\omega \circ F$ is essentially μ-integrable.

Hence we can define the following map:

$$\Psi : \mathcal{N} \ni \omega \mapsto \int \omega(F(x)) \, d\mu(x) \in \mathbb{C},$$

which is linear. Moreover, for any essentially μ-integrable map $H : X \to \mathbb{C}$,

$$\left| \int H(x) \, d\mu(x) \right| \leq \int^{\bullet} |H(x)| \, d|\mu|(x), \tag{2.16}$$

hence by (2.15),

$$\Psi \in \mathcal{N}^*. \tag{2.17}$$

Finally, as $\mathcal{N}^* \subseteq B(G)$, the statement follows from (2.17) and (2.15). ■

Remark 2.3. Let G be a complex Hilbert space. Then the statement of Theorem 2.2 holds if we set $\mathcal{N} := \mathcal{N}_{\mathrm{pd}}(G)$. Indeed, we have the duality property (2.10).

Now we give similar results for $\mathcal{N} = \mathcal{N}_{\mathrm{st}}(G)$.

Lemma 2.4. *Let G be reflexive, that is G^{**} is isometric to G through the natural injective embedding. In addition, let $B : G^* \times G \to \mathbb{C}$ be a bounded bilinear form, that is,*

$$(\exists C > 0)(\forall (\phi, v) \in G^* \times G)(|B(\phi, v)| \leq C \|\phi\|_{G^*} \|v\|_G).$$

Then

$$(\exists! L \in B(G))(\forall \phi \in G^*)(\forall v \in G)(B(\phi, v) = \phi(L(v))$$

and $\|L\|_{B(G)} \leq \|B\|$, *where* $\|B\| := \sup_{\{(\phi,v) \mid \|\phi\|_{G^*}, \|v\|_G \leq 1\}} |B(\phi, v)|$.

Proof. For all $v \in G$ let $T(v) : G^* \ni \phi \mapsto B(\phi, v) \in \mathbb{C}$, so $T(v) \in G^{**}$ with $\|T(v)\|_{(G^*)^*} \leq \|B\| \cdot \|v\|_G$. As G is reflexive, we have $(\forall v \in G)(\exists! L(v) \in G)(\forall \phi \in G^*)(\phi(L(v)) = T(v)(\phi))$, in addition $\|L(v)\|_G = \|T(v)\|_{(G^*)^*} \leq \|B\| \cdot \|v\|_G$. The operator L is linear by the linearity of T and by the fact that G^* separates the points of G by the Hahn–Banach theorem. Thus L is linear and bounded and $\|L\|_{B(G)} \leq \|B\|$. This implies the existence statement. Let now $L' \in B(G)$ be another operator with the same property, so for all $\phi \in G^*$ and $v \in G$, $\phi(L(v)) = \phi(L'(v))$. Thus by the Hahn–Banach theorem for all $v \in G$ $L(v) = L'(v)$, which shows the uniqueness. ■

Theorem 2.5. *Let G be reflexive, $F : X \to B(G)$ be a map such that for all $(\phi, v) \in G^* \times G$ the map $X \ni x \mapsto \phi(F(x)v) \in \mathbb{C}$ is μ-measurable, and assume that* (2.13) *holds. Then the map $F : X \to \langle B(G), \sigma(B(G), \mathcal{N}_{\mathrm{st}}(G)) \rangle$ is scalarly essentially $(\mu, B(G))$-integrable and its weak integral satisfies* (2.14).

Proof. For all $\phi \in G^*$, $v \in G$, and $x \in X$ we have $|\phi(F(x)v)| \leq \|\phi\| \, \|v\| \, \|F(x)\|_{B(G)}$, hence

$$\int^{\bullet} |\phi(F(x)v)| \, d|\mu|(x) \leq \|\phi\| \, \|v\| \int^{\bullet} \|F(x)\|_{B(G)} \, d|\mu|(x). \tag{2.18}$$

Furthermore, $X \ni x \mapsto \phi(F(x)v)$ is μ-measurable by hypothesis, therefore by (2.18) and Proposition 9, §1, n° 3, Ch. 5 of [INT] we infer that $X \ni x \mapsto \phi(F(x)v)$ is essentially μ-integrable.

We can thus define the map

$$B : G^* \times G \ni (\phi, v) \mapsto \int \phi(F(x)v)\, d\mu(x) \in \mathbb{C},$$

which is bilinear. So by (2.16) and (2.18),

$$|B(\phi, v)| \leq \|\phi\|\, \|v\| \int^{\bullet} \|F(x)\|_{B(G)}\, d|\mu|(x).$$

Hence B is a bounded bilinear form with $\|B\| \leq \int^{\bullet} \|F(x)\|_{B(G)}\, d|\mu|(x)$, proving the statement by Lemma 2.4. ∎

Corollary 2.6. *Let G be reflexive, $F : X \to B(G)$ a $\sigma(B(G), \mathcal{N}_{\mathrm{st}}(G))$-continuous map, i.e. $X \ni x \mapsto \phi(F(x)v) \in \mathbb{C}$ is continuous for all $(\phi, v) \in G^* \times G$, and assume that* (2.13) *holds. Then the map $F : X \to \langle B(G), \sigma(B(G), \mathcal{N}_{\mathrm{st}}(G))\rangle$ is scalarly essentially $(\mu, B(G))$-integrable and its weak integral satisfies* (2.14).

Proof. By definition of μ-measurability, the continuity condition implies that for all $(\phi, v) \in G^* \times G$ the map $X \ni x \mapsto \phi(F(x)v) \in \mathbb{C}$ is μ-measurable, proving the statement by Theorem 2.5. ∎

2.3. Commutation and restriction properties. Let $\mathbf{H} : \mathcal{B}_Y \to \mathrm{Pr}(G)$ be a spectral measure in G on $\mathcal{B}_Y$. We shall introduce a special class of subspaces of $B(G)^*$, the class of "$\mathbf{H}$-appropriate sets", which allows one to show two properties important for proving the main Extension Theorem 2.25. These are

- "Commutation" property, Theorem 2.13, for a general E-appropriate set $\mathcal{N}$, and Corollary 2.14 for $\mathcal{N} = \mathcal{N}_{\mathrm{pd}}(G)$ or $\mathcal{N} = \mathcal{N}_{\mathrm{st}}(G)$;
- "Restriction" property, Theorem 2.22, for a general E-appropriate set $\mathcal{N}$.

Lemma 2.7. *Let $A \in B(G)$ be such that $AR \subseteq RA$ and $f \in \mathrm{Bor}(\sigma(R))$. Then*

$$Af(R) \subseteq f(R)A.$$

Proof. By Corollary 18.2.4 of [DS],

$$(\forall \sigma \in \mathcal{B}(\mathbb{C}))([A, E(\sigma)] = \mathbf{0}). \tag{2.19}$$

By (8) of the introduction, for all $T \in B(G)$, $\mathcal{R}(T), \mathcal{L}(T) \in B(B(G))$, so by using the notations in Preliminaries 1.1, for all $n \in \mathbb{N}$ we have

$$\begin{aligned}
\mathbf{I}_{\mathbb{C}}^{E}(f_n)A &= (\mathcal{L}(A) \circ \mathbf{I}_{\mathbb{C}}^{E})(f_n) \\
&= \mathbf{I}_{\mathbb{C}}^{\mathcal{L}(A)\circ E}(f_n) && \text{by (1.3), } \mathcal{L}(A) \in B(B(G)) \\
&= \mathbf{I}_{\mathbb{C}}^{\mathcal{R}(A)\circ E}(f_n) && \text{by (2.19)} \\
&= (\mathcal{R}(A) \circ \mathbf{I}_{\mathbb{C}}^{E})(f_n) && \text{by (1.3), } \mathcal{R}(A) \in B(B(G)) \\
&= A\, \mathbf{I}_{\mathbb{C}}^{E}(f_n).
\end{aligned} \tag{2.20}$$

Let $x \in \mathrm{Dom}(f(R))$. Then by (1.6), the fact that $A \in B(G)$, and (2.20),

$$Af(R)x = \lim_{n\to\infty} \mathbf{I}_{\mathbb{C}}^{E}(f_n)Ax.$$

Hence (1.6) implies $Ax \in \mathrm{Dom}(f(R))$, and

$$f(R)Ax = \lim_{n\to\infty} \mathbf{I}_{\mathbb{C}}^{E}(f_n)Ax = Af(R)x. \ \blacksquare$$

LEMMA 2.8. *Let $\mathcal{N} \subseteq B(G)^*$ be such that $\sigma(B(G),\mathcal{N})$ is a Hausdorff topology, $A \in B(G)$, and let the map $X \ni x \mapsto f_x \in \mathrm{Bor}(\sigma(R))$ be such that $\widetilde{f_x} \in \mathfrak{L}_E^{\infty}(\sigma(R))$, μ-l.a.e.(X). Assume that*

(a) *$X \ni x \mapsto f_x(R) \in \langle B(G), \sigma(B(G),\mathcal{N})\rangle$ is scalarly essentially $(\mu, B(G))$-integrable;*
(b) *$\phi \circ \mathcal{R}(A) \in \mathcal{N}$ and $\phi \circ \mathcal{L}(A) \in \mathcal{N}$ for all $\phi \in \mathcal{N}$;*
(c) *$AR \subseteq RA$.*

Then

$$\left[\int f_x(R)\,d\mu(x),\ A\right] = \mathbf{0}.$$

Proof. By the hypothesis $\widetilde{f_x} \in \mathfrak{L}_E^{\infty}(\sigma(R))$, μ-l.a.e.(X) and Theorem 18.2.11(c) of [DS] applied to the scalar type spectral operator R, we have $f_x(R) \in B(G)$, μ-l.a.e.(X). Set $X_0 := \{x \in X \mid f_x(R) \in B(G)\}$. By (a) there is $F : X \to B(G)$ such that

- $(\forall x \in X_0)(F(x) = f_x(R))$;
- $F : X \to \langle B(G), \sigma(B(G),\mathcal{N})\rangle$ is scalarly essentially $(\mu, B(G))$-integrable.

