Contents

Abstract

Two semigroups are said to be *distinct* if they are neither isomorphic nor anti-isomorphic. Although there exist 1373 distinct monoids of order six, only two are known to be non-finitely based. In the present dissertation, the finite basis property of the other 1371 distinct monoids of order six is verified. Since it is long established that all semigroups of order five or less are finitely based, the two known non-finitely based monoids of order six are the only examples of minimal order.

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1. Introduction

1.1. The finite basis problem for small semigroups. A semigroup is *finitely based* if there exists a finite set of its identities from which all its other identities can be deduced. In 1969, Perkins [28] published the first two examples of non-finitely based finite semigroups, the first of which is the monoid B_2^1 of order six obtained by adjoining an identity element to the Brandt semigroup

$$B_2 = \langle\, a, b \mid a^2 = b^2 = 0,\ aba = a,\ bab = b \,\rangle$$

of order five. The discovery of a non-finitely based semigroup with only six elements focused much attention upon the finite basis problem for semigroups of order five or less. This problem was explicitly raised by Tarski [35] in 1966 and attracted the interest of Bol'bot [4], Edmunds [8, 9], Karnofsky [14], Tishchenko [36], and Trahtman [37]. A solution to this problem was eventually completed by Trahtman [38, 39] in the early 1980s and published in 1991 [41].

Theorem 1.1. *Every semigroup of order five or less is finitely based.*

A more complete historical account of the proof of Theorem 1.1 can be found in the survey of Shevrin and Volkov [34, §10].

By the late 1980s, two more non-finitely based semigroups of order six were discovered: A_2^1 and A_2^g. The semigroup A_2^1, due to Sapir [33] and Trahtman [40] independently, is the monoid obtained by adjoining an identity element to the 0-simple semigroup

$$A_2 = \langle\, a, b \mid a^2 = aba = a,\ b^2 = 0,\ bab = b \,\rangle$$

of order five. The semigroup A_2^g, due to Volkov [42], is obtained by adjoining a new element g to the semigroup A_2 with multiplication defined by $g^2 = 0$ and $gx = xg = g$ for all $x \in A_2$ (1).

Two semigroups are said to be *distinct* if they are neither isomorphic nor anti-isomorphic. It follows from Theorem 1.1 that the semigroups A_2^g, A_2^1, and B_2^1 of order six are minimal with respect to being non-finitely based. In fact, up to the present, these three non-finitely based semigroups are the only known minimal examples. Since there exist 15973 distinct semigroups of order six [29] among which 1373 are monoids [6], it is very natural to question the existence of non-finitely based semigroups or monoids of order six that are distinct from A_2^g, A_2^1, and B_2^1. The objective of the present dissertation is to provide an answer to this question for the case of monoids.

(1) The non-finite basis property of the semigroup A_2^g also follows from Mashevitskiĭ [26]. Refer to Lee and Volkov [24] for an easily described basis of the semigroup A_2^g.

Main Theorem. *Every monoid of order six distinct from* A_2^1 *and* B_2^1 *is finitely based. Consequently, up to isomorphism and anti-isomorphism,* A_2^1 *and* B_2^1 *are the only non-finitely based monoids of minimal order.*

The ultimate goal of completely identifying all minimal non-finitely based semigroups clearly requires the solution of the finite basis problem for all distinct nonunital semigroups of order six. This is a nontrivial and potentially daunting task since the number of such semigroups is $15973 - 1373 = 14600$.

1.2. Organization. The set of all chapters in the present dissertation under the prerequisite relation constitutes the directed tree in Figure 1. Every chapter that follows Chapter 2 can be read independently.

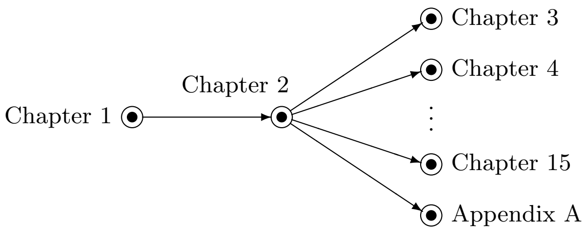

Fig. 1. Prerequisites of chapters

Notation and background information are given in Chapter 2. An outline of the proof of the main theorem is given in Chapter 3, while the finer details are deferred to Chapters 4–15. For completeness, multiplication tables of all 1373 distinct monoids of order six are listed in Appendix A. Each finitely based monoid is associated with a result from Chapter 3 that guarantees its finite basis property.

2. Preliminaries

2.1. Letters and words. Let $\mathscr{X}$ be a countably infinite alphabet. For any subset $\mathscr{Y}$ of $\mathscr{X}$, let $\mathscr{Y}^+$ and $\mathscr{Y}^*$ denote the free semigroup and free monoid over $\mathscr{Y}$ respectively. Elements of $\mathscr{X}$ are referred to as *letters* and elements of $\mathscr{X}^+$ and $\mathscr{X}^*$ are referred to as *words*. Let x be any letter and $\mathbf{w}$ be any word. Then

- the leftmost letter of $\mathbf{w}$ is denoted by $\lambda(\mathbf{w})$;
- the *content* of $\mathbf{w}$, denoted by $\mathsf{con}(\mathbf{w})$, is the set of letters occurring in $\mathbf{w}$;
- the number of occurrences of x in $\mathbf{w}$ is denoted by $\mathsf{occ}(x, \mathbf{w})$;
- x is *simple* in $\mathbf{w}$ if $\mathsf{occ}(x, \mathbf{w}) = 1$;
- the set of simple letters of $\mathbf{w}$ is denoted by $\mathsf{sim}(\mathbf{w})$;
- $\mathbf{w}$ is *simple* if each of its letters is simple in it, that is, $\mathsf{sim}(\mathbf{w}) = \mathsf{con}(\mathbf{w})$;
- the *initial* of $\mathbf{w}$, denoted by $\mathsf{ini}(\mathbf{w})$, is the simple word obtained from $\mathbf{w}$ by retaining the first occurrence of each letter.

Define a relation $\stackrel{\circ}{=}$ on $\mathscr{X}^+$ by $\mathbf{u} \stackrel{\circ}{=} \mathbf{v}$ if $\mathsf{occ}(x, \mathbf{u}) = \mathsf{occ}(x, \mathbf{v})$ for all $x \in \mathscr{X}$. Equivalently, $\mathbf{u} \stackrel{\circ}{=} \mathbf{v}$ if and only if $\mathbf{u}$ and $\mathbf{v}$ are the same word up to letter rearrangement.

2.2. Identities for some semigroups. An identity $\mathbf{u} \approx \mathbf{v}$ is *nontrivial* if $\mathbf{u}$ and $\mathbf{v}$ are distinct words. For any identity $\mathbf{u} \approx \mathbf{v}$, denote by $\mathbf{u} \overset{*}{\approx} \mathbf{v}$ the system of all nontrivial identities that can be obtained from $\mathbf{u} \approx \mathbf{v}$ by removing all occurrences of some letters. For example, $xyxzx \overset{*}{\approx} xyxz$ denotes the system

$$\{xyxzx \approx xyxz,\ xyx^2 \approx xyx,\ x^2zx \approx x^2z,\ x^3 \approx x^2\}.$$

A semigroup S *satisfies* an identity $\mathbf{u} \approx \mathbf{v}$ if $\mathbf{u}\varphi = \mathbf{v}\varphi$ for any substitution φ from $\mathcal{X}$ into S. For any set Σ of identities, the variety *defined* by Σ is the class of all semigroups that satisfy all identities in Σ; in this case, Σ is a *basis* for the variety. A variety is *finitely based* if it possesses a finite basis. All varieties in the present dissertation are varieties of semigroups. Refer to Burris and Sankappanavar [5], Shevrin and Volkov [34], and Volkov [44] for any concept of universal algebra that appears here but is undefined.

For any semigroup S, let S^1 be the monoid obtained from S by adjoining an identity element.

Lemma 2.1 ([2, Lemma 7.1.1]). *Let* $\mathbf{M}$ *be any variety generated by a monoid. Suppose that* S *is any nonunital semigroup in the variety* $\mathbf{M}$. *Then the monoid* S^1 *belongs to the variety* $\mathbf{M}$.

The left-zero semigroup L_2 of order two, the null semigroup N_2 of order two, and the cyclic group $\mathbb{Z}_n$ of order n can be given by the following presentations:

$$\begin{aligned} L_2 &= \langle\, a, b \mid a^2 = ab = a,\, b^2 = ba = b \,\rangle, \\ N_2 &= \langle\, a \mid a^2 = 0 \,\rangle, \\ \mathbb{Z}_n &= \langle\, a \mid a^n = 1 \,\rangle. \end{aligned}$$

Lemma 2.2. *Let* $\mathbf{u} \approx \mathbf{v}$ *be any identity. Then*

(i) L_2^1 *satisfies* $\mathbf{u} \approx \mathbf{v}$ *if and only if* $\mathsf{ini}(\mathbf{u}) = \mathsf{ini}(\mathbf{v})$;
(ii) N_2^1 *satisfies* $\mathbf{u} \approx \mathbf{v}$ *if and only if* $\mathsf{con}(\mathbf{u}) = \mathsf{con}(\mathbf{v})$ *and* $\mathsf{sim}(\mathbf{u}) = \mathsf{sim}(\mathbf{v})$;
(iii) $\mathbb{Z}_n$ *satisfies* $\mathbf{u} \approx \mathbf{v}$ *if and only if* $\mathsf{occ}(x, \mathbf{u}) \equiv \mathsf{occ}(x, \mathbf{v}) \pmod{n}$ *for all* $x \in \mathcal{X}$.

Proof. These results are well known and easy to verify. ∎

Lemma 2.3. *Let* M *be any noncommutative monoid such that the monoid* N_2^1 *is embeddable in* M *and let* $\mathbf{u} \approx \mathbf{v}$ *be any identity satisfied by* M. *If either* $\mathbf{u}$ *or* $\mathbf{v}$ *is a simple word, then the identity* $\mathbf{u} \approx \mathbf{v}$ *is trivial.*

Proof. By symmetry, it suffices to assume that $\mathbf{u}$ is a simple word. By assumption, the identity $\mathbf{u} \approx \mathbf{v}$ is also satisfied by the monoid N_2^1. Therefore the word $\mathbf{v}$ is simple by Lemma 2.2(ii). Since the monoid M is noncommutative, it is easy to see that the words $\mathbf{u}$ and $\mathbf{v}$ are identical. ∎

2.3. Exclusion identities. Let S be any semigroup in a variety $\mathbf{V}$. Suppose that an identity $\mathbf{u} \approx \mathbf{v}$ is satisfied by any subvariety of $\mathbf{V}$ that does not contain the semigroup S. Then $\mathbf{u} \approx \mathbf{v}$ is an *exclusion identity* for S in $\mathbf{V}$.

Lemma 2.4. *Let* S *be any semigroup in a periodic variety* $\mathbf{V}$ *such that the monoid* S^1 *also belongs to* $\mathbf{V}$. *Suppose that* $\mathbf{u} \approx \mathbf{v}$ *is an exclusion identity for* S *in* $\mathbf{V}$. *Then any* *subvariety of* $\mathbf{V}$ *generated by a monoid that does not contain the monoid* S^1 *must satisfy the identity* $\mathbf{u} \approx \mathbf{v}$.

Proof. By assumption, the variety $\mathbf{V}$ satisfies the identity $x^{2n} \approx x^n$ for some $n \geq 1$. Choose any $h \in \mathcal{X} \backslash \mathsf{con}(\mathbf{uv})$. Denote by ψ the substitution $x \mapsto h^n x h^n$ for all $x \in \mathsf{con}(\mathbf{uv})$. Since $\mathbf{u} \approx \mathbf{v}$ is an exclusion identity for S in $\mathbf{V}$, it follows from Lee [18, Theorem 2] that $\mathbf{u}\psi \approx \mathbf{v}\psi$ is an exclusion identity for S^1 in $\mathbf{V}$. Therefore any subvariety $\mathbf{M}$ of $\mathbf{V}$ generated by a monoid that does not contain the monoid S^1 must satisfy the identity $\mathbf{u}\psi \approx \mathbf{v}\psi$; it is easy to see that the variety $\mathbf{M}$ also satisfies the identity $\mathbf{u} \approx \mathbf{v}$. ∎

2.4. Precedence. Let x and y be any letters of a word $\mathbf{w}$. Then

- the number of x in $\mathbf{w}$ that precedes the first y in $\mathbf{w}$ is denoted by $\mathsf{occ}(x, y, \mathbf{w})$;
- write $x \ll_{\mathbf{w}} y$ if every x in $\mathbf{w}$ precedes every y in $\mathbf{w}$.

An identity $\mathbf{u} \approx \mathbf{v}$ is said to *preserve complete precedence* if $\mathsf{con}(\mathbf{u}) = \mathsf{con}(\mathbf{v})$ and the following condition holds: $x \ll_{\mathbf{u}} y$ if and only if $x \ll_{\mathbf{v}} y$ for any $x, y \in \mathcal{X}$.

Lemma 2.5 ([8, Lemma 4.2]). *Let* $\mathbf{pur}$ *and* $\mathbf{pu'r}$ *be any words such that*

(a) $\mathbf{u} \mathrel{\mathring{=}} \mathbf{u}'$;
(b) $\mathsf{occ}(x, \mathbf{pur}) \geq 2$ *for all* $x \in \mathsf{con}(\mathbf{u})$;
(c) $\mathbf{pur} \approx \mathbf{pu'r}$ *preserves complete precedence.*

Then the identity system $\{xhytxy \overset{*}{\approx} xhytyx,\ xyhxty \overset{*}{\approx} yxhxty\}$ *implies the identity* $\mathbf{pur} \approx \mathbf{pu'r}$.

3. Proof of the main theorem

The following nine sufficient conditions for the finite basis property are required:

Condition 1 (Pollák [30, Theorem 1]). *Any semigroup that satisfies the identity*

$$xyx \approx x^2y$$

is finitely based.

Condition 2 (Rasin [31]). *Any finite orthodox completely regular semigroup is finitely based. Specifically, any finite semigroup that satisfies the identities*

$$x^{13} \approx x, \quad x^{12}y^{12}x^{12}y^{12} \approx x^{12}y^{12}$$

is finitely based.

Condition 3 (Edmunds et al. [10, Theorem 7.2]). *Any monoid that satisfies the identities*

$$x^2y \approx yx^2, \quad x^3yx \approx xyx, \quad xyxy \approx x^2y^2$$

is finitely based.

Condition 4. *Any monoid that satisfies the identities*

$$x^2yx \approx xyx, \quad xyx^2 \approx xyx, \quad x^2y^2 \approx y^2x^2$$

is finitely based.

Condition 5 (Lee [21, Corollary 3.4]). *Any monoid that satisfies the identities*

$$xyxy \approx xy^2x, \quad xyxzx \approx xyzx$$

is finitely based.

Condition 6. *Any monoid that satisfies the identities*

$$x^3yx \approx xyx, \quad xy^2x \approx yx^2y, \quad xyxzx \approx x^2yzx$$

is finitely based.

Condition 7 (Luo and Zhang [25, Theorem 1.1 and Corollary 4.6]). *Any monoid that satisfies the identities*

$$x^7yx \approx xyx, \quad xyxy \approx x^2y^2, \quad xyxzx \approx x^2yzx$$

is finitely based.

Condition 8. *Any monoid that satisfies the identities*

$$xyx^3 \approx xyx, \quad xyxzx \approx xyx^2z, \quad xhytxy \approx xhytyx$$

but violates the identity

$$xyxy \approx x^2y^2$$

is finitely based.

Condition 9. *Any semigroup that satisfies the identities*

$$x^4 \approx x^2, \quad x^3yx \approx xyx, \quad xyx^2 \approx x^3y, \quad xyxy \approx xy^2x$$

but violates both of the identities

$$xyxy \approx x^2y^2, \quad xy^3x \approx xyx$$

is finitely based.

Conditions 4, 6, 8, and 9 are established in Chapters 4–7.

With the aid of a computer, it is routine to show that 1360 of the 1373 distinct monoids of order six are finitely based by Conditions 1–9 or their dual conditions. The remaining 13 sporadic cases are the monoids $\mathcal{A}, \mathcal{B}, \ldots, \mathcal{K}, B_2^1$, and A_2^1 given by the following multiplication tables:

$\mathcal{A}$	1	2	3	4	5	6
1	1	1	1	1	1	1
2	1	1	1	1	1	2
3	1	1	1	1	2	3
4	1	2	3	4	3	4
5	1	1	1	1	2	5
6	1	2	3	4	5	6

$\mathcal{B}$	1	2	3	4	5	6
1	1	1	1	1	1	1
2	1	1	1	1	1	2
3	1	1	1	1	2	3
4	1	2	3	4	4	4
5	1	2	3	4	4	5
6	1	2	3	4	5	6

$\mathcal{C}$	1	2	3	4	5	6
1	1	1	1	1	1	1
2	1	1	1	1	1	2
3	1	1	2	1	4	3
4	4	4	4	4	4	4
5	5	5	5	5	5	5
6	1	2	3	4	5	6

$\mathcal{D}$	1	2	3	4	5	6
1	1	1	1	1	1	1
2	1	1	1	1	2	2
3	1	1	1	1	3	3
4	4	4	4	4	4	4
5	1	2	3	4	5	6
6	1	3	2	4	6	5

$\mathcal{E}$	1	2	3	4	5	6
1	1	1	1	1	1	1
2	1	1	1	1	2	2
3	1	1	2	2	3	3
4	1	1	2	2	4	4
5	1	2	3	4	5	6
6	1	2	4	3	6	5

$\mathcal{F}$	1	2	3	4	5	6
1	1	1	1	1	1	1
2	1	1	1	1	2	2
3	1	2	3	3	3	3
4	1	2	4	4	4	4
5	1	2	3	4	5	6
6	1	2	4	3	6	5

$\mathcal{G}$	1	2	3	4	5	6
1	1	1	1	1	1	1
2	1	1	1	1	2	3
3	1	1	1	1	3	2
4	1	2	3	4	4	4
5	1	2	3	4	5	6
6	1	2	3	4	6	5

$\mathcal{H}$	1	2	3	4	5	6
1	1	1	1	1	1	1
2	1	1	1	1	2	3
3	1	1	1	1	3	2
4	4	4	4	4	4	4
5	1	2	3	4	5	6
6	1	2	3	4	6	5

$\mathcal{I}$	1	2	3	4	5	6
1	1	1	1	1	1	1
2	1	1	1	1	2	3
3	1	1	1	3	3	3
4	1	2	3	4	4	4
5	1	2	3	4	5	6
6	1	2	3	6	6	6

$\mathcal{J}$	1	2	3	4	5	6
1	1	1	1	1	1	1
2	1	1	1	1	2	3
3	3	3	3	3	3	3
4	1	2	3	4	4	1
5	1	2	3	4	5	6
6	6	6	6	6	6	6

$\mathcal{K}$	1	2	3	4	5	6
1	1	1	1	1	1	1
2	1	1	1	1	2	3
3	3	3	3	3	3	3
4	1	2	3	4	4	3
5	1	2	3	4	5	6
6	6	6	6	6	6	6

B_2^1	1	2	3	4	5	6
1	1	1	1	1	1	1
2	1	1	1	2	2	3
3	1	2	3	1	3	1
4	1	1	1	4	4	6
5	1	2	3	4	5	6
6	1	4	6	1	6	1

A_2^1	1	2	3	4	5	6
1	1	1	1	1	1	1
2	1	1	1	2	2	3
3	1	2	3	2	3	3
4	1	1	1	4	4	6
5	1	2	3	4	5	6
6	1	4	6	4	6	6

As mentioned in the introduction, the monoids A_2^1 and B_2^1 are non-finitely based. The monoid $\mathcal{C}$ is known to be finitely based [10, Theorem 3.2]. The finite basis property of the other ten monoids is verified in Chapters 8–15. The proof of the main theorem is thus complete.

Multiplication tables of all 1373 distinct monoids of order six are listed in Appendix A. The 13 sporadic cases above, together with other monoids satisfying certain special properties, are explicitly identified.

Remark 3.1.

(i) The finite basis property of the monoids $\mathcal{A}$ and $\mathcal{E}$ was first announced by Edmunds et al. [10, Section 3].

(ii) Conditions 1–9 are all required in the proof of the main theorem since they are distinguished by the monoids $M_1, \ldots, M_9$ with multiplication tables given below, that is, the monoid M_m satisfies Condition n if and only if $m = n$:

M_1	1	2	3	4	5	6
1	1	1	1	1	1	1
2	1	1	1	1	1	2
3	1	1	1	1	1	3
4	1	1	1	1	1	4
5	1	1	1	1	2	5
6	1	2	3	4	5	6

M_2	1	2	3	4	5	6
1	1	1	1	1	1	1
2	1	2	1	1	1	2
3	1	1	3	4	5	3
4	4	4	4	4	4	4
5	4	4	5	1	3	5
6	1	2	3	4	5	6

M_3	1	2	3	4	5	6
1	1	1	1	1	1	1
2	1	1	1	1	2	2
3	1	1	1	1	3	3
4	1	1	1	1	4	4
5	1	2	3	4	5	6
6	1	2	4	3	6	5

M_4	1	2	3	4	5	6
1	1	1	1	1	1	1
2	1	1	1	1	2	2
3	1	1	1	1	3	3
4	1	1	2	1	2	4
5	1	2	2	4	5	5
6	1	2	3	4	5	6

M_5	1	2	3	4	5	6
1	1	1	1	1	1	1
2	1	1	1	1	2	2
3	1	2	3	4	2	3
4	4	4	4	4	4	4
5	1	1	1	1	5	5
6	1	2	3	4	5	6

M_6	1	2	3	4	5	6
1	1	1	1	1	1	1
2	1	1	1	1	2	2
3	1	2	3	4	2	3
4	1	2	4	3	2	4
5	1	1	1	1	5	5
6	1	2	3	4	5	6

M_7	1	2	3	4	5	6
1	1	1	1	1	1	1
2	1	1	1	1	1	2
3	1	1	3	1	5	3
4	4	4	4	4	4	4
5	1	1	3	1	5	5
6	1	2	3	4	5	6

M_8	1	2	3	4	5	6
1	1	1	1	1	1	1
2	1	1	1	1	1	2
3	1	1	1	1	4	3
4	4	4	4	4	4	4
5	5	5	5	5	5	5
6	1	2	3	4	5	6

M_9	1	2	3	4	5	6
1	1	1	1	1	1	1
2	1	1	1	1	1	2
3	1	1	3	4	5	3
4	4	4	4	4	4	4
5	4	4	5	1	3	5
6	1	2	3	4	5	6

4. On Condition 4

The main aim of the present chapter is to establish the finite basis property of any monoid that satisfies the identities

$$x^2yx \approx xyx, \quad xyx^2 \approx xyx, \quad x^2y^2 \approx y^2x^2. \tag{4.1}$$

The semigroup

$$Q = \langle\, a, b, c \mid a^2 = a,\, ab = b,\, ca = c,\, ac = ba = cb = 0 \,\rangle$$

of order five plays a central role. Denote by $\mathbf{Q}^1$ the variety generated by the monoid Q^1.

Remark 4.1. The semigroup Q has appeared in Almeida's investigation of minimal nonpermutative pseudovarieties [2, Chapter 6].

4.1. A basis for the variety $\mathbf{Q}^1$

Lemma 4.2 ([2, Proposition 6.5.10]). *The following statements on any identity* $\mathbf{u} \approx \mathbf{v}$ *satisfied by the semigroup* Q *are equivalent*:

(a) xy *is a factor of the word* $\mathbf{u}$ *with* $x, y \in \mathsf{sim}(\mathbf{u})$;

(b) xy *is a factor of the word* $\mathbf{v}$ *with* $x, y \in \mathsf{sim}(\mathbf{v})$.

PROPOSITION 4.3. *The identities*

$$x^3 \approx x^2, \quad x^2yx \approx xyx, \quad xyx^2 \approx xyx, \tag{4.2a}$$

$$x^2y^2 \approx y^2x^2 \tag{4.2b}$$

constitute a basis for the variety $\mathbf{Q}^1$.

Proof. It is routine to verify that the variety $\mathbf{Q}^1$ satisfies the identities (4.2). It remains to show that any identity $\mathbf{w} \approx \mathbf{w}'$ satisfied by the variety $\mathbf{Q}^1$ is implied by the identities (4.2). Since the monoid Q^1 is noncommutative and its submonoid $\{0, b, 1\}$ is isomorphic to the monoid N_2^1, it follows from Lemma 2.3 that the identity $\mathbf{w} \approx \mathbf{w}'$ is trivial if either $\mathbf{w}$ or $\mathbf{w}'$ is a simple word. Therefore assume that the words $\mathbf{w}$ and $\mathbf{w}'$ are both nonsimple, whence it is possible to write

$$\mathbf{w} = \prod_{i=1}^{m}(\mathbf{s}_i\mathbf{w}_i) \quad \text{and} \quad \mathbf{w}' = \prod_{i=1}^{m'}(\mathbf{s}'_i\mathbf{w}'_i)$$

where

- the letters of $\mathbf{s}_1 \in \mathcal{X}^*$ and $\mathbf{s}_2, \ldots, \mathbf{s}_m \in \mathcal{X}^+$ are all simple in $\mathbf{w}$;
- the letters of $\mathbf{w}_1, \ldots, \mathbf{w}_{m-1} \in \mathcal{X}^+$ and $\mathbf{w}_m \in \mathcal{X}^*$ are all nonsimple in $\mathbf{w}$;
- the letters of $\mathbf{s}'_1 \in \mathcal{X}^*$ and $\mathbf{s}'_2, \ldots, \mathbf{s}'_{m'} \in \mathcal{X}^+$ are all simple in $\mathbf{w}'$;
- the letters of $\mathbf{w}'_1, \ldots, \mathbf{w}'_{m'-1} \in \mathcal{X}^+$ and $\mathbf{w}'_{m'} \in \mathcal{X}^*$ are all nonsimple in $\mathbf{w}'$.

Since the submonoid $\{0, b, 1\}$ of Q^1 is isomorphic to the monoid N_2^1, it follows from Lemma 2.2(ii) that $\bigcup_{i=1}^{m} \mathsf{con}(\mathbf{s}_i) = \bigcup_{i=1}^{m'} \mathsf{con}(\mathbf{s}'_i)$ and $\bigcup_{i=1}^{m} \mathsf{con}(\mathbf{w}_i) = \bigcup_{i=1}^{m'} \mathsf{con}(\mathbf{w}'_i)$. Further, it follows from Lemma 4.2 that $m = m'$ and $\mathbf{s}_i = \mathbf{s}'_i$ for all i, whence

$$\mathbf{w}' = \prod_{i=1}^{m}(\mathbf{s}_i\mathbf{w}'_i).$$

If $m = 1$, then clearly $\mathsf{con}(\mathbf{w}_1) = \mathsf{con}(\mathbf{w}'_1)$. Therefore assume that $m \geq 2$.

CASE 1: $\mathsf{con}(\mathbf{w}_1) \neq \mathsf{con}(\mathbf{w}'_1)$. Without loss of generality, assume that $x \in \mathsf{con}(\mathbf{w}_1)$ and $x \notin \mathsf{con}(\mathbf{w}'_1)$. Let $z = \lambda(\mathbf{s}_2)$ and let φ_1 be the substitution $h \mapsto 1$ for all $h \in \mathcal{X} \setminus \{x, z\}$. Then the monoid Q^1 satisfies the identity $xy(\mathbf{w}x)\varphi_1 \approx xy(\mathbf{w}'x)\varphi_1$, which is of the form $xyx^pzx^q \approx xyzx^r$ for some $p, q \geq 1$ and $r \geq 2$.

CASE 2: $\mathsf{con}(\mathbf{w}_m) \neq \mathsf{con}(\mathbf{w}'_m)$. By an argument symmetrical to Case 1, the monoid Q^1 satisfies an identity of the form $x^pyx^qzx \approx x^ryzx$ for some $p, q \geq 1$ and $r \geq 2$.

CASE 3: $\mathsf{con}(\mathbf{w}_i) \neq \mathsf{con}(\mathbf{w}'_i)$ for some i such that $1 < i < m$. Without loss of generality, assume that $x \in \mathsf{con}(\mathbf{w}_i)$ and $x \notin \mathsf{con}(\mathbf{w}'_i)$. Let $y = \lambda(\mathbf{s}_i)$ and $z = \lambda(\mathbf{s}_{i+1})$, and let φ_2 be the substitution $h \mapsto 1$ for all $h \in \mathcal{X} \setminus \{x, y, z\}$. Then the monoid Q^1 satisfies the identity $(x\mathbf{w}x)\varphi_2 \approx (x\mathbf{w}'x)\varphi_2$, which is of the form $x^pyx^qzx^r \approx x^syzx^t$ for some $p, q, r, s, t \geq 1$.

The three cases just considered are all impossible since by Lemma 4.2, the monoid Q^1 cannot satisfy any identity of the form $x^pyx^qzx^r \approx x^syzx^t$. Therefore $\mathsf{con}(\mathbf{w}_i) = \mathsf{con}(\mathbf{w}'_i)$ for every $i \in \{1, \ldots, m\}$. Let φ_3 be the substitution $x \mapsto x^2$ for all $x \in \bigcup_{i=1}^{m} \mathsf{con}(\mathbf{w}_i) = \bigcup_{i=1}^{m} \mathsf{con}(\mathbf{w}'_i)$. Then the identities (4.2a) imply the identities $\mathbf{w}\varphi_3 \approx \mathbf{w}$ and $\mathbf{w}'\varphi_3 \approx \mathbf{w}'$.

Further, for each i, since $\mathbf{w}_i\varphi_3$ and $\mathbf{w}_i'\varphi_3$ are products of squares with $\mathsf{con}(\mathbf{w}_i) = \mathsf{con}(\mathbf{w}_i')$, it is easy to see that the identities (4.2) imply the identity $\mathbf{w}_i\varphi_3 \approx \mathbf{w}_i'\varphi_3$. Consequently,

$$\mathbf{w} \overset{(4.2\text{a})}{\approx} \mathbf{w}\varphi_3 = \prod_{i=1}^{m}(\mathbf{s}_i(\mathbf{w}_i\varphi_3)) \overset{(4.2)}{\approx} \prod_{i=1}^{m}(\mathbf{s}_i(\mathbf{w}_i'\varphi_3)) = \mathbf{w}'\varphi_3 \overset{(4.2\text{a})}{\approx} \mathbf{w}',$$

that is, the identities (4.2) imply the identity $\mathbf{w} \approx \mathbf{w}'$. ∎

4.2. An exclusion identity

LEMMA 4.4. *The identity*

$$xyxzx \approx xyzx \tag{4.3}$$

is an exclusion identity for the semigroup Q in the variety $\mathbf{Q}^1$.

Proof. Let $\mathbf{V}$ be any subvariety of $\mathbf{Q}^1$ such that $Q \notin \mathbf{V}$. Then it follows from Almeida [2, Lemma 6.5.14] that the variety $\mathbf{V}$ satisfies either the identity (4.3) or the identity

$$xyxyx \approx xyx. \tag{4.4}$$

If $\mathbf{V}$ satisfies (4.4), then it also satisfies (4.3) since

$$xyxzx \overset{(4.4)}{\approx} xyxzxyxzx \overset{(4.2\text{a})}{\approx} xyx^2z^2xy^2x^2zx \overset{(4.2\text{b})}{\approx} xyz^2x^5y^2zx \overset{(4.2\text{a})}{\approx} xyzxyzx \overset{(4.4)}{\approx} xyzx.$$

Therefore (4.3) is an exclusion identity for the semigroup Q in the variety $\mathbf{Q}^1$. ∎

4.3. Proof of Condition 4. Let M be any monoid that satisfies the identities (4.1) and let $\mathbf{M}$ be the variety generated by M. It is clear from Proposition 4.3 that $\mathbf{M}$ is a subvariety of $\mathbf{Q}^1$. If $Q^1 \in \mathbf{M}$, then $\mathbf{M} = \mathbf{Q}^1$ and M is finitely based by Proposition 4.3. Therefore assume that $Q^1 \notin \mathbf{M}$. Then it follows from Lemmas 2.4 and 4.4 that the monoid M satisfies the identity (4.3). It also satisfies the identity $xyxy \approx xy^2x$ since

$$xyxy \overset{(4.2\text{a})}{\approx} xyx^2y^2 \overset{(4.2\text{b})}{\approx} xy^3x^2 \overset{(4.2\text{a})}{\approx} xy^2x.$$

Therefore M satisfies the identities in Condition 5 and is finitely based.

5. On Condition 6

It is easily seen that Condition 6 is a special case of the following result.

PROPOSITION 5.1. *Any monoid that satisfies the identities*

$$x^7yx \approx xyx, \quad xy^2x \approx yx^2y, \quad xyxzx \approx x^2yzx \tag{5.1}$$

is finitely based.

The proof of Proposition 5.1 is given in Section 5.3. The semigroup

$$A_0 = \langle\, a, b \mid a^2 = a,\, b^2 = b,\, ba = 0 \,\rangle$$

of order four plays a central role in this proof. Denote by $\mathbf{A}_0^1$ the variety generated by the monoid A_0^1.

Remark 5.2.

(i) The semigroup A_0 is isomorphic to a subsemigroup of A_2 and has appeared in Almeida [1] as D, in Edmunds [9] and Lee [17] as `S(4,22)`, and in Volkov [43] as $\mathbf{V}$.
(ii) The semigroup A_0 is not completely 0-simple, but it is very important to the study of varieties generated by completely 0-simple semigroups [15, 16, 19, 22, 23, 32].
(iii) The monoid A_0^1 has appeared in Edmunds [8] as $\mathtt{M}_{20}$, while the subvarieties of $\mathbf{A}_0^1$ were extensively investigated by Lee [20].

5.1. A basis for the variety $\mathbf{A}_0^1 \vee \mathbf{Z}_n$

Proposition 5.3. *Let $n \geq 2$ be any integer. Then the identities*

$$xyxzx \overset{*}{\approx} x^2yzx, \tag{5.2a}$$

$$x^{n+1}yx \overset{*}{\approx} xyx, \tag{5.2b}$$

$$xhytxy \overset{*}{\approx} xhytyx, \quad xyhxty \overset{*}{\approx} yxhxty \tag{5.2c}$$

constitute a basis for the variety $\mathbf{A}_0^1 \vee \mathbf{Z}_n$.

In this chapter, a nonsimple word

$$\mathbf{w} = \prod_{i=1}^{m} (\mathbf{s}_i \mathbf{w}_i) \tag{5.3}$$

is said to be in *n-canonical form* if all of the following conditions hold:

(I) the letters of $\mathbf{s}_1 \in \mathcal{X}^*$ and $\mathbf{s}_2, \ldots, \mathbf{s}_m \in \mathcal{X}^+$ are all simple in $\mathbf{w}$;
(II) the letters of $\mathbf{w}_1, \ldots, \mathbf{w}_{m-1} \in \mathcal{X}^+$ and $\mathbf{w}_m \in \mathcal{X}^*$ are all nonsimple in $\mathbf{w}$;
(III) $\mathsf{occ}(x, \mathbf{w}) \leq n + 1$ for all $x \in \mathcal{X}$;
(IV) if $\mathsf{occ}(x, \mathbf{w}) = r + 1$ for some $r \in \{1, \ldots, n\}$, then either $\mathsf{occ}(x, \mathbf{w}_i) = r + 1$ for some i, or $\mathsf{occ}(x, \mathbf{w}_j) = r$ and $\mathsf{occ}(x, \mathbf{w}_k) = 1$ for some j and k with $j < k$.

Note that (I) and (II) imply that $\mathsf{con}(\mathbf{s}_i) \cap \mathsf{con}(\mathbf{w}_j) = \emptyset$ for any i and j.

