I. Classes of L^1-convergence of Fourier series

1.1. Classical and neoclassical results. Denote by $L^1(\mathbb{T})$ the Banach space of all complex, Lebesgue integrable functions on the unit circle $\mathbb{T}$. To every function $f \in L^1(\mathbb{T})$ corresponds the Fourier series of f,

$$S(f) \sim \sum_{|n|<\infty} \widehat{f}(n)e^{int}, \quad \text{where} \quad \widehat{f}(n) = \frac{1}{2\pi}\int_{\mathbb{T}} f(t)e^{-int}\,dt, \quad |n| < \infty,$$

are the Fourier coefficients of f.

The sequence of partial sums will be denoted by

$$S_n(f) = S_n(f,t) = \sum_{|k|\le n} \widehat{f}(k)e^{ikt}, \quad n = 0,1,\ldots,$$

while the $(C,1)$-means (Fejér sums) of the sequence of partial sums will be written as

$$\sigma_n(f) = \sigma_n(f,t) = \frac{1}{n+1}\sum_{k=0}^{n} S_k(f,t), \quad n = 0,1,\ldots.$$

The Dirichlet kernel is denoted by

$$D_n(t) = \frac{1}{2} + \sum_{k=1}^{n} \cos kt = \frac{\sin[(n+1/2)t]}{2\sin(t/2)}$$

and the Fejér kernel by

$$F_n = \frac{1}{n+1}\sum_{k=0}^{n} D_k(t) = \frac{1}{2(n+1)}\left[\frac{\sin[(n+1)t/2]}{\sin(t/2)}\right]^2.$$

Note that

$$\|D_n\|_1 = \frac{4}{\pi^2}\log n + O(1), \quad n \to \infty,$$
$$\|F_n\|_1 = 1 \quad \text{for every } n, \text{ where } \|\ \|_1 \text{ denotes the } L^1(\mathbb{T})\text{-norm.}$$

Let

$$\widetilde{D}_n(t) = \sum_{k=1}^{n} \sin kt = \frac{\cos(t/2) - \cos[(n+1/2)t]}{2\sin(t/2)}$$
$$\overline{D}_n(t) = -\frac{1}{2}\operatorname{ctg}\frac{t}{2} + \widetilde{D}_n(t) = -\frac{\cos[(n+1/2)t]}{2\sin(t/2)},$$
$$\widetilde{K}_n(t) = \frac{1}{n+1}\sum_{k=0}^{n} \widetilde{D}_k(t) = \frac{1}{4\sin^2(t/2)}\left[\sin t - \frac{\sin[(n+1)t]}{n+1}\right]$$

denote the conjugate Dirichlet kernel, modified Dirichlet kernel and conjugate Fejér kernel, respectively.

Let $L^1(0,\pi)$ be the Banach space of all real, Lebesgue integrable functions on $(0,\pi)$. Let

$$\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos nx \tag{C}$$

and

$$\sum_{n=1}^{\infty} a_n \sin nx \tag{S}$$

be cosine and sine trigonometric series. The partial sums of real cosine and sine series will be denoted by $S_n(x)$ and $\widetilde{S}_n(x)$ respectively.

Let f be a 2π-periodic and even function in $L^1(0,\pi)$, and let $\{a_k\}$ be the sequence of its Fourier coefficients. Denote by $\mathcal{F}$ the class of sequences of Fourier coefficients of all such functions. It is well known (see [73, Vol. 1, p. 67]) that, in general, it does not follow from $\{a_k\}\in\mathcal{F}$ that S_n converges to f in the $L^1(0,\pi)$-norm, i.e. it does not follow that $\|S_n-f\|=o(1)$, $n\to\infty$, where $\|\cdot\|$ is the $L^1(0,\pi)$-norm. However, there are examples of subclasses of $\mathcal{F}$ for which $a_n\log n=o(1)$, $n\to\infty$ is a necessary and sufficient condition for $\|S_n-f\|=o(1)$, $n\to\infty$.

A classical result concerning the integrability and L^1-convergence of a cosine series (C) is the following well known theorem of Young.

THEOREM 1.1 (Young [71]). *If $\{a_n\}_{n=0}^{\infty}$ is a convex ($\varDelta^2 a_n=\varDelta(\varDelta a_n)=\varDelta a_n-\varDelta a_{n+1}=a_n-2a_{n+1}+a_{n+2}\geq 0, \forall n$) null sequence, then the cosine series* (C) *is the Fourier series of its sum f, and*

$$\|S_n(f)-f\|=o(1),\ n\to\infty \quad \textit{iff} \quad a_n\log n=o(1),\ n\to\infty. \tag{1.1}$$

The sequences $\{a_n\}$ that satisfy the condition $\sum_{n=1}^{\infty}(n+1)|\varDelta^2 a_n|<\infty$ are called *quasi-convex.* The next theorem of Kolmogorov extends Young's result, since every convex null sequence is also quasi-convex.

THEOREM 1.2 (Kolmogorov [22]). *If $\{a_n\}$ is a quasi-convex null sequence then the cosine series* (C) *is the Fourier series of its sum f and* (1.1) *holds.*

We say that a sequence $\{a_k\}$ is of *bounded variation* and we write $\{a_k\}\in BV$ if $\sum_{k=0}^{\infty}|\varDelta a_k|<\infty$. Several authors (Sidon, Telyakovskiĭ, Fomin, Stanojević and others) have extended these classical results by addressing one or both of the following two questions:

(i) If $\{a_n\}$ is a null sequence of bounded variation, is (C) the Fourier series of its sum f?

(ii) If $\{a_n\}\in BV$, is (C) the Fourier series of some function $f\in L^1$ and does (1.1) hold?

THEOREM 1.3 (Sidon [34]). *Let* $\{\alpha_n\}_{n=1}^{\infty}$ *and* $\{p_n\}_{n=1}^{\infty}$ *be sequences such that* $|\alpha_n| \le 1$ *for every* n *and* $\sum_{n=1}^{\infty} |p_n| < \infty$. *If*

$$a_n = \sum_{k=n}^{\infty} \frac{p_k}{k} \sum_{l=n}^{k} \alpha_l, \qquad n = 1, 2, \ldots,$$

then the cosine series (C) *is the Fourier series of its sum* f.

It is obvious that Sidon's conditions imply that $\{a_n\} \in BV$.

Telyakovskiĭ [45] defined an extension of the class of quasi-convex sequences, denoted by S, as follows: a null sequence $\{a_n\}_{n=0}^{\infty}$ belongs to S if there exists a decreasing sequence $\{A_n\}_{n=0}^{\infty}$ such that $\sum_{n=0}^{\infty} A_n < \infty$ and $|\Delta a_n| \le A_n$ for all n. He proved that the Sidon class is equivalent to the class S. Therefore, the class S is usually called the *Sidon–Telyakovskiĭ class.*

THEOREM 1.4 (Telyakovskiĭ [45]). *Let* $\{a_n\}_{n=0}^{\infty} \in S$. *Then the cosine series* (C) *is the Fourier series of its sum* f *and* (1.1) *holds.*

On the other hand, Kano extended the classical result of Kolmogorov by answering the first question (i).

THEOREM 1.5 (Kano [19]). *If* $\{a_n\}$ *is a null sequence such that*

$$\sum_{n=1}^{\infty} n^2 \left|\Delta^2\left(\frac{a_n}{n}\right)\right| < \infty,$$

then (C) *is a Fourier series*, *or equivalently it represents an integrable function.*

The following lemma was proved by Telyakovskiĭ in [46].