Thus by definition

$$\int f_x(R)\,d\mu(x) = \int F(x)\,d\mu(x). \tag{2.21}$$

Notice that for all $x \in X$ and $\phi \in \mathcal{N}$,

$$\chi_{X_0}(x)\phi \circ \mathcal{L}(A)(F(x)) = \chi_{X_0}(x)\phi \circ \mathcal{R}(A)(F(x)), \tag{2.22}$$

since by Lemma 2.7 for all $x \in X_0$,

$$F(x)A = f_x(R)A = Af_x(R) = AF(x).$$

Moreover, for all $\phi \in \mathcal{N}$,

$$\begin{cases} \int \phi \circ \mathcal{L}(A)(F(x))\,d\mu(x) = \int \chi_{X_0}(x)\phi \circ \mathcal{L}(A)(F(x))\,d\mu(x), \\ \int \phi \circ \mathcal{R}(A)(F(x))\,d\mu(x) = \int \chi_{X_0}(x)\phi \circ \mathcal{R}(A)(F(x))\,d\mu(x). \end{cases} \tag{2.23}$$

Indeed, $\phi \circ \mathcal{L}(A) \in \mathcal{N}$, hence $X \ni x \mapsto \phi \circ \mathcal{L}(A)(F(x))$ is essentially μ-integrable, so by Proposition 9, n° 3, §1, Ch. 5 of [INT]

$$\int^{\bullet} |\chi_{X_0}(x)\phi \circ \mathcal{L}(A)(F(x))|\,d|\mu|(x) \leq \int^{\bullet} |\phi \circ \mathcal{L}(A)(F(x))|\,d|\mu|(x) < \infty.$$

Furthermore, by Proposition 6, n° 2, §5, Ch. 4 of [INT], $X \ni x \mapsto \chi_{X_0}(x)\phi \circ \mathcal{L}(A)(F(x))$ is μ-measurable. Thus by Proposition 9, n°3, §1, Ch. 5 of [INT] the map $X \ni x \mapsto \chi_{X_0}(x)\phi \circ \mathcal{L}(A)(F(x))$ is essentially μ-integrable, and we obtain the first statement of (2.23) by the fact that two essentially μ-integrable maps that are equal μ-l.a.e.(X) have the same integral. In the same way one can show the second statement of (2.23). Therefore for all $\phi \in \mathcal{N}$,

$$\begin{aligned}\phi\Big(\int f_x(R)\,d\mu(x)\,A\Big) &= \phi\circ\mathcal{L}(A)\Big(\int f_x(R)\,d\mu(x)\Big)\\ &= \phi\circ\mathcal{L}(A)\Big(\int F(x)\,d\mu(x)\Big) && \text{by (2.21)}\\ &= \int \phi\circ\mathcal{L}(A)(F(x))\,d\mu(x) && \text{since } \phi\circ\mathcal{L}(A)\in\mathcal{N}\\ &= \int \phi\circ\mathcal{R}(A)(F(x))\,d\mu(x) && \text{by (2.23), (2.22)}\\ &= \phi\circ\mathcal{R}(A)\Big(\int F(x)\,d\mu(x)\Big) && \text{since } \phi\circ\mathcal{R}(A)\in\mathcal{N}\\ &= \phi\Big(A\int f_x(R)\,d\mu(x)\Big) && \text{by (2.21).}\end{aligned}$$

Then the statement follows by (2.1). ∎

REMARK 2.9. By definition of $\mathcal{N}_{\text{st}}(G)$ (see (2.5)), hypothesis (b) of Lemma 2.8 holds for all $A \in B(G)$ and for $\mathcal{N} = \mathcal{N}_{\text{st}}(G)$. Moreover, $\sigma(B(G), \mathcal{N}_{\text{st}}(G))$ is a Hausdorff topology on $B(G)$.

Let G be a Hilbert space. By (2.7) we note that for all $A \in B(G)$ we have $\omega\circ\mathcal{L}(A) \in \mathcal{N}_{\text{pd}}(G)$ and $\omega\circ\mathcal{R}(A) \in \mathcal{N}_{\text{pd}}(G)$. Indeed, if ω is determined by $\{u_n\}_{n\in\mathbb{N}}, \{w_n\}_{n\in\mathbb{N}}$, then $\omega\circ\mathcal{L}(A)$ (respectively $\omega\circ\mathcal{R}(A)$) is determined by $\{u_n\}_{n\in\mathbb{N}}, \{Aw_n\}_{n\in\mathbb{N}}$ (respectively $\{A^*u_n\}_{n\in\mathbb{N}}, \{w_n\}_{n\in\mathbb{N}}$). Hence hypothesis (b) of Lemma 2.8 holds for all $A \in B(G)$ and for $\mathcal{N} = \mathcal{N}_{\text{pd}}(G)$. Furthermore, $\sigma(B(G), \mathcal{N}_{\text{pd}}(G))$ is a Hausdorff topology on $B(G)$.

REMARK 2.10. By Definition 18.2.1 of [DS], $E(\sigma)R \subseteq RE(\sigma)$ for all $\sigma \in \mathcal{B}(\mathbb{C})$, thus hypothesis (c) of Lemma 2.8 holds for $A := E(\sigma)$.

DEFINITION 2.11. Let $\mathbf{H} : \mathcal{B}_Y \to \Pr(G)$ be a spectral measure in G on $\mathcal{B}_Y$ (see Preliminaries 1.1). Then we define $\mathcal{N}$ to be an $\mathbf{H}$-*appropriate set* if

- $\mathcal{N} \subseteq B(G)^*$ is a linear subspace;
- $\mathcal{N}$ separates the points of $B(G)$, that is,

$$(\forall T \in B(G))(T \neq \mathbf{0} \Rightarrow (\exists \phi \in \mathcal{N})(\phi(T) \neq 0));$$

- for all $\phi \in \mathcal{N}$ and $\sigma \in \mathcal{B}_Y$,

$$\phi\circ\mathcal{R}(\mathbf{H}(\sigma)) \in \mathcal{N}, \qquad \phi\circ\mathcal{L}(\mathbf{H}(\sigma)) \in \mathcal{N}. \tag{2.24}$$

Furthermore, $\mathcal{N}$ is an $\mathbf{H}$-*appropriate set with the duality property* if $\mathcal{N}$ is an $\mathbf{H}$-appropriate set such that

$$\mathcal{N}^* \subseteq B(G).$$

Finally, $\mathcal{N}$ is an $\mathbf{H}$-*appropriate set with the isometric duality property* if $\mathcal{N}$ is an $\mathbf{H}$-appropriate set such that

$$\mathcal{N}^* \overset{\|\cdot\|}{\subseteq} B(G).$$

REMARK 2.12. Some comments about the previous definition are in order. The separation property is equivalent to requiring that $\sigma(B(G),\mathcal{N})$ is a Hausdorff topology on $B(G)$, while (2.24) is equivalent to requiring that for all $\sigma \in \mathcal{B}_Y$ the maps $\mathcal{R}(\mathbf{H}(\sigma))$ and $\mathcal{L}(\mathbf{H}(\sigma))$ on $B(G)$ are continuous with respect to the $\sigma(B(G),\mathcal{N})$-topology. Moreover, the duality property $\mathcal{N}^* \subseteq B(G)$ ensures that suitable scalarly essentially μ-integrable maps with respect to the $\sigma(B(G),\mathcal{N})$-topology are $(\mu, B(G))$-integrable (see Theorem 2.2).

Finally, by Remark 2.9, if G is a Hilbert space, $\mathcal{N}_{\mathrm{st}}(G)$ and $\mathcal{N}_{\mathrm{pd}}(G)$ are $\mathbf{H}$-appropriate sets for any spectral measure $\mathbf{H}$, and by (2.10), $\mathcal{N}_{\mathrm{pd}}(G)$ is an $\mathbf{H}$-appropriate set with the isometric duality property.

THEOREM 2.13 (Commutation 1). *Let $\mathcal{N}$ be an E-appropriate set, and let $X \ni x \mapsto f_x \in$ $\mathrm{Bor}(\sigma(R))$ be such that $\widetilde{f}_x \in \mathfrak{L}_E^\infty(\sigma(R))$, μ-l.a.e.(X). Assume that $X \ni x \mapsto f_x(R) \in \langle B(G), \sigma(B(G),\mathcal{N})\rangle$ is scalarly essentially $(\mu, B(G))$-integrable. Then for all $\sigma \in \mathcal{B}(\mathbb{C})$,*

$$\left[\int f_x(R)\, d\mu(x),\, E(\sigma)\right] = \mathbf{0}. \tag{2.25}$$

Proof. $\mathcal{N}$ being an E-appropriate set ensures that hypothesis (b) of Lemma 2.8 is satisfied for $A := E(\sigma)$ for all $\sigma \in \mathcal{B}(\mathbb{C})$, so the statement follows by Remark 2.10 and Lemma 2.8. ■

COROLLARY 2.14 (Commutation 2). (2.25) *holds if we replace $\mathcal{N}$ in Theorem* 2.13 *with $\mathcal{N}_{\mathrm{st}}(G)$ or $\mathcal{N}_{\mathrm{pd}}(G)$ and assume that G is a Hilbert space.*

Proof. By Remark 2.12 and Theorem 2.13. ■

Now we give some results necessary to show the restriction property in Theorem 2.22, namely that the map $X \ni x \mapsto f_x(R_\sigma \upharpoonright G_\sigma) \in \langle B(G_\sigma), \sigma(B(G_\sigma), \mathcal{N}_\sigma)\rangle$ is scalarly essentially $(\mu, B(G_\sigma))$-integrable, where $\mathcal{N}$ is an E-appropriate set, and $\mathcal{N}_\sigma$ could be thought of as the "restriction" of $\mathcal{N}$ to $B(G_\sigma)$ for all $\sigma \in \mathcal{B}(\mathbb{C})$.

In particular, when $\mathcal{N} = \mathcal{N}_{\mathrm{st}}(G)$, respectively $\mathcal{N} = \mathcal{N}_{\mathrm{pd}}(G)$, we can replace $\mathcal{N}_\sigma$ with $\mathcal{N}_{\mathrm{st}}(G_\sigma)$, respectively $\mathcal{N}_{\mathrm{pd}}(G_\sigma)$ (Proposition 2.23).