Lemma 5.4. *Let $\mathbf{w}$ be any nonsimple word. Then there exists some word $\mathbf{w}'$ in n-canonical form such that the identities* (5.2a) *and* (5.2b) *imply the identity $\mathbf{w} \approx \mathbf{w}'$.*

Proof. It suffices to convert the nonsimple word $\mathbf{w}$, using the identities (5.2a) and (5.2b), into a word in n-canonical form. By gathering adjacent simple letters and adjacent non-simple letters in the word $\mathbf{w}$, it is easy to see that $\mathbf{w}$ can be written in the form (5.3) that satisfies (I) and (II). Suppose that $\mathsf{occ}(x, \mathbf{w}) = e > 2$. Then any x in the word $\mathbf{w}$, except the first and last occurrences of x, can be gathered by the identities (5.2a) with the first occurrence of x, resulting in a word of the form $\mathbf{p}x^{e-1}\mathbf{q}x\mathbf{r}$, where $\mathbf{p}, \mathbf{q}, \mathbf{r} \in (\mathcal{X} \setminus \{x\})^*$. If necessary, the identities (5.2b) can then be used to reduce the exponent $e - 1$ to a number in $\{1, \ldots, n\}$. Repeat the same procedure on any letter that occurs more than twice in the word $\mathbf{w}$. The resulting word then satisfies both (III) and (IV). ∎

Lemma 5.5 ([8, Lemma 4.1]). *Any identity satisfied by the monoid A_0^1 preserves complete precedence.*

Proof of Proposition 5.3. It is routine to verify that the variety $\mathbf{A}_0^1 \vee \mathbf{Z}_n$ satisfies the identities (5.2). It remains to show that any identity $\mathbf{w} \approx \mathbf{w}'$ satisfied by the variety $\mathbf{A}_0^1 \vee \mathbf{Z}_n$ is implied by the identities (5.2). Since the monoid A_0^1 is noncommutative and its submonoid $\{0, ab, 1\}$ is isomorphic to the monoid N_2^1, it follows from Lemma 2.3 that the identity $\mathbf{w} \approx \mathbf{w}'$ is trivial if either $\mathbf{w}$ or $\mathbf{w}'$ is a simple word. Hence assume that the words $\mathbf{w}$ and $\mathbf{w}'$ are both nonsimple. In view of Lemma 5.4, these words can be assumed to be in n-canonical form, say

$$\mathbf{w} = \prod_{i=1}^{m}(\mathbf{s}_i\mathbf{w}_i) \quad \text{and} \quad \mathbf{w}' = \prod_{i=1}^{m'}(\mathbf{s}_i'\mathbf{w}_i').$$

It follows from (III) and Lemma 2.2(ii)&(iii) that $\mathsf{occ}(x, \mathbf{w}) = \mathsf{occ}(x, \mathbf{w}') \le n+1$ for all $x \in \mathcal{X}$. Further, since the identity $\mathbf{w} \approx \mathbf{w}'$ preserves complete precedence by Lemma 5.5, it follows that $m = m'$, $\mathbf{s}_i = \mathbf{s}_i'$, and $\mathbf{w}_i \mathrel{\mathring{=}} \mathbf{w}_i'$ for every i, whence

$$\mathbf{w}' = \prod_{i=1}^{m}(\mathbf{s}_i\mathbf{w}_i').$$

If $\mathbf{w}_i = \mathbf{w}_i'$ for all i, then the identity $\mathbf{w} \approx \mathbf{w}'$ is trivial and so is satisfied by the variety $\mathbf{A}_0^1 \vee \mathbf{Z}_n$. Suppose that ℓ is the least integer such that $\mathbf{w}_\ell \neq \mathbf{w}_\ell'$. Let

$$\mathbf{p} = \Big(\prod_{i=1}^{\ell-1}(\mathbf{s}_i\mathbf{w}_i)\Big)\mathbf{s}_\ell, \quad \mathbf{r} = \prod_{i=\ell+1}^{m}(\mathbf{s}_i\mathbf{w}_i), \quad \text{and} \quad \mathbf{r}' = \prod_{i=\ell+1}^{m}(\mathbf{s}_i\mathbf{w}_i').$$

Then $\mathbf{w} = \mathbf{p}\mathbf{w}_\ell\mathbf{r}$ and $\mathbf{w}' = \mathbf{p}\mathbf{w}_\ell'\mathbf{r}'$ with $\mathbf{r} \mathrel{\mathring{=}} \mathbf{r}'$. It is routine to show that the identity $\mathbf{p}\mathbf{w}_\ell\mathbf{r} \approx \mathbf{p}\mathbf{w}_\ell'\mathbf{r}$ preserves complete precedence. Hence by Lemma 2.5, the identities (5.2c) imply the identity $\mathbf{p}\mathbf{w}_\ell\mathbf{r} \approx \mathbf{p}\mathbf{w}_\ell'\mathbf{r}$, that is,

$$\mathbf{w} = \mathbf{s}_1\mathbf{w}_1 \cdots \mathbf{s}_\ell\mathbf{w}_\ell \cdots \mathbf{s}_m\mathbf{w}_m \overset{(5.2c)}{\approx} \mathbf{s}_1\mathbf{w}_1 \cdots \mathbf{s}_\ell\mathbf{w}_\ell' \cdots \mathbf{s}_m\mathbf{w}_m.$$

It is easy to see how the same argument can be repeated on $\mathbf{w}_{\ell+1}, \ldots, \mathbf{w}_m$ to obtain

$$\mathbf{w} \overset{(5.2c)}{\approx} \mathbf{s}_1\mathbf{w}_1 \cdots \mathbf{s}_{\ell-1}\mathbf{w}_{\ell-1}\, \mathbf{s}_\ell\mathbf{w}_\ell'\, \mathbf{s}_{\ell+1}\mathbf{w}_{\ell+1}' \cdots \mathbf{s}_m\mathbf{w}_m' = \mathbf{w}'.$$

Consequently, the identities (5.2) imply the identity $\mathbf{w} \approx \mathbf{w}'$. ∎

5.2. Subvarieties of the variety $\mathbf{A}_0^1 \vee \mathbf{Z}_6$

LEMMA 5.6. *Let $n \ge 2$ be any integer. Then the identity*

$$x^n y^n x^n y^n \approx x^n y^n \tag{5.4}$$

is an exclusion identity for the semigroup A_0 in the variety $\mathbf{A}_0^1 \vee \mathbf{Z}_n$.

Proof. This follows from Volkov [43, Proposition 1.2]. ∎

LEMMA 5.7. *Let $\mathbf{V}$ be the variety generated by any monoid in the variety $\mathbf{A}_0^1 \vee \mathbf{Z}_6$. If either $A_0^1 \notin \mathbf{V}$ or $\mathbb{Z}_6 \notin \mathbf{V}$, then the variety $\mathbf{V}$ is finitely based.*

Proof. By Proposition 5.3, the variety $\mathbf{V}$ satisfies the identities (5.2) with $n = 6$.

CASE 1: $A_0^1 \notin \mathbf{V}$. It follows from Lemmas 2.4 and 5.6 that the variety $\mathbf{V}$ satisfies the identity (5.4). Then $\mathbf{V}$ also satisfies the identity $xyxy \approx x^2y^2$ since

$$xyxy \overset{(5.2)}{\approx} x^2x^6y^6x^6y^6y^2 \overset{(5.4)}{\approx} x^2x^6y^6y^2 \overset{(5.2b)}{\approx} x^2y^2.$$

It is easy to check that $\mathbf{V}$ is finitely based by Condition 7.

CASE 2: $\mathbb{Z}_6 \notin \mathbf{V}$. Then either $\mathbb{Z}_2 \notin \mathbf{V}$ or $\mathbb{Z}_3 \notin \mathbf{V}$. First assume that $\mathbb{Z}_2 \in \mathbf{V}$ and $\mathbb{Z}_3 \notin \mathbf{V}$. Then by Lemma 2.2(iii), the variety $\mathbf{V}$ satisfies the identity $x^4 \approx x^2$. Since

$$x^3yx \approx x^7yx \overset{(5.2b)}{\approx} xyx,$$

$\mathbf{V}$ is a subvariety of $\mathbf{A}_0^1 \vee \mathbf{Z}_2$ by Proposition 5.3. If $A_0^1 \notin \mathbf{V}$, then $\mathbf{V}$ is finitely based by Case 1. If $A_0^1 \in \mathbf{V}$, then $\mathbf{V} = \mathbf{A}_0^1 \vee \mathbf{Z}_2$ and $\mathbf{V}$ is finitely based by Proposition 5.3.

Now if $\mathbb{Z}_2 \notin \mathbf{V}$ and $\mathbb{Z}_3 \in \mathbf{V}$, then $\mathbf{V}$ is finitely based by a similar argument. Hence it remains to assume that $\mathbb{Z}_2 \notin \mathbf{V}$ and $\mathbb{Z}_3 \notin \mathbf{V}$. By Lemma 2.2(iii), $\mathbf{V}$ satisfies the identity $x^3 \approx x^2$; it also satisfies the identity $xyxzx \approx xyzx$ because

$$xyxzx \overset{(5.2a)}{\approx} x^2yzx \approx x^7yzx \overset{(5.2b)}{\approx} xyzx.$$

It is then routine to check that $\mathbf{V}$ is finitely based by Condition 5. ∎

5.3. Proof of Proposition 5.1. Let M be any monoid that satisfies the identities (5.1). Then M satisfies the identities (5.2a) and (5.2b) with $n = 6$. Since

$$xhytxy \overset{(5.2b)}{\approx} x^7hy^7txy \overset{(5.2a)}{\approx} x^6hy^6t(yx^2y) \overset{(5.1)}{\approx} x^6hy^6txy^2x \overset{(5.2a)}{\approx} x^7hy^7tyx \overset{(5.2b)}{\approx} xhytyx,$$

M satisfies the identities $xhytxy \overset{*}{\approx} xhytyx$. By a symmetrical argument, M also satisfies the identities $xyhxty \overset{*}{\approx} yxhxty$. Consequently, M satisfies the identities (5.2c) and so belongs to the variety $\mathbf{A}_0^1 \vee \mathbf{Z}_6$ by Proposition 5.3.

Let $\mathbf{M}$ be the variety generated by M. If $A_0^1, \mathbb{Z}_6 \in \mathbf{M}$, then $\mathbf{M} = \mathbf{A}_0^1 \vee \mathbf{Z}_6$ and the monoid M is finitely based by Proposition 5.3. If either $A_0^1 \notin \mathbf{M}$ or $\mathbb{Z}_6 \notin \mathbf{M}$, then M is finitely based by Lemma 5.7.

6. On Condition 8

Let p denote a fixed prime integer throughout this chapter. It is easily shown that Condition 8 is a special case of the following result.

PROPOSITION 6.1. *Any semigroup that satisfies the identities*

$$xyx^{p+1} \overset{*}{\approx} xyx, \tag{6.1a}$$

$$xyxzx \overset{*}{\approx} xyx^2z, \tag{6.1b}$$

$$xhytxy \overset{*}{\approx} xhytyx \tag{6.1c}$$

but violates the identity

$$h^p(xy)^p \approx h^px^py^p \tag{6.2}$$

is finitely based.

The proof of Proposition 6.1 is given in Section 6.4. The semigroup

$$P_2 = \langle\, a, b \mid a^2 = ab = a,\ b^2a = b^2 \,\rangle$$

of order four plays a central role in this proof. Denote by $\mathbf{P}_2^1$ the variety generated by the monoid P_2^1.

REMARK 6.2.

(i) It is routine to show that the semigroup P_2 satisfies an identity $\mathbf{u} \approx \mathbf{v}$ if and only if the words $\mathbf{u}$ and $\mathbf{v}$ share the same prefix of length two. It follows that the identity $xyz \approx xy$ forms a basis for the semigroup P_2.
(ii) The semigroup P_2 has appeared in Tishchenko [36] as 053 and in Volkov [43] as $L_{3,1}$, while the monoid P_2^1 has appeared in Edmunds et al. [10] as D.

6.1. A basis for the variety $\mathbf{P}_2^1 \vee \mathbf{Z}_p$. In this chapter, a word

$$\mathbf{w} = \prod_{i=1}^{m} (x_i \mathbf{w}_i) \tag{6.3}$$

is said to be in *p-canonical form* if $x_1, \ldots, x_m$ are distinct letters and $\mathbf{w}_1, \ldots, \mathbf{w}_m$ are possibly empty words such that

(I) $\mathsf{ini}(\mathbf{w}) = x_1 \cdots x_m$;
(II) $\mathbf{w}_i \in \{x_1^{e_1} \cdots x_i^{e_i} \mid e_1, \ldots, e_i \in \{0, \ldots, p\}\}$;
(III) $\mathsf{con}(\mathbf{w}_i) \cap \mathsf{con}(\mathbf{w}_j) = \emptyset$ whenever $i \neq j$.

It follows that for any $x \in \mathsf{con}(\mathbf{w})$,

(IV) if x is nonsimple in $\mathbf{w}$, say with $\mathsf{occ}(x, \mathbf{w}) = e \geq 2$, then $\mathbf{w} = \mathbf{a}x\mathbf{b}x^{e-1}\mathbf{c}$ for some $\mathbf{a}, \mathbf{b}, \mathbf{c} \in (\mathscr{X} \setminus \{x\})^*$.

LEMMA 6.3. *Let $\mathbf{w}$ be any word. Then there exists some word $\mathbf{w}'$ in p-canonical form such that the identities* (6.1) *imply the identity $\mathbf{w} \approx \mathbf{w}'$.*

Proof. It suffices to convert the word $\mathbf{w}$, using the identities (6.1), into a word in p-canonical form. The identities (6.1a) and (6.1b) can first be applied to the word $\mathbf{w}$ so that

(a) $\mathsf{occ}(x, \mathbf{w}) \leq p + 1$ for all $x \in \mathscr{X}$;
(b) $\mathbf{w}$ satisfies (IV).

Suppose that $\mathsf{ini}(\mathbf{w}) = x_1 \cdots x_m$. Then the word $\mathbf{w}$ can be written in the form (6.3) that satisfies (I) with $\mathsf{con}(\mathbf{w}_i) \subseteq \{x_1, \ldots, x_i\}$ for all i. For each i, the letters in $\mathbf{w}_i$ are not first occurrences and so can be ordered by the identities (6.1c) according to their indices. Therefore (II) is satisfied in view of (a). It then follows from (b) that (III) is also satisfied. ■

PROPOSITION 6.4. *The identities* (6.1) *constitute a basis for the variety $\mathbf{P}_2^1 \vee \mathbf{Z}_p$.*

Proof. It is routine to verify that $\mathbf{P}_2^1 \vee \mathbf{Z}_p$ satisfies the identities (6.1). It remains to show that any identity $\mathbf{w} \approx \mathbf{w}'$ satisfied by $\mathbf{P}_2^1 \vee \mathbf{Z}_p$ is implied by the identities (6.1). In view of Lemma 6.3, the words $\mathbf{w}$ and $\mathbf{w}'$ can be chosen to be in p-canonical form. Since the submonoid $\{a, b^2, 1\}$ of P_2^1 is isomorphic to L_2^1, it follows from Lemma 2.2(i) that $\mathsf{ini}(\mathbf{w}) = \mathsf{ini}(\mathbf{w}')$. Therefore

$$\mathbf{w} = \prod_{i=1}^{m}(x_i\mathbf{w}_i) \quad \text{and} \quad \mathbf{w}' = \prod_{i=1}^{m}(x_i\mathbf{w}_i').$$

Since $\mathsf{occ}(x, \mathbf{w}), \mathsf{occ}(x, \mathbf{w}') \leq p + 1$ for all $x \in \mathcal{X}$ and the submonoid $\{b, b^2, 1\}$ of P_2^1 is isomorphic to N_2^1, it follows from Lemma 2.2(ii)&(iii) that

(a) $\mathsf{occ}(x, \mathbf{w}) = \mathsf{occ}(x, \mathbf{w}') \leq p + 1$ for all $x \in \mathcal{X}$.

Suppose that ℓ is the least possible integer such that $\mathsf{con}(\mathbf{w}_\ell) \neq \mathsf{con}(\mathbf{w}_\ell')$. Then

(b) $\mathsf{con}(\mathbf{w}_1) = \mathsf{con}(\mathbf{w}_1'), \ldots, \mathsf{con}(\mathbf{w}_{\ell-1}) = \mathsf{con}(\mathbf{w}_{\ell-1}')$

and there exists some $k \leq \ell$ such that x_k belongs to precisely one of $\mathbf{w}_\ell$ and $\mathbf{w}_\ell'$; by symmetry, it suffices to assume that

(c) $x_k \in \mathsf{con}(\mathbf{w}_\ell)$ and $x_k \notin \mathsf{con}(\mathbf{w}_\ell')$.

Since x_k is a nonsimple letter in $\mathbf{w}$, it follows from (a) that $\mathsf{occ}(x_k, \mathbf{w}) = \mathsf{occ}(x_k, \mathbf{w}') = e$ for some $e \in \{2, \ldots, p + 1\}$. By (IV), both $\mathbf{w}$ and $\mathbf{w}'$ are of the form $\cdots x_k \cdots x_k^{e-1} \cdots$. Within the word $\mathbf{w}$, the factor x_k^{e-1} occurs in $\mathbf{w}_\ell$ by (c). However, within the word $\mathbf{w}'$, it follows from (b) and (c) that the factor x_k^{e-1} does not occur in any of $\mathbf{w}_1', \ldots, \mathbf{w}_{\ell-1}', \mathbf{w}_\ell'$. Therefore the factor x_k^{e-1} must occur in $\mathbf{w}_j'$ for some $j > \ell$. Hence the words $\mathbf{w}$ and $\mathbf{w}'$ are of the form

$$\mathbf{w} = \mathbf{a}\, x_k\, \mathbf{b}\, x_k^{e-1}\, \mathbf{c}\, x_{\ell+1}\, \mathbf{d} \quad \text{and} \quad \mathbf{w}' = \mathbf{a}'\, x_k\, \mathbf{b}'\, x_{\ell+1}\, \mathbf{c}'\, x_k^{e-1}\, \mathbf{d}'$$

with $x_{\ell+1} \notin \mathsf{con}(\mathbf{abca}'\mathbf{b}')$. Let φ be the following substitution into the monoid P_2^1:

$$x \mapsto \begin{cases} b & \text{if } x = x_k, \\ a & \text{if } x = x_{\ell+1}, \\ 1 & \text{otherwise.} \end{cases}$$

Then $\mathbf{w}\varphi = b^e((\mathbf{c}x_{\ell+1}\mathbf{d})\varphi) = b^2$ and $\mathbf{w}'\varphi = ba((\mathbf{c}'x_k^{e-1}\mathbf{d}')\varphi) = ba$, which implies that the identity $\mathbf{w} \approx \mathbf{w}'$ is contradictorily not satisfied by the monoid P_2^1.

Therefore the integer ℓ does not exist, whence $\mathsf{con}(\mathbf{w}_i) = \mathsf{con}(\mathbf{w}_i')$ for every i. It then follows from (a) and (IV) that $\mathbf{w}_i = \mathbf{w}_i'$ for every i. Consequently, the identity $\mathbf{w} \approx \mathbf{w}'$ is trivial and is implied by the identities (6.1). ∎

6.2. A basis for the variety $\mathbf{P}_2^1$

Proposition 6.5. *The identities* (6.1) *and*

$$x^3 \approx x^2 \tag{6.4}$$

constitute a basis for the variety $\mathbf{P}_2^1$.

Proof. Let $\mathbf{w}$ be any word. By Lemma 6.3, the identities (6.1) can be used to convert $\mathbf{w}$ into the word (6.3) in p-canonical form that satisfies (I), (II), and (III). Since

$$xyx^2 \overset{(6.4)}{\approx} xyx^{p+1} \overset{(6.1a)}{\approx} xyx,$$

any of the exponents $e_1, \ldots, e_i$ from (II) that is nonzero can be reduced by the identities $xyx^2 \overset{*}{\approx} xyx$ to 1. In this proof, the word $\mathbf{w}$ in (6.3) is said to be in P_2^1-*canonical form* if it satisfies (I), (III), and

(II′) $\mathbf{w}_i \in \{x_1^{e_1} \cdots x_i^{e_i} \mid e_1, \ldots, e_i \in \{0,1\}\}$.

As was just shown, the identities (6.1) and (6.4) convert any word into P_2^1-canonical form.

It is routine to verify that the variety $\mathbf{P}_2^1$ satisfies the identities (6.1) and (6.4). It remains to show that any identity $\mathbf{w} \approx \mathbf{w}'$ satisfied by $\mathbf{P}_2^1$ is implied by the identities (6.1) and (6.4). Convert $\mathbf{w}$ and $\mathbf{w}'$ into words in P_2^1-canonical form by the identities (6.1) and (6.4). By an argument similar to the proof of Proposition 6.4, the identity $\mathbf{w} \approx \mathbf{w}'$ is trivial and hence is satisfied by the variety $\mathbf{P}_2^1$. ∎

Corollary 6.6. *The identities*

$$xyxzx \overset{*}{\approx} xyxz, \quad xhytxy \overset{*}{\approx} xhytyx$$

constitute a basis for the variety $\mathbf{P}_2^1$.

6.3. An exclusion identity

Lemma 6.7. *The identity* (6.2) *is an exclusion identity for the semigroup* P_2^1 *in* $\mathbf{P}_2^1 \vee \mathbf{Z}_p$.

Proof. Let $\mathbf{V}$ be any subvariety of $\mathbf{P}_2^1 \vee \mathbf{Z}_p$ such that $P_2 \notin \mathbf{V}$. First note that the identity

$$(xy)^p \approx x^p y^p \tag{6.5}$$

is not satisfied by the semigroup P_2 since $(ba)^p \neq b^p a^p$. Let $\overline{P}_2$ be the semigroup that is anti-isomorphic to P_2. The semigroups A_0, B_2, and $\overline{P}_2$ do not satisfy the identity $xyxzx \approx xyx^2z$ from (6.1b) so that $A_0, B_2, \overline{P}_2 \notin \mathbf{V}$. It then follows from Volkov [43, Theorem 2.1] that the variety $\mathbf{V}$ satisfies the identity (6.5). Therefore the identity (6.5) is an exclusion identity for P_2 in $\mathbf{P}_2^1 \vee \mathbf{Z}_p$.

Now denote by ψ the substitution $z \mapsto h^p z h^p$ for all $z \in \{x, y\}$. Then it follows from Lee [18, Theorem 2] that $((xy)^p)\psi \approx (x^p y^p)\psi$ is an exclusion identity for P_2^1 in $\mathbf{P}_2^1 \vee \mathbf{Z}_p$; it is easy to show that this identity is equivalent to the identity (6.2) within $\mathbf{P}_2^1 \vee \mathbf{Z}_p$. ∎

6.4. Proof of Proposition 6.1. Let S be any semigroup that satisfies the identities (6.1) but violates the identity (6.2), and let $\mathbf{V}$ be the variety generated by S. Then $\mathbf{V} \subseteq \mathbf{P}_2^1 \vee \mathbf{Z}_p$ by Proposition 6.4, and $P_2^1 \in \mathbf{V}$ by Lemma 6.7. If $\mathbb{Z}_p \in \mathbf{V}$, then $\mathbf{V} = \mathbf{P}_2^1 \vee \mathbf{Z}_p$ and S is finitely based by Proposition 6.4. Therefore assume that $\mathbb{Z}_p \notin \mathbf{V}$. By Lemma 2.2(iii), the variety $\mathbf{V}$ satisfies the identity (6.4) so that $\mathbf{V} \subseteq \mathbf{P}_2^1$ by Proposition 6.5. Since $P_2^1 \in \mathbf{V}$, it follows that $\mathbf{V} = \mathbf{P}_2^1$, whence the semigroup S is finitely based by Proposition 6.5.

7. On Condition 9

The main aim of the present chapter is to establish the finite basis property of any semigroup that satisfies the identities

$$x^4 \approx x^2, \quad x^3yx \approx xyx, \quad xyx^2 \approx x^3y, \quad xyxy \approx xy^2x \tag{7.1}$$

but violates both of the identities

$$xyxy \approx x^2y^2, \tag{7.2a}$$
$$xy^3x \approx xyx. \tag{7.2b}$$

Let $\mathbf{U}$ be the variety defined by the identities

$$xyx^2 \approx x^3y, \tag{7.3a}$$
$$x^2yx^2 \approx x^2y, \tag{7.3b}$$
$$xyx^2zx \overset{*}{\approx} xyzx, \tag{7.3c}$$
$$xhytxy \overset{*}{\approx} xhytyx. \tag{7.3d}$$

7.1. Identities of U. In this chapter, a word

$$\mathbf{w} = x_0^{e_0} \prod_{i=1}^{m} (x_i^{e_i} \mathbf{w}_i) \tag{7.4}$$

is said to be in *canonical form* if $x_0, \ldots, x_m$ are distinct letters and $\mathbf{w}_1, \ldots, \mathbf{w}_m$ are possibly empty words that satisfy all of the following conditions:

(I) $\mathsf{ini}(\mathbf{w}) = x_0 \cdots x_m$;
(II) $\mathbf{w}_i \in \{x_0^{f_0} \cdots x_{i-1}^{f_{i-1}} \mid f_0, \ldots, f_{i-1} \in \{0,1\}\}$;
(III) $e_0, \ldots, e_m \in \{1,2,3\}$;
(IV) if $e_i = 3$, then $x_i \notin \mathsf{con}(\mathbf{w}_{i+1} \cdots \mathbf{w}_m)$.

Note that $x_i \notin \mathsf{con}(\mathbf{w}_1 \cdots \mathbf{w}_i)$ by (II).

LEMMA 7.1. *Let* $\mathbf{w}$ *be any word. Then there exists some word* $\mathbf{w}'$ *in canonical form such that the identities* (7.3) *imply the identity* $\mathbf{w} \approx \mathbf{w}'$.

Proof. It suffices to convert the word $\mathbf{w}$, using the identities (7.3), into a word in canonical form. Without loss of generality, assume that $\mathsf{ini}(\mathbf{w}) = x_0 \cdots x_m$. Then the word $\mathbf{w}$ can be written in the form

$$\mathbf{w} = x_0^{e_0} \prod_{i=1}^{m} (x_i \mathbf{u}_i) \tag{7.5}$$

for some $e_0 \geq 1$ and some $\mathbf{u}_1, \ldots, \mathbf{u}_m \in \mathcal{X}^*$ such that $\mathsf{con}(\mathbf{u}_i) \subseteq \{x_0, \ldots, x_i\}$ for all i. For each i, the letters in $\mathbf{u}_i$ are not first occurrences in the word $\mathbf{w}$ so that the identities (7.3d) can be used to permute them in any manner within $\mathbf{u}_i$. Specifically, any occurrence of x_i in $\mathbf{u}_i$ can be moved to the left and gathered with the x_i that immediately precedes $\mathbf{u}_i$ in (7.5), and any of the letters $x_0, \ldots, x_{i-1}$ in $\mathbf{u}_i$ can be ordered according to their indices. The resulting word is of the form (7.4) that satisfies (I) and the condition that $\mathbf{w}_i \in \{x_0^{f_0} \cdots x_{i-1}^{f_{i-1}} \mid f_0, \ldots, f_{i-1} \geq 0\}$.

Let $f_0, \ldots, f_{i-1} \geq 0$ be such that $\mathbf{w}_i = x_0^{f_0} \cdots x_{i-1}^{f_{i-1}}$. Suppose that $f_j \geq 2$ for some $j \in \{0, \ldots, i-1\}$, say $f_j = 2p + r$ with $p \geq 1$ and $r \in \{0,1\}$. Then the identities (7.3a) can be used to gather all multiples of x_j^2 in $\mathbf{w}_i$ with the first x_j in $\mathbf{w}$:

$$\begin{aligned}\mathbf{w} &= \cdots\ x_j^{e_j}\,\mathbf{w}_j\ \cdots\ x_i^{e_i}\ \underbrace{x_0^{f_0}\cdots x_{j-1}^{f_{j-1}}x_j^{2p+r}x_{j+1}^{f_{j+1}}\cdots x_{i-1}^{f_{i-1}}}_{\mathbf{w}_i}\ \cdots \\ &\overset{(7.3a)}{\approx} \cdots\ x_j^{e_j+2p}\,\mathbf{w}_j\ \cdots\ x_i^{e_i}\ x_0^{f_0}\cdots x_{j-1}^{f_{j-1}}x_j^{r}x_{j+1}^{f_{j+1}}\cdots x_{i-1}^{f_{i-1}}\ \cdots .\end{aligned}$$

For any other $k \neq j$, the same argument can be repeated to gather all multiples of x_k^2 in $\mathbf{w}_i$ with the first x_k in $\mathbf{w}$. Hence (II) is satisfied. It is clear that (III) is satisfied by applying the identity $x^4 \approx x^2$ from (7.3c). If $e_i = 3$ and $x_i \in \mathsf{con}(\mathbf{w}_{i+1}\cdots\mathbf{w}_m)$, then the identities (7.3c) can be used to reduce the exponent e_i to 1. Therefore (IV) is satisfied. ■

Lemma 7.2. *Let* $\mathbf{w}$ *and* $\mathbf{w}'$ *be any words in canonical form such that*

(a) $\mathsf{occ}(x,\mathbf{w}) \equiv \mathsf{occ}(x,\mathbf{w}') \pmod 2$ *for all* $x \in \mathcal{X}$;
(b) $\mathsf{sim}(\mathbf{w}) = \mathsf{sim}(\mathbf{w}')$;
(c) $\mathsf{ini}(\mathbf{w}) = \mathsf{ini}(\mathbf{w}')$;
(d) $\mathsf{occ}(x,y,\mathbf{w}) \equiv \mathsf{occ}(x,y,\mathbf{w}') \pmod 2$ *for all* $x, y \in \mathcal{X}$.

Then the words $\mathbf{w}$ *and* $\mathbf{w}'$ *are identical.*

Proof. By (c), it can be assumed that $\mathsf{ini}(\mathbf{w}) = \mathsf{ini}(\mathbf{w}') = x_0\cdots x_m$. Since the words $\mathbf{w}$ and $\mathbf{w}'$ are in canonical form,

$$\mathbf{w} = x_0^{e_0}\prod_{i=1}^{m}(x_i^{e_i}\mathbf{w}_i) \quad \text{and} \quad \mathbf{w}' = x_0^{e_0'}\prod_{i=1}^{m}(x_i^{e_i'}\mathbf{w}_i')$$

for some $e_i, e_i' \geq 1$ and $\mathbf{w}_i, \mathbf{w}_i' \in \mathcal{X}^*$. Since $\mathsf{occ}(x_m,\mathbf{w}) = e_m$ and $\mathsf{occ}(x_m,\mathbf{w}') = e_m'$ by (II), it follows from (III), (a), and (b) that $e_m = e_m'$ necessarily. Suppose that $\ell < m$ is the least integer such that $e_\ell \neq e_\ell'$. By symmetry, it suffices to assume that $e_\ell > e_\ell'$, whence $(e_\ell, e_\ell') \in \{(2,1),(3,1),(3,2)\}$ by (III). Since (II) implies that $\mathsf{occ}(x_\ell, x_{\ell+1}, \mathbf{w}) = e_\ell$ and $\mathsf{occ}(x_\ell, x_{\ell+1}, \mathbf{w}') = e_\ell'$, it follows from (d) that $(e_\ell, e_\ell') = (3,1)$. Hence (II) and (IV) imply that $\mathsf{occ}(x_\ell,\mathbf{w}) = 3$ and

$$\mathbf{w} = x_0^{e_0}\Big(\prod_{i=1}^{\ell-1}(x_i^{e_i}\mathbf{w}_i)\Big)x_\ell^3\mathbf{w}_\ell\prod_{i=\ell+1}^{m}(x_i^{e_i}\mathbf{w}_i),$$

where x_ℓ does not occur in any of the factors $\mathbf{w}_1,\ldots,\mathbf{w}_m$. Since $\mathsf{occ}(x_\ell,\mathbf{w}) = 3$, it follows from (a) and (b) that $\mathsf{occ}(x_\ell,\mathbf{w}')$ is odd and at least three, say $\mathsf{occ}(x_\ell,\mathbf{w}') = k+1$ for some even $k \geq 2$. Since $e_\ell' = 1$, the factor $\mathbf{w}_\ell'$ of $\mathbf{w}'$ is preceded by the first x_ℓ in $\mathbf{w}'$ and the remaining k occurrences of x_ℓ in $\mathbf{w}'$ are spread out in some $\mathbf{w}_{\ell_1}',\ldots,\mathbf{w}_{\ell_k}'$ with $\ell < \ell_1 < \cdots < \ell_k \leq m$, that is,

$$\mathbf{w}' = \cdots\ x_\ell\mathbf{w}_\ell'\ \cdots\ x_{\ell_1}\underbrace{(\cdots x_\ell \cdots)}_{\mathbf{w}_{\ell_1}'}\cdots\ x_{\ell_k}\underbrace{(\cdots x_\ell \cdots)}_{\mathbf{w}_{\ell_k}'}\cdots .$$

But now $\mathsf{occ}(x_\ell, x_{\ell_2}, \mathbf{w}') = 2$ and $\mathsf{occ}(x_\ell, x_{\ell_2}, \mathbf{w}) = 3$ contradict (d). Therefore the integer ℓ does not exist, whence $(e_0,\ldots,e_m) = (e_0',\ldots,e_m')$.

If $(\mathbf{w}_1,\ldots,\mathbf{w}_{m-1}) = (\mathbf{w}_1',\ldots,\mathbf{w}_{m-1}')$, then it follows from (II) and (a) that $\mathbf{w}_m = \mathbf{w}_m'$, whence the identity $\mathbf{w} \approx \mathbf{w}'$ is trivial and is satisfied by the variety $\mathbf{U}$. Thus it remains to consider the case when $(\mathbf{w}_1,\ldots,\mathbf{w}_{m-1}) \neq (\mathbf{w}_1',\ldots,\mathbf{w}_{m-1}')$. Then there is a least integer $\ell < m$ such that $\mathbf{w}_\ell \neq \mathbf{w}'_\ell$, say $x_r \in \mathsf{con}(\mathbf{w}_\ell) \setminus \mathsf{con}(\mathbf{w}'_\ell)$ for some $r < \ell$. Since $(\mathbf{w}_1, \ldots, \mathbf{w}_{\ell-1}) = (\mathbf{w}'_1, \ldots, \mathbf{w}'_{\ell-1})$, it follows from (II) that

$$\begin{aligned}
\mathsf{occ}(x_r, x_{\ell+1}, \mathbf{w}) &= \mathsf{occ}\Big(x_r,\, x_0^{e_0} \prod_{i=1}^{\ell-1} (x_i^{e_i} \mathbf{w}_i)\Big) + \mathsf{occ}(x_r, x_\ell^{e_\ell} \mathbf{w}_\ell) \\
&= \mathsf{occ}\Big(x_r,\, x_0^{e_0} \prod_{i=1}^{\ell-1} (x_i^{e_i} \mathbf{w}_i)\Big) + 1, \\
\mathsf{occ}(x_r, x_{\ell+1}, \mathbf{w}') &= \mathsf{occ}\Big(x_r,\, x_0^{e'_0} \prod_{i=1}^{\ell-1} (x_i^{e'_i} \mathbf{w}'_i)\Big) + \mathsf{occ}(x_r, x_\ell^{e'_\ell} \mathbf{w}'_\ell) \\
&= \mathsf{occ}\Big(x_r,\, x_0^{e_0} \prod_{i=1}^{\ell-1} (x_i^{e_i} \mathbf{w}_i)\Big) + 0,
\end{aligned}$$

whence (d) is violated. Consequently, the integer ℓ does not exist, whence the identity $\mathbf{w} \approx \mathbf{w}'$ is trivial and is satisfied by the variety $\mathbf{U}$. ■

LEMMA 7.3. *The variety* $\mathbf{U}$ *satisfies an identity* $\mathbf{w} \approx \mathbf{w}'$ *if and only if the four conditions in Lemma* 7.2 *hold.*

Proof. By Lemma 7.1, the words $\mathbf{w}$ and $\mathbf{w}'$ can be chosen to be in canonical form. Suppose that the variety $\mathbf{U}$ satisfies the identity $\mathbf{w} \approx \mathbf{w}'$. It follows from Lemma 2.2 that the variety $\mathbf{U}$ contains the monoids L_2^1, N_2^1, and $\mathbb{Z}_2$ so that (a)–(c) in Lemma 7.2 hold. It is easy to show that the orthodox completely regular monoid

$$O = \langle\, a, b \mid a^2 = ab = a,\, b^2 = 1 \,\rangle$$

of order four belongs to the variety $\mathbf{U}$ and so must satisfy the identity $\mathbf{w} \approx \mathbf{w}'$. If $\mathsf{occ}(x, y, \mathbf{w}) \not\equiv \mathsf{occ}(x, y, \mathbf{w}') \pmod 2$, then letting φ be the substitution

$$z \mapsto \begin{cases} b & \text{if } z = x, \\ a & \text{if } z = y, \\ 1 & \text{if } z \in \mathscr{X} \setminus \{x, y\}, \end{cases}$$

into O, the contradiction $\{\mathbf{w}\varphi, \mathbf{w}'\varphi\} = \{a, ba\}$ is deduced. Thus (d) in Lemma 7.2 holds.