LEMMA 1.1 ([46]). *The condition* $\sum_{n=1}^{\infty} n^2 |\Delta^2(a_n/n)| < \infty$ *is equivalent to the simultaneous fulfillment of the conditions* $\sum_{n=1}^{\infty} |a_n|/n < \infty$ *and* $\sum_{n=1}^{\infty} (n+1)|\Delta^2 a_n| < \infty$.

REMARK 1.1. In view of this lemma, Theorem 1.5 is a corollary of Theorem 1.2.

Later, Kumari and Ram proved the following theorem:

THEOREM 1.6 (Kumari–Ram [23]). *Suppose* $(k+1)^2 |\Delta^2(a_k/k)| \downarrow 0$. *Then*

$$h(x) = \lim_{n\to\infty} \sum_{k=1}^{n} \left[\frac{1}{2}(k+1)^2 \left|\Delta^2\left(\frac{a_k}{k}\right)\right| + \sum_{v=k}^{n} (v+1)^2 \left|\Delta^2\left(\frac{a_v}{v}\right)\right| \cos kx\right]$$

exists for $x \in (0, \pi]$, *and* $h \in L(0, \pi]$ *iff* $\sum_{k=1}^{\infty} (k+1)^2 |\Delta^2(a_k/k)| < \infty$.

The difference of noninteger order $k \ge 0$ of the sequence $\{a_n\}_{n=0}^{\infty}$ is defined as follows:

$$(*) \qquad \Delta^k a_n = \sum_{m=0}^{\infty} \binom{m-k-1}{m} a_{n+m} \qquad (n = 0, 1, 2, \ldots),$$

where

$$\binom{\alpha+m}{m} = \frac{(\alpha+1)(\alpha+2)\ldots(\alpha+m)}{m!}.$$

It is obvious that if $a_n \to 0$ as $n \to \infty$ then the series $(*)$ is convergent and $\lim_{n\to\infty} \Delta^k a_n = 0$. In [29] C. N. Moore generalized quasi-convexity of null sequences in the following way:

$$\text{(M)} \qquad \sum_{n=1}^{\infty} n^k |\Delta^{k+1} a_n| < \infty \qquad \text{for } k > 0,$$

where the order of differences is fractional, and proved the corresponding integrability result. It is known [8] that if $\{a_n\}$ is a null sequence satisfying (M), then $\sum_{n=1}^{\infty} n^r |\Delta^{r+1} a_n| < \infty$ for all $0 \le r < k$. In particular $\{a_n\}$ is of bounded variation.

Singh and Sharma proved the following generalization of a theorem of Kolmogorov.

Theorem 1.7 (Singh–Sharma [36]). *Let $k > 0$. If*

(i) $\lim_{n\to\infty} a_n = 0$,

(ii) $\sum_{n=1}^{\infty} n^k |\Delta^{k+1} a_n| < \infty$,

then the series (C) *converges in L^1 if and only if $a_n \log n = o(1)$, $n \to \infty$.*

Remark 1.2. Theorem 1.7 is a corollary of Theorem 1.4. It suffices to show that the conditions (i) and (ii) of Theorem 1.7 imply the Sidon–Telyakovskiĭ type condition S. First, we suppose that for some $0 < k \le 1$ the series of (M) converges. For $0 < k \le 1$, we define the sequence

$$A_n = \sum_{i=n}^{\infty} \binom{i-n+k-1}{i-n} |\Delta^{k+1} a_i|.$$

Now, we need the following properties of the binomial coefficients $\binom{\alpha+n}{n}$ (see [2, p. 885]):

(a) $\alpha > -1 \Rightarrow \binom{\alpha+n}{n} > 0$,

(b) $\binom{\alpha+n}{n} = \dfrac{n^\alpha}{\Gamma(\alpha+1)} + O(1)$, $0 < \alpha \le 1$,

(c) $\sum_{i=0}^{n} \binom{\alpha+i}{i} = \binom{n+\alpha+1}{n}$, $n \in \mathbb{N}$, $\alpha \in \mathbb{R}$.

We have

$$\begin{aligned}
\sum_{n=0}^{\infty} A_n &= \sum_{n=0}^{\infty}\sum_{i=n}^{\infty} \binom{i-n+k-1}{i-n} |\Delta^{k+1} a_i| \\
&= \sum_{i=0}^{\infty} |\Delta^{k+1} a_i| \sum_{n=0}^{i} \binom{i-n+k-1}{i-n} \\
&= \sum_{i=0}^{\infty} |\Delta^{k+1} a_i| \sum_{n=0}^{i} \binom{n+k-1}{n} = \sum_{i=0}^{\infty} \binom{i+k}{k} |\Delta^{k+1} a_i| \\
&= \frac{1}{\Gamma(k+1)} \sum_{i=0}^{\infty} i^k |\Delta^{k+1} a_i| + O\Big(\sum_{i=0}^{\infty} |\Delta^{k+1} a_i|\Big).
\end{aligned}$$

Since the series $(*)$ is convergent, by condition (M), we obtain

$$\sum_{i=0}^{\infty} |\Delta^{k+1} a_i| = |\Delta^{k+1} a_0| + \sum_{i=1}^{\infty} |\Delta^{k+1} a_i|$$
$$\leq \sum_{m=0}^{\infty} \binom{m-k-2}{m} a_m + \sum_{i=1}^{\infty} i^k |\Delta^{k+1} a_i| < \infty.$$

Thus, $\sum_{n=0}^{\infty} A_n < \infty$ and $A_n \downarrow 0$. Then for $0 < k \leq 1$, we obtain

$$\Delta a_n = \sum_{i=n}^{\infty} \binom{i-n+k-1}{i-n} \Delta^{k+1} a_i,$$

i.e.

$$|\Delta a_n| \leq \sum_{i=n}^{\infty} \binom{i-n+k-1}{i-n} |\Delta^{k+1} a_i| = A_n \quad \text{for all } n.$$

If $k > 1$, by Bosanquet's result [8], we obtain $\sum_{n=1}^{\infty} n|\Delta^2 a_n| < \infty$, i.e. $\{a_n\} \in S$. Finally, $\{a_n\} \in S$ for all $k > 0$.

In [38] Č. V. Stanojević and V. B. Stanojević generalized the Telyakovskiĭ theorem of [45].

They defined a stronger class S_p, $p > 1$, as follows: a null sequence $\{a_n\}$ of real numbers belongs to S_p if for some monotone sequence $\{A_n\}$ such that $\sum_{n=1}^{\infty} A_n < \infty$ the following condition holds:

$$\frac{1}{n} \sum_{k=1}^{n} \frac{|\Delta a_k|^p}{A_k^p} = O(1).$$

There exists a null sequence $\{a_n\}$ such that $\{a_n\} \in S_p$ but $\{a_n\} \notin S$.