LEMMA 2.15. *Let $\mathbf{H} : \mathcal{B}_Y \to \mathrm{Pr}(G)$ be a spectral measure in G on $\mathcal{B}_Y$ (see Preliminaries* 1.1*). Then for all $\sigma \in \mathcal{B}_Y$, $G = G_\sigma \oplus G_{\sigma'}$, where $\sigma' := \complement\sigma$.*

Proof. We have $\mathbf{H}(\sigma) + \mathbf{H}(\sigma') = \mathbf{H}(\sigma \cup \sigma') = \mathbf{1}$ so $\mathbf{H}(\sigma') = \mathbf{1} - \mathbf{H}(\sigma)$ and $\mathbf{H}(\sigma)\mathbf{H}(\sigma') = \mathbf{H}(\sigma')\mathbf{H}(\sigma) = \mathbf{0}$. Hence for all $v \in G$, $v = \mathbf{H}(\sigma)v + \mathbf{H}(\sigma')v$, or $G = G_\sigma + G_{\sigma'}$. But for any $\delta \in \mathcal{B}_Y$ we have $G_\delta = \{y \in G \mid y = \mathbf{H}(\delta)y\}$. Then $G_\sigma \cap G_{\sigma'} = \{y \in G \mid y = \mathbf{H}(\sigma)\mathbf{H}(\sigma')y\} = \{\mathbf{0}\}$. Thus $G_\sigma + G_{\sigma'} = G_\sigma \oplus G_{\sigma'}$. ■

DEFINITION 2.16. Let $\mathbf{H} : \mathcal{B}_Y \to \mathrm{Pr}(G)$ be a spectral measure in G on $\mathcal{B}_Y$, $\sigma \in \mathcal{B}_Y$ and $\sigma' := \complement\sigma$. Then Lemma 2.15 allows us to define the operator $\xi_\sigma^{\mathbf{H}} : B(G_\sigma) \to B(G)$ such that for all $T_\sigma \in B(G_\sigma)$,

$$\xi_\sigma^{\mathbf{H}}(T_\sigma) := T_\sigma \oplus \mathbf{0}_{\sigma'} \in B(G). \tag{2.26}$$

Whenever it causes no confusion we shall denote $\xi_\sigma^{\mathbf{H}}$ simply by ξ_σ. Here $\mathbf{0}_{\sigma'} \in B(G_{\sigma'})$ is the null element of the space $B(G_{\sigma'})$, while the direct sum of two operators $T_\sigma \in B(G_\sigma)$ and $T_{\sigma'} \in B(G_{\sigma'})$ has the following standard definition:

$$(T_\sigma \oplus T_{\sigma'}) : G_\sigma \oplus G_{\sigma'} \ni (v_\sigma \oplus v_{\sigma'}) \mapsto T_\sigma v_\sigma \oplus T_{\sigma'} v_{\sigma'} \in G_\sigma \oplus G_{\sigma'}.$$

Lemma 2.17. *Let* $\mathbf{H} : \mathcal{B}_Y \to \Pr(G)$ *be a spectral measure in* G *on* $\mathcal{B}_Y$*. Then for all* $\sigma \in \mathcal{B}_Y$ *and* $T_\sigma \in B(G_\sigma)$ *we have*

$$\xi_\sigma^{\mathbf{H}}(T_\sigma) = T_\sigma \mathbf{H}(\sigma). \tag{2.27}$$

Hence $\xi_\sigma^{\mathbf{H}}$ *is well-defined, injective,* $\xi_\sigma^{\mathbf{H}} \in B(B(G_\sigma), B(G))$ *and* $\|\xi_\sigma^{\mathbf{H}}\|_{B(B(G_\sigma),B(G))} \leq \|\mathbf{H}(\sigma)\|_{B(G)}$.

Proof. Let $\sigma \in \mathcal{B}_Y$. Then for all $v \in G$ we have

$$(T_\sigma \oplus \mathbf{0}_{\sigma'})v = (T_\sigma \oplus \mathbf{0}_{\sigma'})(\mathbf{H}(\sigma)v \oplus \mathbf{H}(\sigma')v) = (T_\sigma \mathbf{H}(\sigma)v \oplus \mathbf{0}) = T_\sigma \mathbf{H}(\sigma)v,$$

proving the first part. Let $T_\sigma \in B(G_\sigma)$ be such that $\xi_\sigma(T_\sigma) = \mathbf{0}$. Then $T_\sigma \mathbf{H}(\sigma) = \mathbf{0}$, which implies that for all $v_\sigma \in G_\sigma$ we have $T_\sigma v_\sigma = T_\sigma \mathbf{H}(\sigma) v_\sigma = \mathbf{0}$. So $T_\sigma = \mathbf{0}_\sigma$. Let $\mathbf{H}(\sigma) \in B(G, G_\sigma)$ and $T_\sigma \in B(G_\sigma, G)$. Then $T_\sigma \mathbf{H}(\sigma) \in B(G)$ and $\|T_\sigma \mathbf{H}(\sigma)\|_{B(G)} \leq \|T_\sigma\|_{B(G_\sigma,G)} \cdot \|\mathbf{H}(\sigma)\|_{B(G,G_\sigma)} = \|T_\sigma\|_{B(G_\sigma)} \cdot \|\mathbf{H}(\sigma)\|_{B(G)}$. ■

Notice that by (2.27) and the fact that $B(G_\sigma)$ is a Banach space, one can show that $\xi_\sigma(B(G_\sigma))$ is a Banach subspace of $B(G)$, thus ξ_σ has a continuous inverse.

Remark 2.18. Let $\mathbf{H} : \mathcal{B}_Y \to \Pr(G)$ be a spectral measure in G on $\mathcal{B}_Y$, and $\sigma \in \mathcal{B}_Y$. Consider the product space $G_\sigma \times G_{\sigma'}$ with the standard linearization and define

$$\begin{cases} \|(x_\sigma, x_{\sigma'})\|_\oplus := \|x_\sigma + x_{\sigma'}\|_G, \\ I : G_\sigma \times G_{\sigma'} \ni (x_\sigma, x_{\sigma'}) \mapsto x_\sigma + x_{\sigma'} \in G. \end{cases} \tag{2.28}$$

As $G = G_\sigma \oplus G_{\sigma'}$ (see Lemma 2.15), the spaces $\langle G_\sigma \times G_{\sigma'}, \|\cdot\|_\oplus\rangle$ and $\langle G, \|\cdot\|_G\rangle$ are isomorphic, thus isometric and I is an isometry between them. It is not difficult to see that the topology induced by the norm $\|\cdot\|_\oplus$ is the product topology on $G_\sigma \times G_{\sigma'}$ ([2]), which implies the following property that in any case we prefer to show directly.

Proposition 2.19. *Let* $\mathbf{H} : \mathcal{B}_Y \to \Pr(G)$ *be a spectral measure in* G *on* $\mathcal{B}_Y$ *and assume the notations in* (2.28) *and Definition* 2.16*. For all* $T_\sigma \in B(G_\sigma)$ *and* $T_{\sigma'} \in B(G_{\sigma'})$ *set*

$$T_\sigma \times T_{\sigma'} : G_\sigma \times G_{\sigma'} \ni (x_\sigma, x_{\sigma'}) \mapsto (T_\sigma x_\sigma, T_{\sigma'} x_{\sigma'}) \in G_\sigma \times G_{\sigma'}.$$

Then

$$\begin{cases} T_\sigma \oplus T_{\sigma'} = I(T_\sigma \times T_{\sigma'})I^{-1} = T_\sigma \mathbf{H}(\sigma) + T_{\sigma'}\mathbf{H}(\sigma') \in B(G), \\ T_\sigma \times T_{\sigma'} = I^{-1}(T_\sigma \mathbf{H}(\sigma) + T_{\sigma'}\mathbf{H}(\sigma'))I \in B(G_\sigma \times G_{\sigma'}). \end{cases} \tag{2.29}$$

Proof. We have $I(T_\sigma \times T_{\sigma'})I^{-1}(x_\sigma \oplus x_{\sigma'}) = I(T_\sigma x_\sigma, T_{\sigma'} x_{\sigma'}) = T_\sigma x_\sigma \oplus T_{\sigma'} x_{\sigma'}$ for all $x_\sigma \in G_\sigma$ and $x_{\sigma'} \in G_{\sigma'}$, proving the first equality. For all $x \in G$,

$$\begin{aligned} I(T_\sigma \times T_{\sigma'})I^{-1}(x) &= I(T_\sigma \times T_{\sigma'})I^{-1}(\mathbf{H}(\sigma)x + \mathbf{H}(\sigma')x) \\ &= I(T_\sigma \mathbf{H}(\sigma)x, T_{\sigma'}\mathbf{H}(\sigma')x) = T_\sigma \mathbf{H}(\sigma)x + T_{\sigma'}\mathbf{H}(\sigma')x. \end{aligned}$$

This yields the second equality. The third equality follows from the second and the fact that I is an isometry. ■

Notice that by the first statement in (2.29) we obtain (2.27).

([2]) Indeed, let $\sigma \in \mathcal{B}_Y$ be such that $\mathbf{H}(\sigma) \neq \mathbf{0}$, set $M := \max\{\|\mathbf{H}(\sigma)\|, \|\mathbf{H}(\sigma')\|\}$ and for all $r > 0$ define $B_r^\oplus(\mathbf{0}) := \{(x_\sigma, x_{\sigma'}) \in G_\sigma \times G_{\sigma'} \mid \|(x_\sigma, x_{\sigma'})\|_\oplus < r\}$. Thus for all $\varepsilon > 0$ by setting $\eta := \varepsilon/2$ we have $B_\eta(\mathbf{0}_\sigma) \times B_\eta(\mathbf{0}'_\sigma) \subset B_\varepsilon^\oplus(\mathbf{0})$, while for all $\varepsilon_1, \varepsilon_2 > 0$ by setting $\zeta := \min\{\varepsilon_1, \varepsilon_2\}/M$ we have $B_\zeta^\oplus(\mathbf{0}) \subset B_{\varepsilon_1}(\mathbf{0}_\sigma) \times B_{\varepsilon_2}(\mathbf{0}'_\sigma)$.

Definition 2.20. Let $\mathbf{H} : \mathcal{B}_Y \to \Pr(G)$ be a spectral measure in G on $\mathcal{B}_Y$ and $\mathcal{N} \subseteq B(G)^*$. For all $\sigma \in \mathcal{B}_Y$ and $\psi \in \mathcal{N}$ we define

$$\psi_\sigma^{\mathbf{H}} := \psi \circ \xi_\sigma^{\mathbf{H}} \in B(G_\sigma)^*, \qquad \mathcal{N}_\sigma^{\mathbf{H}} := \{\psi_\sigma^{\mathbf{H}} \mid \psi \in \mathcal{N}\}, \tag{2.30}$$

where $\xi_\sigma^{\mathbf{H}}$ has been defined in (2.26). We shall express $\psi_\sigma^{\mathbf{H}}$ and $\mathcal{N}_\sigma^{\mathbf{H}}$ simply by ψ_σ and $\mathcal{N}_\sigma$ respectively, whenever it causes no confusion.