Conversely, if the four conditions in Lemma 7.2 hold, then the identity $\mathbf{w} \approx \mathbf{w}'$ is trivial and so is satisfied by the variety $\mathbf{U}$. ■

7.2. Identities satisfied by proper subvarieties in U. An identity $\mathbf{w} \approx \mathbf{w}'$ *deletes* to an identity $\mathbf{u} \approx \mathbf{u}'$ if, up to letter renaming, the identity $\mathbf{u} \approx \mathbf{u}'$ belongs to the system $\mathbf{w} \overset{*}{\approx} \mathbf{w}'$.

LEMMA 7.4. *Suppose that the identity* $\mathbf{w} \approx \mathbf{w}'$ *deletes to the identity* $\mathbf{u} \approx \mathbf{u}'$. *Then the identities* (7.3) *and* $\mathbf{w} \approx \mathbf{w}'$ *imply the identity* $h^2\mathbf{u} \approx h^2\mathbf{u}'$.

Proof. Suppose that the identity $\mathbf{u} \approx \mathbf{u}'$ is obtained from the identity $\mathbf{w} \approx \mathbf{w}'$ by deleting all occurrences of the letters $x_1, \ldots, x_k$. Denote by φ the substitution $z \mapsto h^2$ for all $z \in \mathscr{X} \setminus \{x_1, \ldots, x_k\}$. Since

$$h^2\mathbf{u} \overset{(7.3\text{b})}{\approx} h^2(\mathbf{w}\varphi) \approx h^2(\mathbf{w}'\varphi) \overset{(7.3\text{b})}{\approx} h^2\mathbf{u}',$$

the identities (7.3) and $\mathbf{w} \approx \mathbf{w}'$ imply the identity $h^2\mathbf{u} \approx h^2\mathbf{u}'$. ■

Lemma 7.5. *Any proper subvariety of* $\mathbf{U}$ *satisfies some identity from* (7.2).

Proof. Let $\mathbf{W}$ be any proper subvariety of $\mathbf{U}$. Then the variety $\mathbf{W}$ satisfies some identity $\mathbf{w} \approx \mathbf{w}'$ that is not satisfied by the variety $\mathbf{U}$. By Lemma 7.3, at least one of the four conditions in Lemma 7.2 does not hold for the identity $\mathbf{w} \approx \mathbf{w}'$.

Case 1: $\mathsf{occ}(x, \mathbf{w}) \not\equiv \mathsf{occ}(x, \mathbf{w}') \pmod 2$ for some $x \in \mathscr{X}$. Then the variety $\mathbf{W}$ satisfies the identity $\sigma_1 : x^3 \approx x^2$; it also satisfies the identity (7.2a) since

$$xyxy \overset{(7.3c)}{\approx} xyx^3y \overset{\sigma_1}{\approx} xyx^2y \overset{(7.3a)}{\approx} x^3y^2 \overset{\sigma_1}{\approx} x^2y^2.$$

Case 2: $\mathsf{sim}(\mathbf{w}) \neq \mathsf{sim}(\mathbf{w}')$.

Subcase 2.1: $\mathsf{con}(\mathbf{w}) = \mathsf{con}(\mathbf{w}')$. Then the identity $\mathbf{w} \approx \mathbf{w}'$ deletes to the identity $x^m \approx x$ for some $m \geq 2$. It follows from Lemma 7.4 that the variety $\mathbf{W}$ satisfies the identity $h^2x^m \approx h^2x$ and so also the identity $\sigma_2 : h^2x^3 \approx h^2x$. Since

$$xy^3x \overset{(7.3c)}{\approx} x^3y^3x \overset{\sigma_2}{\approx} x^3yx \overset{(7.3c)}{\approx} xyx,$$

the variety $\mathbf{W}$ satisfies the identity (7.2b).

Subcase 2.2: $\mathsf{con}(\mathbf{w}) \neq \mathsf{con}(\mathbf{w}')$. Without loss of generality, assume that

$$\mathsf{con}(\mathbf{w}) \setminus \mathsf{con}(\mathbf{w}') = \{x_1, \ldots, x_m\} \quad \text{and} \quad \mathsf{con}(\mathbf{w}') \setminus \mathsf{con}(\mathbf{w}) = \{y_1, \ldots, y_n\}$$

for some $m, n \geq 0$. The assumption $\mathsf{con}(\mathbf{w}) \neq \mathsf{con}(\mathbf{w}')$ implies that $(m, n) \neq (0, 0)$. Then the variety $\mathbf{W}$ satisfies the identity $\mathbf{v} \approx \mathbf{v}'$ where

$$\mathbf{v} = \mathbf{w}x_1 \cdots x_m y_1 \cdots y_n \quad \text{and} \quad \mathbf{v}' = \mathbf{w}'x_1 \cdots x_m y_1 \cdots y_n.$$

It is clear that $\mathsf{con}(\mathbf{v}) = \mathsf{con}(\mathbf{v}')$ and $\mathsf{sim}(\mathbf{v}) \neq \mathsf{sim}(\mathbf{v}')$. Therefore it follows from Subcase 2.1 that the variety $\mathbf{W}$ satisfies the identity (7.2b).

Case 3: $\mathsf{ini}(\mathbf{w}) \neq \mathsf{ini}(\mathbf{w}')$. If $\mathsf{con}(\mathbf{w}) \neq \mathsf{con}(\mathbf{w}')$, say with $x \in \mathsf{con}(\mathbf{w}) \setminus \mathsf{con}(\mathbf{w}')$, then since the variety $\mathbf{W}$ satisfies the identity $x\mathbf{w} \approx x\mathbf{w}'$ with $\mathsf{sim}(x\mathbf{w}) \neq \mathsf{sim}(x\mathbf{w}')$, it follows from Case 2 that this variety also satisfies the identity (7.2b). Therefore it can further be assumed that $\mathsf{con}(\mathbf{w}) = \mathsf{con}(\mathbf{w}')$. The identity $\mathbf{w} \approx \mathbf{w}'$ then deletes to an identity of the form $x^py^q\mathbf{u} \approx y^{p'}x^{q'}\mathbf{u}'$ for some $p, q, p', q' \geq 1$ and $\mathbf{u}, \mathbf{u}' \in \{x, y\}^*$. It follows from Lemma 7.4 that the variety $\mathbf{W}$ satisfies the identity $h^2x^py^q\mathbf{u} \approx h^2y^{p'}x^{q'}\mathbf{u}'$. Denote by φ the substitution $z \mapsto z^2$ for all $z \in \mathscr{X}$. Since

$$(h^2x^py^q\mathbf{u})\varphi \overset{(7.3b)}{\approx} h^2x^2y^2 \quad \text{and} \quad (h^2y^{p'}x^{q'}\mathbf{u}')\varphi \overset{(7.3b)}{\approx} h^2y^2x^2,$$

the variety $\mathbf{W}$ satisfies the identity $\sigma_3 : h^2x^2y^2 \approx h^2y^2x^2$. Since

$$xyxy \overset{(7.3c)}{\approx} x^2(xy)^2y^2 \overset{\sigma_3}{\approx} x^2y^2(xy)^2 \overset{(7.3d)}{\approx} x^2y^2x^2y^2 \overset{(7.3b)}{\approx} x^2y^2,$$

the variety $\mathbf{W}$ satisfies the identity (7.2a).

Case 4: $\mathsf{occ}(x, y, \mathbf{w}) \not\equiv \mathsf{occ}(x, y, \mathbf{w}') \pmod 2$ for some $x, y \in \mathscr{X}$. In view of Cases 1 and 3, it suffices to further assume that

(a) $\mathsf{occ}(z, \mathbf{w}) \equiv \mathsf{occ}(z, \mathbf{w}') \pmod 2$ for all $z \in \mathscr{X}$;
(b) $\mathsf{ini}(\mathbf{w}) = \mathsf{ini}(\mathbf{w}')$.

Let $\mathbf{u} \approx \mathbf{u}'$ be the identity that is obtained from $\mathbf{w} \approx \mathbf{w}'$ by retaining only the letters x and y so that $\mathsf{occ}(x, y, \mathbf{u}) = \mathsf{occ}(x, y, \mathbf{w})$ and $\mathsf{occ}(x, y, \mathbf{u}') = \mathsf{occ}(x, y, \mathbf{w}')$. By Lemma 7.4, the variety $\mathbf{W}$ satisfies the identity $h^2\mathbf{u} \approx h^2\mathbf{u}'$. By (b), there is no loss of generality in assuming that $\mathsf{ini}(\mathbf{u}) = \mathsf{ini}(\mathbf{u}') = xy$, whence $\mathsf{occ}(x, y, \mathbf{u}), \mathsf{occ}(x, y, \mathbf{u}') \geq 1$. Let

$$p = \mathsf{occ}(x, y, \mathbf{u}), \quad q = \mathsf{occ}(x, \mathbf{u}) - p, \quad p' = \mathsf{occ}(x, y, \mathbf{u}'), \quad \text{and} \quad q' = \mathsf{occ}(x, \mathbf{u}') - p'.$$

Then $p \not\equiv p' \pmod 2$ by the assumption of the present case and

$$p + q = \mathsf{occ}(x, \mathbf{u}) = \mathsf{occ}(x, \mathbf{w}) \equiv \mathsf{occ}(x, \mathbf{w}') = \mathsf{occ}(x, \mathbf{u}') = p' + q' \pmod 2$$

by (a), whence $q \not\equiv q' \pmod 2$. Let $r, s \in \{0, 1\}$ be such that

$$r + p \equiv 0 \neq 1 \equiv r + p' \pmod 2 \quad \text{and} \quad s + q \equiv 0 \neq 1 \equiv s + q' \pmod 2.$$

Denote by φ the substitution

$$z \mapsto \begin{cases} y^2 & \text{if } z = y, \\ x & \text{if } z = h. \end{cases}$$

Since

$$x^r((h^2\mathbf{u})\varphi)x^s \overset{(7.3a)}{\approx} x^{r+2+p}y^{2\,\mathsf{occ}(y,\mathbf{u})}x^{q+s} \overset{(7.3c)}{\approx} x^2y^2x^2 \overset{(7.3b)}{\approx} x^2y^2,$$
$$x^r((h^2\mathbf{u}')\varphi)x^s \overset{(7.3a)}{\approx} x^{r+2+p'}y^{2\,\mathsf{occ}(y,\mathbf{u}')}x^{q'+s} \overset{(7.3c)}{\approx} xy^2x \overset{(7.3d)}{\approx} xyxy,$$

the variety $\mathbf{W}$ satisfies the identity (7.2a). ∎

7.3. Proof of Condition 9. Let S be any semigroup that satisfies the identities (7.1) but violates both identities in (7.2). Then it is easy to show that S satisfies the identities (7.3a), (7.3b), and (7.3c). Since

$$xhytxy \overset{(7.3c)}{\approx} xhyty^2xy \overset{(7.3c)}{\approx} xhytyx^2yxy \overset{(7.1)}{\approx} xhytyx^2y^2x \overset{(7.3c)}{\approx} xhyty^3x \overset{(7.3c)}{\approx} xhytyx,$$

the semigroup S also satisfies the identities (7.3d), whence $S \in \mathbf{U}$. Since S does not satisfy any identity in (7.2), it follows from Lemma 7.5 that S must generate the variety $\mathbf{U}$ and is finitely based.

8. The monoid $\mathcal{A}$

8.1. Main result

PROPOSITION 8.1. *The identities*

$$xhxtx \overset{*}{\approx} htx^3, \quad x^4 \approx x^3, \tag{8.1a}$$
$$xhy^3x \overset{*}{\approx} xhxy^3, \quad x^3y^3 \approx y^3x^3, \tag{8.1b}$$
$$xhytxy \overset{*}{\approx} xhytyx, \quad xyhxty \overset{*}{\approx} yxhxty \tag{8.1c}$$

constitute a basis for the monoid $\mathcal{A}$ with the following multiplication table:

$\mathcal{A}$	1	2	3	4	5	6
1	1	1	1	1	1	1
2	1	1	1	1	1	2
3	1	1	1	1	2	3
4	1	2	3	4	3	4
5	1	1	1	1	2	5
6	1	2	3	4	5	6

Note that the identity element of the monoid $\mathcal{A}$ is 6.

8.2. A canonical form. In this chapter, a nonsimple word

$$\mathbf{w} = \prod_{i=1}^{m} (\mathbf{s}_i \mathbf{w}_i) \tag{8.2}$$

is said to be in *canonical form* if all of the following conditions hold:

(I) the letters of $\mathbf{s}_1 \in \mathcal{X}^*$ and $\mathbf{s}_2, \ldots, \mathbf{s}_m \in \mathcal{X}^+$ are all simple in $\mathbf{w}$;
(II) the letters of $\mathbf{w}_1, \ldots, \mathbf{w}_{m-1} \in \mathcal{X}^+$ and $\mathbf{w}_m \in \mathcal{X}^*$ are all nonsimple in $\mathbf{w}$;
(III) $\mathsf{occ}(x, \mathbf{w}) \leq 3$ for all $x \in \mathcal{X}$;
(IV) if $\mathsf{occ}(x, \mathbf{w}) = 3$, then all three occurrences of x in $\mathbf{w}$ form a factor x^3 in some $\mathbf{w}_i$;
(V) if x^3 is a factor of $\mathbf{w}$, then either x^3 is a suffix of $\mathbf{w}$ or x^3 is followed by the first occurrence of some letter of $\mathbf{w}$;
(VI) if $x^3 y^3$ is a factor of $\mathbf{w}$, then x alphabetically precedes y in $\mathcal{X}$.

Note that (I) and (II) imply that $\mathsf{con}(\mathbf{s}_i) \cap \mathsf{con}(\mathbf{w}_j) = \emptyset$ for any i and j.

LEMMA 8.2. *Let* $\mathbf{w}$ *be any nonsimple word. Then there exists some word* $\mathbf{w}'$ *in canonical form such that the identities* (8.1) *imply the identity* $\mathbf{w} \approx \mathbf{w}'$.

Proof. It suffices to convert the nonsimple word $\mathbf{w}$, using the identities (8.1), into a word in canonical form. By gathering adjacent simple letters and adjacent nonsimple letters in $\mathbf{w}$, it is easy to see that $\mathbf{w}$ can be written in the form (8.2) that satisfies (I) and (II). If $\mathsf{occ}(x, \mathbf{w}) = n \geq 3$, then the identities $xhxtx \overset{*}{\approx} htx^3$ from (8.1a) can be used to gather the first $n-1$ occurrences of x with the last x, and the identity $x^4 \approx x^3$ from (8.1a) can be used to eliminate all except three occurrences of x. Hence (III) and (IV) are satisfied. It follows that any factor x^3 in $\mathbf{w}$ that is not a suffix can only be followed by a letter that is either a first or second occurrence. Suppose that some factor x^3 of $\mathbf{w}$ is followed by the second occurrence of y, that is, $\mathbf{w} = \cdots y \cdots x^3 y \cdots$ with $\mathsf{occ}(y, \mathbf{w}) = 2$. Then

$$\mathbf{w} = \cdots y \cdots x^3 y \cdots \overset{(8.1\text{b})}{\approx} \cdots y \cdots y x^3 \cdots .$$

If the factor x^3 in the word $\cdots y \cdots y x^3 \cdots$ is followed by another second occurrence, say of z, then x^3 and this z can be interchanged by the identities (8.1b). This argument can be repeated sufficiently many times until x^3 is either followed by a first occurrence or a suffix of the entire word. Hence (V) is satisfied. It is easy to see that (VI) is satisfied by applying the identity $x^3 y^3 \approx y^3 x^3$ from (8.1b). ■

8.3. Identities of the monoid $\mathcal{A}$

LEMMA 8.3. *The monoid $\mathcal{A}$ does not satisfy any of the following identities*:

$$xy \approx yx, \quad xyx \approx x^2y, \quad xyx \approx yx^2, \quad x^2y \approx yx^2, \quad x^3y \approx yx^3, \tag{8.3a}$$

$$x^2y^2 \approx xyxy, \quad x^2y^2 \approx yxyx, \quad x^2y^2 \approx xy^2x, \quad x^2y^2 \approx yx^2y, \quad x^2y^2 \approx y^2x^2, \tag{8.3b}$$

$$x^3y^2 \approx y^2x^3, \quad x^3y^2 \approx yx^3y. \tag{8.3c}$$

Proof. The third identity from (8.3a) is violated in $\mathcal{A}$ by the substitution $(x, y) \mapsto (5, 4)$. The other four identities in (8.3a) are violated in $\mathcal{A}$ by the substitution $(x, y) \mapsto (4, 2)$. All identities in (8.3b) and (8.3c) are violated in $\mathcal{A}$ by the substitution $(x, y) \mapsto (4, 5)$. ∎

LEMMA 8.4. *Let*

$$\mathbf{w} = \prod_{i=1}^{m}(\mathbf{s}_i\mathbf{w}_i) \quad \text{and} \quad \mathbf{w}' = \prod_{i=1}^{m'}(\mathbf{s}_i'\mathbf{w}_i')$$

be nonsimple words in canonical form. Suppose that the monoid $\mathcal{A}$ satisfies the identity $\mathbf{w} \approx \mathbf{w}'$. Then

(i) *$m = m'$ with $\mathbf{s}_i = \mathbf{s}_i'$ and $\mathbf{w}_i \stackrel{\circ}{=} \mathbf{w}_i'$ for all i;*

(ii) *the identity $\mathbf{w} \approx \mathbf{w}'$ preserves complete precedence.*

Proof. (i) Since the submonoid $\{1, 2, 6\}$ of $\mathcal{A}$ is isomorphic to the monoid N_2^1, it follows from Lemma 2.2(ii) that $\mathsf{con}(\mathbf{w}) = \mathsf{con}(\mathbf{w}')$ and $\mathsf{sim}(\mathbf{w}) = \mathsf{sim}(\mathbf{w}')$. Since $\mathcal{A}$ does not satisfy the identity $x^3 \approx x^2$, it follows from (III) that $\mathsf{occ}(x, \mathbf{w}) = \mathsf{occ}(x, \mathbf{w}')$ for all $x \in \mathcal{X}$. Further, $\mathcal{A}$ does not satisfy any of the identities in (8.3a) so that $m = m'$ with $\mathbf{s}_i = \mathbf{s}_i'$ and $\mathbf{w}_i \stackrel{\circ}{=} \mathbf{w}_i'$ for all i.

(ii) Suppose that $x \ll_{\mathbf{w}} y$ for some $x, y \in \mathcal{X}$ with $p = \mathsf{occ}(x, \mathbf{w}) = \mathsf{occ}(x, \mathbf{w}')$ and $q = \mathsf{occ}(y, \mathbf{w}) = \mathsf{occ}(y, \mathbf{w}')$ (so that $p, q \in \{1, 2, 3\}$ by (III)). Then there are four cases to consider.

CASE 1: $(p, q) \in \{(1, 1), (2, 2), (1, 2), (2, 1), (1, 3), (3, 1)\}$. Since the monoid $\mathcal{A}$ does not satisfy any of the identities in (8.3a) and (8.3b), it follows that $x \ll_{\mathbf{w}'} y$.

CASE 2: $(p, q) = (3, 2)$. Then $\mathbf{w} = \cdots x^3 \cdots y \cdots y \cdots$. If $x \not\ll_{\mathbf{w}'} y$, then the word $\mathbf{w}'$ is of the form $\cdots y \cdots y \cdots x^3 \cdots$ or of the form $\cdots y \cdots x^3 \cdots y \cdots$, whence the monoid $\mathcal{A}$ contradictorily satisfies some identity from (8.3c). Therefore $x \ll_{\mathbf{w}'} y$ necessarily.

CASE 3: $(p, q) = (2, 3)$. Then $\mathbf{w} = \mathbf{a}x\mathbf{b}x\mathbf{c}y^3\mathbf{d}$ for some $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \in (\mathcal{X} \setminus \{x, y\})^*$. Working toward a contradiction, suppose that $x \not\ll_{\mathbf{w}'} y$. If $y \ll_{\mathbf{w}'} x$, then the monoid $\mathcal{A}$ satisfies the identity $x^2y^3 \approx y^3x^2$ in (8.3c) and so contradicts Lemma 8.3. Therefore $y \not\ll_{\mathbf{w}'} x$ and it follows that $\mathbf{w}' = \mathbf{a}'x\mathbf{b}'y^3\mathbf{c}'x\mathbf{d}'$ for some $\mathbf{a}', \mathbf{b}', \mathbf{c}', \mathbf{d}' \in (\mathcal{X} \setminus \{x, y\})^*$. By (V), the factor $\mathbf{c}'$ can be neither empty nor of the form $z_1^3 \cdots z_r^3$ with $z_1, \ldots, z_r \in \mathcal{X}$. It follows that there exists some letter z with $\mathsf{occ}(z, \mathbf{w}') \in \{1, 2\}$ such that the first occurrence of z is in $\mathbf{c}'$. Then $y \ll_{\mathbf{w}'} z$ so that $y \ll_{\mathbf{w}} z$ by Case 2. It follows that $x \ll_{\mathbf{w}} z$ and $x \not\ll_{\mathbf{w}'} z$, which is impossible by Case 1.

CASE 4: $(p, q) = (3, 3)$. Then $\mathbf{w} = \mathbf{a}x^3\mathbf{b}y^3\mathbf{c}$ for some $\mathbf{a}, \mathbf{b}, \mathbf{c} \in (\mathcal{X} \setminus \{x, y\})^*$. Working toward a contradiction, suppose that $x \not\ll_{\mathbf{w}'} y$. Then $\mathbf{w}' = \mathbf{a}'y^3\mathbf{b}'x^3\mathbf{c}'$ for some $\mathbf{a}', \mathbf{b}', \mathbf{c}' \in$

$(\mathcal{X} \setminus \{x, y\})^*$. If $\mathbf{b} = h_1^3 \cdots h_r^3$ and $\mathbf{b}' = t_1^3 \cdots t_s^3$ for some $h_1, \ldots, h_r, t_1, \ldots, t_s \in \mathcal{X}$ with $r, s \geq 0$, then the following consequences of (IV) constitute a contradiction:

(a) x alphabetically precedes y because $\mathbf{w} = \mathbf{a}x^3h_1^3 \cdots h_r^3y^3\mathbf{c}$;
(b) y alphabetically precedes x because $\mathbf{w}' = \mathbf{a}'y^3t_1^3 \cdots t_s^3x^3\mathbf{c}'$.

Therefore by symmetry, assume that $\mathbf{b}$ contains the first occurrence of some letter z with $\mathsf{occ}(z, \mathbf{w}) \in \{1, 2\}$. Then $x \ll_{\mathbf{w}} z$ and so $x \ll_{\mathbf{w}'} z$ by Case 1 or 2. It follows that $y \not\ll_{\mathbf{w}} z$ and $y \ll_{\mathbf{w}'} z$, contradicting Case 1 or 2. ∎

8.4. Proof of Proposition 8.1. It is routine to verify that the monoid $\mathcal{A}$ satisfies the identities (8.1). It remains to show that any identity $\mathbf{w} \approx \mathbf{w}'$ satisfied by $\mathcal{A}$ is implied by the identities (8.1). Since $\mathcal{A}$ is noncommutative and its submonoid $\{1, 2, 6\}$ is isomorphic to the monoid N_2^1, it follows from Lemma 2.3 that the identity $\mathbf{w} \approx \mathbf{w}'$ is trivial if either $\mathbf{w}$ or $\mathbf{w}'$ is a simple word. Hence assume that the words $\mathbf{w}$ and $\mathbf{w}'$ are both nonsimple. In view of Lemma 8.2, these words can be assumed to be in canonical form. Therefore by Lemma 8.4(i),

$$\mathbf{w} = \prod_{i=1}^{m}(\mathbf{s}_i\mathbf{w}_i) \quad \text{and} \quad \mathbf{w}' = \prod_{i=1}^{m}(\mathbf{s}_i\mathbf{w}_i')$$

with $\mathbf{w}_i \stackrel{\circ}{=} \mathbf{w}_i'$ for all i. By Lemma 8.4(ii), the identity $\mathbf{w} \approx \mathbf{w}'$ preserves complete precedence. If $\mathbf{w}_i = \mathbf{w}_i'$ for all i, then the identity $\mathbf{w} \approx \mathbf{w}'$ is implied by the identities (8.1). Therefore suppose that ℓ is the least integer such that $\mathbf{w}_\ell \neq \mathbf{w}_\ell'$. Let

$$\mathbf{p} = \Big(\prod_{i=1}^{\ell-1}(\mathbf{s}_i\mathbf{w}_i)\Big)\mathbf{s}_\ell, \quad \mathbf{r} = \prod_{i=\ell+1}^{m}(\mathbf{s}_i\mathbf{w}_i), \quad \text{and} \quad \mathbf{r}' = \prod_{i=\ell+1}^{m}(\mathbf{s}_i\mathbf{w}_i').$$

Then $\mathbf{w} = \mathbf{p}\mathbf{w}_\ell\mathbf{r}$ and $\mathbf{w}' = \mathbf{p}\mathbf{w}_\ell'\mathbf{r}'$ with $\mathbf{r} \stackrel{\circ}{=} \mathbf{r}'$. It is routine to show that the identity $\mathbf{p}\mathbf{w}_\ell\mathbf{r} \approx \mathbf{p}\mathbf{w}_\ell'\mathbf{r}$ preserves complete precedence. Hence by Lemma 2.5, the identities (8.1c) imply the identity $\mathbf{p}\mathbf{w}_\ell\mathbf{r} \approx \mathbf{p}\mathbf{w}_\ell'\mathbf{r}$, that is,

$$\mathbf{w} = \mathbf{s}_1\mathbf{w}_1 \cdots \mathbf{s}_\ell\mathbf{w}_\ell \cdots \mathbf{s}_m\mathbf{w}_m \stackrel{(8.1c)}{\approx} \mathbf{s}_1\mathbf{w}_1 \cdots \mathbf{s}_\ell\mathbf{w}_\ell' \cdots \mathbf{s}_m\mathbf{w}_m.$$

It is easy to see that the same argument can be repeated on $\mathbf{w}_{\ell+1}, \ldots, \mathbf{w}_m$ to obtain

$$\mathbf{w} \stackrel{(8.1c)}{\approx} \mathbf{s}_1\mathbf{w}_1 \cdots \mathbf{s}_{\ell-1}\mathbf{w}_{\ell-1}\,\mathbf{s}_\ell\mathbf{w}_\ell'\,\mathbf{s}_{\ell+1}\mathbf{w}_{\ell+1}' \cdots \mathbf{s}_m\mathbf{w}_m' = \mathbf{w}'.$$

Consequently, the identities (8.1) imply the identity $\mathbf{w} \approx \mathbf{w}'$.

9. The monoid $\mathcal{B}$

9.1. Main result

PROPOSITION 9.1. *The identities*

$$xhxtx \stackrel{*}{\approx} hxtx, \tag{9.1a}$$

$$xhytxy \stackrel{*}{\approx} xhytyx, \quad xhxyty \stackrel{*}{\approx} xhyxty, \quad xyhxty \stackrel{*}{\approx} yxhxty \tag{9.1b}$$

constitute a basis for the monoid $\mathcal{B}$ with the following multiplication table:

$\mathcal{B}$	1	2	3	4	5	6
1	1	1	1	1	1	1
2	1	1	1	1	1	2
3	1	1	1	1	2	3
4	1	2	3	4	4	4
5	1	2	3	4	4	5
6	1	2	3	4	5	6

Note that the identity element of the monoid $\mathcal{B}$ is 6.

9.2. A canonical form. In this chapter, a nonsimple word

$$\mathbf{w} = \prod_{i=1}^{m} (\mathbf{s}_i \mathbf{w}_i) \tag{9.2}$$

is said to be in *canonical form* if

(I) the letters of $\mathbf{s}_1 \in \mathcal{X}^*$ and $\mathbf{s}_2, \ldots, \mathbf{s}_m \in \mathcal{X}^+$ are all simple in $\mathbf{w}$;
(II) the letters of $\mathbf{w}_1, \ldots, \mathbf{w}_{m-1} \in \mathcal{X}^+$ and $\mathbf{w}_m \in \mathcal{X}^*$ are all nonsimple in $\mathbf{w}$;
(III) for each i, the letters of $\mathbf{w}_i$ are in alphabetical order;
(IV) $\mathsf{occ}(x, \mathbf{w}) \leq 2$ for all $x \in \mathcal{X}$.

Note that (I) and (II) imply that $\mathsf{con}(\mathbf{s}_i) \cap \mathsf{con}(\mathbf{w}_j) = \emptyset$ for any i and j.

LEMMA 9.2. *Let* $\mathbf{w}$ *be any nonsimple word. Then there exists some word* $\mathbf{w}'$ *in canonical form such that the identities* (9.1) *imply the identity* $\mathbf{w} \approx \mathbf{w}'$.

Proof. It suffices to convert the nonsimple word $\mathbf{w}$, using the identities (9.1), into a word in canonical form. By gathering adjacent simple letters and adjacent nonsimple letters in $\mathbf{w}$, it is easy to see that $\mathbf{w}$ can be written in the form (9.2) that satisfies (I) and (II). For each i, since the letters in $\mathbf{w}_i$ are nonsimple in $\mathbf{w}$, they can be ordered by the identities (9.1b) so that (III) is satisfied. For any letter x such that $\mathsf{occ}(x, \mathbf{w}) \geq 3$, the identities (9.1a) can be used to eliminate all except the last two occurrences of x from $\mathbf{w}$ so that (IV) is satisfied. ∎

9.3. Proof of Proposition 9.1. It is routine to verify that the monoid $\mathcal{B}$ satisfies the identities (9.1). It remains to show that any identity $\mathbf{w} \approx \mathbf{w}'$ satisfied by $\mathcal{B}$ is implied by the identities (9.1). Since $\mathcal{B}$ is noncommutative and its submonoid $\{1, 2, 6\}$ is isomorphic to N_2^1, it follows from Lemma 2.3 that the identity $\mathbf{w} \approx \mathbf{w}'$ is trivial if either $\mathbf{w}$ or $\mathbf{w}'$ is a simple word. Hence assume that the words $\mathbf{w}$ and $\mathbf{w}'$ are both nonsimple. In view of Lemma 9.2, these words can be assumed to be in canonical form, say

$$\mathbf{w} = \prod_{i=1}^{m} (\mathbf{s}_i \mathbf{w}_i) \quad \text{and} \quad \mathbf{w}' = \prod_{i=1}^{m'} (\mathbf{s}'_i \mathbf{w}'_i).$$

It follows from Lemma 2.2(ii) that $\mathsf{con}(\mathbf{w}) = \mathsf{con}(\mathbf{w}')$ and $\mathsf{sim}(\mathbf{w}) = \mathsf{sim}(\mathbf{w}')$. Further, it is routine to show that the monoid $\mathcal{B}$ does not satisfy any of the following identities:

$$xy \approx yx, \quad xyx \approx x^2y, \quad xyx \approx yx^2, \quad x^2y \approx yx^2.$$

It follows that $m = m'$ with $\mathbf{s}_i = \mathbf{s}_i'$ and $\mathbf{w}_i = \mathbf{w}_i'$ for all i. The identity $\mathbf{w} \approx \mathbf{w}'$ is thus trivial and implied by the identities (9.1).

10. The monoids $\mathcal{D}$ and $\mathcal{H}$

10.1. Main result

PROPOSITION 10.1. *The identities*

$$xyx^2 \approx x^3y, \tag{10.1a}$$

$$xyx^2zx \overset{*}{\approx} xyzx, \tag{10.1b}$$

$$xhxyty \overset{*}{\approx} xhyxty, \tag{10.1c}$$

$$xhytxy \overset{*}{\approx} xhytyx \tag{10.1d}$$

constitute a basis for the monoids $\mathcal{D}$ and $\mathcal{H}$ with the following multiplication tables:

$\mathcal{D}$	1	2	3	4	5	6
1	1	1	1	1	1	1
2	1	1	1	1	2	2
3	1	1	1	1	3	3
4	4	4	4	4	4	4
5	1	2	3	4	5	6
6	1	3	2	4	6	5

$\mathcal{H}$	1	2	3	4	5	6
1	1	1	1	1	1	1
2	1	1	1	1	2	3
3	1	1	1	1	3	2
4	4	4	4	4	4	4
5	1	2	3	4	5	6
6	1	2	3	4	6	5

Note that the identity elements of the monoids $\mathcal{D}$ and $\mathcal{H}$ are both 5.

10.2. A canonical form. In this chapter, a word

$$\mathbf{w} = \prod_{i=1}^{m}(x_i\mathbf{w}_i) \tag{10.2}$$

is said to be in *canonical form* if $x_1, \ldots, x_m$ are distinct letters and $\mathbf{w}_1, \ldots, \mathbf{w}_m$ are possibly empty words that satisfy all of the following conditions:

(I) $\mathsf{ini}(\mathbf{w}) = x_1 \cdots x_m$;
(II) $\mathbf{w}_i \in \{x_1^{e_1} \cdots x_i^{e_i} \mid e_1, \ldots, e_{i-1} \in \{0,1\},\ e_i \in \{0,1,2\}\}$;
(III) if $\mathsf{occ}(x_i, \mathbf{w}_i) = 2$, then $x_i \notin \mathsf{con}(\mathbf{w}_{i+1} \cdots \mathbf{w}_m)$;
(IV) if $\mathbf{w}_i \neq \emptyset$, then x_{i+1} is simple in $\mathbf{w}$.

Note that $x_i \notin \mathsf{con}(\mathbf{w}_1 \cdots \mathbf{w}_{i-1})$ by (II).

LEMMA 10.2. *Let $\mathbf{w}$ be any word. Then there exists some word $\mathbf{w}'$ in canonical form such that the identities* (10.1) *imply the identity $\mathbf{w} \approx \mathbf{w}'$.*

Proof. It suffices to convert the word $\mathbf{w}$, using the identities (10.1), into a word in canonical form. Observe that (I), (II), and (III) are identical to (I), (II), and (III) in Section 7. Since the identities (10.1) contain the identities (7.3), it follows from Lemma 7.1 that by applying the identities (10.1), the word $\mathbf{w}$ can be converted into the word in (10.2) that satisfies (I), (II), and (III).

Suppose that $\mathbf{w}_i \neq \emptyset$ and x_{i+1} is nonsimple in $\mathbf{w}$. Let $e_1, \ldots, e_{i-1}, f_1, \ldots, f_i \in \{0,1\}$ and $e_i, f_{i+1} \in \{0,1,2\}$ be such that $\mathbf{w}_i = x_1^{e_1} \cdots x_i^{e_i}$ and $\mathbf{w}_{i+1} = x_1^{f_1} \cdots x_i^{f_i} x_{i+1}^{f_{i+1}}$. Since the letters in $\mathbf{w}_i$ and $\mathbf{w}_{i+1}$ are not first occurrences in $\mathbf{w}$,

$$\begin{aligned}
\mathbf{w} &= \cdots\ x_i \mathbf{w}_i\ x_{i+1} \mathbf{w}_{i+1}\ \cdots \\
&= \cdots\ x_i (x_1^{e_1} \cdots x_i^{e_i}) x_{i+1} (x_1^{f_1} \cdots x_i^{f_i} x_{i+1}^{f_{i+1}})\ \cdots \\
&\overset{(10.1c)}{\approx} \cdots\ x_i x_{i+1} (x_1^{e_1} \cdots x_i^{e_i})(x_1^{f_1} \cdots x_i^{f_i} x_{i+1}^{f_{i+1}})\ \cdots \\
&\overset{(10.1d)}{\approx} \cdots\ x_i x_{i+1} x_1^{e_1+f_1} \cdots x_i^{e_i+f_i} x_{i+1}^{f_{i+1}}\ \cdots . \qquad (10.3)
\end{aligned}$$

Hence the factor $\mathbf{w}_i$ is combined with the factor $\mathbf{w}_{i+1}$. This procedure can be repeated on any nonempty factor $\mathbf{w}_j$ that precedes a nonsimple letter x_{j+1}. Hence the word $\mathbf{w}$ can be converted into a word $\mathbf{w}'$ that satisfies (IV). However, the word $\mathbf{w}'$ may no longer satisfy (II) since, for example, some of the exponents $e_1 + f_1, \ldots, e_i + f_i$ in (10.3) may not be in $\{0,1\}$. The arguments in the proof of Lemma 7.1 can then be used to convert the word $\mathbf{w}'$ into a word that satisfies (II). ∎

10.3. Identities of the monoids $\mathcal{D}$ and $\mathcal{H}$

LEMMA 10.3. *Let $M \in \{\mathcal{D}, \mathcal{H}\}$ and let $\mathbf{w} \approx \mathbf{w}'$ be any identity satisfied by the monoid M. Then*

(i) $\mathsf{ini}(\mathbf{w}) = \mathsf{ini}(\mathbf{w}')$ (*so that* $\mathsf{con}(\mathbf{w}) = \mathsf{con}(\mathbf{w}')$);
(ii) $\mathsf{sim}(\mathbf{w}) = \mathsf{sim}(\mathbf{w}')$;
(iii) $\mathsf{occ}(x, \mathbf{w}) \equiv \mathsf{occ}(x, \mathbf{w}') \pmod 2$ *for all* $x \in \mathcal{X}$.