Example. Define a sequence $\{a_n\}$ as follows: let $\Delta a_n = 1/m^2$ for $n = m^2$ and $\Delta a_n = 0$ for $n \neq m^2$. First, we shall show that $\{a_n\} \notin S$. We have

$$a_{m^2} = \sum_{i=m^2}^{\infty} \Delta a_i = \Delta a_{m^2} + \Delta a_{m^2+1} + \ldots + \Delta a_{(m+1)^2} + \ldots = \sum_{i=m}^{\infty} \Delta a_{i^2}$$
$$= \sum_{i=m}^{\infty} \frac{1}{i^2} \to 0, \qquad m \to \infty,$$

i.e. $a_n \to 0$ as $n \to \infty$. Set $A_n^* = \max_{i \geq n} |\Delta a_i|$. Then $A_n^* \downarrow 0$ and $\sum_{n=1}^{\infty} A_n^* = \infty$. Indeed, $A_n^* = 1/m^2$ for $(m-1)^2 + 1 \leq n \leq m^2$ and

$$\sum_{k=1}^{\infty} A_k^* = \sum_{m=1}^{\infty} \sum_{k=(m-1)^2+1}^{m^2} A_k^* = \sum_{m=1}^{\infty} \sum_{k=(m-1)^2+1}^{m^2} A_{m^2}^*$$
$$= \sum_{m=1}^{\infty} \frac{1}{m^2} [m^2 - (m-1)^2] = \sum_{m=1}^{\infty} \frac{2m-1}{m^2} = \infty.$$

Therefore for every positive sequence $\{A_n\}$ such that $A_n \geq A_n^*$, we have $\sum_{n=1}^{\infty} A_n = \infty$, i.e. $\{a_n\} \notin S$.

Now, let $A_n = 1/n^{1+1/2p}$ for all n. Then $A_n \downarrow 0$, $\sum_{n=1}^{\infty} A_n < \infty$, and for $n = m^2$ we have

$$\frac{1}{m^2}\sum_{i=1}^{m^2}\frac{|\Delta a_i|^p}{A_i^p} = \frac{1}{m^2}\sum_{k=1}^{m}\left(\frac{|\Delta a_{k^2}|}{A_{k^2}}\right)^p = \frac{1}{m^2}\sum_{k=1}^{m}\left(\frac{1/k^2}{1/k^{2+1/p}}\right)^p$$
$$= \frac{1}{m^2}\sum_{k=1}^{m}(k^{1/p})^p = O(1).$$

Theorem 1.8 (Č. V. Stanojević and V. B. Stanojević [38]). *Let $\{a_n\} \in S_p$ for some $1 < p \le 2$. Then the cosine series* (C) *is the Fourier series of its sum f and* (1.1) *holds.*

Fomin [13] also extended the Sidon–Telyakovskiĭ class. He defined a class F_p, $1<p\le 2$, of Fourier coefficients as follows: a sequence $\{a_n\}$ belongs to F_p if $a_k \to 0$ as $k \to \infty$ and

$$\sum_{k=1}^{\infty}\left(\frac{1}{k}\sum_{i=k}^{\infty}|\Delta a_i|^p\right)^{1/p} < \infty. \tag{1.2}$$

He also gave an equivalent form of the condition (1.2) by proving the following lemma.

Lemma 1.2 ([13]). *Let $p > 1$. Then a sequence $\{a_n\}$ is in F_p iff $\sum_{s=1}^{\infty} 2^s \Delta_s^{(p)} < \infty$, where*

$$\Delta_s^{(p)} = \left\{\frac{1}{2^{s-1}}\sum_{k=2^{s-1}+1}^{2^s}|\Delta a_k|^p\right\}^{1/p}.$$

In [13], Fomin noted that the class F_p is wider when p is closer to 1. Now we present the proof of this fact.

Corollary 1.1 ([61]). *For any $1 < r < p$ we have the embedding $F_p \subset F_r$.*

Proof. Since $1/r > 1/p$, we have $1/r = 1/p + 1/q$ for some $q > 0$. This implies that $1/p' + 1/q' = 1$, where $p' = p/r$ and $q' = q/r$. Applying the Hölder inequality, we have

$$\sum_{k=2^s+1}^{2^{s+1}}|\Delta a_k|^r = \sum_{k=2^s+1}^{2^{s+1}}|\Delta a_k|^r\cdot 1 \le \Big(\sum_{k=2^s+1}^{2^{s+1}}|\Delta a_k|^{rp'}\Big)^{1/p'}\Big(\sum_{k=2^s+1}^{2^{s+1}}1^{q'}\Big)^{1/q'}$$
$$= (2^s)^{1/q'}\Big(\sum_{k=2^s+1}^{2^{s+1}}|\Delta a_k|^p\Big)^{1/p'}.$$

Then

$$\sum_{s=1}^{\infty}2^s\Delta_s^{(r)} \le \sum_{s=1}^{\infty}2^s\cdot 2^{-s/r}\cdot 2^{s/q'r}\Big(\sum_{k=2^s+1}^{2^{s+1}}|\Delta a_k|^p\Big)^{1/rp'}$$
$$= \sum_{s=1}^{\infty}2^s\left(\frac{1}{2^s}\right)^{1/r-1/q}\Big(\sum_{k=2^s+1}^{2^{s+1}}|\Delta a_k|^p\Big)^{1/p} = \sum_{s=1}^{\infty}2^s\Delta_s^{(p)}.$$

Applying Lemma 1.2, Fomin proved that for the class F_p, $1 < p \le 2$, we have positive answers to both questions (i) and (ii).

Theorem 1.9 (Fomin [13]). *Let $\{a_n\} \in F_p$ for some $1 < p \le 2$. Then the cosine series* (C) *is the Fourer series of its sum f and* (1.1) *holds.*

Next we shall prove that S_p is a subclass of F_p for all $p>1$.

THEOREM 1.10 ([61]). *For every $p>1$ we have the embedding $S_p \subset F_p$.*

Proof. Applying the Abel transformation we have

$$\begin{aligned}
\sum_{k=2^{s-1}+1}^{2^s} |\Delta a_k|^p &= \sum_{k=2^{s-1}+1}^{2^s} A_k^p \frac{|\Delta a_k|^p}{A_k^p} \\
&= \sum_{k=2^{s-1}+1}^{2^s-1} \Delta(A_k^p) \sum_{j=1}^{k} \frac{|\Delta a_j|^p}{A_j^p} + A_{2^s}^p \sum_{j=1}^{2^s} \frac{|\Delta a_j|^p}{A_j^p} - A_{2^{s-1}+1}^p \sum_{j=1}^{2^{s-1}} \frac{|\Delta a_j|^p}{A_j^p} \\
&= \sum_{k=2^{s-1}+1}^{2^s-1} k\,\Delta(A_k^p)\Big(\frac{1}{k}\sum_{j=1}^{k} \frac{|\Delta a_j|^p}{A_j^p}\Big) + 2^s A_{2^s}^p\Big(\frac{1}{2^s}\sum_{j=1}^{2^s} \frac{|\Delta a_j|^p}{A_j^p}\Big) \\
&\quad - 2^{s-1} A_{2^{s-1}+1}^p\Big(\frac{1}{2^{s-1}}\sum_{j=1}^{2^{s-1}} \frac{|\Delta a_j|^p}{A_j^p}\Big) \\
&= O(1)\Big[\sum_{k=2^{s-1}+1}^{2^s-1} k\,\Delta(A_k^p) + 2^s A_{2^s}^p + 2^{s-1} A_{2^{s-1}+1}^p\Big] \\
&= O(1)\Big(\sum_{k=2^{s-1}+1}^{2^s} A_k^p + 2^{s-1}A_{2^{s-1}+1}^p - 2^s A_{2^s}^p + 2^s A_{2^s}^p + 2^{s-1}A_{2^{s-1}+1}^p\Big) \\
&= O(1)\Big(\sum_{k=2^{s-1}+1}^{2^s} A_k^p + 2^s A_{2^{s-1}+1}^p\Big) = O(2^{s-1}A_{2^{s-1}}^p).
\end{aligned}$$

First applying the Fomin lemma, and then the Cauchy type theorem, we obtain

$$\sum_{s=1}^{\infty} 2^s \Delta_s^{(p)} \le O(1)\sum_{s=1}^{\infty} 2^s \Big(\frac{1}{2^{s-1}}\,2^{s-1}A_{2^{s-1}}^p\Big)^{1/p} = O\Big(\sum_{s=1}^{\infty} 2^{s-1}A_{2^{s-1}}\Big) < \infty.$$

Recently, Leindler proved the important result that, conversely, the Fomin class F_p is a subclass of S_p; he also gave another proof of Theorem 1.10. Precisely he proved the following theorem.