Proposition 2.21. *Let $\mathbf{H} : \mathcal{B}_Y \to \Pr(G)$ be a spectral measure in G on $\mathcal{B}_Y$, $\mathcal{N} \subseteq B(G)^*$ a subset that separates the points of $B(G)$, and $\sigma \in \mathcal{B}_Y$. Then $\mathcal{N}_\sigma$ separates the points of $B(G_\sigma)$.*

Proof. Let $T_\sigma \in B(G_\sigma) - \{\mathbf{0}_\sigma\}$. By Lemma 2.17, ξ_σ is injective so $\xi_\sigma(T_\sigma) \neq \mathbf{0}$. But $\mathcal{N}$ separates the points of $B(G)$, so there is $\psi \in \mathcal{N}$ such that $\psi(\xi_\sigma(T_\sigma)) \neq 0$. ∎

Theorem 2.22 (Restriction). *Let $\mathcal{N}$ be an E-appropriate set, and let $X \ni x \mapsto f_x \in \mathrm{Bor}(\sigma(R))$ be such that $\widetilde{f_x} \in \mathfrak{L}_E^\infty(\sigma(R))$, μ-l.a.e.(X) Assume that the map $X \ni x \mapsto f_x(R) \in \langle B(G), \sigma(B(G), \mathcal{N})\rangle$ is scalarly essentially $(\mu, B(G))$-integrable. Then for all $\sigma \in \mathcal{B}(\mathbb{C})$ the map $X \ni x \mapsto f_x(R_\sigma \restriction G_\sigma) \in \langle B(G_\sigma), \sigma(B(G_\sigma), \mathcal{N}_\sigma)\rangle$ is scalarly essentially $(\mu, B(G_\sigma))$-integrable and*

$$\int f_x(R_\sigma \restriction G_\sigma)\, d\mu(x) = \int f_x(R)\, d\mu(x) \restriction G_\sigma. \tag{2.31}$$

Proof. Let $\sigma \in \mathcal{B}(\mathbb{C})$. Then (1.14) implies that for all $x \in X$ the operator $f_x(R_\sigma \restriction G_\sigma)$ is well-defined. By the hypothesis $\widetilde{f_x} \in \mathfrak{L}_E^\infty(\sigma(R))$, μ-l.a.e.(X) and Theorem 18.2.11(c) of [DS] applied to R, we have $f_x(R) \in B(G)$, μ-l.a.e.(X). Set

$$X_0 := \{x \in X \mid f_x(R) \in B(G)\}.$$

Then by Lemma 1.7(2) we obtain

$$(\forall x \in X_0)(f_x(R_\sigma \restriction G_\sigma) \in B(G_\sigma)). \tag{2.32}$$

Hence $f_x(R_\sigma \restriction G_\sigma) \in B(G_\sigma)$, μ-l.a.e.(X). So by Proposition 2.21 and (2.1) it is well-defined the statement that $X \ni x \mapsto f_x(R_\sigma \restriction G_\sigma) \in \langle B(G_\sigma), \sigma(B(G_\sigma), \mathcal{N}_\sigma)\rangle$ is scalarly essentially $(\mu, B(G_\sigma))$-integrable is meaningful. By hypothesis we deduce that there is $F : X \to B(G)$ such that

- $(\forall x \in X_0)(F(x) = f_x(R))$;
- $F : X \to \langle B(G), \sigma(B(G), \mathcal{N})\rangle$ is scalarly essentially $(\mu, B(G))$-integrable.

Thus by (2.12),

$$\int f_x(R)\, d\mu(x) = \int F(x)\, d\mu(x). \tag{2.33}$$

Now for all $\sigma \in \mathcal{B}(\mathbb{C})$ define $F^\sigma : X \to B(G_\sigma)$ by setting, for all $x \in X$,

$$F^\sigma(x) := E(\sigma)F(x) \restriction G_\sigma.$$

By (2.32) we can claim that

(1) $(\forall x \in X_0)(F^\sigma(x) = f_x(R_\sigma \restriction G_\sigma))$;

(2) the map $F^{\sigma} : X \to \langle B(G_{\sigma}), \sigma(B(G_{\sigma}), \mathcal{N}_{\sigma})\rangle$ is scalarly essentially $(\mu, B(G_{\sigma}))$-integrable, and

$$\int F^{\sigma}(x)\, d\mu(x) = \int f_x(R)\, d\mu(x) \restriction G_{\sigma}. \tag{2.34}$$

Then the statement will follow by setting, according to (2.12),

$$\int f_x(R_{\sigma} \restriction G_{\sigma})\, d\mu(x) := \int F^{\sigma}(x)\, d\mu(x).$$

For all $x \in X_0$,

$$\begin{aligned} F^{\sigma}(x) &= E(\sigma) f_x(R) \restriction G_{\sigma} \\ &= f_x(R) E(\sigma) \restriction G_{\sigma} \qquad \text{since } [f_x(R), E(\sigma)] = \mathbf{0} \\ &= f_x(R_{\sigma} \restriction G_{\sigma}) \qquad \text{by Key Lemma 1.7.} \end{aligned}$$

Hence (1) of our claim follows. For all $\psi \in \mathcal{N}$ and $x \in X$,

$$\begin{aligned} \psi \circ \mathcal{L}(E(\sigma)) \circ \mathcal{R}(E(\sigma))(F(x)) &\doteq \psi(E(\sigma)F(x)E(\sigma)) = \psi_{\sigma}(E(\sigma)F(x) \restriction G_{\sigma}) \\ &\doteq \psi_{\sigma}(F^{\sigma}(x)). \end{aligned} \tag{2.35}$$

From the second equality we deduce by Lemma 2.17 that for all $T \in B(G)$ we have $\xi_{\sigma}(E(\sigma)T \restriction G_{\sigma}) = E(\sigma)TE(\sigma)$. Since $F : X \to \langle B(G), \sigma(B(G), \mathcal{N})\rangle$ is scalarly essentially μ-integrable, and $\psi \circ \mathcal{L}(E(\sigma)) \circ \mathcal{R}(E(\sigma)) \in \mathcal{N}$ for all $\psi \in \mathcal{N}$, by (2.35) the map

$$F^{\sigma} : X \to \langle B(G_{\sigma}), \sigma(B(G_{\sigma}), \mathcal{N}_{\sigma})\rangle \text{ is scalarly essentially } \mu\text{-integrable.}$$

Now by (2.25) we have, for all $v \in G_{\sigma}$,

$$\int f_x(R)\, d\mu(x) v = \int f_x(R)\, d\mu(x) E(\sigma) v = E(\sigma) \int f_x(R)\, d\mu(x) v \in G_{\sigma}. \tag{2.36}$$

Moreover, $\int f_x(R)\, d\mu(x) \in B(G)$ so $\int f_x(R)\, d\mu(x) \restriction G_{\sigma} \in B(G_{\sigma})$. Therefore for all $\psi \in \mathcal{N}$,

$$\begin{aligned} &\psi_{\sigma}\Big(\int f_x(R)\, d\mu(x) \restriction G_{\sigma}\Big) \\ &\quad = \psi_{\sigma}\Big(E(\sigma) \int f_x(R)\, d\mu(x) \restriction G_{\sigma}\Big) \qquad \text{by (2.36)} \\ &\quad = \psi\Big(E(\sigma) \int f_x(R)\, d\mu(x)\, E(\sigma)\Big) \qquad \text{by Lemma 2.17} \\ &\quad \doteq \psi \circ \mathcal{L}(E(\sigma)) \circ \mathcal{R}(E(\sigma))\Big(\int f_x(R)\, d\mu(x)\Big) \\ &\quad \doteq \psi \circ \mathcal{L}(E(\sigma)) \circ \mathcal{R}(E(\sigma))\Big(\int F(x)\, d\mu(x)\Big) \qquad \text{by (2.33)} \\ &\quad = \int \psi \circ \mathcal{L}(E(\sigma)) \circ \mathcal{R}(E(\sigma))(F(x))\, d\mu(x) \qquad \text{as } \psi \circ \mathcal{L}(E(\sigma)) \circ \mathcal{R}(E(\sigma)) \in \mathcal{N} \\ &\quad = \int \psi_{\sigma}(F^{\sigma}(x))\, d\mu(x) \qquad \text{by (2.35).} \end{aligned}$$

Hence (2.3) and (2.4) yield (2.34), and the statement follows. ■

PROPOSITION 2.23. *For all $\sigma \in \mathcal{B}(\mathbb{C})$,*

$$(\mathcal{N}_{\mathrm{st}}(G))_\sigma = \mathcal{N}_{\mathrm{st}}(G_\sigma) \quad \textit{and} \quad (\mathcal{N}_{\mathrm{pd}}(G))_\sigma = \mathcal{N}_{\mathrm{pd}}(G_\sigma). \tag{2.37}$$

Proof. By the Hahn–Banach theorem,

$$(G_\sigma)^* = \{\phi\restriction G_\sigma \mid \phi \in G^*\}. \tag{2.38}$$

Then we have

$$\begin{aligned}(\mathcal{N}_{\mathrm{st}}(G))_\sigma &\doteq \mathfrak{L}_{\mathbb{C}}(\{\psi_{(\phi,v)} \circ \xi_\sigma \mid (\phi, v) \in G^* \times G\}) = \mathfrak{L}_{\mathbb{C}}(\{\psi_{(\phi\restriction G_\sigma,w)} \mid (\phi, w) \in G^* \times G_\sigma\}) \\ &= \mathfrak{L}_{\mathbb{C}}(\{\psi_{(\rho,w)} \mid (\rho, w) \in (G_\sigma)^* \times G_\sigma\}) \doteq \mathcal{N}_{\mathrm{st}}(G_\sigma).\end{aligned}$$

Here in the third equality we used (2.38), while in the second equality we applied the fact that for all $(\phi, v) \in G^* \times G$ and for all $T_\sigma \in B(G_\sigma)$,

$$\begin{aligned}\psi_{(\phi,v)} \circ \xi_\sigma(T_\sigma) &= \phi(T_\sigma E(\sigma)v) \quad \text{by (2.27)} \\ &= (\phi\restriction G_\sigma)(T_\sigma E(\sigma)v) \\ &= \psi_{(\phi\restriction G_\sigma,E(\sigma)v)}(T_\sigma).\end{aligned} \tag{2.39}$$