Further, if $x \in \mathsf{con}(\mathbf{w}) = \mathsf{con}(\mathbf{w}')$ *and* $y \in \mathsf{sim}(\mathbf{w}) = \mathsf{sim}(\mathbf{w}')$, *then*

(iv) $\mathsf{occ}(x, y, \mathbf{w}) \equiv \mathsf{occ}(x, y, \mathbf{w}') \pmod 2$.

Proof. The submonoids $\{1,4,5\}$, $\{1,2,5\}$ and $\{5,6\}$ of M are isomorphic to L_2^1, N_2^1, and $\mathbb{Z}_2$ respectively. Therefore parts (i), (ii), and (iii) follow from Lemma 2.2. Suppose that $\mathsf{occ}(x, y, \mathbf{w}) \not\equiv \mathsf{occ}(x, y, \mathbf{w}') \pmod 2$ for some $x \in \mathsf{con}(\mathbf{w}) = \mathsf{con}(\mathbf{w}')$ and $y \in \mathsf{sim}(\mathbf{w}) = \mathsf{sim}(\mathbf{w}')$, say $\mathsf{occ}(x, y, \mathbf{w}) = 2p+1$ is odd and $\mathsf{occ}(x, y, \mathbf{w}') = 2q$ is even. By part (iii), there exists some $r \in \{1,2\}$ such that $\mathsf{occ}(x, \mathbf{w}) + r \equiv 0 \equiv \mathsf{occ}(x, \mathbf{w}') + r \pmod 2$. Denote by φ the following substitution into the monoid M:

$$z \mapsto \begin{cases} 6 & \text{if } z = x, \\ 2 & \text{if } z = y, \\ 5 & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned}
(\mathbf{w} x^r)\varphi &= 6^{2p+1} \cdot 2 \cdot 6^{\mathsf{occ}(x,\mathbf{w}) - (2p+1) + r} = 6 \cdot 2 \cdot 6 = 3, \\
(\mathbf{w}' x^r)\varphi &= 6^{2q} \cdot 2 \cdot 6^{\mathsf{occ}(x,\mathbf{w}') - 2q + r} = 5 \cdot 2 \cdot 5 = 2,
\end{aligned}$$

which is impossible. Hence part (iv) holds. ∎

10.4. Proof of Proposition 10.1. Let $M \in \{\mathcal{D}, \mathcal{H}\}$. It is routine to verify that the monoid M satisfies the identities (10.1). It remains to show that any identity $\mathbf{w} \approx \mathbf{w}'$ satisfied by M is implied by the identities (10.1). In view of Lemma 10.2, the words

$\mathbf{w}$ and $\mathbf{w}'$ can be assumed to be in canonical form. It follows from Lemma 10.3(i) that

$$\mathbf{w} = \prod_{i=1}^{m}(x_i\mathbf{w}_i) \quad \text{and} \quad \mathbf{w}' = \prod_{i=1}^{m}(x_i\mathbf{w}_i').$$

First assume that $(\mathbf{w}_1, \ldots, \mathbf{w}_{m-1}) = (\mathbf{w}_1', \ldots, \mathbf{w}_{m-1}')$. Then by (II), there exist exponents $e_1, e_1', \ldots, e_{m-1}, e_{m-1}' \in \{0,1\}$ and $e_m, e_m' \in \{0,1,2\}$ such that $\mathbf{w}_m = x_1^{e_1} \cdots x_m^{e_m}$ and $\mathbf{w}_m' = x_1^{e_1'} \cdots x_m^{e_m'}$. Hence $(e_1, \ldots, e_{m-1}) = (e_1', \ldots, e_{m-1}')$ by Lemma 10.3(iii). Since $\mathsf{occ}(x_m, \mathbf{w}) = 1 + e_m$ and $\mathsf{occ}(x_m, \mathbf{w}') = 1 + e_m'$, it follows from Lemma 10.3(ii)&(iii) that $e_m = e_m'$. Therefore $\mathbf{w}_m = \mathbf{w}_m'$. The identity $\mathbf{w} \approx \mathbf{w}'$ is thus trivial and implied by the identities (10.1).

It remains to assume that $\mathbf{w}_\ell \neq \mathbf{w}_\ell'$ for some least possible $\ell < m$, say

$$\mathbf{w}_\ell = x_1^{e_1} \cdots x_\ell^{e_\ell} \quad \text{and} \quad \mathbf{w}_\ell' = x_1^{e_1'} \cdots x_\ell^{e_\ell'}$$

for some $e_1, e_1', \ldots, e_{\ell-1}, e_{\ell-1}' \in \{0,1\}$ and $e_\ell, e_\ell' \in \{0,1,2\}$ with $(e_1, \ldots, e_\ell) \neq (e_1', \ldots, e_\ell')$. By symmetry, it suffices to assume that

(a) $e_j > e_j'$

for some j. Then $\mathbf{w}_\ell \neq \emptyset$ so that by (IV), the letter $x_{\ell+1}$ is simple in $\mathbf{w}$. Hence $x_{\ell+1}$ is also simple in $\mathbf{w}'$ by Lemma 10.3(ii). Since

$$\begin{aligned}
\mathsf{occ}(x_j, x_{\ell+1}, \mathbf{w}) &= \mathsf{occ}(x_j, x_1\mathbf{w}_1 \cdots x_{\ell-1}\mathbf{w}_{\ell-1}x_\ell) + \mathsf{occ}(x_j, \mathbf{w}_\ell) \\
&= \mathsf{occ}(x_j, x_1\mathbf{w}_1 \cdots x_{\ell-1}\mathbf{w}_{\ell-1}x_\ell) + e_j, \\
\mathsf{occ}(x_j, x_{\ell+1}, \mathbf{w}') &= \mathsf{occ}(x_j, x_1\mathbf{w}_1' \cdots x_{\ell-1}\mathbf{w}_{\ell-1}'x_\ell) + \mathsf{occ}(x_j, \mathbf{w}_\ell') \\
&= \mathsf{occ}(x_j, x_1\mathbf{w}_1 \cdots x_{\ell-1}\mathbf{w}_{\ell-1}x_\ell) + e_j',
\end{aligned}$$

it follows from Lemma 10.3(iv) that

(b) $e_j \equiv e_j' \pmod 2$.

By (II), the conditions (a) and (b) imply that $j = \ell$ with $e_\ell = 2$ and $e_\ell' = 0$. It follows from (III) that $x_\ell \notin \mathsf{con}(\mathbf{w}_{\ell+1} \cdots \mathbf{w}_m)$, whence $\mathsf{occ}(x_\ell, \mathbf{w}) = \mathsf{occ}(x_\ell, x_\ell\mathbf{w}_\ell) = 3$. Therefore $\mathsf{occ}(x_\ell, \mathbf{w}') \in \{3, 5, \ldots\}$ by Lemma 10.3(ii)&(iii). Now $e_\ell' = 0$ implies that $x_\ell \notin \mathsf{con}(\mathbf{w}_\ell')$, whence $\mathsf{occ}(x_\ell, \mathbf{w}') = 1 + \mathsf{occ}(x_\ell, \mathbf{w}_{\ell+1}' \cdots \mathbf{w}_m')$. Therefore there exist least possible integers p and q such that $\mathsf{occ}(x_\ell, \mathbf{w}_p') = \mathsf{occ}(x_\ell, \mathbf{w}_q') = 1$ and $\ell < p < q$. Then $\mathbf{w}_p' \neq \emptyset$ so that by (IV), the letter x_{p+1} is simple in $\mathbf{w}'$. Hence x_{p+1} is also simple in $\mathbf{w}$ by Lemma 10.3(ii). Now $\mathsf{occ}(x_\ell, x_{p+1}, \mathbf{w}) = 3$ and $\mathsf{occ}(x_\ell, x_{p+1}, \mathbf{w}') = 2$ so that Lemma 10.3(iv) is violated. Consequently, the assumption $\mathbf{w}_\ell \neq \mathbf{w}_\ell'$ is impossible.

11. The monoid $\mathcal{E}$

11.1. Main result

PROPOSITION 11.1. *The identities*

$$x^2y \approx yx^2, \tag{11.1a}$$

$$xhytxy \overset{*}{\approx} xhytyx, \quad xhxyty \overset{*}{\approx} xhyxty, \quad xyhxty \overset{*}{\approx} yxhxty, \tag{11.1b}$$

$$x^2hxyxtx \overset{*}{\approx} hxyxtx \tag{11.1c}$$

constitute a basis for the monoid $\mathcal{E}$ with the following multiplication table:

$\mathcal{E}$	1	2	3	4	5	6
1	1	1	1	1	1	1
2	1	1	1	1	2	2
3	1	1	2	2	3	3
4	1	1	2	2	4	4
5	1	2	3	4	5	6
6	1	2	4	3	6	5

Note that the identity element of the monoid $\mathcal{E}$ is 5.

11.2. A canonical form. In this chapter, a nonsimple word

$$\mathbf{w} = \mathbf{w}_0 \prod_{i=1}^{m} (\mathbf{s}_i \mathbf{w}_i) \tag{11.2}$$

with nonsimple letters $x_1, \ldots, x_r$ is said to be in *canonical form* if all of the following conditions hold:

(I) the letters of $\mathbf{s}_1 \in \mathcal{X}^*$ and $\mathbf{s}_2, \ldots, \mathbf{s}_m \in \mathcal{X}^+$ are all simple in $\mathbf{w}$;
(II) $\mathbf{w}_1, \ldots, \mathbf{w}_m \in \{x_1^{f_1} \cdots x_r^{f_r} \mid f_1, \ldots, f_r \in \{0,1\}\}$ with $\mathbf{w}_1, \ldots, \mathbf{w}_{m-1} \neq \emptyset$;
(III) $\mathbf{w}_0 = x_1^{e_1} \cdots x_r^{e_r}$ for some $e_1, \ldots, e_r \in \{0, 2, 4\}$;
(IV) if $\mathsf{occ}(x_i, \mathbf{w}_1 \cdots \mathbf{w}_m) \in \{1, 2\}$, then $e_i \in \{0, 2\}$;
(V) if $\mathsf{occ}(x_i, \mathbf{w}_1 \cdots \mathbf{w}_m) \geq 3$, then $e_i = 0$.

Note that $\mathsf{con}(\mathbf{s}_i) \cap \mathsf{con}(\mathbf{w}_j) = \emptyset$ for any i and j.

Lemma 11.2. *Let* $\mathbf{w}$ *be any nonsimple word. Then there exists some word* $\mathbf{w}'$ *in canonical form such that the identities* (11.1) *imply the identity* $\mathbf{w} \approx \mathbf{w}'$.

Proof. It suffices to convert the nonsimple word $\mathbf{w}$, using the identities (11.1), into a word in canonical form. Let $x_1, \ldots, x_r$ be the nonsimple letters of $\mathbf{w}$. By gathering adjacent simple letters and adjacent nonsimple letters, the word $\mathbf{w}$ can be written in the form $\mathbf{w} = \prod_{i=1}^{m} (\mathbf{s}_i \mathbf{w}_i)$ where $\mathbf{w}_1, \ldots, \mathbf{w}_{m-1} \in \{x_1, \ldots, x_r\}^+$, $\mathbf{w}_m \in \{x_1, \ldots, x_r\}^*$, and the factors $\mathbf{s}_1, \ldots, \mathbf{s}_m$ satisfy (I). Since the letters in each $\mathbf{w}_i$ are nonsimple in $\mathbf{w}$, they can be ordered by the identities (11.1b) so that $\mathbf{w}_i = x_1^{f_1} \cdots x_r^{f_r}$ for some $f_1, \ldots, f_r \geq 0$. Any factor x_j^2 can be moved by the identity (11.1a) to the left of $\mathbf{s}_1$ so that the word $\mathbf{w}$ is of the form (11.2) that satisfies (I) and (II), with $\mathbf{w}_0 = x_1^{e_1} \cdots x_r^{e_r}$ for some $e_1, \ldots, e_r \in \{0, 2, 4, \ldots\}$. It is easy to show that (III), (IV), and (V) are satisfied by applying the identities (11.1c). ∎

11.3. Identities of the monoid $\mathcal{E}$

Lemma 11.3. *Let* $\mathbf{w} \approx \mathbf{w}'$ *be any identity satisfied by the monoid* $\mathcal{E}$. *Then*

(i) $\mathsf{con}(\mathbf{w}) = \mathsf{con}(\mathbf{w}')$ *and* $\mathsf{sim}(\mathbf{w}) = \mathsf{sim}(\mathbf{w}')$;
(ii) $\mathsf{occ}(x, \mathbf{w}) \equiv \mathsf{occ}(x, \mathbf{w}') \pmod 2$ *for all* $x \in \mathcal{X}$.

Further, if $x \in \mathsf{con}(\mathbf{w}) = \mathsf{con}(\mathbf{w}')$ *and* $y \in \mathsf{sim}(\mathbf{w}) = \mathsf{sim}(\mathbf{w}')$, *then*

(iii) $\mathsf{occ}(x, y, \mathbf{w}) \equiv \mathsf{occ}(x, y, \mathbf{w}') \pmod 2$.

Proof. Since the submonoids $\{1,2,5\}$ and $\{5,6\}$ of $\mathcal{E}$ are isomorphic to N_2^1 and $\mathbb{Z}_2$ respectively, parts (i) and (ii) follow from Lemma 2.2. To establish part (iii), suppose that $\mathsf{occ}(x,y,\mathbf{w}) \not\equiv \mathsf{occ}(x,y,\mathbf{w}') \pmod 2$ for some $x \in \mathsf{con}(\mathbf{w}) = \mathsf{con}(\mathbf{w}')$ and some $y \in \mathsf{sim}(\mathbf{w}) = \mathsf{sim}(\mathbf{w}')$, say $\mathsf{occ}(x,y,\mathbf{w})$ is even and $\mathsf{occ}(x,y,\mathbf{w}')$ is odd. Denote by φ the following substitution into the monoid $\mathcal{E}$:

$$z \mapsto \begin{cases} 6 & \text{if } z = x, \\ 3 & \text{if } z = y, \\ 5 & \text{otherwise.} \end{cases}$$

Then $\mathbf{w}\varphi \in 5 \cdot 3 \cdot \{5,6\} = \{3\}$ and $\mathbf{w}'\varphi \in 6 \cdot 3 \cdot \{5,6\} = \{4\}$, which is impossible. ∎

11.4. Proof of Proposition 11.1. It is routine to verify that the monoid $\mathcal{E}$ satisfies the identities (11.1). It remains to show that any identity $\mathbf{w} \approx \mathbf{w}'$ satisfied by $\mathcal{E}$ is implied by the identities (11.1). Since $\mathcal{E}$ is noncommutative and its submonoid $\{1,2,5\}$ is isomorphic to N_2^1, it follows from Lemma 2.3 that the identity $\mathbf{w} \approx \mathbf{w}'$ is trivial if either $\mathbf{w}$ or $\mathbf{w}'$ is a simple word. Hence assume that they are both nonsimple. In view of Lemma 11.2, these words can be assumed to be in canonical form, say

$$\mathbf{w} = \mathbf{w}_0 \prod_{i=1}^{m} (\mathbf{s}_i \mathbf{w}_i) \quad \text{and} \quad \mathbf{w}' = \mathbf{w}_0' \prod_{i=1}^{m'} (\mathbf{s}_i' \mathbf{w}_i').$$

It follows from Lemma 11.3(i) that $\mathsf{con}(\mathbf{w}) = \mathsf{con}(\mathbf{w}')$ and $\mathsf{sim}(\mathbf{w}) = \mathsf{sim}(\mathbf{w}')$.

Case 1: $\mathsf{sim}(\mathbf{w}) = \mathsf{sim}(\mathbf{w}') = \emptyset$. Then $m = m' = 1$ with $\mathbf{s}_1 = \mathbf{s}_1' = \emptyset$ so that $\mathbf{w} = \mathbf{w}_0\mathbf{w}_1$ and $\mathbf{w}' = \mathbf{w}_0'\mathbf{w}_1'$. For each $x \in \mathcal{X}$, since $\mathsf{occ}(x,\mathbf{w}) \equiv \mathsf{occ}(x,\mathbf{w}_1) \pmod 2$ and $\mathsf{occ}(x,\mathbf{w}') \equiv \mathsf{occ}(x,\mathbf{w}_1') \pmod 2$ by (III), it follows from Lemma 11.3(ii) that $\mathsf{occ}(x,\mathbf{w}_1) \equiv \mathsf{occ}(x,\mathbf{w}_1') \pmod 2$. Therefore $\mathbf{w}_1 = \mathbf{w}_1'$ by (II). It is then easy to deduce from (III), (IV), (V), and Lemma 11.3(ii) that $\mathbf{w}_0 = \mathbf{w}_0'$. Hence the identity $\mathbf{w} \approx \mathbf{w}'$ is trivial and is implied by the identities (11.1).

Case 2: $\mathsf{sim}(\mathbf{w}) = \mathsf{sim}(\mathbf{w}') \neq \emptyset$. Since the monoid $\mathcal{E}$ is noncommutative, it follows that $\mathbf{s}_1 \cdots \mathbf{s}_m = \mathbf{s}_1' \cdots \mathbf{s}_{m'}'$. It is first shown that the negation of either $\mathbf{w}_m = \mathbf{w}_{m'}'$ or $\mathbf{s}_m = \mathbf{s}_{m'}'$ leads to a contradiction.

Subcase 2.1: $\mathbf{w}_m \neq \mathbf{w}_{m'}'$. By symmetry, it suffices to assume that $x \in \mathsf{con}(\mathbf{w}_m) \setminus \mathsf{con}(\mathbf{w}_{m'}')$. Then $\mathsf{occ}(x,\mathbf{w}_m) = 1$ by (II) and $\mathsf{occ}(x,\mathbf{w}_{m'}') = 0$. Let z be the last letter of $\mathbf{s}_m$ and $\mathbf{s}_{m'}'$. Since $\mathsf{occ}(x,\mathbf{w}) \equiv \mathsf{occ}(x,\mathbf{w}') \pmod 2$ by Lemma 11.3(ii), it follows that

$$\begin{aligned} \mathsf{occ}(x,z,\mathbf{w}) &= \mathsf{occ}(x,\mathbf{w}) - \mathsf{occ}(x,\mathbf{w}_m) \not\equiv \mathsf{occ}(x,\mathbf{w}') - \mathsf{occ}(x,\mathbf{w}_{m'}') \\ &= \mathsf{occ}(x,z,\mathbf{w}') \pmod 2, \end{aligned}$$

whence Lemma 11.3(iii) is contradicted.

Subcase 2.2: $\mathbf{s}_m \neq \mathbf{s}_{m'}'$. By symmetry, it suffices to assume that $|\mathbf{s}_m| > |\mathbf{s}_{m'}'|$. Since $\mathbf{s}_1 \cdots \mathbf{s}_m = \mathbf{s}_1' \cdots \mathbf{s}_{m'}'$, there exists some $s \in \mathsf{con}(\mathbf{s}_m) \cap \mathsf{con}(\mathbf{s}_{m'-1}')$. It follows from (II) that $\mathsf{occ}(x,\mathbf{w}_{m'-1}') = 1$ for some $x \in \mathcal{X}$. Hence $\mathsf{occ}(x,s,\mathbf{w}) = \mathsf{occ}(x,\mathbf{w}) - \mathsf{occ}(x,\mathbf{w}_m)$ and

$$\begin{aligned} \mathsf{occ}(x,s,\mathbf{w}') &= \mathsf{occ}(x,\mathbf{w}') - \mathsf{occ}(x,\mathbf{w}_{m'-1}') - \mathsf{occ}(x,\mathbf{w}_{m'}') \\ &= \mathsf{occ}(x,\mathbf{w}') - 1 - \mathsf{occ}(x,\mathbf{w}_{m'}'). \end{aligned}$$

Now since $\mathsf{occ}(x, \mathbf{w}) \equiv \mathsf{occ}(x, \mathbf{w}') \pmod 2$ by Lemma 11.3(ii) and $\mathbf{w}_m = \mathbf{w}'_{m'}$ as established in Subcase 2.1, it follows that $\mathsf{occ}(x, s, \mathbf{w}) \not\equiv \mathsf{occ}(x, s, \mathbf{w}') \pmod 2$, contradicting Lemma 11.3(iii).

Arguments similar to Subcases 2.1 and 2.2 can be repeated to successively show that $\mathbf{w}_{m-j} = \mathbf{w}'_{m'-j}$ and $\mathbf{s}_{m-j} = \mathbf{s}'_{m'-j}$ for each $j \geq 1$, whence $\prod_{i=1}^{m}(\mathbf{s}_i\mathbf{w}_i) = \prod_{i=1}^{m'}(\mathbf{s}'_i\mathbf{w}'_i)$. It is then easy to deduce from (III), (IV), and (V) that $\mathbf{w}_0 = \mathbf{w}'_0$. Hence the identity $\mathbf{w} \approx \mathbf{w}'$ is trivial and is implied by the identities (11.1).

12. The monoid $\mathcal{F}$

12.1. Main result

PROPOSITION 12.1. *The identities*

$$xhx^2tx \overset{*}{\approx} xhtx, \tag{12.1a}$$

$$xhyx^2ty \overset{*}{\approx} xhx^2yty, \tag{12.1b}$$

$$xhytxy \overset{*}{\approx} xhytyx \tag{12.1c}$$

constitute a basis for the monoid $\mathcal{F}$ with the following multiplication table:

$\mathcal{F}$	1	2	3	4	5	6
1	1	1	1	1	1	1
2	1	1	1	1	2	2
3	1	2	3	3	3	3
4	1	2	4	4	4	4
5	1	2	3	4	5	6
6	1	2	4	3	6	5

Note that the identity element of the monoid $\mathcal{F}$ is 5.

12.2. A canonical form. In this chapter, a word

$$\mathbf{w} = \prod_{i=1}^{m}(x_i\mathbf{w}_i) \tag{12.2}$$

is said to be in *canonical form* if $x_1, \ldots, x_m$ are distinct letters and $\mathbf{w}_1, \ldots, \mathbf{w}_m$ are possibly empty words that satisfy all of the following conditions:

(I) $\mathsf{ini}(\mathbf{w}) = x_1 \cdots x_m$;
(II) $\mathbf{w}_i \in \{x_1^{e_1} \cdots x_i^{e_i} \mid e_1, \ldots, e_i \in \{0, 1, 2\}\}$;
(III) if $\mathsf{occ}(x, \mathbf{w}_i) = 2$, then $x \notin \mathsf{con}(\mathbf{w}_{i+1} \cdots \mathbf{w}_m)$;
(IV) if $\mathsf{occ}(x_j, \mathbf{w}_i) = 2$ for some $j < i$, then x_i is simple in $\mathbf{w}$.

Note that $x_i \notin \mathsf{con}(\mathbf{w}_1 \cdots \mathbf{w}_{i-1})$ by (II).

LEMMA 12.2. *Let $\mathbf{w}$ be any word. Then there exists some word $\mathbf{w}'$ in canonical form such that the identities* (12.1) *imply the identity $\mathbf{w} \approx \mathbf{w}'$.*

Proof. It suffices to convert the word $\mathbf{w}$, using the identities (12.1), into a word in canonical form. Without loss of generality, assume that $\mathsf{ini}(\mathbf{w}) = x_1 \cdots x_m$. Then $\mathbf{w}$ can be written in the form (12.2) for some $\mathbf{w}_1, \ldots, \mathbf{w}_m \in \mathcal{X}^*$ such that $\mathsf{con}(\mathbf{w}_i) \subseteq \{x_1, \ldots, x_i\}$ for all i. For each i, the letters in $\mathbf{w}_i$ are not first occurrences in $\mathbf{w}$ so that the identities (12.1c) can be used to permute them in any manner within $\mathbf{w}_i$. Hence for each i, the factor $\mathbf{w}_i$ is of the form $x_1^{e_1} \cdots x_i^{e_i}$ where $e_1, \ldots, e_i \geq 0$. The letters $x_1, \ldots, x_i$ in $\mathbf{w}_i$ are not first occurrences in $\mathbf{w}$ so that if $e_j \geq 3$ for some $j \leq i$, then the identities (12.1a) can be used to reduce e_j to a number in $\{1, 2\}$. Therefore (II) is satisfied by every $\mathbf{w}_i$. If $\mathsf{occ}(x_j, \mathbf{w}_i) = 2$ for some $j \leq i$ and $x_j \in \mathsf{con}(\mathbf{w}_{i+1} \cdots \mathbf{w}_m)$, say $x_j \in \mathsf{con}(\mathbf{w}_k)$ for some $k > i$, then

$$\mathbf{w} = \cdots x_j \cdots \underbrace{\cdots x_j^2 \cdots}_{\mathbf{w}_i} \cdots x_k \underbrace{\cdots x_j \cdots}_{\mathbf{w}_k} \cdots$$

and the identities (12.1a) can be used to eliminate the factor x_j^2 from $\mathbf{w}_i$. Hence (III) is satisfied by $\mathbf{w}$.

It is clear that the factor $\mathbf{w}_1$ satisfies (IV) vacuously. Suppose that $\ell > 1$ and that the factors $\mathbf{w}_{\ell+1}, \ldots, \mathbf{w}_m$ have been converted to words that satisfy (IV). It suffices to convert the factor $\mathbf{w}_\ell$ into a word that satisfies (IV). Let $e_1, \ldots, e_\ell \in \{0, 1, 2\}$ be such that $\mathbf{w}_\ell = x_1^{e_1} \cdots x_\ell^{e_\ell}$. Suppose that $\mathsf{occ}(x_j, \mathbf{w}_\ell) = 2$ for some $j < \ell$ and that x_ℓ is nonsimple in $\mathbf{w}$. Then

$$\mathbf{w} = \cdots x_j \cdots x_{\ell-1}\mathbf{w}_{\ell-1} \cdot x_\ell \cdot \underbrace{x_1^{e_1} \cdots x_{j-1}^{e_{j-1}} x_j^2 x_{j+1}^{e_{j+1}} \cdots x_\ell^{e_\ell}}_{\mathbf{w}_\ell} \cdots x_\ell^r \cdots$$

for some $r \geq 0$ such that $e_\ell + r > 0$. Now since the letters in $\mathbf{w}_\ell$ are not first occurrences,

$$\begin{aligned}
\mathbf{w} &= \cdots x_j \cdots x_{\ell-1}\mathbf{w}_{\ell-1} \cdot x_\ell \cdot \underbrace{x_1^{e_1} \cdots x_{j-1}^{e_{j-1}} x_j^2 x_{j+1}^{e_{j+1}} \cdots x_\ell^{e_\ell}}_{\mathbf{w}_\ell} \cdots x_\ell^r \cdots \\
&\overset{(12.1c)}{\approx} \cdots x_j \cdots x_{\ell-1}\mathbf{w}_{\ell-1} \cdot x_\ell x_j^2 \cdot x_1^{e_1} \cdots x_{j-1}^{e_{j-1}} x_{j+1}^{e_{j+1}} \cdots x_\ell^{e_\ell} \cdots x_\ell^r \cdots \\
&\overset{(12.1b)}{\approx} \cdots x_j \cdots x_{\ell-1}(\mathbf{w}_{\ell-1} x_j^2) \cdot x_\ell \cdot x_1^{e_1} \cdots x_{j-1}^{e_{j-1}} x_{j+1}^{e_{j+1}} \cdots x_\ell^{e_\ell} \cdots x_\ell^r \cdots . \qquad (12.3)
\end{aligned}$$

Since the letters in $\mathbf{w}_{\ell-1}$ are not first occurrences, the factor x_j^2 in (12.3) can be moved by the identities (12.1c) to the left until it is combined with any x_j in $\mathbf{w}_{\ell-1}$. (If this results in $\mathbf{w}_{\ell-1}$ containing more than two occurrences of x_j, then the identities (12.1a) can be used to eliminate x_j^2 from $\mathbf{w}_{\ell-1}$.) Hence the factor x_j^2 is eliminated from the factor $\mathbf{w}_\ell$. The same argument can be repeated to eliminate any square factors from $\mathbf{w}_\ell$. Therefore the factor $\mathbf{w}_\ell$ can be converted into a word that satisfies (IV). ∎

12.3. Identities of the monoid $\mathcal{F}$

LEMMA 12.3. *Let $\mathbf{w} \approx \mathbf{w}'$ be any identity satisfied by the monoid $\mathcal{F}$. Then*

(i) $\mathsf{ini}(\mathbf{w}) = \mathsf{ini}(\mathbf{w}')$ (*so that* $\mathsf{con}(\mathbf{w}) = \mathsf{con}(\mathbf{w}')$);

(ii) $\mathsf{sim}(\mathbf{w}) = \mathsf{sim}(\mathbf{w}')$;

(iii) $\mathsf{occ}(x, \mathbf{w}) \equiv \mathsf{occ}(x, \mathbf{w}') \pmod 2$ *for all* $x \in \mathcal{X}$.

Further, if $x \in \mathsf{con}(\mathbf{w}) = \mathsf{con}(\mathbf{w}')$ *and* $y \in \mathsf{sim}(\mathbf{w}) = \mathsf{sim}(\mathbf{w}')$, *then*

(iv) $\mathsf{occ}(x, y, \mathbf{w}) \equiv \mathsf{occ}(x, y, \mathbf{w}') \pmod 2$;
(v) $x \ll_{\mathbf{w}} y$ *if and only if* $x \ll_{\mathbf{w}'} y$.

Proof. The submonoids $\{3, 4, 5\}$, $\{1, 2, 5\}$, and $\{5, 6\}$ of $\mathcal{F}$ are isomorphic to L_2^1, N_2^1, and $\mathbb{Z}_2$ respectively. Therefore parts (i), (ii), and (iii) follow from Lemma 2.2. Suppose that $\mathsf{occ}(x, y, \mathbf{w}) \not\equiv \mathsf{occ}(x, y, \mathbf{w}') \pmod 2$ for some $x \in \mathsf{con}(\mathbf{w}) = \mathsf{con}(\mathbf{w}')$ and some $y \in \mathsf{sim}(\mathbf{w}) = \mathsf{sim}(\mathbf{w}')$, say $\mathsf{occ}(x, y, \mathbf{w})$ is odd and $\mathsf{occ}(x, y, \mathbf{w}')$ is even. Denote by φ_1 the following substitution into the monoid $\mathcal{F}$:

$$z \mapsto \begin{cases} 6 & \text{if } z = x, \\ 3 & \text{if } z = y, \\ 5 & \text{otherwise.} \end{cases}$$

Then $\mathbf{w}\varphi_1 = 6 \cdot 3 = 4$ and $\mathbf{w}'\varphi_1 = 5 \cdot 3 = 3$, which is impossible. Hence part (iv) holds.

Suppose that $x \in \mathsf{con}(\mathbf{w}) = \mathsf{con}(\mathbf{w}')$ and $y \in \mathsf{sim}(\mathbf{w}) = \mathsf{sim}(\mathbf{w}')$ are such that $x \ll_{\mathbf{w}} y$ and $x \not\ll_{\mathbf{w}'} y$. Then it follows from part (i) that within $\mathbf{w}$, every x occurs before y, while within $\mathbf{w}'$, two occurrences of x sandwich y. Denote by φ_2 the following substitution into $\mathcal{F}$:

$$z \mapsto \begin{cases} 3 & \text{if } z = x, \\ 2 & \text{if } z = y, \\ 5 & \text{otherwise.} \end{cases}$$

Then $\mathbf{w}\varphi_2 = 3 \cdot 2 = 2$ and $\mathbf{w}'\varphi_2 = 3 \cdot 2 \cdot 3 = 1$, which is impossible. Therefore part (v) holds. ∎

12.4. Proof of Proposition 12.1. It is routine to verify that the monoid $\mathcal{F}$ satisfies the identities (12.1). It remains to show that any identity $\mathbf{w} \approx \mathbf{w}'$ satisfied by $\mathcal{F}$ is implied by the identities (12.1). In view of Lemma 12.2, the words $\mathbf{w}$ and $\mathbf{w}'$ can be assumed to be in canonical form. Therefore by Lemma 12.3(i),

$$\mathbf{w} = \prod_{i=1}^{m} (x_i \mathbf{w}_i) \quad \text{and} \quad \mathbf{w}' = \prod_{i=1}^{m} (x_i \mathbf{w}_i').$$

First assume that $(\mathbf{w}_1, \ldots, \mathbf{w}_{m-1}) = (\mathbf{w}_1', \ldots, \mathbf{w}_{m-1}')$. By (II),

$$\mathbf{w}_m = x_1^{e_1} \cdots x_m^{e_m} \quad \text{and} \quad \mathbf{w}_m' = x_1^{e_1'} \cdots x_m^{e_m'}$$

for some $e_1, e_1', \ldots, e_m, e_m' \in \{0, 1, 2\}$. Since $\mathsf{occ}(x_m, \mathbf{w}) = 1 + e_m$ and $\mathsf{occ}(x_m, \mathbf{w}') = 1 + e_m'$, it follows from Lemma 12.3(ii)&(iii) that $e_m = e_m'$. Suppose that $e_i > e_i'$ for some $i < m$. Then $e_i = 2$ and $e_i' = 0$ by Lemma 12.3(iii). The letter x_m is then simple in $\mathbf{w}$ by (IV), whence x_m is also simple in $\mathbf{w}'$ by Lemma 12.3(ii). Then $x_i \not\ll_{\mathbf{w}} x_m$ and $x_i \ll_{\mathbf{w}'} x_m$, contradicting Lemma 12.3(v). Therefore i does not exist, whence $\mathbf{w}_m = \mathbf{w}_m'$. It follows that the identity $\mathbf{w} \approx \mathbf{w}'$ is trivial and so is clearly implied by the identities (12.1).

It remains to assume that $\mathbf{w}_\ell \neq \mathbf{w}_\ell'$ for some least possible $\ell < m$, say

$$\mathbf{w}_\ell = x_1^{e_1} \cdots x_\ell^{e_\ell} \quad \text{and} \quad \mathbf{w}_\ell' = x_1^{e_1'} \cdots x_\ell^{e_\ell'}$$

for some $e_1, e_1', \ldots, e_\ell, e_\ell' \in \{0, 1, 2\}$ with $(e_1, \ldots, e_\ell) \neq (e_1', \ldots, e_\ell')$. Let

$$\mathbf{w} = \mathbf{p}\, x_\ell \mathbf{w}_\ell\, \mathbf{q} \quad \text{and} \quad \mathbf{w}' = \mathbf{p}\, x_\ell \mathbf{w}_\ell'\, \mathbf{q}'$$

where $\mathbf{p} = \prod_{i=1}^{\ell-1}(x_i\mathbf{w}_i) = \prod_{i=1}^{\ell-1}(x_i\mathbf{w}'_i)$, $\mathbf{q} = \prod_{i=\ell+1}^{m}(x_i\mathbf{w}_i)$, and $\mathbf{q}' = \prod_{i=\ell+1}^{m}(x_i\mathbf{w}'_i)$. Since $\mathbf{w}_\ell \neq \mathbf{w}'_\ell$, there is some $k \leq \ell$ such that $e_k \neq e'_k$. By symmetry, it suffice to assume that $e_k > e'_k$. Since

$$\mathsf{occ}(x_k, x_{\ell+1}, \mathbf{w}) = \mathsf{occ}(x_k, \mathbf{p}x_\ell) + \mathsf{occ}(x_k, \mathbf{w}_\ell) = \mathsf{occ}(x_k, \mathbf{p}x_\ell) + e_k,$$

and similarly $\mathsf{occ}(x_k, x_{\ell+1}, \mathbf{w}') = \mathsf{occ}(x_k, \mathbf{p}x_\ell) + e'_k$, it follows from Lemma 12.3(iv) that $e_k = 2$ and $e'_k = 0$. Then there are two cases to consider: $k < \ell$ and $k = \ell$. It is shown below that these cases lead to contradictions. Therefore ℓ does not exist, whence the identity $\mathbf{w} \approx \mathbf{w}'$ is trivial and implied by the identities (12.1).