THEOREM 1.11 (Leindler [24]). *For all $p>1$, the classes F_p and S_p are identical.*

A still larger class that answers both questions, but is expressed in terms of a condition difficult to apply, is the class $BV\cap C$, where C was defined by Garrett and Stanojević [17] as follows: a null sequence $\{a_n\}$ of real numbers satisfies the condition C if for every $\varepsilon>0$ there exists $\delta(\varepsilon)>0$, independent of n, such that

$$\int_0^{\delta}\Big|\sum_{k=n}^{\infty}\Delta a_k D_k(x)\Big|\,dx < \varepsilon \quad \text{for every } n.$$

Singh and Sharma [36] proved that the Garrett–Stanojević class C is stronger than the Moore class (M).

THEOREM 1.12 (Garrett–Stanojević [17]). *Let $\{a_n\}\in BV\cap C$. Then the series* (C) *is the Fourier series of its sum f and* (1.1) *holds.*

In [16] Garrett, Rees and Stanojević proved the following theorem.

THEOREM 1.13. *We have the embedding*

$$S \subset BV \cap C.$$

Now, we shall prove an extension of this theorem.

THEOREM 1.14 ([50]). *For all $p > 1$, we have the embedding*

$$S_p \subset BV \cap C.$$

For the proof we need the following lemma.

LEMMA 1.3 (Hausdorff–Young [73]). *Let $1 < p \le 2$ and let $\{c_n\} \in l^p$ be a sequence of complex numbers. Then $\{c_n\}$ is the sequence of Fourier coefficients of some $\varphi \in L^q$ $(1/p + 1/q = 1)$ and*

$$\left(\frac{1}{2\pi}\int_0^{2\pi} |\varphi(x)|^q\,dx\right)^{1/q} \le \Big(\sum_{n=-\infty}^{\infty} |c_n|^p\Big)^{1/p}.$$

Proof of Theorem 1.14. It suffices to show that

$$T_n = \int_0^{\pi}\Big|\sum_{k=n}^{\infty} \Delta a_k D_k(x)\Big|\,dx = o(1), \qquad n \to \infty.$$

For each n, let k_n be the least natural number such that $n \le 2^{k_n} - 1$. Then T_n can be majorized by

$$T_n \le \int_0^{\pi}\Big|\sum_{j=n}^{2^{k_n}-1} \Delta a_j D_j(x)\Big|\,dx + \sum_{l=k_n}^{\infty}\int_0^{\pi}\Big|\sum_{j=2^l}^{2^{l+1}-1} \Delta a_j D_j(x)\Big|\,dx = I_1 + I_2.$$

The second term is written as follows:

$$I_2 = \sum_{l=k_n}^{\infty}\Big\{\int_0^{1/2^{l+1}} + \int_{1/2^{l+1}}^{\pi}\Big\}\Big|\sum_{j=2^l}^{2^{l+1}-1} \Delta a_j D_j(x)\Big|\,dx = \Sigma_1 + \Sigma_2.$$

For the first term, the uniform estimate $|D_n(x)| \le n + 1/2$ can be applied, i.e.

$$\begin{aligned}\Sigma_1 &\le \sum_{l=k_n}^{\infty}\frac{1}{2^{l+1}}\sum_{j=2^l}^{2^{l+1}-1} |\Delta a_j|(j+1/2) \le \sum_{l=k_n}^{\infty}\frac{1}{2^{l+1}}\sum_{j=2^l}^{2^{l+1}-1} |\Delta a_j| 2^{l+1}\\ &= \sum_{l=k_n}^{\infty}\sum_{j=2^l}^{2^{l+1}-1} |\Delta a_j| = \sum_{j=2^{k_n}}^{\infty} |\Delta a_j|.\end{aligned}$$

By summation by parts, and by Hölder's inequality, we have

$$\sum_{i=2^{k_n}}^{\infty} |\Delta a_i| = \sum_{i=2^{k_n}}^{\infty}\frac{|\Delta a_i|}{A_i}A_i = \sum_{i=2^{k_n}}^{\infty}\Delta A_i\sum_{j=1}^{i}\frac{|\Delta a_j|}{A_j} - A_{2^{k_n}}\sum_{j=1}^{2^{k_n}-1}\frac{|\Delta a_j|}{A_j}$$

$$\leq \sum_{i=2^{k_n}}^{\infty} i(\Delta A_i)\biggl(\frac{1}{i}\sum_{j=1}^{i}\frac{|\Delta a_j|^p}{A_j^p}\biggr)^{1/p} + 2^{k_n}A_{2^{k_n}}\biggl(\frac{1}{2^{k_n}}\sum_{j=1}^{2^{k_n}}\frac{|\Delta a_j|^p}{A_j^p}\biggr)^{1/p}$$

$$= O(1)\Bigl[\sum_{i=2^{k_n}}^{\infty} i(\Delta A_i) + 2^{k_n}A_{2^{k_n}}\Bigr].$$

Since $\sum_{n=1}^{\infty} A_n < \infty$ and $A_n \downarrow 0$, both terms on the right-hand side of the above inequality are $o(1)$ as $n \to \infty$. Thus $\Sigma_1 = o(1)$, $n \to \infty$.

Let

$$\Sigma_2 = \sum_{l=k_n}^{\infty}\int_{1/2^{l+1}}^{\pi}\biggl|\sum_{j=2^l}^{2^{l+1}-1}\frac{\Delta a_j}{A_j}A_jD_j(x)\biggr|\,dx.$$

Applying the Abel transformation, we get

$$\int_{1/2^{l+1}}^{\pi}\biggl|\sum_{j=2^l}^{2^{l+1}-1}\frac{\Delta a_j}{A_j}A_jD_j(x)\biggr| \leq \sum_{j=2^l}^{2^{l+1}-2}\Delta A_j\int_{1/2^{l+1}}^{\pi}\biggl|\sum_{r=1}^{j}\frac{\Delta a_r}{A_r}D_r(x)\biggr|\,dx$$
$$+ A_{2^l}\int_{1/2^{l+1}}^{\pi}\biggl|\sum_{r=1}^{2^l-1}\frac{\Delta a_r}{A_r}D_r(x)\biggr|\,dx + A_{2^{l+1}-1}\int_{1/2^{l+1}}^{\pi}\biggl|\sum_{r=1}^{2^{l+1}-1}\frac{\Delta a_r}{A_r}D_r(x)\biggr|\,dx.$$