Let G be a complex Hilbert space. Then

$$\begin{aligned}&(\mathcal{N}_{\mathrm{pd}}(G))_\sigma \\ &= \Big\{\Big(\sum_{n=0}^{\infty} \psi_{(u_n^\dagger,w_n)}\Big) \circ \xi_\sigma \,\Big|\, \{u_n\}_{n\in\mathbb{N}}, \{w_n\}_{n\in\mathbb{N}} \subset G,\ \sum_{n=0}^{\infty} \|u_n\|_G^2, \sum_{n=0}^{\infty} \|w_n\|_G^2 < \infty\Big\} \\ &= \Big\{\sum_{n=0}^{\infty} (\psi_{(u_n^\dagger,w_n)} \circ \xi_\sigma) \,\Big|\, \{u_n\}_{n\in\mathbb{N}}, \{w_n\}_{n\in\mathbb{N}} \subset G,\ \sum_{n=0}^{\infty} \|u_n\|_G^2, \sum_{n=0}^{\infty} \|w_n\|_G^2 < \infty\Big\} \\ &= \Big\{\sum_{n=0}^{\infty} \psi_{(u_n^\dagger\restriction G_\sigma,E(\sigma)w_n)} \,\Big|\, \{u_n\}_{n\in\mathbb{N}}, \{w_n\}_{n\in\mathbb{N}} \subset G,\ \sum_{n=0}^{\infty} \|u_n\|_G^2, \sum_{n=0}^{\infty} \|w_n\|_G^2 < \infty\Big\} \\ &= \Big\{\sum_{n=0}^{\infty} \psi_{(E(\sigma)^*u_n)^\dagger\restriction G_\sigma,E(\sigma)w_n)} \,\Big|\, \{u_n\}_{n\in\mathbb{N}}, \{w_n\}_{n\in\mathbb{N}} \subset G,\ \sum_{n=0}^{\infty} \|u_n\|_G^2, \sum_{n=0}^{\infty} \|w_n\|_G^2 < \infty\Big\} \\ &= \Big\{\sum_{n=0}^{\infty} \psi_{(a_n^\dagger,b_n)} \,\Big|\, \{a_n\}_{n\in\mathbb{N}}, \{b_n\}_{n\in\mathbb{N}} \subset G_\sigma,\ \sum_{n=0}^{\infty} \|a_n\|_{G_\sigma}^2, \sum_{n=0}^{\infty} \|b_n\|_{G_\sigma}^2 < \infty\Big\} \\ &= \mathcal{N}_{\mathrm{pd}}(G_\sigma).\end{aligned}$$

Here for any Hilbert space F we let $u^\dagger \in F^*$ be such that $u^\dagger(v) := \langle u, v\rangle$ for all $u, v \in F$, and the series in the first equality is converging with respect to the strong operator topology on $B(G)^*$, while all the others are converging with respect to the strong operator topology on $B(G_\sigma)^*$.

The first equality follows by (2.7), the third by (2.39), the fourth by the fact that $E(\sigma)\restriction G_\sigma = \mathbf{1}_\sigma$, the identity operator on G_σ. Now we show the fifth equality. Notice that

$$\sum_{n=0}^{\infty} \|E(\sigma)w_n\|_{G_\sigma}^2 \doteq \sum_{n=0}^{\infty} \|E(\sigma)w_n\|_G^2 \le \|E(\sigma)\|^2 \sum_{n=0}^{\infty} \|w_n\|_G^2 < \infty.$$

As $\dagger : H \to H^*$ is a semilinear isometry, for all $n \in \mathbb{N}$ there exists only one $a_n \in G_\sigma$ such that $a_n^\dagger = (E(\sigma)^* u_n)^\dagger \lceil G_\sigma$, and moreover

$$\|a_n\|_{G_\sigma} = \|(E(\sigma)^* u_n)^\dagger \lceil G_\sigma\|_{G_\sigma^*}.$$

Next,

$$\begin{aligned}\|(E(\sigma)^* u_n)^\dagger \lceil G_\sigma\|_{G_\sigma^*} &= \sup_{\{v \in G_\sigma \,|\, \|v\|_{G_\sigma} \le 1\}} |\langle E(\sigma)^* u_n, v\rangle| = \sup_{\{v \in G_\sigma \,|\, \|v\|_{G_\sigma} \le 1\}} |\langle u_n, v\rangle| \\ &\le \sup_{\{v \in G \,|\, \|v\|_G \le 1\}} |\langle u_n, v\rangle| = \|u_n^\dagger\|_{G^*} = \|u_n\|_G.\end{aligned}$$

Hence $\sum_{n=0}^\infty \|a_n\|_{G_\sigma}^2 \le \sum_{n=0}^\infty \|u_n\|_G^2 < \infty$ and the fifth equality follows. ∎

2.4. Extension theorem for integral equalities with respect to the $\sigma(B(G), \mathcal{N})$-topology. In the present section we shall prove the Extension Theorems for integration with respect to the $\sigma(B(G), \mathcal{N})$-topology, when $\mathcal{N}$ is an E-appropriate set (Theorem 2.25) and when $\mathcal{N}$ is an E-appropriate set with the duality property (Corollary 2.26). As an application we shall consider the cases of the sigma-weak topology (Corollaries 2.28 and 2.29) and weak operator topology (Corollaries 2.27 and 2.30). In this section we adopt all the notations defined in Section 2.2.

THEOREM 2.24. *Let $\mathcal{N}$ be an E-appropriate set and $\{\sigma_n\}_{n \in \mathbb{N}}$ be an E-sequence* (*see Definition* 1.10) *and let $X \ni x \mapsto f_x \in \mathrm{Bor}(\sigma(R))$ be such that $\widetilde{f_x} \in \mathfrak{L}_E^\infty(\sigma(R))$, μ-l.a.e.(X). Let $X \ni x \mapsto f_x(R) \in \langle B(G), \sigma(B(G), \mathcal{N})\rangle$ be scalarly essentially $(\mu, B(G))$-integrable and $g, h \in \mathrm{Bor}(\sigma(R))$. If for all $n \in \mathbb{N}$,*

$$g(R_{\sigma_n} \lceil G_{\sigma_n}) \int f_x(R_{\sigma_n} \lceil G_{\sigma_n})\, d\mu(x) \subseteq h(R_{\sigma_n} \lceil G_{\sigma_n}) \tag{2.40}$$

then

$$g(R) \int f_x(R)\, d\mu(x) \lceil \Theta = h(R) \lceil \Theta. \tag{2.41}$$

In (2.40) *the weak integral is with respect to the measure μ and the $\sigma(B(G_{\sigma_n}), \mathcal{N}_{\sigma_n})$-topology, while in* (2.41),

$$\Theta := \mathrm{Dom}\Big(g(R) \int f_x(R)\, d\mu(x)\Big) \cap \mathrm{Dom}(h(R))$$

and the weak integral is with respect to the measure μ and the $\sigma(B(G), \mathcal{N})$-topology.

Proof. (2.40) is meaningful by Theorem 2.22.

By (1.18), for all $y \in \Theta$,

$$g(R) \int f_x(R)\, d\mu(x)\, y = \lim_{n \in \mathbb{N}} E(\sigma_n) g(R) \int f_x(R)\, d\mu(x)\, y$$

by Theorem 18.2.11(g) of [DS] and (2.25)

$$= \lim_{n \in \mathbb{N}} g(R) \int f_x(R)\, d\mu(x)\, E(\sigma_n) y$$

by (2.31) and Lemma 1.7 applied to $g(R)$

$$= \lim_{n\in\mathbb{N}} g(R_{\sigma_n}\lceil G_{\sigma_n}) \int f_x(R_{\sigma_n}\lceil G_{\sigma_n})\, d\mu(x)\, E(\sigma_n)y$$

by hypothesis (2.40)

$$= \lim_{n\in\mathbb{N}} h(R_{\sigma_n}\lceil G_{\sigma_n})\, E(\sigma_n)y$$

by Lemma 1.7 and Theorem 18.2.11(g) of [DS]

$$\begin{aligned} &= \lim_{n\in\mathbb{N}} E(\sigma_n)h(R)\, y \\ &= h(R)\, y. \end{aligned} \tag{2.42}$$

In the last equality we have applied (1.18). ■

Main Theorem 2.25 ($\sigma(B(G),\mathcal{N})$-Extension Theorem). *Let R be a possibly unbounded scalar type spectral operator in G, E its resolution of the identity, and $\mathcal{N}$ an E-appropriate set. Let $X \ni x \mapsto f_x \in \mathrm{Bor}(\sigma(R))$ be such that $\widetilde{f}_x \in \mathfrak{L}_E^\infty(\sigma(R))$, μ-l.a.e.(X), and $X \ni x \mapsto f_x(R) \in \langle B(G), \sigma(B(G),\mathcal{N})\rangle$ is scalarly essentially $(\mu, B(G))$-integrable. Finally, let $g, h \in \mathrm{Bor}(\sigma(R))$ and $\widetilde{h} \in \mathfrak{L}_E^\infty(\sigma(R))$. If $\{\sigma_n\}_{n\in\mathbb{N}}$ is an E-sequence and for all $n \in \mathbb{N}$,*

$$g(R_{\sigma_n}\lceil G_{\sigma_n}) \int f_x(R_{\sigma_n}\lceil G_{\sigma_n})\, d\mu(x) \subseteq h(R_{\sigma_n}\lceil G_{\sigma_n}), \tag{2.43}$$

then $h(R) \in B(G)$ and

$$g(R) \int f_x(R)\, d\mu(x) = h(R). \tag{2.44}$$

In (2.43) the weak integral is with respect to the measure μ and the $\sigma(B(G_{\sigma_n}),\mathcal{N}_{\sigma_n})$-topology, while in (2.44) it is with respect to μ and the $\sigma(B(G),\mathcal{N})$-topology.

Notice that $g(R)$ is a possibly *unbounded* operator in G.