Case 1: $k < \ell$ (so that $\ell > 1$). Then

$$\mathbf{w} = \mathbf{p}\, x_\ell x_1^{e_1} \cdots x_{k-1}^{e_{k-1}}\, x_k^2\, x_{k+1}^{e_{k+1}} \cdots x_\ell^{e_\ell}\, \mathbf{q}$$

where $x_k \notin \mathsf{con}(\mathbf{q})$ by (III) and x_ℓ is simple in $\mathbf{w}$ by (IV) (so that $e_\ell = 0$), and

$$\mathbf{w}' = \mathbf{p}\, x_\ell \mathbf{w}'_\ell\, \mathbf{q}'$$

where $x_k \notin \mathsf{con}(\mathbf{w}'_\ell)$ by assumption and the letter x_ℓ is simple in $\mathbf{w}'$ by Lemma 12.3(ii). Since $x_k \in \mathsf{con}(\mathbf{p})$, the simple letter x_ℓ in $\mathbf{w}$ is sandwiched between two occurrences of x_k. By Lemma 12.3(i)&(v), the simple letter x_ℓ in $\mathbf{w}'$ is also sandwiched between two occurrences of x_k. Hence $x_k \in \mathsf{con}(\mathbf{q}')$. Further,

$$\mathsf{occ}(x_k, \mathbf{w}) = \mathsf{occ}(x_k, \mathbf{p}x_\ell) + \mathsf{occ}(x_k, \mathbf{w}_\ell) + \mathsf{occ}(x_k, \mathbf{q}) = \mathsf{occ}(x_k, \mathbf{p}x_\ell) + 2,$$
$$\mathsf{occ}(x_k, \mathbf{w}') = \mathsf{occ}(x_k, \mathbf{p}x_\ell) + \mathsf{occ}(x_k, \mathbf{w}'_\ell) + \mathsf{occ}(x_k, \mathbf{q}') = \mathsf{occ}(x_k, \mathbf{p}x_\ell) + \mathsf{occ}(x_k, \mathbf{q}').$$

Therefore $\mathsf{occ}(x_k, \mathbf{q}') \in \{2, 4, \ldots, \}$ by Lemma 12.3(iii). Since $\mathbf{q}' = \prod_{i=\ell+1}^{m}(x_i\mathbf{w}'_i)$, it follows that $x_k \in \mathsf{con}(\mathbf{w}'_r)$ for some $r \geq \ell + 1$. There are two subcases.

Subcase 1.1: $\mathsf{occ}(x_k, \mathbf{w}'_r) = 2$ for some $r \geq \ell + 1$. Then the letter x_r is simple in $\mathbf{w}'$ by (IV) and so is sandwiched between two occurrences of x_k. However, x_r is simple in $\mathbf{w}$ but is not sandwiched between two occurrences of x_k, contradicting Lemma 12.3(v).

Subcase 1.2: $\mathsf{occ}(x_k, \mathbf{w}'_r) \leq 1$ for all $r \geq \ell+1$. Since $\mathsf{occ}(x_k, \mathbf{q}') \in \{2, 4, \ldots, \}$, there exist least possible integers p and q such that $\mathsf{occ}(x_k, \mathbf{w}'_p) = \mathsf{occ}(x_k, \mathbf{w}'_q) = 1$ and $\ell+1 \leq p < q$. Then

$$\begin{aligned}\mathsf{occ}(x_k, x_q, \mathbf{w}') &= \mathsf{occ}(x_k, \mathbf{p}) + \mathsf{occ}(x_k, x_\ell\mathbf{w}'_\ell) + \mathsf{occ}(x_k, x_{\ell+1}\mathbf{w}'_{\ell+1} \cdots x_p\mathbf{w}'_p) \\ &= \mathsf{occ}(x_k, \mathbf{p}) + 0 + 1.\end{aligned}$$

However, since $x_k \notin \mathsf{con}(\mathbf{q})$,

$$\mathsf{occ}(x_k, x_q, \mathbf{w}) = \mathsf{occ}(x_k, \mathbf{p}) + \mathsf{occ}(x_k, x_\ell\mathbf{w}_\ell) = \mathsf{occ}(x_k, \mathbf{p}) + 2.$$

Therefore $\mathsf{occ}(x_k, x_q, \mathbf{w}') \not\equiv \mathsf{occ}(x_k, x_q, \mathbf{w}) \pmod 2$, contradicting Lemma 12.3(iv).

Case 2: $k = \ell$. Then

$$\mathbf{w} = \mathbf{p}\, x_\ell x_1^{e_1} \cdots x_{\ell-1}^{e_{\ell-1}}\, x_\ell^2\, \mathbf{q}$$

where $x_\ell \notin \mathsf{con}(\mathbf{q})$ by (III), and

$$\mathbf{w}' = \mathbf{p}\, x_\ell \mathbf{w}'_\ell\, \mathbf{q}'$$

where $x_\ell \notin \mathsf{con}(\mathbf{w}'_\ell)$ by assumption. Since the letter x_ℓ is nonsimple in $\mathbf{w}$, it is also nonsimple in $\mathbf{w}'$ by Lemma 12.3(ii), whence $x_\ell \in \mathsf{con}(\mathbf{q}')$. Further, since $\mathsf{occ}(x_\ell, \mathbf{w}) = 3$, it follows from Lemma 12.1(iii) that $\mathsf{occ}(x_\ell, \mathbf{q}') \in \{2, 4, \ldots\}$. Since $\mathbf{q}' = \prod_{i=\ell+1}^{m}(x_i \mathbf{w}_i')$, it follows that $x_\ell \in \mathsf{con}(\mathbf{w}_r')$ for some $r \geq \ell + 1$. Contradictions can then be deduced by applying the arguments in Subcases 1.1 and 1.2.

13. The monoid $\mathcal{G}$

13.1. Main result

PROPOSITION 13.1. *The identities*

$$x^2yx \approx yx^3, \tag{13.1a}$$

$$xyx^2zx \overset{*}{\approx} xyzx, \tag{13.1b}$$

$$xyhxty \overset{*}{\approx} yxhxty, \quad xhxyty \overset{*}{\approx} xhyxty, \quad xhytxy \overset{*}{\approx} xhytyx \tag{13.1c}$$

constitute a basis for the monoid $\mathcal{G}$ *with the following multiplication table*:

$\mathcal{G}$	1	2	3	4	5	6
1	1	1	1	1	1	1
2	1	1	1	1	2	3
3	1	1	1	1	3	2
4	1	2	3	4	4	4
5	1	2	3	4	5	6
6	1	2	3	4	6	5

Note that the identity element of the monoid $\mathcal{G}$ is 5.

13.2. A canonical form. In this chapter, a nonsimple word

$$\mathbf{w} = \prod_{i=1}^{m}(\mathbf{s}_i \mathbf{w}_i) \tag{13.2}$$

is said to be in *canonical form* if all of the following conditions are satisfied:

(I) the letters of $\mathbf{s}_1 \in \mathcal{X}^*$ and $\mathbf{s}_2, \ldots, \mathbf{s}_m \in \mathcal{X}^+$ are all simple in $\mathbf{w}$;
(II) the letters of $\mathbf{w}_1, \ldots, \mathbf{w}_{m-1} \in \mathcal{X}^+$ and $\mathbf{w}_m \in \mathcal{X}^*$ are all nonsimple in $\mathbf{w}$;
(III) for each i, the letters of $\mathbf{w}_i$ are in alphabetical order with $\mathsf{occ}(x, \mathbf{w}_i) \leq 3$ for any $x \in \mathcal{X}$;
(IV) if $\mathsf{occ}(x, \mathbf{w}_i) = 1$, then $\mathsf{occ}(x, \mathbf{w}_1), \ldots, \mathsf{occ}(x, \mathbf{w}_{i-1}) \leq 1$;
(V) if $\mathsf{occ}(x, \mathbf{w}_i) = 2$, then $\mathsf{occ}(x, \mathbf{w}_{i+1}) = \cdots = \mathsf{occ}(x, \mathbf{w}_m) = 0$;
(VI) if $\mathsf{occ}(x, \mathbf{w}_i) = 3$, then $\mathsf{occ}(x, \mathbf{w}_j) = 0$ for all $j \neq i$.

Note that (I) and (II) imply that $\mathsf{con}(\mathbf{s}_i) \cap \mathsf{con}(\mathbf{w}_j) = \emptyset$ for any i and j.

LEMMA 13.2. *Let* $\mathbf{w}$ *be any nonsimple word. Then there exists some word* $\mathbf{w}'$ *in canonical form such that the identities* (13.1) *imply the identity* $\mathbf{w} \approx \mathbf{w}'$.

Proof. It suffices to convert the nonsimple word $\mathbf{w}$, using the identities (13.1), into a word in canonical form. By gathering adjacent simple letters and adjacent nonsimple letters in the word $\mathbf{w}$, it is easy to see that $\mathbf{w}$ can be written in the form (13.2) that satisfies (I) and (II). For each i, since the letters in $\mathbf{w}_i$ are nonsimple in $\mathbf{w}$, they can be ordered by the identities (13.1c); then (III) is satisfied by applying the identity $x^4 \approx x^2$ from (13.1b). Specifically, each $\mathbf{w}_i$ is of the form $x_1^{e_1} \cdots x_r^{e_r}$ where $e_1, \ldots, e_r \in \{1, 2, 3\}$.

If $\mathsf{occ}(x, \mathbf{w}_i) = 1$ and $\mathsf{occ}(x, \mathbf{w}_j) \in \{2, 3\}$ for some $j < i$, say $\mathbf{w}_i = \mathbf{w}_i' \, x \, \mathbf{w}_i''$ and $\mathbf{w}_j = \mathbf{w}_j' \, x^{\mathsf{occ}(x,\mathbf{w}_j)} \, \mathbf{w}_j''$ where $\mathbf{w}_i', \mathbf{w}_i'', \mathbf{w}_j', \mathbf{w}_j'' \in (\mathscr{X} \setminus \{x\})^*$, then the identity (13.1a) can be used to gather a factor x^2 from $\mathbf{w}_j$ with the x in $\mathbf{w}_i$:

$$\mathbf{w} = \cdots \mathbf{w}_j' \, x^{\mathsf{occ}(x,\mathbf{w}_j)} \, \mathbf{w}_j'' \cdots \mathbf{w}_i' \, x \, \mathbf{w}_i'' \cdots \overset{(13.1\text{a})}{\approx} \cdots \mathbf{w}_j' \, x^{\mathsf{occ}(x,\mathbf{w}_j)-2} \, \mathbf{w}_j'' \cdots \mathbf{w}_i' \, x^3 \, \mathbf{w}_i'' \cdots .$$

Similarly, if $\mathsf{occ}(x, \mathbf{w}_i) = 2$ and $\mathsf{occ}(x, \mathbf{w}_j) \in \{1, 2, 3\}$ for some $j > i$, say $\mathbf{w}_i = \mathbf{w}_i' \, x^2 \, \mathbf{w}_i''$ and $\mathbf{w}_j = \mathbf{w}_j' \, x^{\mathsf{occ}(x,\mathbf{w}_j)} \, \mathbf{w}_j''$ where $\mathbf{w}_i', \mathbf{w}_i'', \mathbf{w}_j', \mathbf{w}_j'' \in (\mathscr{X} \setminus \{x\})^*$, then the identity (13.1a) can be used to gather the factor x^2 in $\mathbf{w}_i$ with the factor $x^{\mathsf{occ}(x,\mathbf{w}_j)}$ in $\mathbf{w}_j$:

$$\begin{aligned} \mathbf{w} \quad &= \quad \cdots \mathbf{w}_i' \, x^2 \, \mathbf{w}_i'' \cdots \mathbf{w}_j' \, x^{\mathsf{occ}(x,\mathbf{w}_j)} \, \mathbf{w}_j'' \cdots \\ &\overset{(13.1\text{a})}{\approx} \cdots \mathbf{w}_i' \, \mathbf{w}_i'' \cdots \mathbf{w}_j' \, x^{\mathsf{occ}(x,\mathbf{w}_j)+2} \, \mathbf{w}_j'' \cdots \\ &\overset{(13.1\text{b})}{\approx} \begin{cases} \cdots \mathbf{w}_i' \, \mathbf{w}_i'' \cdots \mathbf{w}_j' \, x^3 \, \mathbf{w}_j'' \cdots & \text{if } \mathsf{occ}(x, \mathbf{w}_j) = 1, \\ \cdots \mathbf{w}_i' \, \mathbf{w}_i'' \cdots \mathbf{w}_j' \, x^{\mathsf{occ}(x,\mathbf{w}_j)} \, \mathbf{w}_j'' \cdots & \text{if } \mathsf{occ}(x, \mathbf{w}_j) \in \{2, 3\}. \end{cases} \end{aligned}$$

Hence (IV) and (V) are satisfied. If $\mathsf{occ}(x, \mathbf{w}_i) = 3$ and $\mathsf{occ}(x, \mathbf{w}_j) > 0$ for some $j \neq i$, then it is easy to see that the identities (13.1b) can be used to reduce the exponent of x in $\mathbf{w}_i$ from 3 to 1. Hence (VI) is satisfied. ∎

13.3. Identities of the monoid $\mathcal{G}$

LEMMA 13.3. *Let* $\mathbf{w}$ *and* $\mathbf{w}'$ *be any words in canonical form such that the identity* $\mathbf{w} \approx \mathbf{w}'$ *is satisfied by the monoid* $\mathcal{G}$. *Then*

(i) $\mathsf{con}(\mathbf{w}) = \mathsf{con}(\mathbf{w}')$ *and* $\mathsf{sim}(\mathbf{w}) = \mathsf{sim}(\mathbf{w}')$.

Further, if $x \in \mathsf{con}(\mathbf{w}) = \mathsf{con}(\mathbf{w}')$ *and* $y, z \in \mathsf{sim}(\mathbf{w}) = \mathsf{sim}(\mathbf{w}')$, *then*

(ii) $\mathsf{occ}(x, \mathbf{w}) \equiv \mathsf{occ}(x, \mathbf{w}') \pmod 2$;
(iii) $\mathsf{occ}(x, y, \mathbf{w}) \equiv \mathsf{occ}(x, y, \mathbf{w}') \pmod 2$;
(iv) $x \ll_{\mathbf{w}} y$ *if and only if* $x \ll_{\mathbf{w}'} y$;
(v) yz *is a factor of* $\mathbf{w}$ *if and only if* yz *is a factor of* $\mathbf{w}'$.

Proof. Parts (i) and (ii) follow from Lemma 2.2 since the submonoids $\{1, 2, 5\}$ and $\{5, 6\}$ of $\mathcal{G}$ are isomorphic to N_2^1 and $\mathbb{Z}_2$ respectively.

(iii) Suppose that $\mathsf{occ}(x, y, \mathbf{w}) \not\equiv \mathsf{occ}(x, y, \mathbf{w}') \pmod 2$ for some $x \in \mathsf{con}(\mathbf{w}) = \mathsf{con}(\mathbf{w}')$ and $y \in \mathsf{sim}(\mathbf{w}) = \mathsf{sim}(\mathbf{w}')$, say $\mathsf{occ}(x, y, \mathbf{w}) = 2p+1$ is odd and $\mathsf{occ}(x, y, \mathbf{w}') = 2q$ is even. By part (ii), there exists some $r \in \{1, 2\}$ such that $\mathsf{occ}(x, \mathbf{w}) + r \equiv 0 \equiv \mathsf{occ}(x, \mathbf{w}') + r \pmod 2$. Denote by φ_1 the following substitution into $\mathcal{G}$:

$$z \mapsto \begin{cases} 6 & \text{if } z = x, \\ 2 & \text{if } z = y, \\ 5 & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned} (x^2 \mathbf{w} x^r)\varphi_1 &= 6^{2p+3} \cdot 2 \cdot 6^{\mathsf{occ}(x,\mathbf{w})-(2p+1)+r} = 6 \cdot 2 \cdot 6 = 3, \\ (x^2 \mathbf{w}' x^r)\varphi_1 &= 6^{2q+2} \cdot 2 \cdot 6^{\mathsf{occ}(x,\mathbf{w}')-2q+r} = 5 \cdot 2 \cdot 5 = 2, \end{aligned}$$

which is impossible.

(iv) Suppose that $x \in \mathsf{con}(\mathbf{w}) = \mathsf{con}(\mathbf{w}')$ and $y \in \mathsf{sim}(\mathbf{w}) = \mathsf{sim}(\mathbf{w}')$ are such that $x \ll_{\mathbf{w}} y$ and $x \not\ll_{\mathbf{w}'} y$. Then $\mathbf{w} = \mathbf{w}_1 y \mathbf{w}_2$ for some $\mathbf{w}_1, \mathbf{w}_2 \in (\mathscr{X} \setminus \{y\})^*$ such that $x \in \mathsf{con}(\mathbf{w}_1) \setminus \mathsf{con}(\mathbf{w}_2)$, and $\mathbf{w}' = \mathbf{w}_1' y \mathbf{w}_2'$ for some $\mathbf{w}_1', \mathbf{w}_2' \in (\mathscr{X} \setminus \{y\})^*$ such that $x \in \mathsf{con}(\mathbf{w}_2')$. Denote by φ_2 the following substitution into $\mathcal{G}$:

$$z \mapsto \begin{cases} 4 & \text{if } z = x, \\ 2 & \text{if } z = y, \\ 5 & \text{otherwise.} \end{cases}$$

Then $\mathbf{w}\varphi_2 = 4 \cdot 2 = 2$ and $\mathbf{w}'\varphi_2 = 4 \cdot 2 \cdot 4 = 1$, which is impossible.

(v) Suppose that $y, z \in \mathsf{sim}(\mathbf{w}) = \mathsf{sim}(\mathbf{w}')$ are such that yz is a factor of the word $\mathbf{w}$ but is not a factor of the word $\mathbf{w}'$. Then since the monoid $\mathcal{G}$ is noncommutative,

$$\mathbf{w} = \mathbf{p} yz \mathbf{q} \quad \text{and} \quad \mathbf{w}' = \mathbf{p}' y \mathbf{u} z \mathbf{q}'$$

for some $\mathbf{p}, \mathbf{q}, \mathbf{p}', \mathbf{q}' \in \mathscr{X}^*$ and $\mathbf{u} \in \mathscr{X}^+$. It is easy to see that if the factor $\mathbf{u}$ contains some simple letter of the word $\mathbf{w}'$, then the monoid $\mathcal{G}$ is contradictorily commutative. Hence every letter in the factor $\mathbf{u}$ is nonsimple in the word $\mathbf{w}'$. Since the word $\mathbf{w}'$ is in canonical form, it follows from (III) that $\mathbf{u} = x_1^{e_1} \cdots x_r^{e_r}$ for some $x_1, \ldots, x_r \in \mathsf{con}(\mathbf{w}') \setminus \mathsf{sim}(\mathbf{w}')$ and some $e_1, \ldots, e_r \in \{1, 2, 3\}$. Since $\mathsf{occ}(x_i, z, \mathbf{w}) \equiv \mathsf{occ}(x_i, z, \mathbf{w}') \pmod 2$ by part (iii),

$$\mathsf{occ}(x_i, y, \mathbf{w}) = \mathsf{occ}(x_i, z, \mathbf{w}) \equiv \mathsf{occ}(x_i, z, \mathbf{w}') = \mathsf{occ}(x_i, y, \mathbf{w}') + \mathsf{occ}(x_i, \mathbf{u}) \pmod 2.$$

But $\mathsf{occ}(x_i, y, \mathbf{w}) \equiv \mathsf{occ}(x_i, y, \mathbf{w}') \pmod 2$ by (iii) so that $\mathsf{occ}(x_i, \mathbf{u}) \equiv 0 \pmod 2$. Hence $e_i = 2$ for all i. It follows from (V) that $x_i \notin \mathsf{con}(\mathbf{q}')$. Therefore $x_i \ll_{\mathbf{w}'} z$, whence $x_i \ll_{\mathbf{w}} z$ by (iv). It is then obvious that $x_i \ll_{\mathbf{w}} y$, but (iv) is violated since $x_i \not\ll_{\mathbf{w}'} y$. ∎

13.4. Proof of Proposition 13.1. It is routine to verify that the monoid $\mathcal{G}$ satisfies the identities (13.1). It remains to show that any identity $\mathbf{w} \approx \mathbf{w}'$ satisfied by $\mathcal{G}$ is implied by the identities (13.1). Since $\mathcal{G}$ is noncommutative and its submonoid $\{1, 2, 5\}$ is isomorphic to N_2^1, it follows from Lemma 2.3 that the identity $\mathbf{w} \approx \mathbf{w}'$ is trivial if either $\mathbf{w}$ or $\mathbf{w}'$ is a simple word. Hence assume that they are both nonsimple. In view of Lemma 13.2, they can be assumed to be in canonical form, say

$$\mathbf{w} = \prod_{i=1}^{m} (\mathbf{s}_i \mathbf{w}_i) \quad \text{and} \quad \mathbf{w}' = \prod_{i=1}^{m'} (\mathbf{s}_i' \mathbf{w}_i').$$

It follows from Lemma 13.3(v) that $m = m'$ and $\mathbf{s}_i = \mathbf{s}_i'$ for every i. Hence

$$\mathbf{w}' = \prod_{i=1}^{m} (\mathbf{s}_i \mathbf{w}_i').$$

It is easy to show that if $m = 1$, then $\mathbf{w}_1 = \mathbf{w}_1'$ by (II), (III), and Lemma 13.3(i)&(ii), whence the identity $\mathbf{w} \approx \mathbf{w}'$ is trivial and so is satisfied by the monoid $\mathcal{G}$. Therefore assume that $m > 1$ and let ℓ be the least integer such that $\mathbf{w}_\ell \neq \mathbf{w}_\ell'$. Let

$$\mathbf{w} = \mathbf{p} \mathbf{w}_\ell \mathbf{r} \quad \text{and} \quad \mathbf{w}' = \mathbf{p} \mathbf{w}_\ell' \mathbf{r}'$$

where $\mathbf{p} = (\prod_{i=1}^{\ell-1} (\mathbf{s}_i \mathbf{w}_i)) \mathbf{s}_\ell$, $\mathbf{r} = \prod_{i=\ell+1}^{m} (\mathbf{s}_i \mathbf{w}_i)$, and $\mathbf{r}' = \prod_{i=\ell+1}^{m} (\mathbf{s}_i \mathbf{w}_i')$.

Case 1: $\ell < m$. Then $\mathbf{s}_{\ell+1} \neq \emptyset$. Since

$$\mathsf{occ}(x, \lambda(\mathbf{s}_{\ell+1}), \mathbf{w}) = \mathsf{occ}(x, \mathbf{p}) + \mathsf{occ}(x, \mathbf{w}_\ell),$$
$$\mathsf{occ}(x, \lambda(\mathbf{s}_{\ell+1}), \mathbf{w}') = \mathsf{occ}(x, \mathbf{p}) + \mathsf{occ}(x, \mathbf{w}'_\ell),$$

Lemma 13.3(iii) implies that $\mathsf{occ}(x, \mathbf{w}_\ell) \equiv \mathsf{occ}(x, \mathbf{w}'_\ell) \pmod 2$ for all $x \in \mathscr{X}$. By symmetry, it suffices to assume that $\mathsf{occ}(x, \mathbf{w}_\ell) > \mathsf{occ}(x, \mathbf{w}'_\ell)$. Then $(\mathsf{occ}(x, \mathbf{w}_\ell), \mathsf{occ}(x, \mathbf{w}'_\ell)) \in \{(2, 0), (3, 1)\}$ by (III).

Subcase 1.1: $\mathsf{occ}(x, \mathbf{w}_\ell) = 2$ and $\mathsf{occ}(x, \mathbf{w}'_\ell) = 0$. Then it follows from (V) that $\mathsf{occ}(x, \mathbf{r}) = 0$ so that $x \ll_{\mathbf{w}} \lambda(\mathbf{s}_{\ell+1})$. Hence $x \ll_{\mathbf{w}'} \lambda(\mathbf{s}_{\ell+1})$ by Lemma 13.3(iv) so that $x \notin \mathsf{con}(\mathbf{r}')$. Now $x \notin \mathsf{con}(\mathbf{w}'_\ell)$ by assumption and $x \notin \mathsf{con}(\mathbf{s}_\ell)$ since x is nonsimple in $\mathbf{w}'$, and it follows that $x \ll_{\mathbf{w}'} \lambda(\mathbf{s}_\ell)$. However, the assumption $\mathsf{occ}(x, \mathbf{w}_\ell) = 2$ implies that $x \not\ll_{\mathbf{w}} \lambda(\mathbf{s}_\ell)$, whence Lemma 13.3(iv) is violated.

Subcase 1.2: $\mathsf{occ}(x, \mathbf{w}_\ell) = 3$ and $\mathsf{occ}(x, \mathbf{w}'_\ell) = 1$. Then it follows from (VI) that $\mathsf{occ}(x, \mathbf{w}_i) = 0$ for all $i \neq \ell$, whence $\mathsf{occ}(x, \mathbf{p}) = \mathsf{occ}(x, \mathbf{r}) = 0$ by (I) and (II). Specifically, $x \ll_{\mathbf{w}} \lambda(\mathbf{s}_{\ell+1})$. Now the letter x is nonsimple in $\mathbf{w}'$ by Lemma 13.3(i) so that

$$1 < \mathsf{occ}(x, \mathbf{w}') = \mathsf{occ}(x, \mathbf{p}) + \mathsf{occ}(x, \mathbf{w}'_\ell) + \mathsf{occ}(x, \mathbf{r}') = 1 + \mathsf{occ}(x, \mathbf{r}'),$$

that is, $x \in \mathsf{con}(\mathbf{r}')$. Hence $x \not\ll_{\mathbf{w}'} \lambda(\mathbf{s}_{\ell+1})$ in violation of Lemma 13.3(iv).

Case 2: $\ell = m$ so that $\mathbf{w} = \mathbf{p}\mathbf{w}_m$ and $\mathbf{w}' = \mathbf{p}\mathbf{w}'_m$. Then it follows from Lemma 13.3(ii) that $\mathsf{occ}(x, \mathbf{w}_m) \equiv \mathsf{occ}(x, \mathbf{w}'_m) \pmod 2$ for all $x \in \mathscr{X}$. Without loss of generality, assume that $\mathsf{occ}(x, \mathbf{w}_m) > \mathsf{occ}(x, \mathbf{w}'_m)$. Then $(\mathsf{occ}(x, \mathbf{w}_m), \mathsf{occ}(x, \mathbf{w}'_m)) \in \{(2, 0), (3, 1)\}$ by (III).

Subcase 2.1: $\mathsf{occ}(x, \mathbf{w}_m) = 2$ and $\mathsf{occ}(x, \mathbf{w}'_m) = 0$. Since $x \in \mathsf{con}(\mathbf{w}) = \mathsf{con}(\mathbf{w}')$ by Lemma 13.3(i),

$$0 < \mathsf{occ}(x, \mathbf{w}') = \mathsf{occ}(x, \mathbf{p}) + \mathsf{occ}(x, \mathbf{w}'_m) = \mathsf{occ}(x, \mathbf{p}).$$

Therefore $x \not\ll_{\mathbf{w}} \lambda(\mathbf{s}_m)$ and $x \ll_{\mathbf{w}'} \lambda(\mathbf{s}_m)$ in violation of Lemma 13.3(iv).

Subcase 2.2: $\mathsf{occ}(x, \mathbf{w}_m) = 3$ and $\mathsf{occ}(x, \mathbf{w}'_m) = 1$. Then it follows from (VI) that $\mathsf{occ}(x, \mathbf{w}_i) = 0$ for all $i < m$, whence $\mathsf{occ}(x, \mathbf{p}) = 0$ by (I) and (II). Since

$$\mathsf{occ}(x, \mathbf{w}) = \mathsf{occ}(x, \mathbf{p}) + \mathsf{occ}(x, \mathbf{w}_m) = 3,$$
$$\mathsf{occ}(x, \mathbf{w}') = \mathsf{occ}(x, \mathbf{p}) + \mathsf{occ}(x, \mathbf{w}'_m) = 1,$$

it follows that $\mathsf{sim}(\mathbf{w}) \neq \mathsf{sim}(\mathbf{w}')$, whence Lemma 13.3(i) is violated.

Consequently, Cases 1 and 2 are both impossible, whence the integer ℓ does not exist. The identity $\mathbf{w} \approx \mathbf{w}'$ is trivial and thus satisfied by the monoid $\mathcal{G}$.

14. The monoid $\mathcal{I}$

14.1. Main result

Proposition 14.1. *The identities*

$$x^2yx \approx xyx^2 \approx xyx, \quad x^3 \approx x^2, \tag{14.1a}$$
$$x^2y^2x \approx x^2y^2 \tag{14.1b}$$

constitute a basis for the monoid $\mathcal{I}$ with the following multiplication table:

$\mathcal{I}$	1	2	3	4	5	6
1	1	1	1	1	1	1
2	1	1	1	1	2	3
3	1	1	1	3	3	3
4	1	2	3	4	4	4
5	1	2	3	4	5	6
6	1	2	3	6	6	6

Note that the identity element of the monoid $\mathcal{I}$ is 5.

14.2. A canonical form. In this chapter, a word

$$\mathbf{w} = \mathbf{w}_0 \prod_{i=1}^{m} (x_i \mathbf{w}_i) \tag{14.2}$$

is said to be in *canonical form* if $x_1, \ldots, x_m$ are all the simple letters of $\mathbf{w}$ and for each i, the factor $\mathbf{w}_i$ is either empty or of the form $y_1^2 \cdots y_r^2$ for some distinct nonsimple letters $y_1, \ldots, y_r$ of $\mathbf{w}$.

LEMMA 14.2. *Let $\mathbf{w}$ be any word. Then there exists some word $\mathbf{w}'$ in canonical form such that the identities* (14.1) *imply the identity $\mathbf{w} \approx \mathbf{w}'$.*

Proof. It suffices to convert the word $\mathbf{w}$, using the identities (14.1), into a word in canonical form. It is easy to show by induction that the identity (14.1b) implies the identity $\alpha_n : x^2 y_1^2 \cdots y_n^2 x^2 \approx x^2 y_1^2 \cdots y_n^2$ for any $n \geq 1$. Let $x_1, \ldots, x_m$ be the simple letters of $\mathbf{w}$ in order of first occurrence. Then the word $\mathbf{w}$ can be written in the form (14.2) where each factor $\mathbf{w}_i$, if nonempty, contains only nonsimple letters of $\mathbf{w}$. Apply the identities (14.1a) to replace each letter y in each $\mathbf{w}_i$ by y^2. Therefore by applying the identity (14.1b) and its consequences α_n, each factor $\mathbf{w}_i$ is converted to the form $y_1^2 \cdots y_r^2$ where the letters $y_1, \ldots, y_r$ are distinct. ∎

14.3. Proof of Proposition 14.1. It is routine to verify that the monoid $\mathcal{I}$ satisfies the identities (14.1). It remains to show that any identity $\mathbf{w} \approx \mathbf{w}'$ satisfied by $\mathcal{I}$ is implied by the identities (14.1). In view of Lemma 14.2, the words $\mathbf{w}$ and $\mathbf{w}'$ can be assumed to be in canonical form. Since the submonoid $\{1, 2, 5\}$ of $\mathcal{I}$ is isomorphic to N_2^1, it follows from Lemma 2.2(ii) that $\mathsf{con}(\mathbf{w}) = \mathsf{con}(\mathbf{w}')$ and $\mathsf{sim}(\mathbf{w}) = \mathsf{sim}(\mathbf{w}')$. Further, since $\mathcal{I}$ is noncommutative, the order of appearance of the simple letters of $\mathbf{w}$ and $\mathbf{w}'$ is the same. Therefore

$$\mathbf{w} = \mathbf{w}_0 \prod_{i=1}^{m} (x_i \mathbf{w}_i) \quad \text{and} \quad \mathbf{w}' = \mathbf{w}'_0 \prod_{i=1}^{m} (x_i \mathbf{w}'_i)$$

where $\mathsf{sim}(\mathbf{w}) = \mathsf{sim}(\mathbf{w}') = \{x_1, \ldots, x_m\}$. Since the submonoid $\{4, 5, 6\}$ of $\mathcal{I}$ is isomorphic to L_2^1, it follows from Lemma 2.2(i) that $\mathsf{ini}(\mathbf{w}) = \mathsf{ini}(\mathbf{w}')$, whence $\mathbf{w}_0 = \mathbf{w}'_0$.

Suppose that $\mathsf{con}(\mathbf{w}_m) \neq \mathsf{con}(\mathbf{w}'_m)$, say $y \in \mathsf{con}(\mathbf{w}_m) \setminus \mathsf{con}(\mathbf{w}'_m)$. Denote by φ_1 the following substitution into the monoid $\mathcal{I}$:

$$t \mapsto \begin{cases} 2 & \text{if } t = x_m, \\ 4 & \text{if } t = y, \\ 5 & \text{otherwise.} \end{cases}$$

Then $\mathbf{w}\varphi_1 \in \{4,5\} \cdot 2 \cdot 4 = \{1\}$ and $\mathbf{w}'\varphi_1 = 4 \cdot 2 = 2$, which is impossible. Therefore $\mathsf{con}(\mathbf{w}_m) = \mathsf{con}(\mathbf{w}'_m)$. Suppose that $\mathsf{con}(\mathbf{w}_i) \neq \mathsf{con}(\mathbf{w}'_i)$ for some i with $0 < i < m$, say $y \in \mathsf{con}(\mathbf{w}_i) \setminus \mathsf{con}(\mathbf{w}'_i)$. Denote by φ_2 the following substitution into $\mathcal{I}$:

$$t \mapsto \begin{cases} 2 & \text{if } t = x_i, \\ 6 & \text{if } t = x_{i+1}, \\ 4 & \text{if } t = y, \\ 5 & \text{otherwise.} \end{cases}$$

Then $\mathbf{w}\varphi_2 \in \{4,5\} \cdot 2 \cdot 4 \cdot 6 \cdot \{4,5\} = \{1\}$ and $\mathbf{w}'\varphi_2 \in \{4,5\} \cdot 2 \cdot 6 \cdot \{4,5\} = \{3\}$, which is impossible. Therefore $\mathsf{con}(\mathbf{w}_i) = \mathsf{con}(\mathbf{w}'_i)$ whenever $0 < i \leq m$.

Suppose that $\mathbf{w}_i \neq \mathbf{w}'_i$ for some i such that $0 < i \leq m$. Then there exist some nonsimple letters y and z such that $\mathbf{w}_i = \cdots y^2 \cdots z^2 \cdots$ and $\mathbf{w}'_i = \cdots z^2 \cdots y^2 \cdots$. Denote by φ_3 the following substitution into $\mathcal{I}$:

$$t \mapsto \begin{cases} 2 & \text{if } t = x_i, \\ 4 & \text{if } t = y, \\ 6 & \text{if } t = z, \\ 5 & \text{otherwise.} \end{cases}$$

Then $\mathbf{w}\varphi_3 \in \{4,5,6\} \cdot 2 \cdot 4 \cdot 6 \cdot \{4,5,6\} = \{1\}$ and $\mathbf{w}'\varphi_3 \in \{4,5,6\} \cdot 2 \cdot 6 \cdot 4 \cdot \{4,5,6\} = \{3\}$, which is impossible. Therefore $\mathbf{w}_i = \mathbf{w}'_i$. Consequently, the identity $\mathbf{w} \approx \mathbf{w}'$ is trivial and so is implied by the identities (14.1).