Applying the Hölder type inequality, we get

$$V_l = \int_{1/2^{l+1}}^{\pi}\biggl|\sum_{r=1}^{2^l-1}\frac{\Delta a_r}{A_r}D_r(x)\biggr|\,dx = \int_{1/2^{l+1}}^{\pi}\frac{1}{2\sin(x/2)}\biggl|\sum_{r=1}^{2^l-1}\frac{\Delta a_r}{A_r}\sin[(r+1/2)x]\biggr|\,dx$$
$$\leq \biggl[\int_{1/2^{l+1}}^{\pi}\frac{dx}{[2\sin(x/2)]^p}\biggr]^{1/p}\biggl[\int_{1/2^{l+1}}^{\pi}\biggl|\sum_{r=1}^{2^l-1}\frac{\Delta a_r}{A_r}\sin[(r+1/2)x]\biggr|^q dx\biggr]^{1/q},$$

where $1/p + 1/q = 1$. Since

$$\int_{1/2^{l+1}}^{\pi}\frac{dx}{[2\sin(x/2)]^p} \leq \frac{\pi^p}{2^p}\int_{1/2^{l+1}}^{\pi}\frac{dx}{x^p} \leq M_p(2^{l+1})^{p-1},$$

where M_p is an absolute constant depending on p, it follows that

$$V_l \leq (2^{l+1})^{1/q}(M_p)^{1/p}\biggl[\int_0^{\pi}\biggl|\sum_{r=1}^{2^l}\frac{\Delta a_r}{A_r}\sin[(r+1/2)x]\biggr|^q dx\biggr]^{1/q}.$$

Applying the Hausdorff–Young inequality to the last integral, we get

$$\biggl[\int_0^{\pi}\biggl|\sum_{r=1}^{2^l}\frac{\Delta a_r}{A_r}\sin[(r+1/2)x]\biggr|^q dx\biggr]^{1/q} \leq B_p\biggl(\sum_{r=1}^{2^l-1}\frac{|\Delta a_r|^p}{A_r^p}\biggr)^{1/p}.$$

Thus

$$V_l \leq 2^{l+1}C_p\biggl(\frac{1}{2^{l+1}}\sum_{r=1}^{2^{l+1}}\frac{|\Delta a_r|^p}{A_r^p}\biggr)^{1/p},\quad C_p > 0.$$

Then

$$\begin{aligned}\varSigma_2 \le& \sum_{l=k_n}^{\infty}\sum_{j=2^l}^{2^{l+1}-2} \varDelta A_j \int\limits_{1/2^{l+1}}^{\pi} \left|\sum_{r=1}^{j} \frac{\varDelta a_r}{A_r}\, D_r(x)\right| dx \\ &+ \sum_{l=k_n}^{\infty} A_{2^l} \int\limits_{1/2^{l+1}}^{\pi} \left|\sum_{r=1}^{2^l-1} \frac{\varDelta a_r}{A_r}\, D_r(x)\right| dx \\ &+ \sum_{l=k_n}^{\infty} A_{2^{l+1}-1} \int\limits_{1/2^{l+1}}^{\pi} \left|\sum_{r=1}^{2^{l+1}-1} \frac{\varDelta a_r}{A_r}\, D_r(x)\right| dx \\ =& \, O_p(1)\Big[\sum_{l=k_n}^{\infty}\sum_{j=2^l}^{2^{l+1}-2} j\,\varDelta A_j + 4\sum_{l=k_n}^{\infty} 2^l A_{2^l}\Big].\end{aligned}$$

Now, applying the Cauchy condensation test, we get

$$\sum_{l=k_n}^{\infty} 2^l A_{2^l} = o(1), \qquad n\to\infty.$$

But

$$\sum_{j=2^l}^{2^{l+1}-2} j\,\varDelta A_j = \sum_{j=2^l+1}^{2^{l+1}-1} A_j - 2^{l+1}A_{2^{l+1}-1} + 2^l A_{2^l} + A_{2^{l+1}-1} \le 2^l A_{2^l} + 2^l A_{2^l} + A_{2^l}.$$

Thus

$$\sum_{l=k_n}^{\infty}\sum_{j=2^l}^{2^{l+1}-2} j\varDelta A_j \le 2\sum_{l=k_n}^{\infty} 2^l A_{2^l} + \sum_{l=k_n}^{\infty} A_{2^l} = o(1), \quad n\to\infty,$$

i.e. $\varSigma_2 = o(1)$, $n\to\infty$. Finally, $I_2 = o(1)$, $n\to\infty$.

The same method applied to I_1 yields the estimate

$$I_1 \le O(1)\sum_{l=2^{k_n-1}}^{2^{k_n}-1} |\varDelta a_j| + O_p(1)\Big(\sum_{j=2^{k_n-1}}^{2^{k_n}-2} j\,\varDelta A_j + 4(2^{k_n-1}A_{2^{k_n-1}})\Big).$$

Letting $n\to\infty$ completes the proof of the theorem.

Remark 1.3. Theorem 1.8 is a corollary of Theorems 1.14 and 1.12. Thus by proving Theorem 1.14 we obtained a new proof of Theorem 1.8.

On the other hand, Stanojević [37] proved the following inclusion connecting the classes F_p, C and BV.

Theorem 1.15. *For all $1 < p \le 2$ we have the embedding*

$$F_p \subset BV \cap C.$$

In [16] Garrett, Rees and Stanojević defined an extension of the class of null sequences of bounded variation. Namely, a null sequence $\{a_k\}$ belongs to the class $(BV)^{(m)}$, $m\ge 1$, if $\sum_{k=1}^{\infty} |\varDelta^m a_k| < \infty$, where $\varDelta^m a_k = \varDelta(\varDelta^{m-1}a_k) = \varDelta^{m-1}a_k - \varDelta^{m-1}a_{k+1}$. For $m=1$, $(BV)^1 = BV$.

Theorem 1.16 (Garrett–Rees–Stanojević [16]). *Let* $\{a_n\} \in (BV)^{(m)}$ *for some* $m \geq 1$ *and* $a_n \log n = o(1)$, $n \to \infty$. *Then* $\|S_n - f\| = o(1)$, $n \to \infty$ *iff* $\{a_n\} \in C$.

Both Fomin [12] and Stanojević [37] considered the following natural extension of the class F_p. Let $p \geq 1$. A sequence $\{a_k\}$ belongs to C_p if $a_k \to 0$ as $k \to \infty$ and

$$n^{p-1} \sum_{k=n}^{\infty} |\Delta a_k|^p = o(1) \quad \text{as } n \to \infty. \tag{1.3}$$

Answering question (ii) Fomin and Stanojević proved the following result:

Theorem 1.17 (Fomin [12], Stanojević [37]). *If* (C) *is a Fourier series of* $f \in L^1$ *and* $\{a_n\} \in C_p \cap BV$ *for some* $1 < p \leq 2$ *then* (1.1) *holds.*

Later, Fomin extended the above result by considering a still larger class:

Theorem 1.18 (Fomin [14]). *If* (C) *is the Fourier series of* $f \in L^1$ *and for each sequence* $\{m_n\}$ *of natural numbers such that* $m_n/n \to 0$ *as* $n \to \infty$ *there exists* p, $1 < p \leq 2$, *independent of* $\{m_n\}$ *such that*

$$m_n^{p-1} \sum_{k=n}^{n+m_n} |\Delta a_k|^p = o(1), \quad n \to \infty, \tag{1.4}$$

then (1.1) *holds.*

The same statement holds for the sine series (S), i.e. the Fourier series of odd functions.