Proof. Theorem 18.2.11 of [DS] and the hypothesis $\widetilde{h} \in \mathfrak{L}_E^\infty(\sigma(R))$ imply $h(R) \in B(G)$, so by (2.41) we can deduce that

$$g(R) \int f_x(R)\, d\mu(x) \subseteq h(R). \tag{2.45}$$

Set

$$(\forall n \in \mathbb{N})(\delta_n := |g|^{-1}([0,n])). \tag{2.46}$$

We claim that

$$\begin{cases} \bigcup_{n\in\mathbb{N}} \delta_n = \sigma(R), \\ n \geq m \Rightarrow \delta_n \supseteq \delta_m, \\ (\forall n \in \mathbb{N})(g(\delta_n) \text{ is bounded}). \end{cases} \tag{2.47}$$

Since $|g| \in \mathrm{Bor}(\sigma(R))$ we have $\delta_n \in \mathcal{B}(\mathbb{C})$ for all $n \in \mathbb{N}$, so $\{\delta_n\}_{n\in\mathbb{N}}$ is an E-sequence, hence by (1.18),

$$\lim_{n\in\mathbb{N}} E(\delta_n) = \mathbf{1} \tag{2.48}$$

in the strong operator topology on $B(G)$.

Indeed, the first equality of (2.47) follows from $\bigcup_{n\in\mathbb{N}}\delta_n \doteq \bigcup_{n\in\mathbb{N}}|g|^{-1}([0,n]) = |g|^{-1}(\bigcup_{n\in\mathbb{N}}[0,n]) = |g|^{-1}(\mathbb{R}^+) = \mathrm{Dom}(g) := \sigma(R)$, the second from the fact that $|g|^{-1}$ preserves the inclusion, and the third from the inclusion $|g|(\delta_n) \subseteq [0,n]$. By the third statement of (2.47), $\delta_n \in \mathcal{B}(\mathbb{C})$ and Lemma 1.7(3),

$$(\forall n \in \mathbb{N})(E(\delta_n)G \subseteq \mathrm{Dom}(g(R))). \tag{2.49}$$

By (2.25) and (2.49), for all $n \in \mathbb{N}$,

$$\int f_x(R)\,d\mu(x)\,E(\delta_n)G \subseteq E(\delta_n)G \subseteq \mathrm{Dom}(g(R)).$$

Therefore

$$(\forall n \in \mathbb{N})(\forall v \in G)\Big(E(\delta_n)v \in \mathrm{Dom}\Big(g(R)\int f_x(R)\,d\mu(x)\Big)\Big).$$

Hence by (2.48),

$$\mathbf{D} := \mathrm{Dom}\Big(g(R)\int f_x(R)\,d\mu(x)\Big) \text{ is dense in } G. \tag{2.50}$$

But $\int f_x(R)\,d\mu(x) \in B(G)$ and $g(R)$ is closed by Theorem 18.2.11 of [DS], so by Lemma 1.15 we find that

$$g(R)\int f_x(R)\,d\mu(x) \text{ is closed.} \tag{2.51}$$

But we know that $h(R) \in B(G)$ so by (2.45) we deduce

$$g(R)\int f_x(R)\,d\mu(x) \in B(\mathbf{D}, G). \tag{2.52}$$

Now (2.51), (2.52) and Lemma 1.16 imply that $\mathbf{D}$ is closed in G, therefore by (2.50),

$$\mathbf{D} = G.$$

Hence by (2.45) we conclude that the statement holds. ∎

Now we give conditions on the maps f_x ensuring that $f_x(R) \in B(G)$, and that $X \ni x \mapsto f_x(R) \in B(G)$ is scalarly essentially $(\mu, B(G))$-integrable with respect to the $\sigma(B(G), \mathcal{N})$-topology.

Corollary 2.26 ($\sigma(B(G),\mathcal{N})$-Extension Theorem—duality case). *Let $\mathcal{N}$ be an E-appropriate set with the duality property and $X \ni x \mapsto f_x \in \mathrm{Bor}(\sigma(R))$. Assume that there is $X_0 \subseteq X$ such that $\complement X_0$ is μ-locally negligible and $\widetilde{f}_x \in \mathfrak{L}_E^{\infty}(\sigma(R))$ for all $x \in X_0$. Moreover, suppose there exists $F : X \to B(G)$ extending $X_0 \ni x \mapsto f_x(R) \in B(G)$ such that for all $\omega \in \mathcal{N}$ the map $X \ni x \mapsto \omega(F(x)) \in \mathbb{C}$ is μ-measurable and*

$$(X \ni x \mapsto \|F(x)\|_{B(G)}) \in \mathfrak{F}_{\mathrm{ess}}(X;\mu). \tag{2.53}$$

If $g, h \in \mathrm{Bor}(\sigma(R))$ are such that $\widetilde{h} \in \mathfrak{L}_E^{\infty}(\sigma(R))$ and $\{\sigma_n\}_{n\in\mathbb{N}}$ is an E-sequence such that (2.43) holds for all $n \in \mathbb{N}$ then the statement of Theorem 2.25 holds. Moreover, if $\mathcal{N}$ is an E-appropriate set with the isometric duality property, then

$$\Big\|\int f_x(R)\,d\mu(x)\Big\|_{B(G)} \le \int^{\bullet} \|f_x(R)\|_{B(G)}\,d|\mu|(x).$$

Proof. The map $X \ni x \mapsto f_x(R) \in \langle B(G), \sigma(B(G), \mathcal{N})\rangle$ is scalarly essentially $(\mu, B(G))$-integrable by the duality property and Theorem 2.2. Hence the first part of the statement follows from Theorem 2.25. The inequality follows by (2.14), (2.12) and (2.11). ∎

Now we will give corollaries in which $\mathcal{N} = \mathcal{N}_{\mathrm{st}}(G)$ or $\mathcal{N} = \mathcal{N}_{\mathrm{pd}}(G)$ and G is a Hilbert space.

COROLLARY 2.27. *The statement of Theorem* 2.25 *holds if* $\mathcal{N}$ *is replaced by* $\mathcal{N}_{\mathrm{st}}(G)$ *and* $\mathcal{N}_{\sigma_n}$ *is replaced by* $\mathcal{N}_{\mathrm{st}}(G_{\sigma_n})$ *for all* $n \in \mathbb{N}$.

Proof. By Remark 2.12 we know that $\mathcal{N}_{\mathrm{st}}(G)$ is an E-appropriate set, therefore the statement follows from (2.37) and Theorem 2.25. ∎

COROLLARY 2.28. *The statement of Theorem* 2.25 *holds if* G *is a complex Hilbert space,* $\mathcal{N}$ *is replaced by* $\mathcal{N}_{\mathrm{pd}}(G)$, *and* $\mathcal{N}_{\sigma_n}$ *is replaced by* $\mathcal{N}_{\mathrm{pd}}(G_{\sigma_n})$ *for all* $n \in \mathbb{N}$.

Proof. By Remark 2.12 we know that $\mathcal{N}_{\mathrm{pd}}(G)$ is in particular an E-appropriate set, therefore the statement follows from (2.37) and Theorem 2.25. ∎

THEOREM 2.29 (Sigma-weak Extension Theorem). *Let* G *be a Hilbert space. Then the statement of Corollary* 2.26 *holds if we set* $\mathcal{N} := \mathcal{N}_{\mathrm{pd}}(G)$ *and* $\mathcal{N}_{\sigma_n} := \mathcal{N}_{\mathrm{pd}}(G_{\sigma_n})$ *for all* $n \in \mathbb{N}$.

Proof. By Remark 2.12, $\mathcal{N}_{\mathrm{pd}}(G)$ is an E-appropriate set with the isometric duality property, so we obtain the statement by Corollary 2.26 and (2.37). ∎

COROLLARY 2.30 (Weak Extension Theorem). *Let* G *be reflexive. Then the statement of Corollary* 2.26 *holds if we set* $\mathcal{N} := \mathcal{N}_{\mathrm{st}}(G)$ *and* $\mathcal{N}_{\sigma_n} := \mathcal{N}_{\mathrm{st}}(G_{\sigma_n})$ *for all* $n \in \mathbb{N}$.

Proof. By Theorem 2.5 the map $X \ni x \mapsto f_x(R) \in \langle B(G), \sigma(B(G), \mathcal{N}_{\mathrm{st}}(G))\rangle$ is scalarly essentially $(\mu, B(G))$-integrable. Hence the first part of the statement follows from Corollary 2.27, while the inequality follows by (2.14), (2.12) and (2.11). ∎

REMARK 2.31. If G is a Hilbert space we can obtain Corollary 2.30 as an application of the duality property of the predual $\mathcal{N}_{\mathrm{pd}}(G)$. Indeed, as we know, $\mathcal{N}_{\mathrm{st}}(G) \subset \mathcal{N}_{\mathrm{pd}}(G)$, hence by the Hahn–Banach theorem for all $\Psi_0 \in \mathcal{N}_{\mathrm{st}}(G)^*$ there exists $\Psi \in \mathcal{N}_{\mathrm{pd}}(G)^*$ such that $\Psi \upharpoonright \mathcal{N}_{\mathrm{st}}(G) = \Psi_0$. Thus by the duality property $\mathcal{N}_{\mathrm{pd}}(G)^* = B(G)$ we obtain $(\forall \Psi_0 \in \mathcal{N}_{\mathrm{st}}(G)^*)(\exists B \in B(G))(\forall \omega \in \mathcal{N}_{\mathrm{st}}(G))(\Psi_0(\omega) = \omega(B))$, which ensures that the weak integral with respect to μ and the weak operator topology of the map $X \ni x \mapsto f_x(R) \in B(G)$ belongs to $B(G)$.