15. The monoids $\mathcal{J}$ and $\mathcal{K}$

15.1. Main result

PROPOSITION 15.1. *The identities*

$$xyx^2 \overset{*}{\approx} xyx, \tag{15.1a}$$

$$xhytxy \overset{*}{\approx} xhytyx, \tag{15.1b}$$

$$x^2y^2x \overset{*}{\approx} x^2y^2 \tag{15.1c}$$

constitute a basis for the monoids $\mathcal{J}$ and $\mathcal{K}$ with the following multiplication tables:

$\mathcal{J}$	1	2	3	4	5	6
1	1	1	1	1	1	1
2	1	1	1	1	2	3
3	3	3	3	3	3	3
4	1	2	3	4	4	1
5	1	2	3	4	5	6
6	6	6	6	6	6	6

$\mathcal{K}$	1	2	3	4	5	6
1	1	1	1	1	1	1
2	1	1	1	1	2	3
3	3	3	3	3	3	3
4	1	2	3	4	4	3
5	1	2	3	4	5	6
6	6	6	6	6	6	6

Note that the identity elements of the monoids $\mathscr{J}$ and $\mathscr{K}$ are both 5.

15.2. Identities of the monoids $\mathscr{J}$ and $\mathscr{K}$

LEMMA 15.2. *Let $x, y \in \mathscr{X}$ and $\mathbf{u} \in \mathscr{X}^+$ be such that $x, y \notin \mathsf{con}(\mathbf{u})$ and $\mathsf{sim}(\mathbf{u}) = \emptyset$.*

(i) *The identities* (15.1) *imply the identity $\mathbf{u} \approx \mathbf{v}^2$ for some $\mathbf{v} \in \mathscr{X}^+$.*
(ii) *The identities* (15.1) *imply the identities*

$$xyx\mathbf{u}x \overset{*}{\approx} xyx\mathbf{u}. \tag{15.2}$$

Proof. (i) Let $\mathsf{ini}(\mathbf{u}) = x_1 \cdots x_m$. By assumption, the letters $x_1, \ldots, x_m$ are all nonsimple in $\mathbf{u}$ so that the result clearly holds if $m = 1$. Therefore assume that $m \geq 2$. Write the word $\mathbf{u}$ in the form

$$\mathbf{u} = \prod_{i=1}^{m} (x_i \mathbf{u}_i)$$

where $\mathbf{u}_1, \ldots, \mathbf{u}_m$ are possibly empty words such that $\mathsf{con}(\mathbf{u}_i) \subseteq \{x_1, \ldots, x_i\}$. Since the letter x_m is nonsimple in the word $\mathbf{u}$, all occurrences of x_m in $\mathbf{u}$ must belong to the suffix $x_m \mathbf{u}_m$, whence $\mathbf{u}_m \neq \emptyset$. Let $\mathbf{v} = (\prod_{i=1}^{m-1} (x_i \mathbf{u}_i)) x_m$ so that

$$\mathbf{u} = \mathbf{v}\mathbf{u}_m.$$

For any word $\mathbf{w}$, let $\widetilde{\mathbf{w}}$ be the word obtained from $\mathbf{w}$ by putting its letters in alphabetical order. (For example, if $\mathbf{w} = x_7 x_4^2 x_1^4 x_4 x_2^2 x_7 x_1^3 x_4^3 x_2 x_1$ then $\widetilde{\mathbf{w}} = x_1^8 x_2^3 x_4^6 x_7^2$.) It is clear that $\mathsf{con}(\mathbf{v}) = \{x_1, \ldots, x_m\}$, whence $\widetilde{\mathbf{v}} = x_1^{e_1} \cdots x_m^{e_m}$ for some $e_1, \ldots, e_m \geq 1$.

CASE 1: $\mathsf{con}(\mathbf{u}_m) = \mathsf{con}(\mathbf{v})$, say $\widetilde{\mathbf{u}}_m = x_1^{f_1} \cdots x_m^{f_m}$ for some $f_1, \ldots, f_m \geq 1$. Since any letter in $\mathbf{u}_m$ also occurs in $\mathbf{v}$,

$$\mathbf{u} = \mathbf{v}\mathbf{u}_m \overset{(15.1\text{b})}{\approx} \mathbf{v} x_1^{f_1} \cdots x_m^{f_m} \overset{(15.1\text{a})}{\approx} \mathbf{v} x_1^{e_1} \cdots x_m^{e_m} = \mathbf{v}\widetilde{\mathbf{v}} \overset{(15.1\text{b})}{\approx} \mathbf{v}\mathbf{v}.$$

CASE 2: $\mathsf{con}(\mathbf{u}_m) \neq \mathsf{con}(\mathbf{v})$. Let x_{ℓ_1} be the last letter in $\mathbf{v}$ that does not occur in $\mathbf{u}_m$. Then $\mathbf{v} = \mathbf{v}' x_{\ell_1} \mathbf{v}''$ for some $\mathbf{v}', \mathbf{v}'' \in \mathscr{X}^*$ such that $x_{\ell_1} \in \mathsf{con}(\mathbf{v}') \setminus \mathsf{con}(\mathbf{v}'')$ and $\mathsf{con}(\mathbf{v}'') \subseteq \mathsf{con}(\mathbf{u}_m)$. Suppose that all distinct letters of $\mathbf{v}''$, when listed in alphabetical order, are $x_{j_1}, \ldots, x_{j_r}$ (so that $\widetilde{\mathbf{v}}'' = x_{j_1}^{g_1} \cdots x_{j_r}^{g_r}$ for some $g_1, \ldots, g_r \geq 1$). Then $\mathbf{u}_m \mathrel{\mathring{=}} x_{j_1} \cdots x_{j_r} \mathbf{u}'_m$ for some $\mathbf{u}'_m \in \mathscr{X}^*$. Since any letter in $\mathbf{u}_m$ also occurs in $\mathbf{v}$,

$$\begin{aligned}
\mathbf{v}\mathbf{u}_m &= \mathbf{v}' x_{\ell_1} \mathbf{v}'' \mathbf{u}_m \overset{(15.1\text{b})}{\approx} \mathbf{v}' x_{\ell_1} \mathbf{v}'' x_{j_1} \cdots x_{j_r} \mathbf{u}'_m \overset{(15.1\text{a})}{\approx} \mathbf{v}' x_{\ell_1}^2 \mathbf{v}'' \widetilde{\mathbf{v}}'' \mathbf{u}'_m \\
&\overset{(15.1\text{b})}{\approx} \mathbf{v}' (x_{\ell_1}^2 \mathbf{v}'' \mathbf{v}'') \mathbf{u}'_m \overset{(15.1\text{c})}{\approx} \mathbf{v}' x_{\ell_1}^2 \mathbf{v}'' \mathbf{v}'' x_{\ell_1} \mathbf{u}'_m \overset{(15.1\text{b})}{\approx} \mathbf{v}' x_{\ell_1}^2 \mathbf{v}'' \widetilde{\mathbf{v}}'' \mathbf{u}'_m x_{\ell_1} \\
&\overset{(15.1\text{a})}{\approx} \mathbf{v}' x_{\ell_1} \mathbf{v}'' (x_{j_1} \cdots x_{j_r} \mathbf{u}'_m) x_{\ell_1} \overset{(15.1\text{b})}{\approx} \mathbf{v}' x_{\ell_1} \mathbf{v}'' \mathbf{u}_m x_{\ell_1} = \mathbf{v}\mathbf{u}_m x_{\ell_1}.
\end{aligned}$$

Therefore $\mathbf{v}\mathbf{u}_m \overset{(15.1)}{\approx} \mathbf{v}\mathbf{u}_m x_{\ell_1}$. If x_{ℓ_2} is the last letter in $\mathbf{v}$ that does not occur in $\mathbf{u}_m x_{\ell_1}$, then repeat the same argument to deduce that $\mathbf{v}\mathbf{u}_m x_{\ell_1} \overset{(15.1)}{\approx} \mathbf{v}\mathbf{u}_m x_{\ell_1} x_{\ell_2}$. It is easy to see how this can be continued so that $\mathbf{v}\mathbf{u}_m x_{\ell_1} \cdots x_{\ell_{k-1}} \overset{(15.1)}{\approx} \mathbf{v}\mathbf{u}_m x_{\ell_1} \cdots x_{\ell_{k-1}} x_{\ell_k}$ where every letter from $\mathbf{v}$ belongs to $\mathbf{u}_m^\star = \mathbf{u}_m x_{\ell_1} \cdots x_{\ell_{k-1}} x_{\ell_k}$. It follows that $\mathbf{u} \overset{(15.1)}{\approx} \mathbf{v}\mathbf{u}_m^\star$ with $\mathsf{con}(\mathbf{u}_m^\star) = \mathsf{con}(\mathbf{v})$. Repeat the argument in Case 1 to deduce that $\mathbf{v}\mathbf{u}_m^\star \overset{(15.1)}{\approx} \mathbf{v}\mathbf{v}$. Consequently, $\mathbf{u} \overset{(15.1)}{\approx} \mathbf{v}\mathbf{v}$.

(ii) By part (i), there exists some $\mathbf{v} \in \mathscr{X}^+$ such that $\mathbf{u} \overset{(15.1)}{\approx} \mathbf{v}^2$. Since

$$xyx\mathbf{u}x \overset{(15.1)}{\approx} xyx\mathbf{v}^2x \overset{(15.1a)}{\approx} xyx^2\mathbf{v}^2x \overset{(15.1c)}{\approx} xyx^2\mathbf{v}^2 \overset{(15.1a)}{\approx} xyx\mathbf{v}^2 \overset{(15.1)}{\approx} xyx\mathbf{u},$$

the identities (15.1) imply the identities (15.2). ∎

LEMMA 15.3. *Let $M \in \{\mathcal{J}, \mathcal{K}\}$ and let $\mathbf{w} \approx \mathbf{w}'$ be any identity satisfied by the monoid M. Then*

(i) $\mathsf{ini}(\mathbf{w}) = \mathsf{ini}(\mathbf{w}')$ (*so that* $\mathsf{con}(\mathbf{w}) = \mathsf{con}(\mathbf{w}')$);
(ii) $\mathsf{sim}(\mathbf{w}) = \mathsf{sim}(\mathbf{w}')$.

Further, if $x, y \in \mathsf{con}(\mathbf{w}) = \mathsf{con}(\mathbf{w}')$, then

(iii) $\mathsf{occ}(x, y, \mathbf{w}) = 1$ *if and only if* $\mathsf{occ}(x, y, \mathbf{w}') = 1$.

Proof. The submonoids $\{1, 3, 5\}$ and $\{1, 2, 5\}$ of M are isomorphic to L_2^1 and N_2^1 respectively. Hence parts (i) and (ii) follow from Lemma 2.2. Suppose that $\mathsf{occ}(x, y, \mathbf{w}) = 1$ and $\mathsf{occ}(x, y, \mathbf{w}') = k \neq 1$ for some $x, y \in \mathsf{con}(\mathbf{w}) = \mathsf{con}(\mathbf{w}')$. If $k = 0$, then $\mathsf{ini}(\mathbf{w}) \neq \mathsf{ini}(\mathbf{w}')$ and part (i) is violated. Therefore further assume that $k \geq 2$. Denote by φ the following substitution into the monoid M:

$$z \mapsto \begin{cases} 2 & \text{if } z = x, \\ 6 & \text{if } z = y, \\ 5 & \text{otherwise.} \end{cases}$$

Then $\mathbf{w}\varphi = 2 \cdot 6 \cdots = 3$ and $\mathbf{w}'\varphi = 2^k \cdot 6 \cdots = 1$, which is impossible. Hence part (iii) holds. ∎

LEMMA 15.4. *Let $\mathbf{u}, \mathbf{u}' \in \{x, y, z\}^*$ be such that either $\mathbf{u} = \mathbf{u}' = \emptyset$ or $\lambda(\mathbf{u}) = \lambda(\mathbf{u}') = z$. Then the monoids $\mathcal{J}$ and $\mathcal{K}$ do not satisfy the identity $x^n yx\mathbf{u} \approx x^n y\mathbf{u}'$ for any $n \geq 1$.*

Proof. Let $M \in \{\mathcal{J}, \mathcal{K}\}$ and let φ be the substitution $(x, y, z) \mapsto (4, 2, 6)$ into M. Since $(x^n yx\mathbf{u})\varphi = 1 \cdot (\mathbf{u}\varphi) = 1$ and

$$(x^n y\mathbf{u}')\varphi = 2 \cdot (\mathbf{u}'\varphi) = \begin{cases} 2 & \text{if } \mathbf{u}' = \emptyset, \\ 3 & \text{otherwise,} \end{cases}$$

the monoid M does not satisfy the identity $x^2yxz\mathbf{u} \approx x^2yz\mathbf{u}'$. ∎

15.3. A canonical form. In this chapter, a word

$$\mathbf{w} = x_0^{e_0} \prod_{i=1}^{m} (x_i^{e_i} \mathbf{w}_i) \tag{15.3}$$

is said to be in *canonical form* if $x_0, \ldots, x_m$ are distinct letters and $\mathbf{w}_1, \ldots, \mathbf{w}_m$ are possibly empty words such that

(I) $\mathsf{ini}(\mathbf{w}) = x_0 \cdots x_m$;
(II) $e_0, \ldots, e_m \in \{1, 2\}$;
(III) $\mathbf{w}_i \in \{x_0^{f_0} \cdots x_{i-1}^{f_{i-1}} \mid f_0, \ldots, f_{i-1} \in \{0, 1\}\}$;
(IV) for any i with $\mathsf{occ}(x_i, \mathbf{w}) \geq 3$ and any $j > 1$, if the jth occurrence of x_i is in $x_p^{e_p}\mathbf{w}_p$ and the $(j+1)$st occurrence of x_i is in $x_r^{e_r}\mathbf{w}_r$ for some $r > p$, then there exists q with $p < q \leq r$ such that $e_q = 1$ and $x_q \notin \mathsf{con}(\mathbf{w}_{q+1} \cdots \mathbf{w}_r)$.

Note that since $x_i \notin \mathsf{con}(\mathbf{w}_1 \cdots \mathbf{w}_i)$ by (III), it follows from (IV) that

(V) $x_q \notin \mathsf{con}(\mathbf{w}_1 \cdots \mathbf{w}_r)$ in (IV).

LEMMA 15.5. *Let* $\mathbf{w}$ *be any word. Then there exists some word* $\mathbf{w}'$ *in canonical form such that the identities* (15.1) *imply the identity* $\mathbf{w} \approx \mathbf{w}'$.

Proof. It suffices to convert the word $\mathbf{w}$, using the identities (15.1), into a word in canonical form. Without loss of generality, assume that $\mathsf{ini}(\mathbf{w}) = x_0 \cdots x_m$. Then $\mathbf{w}$ can be written in the form

$$\mathbf{w} = x_0^{e_0} \prod_{i=1}^{m} (x_i^{e_i} \mathbf{w}_i)$$

for some $e_0, \ldots, e_m \geq 1$ and some $\mathbf{w}_1, \ldots, \mathbf{w}_m \in \mathcal{X}^*$ such that $\mathsf{con}(\mathbf{w}_i) \subseteq \{x_1, \ldots, x_i\}$ for all i. For each i, the letters in $\mathbf{w}_i$ are not first occurrences in $\mathbf{w}$ so that the identities (15.1b) can be used to permute them in any manner within $\mathbf{w}_i$. Specifically, any x_i in the factor $\mathbf{w}_i$ can be moved to the left and gathered with the factor $x_i^{e_i}$ that precedes $\mathbf{w}_i$, and the rest of the letters in $\mathbf{w}_i$ can be ordered alphabetically. Hence for each i, it can be assumed that $\mathbf{w}_i = x_0^{f_0} \cdots x_{i-1}^{f_{i-1}}$. It is then easy to see that (II) and (III) are satisfied by applying the identities (15.1a).

It remains to show that (IV) can be satisfied. Let $\mathsf{occ}(x_i, \mathbf{w}) \geq 3$ and $j > 1$. Suppose that the jth occurrence of x_i is in $x_p^{e_p} \mathbf{w}_p$ and the $(j+1)$st occurrence of x_i is in $x_r^{e_r} \mathbf{w}_r$ for some $r > p$. Then it suffices to achieve either one of the following:

(a) $e_q = 1$ and $x_q \notin \mathsf{con}(\mathbf{w}_{q+1} \cdots \mathbf{w}_r)$ for some q with $p < q \leq r$;
(b) eliminate the $(j+1)$st occurrence of x_i from $\mathbf{w}$ by the identities (15.1).

First observe that

- x_i cannot be x_r, since $i = r > p$ implies that x_i is, by (III), contradictorily not in $x_p^{e_p} \mathbf{w}_p$;
- if $x_i = x_p$ (that is, the factor $x_p^{e_p} \mathbf{w}_p$ contains the first and jth occurrences of x_i with $j > 1$), then it follows from (III) that $e_p = 2$, $j = 2$, and the factor $x_p^{e_p}$ consists of the first and second x_i of $\mathbf{w}$.

Therefore $x_p^{e_p} \mathbf{w}_p = \mathbf{w}_p' x_i \mathbf{w}_p''$ and $\mathbf{w}_r = \mathbf{w}_r' x_i \mathbf{w}_r''$ for some $\mathbf{w}_p', \mathbf{w}_p'', \mathbf{w}_r', \mathbf{w}_r'' \in \mathcal{X}^*$. Denote by φ the substitution $x \mapsto x^2$ for all $x \in \mathcal{X}$. Since the letters in $\mathbf{w}_1, \ldots, \mathbf{w}_m$ are all nonsimple in $\mathbf{w}$,

$$\begin{aligned}
\mathbf{w} \quad &= \quad \cdots \underbrace{(\mathbf{w}_p' x_i \mathbf{w}_p'')}_{x_p^{e_p} \mathbf{w}_p} (x_{p+1}^{e_{p+1}} \mathbf{w}_{p+1}) \cdots (x_{r-1}^{e_{r-1}} \mathbf{w}_{r-1}) (x_r^{e_r} \underbrace{\mathbf{w}_r' x_i \mathbf{w}_r''}_{\mathbf{w}_r}) \cdots \\
&\overset{(15.1b)}{\approx} \cdots (\mathbf{w}_p' x_i \mathbf{w}_p'') (x_{p+1}^{e_{p+1}} \mathbf{w}_{p+1}) \cdots (x_{r-1}^{e_{r-1}} \mathbf{w}_{r-1}) (x_r^{e_r} \mathbf{w}_r' \mathbf{w}_r'' x_i) \cdots \\
&\overset{(15.1a)}{\approx} \cdots \mathbf{w}_p' x_i \underbrace{(\mathbf{w}_p'' \varphi)(x_{p+1}^{e_{p+1}} (\mathbf{w}_{p+1} \varphi)) \cdots (x_{r-1}^{e_{r-1}} (\mathbf{w}_{r-1} \varphi))(x_r^{e_r} (\mathbf{w}_r' \varphi)(\mathbf{w}_r'' \varphi))}_{\mathbf{z}} x_i \cdots .
\end{aligned}$$

CASE 1: $\mathsf{sim}(\mathbf{z}) \neq \emptyset$. Since the letters in $\mathbf{w}_p'' \varphi, \mathbf{w}_{p+1} \varphi, \ldots, \mathbf{w}_{r-1} \varphi, \mathbf{w}_r' \varphi$, and $\mathbf{w}_r'' \varphi$ are all nonsimple in $\mathbf{z}$, at least one of the letters $x_{p+1}, \ldots, x_r$ is simple in $\mathbf{z}$. Therefore there exists q with $p < q \leq r$ such that $e_q = 1$ and $x_q \notin \mathsf{con}(\mathbf{w}_{q+1} \cdots \mathbf{w}_r)$, whence (a) holds.

Case 2: $\mathsf{sim}(\mathbf{z}) = \emptyset$. Since the x_i that immediately precedes $\mathbf{z}$ is the jth occurrence in $\mathbf{w}$ with $j > 1$, the factor $\mathbf{z}$ is preceded by two or more occurrences of x_i. It then follows from Lemma 15.2 that

$$\begin{aligned}\mathbf{w} &\overset{(15.1)}{\approx} \cdots \mathbf{w}'_p x_i \, \mathbf{z} \, x_i \cdots \\ &\overset{(15.2)}{\approx} \cdots \mathbf{w}'_p x_i \, \mathbf{z} \cdots \\ &\overset{(15.1a)}{\approx} \cdots (\mathbf{w}'_p x_i \mathbf{w}''_p)(x_{p+1}^{e_{p+1}} \mathbf{w}_{p+1}) \cdots (x_{r-1}^{e_{r-1}} \mathbf{w}_{r-1})(x_r^{e_r} \mathbf{w}'_r \mathbf{w}''_r) \cdots \\ &= \cdots (x_p^{e_p} \mathbf{w}_p)(x_{p+1}^{e_{p+1}} \mathbf{w}_{p+1}) \cdots (x_{r-1}^{e_{r-1}} \mathbf{w}_{r-1})(x_r^{e_r} \mathbf{w}'_r \mathbf{w}''_r) \cdots .\end{aligned}$$

Hence the $(j+1)$st occurrence of x_i in $\mathbf{w}$ is eliminated and (b) holds. ∎

15.4. Proof of Proposition 15.1. Let $M \in \{\mathcal{J}, \mathcal{K}\}$. It is routine to verify that the monoid M satisfies the identities (15.1). It remains to show that any identity $\mathbf{w} \approx \mathbf{w}'$ satisfied by M is implied by the identities (15.1). In view of Lemma 15.5, the words $\mathbf{w}$ and $\mathbf{w}'$ can be assumed to be in canonical form. Therefore by Lemma 15.3(i),

$$\mathbf{w} = x_0^{e_0} \prod_{i=1}^{m} (x_i^{e_i} \mathbf{w}_i) \quad \text{and} \quad \mathbf{w}' = x_0^{e'_0} \prod_{i=1}^{m} (x_i^{e'_i} \mathbf{w}'_i).$$

It follows from (II) and (III) that $\mathsf{occ}(x_m, \mathbf{w}) = e_m$ and $\mathsf{occ}(x_m, \mathbf{w}') = e'_m$ are in $\{1, 2\}$, whence $e_m = e'_m$ by Lemma 15.3(ii). If $i < m$, then $\mathsf{occ}(x_i, x_{i+1}, \mathbf{w}) = e_i$ and $\mathsf{occ}(x_i, x_{i+1}, \mathbf{w}') = e'_i$ are in $\{1, 2\}$, whence $e_i = e'_i$ by Lemma 15.3(iii). Hence $(e_0, \ldots, e_m) = (e'_0, \ldots, e'_m)$. Working toward a contradiction, suppose that there exists a least integer r such that $\mathbf{w}_r \neq \mathbf{w}'_r$. Then

$$\mathbf{w} = \mathbf{h} x_r^{e_r} \mathbf{w}_r \mathbf{t} \quad \text{and} \quad \mathbf{w}' = \mathbf{h} x_r^{e_r} \mathbf{w}'_r \mathbf{t}'$$

with $\mathbf{h} = x_0^{e_0} \prod_{i=1}^{r-1} (x_i^{e_i} \mathbf{w}_i)$, $\mathbf{t} = \prod_{i=r+1}^{m} (x_i^{e_i} \mathbf{w}_i)$, and $\mathbf{t}' = \prod_{i=r+1}^{m} (x_i^{e_i} \mathbf{w}'_i)$, where

(a) $\mathbf{t} = \mathbf{t}' = \emptyset$ if $r = m$ (and $\lambda(\mathbf{t}) = \lambda(\mathbf{t}') = x_{r+1}$ if $r < m$).

Since $\mathbf{w}_r \neq \mathbf{w}'_r$, there exists some ℓ such that x_ℓ belongs to either $\mathbf{w}_r$ or $\mathbf{w}'_r$ but not both. By symmetry, it suffices to assume that $x_\ell \in \mathsf{con}(\mathbf{w}_r) \setminus \mathsf{con}(\mathbf{w}'_r)$. It then follows from (III) that

(b) $\ell < r$, $\mathsf{occ}(x_\ell, \mathbf{w}_r) = 1$, and $\mathsf{occ}(x_\ell, \mathbf{w}'_r) = 0$.

Since every letter of $\mathbf{w}_r$ is nonsimple in $\mathbf{w}$, the factor $\mathbf{h}$ contains some x_ℓ, say the last x_ℓ in $\mathbf{h}$ occurs in the factor $x_p^{e_p} \mathbf{w}_p$. Then by (IV), there exists some q with $p < q \leq r$ such that $e_q = 1$ and $x_q \notin \mathsf{con}(\mathbf{w}_{q+1} \cdots \mathbf{w}_r)$. Further, since $x_q \notin \mathsf{con}(\mathbf{w}_1 \cdots \mathbf{w}_r)$ by (V),

(c) $x_\ell \ll_{\mathbf{h}} x_q$, $\mathsf{occ}(x_q, \mathbf{h}) = 1$, and $\mathsf{occ}(x_q, \mathbf{w}_r) = 0$.

Suppose that $x_q \in \mathsf{con}(\mathbf{w}'_r)$. Then

$$\mathsf{occ}(x_q, \mathbf{w}') \geq \mathsf{occ}(x_q, \mathbf{h}) + \mathsf{occ}(x_q, \mathbf{w}'_r) = 1 + 1 = 2$$

by (c) so that $\mathsf{occ}(x_q, \mathbf{w}) \geq 2$ by Lemma 15.3(ii). Since

$$2 \leq \mathsf{occ}(x_q, \mathbf{w}) = \mathsf{occ}(x_q, \mathbf{h}) + \mathsf{occ}(x_q, \mathbf{w}_r) + \mathsf{occ}(x_q, \mathbf{t}) = 1 + 0 + \mathsf{occ}(x_q, \mathbf{t})$$

by (c), it follows that $\mathsf{occ}(x_q, \mathbf{t}) \geq 1$. Hence $\mathbf{t}$ is nonempty with $\lambda(\mathbf{t}) = \lambda(\mathbf{t}') = x_{r+1}$. It follows from (c) that

$$\mathsf{occ}(x_q, x_{r+1}, \mathbf{w}) = \mathsf{occ}(x_q, \mathbf{h}) + \mathsf{occ}(x_q, \mathbf{w}_r) = 1 + 0 = 1,$$
$$\mathsf{occ}(x_q, x_{r+1}, \mathbf{w}') = \mathsf{occ}(x_q, \mathbf{h}) + \mathsf{occ}(x_q, \mathbf{w}'_r) = 1 + 1 = 2,$$

contradicting Lemma 15.3(iii). Thus the condition $x_q \in \mathsf{con}(\mathbf{w}'_r)$ is impossible, whence

(d) $\mathsf{occ}(x_q, \mathbf{w}'_r) = 0$.

Let φ be the substitution $x \mapsto 1$ for all $x \in \mathscr{X} \setminus \{x_\ell, x_q, x_{r+1}\}$ and let $n = \mathsf{occ}(x_\ell, \mathbf{h})$. Then $\mathbf{h}\varphi = x_\ell^n x_q$ by (c), $\mathbf{w}_r\varphi = x_\ell$ by (b) and (c), and $\mathbf{w}'_r\varphi = 1$ by (b) and (d). Therefore

$$\mathbf{w}\varphi = (\mathbf{h}\varphi)(x_r^{e_r}\varphi)(\mathbf{w}_r\varphi)(\mathbf{t}\varphi) = x_\ell^n x_q \cdot 1 \cdot x_\ell \cdot (\mathbf{t}\varphi) = x_\ell^n x_q x_\ell(\mathbf{t}\varphi),$$
$$\mathbf{w}'\varphi = (\mathbf{h}\varphi)(x_r^{e_r}\varphi)(\mathbf{w}'_r\varphi)(\mathbf{t}'\varphi) = x_\ell^n x_q \cdot 1 \cdot 1 \cdot (\mathbf{t}'\varphi) = x_\ell^n x_q(\mathbf{t}'\varphi),$$

where $\mathbf{t}\varphi, \mathbf{t}'\varphi \in \{x_\ell, x_q, x_{r+1}\}^*$. Since the identity $x_\ell^n x_q x_\ell(\mathbf{t}\varphi) \approx x_\ell^n x_q(\mathbf{t}'\varphi)$ is obtained from $\mathbf{w} \approx \mathbf{w}'$ by eliminating all occurrences of some letters, it is satisfied by M. However, it follows from (a) that either $\mathbf{t}\varphi = \mathbf{t}'\varphi = \emptyset$ or $\lambda(\mathbf{t}\varphi) = \lambda(\mathbf{t}'\varphi) = x_{r+1}$, which is impossible in view of Lemma 15.4.

A. Multiplication tables of monoids of order six

The multiplication tables of all 1373 distinct monoids of order six are lexicographically listed in rows (A.0)–(A.137) below. The underlying set of each monoid is $\{1, 2, 3, 4, 5, 6\}$, and each table is given by a 6×6 array where the (i, j)-entry denotes the product of the elements i and j. Each monoid is identified in one of the following ways:

Commutativity or idempotency ([1])

- The table of a monoid that is commutative or idempotent is labeled with (`Com`) or (`Idem`) respectively.
- The table of a semilattice is labeled with (`S.L.`). Recall that a *semilattice* is a semigroup that is both commutative and idempotent.
- The table of the commutative (cyclic) group of order six is labeled with $\mathbb{Z}_6$.

Conditions 1–9

- If a noncommutative, nonidempotent monoid is finitely based by Condition m or its dual condition, where $m \in \{1, 2, \ldots, 9\}$, then its table is labeled with (m).
- The symmetric group on three letters is finitely based by Condition 2 and its table is labeled with S_3 ([2]).

([1]) A commutative monoid is finitely based by Condition 1 and a finite idempotent monoid is finitely based by Condition 2. However, it has been known since the late 1960s that a semigroup is finitely based if it is either commutative [28] or idempotent [3, 11, 12].

([2]) Although the groups S_3 and $\mathbb{Z}_6$ are finitely based by Condition 2, the finite basis property of every finite group was already established by Oates and Powell in the early 1960s [27].

Sporadic cases

- The table of each of the 13 sporadic monoids in Chapter 3 is labeled with the monoid's symbol.

The following records the rows in which the tables of the 13 sporadic monoids are located:

Monoid	$\mathcal{A}$	$\mathcal{B}$	$\mathcal{C}$	$\mathcal{D}$	$\mathcal{E}$
Row	(A.14)	(A.15)	(A.25)	(A.46)	(A.50)

Monoid	$\mathcal{F}$	$\mathcal{G}$, $\mathcal{H}$, $\mathcal{I}$	$\mathcal{J}$, $\mathcal{K}$	B_2^1	A_2^1
Row	(A.57)	(A.62)	(A.63)	(A.72)	(A.73)

The tables of the groups $\mathbb{Z}_6$ and S_3 are located in row (A.137).

111111	111111	111111	111111	111111	111111	111111	111111	111111	111111	
111112	111112	111112	111112	111112	111112	111112	111112	111112	111112	
111113	111113	111113	111113	111113	111113	111113	111113	111113	111113	
111114	111114	111114	111114	111114	111114	111114	111114	111114	111114	(A.0)
111115	111125	111155	111215	111225	111235	111455	112215	112225	113355	
123456	123456	123456	123456	123456	123456	123456	123456	123456	123456	
(Com)	(Com)	(Com)	(1)	(1)	(1)	(1)	(1)	(1)	(1)	
111111	111111	111111	111111	111111	111111	111111	111111	111111	111111	
111112	111112	111112	111112	111112	111112	111112	111112	111112	111112	
111113	111113	111113	111113	111113	111113	111113	111113	111113	111113	
111114	111114	111114	111114	111114	111124	111124	111124	111124	111124	(A.1)
113455	122255	122455	123455	555555	111215	111225	111235	111245	111315	
123456	123456	123456	123456	123456	123456	123456	123456	123456	123456	
(1)	(1)	(1)	(1)	(1)	(Com)	(Com)	(Com)	(Com)	(1)	
111111	111111	111111	111111	111111	111111	111111	111111	111111	111111	
111112	111112	111112	111112	111112	111112	111112	111112	111112	111112	
111113	111113	111113	111113	111113	111113	111113	111113	111113	111113	
111124	111124	111124	111124	111124	111124	111144	111144	111144	111144	(A.2)
111325	112115	112125	112215	112225	112245	111455	113155	113455	122155	
123456	123456	123456	123456	123456	123456	123456	123456	123456	123456	
(1)	(1)	(1)	(1)	(1)	(1)	(Com)	(4)	(1)	(4)	
111111	111111	111111	111111	111111	111111	111111	111111	111111	111111	
111112	111112	111112	111112	111112	111112	111112	111112	111112	111112	
111113	111113	111113	111113	111113	111113	111113	111113	111113	111113	
111144	111144	111144	111214	111214	111214	111214	111214	111214	111214	(A.3)
122455	123155	123455	111125	111135	111155	111225	111235	111325	112115	
123456	123456	123456	123456	123456	123456	123456	123456	123456	123456	
(1)	(4)	(1)	(Com)	(Com)	(Com)	(1)	(1)	(1)	(1)	
111111	111111	111111	111111	111111	111111	111111	111111	111111	111111	
111112	111112	111112	111112	111112	111112	111112	111112	111112	111112	
111113	111113	111113	111113	111113	111113	111113	111113	111113	111113	
111214	111214	111214	111214	111214	111214	111224	111224	111224	111224	(A.4)
112125	112215	112225	113155	113355	555555	111225	111235	111325	111335	
123456	123456	123456	123456	123456	123456	123456	123456	123456	123456	
(1)	(1)	(1)	(1)	(1)	(1)	(Com)	(Com)	(1)	(1)	
111111	111111	111111	111111	111111	111111	111111	111111	111111	111111	
111112	111112	111112	111112	111112	111112	111112	111112	111112	111112	
111113	111113	111113	111113	111113	111113	111113	111113	111113	111113	
111224	111224	111224	111224	111234	111414	111414	111414	111414	111414	(A.5)
112115	112125	112215	112225	111325	111155	112115	112125	113155	122155	
123456	123456	123456	123456	123456	123456	123456	123456	123456	123456	
(1)	(1)	(1)	(1)	(Com)	(Com)	(1)	(1)	(1)	(1)	
111111	111111	111111	111111	111111	111111	111111	111111	111111	111111	
111112	111112	111112	111112	111112	111112	111112	111112	111112	111112	
111113	111113	111113	111113	111113	111113	111113	111113	111113	111113	
111414	111414	111444	111444	111444	111444	111444	111444	111444	111454	(A.6)
123155	555555	111445	111455	111555	112445	113455	122455	123455	111545	
123456	123456	123456	123456	123456	123456	123456	123456	123456	123456	
(1)	(1)	(Com)	(Com)	(1)	(1)	(1)	(1)	(1)	(Com)	

111111	111111	111111	111111	111111	111111	111111	111111	111111	111111
111112	111112	111112	111112	111112	111112	111112	111112	111112	111112
111113	111113	111113	111113	111113	111113	111113	111113	111113	111113
111454	112114	112114	112114	112114	112114	112124	112124	112124	112144
555555	112215	112225	121455	122455	555555	112125	112215	112225	113155
123456	123456	123456	123456	123456	123456	123456	123456	123456	123456
(1)	(1)	(1)	(1)	(1)	(1)	(1)	(1)	(1)	(4)

(A.7)

111111	111111	111111	111111	111111	111111	111111	111111	111111	111111
111112	111112	111112	111112	111112	111112	111112	111112	111112	111112
111113	111113	111113	111113	111113	111113	111113	111113	111113	111113
112144	112214	112214	112214	112224	113414	113414	113444	113444	113444
123455	112125	112225	555555	112225	121155	555555	113445	113455	113555
123456	123456	123456	123456	123456	123456	123456	123456	123456	123456
(1)	(1)	(1)	(1)	(1)	(1)	(5)	(1)	(1)	(5)

(A.8)

111111	111111	111111	111111	111111	111111	111111	111111	111111	111111
111112	111112	111112	111112	111112	111112	111112	111112	111112	111112
111113	111113	111113	111113	111113	111113	111113	111113	111113	111113
113444	113454	113454	113454	113454	122414	122444	122444	122444	122444
123455	113455	113545	133455	555555	555555	122445	122455	122555	123455
123456	123456	123456	123456	123456	123456	123456	123456	123456	123456
(1)	(1)	(1)	(1)	(5)	(5)	(1)	(1)	(5)	(1)

(A.9)