Remark 1.4. It is trivial to see that Theorem 1.17 is a corollary of Theorem 1.18, that is, that (1.3) implies (1.4) for each sequence $\{m_n\}$ of natural numbers such that $m_n/n \to 0$, $n \to \infty$.

The class C_p has an interesting subclass C_p^*. A null sequence $\{a_k\}$ belongs to C_p^*, $1 < p \leq 2$, if

$$\sum_{k=1}^{\infty} k^{p-1} |\Delta a_k|^p < \infty.$$

The next theorem is a corollary to Theorem 1.17.

Theorem 1.19 (Fomin [12], Stanojević [37]). *Let* (C) *be the Fourier series of some* $f \in L^1(0,\pi)$, *let* $1 < p \leq 2$ *and let* $\{a_n\} \in C_p^* \cap BV$. *Then* (1.1) *holds.*

A natural extension of BV is the following class: a null sequence $\{a_k\}$ belongs to the class P if

$$\frac{1}{n} \sum_{k=1}^{n} k|\Delta a_k| = o(1), \quad n \to \infty.$$

Combining the class P with the condition $n\,\Delta a_n = O(1)$, Stanojević obtained a theorem on L^1-convergence of Fourier–Stieltjes series.

Theorem 1.20 (Stanojević [37]). *Let* (C) *be a Fourier–Stieltjes seris with* $\{a_k\} \in P$ *and suppose that* $n\Delta a_n = O(1)$. *Then* (C) *converges in* L^1 *iff* $a_n \log n = o(1)$, $n \to \infty$.

Bojanić and Stanojević [5] defined a subclass of P as follows: a null sequence $\{a_k\}$ belongs to the class V_p, $p > 1$, if

$$\frac{1}{n}\sum_{k=1}^{n} k^p |\Delta a_k|^p = o(1), \quad n \to \infty.$$

They proved the following theorems.

Theorem 1.21 (Bojanić–Stanojević [5]). *If* (C) *is the Fourier series of* $f \in L^1$ *and* $\{a_k\} \in V_p$ *for some* $1 < p \le 2$ *then* (1.1) *holds.*

Theorem 1.22 (Bojanić–Stanojević [5]). *If* $\{a_k\} \in V_p \cap BV$ *for some* $1 < p \le 2$, *then* (C) *is the Fourier series iff* $\{a_k\} \in C$.

Tanović-Miller considered the problem of integrability of the series (C) with regard to the classes C_p, $p > 1$, and $C_1 = BV$.

Theorem 1.23 (Tanović-Miller [40]). (i) *If* $\{a_k\} \in \bigcup\{C_p : p \ge 1\}$ *then* (C) *converges a.e. to the function*

$$f(x) = \sum_{k=0}^{\infty} \Delta a_k D_k(x);$$

moreover, *in that case* (C) *is a Fourier series iff for some* $\delta > 0$,

$$\int_0^{\delta} \left| \sum_{k=0}^{\infty} \Delta a_k D_k(x) \right| dx < \infty,$$

in which case (C) *is the Fourier series of* f.

(ii) *If* $\{a_k\} \in \bigcup\{C_p : p > 1\}$ *then* (C) *is a Fourier series iff* $\{a_k\} \in C$.

These results extend Theorem 1.22 and show that the classical question of integrability of the series (C) need not be restricted to series with coefficients of bounded variation.

Garrett and Stanojević obtained a theorem on L^1-convergence of Fourier series with monotone coefficients.

Theorem 1.24 (Garrett–Stanojević [17]). *Let*

$$\text{(CS)} \qquad \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

be a Fourier series with monotone coefficients. Then (1.1) *holds*, *where* S_n *is the partial sums of this series.*

Telyakovskiĭ and Fomin obtained a similar result for Fourier series with quasi-monotone coefficients. A null sequence $\{a_n\}$ of positive numbers is called *quasi-monotone* if for some $\alpha \ge 0$, $a_n/n^\alpha \downarrow 0$, $n \to \infty$ or equivalently $a_{n+1} \le a_n(1 + \alpha/n)$.

Theorem 1.25 (Fomin–Telyakovskiĭ [48]). *Let* $\{a_n\}$ *be a quasi-monotone sequence. If* (C) *is the Fourier series of its sum* f, *then* (1.1) *holds.*

The proof of sufficiency of the theorem of Fomin–Telyakovskiĭ was simplified by Garrett–Rees–Stanojević [15] using a more refined estimate of $\|S_n - \sigma_n\|$. Telyakovskiĭ and Fomin [48] also proved a corresponding result for the sine series, namely if $\{a_k\}$

is a quasi-monotone sequence and (S) is the Fourier series of its sum g then the same conclusion holds for the sine series.

Theorem 1.26 (Garrett–Rees–Stanojević [15]). *Let* (CS) *be a Fourier series with quasi-monotone coefficients. Then* $\|S_n - \sigma_n\| = o(1)$, $n \to \infty$ *iff* $(a_n + b_n)\log n = o(1)$, $n \to \infty$.

The class P extends not only BV, but also the class of quasi-monotone sequences. The next theorem is a slightly weaker form of a theorem of Telyakovskiĭ and Fomin.

Theorem 1.27 (Stanojević [37]). *Let* (C) *be a Fourier series with quasi-monotone coefficients and suppose that* $n\Delta a_n = O(1)$. *Then* (1.1) *holds.*

Later, Bray and Stanojević [9] considered the question of L^1-convergence for more general Fourier series of so called asymptotically even functions. Concerning the Fourier series of even functions one of the results in [9] can be stated as follows:

Theorem 1.28 (Bray–Stanojević). *If* (C) *is the Fourier series of* $f \in L^1$ *and for some* $1 < p \le 2$,

$$\lim_{\lambda\to 1^+} \limsup_{n\to\infty} \sum_{k=n}^{[\lambda n]} k^{p-1}|\Delta a_k|^p = 0,$$

then (1.1) *holds.*

Remark 1.5. Theorem 1.28 is corollary of Theorem 1.18.

1.2. Generalizations of the Sidon–Fomin lemma. Sidon [34] proved the inequality named after him in 1939. It is an upper estimate for the integral norm of a linear combination of trigonometric Dirichlet kernels expressed in terms of the coefficients. Since the estimate has many applications, for instance in L^1-convergence problems and summation methods for trigonometric series, newer and newer improvements of the original inequality has been proved by several authors. Fomin [10] gave another proof of this inequality by applying the linear method for summing Fourier series. Therefore the inequality is known as the Sidon–Fomin inequality.

Also, Telyakovskiĭ [45] gave an elegant proof of the Sidon–Fomin inequality.

Lemma 1.4 (Sidon–Fomin). *Let* $\{\alpha_k\}_{k=0}^n$ *be a sequence of real numbers such that* $|\alpha_k| \le 1$ *for all* k. *Then there exists a positive constant* C *such that for any* $n \ge 0$,

$$\Big\|\sum_{k=0}^{n} \alpha_k D_k(x)\Big\| \le C(n+1).$$

For the proof of our new result we need the following lemma.

Lemma 1.5. *If* $T_n(x)$ *is a trigonometric polynomial of order* n, *then*

$$\|T_n^{(r)}\| \le n^r \|T_n\|.$$

This is S. Bernstein's inequality in the $L^1(0,\pi)$-metric (see [73, Vol. 2, p. 11]).