REMARK 2.32. Let $\mathsf{D} \subset G$ be a linear subspace of G and $E : \mathcal{B}(\mathbb{C}) \to \mathrm{Pr}(G)$ be a countably additive spectral measure. Then by (1.3) for all $f \in \mathbf{TM}$, $\phi \in G^*$ and $v \in \mathsf{D}$,

$$|\phi(\mathbf{I}_{\mathbb{C}}^{E}(f)v)| = \left|\int f(\lambda)\, d(\psi_{\phi,v} \circ E)(\lambda)\right| \leq 4M\|f\|_{\sup}\|\phi\|\,\|v\|, \tag{2.54}$$

where $M := \sup_{\delta \in \mathcal{B}(\mathbb{C})} \|E(\delta)\|$, $\psi_{\phi,v} : B(G) \ni A \mapsto \phi(Av) \in \mathbb{C}$ and $\mathbf{TM}$ is the space of all totally $\mathcal{B}(\mathbb{C})$-measurable complex maps on $\mathbb{C}$. Next we know that

$$H(\mathbb{C}) \subset \mathbf{TM}. \tag{2.55}$$

Here $H(\mathbb{C})$ is the space of all compactly supported continuous complex functions on $\mathbb{C}$, with the topology of the direct limit of the spaces $H(\mathbb{C}; K)$ with K running over the class of all compact subsets of $\mathbb{C}$; here $H(\mathbb{C};K)$ is the space of all continuous functions $f:\mathbb{C}\to\mathbb{C}$ such that $\operatorname{supp}(f)\subset K$ with the topology of uniform convergence. Set

$$F_w^{\mathsf{D}} := \overline{B(\mathsf{D},G)} \quad \text{in } \mathfrak{L}_w(\mathsf{D},G),$$

where $\mathfrak{L}_w(\mathsf{D},G)$ is the Hausdorff locally convex space of all linear operators on D with values in G and with the topology generated by the following set of seminorms:

$$\{\mathfrak{L}_w(\mathsf{D},G)\ni B\mapsto |q_{\phi,v}(B)| \mid (\phi,v)\in G^*\times\mathsf{D}\},$$

where $q_{\phi,v}(B):=\phi(Bv)$ for all $(\phi,v)\in G^*\times\mathsf{D}$ and $B\in\mathfrak{L}_w(\mathsf{D},G)$, while $B(\mathsf{D},G)$ is the space of all bounded operators belonging to $\mathfrak{L}_w(\mathsf{D},G)$. By (2.55) we can define

$$\mathbf{m}_E : H(\mathbb{C})\ni f\mapsto (\mathbf{I}_{\mathbb{C}}^E(f)\upharpoonright\mathsf{D})\in F_w^{\mathsf{D}}.$$

Moreover, by (2.54), with the notations of 1.9, the operator $\mathbf{m}_E\upharpoonright H(\mathbb{C};K)$ is continuous for all compact K. Therefore as a corollary of the general result in Proposition 5(ii), n° 4, §4, Ch. 2 of [TVS] about locally convex final topologies, so in particular for the inductive limit topology, we deduce that $\mathbf{m}_E$ is continuous on $H(\mathbb{C})$, i.e.

$$\mathbf{m}_E \text{ is a vector measure on } \mathbb{C} \text{ with values in } F_w^{\mathsf{D}}.$$

Here, by generalizing to the complex case Definition 1, n° 1, §2, Ch. 6 of [INT], a *vector measure* on a locally compact space X with values in a complex Hausdorff locally convex space Y is any $\mathbb{C}$-linear continuous map $\mathbf{m}:H(X)\to Y$. Furthermore, for all $(\phi,v)\in G^*\times\mathsf{D}$,

$$q_{\phi,v}\circ\mathbf{m}_E = \psi_{\phi,v}\circ\mathbf{I}_{\mathbb{C}}^E\upharpoonright H(\mathbb{C}) = \mathbf{I}_{\mathbb{C}}^{\psi_{\phi,v}\circ E}\upharpoonright H(\mathbb{C}) \quad \text{by (1.3).}$$

Hence

$$\mathfrak{L}_1(\mathbb{C};q_{\phi,v}\circ\mathbf{m}_E)=\mathfrak{L}_1(\mathbb{C};\psi_{\phi,v}\circ E),$$

where the left hand side is understood in the sense of Ch. 4 of [INT], while the right hand side is in the standard sense, and for all $f\in\mathfrak{L}_1(\mathbb{C};q_{\phi,v}\circ\mathbf{m}_E)$,

$$\int f(\lambda)\,d(q_{\phi,v}\circ\mathbf{m}_E)(\lambda)=\int f(\lambda)\,d(\psi_{\phi,v}\circ E)(\lambda) \tag{2.56}$$

Finally, assume that D is dense. Then for all $f\in\operatorname{Bor}(\operatorname{supp}E)$ such that $\operatorname{Dom}(f(E))=\mathsf{D}$ by (1.6) we have

$$f(E)\in F_w^{\mathsf{D}},$$

and by Theorem 18.2.11 of [DS] for all $(\phi,v)\in G^*\times\mathsf{D}$ we have $f\in\mathfrak{L}_1(\mathbb{C};\psi_{\phi,v}\circ E)$ and

$$q_{\phi,v}(f(E))=\int f(\lambda)\,d(\psi_{\phi,v}\circ E)(\lambda). \tag{2.57}$$

Therefore by adopting the definitions in n° 2, §2, Ch. 6 of [INT], we deduce by (2.56) that each $f\in\operatorname{Bor}(\operatorname{supp}E)$ such that $\operatorname{Dom}(f(E))=\mathsf{D}$ is *essentially integrable for* $\mathbf{m}_E$ and

$$f(E)=\int f(\lambda)\,d\mathbf{m}_E(\lambda).$$

Here $\int f(\lambda)\,d\mathbf{m}_E(\lambda)$ is the *integral of f with respect to* $\mathbf{m}_E$. Thus if R is an unbounded scalar type spectral operator in G, then for all $f\in\operatorname{Bor}(\sigma(R))$ such that $\operatorname{Dom}(f(R))=\mathsf{D}$,

f is essentially integrable for $\mathbf{m}_E$ and

$$f(R) = \int f(\lambda)\, d\mathbf{m}_E(\lambda).$$

2.5. Generalization of the Newton–Leibniz formula. In this section we shall apply the results of the previous one to prove Newton–Leibniz formulas for integration with respect to the $\sigma(B(G),\mathcal{N})$-topology when $\mathcal{N}$ is an E-appropriate set with the duality property, for integration with respect to the sigma-weak topology, and for integration with respect to the weak operator topology.

COROLLARY 2.33 ($\sigma(B(G),\mathcal{N})$-Newton–Leibniz formula 1). *Let R be a possibly unbounded scalar type spectral operator in G, U an open neighborhood of $\sigma(R)$, $S : U \to \mathbb{C}$ an analytic map, and $\mathcal{N}$ an E-appropriate set with the duality property. Assume that $S : U \to \mathbb{C}$ is an analytic map and there is $L > 0$ such that $]-L, L[\cdot U \subseteq U$ and*

(1) $\widetilde{S}_t \in \mathfrak{L}_E^\infty(\sigma(R))$, $\widetilde{\left(\frac{dS}{d\lambda}\right)_t} \in \mathfrak{L}_E^\infty(\sigma(R))$ *for all* $t \in]-L, L[$;
(2) $\int^* \|\widetilde{\left(\frac{dS}{d\lambda}\right)_t}\|_\infty^E\, dt < \infty$ *(the upper integral is with respect to the Lebesgue measure on* $]-L, L[$*);*
(3) *for all $\omega \in \mathcal{N}$ the map $]-L, L[\ni t \mapsto \omega\left(\frac{dS}{d\lambda}(tR)\right) \in \mathbb{C}$ is Lebesgue measurable.*

Then for all $u_1, u_2 \in]-L, L[$,

$$R \int_{u_1}^{u_2} \frac{dS}{d\lambda}(tR)\, dt = S(u_2 R) - S(u_1 R) \in B(G).$$

Here the integral is the weak integral of the map $[u_1, u_2] \ni t \mapsto \frac{dS}{d\lambda}(tR) \in B(G)$ with respect to the Lebesgue measure on $[u_1, u_2]$ and the $\sigma(B(G),\mathcal{N})$-topology. Moreover, if $\mathcal{N}$ is an E-appropriate set with the isometric duality property and $M := \sup_{\sigma \in \mathcal{B}(\mathbb{C})} \|E(\sigma)\|_{B(G)}$ then

$$\left\| \int_{u_1}^{u_2} \frac{dS}{d\lambda}(tR)\, dt \right\|_{B(G)} \le 4M \int_{[u_1,u_2]}^* \left\| \widetilde{\left(\frac{dS}{d\lambda}\right)_t} \right\|_\infty^E dt. \tag{2.58}$$

Proof. Let μ be the Lebesgue measure on $[u_1, u_2]$. By (1.42), the hypotheses, and Theorem 18.2.11(c) of [DS] we have

(a) $(\forall t \in [u_1, u_2])(S(tR) \in B(G))$;
(b) $(\forall t \in [u_1, u_2])\left(\frac{dS}{d\lambda}(tR) \in B(G)\right)$;
(c) $([u_1, u_2] \ni t \mapsto \|\frac{dS}{d\lambda}(tR)\|_{B(G)}) \in \mathfrak{F}_1([u_1, u_2]; \mu)$,

So by hypothesis (3), (c) and Theorem 2.2 the map

$$[u_1, u_2] \ni t \mapsto \frac{dS}{d\lambda}(tR) \in \langle B(G), \sigma(B(G), \mathcal{N})\rangle \tag{2.59}$$

is scalarly essentially $(\mu, B(G))$-integrable, and if $\mathcal{N}$ is an E-appropriate set with the isometric duality property then its weak integral satisfies (2.58).

This means that, except for (2.43), all the hypotheses of Theorem 2.25 hold for $X := [u_1, u_2]$, $h := (S_{u_2} - S_{u_1}) \upharpoonright \sigma(R)$, $g : \sigma(R) \ni \lambda \mapsto \lambda \in \mathbb{C}$ and the map $[u_1, u_2] \ni t \mapsto f_t := \left(\frac{dS}{d\lambda}\right)_t \upharpoonright \sigma(R)$.