111111	111111	111111	111111	111111	111111	111111	111111	111111	111111
111112	111112	111112	111112	111112	111112	111112	111112	111112	111112
111113	111113	111113	111113	111113	111113	111113	111113	111113	111113
122444	122454	122454	122454	123414	123444	123444	123444	123454	123454
133555	122455	122545	555555	555555	123445	123455	123555	123455	123545
123456	123456	123456	123456	123456	123456	123456	123456	123456	123456
(5)	(1)	(1)	(5)	(5)	(1)	(1)	(5)	(1)	(1)

(A.10)

111111	111111	111111	111111	111111	111111	111111	111111	111111	111111
111112	111112	111112	111112	111112	111112	111112	111112	111112	111112
111113	111113	111113	111113	111113	111123	111123	111123	111123	111123
123454	123454	444444	444444	444444	111124	111124	111124	111214	111214
132545	555555	444445	444455	555555	112215	112225	112235	112115	112125
123456	123456	123456	123456	123456	123456	123456	123456	123456	123456
(1)	(5)	(1)	(1)	(1)	(Com)	(Com)	(Com)	(Com)	(Com)

(A.11)

111111	111111	111111	111111	111111	111111	111111	111111	111111	111111
111112	111112	111112	111112	111112	111112	111112	111112	111112	111112
111123	111123	111123	111123	111123	111123	111123	111123	111123	111123
111214	111214	111214	111214	111224	111224	111224	111234	111414	111414
112135	112215	112225	112235	112215	112225	112235	112345	112115	112125
123456	123456	123456	123456	123456	123456	123456	123456	123456	123456
(Com)	(1)	(1)	(1)	(Com)	(Com)	(Com)	(Com)	(Com)	(Com)

(A.12)

111111	111111	111111	111111	111111	111111	111111	111111	111111	111111
111112	111112	111112	111112	111112	111112	111112	111112	111112	111112
111123	111123	111123	111123	111123	111123	111123	111123	111123	111123
111414	111444	112114	112114	112114	112114	112124	112124	112214	112214
112135	112445	111215	111225	112215	112225	112215	112225	111125	111225
123456	123456	123456	123456	123456	123456	123456	123456	123456	123456
(Com)	(Com)	(1)	(1)	(1)	(1)	(1)	(1)	(1)	(1)

(A.13)

111111	111111	111111	111111	111111	111111	111111	111111	111111	111111
111112	111112	111112	111112	111112	111112	111112	111112	111112	111112
111123	111123	111123	111123	111123	111123	111123	111123	111123	111123
112214	112214	112214	112224	112224	112224	112224	112224	112224	123434
112125	112215	112225	111125	111225	112115	112125	112215	112225	111125
123456	123456	123456	123456	123456	123456	123456	123456	123456	123456
(1)	(1)	(1)	(1)	(1)	(1)	(1)	(1)	(1)	$\mathcal{A}$

(A.14)

111111	111111	111111	111111	111111	111111	111111	111111	111111	111111
111112	111112	111112	111112	111112	111112	111112	111112	111112	111112
111123	111123	111123	111123	111123	111123	111133	111133	111133	111133
123444	444444	444444	444444	444444	444444	111134	111134	111134	111134
123445	111125	112115	112125	112135	444445	113355	113455	121255	123355
123456	123456	123456	123456	123456	123456	123456	123456	123456	123456
$\mathcal{B}$	(1)	(1)	(1)	(1)	(1)	(Com)	(1)	(4)	(1)

(A.15)

111111	111111	111111	111111	111111	111111	111111	111111	111111	111111
111112	111112	111112	111112	111112	111112	111112	111112	111112	111112
111133	111133	111133	111133	111133	111133	111133	111133	111133	111133
111134	111144	111144	111144	111144	111214	111234	111344	111344	111414
123455	113455	121455	123355	123455	113155	113355	113455	123455	113155
123456	123456	123456	123456	123456	123456	123456	123456	123456	123456
(1)	(Com)	(4)	(4)	(1)	(Com)	(Com)	(Com)	(1)	(Com)

(A.16)

111111	111111	111111	111111	111111	111111	111111	111111	111111	111111
111112	111112	111112	111112	111112	111112	111112	111112	111112	111112
111133	111133	111133	111133	111133	111133	111133	111133	111133	111133
111414	111414	111444	111444	111444	113414	113414	113434	113434	113444
121155	123155	113455	121455	123455	111155	121155	111155	121155	113455
123456	123456	123456	123456	123456	123456	123456	123456	123456	123456
(4)	(1)	(Com)	(4)	(1)	(4)	(4)	(5)	(5)	(1)

(A.17)

111111	111111	111111	111111	111111	111111	111111	111111	111111	111111
111112	111112	111112	111112	111112	111112	111112	111112	111112	111112
111133	111133	111133	111133	111133	111133	111133	111133	111133	111133
113444	121414	121414	121444	121444	123414	123434	123444	444444	444444
123455	111155	113155	121455	123455	111155	111155	123455	111155	111455
123456	123456	123456	123456	123456	123456	123456	123456	123456	123456
(1)	(4)	(1)	(4)	(1)	(4)	(5)	(1)	(1)	(1)

(A.18)

111111	111111	111111	111111	111111	111111	111111	111111	111111	111111
111112	111112	111112	111112	111112	111112	111112	111112	111112	111112
111133	111133	111133	111133	111133	111133	111133	111143	111223	111223
444444	444444	444444	444444	444444	444444	444444	444444	112124	112124
113155	113455	121155	121455	123155	123455	444455	555555	112215	112225
123456	123456	123456	123456	123456	123456	123456	123456	123456	123456
(1)	(1)	(5)	(5)	(5)	(5)	(1)	(8)	(Com)	(Com)

(A.19)

111111	111111	111111	111111	111111	111111	111111	111111	111111	111111
111112	111112	111112	111112	111112	111112	111112	111112	111112	111112
111223	111223	111223	111223	111233	111333	111333	111333	111333	111333
112214	112214	112224	112334	111234	111444	113444	113444	113444	113444
112125	112225	112225	112335	123455	123455	113445	113455	113555	123455
123456	123456	123456	123456	123456	123456	123456	123456	123456	123456
(Com)	(1)	(Com)	(Com)	(1)	(4)	(Com)	(Com)	(1)	(1)

(A.20)

111111	111111	111111	111111	111111	111111	111111	111111	111111	111111
111112	111112	111112	111112	111112	111112	111112	111112	111112	111112
111333	111333	111333	111333	111333	111333	111333	111333	111333	111333
113454	121444	121444	121444	121444	121454	123444	123444	123444	123454
113545	121445	121455	121555	123455	121545	123445	123455	123555	123455
123456	123456	123456	123456	123456	123456	123456	123456	123456	123456
(Com)	(4)	(4)	(5)	(4)	(6)	(1)	(1)	(5)	(1)

(A.21)

111111	111111	111111	111111	111111	111111	111111	111111	111111	111111
111112	111112	111112	111112	111112	111112	111112	111112	111112	111112
111333	112113	112113	112113	112113	112113	112113	112113	112113	112113
123454	111214	111214	111214	111214	111214	111224	111224	111224	111414
123545	111125	111155	111225	112225	555555	111225	112125	112225	111155
123456	123456	123456	123456	123456	123456	123456	123456	123456	123456
(1)	(Com)	(Com)	(1)	(1)	(1)	(Com)	(1)	(1)	(Com)

(A.22)

111111	111111	111111	111111	111111	111111	111111	111111	111111	111111
111112	111112	111112	111112	111112	111112	111112	111112	111112	111112
112113	112113	112113	112113	112113	112113	112113	112113	112113	112113
111414	111414	111444	111444	111444	111444	111454	111454	112214	112214
112125	555555	111445	111455	111555	112445	111545	555555	112225	555555
123456	123456	123456	123456	123456	123456	123456	123456	123456	123456
(1)	(1)	(Com)	(Com)	(1)	(1)	(Com)	(1)	(1)	(1)

(A.23)

111111	111111	111111	111111	111111	111111	111111	111111	111111	111111
111112	111112	111112	111112	111112	111112	111112	111112	111112	111112
112113	112113	112113	112113	112123	112123	112123	112123	112123	112123
112224	444444	444444	444444	111224	111414	111444	112214	112214	112224
112225	444445	444455	555555	112225	112125	112445	111225	112225	112225
123456	123456	123456	123456	123456	123456	123456	123456	123456	123456
(1)	(1)	(1)	(1)	(Com)	(Com)	(Com)	(1)	(1)	(1)

(A.24)

111111	111111	111111	111111	111111	111111	111111	111111	111111	111111	
111112	111112	111112	111112	111112	111112	111112	111112	111112	111112	
112123	112123	112143	112223	113113	113113	113113	113113	113113	113113	
444444	444444	444444	112224	111414	111414	111414	111444	111444	111444	(A.25)
112125	444445	555555	112225	111155	121155	555555	111445	111455	111555	
123456	123456	123456	123456	123456	123456	123456	123456	123456	123456	
(1)	(1)	$\mathcal{C}$	(Com)	(Com)	(1)	(1)	(Com)	(Com)	(1)	

111111	111111	111111	111111	111111	111111	111111	111111	111111	111111	
111112	111112	111112	111112	111112	111112	111112	111112	111112	111112	
113113	113113	113113	113113	113113	113113	113113	113113	113113	113113	
111444	111454	111454	121414	121444	121444	121444	121454	121454	121454	(A.26)
121455	111545	555555	555555	121445	121455	121555	121455	121545	555555	
123456	123456	123456	123456	123456	123456	123456	123456	123456	123456	
(1)	(Com)	(1)	(5)	(1)	(1)	(5)	(1)	(1)	(5)	

111111	111111	111111	111111	111111	111111	111111	111111	111111	111111	
111112	111112	111112	111112	111112	111112	111112	111112	111112	111112	
113113	113113	113113	113113	113133	113133	113133	113133	113133	113133	
444444	444444	444444	444444	111444	111444	121414	121414	121414	121424	(A.27)
444445	444455	445445	555555	113455	123455	113135	113155	115155	113135	
123456	123456	123456	123456	123456	123456	123456	123456	123456	123456	
(1)	(1)	(1)	(1)	(Com)	(1)	(1)	(1)	(5)	(1)	

111111	111111	111111	111111	111111	111111	111111	111111	111111	111111	
111112	111112	111112	111112	111112	111112	111112	111112	111112	111112	
113133	113133	113133	113133	113133	113133	113133	113133	113153	113153	
121444	444444	444444	444444	444444	444444	444444	444444	121414	121414	(A.28)
123455	113135	113155	113455	115155	123155	123455	445455	113155	115135	
123456	123456	123456	123456	123456	123456	123456	123456	123456	123456	
(1)	(1)	(1)	(1)	(1)	(5)	(5)	(1)	(1)	(1)	

111111	111111	111111	111111	111111	111111	111111	111111	111111	111111	
111112	111112	111112	111112	111112	111112	111112	111112	111112	111112	
113153	113153	113153	113153	113333	113333	113333	113333	113333	113333	
121414	444444	444444	444444	113334	113334	113334	113334	113334	113334	(A.29)
555555	113155	115135	555555	113335	113345	113355	113455	115555	123355	
123456	123456	123456	123456	123456	123456	123456	123456	123456	123456	
(5)	(7)	(1)	(1)	(Com)	(Com)	(Com)	(1)	(1)	(1)	

111111	111111	111111	111111	111111	111111	111111	111111	111111	111111	
111112	111112	111112	111112	111112	111112	111112	111112	111112	111112	
113333	113333	113333	113333	113333	113333	113333	113333	113333	113333	
113334	113344	113344	113344	113434	113434	113434	113444	113444	113444	(A.30)
123455	113455	123355	123455	113355	115555	123355	113445	113455	113555	
123456	123456	123456	123456	123456	123456	123456	123456	123456	123456	
(1)	(Com)	(4)	(1)	(Com)	(1)	(1)	(Com)	(Com)	(1)	

111111	111111	111111	111111	111111	111111	111111	111111	111111	111111	
111112	111112	111112	111112	111112	111112	111112	111112	111112	111112	
113333	113333	113333	113333	113333	113333	113333	113333	113333	113333	
113444	113454	113454	114444	114444	114444	123444	123444	123444	123454	(A.31)
123455	113545	115555	115555	123355	123455	123445	123455	123555	123455	
123456	123456	123456	123456	123456	123456	123456	123456	123456	123456	
(1)	(Com)	(1)	(1)	(5)	(5)	(1)	(1)	(5)	(1)	

111111	111111	111111	111111	111111	111111	111111	111111	111111	111111	
111112	111112	111112	111112	111112	111112	111112	111112	111112	111112	
113333	113353	113353	113353	113353	113353	113353	113353	113353	113353	
123454	113354	113354	113454	113454	114454	123454	123454	123454	123454	(A.32)
123545	115535	555555	115535	555555	555555	113355	113555	115535	555555	
123456	123456	123456	123456	123456	123456	123456	123456	123456	123456	
(1)	(Com)	(1)	(Com)	(1)	(1)	(1)	(1)	(1)	(5)	

111111	111111	111111	111111	111111	111111	111111	111111	111111	111111	
111112	111112	111112	111112	111112	111112	111112	111112	111112	111112	
113443	113443	113443	113443	113443	113453	113453	113453	113453	113453	
114334	444444	444444	444444	444444	113454	114354	114534	444444	444444	(A.33)
114335	444445	444455	445445	555555	555555	555555	115345	444445	445135	
123456	123456	123456	123456	123456	123456	123456	123456	123456	123456	
(Com)	(1)	(1)	(1)	(1)	(7)	(1)	(Com)	(5)	(9)	

111111	111111	111111	111111	111111	111111	111111	111111	111111	111111
111112	111112	111112	111112	111112	111112	111112	111112	111112	111112
113453	113453	113453	123113	123113	123113	123113	123123	123133	123133
444444	444444	444444	444444	444444	444444	444444	444444	444444	444444
445445	445455	555555	444445	444455	445445	555555	444445	123135	123155
123456	123456	123456	123456	123456	123456	123456	123456	123456	123456
(1)	(1)	(1)	(5)	(5)	(5)	(5)	(5)	(5)	(5)

(A.34)

111111	111111	111111	111111	111111	111111	111111	111111	111111	111111
111112	111112	111112	111112	111112	111112	111112	111112	111112	111112
123133	123133	123153	123153	123153	123333	123333	123333	123333	123333
444444	444444	444444	444444	444444	123334	123334	123334	123334	123334
123455	125155	123155	125135	555555	123335	123345	123355	123455	125555
123456	123456	123456	123456	123456	123456	123456	123456	123456	123456
(5)	(5)	(7)	(7)	(5)	(1)	(1)	(1)	(1)	(5)

(A.35)

111111	111111	111111	111111	111111	111111	111111	111111	111111	111111
111112	111112	111112	111112	111112	111112	111112	111112	111112	111112
123333	123333	123333	123333	123333	123333	123333	123333	123333	123333
123344	123344	123434	123434	123444	123444	123444	123454	123454	123454
123355	123455	123355	125555	123445	123455	123555	123455	123545	125555
123456	123456	123456	123456	123456	123456	123456	123456	123456	123456
(4)	(1)	(1)	(5)	(1)	(1)	(5)	(1)	(1)	(5)

(A.36)

111111	111111	111111	111111	111111	111111	111111	111111	111111	111111
111112	111112	111112	111112	111112	111112	111112	111112	111112	111112
123333	123353	123353	123353	123353	123353	123353	123353	123353	123443
124444	123354	123354	123354	123454	123454	123454	123454	124454	124334
125555	123355	125535	555555	123355	123555	125535	555555	555555	124335
123456	123456	123456	123456	123456	123456	123456	123456	123456	123456
(5)	(1)	(1)	(5)	(1)	(1)	(1)	(5)	(5)	(1)

(A.37)

111111	111111	111111	111111	111111	111111	111111	111111	111111	111111
111112	111112	111112	111112	111112	111112	111112	111112	111112	111112
123443	123443	123443	123453	123453	123453	123453	123453	123453	123453
444444	444444	444444	123454	123454	124354	124534	444444	444444	444444
444455	445445	555555	123455	555555	555555	125345	444445	445445	445455
123456	123456	123456	123456	123456	123456	123456	123456	123456	123456
(5)	(5)	(5)	(1)	(7)	(7)	(1)	(5)	(5)	(5)

(A.38)

111111	111111	111111	111111	111111	111111	111111	111111	111111	111111
111112	111112	111112	111112	111112	111112	111112	111112	111112	111112
123453	333333	333333	333333	333333	333333	333333	333333	333333	333333
444444	333314	333334	333344	333344	333434	333434	333444	333444	333444
555555	555555	555555	333355	333455	333355	555555	333445	333455	333555
123456	123456	123456	123456	123456	123456	123456	123456	123456	123456
(5)	(8)	(1)	(1)	(1)	(1)	(1)	(1)	(1)	(1)

(A.39)

111111	111111	111111	111111	111111	111111	111111	111111	111111	111111
111112	111112	111112	111112	111112	111122	111122	111122	111122	111122
333333	333333	333333	333333	333333	111123	111123	111123	111123	111123
333454	333454	333454	335554	444444	111124	111124	111124	111124	111144
333455	333545	555555	555555	555555	122255	122455	123355	123455	122455
123456	123456	123456	123456	123456	123456	123456	123456	123456	123456
(7)	(1)	(1)	(8)	(1)	(Com)	(1)	(1)	(1)	(Com)

(A.40)

111111	111111	111111	111111	111111	111111	111111	111111	111111	111111
111122	111122	111122	111122	111122	111122	111122	111122	111122	111122
111123	111123	111123	111123	111123	111123	111123	111123	111123	111123
111144	111144	111244	111244	111414	111414	111444	111444	122414	122424
123255	123455	122455	123455	122155	123155	122455	123455	111155	111155
123456	123456	123456	123456	123456	123456	123456	123456	123456	123456
(4)	(1)	(Com)	(1)	(Com)	(1)	(Com)	(1)	(4)	(5)

(A.41)

111111	111111	111111	111111	111111	111111	111111	111111	111111	111111
111122	111122	111122	111122	111122	111122	111122	111122	111122	111122
111123	111123	111123	111123	111123	111123	111123	111123	111123	111123
122444	122444	123414	123424	123444	444444	444444	444444	444444	444444
122455	123455	111155	111155	123455	111155	111455	122155	122455	123155
123456	123456	123456	123456	123456	123456	123456	123456	123456	123456
(1)	(1)	(4)	(5)	(1)	(1)	(1)	(1)	(1)	(5)

(A.42)

111111	111111	111111	111111	111111	111111	111111	111111	111111	111111	
111122	111122	111122	111122	111122	111122	111122	111122	111122	111122	
111123	111123	111133	111133	111133	111133	111133	111133	111133	111133	
444444	444444	111144	111144	111144	111244	111244	111414	111444	111444	(A.43)
123455	444455	123455	123456	123456	123455	123456	123155	123455	123456	
123456	123456	123456	123465	124365	123456	123465	123456	123456	123465	
(5)	(1)	(Com)	(Com)	(3)	(Com)	(Com)	(Com)	(Com)	(Com)	

111111	111111	111111	111111	111111	111111	111111	111111	111111	111111	
111122	111122	111122	111122	111122	111122	111122	111122	111122	111122	
111133	111133	111133	111133	111133	111133	111133	111133	111133	111133	
111444	112114	112114	112124	112144	112144	112244	112244	113414	113434	(A.44)
123456	121455	122455	122455	123455	123456	123455	123456	121155	121155	
132465	123456	123456	123456	123456	123465	123456	123465	123456	123456	
(3)	(4)	(4)	(4)	(1)	(1)	(1)	(1)	(4)	(5)	

111111	111111	111111	111111	111111	111111	111111	111111	111111	111111	
111122	111122	111122	111122	111122	111122	111122	111122	111122	111122	
111133	111133	111133	111133	111133	111133	111133	111133	111133	111133	
113444	113444	122444	122444	122444	123414	123424	123444	123444	444444	(A.45)
123455	123456	122455	123455	123456	111155	111155	123455	123456	111155	
123456	123465	123456	123456	123465	123456	123456	123456	123465	123456	
(1)	(1)	(4)	(1)	(1)	(4)	(5)	(1)	(1)	(1)	

111111	111111	111111	111111	111111	111111	111111	111111	111111	111111	
111122	111122	111122	111122	111122	111122	111122	111122	111122	111122	
111133	111133	111133	111133	111133	111133	111133	111133	111133	111133	
444444	444444	444444	444444	444444	444444	444444	444444	444444	444444	(A.46)
111455	113155	113455	122155	122455	123155	123455	123456	123456	123456	
123456	123456	123456	123456	123456	123456	123456	123465	132465	444466	
(1)	(1)	(1)	(1)	(1)	(1)	(1)	(1)	$\mathcal{D}$	(1)	

111111	111111	111111	111111	111111	111111	111111	111111	111111	111111	
111122	111122	111122	111122	111122	111122	111122	111122	111122	111122	
111134	111233	111233	111233	111233	111233	111233	111313	111313	111313	
111143	112144	112144	112244	112244	112344	112344	113414	121414	121414	(A.47)
123456	123455	123456	123455	123456	123455	123456	121155	113155	113355	
124365	123456	123465	123456	123465	123456	123465	123456	123456	123456	
(Com)	(Com)	(Com)	(Com)	(Com)	(Com)	(Com)	(Com)	(4)	(5)	

111111	111111	111111	111111	111111	111111	111111	111111	111111	111111	
111122	111122	111122	111122	111122	111122	111122	111122	111122	111122	
111313	111333	111333	111333	111333	111333	111333	112133	112133	112133	
121424	113444	113444	121444	121444	123444	123444	111244	111244	111414	(A.48)
113355	123455	123456	123455	123456	123455	123456	123455	123456	123155	
123456	123456	123465	123456	123465	123456	123465	123456	123465	123456	
(5)	(Com)	(Com)	(4)	(6)	(1)	(1)	(Com)	(Com)	(Com)	

111111	111111	111111	111111	111111	111111	111111	111111	111111	111111	
111122	111122	111122	111122	111122	111122	111122	111122	111122	111122	
112133	112133	112133	112133	112133	112133	112133	112233	112233	112233	
111444	111444	112244	112244	444444	444444	444444	112234	112234	112244	(A.49)
123455	123456	123455	123456	123155	123455	123456	123355	123455	123455	
123456	123465	123456	123465	123456	123456	123465	123456	123456	123456	
(Com)	(Com)	(1)	(1)	(1)	(1)	(1)	(Com)	(1)	(Com)	

111111	111111	111111	111111	111111	111111	111111	111111	111111	111111	
111122	111122	111122	111122	111122	111122	111122	111122	111122	111122	
112233	112233	112234	113113	113113	113113	113113	113113	113113	113113	
112244	112244	112243	111414	111444	121414	121424	121444	444444	444444	(A.50)
123456	123456	123456	121155	121455	111155	111155	121455	111155	111455	
123465	124365	124365	123456	123456	123456	123456	123456	123456	123456	
(Com)	$\mathcal{E}$	(Com)	(Com)	(Com)	(4)	(5)	(1)	(1)	(1)	

111111	111111	111111	111111	111111	111111	111111	111111	111111	111111	
111122	111122	111122	111122	111122	111122	111122	111122	111122	111122	
113113	113113	113113	113133	113133	113133	113133	113133	113133	113133	
444444	444444	444444	111444	111444	121414	121424	121444	121444	444444	(A.51)
121155	121455	444455	123455	123456	113155	113155	123455	123456	113155	
123456	123456	123456	123456	123465	123456	123456	123456	123465	123456	
(1)	(1)	(1)	(Com)	(Com)	(4)	(5)	(1)	(1)	(1)	

111111	111111	111111	111111	111111	111111	111111	111111	111111	111111	
111122	111122	111122	111122	111122	111122	111122	111122	111122	111122	
113133	113133	113133	113133	113313	113313	113313	113313	113333	113333	
444444	444444	444444	444444	113314	113324	113414	114414	113334	113334	(A.52)
113455	123155	123455	123456	121155	121255	121155	121155	123355	123455	
123456	123456	123456	123465	123456	123456	123456	123456	123456	123456	
(1)	(1)	(1)	(1)	(Com)	(Com)	(Com)	(1)	(Com)	(1)	

111111	111111	111111	111111	111111	111111	111111	111111	111111	111111	
111122	111122	111122	111122	111122	111122	111122	111122	111122	111122	
113333	113333	113333	113333	113333	113333	113333	113333	113333	113333	
113344	113344	113434	113444	113444	114444	114444	114444	114444	123434	(A.53)
123455	123456	123355	123455	123456	123355	123455	123456	123456	113355	
123456	123465	123456	123456	123465	123456	123456	123465	124365	123456	
(Com)	(Com)	(Com)	(Com)	(Com)	(1)	(1)	(1)	(9)	(4)	

111111	111111	111111	111111	111111	111111	111111	111111	111111	111111	
111122	111122	111122	111122	111122	111122	111122	111122	111122	111122	
113333	113333	113413	113413	113413	113433	113433	113433	113433	113433	
123444	123444	114314	444444	444444	114344	114344	444444	444444	444444	(A.54)
123455	123456	121155	111155	121155	123455	123456	113455	123455	123456	
123456	123465	123456	123456	123456	123456	123465	123456	123456	123465	
(1)	(1)	(Com)	(1)	(1)	(Com)	(Com)	(1)	(1)	(1)	

111111	111111	111111	111111	111111	111111	111111	111111	111111	111111	
111122	111122	111122	111122	111122	111122	111122	111122	111122	111122	
113434	113434	123113	123113	123113	123123	123123	123123	123133	123133	
114343	444444	444444	444444	444444	444444	444444	444444	444444	444444	(A.55)
123456	123456	111155	111455	444455	111155	111455	444455	123155	123455	
124365	444466	123456	123456	123456	123456	123456	123456	123456	123456	
(Com)	(1)	(5)	(5)	(5)	(5)	(5)	(5)	(5)	(5)	

111111	111111	111111	111111	111111	111111	111111	111111	111111	111111	
111122	111122	111122	111122	111122	111122	111122	111122	111122	111122	
123133	123313	123313	123313	123323	123323	123323	123333	123333	123333	
444444	123314	123414	124414	123324	123424	124424	123334	123334	123344	(A.56)
123456	111155	111155	111155	111155	111155	111155	123355	123455	123355	
123465	123456	123456	123456	123456	123456	123456	123456	123456	123456	
(7)	(4)	(4)	(5)	(5)	(5)	(5)	(1)	(1)	(4)	

111111	111111	111111	111111	111111	111111	111111	111111	111111	111111	
111122	111122	111122	111122	111122	111122	111122	111122	111122	111122	
123333	123333	123333	123333	123333	123333	123333	123333	123333	123413	
123344	123344	123434	123444	123444	124444	124444	124444	124444	123414	(A.57)
123455	123456	123355	123455	123456	123355	123455	123456	123456	111155	
123456	123465	123456	123456	123465	123456	123456	123465	124365	123456	
(1)	(1)	(1)	(1)	(1)	(5)	(5)	(7)	$\mathcal{F}$	(5)	

111111	111111	111111	111111	111111	111111	111111	111111	111111	111111	
111122	111122	111122	111122	111122	111122	111122	111122	111122	111122	
123413	123413	123413	123423	123423	123423	123433	123433	123433	123433	
123424	124314	444444	123424	124324	444444	123434	123444	123444	124344	(A.58)
111155	111155	111155	111155	111155	111155	123455	123455	123456	123455	
123456	123456	123456	123456	123456	123456	123456	123456	123465	123456	
(5)	(6)	(5)	(5)	(6)	(5)	(1)	(1)	(1)	(1)	

111111	111111	111111	111111	111111	111111	111111	111111	111111	111111	
111122	111122	111122	111122	111122	111122	111122	111122	111122	111122	
123433	123433	123433	123434	123434	123434	333333	333333	333333	333333	
124344	444444	444444	123443	124343	444444	333344	333344	333344	333344	(A.59)
123456	123455	123456	123456	123456	123456	111155	113355	113455	121155	
123465	123456	123465	123465	124365	444466	123456	123456	123456	123456	
(1)	(5)	(7)	(9)	(1)	(5)	(1)	(1)	(1)	(1)	

111111	111111	111111	111111	111111	111111	111111	111111	111111	111111	
111122	111122	111122	111122	111122	111122	111122	111122	111122	111122	
333333	333333	333333	333333	333333	333333	333333	333333	333333	333333	
333344	333344	333344	333344	333434	333434	333434	333434	333434	333444	(A.60)
121255	123455	123456	123456	111155	113355	121155	123355	333355	113455	
123456	123456	123465	341265	123456	123456	123456	123456	123456	123456	
(1)	(1)	(1)	(9)	(1)	(1)	(1)	(1)	(1)	(1)	

111111	111111	111111	111111	111111	111111	111111	111111	111111	111111	
111122	111122	111122	111122	111122	111122	111122	111122	111122	111122	
333333	333333	333333	333333	333333	333333	333333	333333	333333	333333	
333444	333444	333444	444444	444444	444444	444444	444444	444444	444444	(A.61)
123455	123456	123456	111155	111455	113355	113455	121155	121455	123355	
123456	123465	333466	123456	123456	123456	123456	123456	123456	123456	
(1)	(1)	(1)	(1)	(1)	(1)	(1)	(1)	(1)	(1)	

111111	111111	111111	111111	111111	111111	111111	111111	111111	111111	
111122	111122	111122	111122	111122	111123	111123	111123	111123	111123	
333333	333333	333333	333333	333333	111132	111132	111132	111132	111333	
444444	444444	444444	444444	444444	111444	123444	444444	444444	123444	(A.62)
123455	123456	123456	123456	123456	123456	123456	123456	123456	123456	
123456	123465	124365	333366	333466	132465	123465	123465	132465	123666	
(1)	(1)	(9)	(1)	(1)	(Com)	$\mathcal{G}$	$\mathcal{H}$	(1)	$\mathcal{I}$	

111111	111111	111111	111111	111111	111111	111111	111111	111111	111111	
111123	111123	111123	111123	111123	111123	111123	111222	111222	111222	
333333	333333	333333	333333	333333	333333	333333	111223	111223	111223	
111143	111441	123441	123443	333341	333443	444444	122444	122444	122444	(A.63)
123456	123456	123456	123456	123456	123456	123456	122445	122455	122555	
666666	666666	666666	666666	666666	666666	666666	123456	123456	123456	
(8)	(8)	$\mathcal{J}$	$\mathcal{K}$	(8)	(8)	(8)	(Com)	(Com)	(1)	

111111	111111	111111	111111	111111	111111	111111	111111	111111	111111	
111222	111222	111222	111222	111222	111222	111222	111222	111222	111222	
111223	111223	111223	111223	111223	111223	111223	111233	111233	111233	
122444	122454	123444	123444	123444	123454	123454	122444	122444	123444	(A.64)
123455	122545	123445	123455	123555	123455	123545	123455	123456	123455	
123456	123456	123456	123456	123456	123456	123456	123456	123465	123456	
(1)	(Com)	(1)	(1)	(5)	(1)	(1)	(Com)	(Com)	(1)	

111111	111111	111111	111111	111111	111111	111111	111111	111111	111111	
111222	111222	111222	111222	111222	111222	111222	111222	111222	111222	
111233	111333	111333	111333	111333	111333	111333	111333	111333	112333	
123444	122444	123444	123444	123444	123444	123446	123446	123456	123444	(A.65)
123456	123456	123445	123455	123456	123456	123456	123456	123564	123445	
123465	133666	123456	123456	123465	123666	123664	132664	123645	123456	
(1)	(1)	(Com)	(Com)	(Com)	(1)	(Com)	(3)	(Com)	(Com)	

111111	111111	111111	111111	111111	111111	111111	111111	111111	111111	
111222	111222	111222	111222	111222	111222	111222	111222	111222	111222	
112333	112333	112333	112333	112333	113113	113113	113113	113113	113133	
123444	123444	123444	123446	123456	121444	121444	121444	121454	121444	(A.66)
123455	123456	123456	123456	123564	121445	121455	121555	121545	123455	
123456	123465	123666	123664	123645	123456	123456	123456	123456	123456	
(Com)	(Com)	(1)	(Com)	(Com)	(Com)	(Com)	(1)	(Com)	(Com)	

111111	111111	111111	111111	111111	111111	111111	111111	111111	111111	
111222	111222	111222	111222	111222	111222	111222	111222	111222	111222	
113133	113333	113333	113333	113333	113333	113333	123133	123133	123233	
121444	123444	123444	123444	123444	123446	123456	111444	111444	111444	(A.67)
123456	123445	123455	123456	123456	123456	123564	123455	123456	123455	
123465	123456	123456	123465	123666	123664	123645	123456	123465	123456	
(Com)	(Com)	(Com)	(Com)	(1)	(Com)	(Com)	(4)	(6)	(5)	

111111	111111	111111	111111	111111	111111	111111	111111	111111	111111	
111222	111222	111222	111222	111222	111222	111222	111222	111222	111222	
123233	123333	123333	123333	123333	123333	123333	123333	333333	333333	
111444	123444	123444	123444	123444	123446	123446	123456	111444	111444	(A.68)
123456	123445	123455	123456	123456	123456	123456	123564	111445	111455	
123465	123456	123456	123465	123666	123466	123664	123645	123456	123456	
(6)	(1)	(1)	(1)	(5)	(1)	(1)	(1)	(1)	(1)	

111111	111111	111111	111111	111111	111111	111111	111111	111111	111111	
111222	111222	111222	111222	111222	111222	111222	111222	111222	111222	
333333	333333	333333	333333	333333	333333	333333	333333	333333	333333	
111444	111444	111444	111444	111444	111444	111446	111446	113444	113444	(A.69)
111555	113455	121455	123455	123456	123456	123456	123456	113445	113455	
123456	123456	123456	123456	123465	333666	111466	111664	123456	123456	
(1)	(1)	(1)	(1)	(1)	(1)	(7)	(1)	(1)	(1)	

111111	111111	111111	111111	111111	111111	111111	111111	111111	111111
111222	111222	111222	111222	111222	111222	111222	111222	111222	111222
333333	333333	333333	333333	333333	333333	333333	333333	333333	333333
113444	113444	113444	113446	113446	113446	113446	121444	121444	121444
113555	123455	123456	123456	123456	123456	123456	121445	121455	121555
123456	123456	123465	113466	113664	331664	333666	123456	123456	123456
(1)	(1)	(1)	(7)	(1)	(9)	(1)	(1)	(1)	(1)

(A.70)

111111	111111	111111	111111	111111	111111	111111	111111	111111	111111
111222	111222	111222	111222	111222	111222	111222	111222	111222	111222
333333	333333	333333	333333	333333	333333	333333	333333	333333	333333
121444	121444	121446	121446	123444	123444	123444	123444	123446	123446
123455	123456	123456	123456	123445	123455	123456	123456	123456	123456
123456	123465	121466	121664	123456	123456	123465	123666	123466	123664
(1)	(1)	(7)	(1)	(1)	(1)	(1)	(1)	(7)	(1)

(A.71)

111111	111111	111111	111111	111111	111111	111111	111111	111111	111111
111222	111222	111222	111222	111222	111222	111222	111222	111223	111223
333333	333333	333333	333333	333333	333333	333333	333333	111332	123131
123446	123456	123456	123456	123456	123456	123456	123456	123446	111446
123456	123546	123564	333555	333555	333555	333556	333556	123456	123456
333666	333666	123645	333655	333656	333666	333656	333665	132664	146161
(1)	(1)	(1)	(1)	(1)	(1)	(7)	(1)	(Com)	B_2^1

(A.72)

111111	111111	111111	111111	111111	111111	111111	111111	111111	111111
111223	111223	111223	111223	111223	111233	111233	112222	112222	112222
123233	333333	333333	333333	333333	333333	333333	123333	123333	123333
111446	111446	123446	123456	123456	123456	123456	123334	123334	123334
123456	123456	123456	123546	333556	555555	555555	123335	123345	123355
146466	666666	666666	666666	666666	555656	666666	123456	123456	123456
A_2^1	(8)	(8)	(8)	(8)	(8)	(8)	(Com)	(Com)	(Com)

(A.73)

111111	111111	111111	111111	111111	111111	111111	111111	111111	111111
112222	112222	112222	112222	112222	112222	112222	112222	112222	112222
123333	123333	123333	123333	123333	123333	123333	123333	123333	123333
123334	123334	123344	123344	123434	123434	123444	123444	123444	123444
123455	125555	123455	123456	123355	125555	123445	123455	123456	123456
123456	123456	123456	123465	123456	123456	123456	123456	123465	123666
(1)	(1)	(Com)	(Com)	(Com)	(1)	(Com)	(Com)	(Com)	(1)