Lemma 1.6 ([52]). *Let* $\{\alpha_k\}_{k=0}^n$ *be a sequence of real numbers such that* $|\alpha_k| \le 1$ *for all* k. *Then there exists a constant* $C > 0$ *such that for any* $n \ge 0$,

$$\Big\| \sum_{k=0}^{n} \alpha_k D_k^{(r)}(x) \Big\| \le C(n+1)^{r+1},$$

where $D_k^{(r)}(x)$, $k = 0, 1, \dots, n$, *is the* r*th derivative of the Dirichlet kernel.*

Proof. Since

$$\sum_{k=0}^{n} \alpha_k D_k(x) = \frac{1}{2} \sum_{i=0}^{n} \alpha_i + \sum_{k=1}^{n} \Big(\sum_{i=k}^{n} \alpha_i \Big) \cos kx,$$

we see that $\sum_{k=0}^{n} \alpha_k D_k(x)$ is a cosine trigonometric polynomial of order n. Applying first the Bernstein inequality, and then the Sidon–Fomin lemma yields

$$\Big\| \sum_{k=0}^{n} \alpha_k D_k^{(r)}(x) \Big\| \le (n+1)^r \Big\| \sum_{k=0}^{n} \alpha_k D_k(x) \Big\| \le C(n+1)^{r+1}, \quad C > 0.$$

Lemma 1.7 (Fomin–Stečkin [11]). *Let* $1 < p \le 2$ *and* $\{\alpha_k\}_{k=0}^n$ *be a sequence of real numbers such that* $\sum_{k=0}^{n} \alpha_k^p \le A^p(n+1)$. *Then there exists a positive constant* C_p *depending only on* p *such that*

$$\Big\| \sum_{i=0}^{n} \alpha_i D_i(x) \Big\| \le C_p A(n+1).$$

Lemma 1.8 (Bojanić–Stanojević [5]). *Let* $\{\alpha_k\}_{k=0}^n$ *be a sequence of real numbers. Then for any* $1 < p \le 2$ *and* $n \ge 0$,

$$\Big\| \sum_{k=0}^{n} \alpha_k D_k(x) \Big\| \le C_p(n+1) \Big(\frac{1}{n+1} \sum_{k=0}^{n} |\alpha_k|^p \Big)^{1/p}, \tag{1.5}$$

where the constant C_p *depends only on* p.

Remark 1.6. We note that this estimate is essentially contained (case $p = 2$) in Fomin [10].

Remark 1.7. It is easy to see that a Bojanić–Stanojević type inequality is not valid for $p = 1$. Indeed, if $\alpha_n = 1$ and $\alpha_k = 0$ ($k \ne n$, $k \in \mathbb{N}$) then the left side of (1.5) is of order $(\log n)/n$ while the right side is of order $1/n$ as $n \to \infty$.

Remark 1.8. The Sidon–Fomin inequality is a special case of the Bojanić–Stanojević inequality, i.e. it can easily be deduced from Lemma 1.8.

Now, we will prove a counterpart of inequality (1.5) for $D_k^{(r)}$ in place of $D_k(x)$.

Lemma 1.9 ([58]). *Let* $\{\alpha_k\}_{k=0}^n$ *be a sequence of real numbers. Then for any* $1 < p \le 2$, $r \in \mathbb{N} \cup \{0\}$ *and* $n \ge 0$,

$$\Big\| \sum_{k=0}^{n} \alpha_k D_k^{(r)}(x) \Big\| \le C_p(n+1)^{r+1} \Big(\frac{1}{n+1} \sum_{k=0}^{n} |\alpha_k|^p \Big)^{1/p},$$

where the constant C_p *depends only on* p.

Proof. Applying first the Bernstein inequality, and then the Bojanić–Stanojević inequality yields

$$\Big\|\sum_{k=0}^{n}\alpha_k D_k^{(r)}(x)\Big\| \le (n+1)^r\Big\|\sum_{k=0}^{n}\alpha_k D_k(x)\Big\| \le C_p(n+1)^{r+1}\bigg(\frac{1}{n+1}\sum_{k=0}^{n}|\alpha_k|^p\bigg)^{1/p}.$$

1.3. Extensions of some classes of Fourier coefficients. In this section we shall give the extensions of the Garrett–Stanojević class C, Sidon–Telyakovskiĭ class S and the class S_p, $p>1$, defined by V. B. Stanojević and Č. V. Stanojević.

A null sequence $\{a_k\}$ belongs to the class C_r, $r\in\mathbb{N}\cup\{0\}$, if for every $\varepsilon>0$ there is a $\delta>0$ such that

$$\int_0^\delta\Big|\sum_{k=n}^{\infty}\Delta a_k D_k^{(r)}(x)\Big|\,dx<\varepsilon\quad\text{for all } n,$$

where $D_k^{(r)}(x)$ is the rth derivative of the Dirichlet kernel. When $r=0$, we set $C_r=C$.

A null sequence $\{a_k\}$ belongs to the class $\Im_r$, $r\in\mathbb{N}\cup\{0\}$, if there exists a decreasing sequence $\{A_k\}$ such that $\sum_{k=0}^{\infty}k^rA_k<\infty$ and $|\Delta a_k|\le A_k$ for all k. When $r=0$ it is clear that $\Im_r=S$.

A null sequence $\{a_k\}$ belongs to the class S_{pr}, $1<p\le 2$, $r\in\mathbb{N}\cup\{0\}$, if there exists a decreasing sequence $\{A_k\}$ such that $\sum_{k=1}^{\infty}k^rA_k<\infty$ and

$$\frac{1}{n}\sum_{k=1}^{n}\frac{|\Delta a_k|^p}{A_k^p}=O(1).$$

When $r=0$, we define $S_p=S_{pr}$. The following lemma was proved by Ch. J. de la Vallée Poussin (see [69]), but we shall present two other proofs.

Lemma 1.10. *If $A_n\downarrow 0$ with $\sum_{n=1}^{\infty}n^rA_n<\infty$ for some $r\ge 0$ then $n^{r+1}A_n=o(1)$, $n\to\infty$.*

Proof 1. Let $0<m<n$. Adding the inequalities

$$\begin{aligned} n^{r+1}\Delta A_{n-1}&\ge 0,\\ (n-1)^{r+1}\Delta A_{n-2}&\ge 0,\\ (n-2)^{r+1}\Delta A_{n-3}&\ge 0,\\ \dots\dots\dots&\dots\\ (m+1)^{r+1}\Delta A_m&\ge 0, \end{aligned}$$

we obtain

$$-A_nn^{r+1}+\sum_{k=m+1}^{n-1}A_k[(k+1)^{r+1}-k^{r+1}]+A_m(m+1)^{r+1}\ge 0.$$

The sum on the left is $o(1)$ because $\sum_{n=1}^{\infty}n^rA_n<\infty$. Hence,

$$A_m(m+1)^{r+1}-A_nn^{r+1}\ge o(1),\quad m,n\to\infty.$$

Since $m^rA_m\to 0$, this means that

$$A_mm^{r+1}-A_nn^{r+1}\ge o(1),\quad m,n\to\infty. \tag{1.6}$$

We cannot have $\liminf_{n\to\infty} n^{r+1}A_n > 0$, since otherwise $\sum_{n=1}^{\infty} n^r A_n$ could not converge. Hence, in particular, for each $\varepsilon > 0$ there is an infinite sequence of indices m for which

$$m^{r+1}A_m < \varepsilon. \tag{1.7}$$

Now suppose that $\limsup_{n\to\infty} n^{r+1}A_n > 0$. Then there exists $\varepsilon > 0$ and an infinite sequence of indices n such that

$$n^{r+1}A_n > 2\varepsilon > 0. \tag{1.8}$$

For each m satisfying (1.7) take a larger n satisfying (1.8); then we get a contradiction to (1.6). Hence $\limsup_{n\to\infty} n^{r+1}A_n = 0$, i.e. $n^{r+1}A_n = o(1)$, $n \to \infty$.