Next let $\sigma \in \mathcal{B}(\mathbb{C})$ be bounded, so by Key Lemma 1.7, $R_\sigma \upharpoonright G_\sigma$ is a scalar type spectral operator such that $R_\sigma \upharpoonright G_\sigma \in B(G_\sigma)$, and by (1.14), U is an open neighborhood of $\sigma(R_\sigma \upharpoonright G_\sigma)$. Thus we can apply Theorem 1.21(3) to the Banach space G_σ, the analytic map S and the operator $R_\sigma \upharpoonright G_\sigma$. In particular, the map $[u_1, u_2] \ni t \mapsto \frac{dS}{d\lambda}(t(R_\sigma \upharpoonright G_\sigma)) \in B(G_\sigma)$ is Lebesgue integrable in the $\|\cdot\|_{B(G_\sigma)}$-topology, that is, in the sense of Definition 2, n° 4, §3, Ch. IV of [INT]. By Lemma 2.17, $\xi_\sigma \in B(B(G_\sigma), B(G))$, so

$$\mathcal{N}_\sigma \subset B(G_\sigma)^*.$$

Therefore by using Theorem 1, IV.35 of [INT] we deduce that for all $\omega_\sigma \in \mathcal{N}_\sigma$ the map $[u_1, u_2] \ni t \mapsto \omega_\sigma\big(\frac{dS}{d\lambda}(t(R_\sigma \upharpoonright G_\sigma))\big) \in \mathbb{C})$ is Lebesgue integrable, and for all $\omega_\sigma \in \mathcal{N}_\sigma$,

$$\int_{u_1}^{u_2} \omega_\sigma\bigg(\frac{dS}{d\lambda}(t(R_\sigma \upharpoonright G_\sigma))\bigg)\,dt = \omega_\sigma\bigg(\oint_{u_1}^{u_2} \frac{dS}{d\lambda}(t(R_\sigma \upharpoonright G_\sigma))\,dt\bigg).$$

Thus $[u_1, u_2] \ni t \mapsto \frac{dS}{d\lambda}(t(R_\sigma \upharpoonright G_\sigma)) \in \langle B(G_\sigma), \sigma(B(G_\sigma), \mathcal{N}_\sigma)\rangle$ is scalarly essentially $(\mu, B(G_\sigma))$-integrable and its weak integral is such that

$$\int_{u_1}^{u_2} \frac{dS}{d\lambda}(t(R_\sigma \upharpoonright G_\sigma))\,dt = \oint_{u_1}^{u_2} \frac{dS}{d\lambda}(t(R_\sigma \upharpoonright G_\sigma))\,dt. \tag{2.60}$$

Moreover, by Theorem 1.21(3),

$$(R_\sigma \upharpoonright G_\sigma) \oint_{u_1}^{u_2} \frac{dS}{d\lambda}(t(R_\sigma \upharpoonright G_\sigma))\,dt = S(u_2(R_\sigma \upharpoonright G_\sigma)) - S(u_1(R_\sigma \upharpoonright G_\sigma)).$$

Thus by (2.60),

$$(R_\sigma \upharpoonright G_\sigma) \int_{u_1}^{u_2} \frac{dS}{d\lambda}(t(R_\sigma \upharpoonright G_\sigma))\,dt = S(u_2(R_\sigma \upharpoonright G_\sigma)) - S(u_1(R_\sigma \upharpoonright G_\sigma)). \tag{2.61}$$

This implies exactly the hypothesis (2.43) of Theorem 2.25, by choosing for example $\sigma_n := B_n(\mathbf{0})$ for all $n \in \mathbb{N}$. Therefore by Theorem 2.25 we obtain the statement. ■

Corollary 2.34 ($\sigma(B(G), \mathcal{N})$-Newton–Leibniz formula 2). *Let R be a possibly unbounded scalar type spectral operator in G, U an open neighborhood of $\sigma(R)$, $S : U \to \mathbb{C}$ an analytic map, and $\mathcal{N}$ an E-appropriate set with the duality property. Assume that there exists $L > 0$ such that $]-L, L[\cdot U \subseteq U$ and for all $t \in]-L, L[$, $\widetilde{S}_t \in \mathfrak{L}_E^\infty(\sigma(R))$ and there exists $K_0 \subset]-L, L[$ such that $\complement K_0$ is a Lebesgue negligible set and for all $t \in K_0$, $\widetilde{\big(\frac{dS}{d\lambda}\big)_t} \in \mathfrak{L}_E^\infty(\sigma(R))$. Moreover, suppose that*

- *there is $F :]-L, L[\to B(G)$ extending $K_0 \ni t \mapsto \frac{dS}{d\lambda}(tR) \in B(G)$ such that*

$$\int^* \|F(t)\|_{B(G)}\,dt < \infty$$

(the upper integral is with respect to the Lebesgue measure on $]-L, L[$),
- *for all $\omega \in \mathcal{N}$ the map $]-L, L[\ni t \mapsto \omega(F(t)) \in \mathbb{C}$ is Lebesgue measurable.*

Then for all $u_1, u_2 \in]-L, L[$,

$$R \int_{u_1}^{u_2} \frac{dS}{d\lambda}(tR)\,dt = S(u_2 R) - S(u_1 R) \in B(G).$$

Here the integral is the weak integral of the map $[u_1, u_2] \ni t \mapsto \frac{dS}{d\lambda}(tR) \in B(G)$ *with respect to the Lebesgue measure on* $[u_1, u_2]$ *and the* $\sigma(B(G), \mathcal{N})$*-topology. Moreover, if* $\mathcal{N}$ *is an* E*-appropriate set with the isometric duality property then*

$$\left\| \int_{u_1}^{u_2} \frac{dS}{d\lambda}(tR)\, dt \right\|_{B(G)} \leq \int_{[u_1,u_2]}^{*} \left\| \frac{dS}{d\lambda}(tR) \right\|_{B(G)} dt.$$

Proof. By Theorem 2.2 and (2.12),

$$[u_1, u_2] \ni t \mapsto \frac{dS}{d\lambda}(tR) \in \langle B(G), \sigma(B(G), \mathcal{N}) \rangle$$

is scalarly essentially $(\mu, B(G))$-integrable, and if $\mathcal{N}$ is an E-appropriate set with the isometric duality property its weak integral satisfies the inequality in the statement by (2.11). Thus the proof goes as that of Corollary 2.33. ∎

Corollary 2.35 (Sigma-Weak Newton–Leibniz formula). *The statement of Corollary* 2.33 (*respectively Corollary* 2.34) *holds if* G *is a complex Hilbert space and* $\mathcal{N}$ *is replaced by* $\mathcal{N}_{\mathrm{pd}}(G)$.

Proof. By Remark 2.12, $\mathcal{N}_{\mathrm{pd}}(G)$ is an E-appropriate set with the isometric duality property, which yields the statement by Corollary 2.33 (respectively Corollary 2.34). ∎

Corollary 2.36 (Weak Newton–Leibniz formula). *The statement of Corollary* 2.33 (*respectively Corollary* 2.34) *holds if* G *is a reflexive complex Banach space and* $\mathcal{N}$ *is replaced by* $\mathcal{N}_{\mathrm{st}}(G)$.

Proof. By using Corollary 2.6 instead of Theorem 2.2, we obtain (2.59) and (2.58) by replacing $\mathcal{N}$ with $\mathcal{N}_{\mathrm{st}}(G)$. Then the proof proceeds similarly to that of Corollary 2.33 (respectively Corollary 2.34). ∎

References

[GT] N. Bourbaki, *General Topology, Parts 1, 2*, Springer, 1989.

[TVS] —, *Topological Vector Spaces*, Springer, 1989.

[FVR] —, *Functions of a Real Variable*, Springer, 2003.

[INT] —, *Integration I, II*, Springer, 2003.

[BR] O. Bratteli and D. W. Robinson, *Operator Algebras and Quantum Statistical Mechanics I. C^*- and W^*-Algebras, Symmetry Groups, Decomposition of States*, 2nd ed., Springer, 1987.

[Coj] P. A. Cojuhari and A. M. Gomilko, *On the characterization of scalar type spectral operators*, Studia Math. 184 (2008), 121–132.

[Dal] H. G. Dales, *Banach Algebras and Automatic Continuity*, London Math. Soc. Monogr. 24, Cambridge Univ. Press, 2000.

[Dav] E. B. Davies, *One-Parameter Semigroups*, London Math. Soc. Monogr. 15, Academic Press, 1980.

[deL] R. deLaubenfels, *Scalar-type spectral operators and holomorphic semigroups*, Semigroup Forum 33 (1986), 257–263.

[DU] J. Diestel and J. J. Uhl, Jr., *Vector Measures*, Amer. Math. Soc., 1977.

[Dieu] J. Dieudonné, *Foundations of Modern Analysis 1*, Academic Press, 1969.

[Din1] N. Dinculeanu, *Integration on Locally Compact Spaces*, Noordhoff, 1974.

[Din2] —, *Vector Integration and Stochastic Integration in Banach Spaces*, Wiley-Interscience, 2000.

[Dod] P. G. Dodds and B. de Pagter, *Algebras of unbounded scalar-type spectral operators*, Pacific J. Math. 130 (1987), 41–74.

[Dow] H. R. Dowson, *Spectral Theory of Linear Operators*, Academic Press, 1978.

[DS] N. Dunford and J. T. Schwartz, *Linear Operators, Vols. 1, 2, 3*, Wiley-Interscience, 1988.

[KR] R. V. Kadison and J. R. Ringrose, *Fundamentals of the Theory of Operator Algebras, Vols. 1, 2*, Amer. Math. Soc., 1997.

[Kat] T. Kato, *Perturbation Theory for Linear Operators*, Springer, 1980.

[Jar] H. Jarchow, *Locally Convex Spaces*, B.G. Teubner, 1981.

[LN] K. B. Laursen and M. M. Neumann, *An Introduction to Local Spectral Theory*, London Math. Soc. Monogr. 20, Cambridge Univ. Press, 2000.

[Mark1] M. V. Markin, *On the Carleman classes of vectors of a scalar type spectral operator*, Int. J. Math. Math. Sci. 2004, no. 57–60, 3219–3235.

[Mark2] —, *A characterization of the generators of analytic C_0-semigroups in the class of scalar type spectral operators*, Abstr. Appl. Anal. 2004, no. 12, 1007–1018.

[Mark3] —, *On scalar type spectral operators, infinite differentiable and Gevrey ultradifferentiable C_0-semigroups*, Int. J. Math. Math. Sci. 2004, no. 45–48, 2401–2422.

[Oka] S. Okada, *Spectra of scalar-type spectral operators and Schauder decompositions*, Math. Nachr. 139 (1988), 167–174.

[Pal] T. W. Palmer, *Banach Algebras and the General Theory of $*$Algebras, Vol. 1*, Cambridge Univ. Press, 1994.

[Ric1] W. Ricker, *A spectral mapping theorem for scalar-type spectral operators in locally convex spaces*, Integral Equations Operator Theory 8 (1985), 276–293.

[Ric2] —, *Scalar-type spectral operators and $C(\Omega)$-operational calculi*, ibid. 13 (1990), 901–905.

[Rud] W. Rudin, *Functional Analysis*, McGraw-Hill, 1973.

[Schw] L. Schwartz, *Analyse, I–IV*, Hermann, 1993.

[Tak] M. Takesaki, *Theory of Operator Algebras I*, Springer, 2001.

[Yos] K. Yosida, *Functional Analysis*, Springer, 1980.