(A.74)

111111	111111	111111	111111	111111	111111	111111	111111	111111	111111
112222	112222	112222	112222	112222	112222	112222	112222	112222	112222
123333	123333	123333	123333	123333	123333	123336	123336	123336	123355
123446	123446	123456	123456	123456	123456	123346	123446	123456	123456
123456	123456	123546	123564	125555	125555	123456	123456	123546	125533
123664	126666	126666	123645	125634	126666	126663	126663	126663	125633
(Com)	(1)	(1)	(Com)	(9)	(1)	(Com)	(Com)	(Com)	(Com)

(A.75)

111111	111111	111111	111111	111111	111111	111111	111111	111111	111111
112222	112222	112222	112222	121112	121112	121112	121112	121112	121112
123355	123356	123456	123456	113113	113113	113113	113113	113113	113113
123456	123456	124365	124365	111414	111414	111444	111444	111444	111454
125533	125563	125634	125643	111155	555555	111445	111455	111555	111545
125634	126635	126543	126534	123456	123456	123456	123456	123456	123456
(Com)	(Com)	(Com)	(Com)	(S.L.)	(Idem)	(Com)	(S.L.)	(Idem)	(Com)

(A.76)

111111	111111	111111	111111	111111	111111	111111	111111	111111	111111
121112	121112	121112	121112	121112	121112	121112	121112	121112	121112
113113	113113	113113	113133	113133	113133	113133	113133	113133	113153
111454	444444	444444	111444	444444	444444	444444	444444	444444	444444
555555	444455	555555	113455	113135	113155	113455	115155	445455	113155
123456	123456	123456	123456	123456	123456	123456	123456	123456	123456
(Idem)	(Idem)	(Idem)	(S.L.)	(1)	(Idem)	(Idem)	(Idem)	(Idem)	(Idem)

(A.77)

111111	111111	111111	111111	111111	111111	111111	111111	111111	111111
121112	121112	121112	121112	121112	121112	121112	121112	121112	121112
113153	113153	113333	113333	113333	113333	113333	113333	113333	113333
444444	444444	113334	113334	113334	113334	113334	113344	113434	113434
115135	555555	113335	113345	113355	113455	115555	113455	113355	115555
123456	123456	123456	123456	123456	123456	123456	123456	123456	123456
(1)	(Idem)	(Com)	(Com)	(Com)	(1)	(1)	(Com)	(S.L.)	(Idem)

(A.78)

111111	111111	111111	111111	111111	111111	111111	111111	111111	111111	
121112	121112	121112	121112	121112	121112	121112	121112	121112	121112	
113333	113333	113333	113333	113333	113333	113353	113353	113353	113353	
113444	113444	113444	113454	113454	114444	113354	113354	113454	113454	(A.79)
113445	113455	113555	113545	115555	115555	115535	555555	115535	555555	
123456	123456	123456	123456	123456	123456	123456	123456	123456	123456	
(Com)	(S.L.)	(Idem)	(Com)	(Idem)	(Idem)	(Com)	(1)	(Com)	(Idem)	

111111	111111	111111	111111	111111	111111	111111	111111	111111	111111	
121112	121112	121112	121112	121112	121112	121112	121112	121112	121112	
113353	113443	113443	113443	113453	113453	113453	113453	113453	113453	
114454	114334	444444	444444	113454	114354	114534	444444	444444	444444	(A.80)
555555	114335	444455	555555	555555	555555	115345	445135	445455	555555	
123456	123456	123456	123456	123456	123456	123456	123456	123456	123456	
(Idem)	(Com)	(Idem)	(Idem)	(Idem)	(1)	(Com)	(2)	(Idem)	(Idem)	

111111	111111	111111	111111	111111	111111	111111	111111	111111	111111	
121112	121112	121112	121112	121112	121112	121112	121112	121122	121122	
333333	333333	333333	333333	333333	333333	333333	333333	113133	113133	
333434	333444	333444	333444	333454	333454	333454	444444	111444	111444	(A.81)
555555	333445	333455	333555	333455	333545	555555	555555	123455	123456	
123456	123456	123456	123456	123456	123456	123456	123456	123456	123465	
(Idem)	(1)	(Idem)	(Idem)	(Idem)	(1)	(Idem)	(Idem)	(S.L.)	(Com)	

111111	111111	111111	111111	111111	111111	111111	111111	111111	111111	
121122	121122	121122	121122	121122	121122	121122	121122	121122	121122	
113133	113133	113133	113313	113313	113313	113313	113313	113313	113333	
444444	444444	444444	113314	113314	113314	113414	113414	114414	113334	(A.82)
123155	123455	123456	121125	121155	151155	121155	151155	151155	123355	
123456	123456	123465	123456	123456	123456	123456	123456	123456	123456	
(Idem)	(Idem)	(1)	(Com)	(Com)	(1)	(S.L.)	(Idem)	(Idem)	(Com)	

111111	111111	111111	111111	111111	111111	111111	111111	111111	111111	
121122	121122	121122	121122	121122	121122	121122	121122	121122	121122	
113333	113333	113333	113333	113333	113333	113333	113333	113333	113333	
113334	113344	113344	113434	113444	113444	114444	114444	114444	114444	(A.83)
123455	123455	123456	123355	123455	123456	123355	123455	123456	123456	
123456	123456	123465	123456	123456	123465	123456	123456	123465	124365	
(1)	(Com)	(Com)	(S.L.)	(S.L.)	(Com)	(Idem)	(Idem)	(1)	(2)	

111111	111111	111111	111111	111111	111111	111111	111111	111111	111111	
121122	121122	121122	121122	121122	121122	121122	121122	121122	121122	
113413	113413	113413	113413	113413	113413	113413	113433	113433	113433	
113414	114314	114314	114314	444444	444444	444444	114344	114344	444444	(A.84)
151155	121125	121155	151155	121125	121155	151155	123455	123456	123455	
123456	123456	123456	123456	123456	123456	123456	123456	123465	123456	
(Idem)	(Com)	(Com)	(1)	(1)	(Idem)	(Idem)	(Com)	(Com)	(Idem)	

111111	111111	111111	111111	111111	111111	111111	111111	111111	111111	
121122	121122	121122	121122	121122	121122	121122	121122	121122	121122	
113433	113434	113434	333333	333333	333333	333333	333333	333333	333333	
444444	114343	444444	333444	333444	444444	444444	444444	444444	444444	(A.85)
123456	123456	123456	123455	123456	121125	121155	121325	121455	123355	
123465	124365	464466	123456	123465	123456	123456	123456	123456	123456	
(1)	(Com)	(Idem)	(Idem)	(1)	(1)	(Idem)	(8)	(Idem)	(Idem)	

111111	111111	111111	111111	111111	111111	111111	111111	111111	111111	
121122	121122	121122	121122	121122	121126	121126	121126	121126	121126	
333333	333333	333333	333333	333333	113431	113431	113431	113431	333333	
444444	444444	444444	444444	444444	113441	114341	114341	444444	444444	(A.86)
123455	123456	123456	123456	123456	123456	123456	123456	123456	123456	
123456	123465	124365	161166	363366	666666	161162	666666	666666	121166	
(Idem)	(1)	(2)	(Idem)	(Idem)	(Idem)	(Com)	(1)	(Idem)	(Idem)	

111111	111111	111111	111111	111111	111111	111111	111111	111111	111111	
121126	121126	121222	121222	121222	121222	121222	121222	121222	121222	
333333	333333	113333	113333	113333	113333	113333	113333	333333	333333	
444444	444444	123444	123444	123444	123444	123446	123456	121224	121224	(A.87)
123456	123456	123445	123455	123456	123456	123456	123564	121225	121245	
161162	666666	123456	123456	123465	123666	123664	123645	123456	123456	
(1)	(Idem)	(Com)	(S.L.)	(Com)	(Idem)	(Com)	(Com)	(1)	(1)	

111111	111111	111111	111111	111111	111111	111111	111111	111111	111111	
121222	121222	121222	121222	121222	121222	121222	121222	121222	121222	
333333	333333	333333	333333	333333	333333	333333	333333	333333	333333	
121224	121224	121224	121224	121224	121224	121244	121244	121244	121244	(A.88)
121255	121455	123255	123455	151555	353555	121255	121455	123255	123455	
123456	123456	123456	123456	123456	123456	123456	123456	123456	123456	
(1)	(5)	(1)	(5)	(1)	(1)	(1)	(1)	(1)	(1)	

111111	111111	111111	111111	111111	111111	111111	111111	111111	111111	
121222	121222	121222	121222	121222	121222	121222	121222	121222	121222	
333333	333333	333333	333333	333333	333333	333333	333333	333333	333333	
121244	121424	121424	121424	121424	121444	121444	121444	121444	121444	(A.89)
123456	121255	123255	151555	353555	121445	121455	121555	123455	123456	
123465	123456	123456	123456	123456	123456	123456	123456	123456	123465	
(1)	(Idem)	(Idem)	(Idem)	(Idem)	(1)	(Idem)	(Idem)	(Idem)	(1)	

111111	111111	111111	111111	111111	111111	111111	111111	111111	111111	
121222	121222	121222	121222	121222	121222	121222	121222	121222	121222	
333333	333333	333333	333333	333333	333333	333333	333333	333333	333333	
121446	121446	121446	123424	123444	123444	123444	123444	123446	123446	(A.90)
123456	123456	123456	151555	123445	123455	123456	123456	123456	123456	
121466	121664	161666	123456	123456	123456	123465	123666	123466	123664	
(Idem)	(1)	(Idem)	(Idem)	(1)	(Idem)	(1)	(Idem)	(Idem)	(1)	

111111	111111	111111	111111	111111	111111	111111	111111	111111	111111	
121222	121222	121222	121222	121222	121222	121222	121222	121222	121226	
333333	333333	333333	333333	333333	333333	333333	333333	333333	333333	
123446	123446	123456	123456	123456	123456	123456	123456	123456	121246	(A.91)
123456	123456	123546	123546	123564	151555	151555	151555	353555	123456	
161666	363666	161666	363666	123645	153624	161666	363666	351624	121266	
(Idem)	(Idem)	(1)	(1)	(1)	(2)	(Idem)	(Idem)	(2)	(7)	

111111	111111	111111	111111	111111	111111	111111	111111	111111	111111	
121226	121226	121226	121226	121226	121226	121226	121226	121226	121226	
333333	333333	333333	333333	333333	333333	333333	333333	333333	333333	
121246	121246	121446	121446	121446	121446	123446	123446	123446	123446	(A.92)
123456	123456	123456	123456	123456	123456	123456	123456	123456	123456	
161662	666666	121266	121666	161662	666666	121266	121666	161662	666666	
(1)	(1)	(Idem)	(Idem)	(1)	(Idem)	(Idem)	(Idem)	(1)	(Idem)	

111111	111111	111111	111111	111111	111111	111111	111111	111111	111111	
121226	121226	121226	121226	121226	121255	121255	121255	121255	121256	
333333	333333	333333	333333	333333	333333	333333	333333	333333	333333	
123456	123456	123456	123456	123456	123456	123456	123456	123456	123456	(A.93)
123546	123546	123546	151556	353556	121552	151522	151522	555555	121556	
121666	161662	666666	666666	666666	123654	151622	153624	666666	121656	
(2)	(1)	(1)	(Idem)	(Idem)	(2)	(1)	(1)	(Idem)	(Idem)	

111111	111111	111111	111111	111111	111111	111111	111111	111111	111111	
121256	121256	121256	121256	121256	121256	122222	122222	122222	122222	
333333	333333	333333	333333	333333	333333	122223	122223	122223	122223	
123456	123456	123456	123456	123456	123456	122224	122224	122224	122224	(A.94)
121556	126556	151526	151562	555555	555555	122225	122235	122255	122325	
666666	666666	666666	161625	565612	666666	123456	123456	123456	123456	
(Idem)	(Idem)	(1)	(1)	(2)	(Idem)	(Com)	(Com)	(Com)	(1)	

111111	111111	111111	111111	111111	111111	111111	111111	111111	111111	
122222	122222	122222	122222	122222	122222	122222	122222	122222	122222	
122223	122223	122223	122223	122223	122223	122223	122223	122223	122223	
122224	122224	122224	122224	122224	122234	122234	122234	122244	122244	(A.95)
122335	122455	123355	123455	155555	122325	122335	122345	122455	123255	
123456	123456	123456	123456	123456	123456	123456	123456	123456	123456	
(1)	(1)	(1)	(1)	(1)	(Com)	(Com)	(Com)	(Com)	(4)	

111111	111111	111111	111111	111111	111111	111111	111111	111111	111111	
122222	122222	122222	122222	122222	122222	122222	122222	122222	122222	
122223	122223	122223	122223	122223	122223	122223	122223	122223	122223	
122244	122324	122324	122324	122324	122334	122424	122424	122424	122444	(A.96)
123455	122235	122255	122335	155555	122335	122255	123255	155555	122445	
123456	123456	123456	123456	123456	123456	123456	123456	123456	123456	
(1)	(Com)	(Com)	(1)	(1)	(Com)	(Com)	(1)	(1)	(Com)	

111111	111111	111111	111111	111111	111111	111111	111111	111111	111111	
122222	122222	122222	122222	122222	122222	122222	122222	122222	122222	
122223	122223	122223	122223	122223	122223	122223	122223	122223	122223	
122444	122444	122444	122454	122454	123424	123444	123444	123444	123454	(A.97)
122455	122555	123455	122545	155555	155555	123445	123455	123555	123455	
123456	123456	123456	123456	123456	123456	123456	123456	123456	123456	
(Com)	(1)	(1)	(Com)	(1)	(5)	(1)	(1)	(5)	(1)	

111111	111111	111111	111111	111111	111111	111111	111111	111111	111111	
122222	122222	122222	122222	122222	122222	122222	122222	122222	122222	
122223	122223	122223	122223	122223	122233	122233	122233	122233	122233	
123454	123454	144444	144444	144444	122234	122234	122244	122244	122244	(A.98)
123545	155555	144445	144455	155555	123355	123455	123455	123456	123456	
123456	123456	123456	123456	123456	123456	123456	123456	123465	124365	
(1)	(5)	(1)	(1)	(1)	(Com)	(1)	(Com)	(Com)	(3)	

111111	111111	111111	111111	111111	111111	111111	111111	111111	111111	
122222	122222	122222	122222	122222	122222	122222	122222	122222	122222	
122233	122233	122233	122233	122233	122233	122233	122233	122233	122233	
122344	122344	122424	122444	122444	123424	123434	123444	123444	144444	(A.99)
123455	123456	123255	123455	123456	122255	122255	123455	123456	122255	
123456	123465	123456	123456	123465	123456	123456	123456	123465	123456	
(Com)	(Com)	(Com)	(Com)	(Com)	(4)	(5)	(1)	(1)	(1)	

111111	111111	111111	111111	111111	111111	111111	111111	111111	111111	
122222	122222	122222	122222	122222	122222	122222	122222	122222	122222	
122233	122233	122233	122233	122233	122234	122234	122333	122333	122333	
144444	144444	144444	144444	144444	122243	144444	123444	123444	123444	(A.100)
122455	123255	123455	123456	123456	123456	123456	123445	123455	123456	
123456	123456	123456	123465	144466	124365	166666	123456	123456	123465	
(1)	(1)	(1)	(1)	(1)	(Com)	(8)	(Com)	(Com)	(Com)	

111111	111111	111111	111111	111111	111111	111111	111111	111111	111111	
122222	122222	122222	122222	122222	122222	122222	122222	122222	122222	
122333	122333	122333	123223	123223	123223	123223	123223	123223	123223	
123444	123446	123456	122424	122424	122444	122444	122444	122454	122454	(A.101)
123456	123456	123564	122255	155555	122445	122455	122555	122545	155555	
123666	123664	123645	123456	123456	123456	123456	123456	123456	123456	
(1)	(Com)	(Com)	(S.L.)	(Idem)	(Com)	(S.L.)	(Idem)	(Com)	(Idem)	

111111	111111	111111	111111	111111	111111	111111	111111	111111	111111	
122222	122222	122222	122222	122222	122222	122222	122222	122222	122222	
123223	123223	123233	123233	123233	123233	123233	123233	123233	123233	
144444	144444	122444	122444	144444	144444	144444	144444	144444	144444	(A.102)
144455	155555	123455	123456	123235	123255	123455	123456	123456	123456	
123456	123456	123456	123465	123456	123456	123456	123465	126266	146466	
(Idem)	(Idem)	(S.L.)	(Com)	(1)	(Idem)	(Idem)	(1)	(Idem)	(Idem)	

111111	111111	111111	111111	111111	111111	111111	111111	111111	111111	
122222	122222	122222	122222	122222	122222	122222	122222	122222	122222	
123236	123236	123236	123333	123333	123333	123333	123333	123333	123333	
144444	144444	144444	123334	123334	123334	123334	123334	123344	123344	(A.103)
123456	123456	123456	123335	123345	123355	123455	125555	123455	123456	
123266	126263	166666	123456	123456	123456	123456	123456	123456	123465	
(Idem)	(1)	(Idem)	(Com)	(Com)	(Com)	(1)	(1)	(Com)	(Com)	

111111	111111	111111	111111	111111	111111	111111	111111	111111	111111	
122222	122222	122222	122222	122222	122222	122222	122222	122222	122222	
123333	123333	123333	123333	123333	123333	123333	123333	123333	123333	
123434	123434	123444	123444	123444	123444	123446	123446	123456	123456	(A.104)
123355	125555	123445	123455	123456	123456	123456	123456	123546	123564	
123456	123456	123456	123456	123465	123666	123664	126666	126666	123645	
(S.L.)	(Idem)	(Com)	(S.L.)	(Com)	(Idem)	(Com)	(Idem)	(1)	(Com)	

111111	111111	111111	111111	111111	111111	111111	111111	111111	111111	
122222	122222	122222	122222	122222	122222	122222	122222	122222	122222	
123333	123333	123336	123336	123336	123336	123336	123336	123336	123355	
123456	123456	123346	123346	123446	123446	123456	123456	123456	123456	(A.105)
125555	125555	123456	123456	123456	123456	123546	123546	125556	125533	
125634	126666	126663	166666	126663	166666	126663	166666	166666	125633	
(2)	(Idem)	(Com)	(1)	(Com)	(Idem)	(Com)	(1)	(Idem)	(Com)	

111111	111111	111111	111111	111111	111111	111111	111111	111111	111111	
122222	122222	122222	122222	122222	122222	122222	122222	122222	122222	
123355	123356	123356	123356	123356	123356	123456	123456	123456	123456	
123456	123456	123456	123456	123456	123456	124356	124365	124365	124365	(A.106)
125533	123556	125536	125563	155555	155555	155555	125634	125643	155555	
125634	166666	166666	126635	156623	166666	166666	126543	126534	166666	
(Com)	(Idem)	(1)	(Com)	(2)	(Idem)	(1)	(Com)	(Com)	(2)	

111111	111111	111111	111111	111111	111111	111111	111111	111111	111111	
122222	122222	122226	122226	122226	122226	122226	122226	122226	122226	
123456	123456	122236	122236	122236	122236	122236	122236	122236	122236	
124536	144444	122246	122246	122346	122346	122446	122446	123446	123446	(A.107)
125346	155555	123456	123456	123456	123456	123456	123456	123456	123456	
166666	166666	166662	666666	166662	666666	166662	666666	166662	666666	
(1)	(Idem)	(Com)	(1)	(Com)	(1)	(Com)	(1)	(1)	(5)	

111111	111111	111111	111111	111111	111111	111111	111111	111111	111111	
122226	122226	122226	122226	122226	122226	122226	122226	122226	122226	
122236	122336	122336	122336	122336	122336	123236	123236	123236	123336	
144446	122446	123446	123446	123456	123456	122446	122446	144446	123346	(A.108)
123456	123456	123456	123456	123546	123546	123456	123456	123456	123456	
666666	666666	166662	666666	166662	666666	166662	666666	666666	166662	
(1)	(1)	(Com)	(1)	(Com)	(1)	(Com)	(Idem)	(Idem)	(Com)	

111111	111111	111111	111111	111111	111111	111111	111111	111111	111111	
122226	122226	122226	122226	122226	122226	122226	122226	122226	122226	
123336	123336	123336	123336	123336	123336	123336	123356	123356	123356	
123346	123446	123446	123456	123456	123456	123456	123456	123456	123456	(A.109)
123456	123456	123456	123546	123546	125556	125556	123556	125536	125536	
666666	166662	666666	166662	666666	166662	666666	666666	166662	666666	
(1)	(Com)	(Idem)	(Com)	(1)	(1)	(Idem)	(Idem)	(Com)	(1)	

111111	111111	111111	111111	111111	111111	111111	111111	111111	111111	
122226	122226	122226	122226	122226	122226	122255	122255	122255	122255	
123356	123456	123456	123456	123456	123456	122355	122355	123355	123355	
123456	124356	124536	124536	144446	144446	123456	123456	123456	123456	(A.110)
155556	155556	125346	125346	145236	155556	155522	155522	155522	155522	
666666	666666	166662	666666	666666	666666	155622	155623	155622	156622	
(Idem)	(1)	(Com)	(1)	(2)	(Idem)	(Com)	(Com)	(Com)	(1)	

111111	111111	111111	111111	111111	111111	111111	111111	111111	111111	
122255	122255	122255	122255	122255	122256	122256	122256	122256	122256	
123356	123356	123456	123456	123456	122356	122356	122356	122356	122356	
123456	123456	124356	144466	144466	123456	123456	123456	123456	123456	(A.111)
155522	155522	155522	125255	155522	122556	155526	155562	555555	555555	
156622	156623	156622	146466	166644	666666	666666	166625	566612	666666	
(Com)	(Com)	(Com)	(Idem)	(1)	(7)	(1)	(Com)	(9)	(1)	

111111	111111	111111	111111	111111	111111	111111	111111	111111	111111	
122256	122256	122256	122256	122256	122256	122256	122256	122256	122256	
123356	123356	123356	123356	123356	123356	123456	123456	123456	123456	
123456	123456	123456	123456	123456	123456	124356	124356	124356	124356	(A.112)
122556	125556	155526	155562	555555	555555	125556	155526	155562	555555	
666666	666666	666666	166625	566612	666666	666666	666666	166625	566612	
(Idem)	(Idem)	(1)	(Com)	(2)	(Idem)	(2)	(1)	(Com)	(2)	

111111	111111	111111	111111	111111	111111	111111	111111	111111	111111	
122256	122256	122444	122446	122446	122446	122446	122446	122446	122456	
123456	123456	123456	123456	123456	123456	123456	123456	123456	123456	
124356	144456	144222	124426	144226	144226	144662	644221	644221	124456	(A.113)
555555	555555	145222	125436	145226	145236	145662	645221	645231	125456	
666666	666666	146222	666666	666666	666666	166224	666666	666666	666666	
(1)	(Idem)	(Com)	(2)	(1)	(1)	(Com)	(9)	(2)	(Idem)	

111111	111111	111111	111111	111111	111111	111111	111111	111111	111111	
122456	122456	122456	122456	122456	122456	122456	123456	123456	123456	
123456	123456	123456	123456	123456	123456	123456	132456	132465	132465	
124456	144256	144265	144265	144265	144526	444444	444444	444444	444444	(A.114)
555555	555555	155624	155642	555555	155246	555555	555555	156423	156432	
666666	666666	166542	166524	666666	666666	666666	666666	165432	165423	
(Idem)	(1)	(Com)	(Com)	(2)	(1)	(Idem)	(1)	(1)	(1)	

111111	111111	111111	111111	111111	111111	111111	111111	111116	111116	
123456	123456	123456	123456	123456	123456	123456	123456	111126	111126	
132465	132465	132465	134256	134562	333333	333333	333333	111136	111136	
444444	444444	444444	142356	145623	341265	345612	444444	111146	111246	(A.115)
456123	456132	555555	555555	156234	555555	555555	555555	123456	123456	
465132	465123	666666	666666	162345	666666	561234	666666	666661	666661	
(2)	(2)	(2)	(1)	(Com)	(2)	(2)	(Idem)	(Com)	(Com)	

111116	111116	111116	111116	111116	111116	111116	111116	111116	111116	
111126	111126	111126	111126	111126	111126	111126	111126	111126	111126	
111136	111136	111136	111136	111136	111136	111236	111236	111236	111336	
111446	112146	112246	113446	122446	123446	112146	112246	112346	113446	(A.116)
123456	123456	123456	123456	123456	123456	123456	123456	123456	123456	
666661	666661	666661	666661	666661	666661	666661	666661	666661	666661	
(Com)	(1)	(1)	(1)	(1)	(1)	(Com)	(Com)	(Com)	(Com)	

111116	111116	111116	111116	111116	111116	111116	111116	111116	111116	
111126	111126	111126	111126	111126	111126	111126	111126	111126	111126	
111336	111336	112136	112136	112136	112236	113136	113136	113336	113336	
121446	123446	111246	111446	112246	112246	111446	121446	113346	113446	(A.117)
123456	123456	123456	123456	123456	123456	123456	123456	123456	123456	
666661	666661	666661	666661	666661	666661	666661	666661	666661	666661	
(6)	(1)	(Com)	(Com)	(1)	(Com)	(Com)	(1)	(Com)	(Com)	

111116	111116	111116	111116	111116	111116	111116	111116	111116	111116	
111126	111126	111126	111126	111126	111126	111126	111126	111226	111226	
113336	113336	113436	123336	123336	123336	123436	123436	111236	111236	
114446	123446	114346	123346	123446	124446	123446	124346	122446	123446	(A.118)
123456	123456	123456	123456	123456	123456	123456	123456	123456	123456	
666661	666661	666661	666661	666661	666661	666661	666661	666661	666661	
(1)	(1)	(Com)	(1)	(1)	(7)	(1)	(1)	(Com)	(1)	

111116	111116	111116	111116	111116	111116	111116	111116	111116	111116	
111226	111226	111226	111226	111226	111226	111226	111226	111226	111226	
111336	111336	111336	112336	112336	113136	113336	113336	123136	123236	
123446	123456	123456	123446	123456	121446	123446	123456	111446	111446	(A.119)
123456	123546	132546	123456	123546	123456	123456	123546	123456	123456	
666661	666661	666661	666661	666661	666661	666661	666661	666661	666661	
(Com)	(Com)	(3)	(Com)	(Com)	(Com)	(Com)	(Com)	(6)	(6)	

111116	111116	111116	111116	111116	111116	111116	111116	111116	111116	
111226	111226	111236	112226	112226	112226	112226	112226	112226	121126	
123336	123336	111326	123336	123336	123336	123336	123356	123456	113136	
123446	123456	123456	123346	123446	123456	123456	123456	124536	111446	(A.120)
123456	123546	132546	123456	123456	123546	125556	125536	125346	123456	
666661	666661	666661	666661	666661	666661	666661	666661	666661	666661	
(1)	(1)	(Com)	(Com)	(Com)	(Com)	(1)	(Com)	(Com)	(Com)	

111116	111116	111116	111116	111116	111116	111116	111116	111116	111116	
121126	121126	121126	121126	121226	121226	122226	122226	122226	122226	
113336	113336	113336	113436	113336	113336	122236	122236	122236	122236	
113346	113446	114446	114346	123446	123456	122246	122346	122446	123446	(A.121)
123456	123456	123456	123456	123456	123546	123456	123456	123456	123456	
666661	666661	666661	666661	666661	666661	666661	666661	666661	666661	
(Com)	(Com)	(1)	(Com)	(Com)	(Com)	(Com)	(Com)	(Com)	(1)	

111116	111116	111116	111116	111116	111116	111116	111116	111116	111116	
122226	122226	122226	122226	122226	122226	122226	122226	122226	122226	
122236	122336	122336	123236	123236	123336	123336	123336	123336	123356	
144446	123446	123456	122446	144446	123346	123446	123456	123456	123456	(A.122)
123456	123456	123546	123456	123456	123456	123456	123546	125556	125536	
666661	666661	666661	666661	666661	666661	666661	666661	666661	666661	
(1)	(Com)	(Com)	(Com)	(1)	(Com)	(Com)	(Com)	(1)	(Com)	

111116	111116	111116	111116	111116	111116	111116	111116	111116	111116	
122226	122226	122226	122226	122226	122256	122256	122256	122446	122446	
123356	123456	123456	123456	123456	122356	123356	123456	123456	123456	
123456	124356	124536	144446	144446	123456	123456	124356	144226	144226	(A.123)
155556	155556	125346	145236	155556	155526	155526	155526	145226	145236	
666661	666661	666661	666661	666661	666661	666661	666661	666661	666661	
(1)	(1)	(Com)	(2)	(1)	(Com)	(Com)	(Com)	(Com)	(Com)	

111116	111116	111116	111155	111155	111155	111155	111155	111155	111155	
122456	123456	123456	111255	111255	111255	111255	111255	111255	111255	
123456	132546	132546	111355	111355	112355	112355	113355	113355	113355	
144526	145236	145326	123456	123456	123456	123456	123456	123456	123456	(A.124)
155246	154326	154236	555511	555511	555511	555511	555511	555511	555511	
666661	666661	666661	555611	555612	555611	555612	555611	555612	556611	
(Com)	(Com)	(Com)	(Com)	(Com)	(Com)	(Com)	(Com)	(Com)	(1)	

111155	111155	111155	111155	111155	111155	111155	111155	111155	111155	
111255	111255	111255	111255	111255	111255	111255	111255	112255	112255	
113356	113356	123355	123355	123356	123356	333366	333366	123355	123355	
123456	123456	123456	123456	123456	123456	123456	123456	123456	123456	(A.125)
555511	555511	555511	555511	555511	555511	111555	555511	555511	555511	
556611	556613	555611	556611	555611	556611	333666	666633	555611	556611	
(Com)	(Com)	(1)	(6)	(1)	(1)	(7)	(1)	(Com)	(1)	

111155	111155	111155	111155	111155	111155	111155	111155	111155	111155	
112255	112255	112255	112255	121255	121255	121255	121255	121255	121255	
123356	123356	123456	123456	113355	113355	113356	113356	113356	333366	
123456	123456	124356	124356	123456	123456	123456	123456	123456	123456	(A.126)
555511	555511	555511	555511	555511	555511	555511	555511	555511	111555	
556611	556612	556611	556612	555611	556611	556611	556613	565611	333666	
(Com)	(Com)	(Com)	(Com)	(Com)	(1)	(Com)	(Com)	(6)	(Idem)	

111155	111155	111155	111155	111155	111155	111155	111155	111155	111155	
121255	121255	122255	122255	122255	122255	122255	122255	122255	122255	
333366	333366	122355	122355	123355	123355	123355	123356	123356	123456	
123456	123456	123456	123456	123456	123456	123456	123456	123456	124356	(A.127)
151555	555511	555511	555511	555511	555511	555511	555511	555511	555511	
363666	666633	555611	566611	555611	556611	566611	556611	566611	556611	
(Idem)	(1)	(Com)	(1)	(Com)	(1)	(1)	(Com)	(1)	(Com)	

111155	111155	111155	111155	111155	111155	111155	111155	111155	111155	
122255	122255	122255	122256	122256	122256	122256	122256	122256	122256	
123456	123456	123456	122356	122356	123356	123356	123456	123456	123456	
124356	144455	144455	123456	123456	123456	123456	124356	124356	144456	(A.128)
555511	555511	555511	555511	555511	555511	555511	555511	555511	555511	
566611	556511	566611	566611	566612	566611	566612	566611	566612	556511	
(1)	(1)	(1)	(Com)	(Com)	(Com)	(Com)	(Com)	(Com)	(7)	

111155	111155	111155	111155	111155	111155	111155	111155	111155	111155	
122256	122455	122455	122456	122456	122456	123456	123456	123456	123456	
123456	123456	123456	123456	123456	123456	132456	132456	134256	333366	
144456	144255	144255	144256	444466	444466	444466	444466	142356	341265	(A.129)
555511	555511	555511	555511	155155	555511	155155	555511	555511	151555	
566611	556511	566611	566611	466466	666644	466466	666644	566611	363666	
(1)	(Com)	(1)	(Com)	(Idem)	(1)	(2)	(1)	(Com)	(2)	

111155	111156	111156	111156	111156	111156	111156	111156	111156	111156	
123456	111256	111256	111256	111256	112256	112256	121256	122256	122256	
333366	111356	112356	113356	123356	123356	123456	113356	122356	123356	
341265	123456	123456	123456	123456	123456	124356	123456	123456	123456	(A.130)
555511	555561	555561	555561	555561	555561	555561	555561	555561	555561	
666633	666615	666615	666615	666615	666615	666615	666615	666615	666615	
(2)	(Com)	(Com)	(Com)	(1)	(Com)	(Com)	(Com)	(Com)	(Com)	

111156	111156	111156	111156	111444	111444	111444	111444	111444	111444	
122256	122256	122456	123456	112444	112444	112444	112444	112444	112444	
123456	123456	123456	134256	123456	123456	123456	123456	123456	123456	
124356	144456	144256	142356	444111	444111	444111	444111	444111	444111	(A.131)
555561	555561	555561	555561	445111	445111	445111	445111	445111	445112	
666615	666615	666615	666615	446111	446112	446121	446122	456111	446121	
(Com)	(1)	(Com)	(Com)	(Com)	(Com)	(1)	(1)	(1)	(Com)	

111444	111444	111444	111444	111444	111444	111444	111444	111444	111444	
112444	112444	112444	112444	112445	112445	112445	122444	122444	122444	
123456	123456	123456	123456	123456	123456	123456	123456	123456	123456	
444111	444111	444111	444111	444111	444111	444111	444111	444111	444111	(A.132)
445112	445121	445121	445122	445111	445111	445112	445111	445111	455111	
446122	446112	446122	446122	456111	456112	456123	446111	466111	456111	
(Com)	(Com)	(1)	(Com)	(Com)	(Com)	(Com)	(Com)	(1)	(1)	

111444	111444	111444	111444	111444	111444	111444	111444	111444	111444	
122444	122446	122446	122446	122446	122455	122455	122455	122455	122456	
123456	123456	123456	123456	123456	123456	123456	123456	123456	123456	
444111	444111	444111	444111	444111	444111	444111	444111	444111	444111	(A.133)
455111	445111	445111	455111	455111	455111	455111	455122	455122	455111	
466111	466111	466112	446111	466111	456111	466111	456122	456123	466111	
(1)	(Com)	(Com)	(6)	(1)	(Com)	(1)	(Com)	(Com)	(Com)	

111444	111444	111444	111444	111444	111444	111444	111444	111444	111446	
123456	123456	123456	123456	123456	123456	123456	123456	123456	112446	
132456	132456	132465	132465	132465	333555	333555	333555	333555	123456	
444111	444111	444111	444111	444111	141441	444111	444111	444111	444661	(A.134)
455111	456111	456111	456123	456132	353553	555333	555333	555333	445661	
466111	465111	465111	465132	465123	361542	464111	465132	564312	666114	
(Com)	(3)	(Com)	(Com)	(Com)	(2)	(1)	(1)	(2)	(Com)	

111446	111446	111446	111446	111456	111456	111456	111456	111456	111456	
122446	122446	122456	123456	112456	112456	122456	122456	123456	123456	
123456	123456	123456	132456	123456	123456	123456	123456	132456	132456	
444661	444661	444661	444661	444165	444165	444165	444165	444165	444165	(A.135)
445661	455661	455661	455661	555614	555641	555614	555641	555614	555641	
666114	666114	666114	666114	666541	666514	666541	666514	666541	666514	
(Com)	(1)	(Com)	(Com)	(Com)	(Com)	(Com)	(Com)	(Com)	(Com)	

113333	113336	113355	113355	113355	113356	113356	113356	113356	113356	
123456	123456	123456	123456	123456	123456	123456	123456	123456	123456	
331111	336661	335511	335511	335511	331165	331165	331165	331165	335561	
341111	346661	345511	345511	345612	341165	341165	341265	341265	345561	(A.136)
351111	356661	551133	551133	551133	556613	556631	556613	556631	556613	
361111	661113	561133	561134	561234	665531	665513	665531	665513	661135	
(Com)	(Com)	(Com)	(Com)	(Com)	(Com)	(Com)	(Com)	(Com)	(Com)	

113456	123456	123456	
123456	214365	214365	
334561	345612	351624	
445613	436521	462513	(A.137)
556134	561234	536142	
661345	652143	645231	
(Com)	$\mathbb{Z}_6$	S_3	

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