Proof 2. By the inequalities

$$n^{r+1}A_{2n} \le n^r(A_{n+1} + A_{n+2} + \ldots + A_{2n}) \le \sum_{i=n+1}^{\infty} i^r A_i,$$

we obtain

$$(2n)^{r+1}A_{2n} \le 2^{r+1} \sum_{i=n+1}^{\infty} i^r A_i = o(1), \qquad n \to \infty.$$

Similarly, we can get

$$(2n+1)^{r+1}A_{2n+1} \le \left(2 + \frac{1}{n}\right)^{r+1} \sum_{i=n+1}^{\infty} i^r A_i = o(1), \quad n \to \infty.$$

Finally $n^{r+1}A_n = o(1)$, $n \to \infty$.

Lemma 1.11. *If $A_n \downarrow 0$ with $\sum_{n=1}^{\infty} n^r A_n < \infty$ for some $r \ge 0$, then $\sum_{n=1}^{\infty} n^{r+1}(\Delta A_n) < \infty$.*

Proof. By partial summation,

$$\sum_{k=1}^{n-1} k^{r+1}(\Delta A_k) = \sum_{k=1}^{n}[k^{r+1} - (k-1)^{r+1}]A_k - n^{r+1}A_n = O\Big(\sum_{k=1}^{n} k^r A_k\Big) - n^{r+1}A_n.$$

The series on the right converges; $n^{r+1}A_n = o(1)$, $n \to \infty$, by Lemma 1.10; so the partial sums on the left converge as $n \to \infty$.

It is trivial to see that $\Im_{r+1} \subset \Im_r$ for all $r = 1, 2, 3, \ldots$ Now, let $\{a_n\}_{n=1}^{\infty} \in \Im_1$. For any real number a_0, we shall prove that the sequence $\{a_n\}_{n=0}^{\infty}$ belongs to S. We define $A_0 = \max(|\Delta a_0|, A_1)$. Then $|\Delta a_0| \le A_0$, i.e. $|\Delta a_n| \le A_n$ for all $n \in \{0, 1, 2, \ldots\}$ and $\{A_n\}_{n=0}^{\infty}$ is decreasing sequence. On the other hand,

$$\sum_{n=0}^{\infty} A_n \le A_0 + \sum_{n=1}^{\infty} nA_n < \infty.$$

Thus, $\{a_n\}_{n=0}^{\infty} \in S$, i.e. $\Im_{r+1} \subset \Im_r$ for all $r \in \mathbb{N} \cup \{0\}$. The next example shows that the implication

$$\{a_n\} \in \Im_{r+1} \;\Rightarrow\; \{a_n\} \in \Im_r, \qquad r \in \mathbb{N} \cup \{0\},$$

is not reversible.

Example ([55]). For $n \in \mathbb{N} \cup \{0\}$ define $a_n = \sum_{k=n+1}^{\infty} 1/k^2$. Then $a_n \to 0$ as $n \to \infty$ and $\Delta a_n = 1/(n+1)^2$ for $n \in \mathbb{N} \cup \{0\}$. First we shall show that $\{a_n\} \notin \Im_1$. Let $\{A_n\}_{n=1}^{\infty}$

be an arbitrary positive sequence such that $A \downarrow 0$ and $\Delta a_n \le A_n$. Then $\sum_{n=1}^{\infty} nA_n \ge \sum_{n=1}^{\infty} n/(n+1)^2$ is divergent, i.e. $\{a_n\} \notin \Im_1$.

Now, for all $n \in \mathbb{N} \cup \{0\}$ let $A_n = 1/(n+1)^2$. Then $A_n \downarrow 0$, $|\Delta a_n| \le A_n$ and $\sum_{n=0}^{\infty} A_n = \sum_{n=1}^{\infty} 1/n^2 < \infty$, i.e. $\{a_n\} \in S$.

Our next example will show that there exists a sequence $\{a_n\}_{n=1}^{\infty}$ such that $\{a_n\}_{n=1}^{\infty} \in \Im_r$ but $\{a_n\}_{n=1}^{\infty} \notin \Im_{r+1}$, for all $r \in \mathbb{N}$. Namely, for all $n \in \mathbb{N}$ let $a_n = \sum_{k=n}^{\infty} 1/k^{r+2}$. Then $a_n \to 0$ as $n \to \infty$, and $\Delta a_n = 1/n^{r+2}$ for $n \in \mathbb{N}$. Let $\{A_n\}_{n=1}^{\infty}$ be an arbitrary positive sequence such that $A_n \downarrow 0$ and $\Delta a_n \le A_n$. Then

$$\sum_{n=1}^{\infty} n^{r+1} A_n \ge \sum_{n=1}^{\infty} n^{r+1} \frac{1}{n^{r+2}} = \sum_{n=1}^{\infty} \frac{1}{n}$$

is divergent, i.e. $\{a_n\} \notin \Im_{r+1}$. On the other hand, for all $n \in \mathbb{N}$ let $A_n = 1/n^{r+2}$. Then $A_n \downarrow 0$, $|\Delta a_n| \le A_n$ and $\sum_{n=1}^{\infty} n^r A_n = \sum_{n=1}^{\infty} 1/n^2 < \infty$, i.e. $\{a_n\} \in \Im_r$.

THEOREM 1.29 ([52]). *For all $r \in \mathbb{N} \cup \{0\}$ we have the embedding*

$$\Im_r \subset BV \cap C_r.$$

Proof. It is clear that $\{a_n\} \in \Im_r$ implies $\{a_n\} \in BV$. Now for $x \ne 0$ we consider the identity

$$\sum_{k=n}^{\infty} \Delta a_k D_k(x) = \sum_{k=n}^{\infty} (\Delta A_k) \sum_{j=0}^{k} \frac{\Delta a_j}{A_j} D_j(x) - A_n \sum_{j=0}^{n-1} \frac{\Delta a_j}{A_j} D_j(x).$$

Later (see proof of Theorem 3.8) we shall prove that the series $\sum_{k=1}^{\infty} \Delta a_k D_k^{(r)}(x)$ is uniformly convergent on any compact subset of $(0, \pi)$. This implies that

$$\int_0^{\delta} \Big| \sum_{k=n}^{\infty} \Delta a_k D_k^{(r)}(x) \Big| \, dx \le \sum_{k=n}^{\infty} (\Delta A_k) \int_0^{\pi} \Big| \sum_{j=0}^{k} \frac{\Delta a_j}{A_j} D_j^{(r)}(x) \Big| \, dx + A_n \int_0^{\pi} \Big| \sum_{j=0}^{n-1} \frac{\Delta a_j}{A_j} D_j^{(r)}(x) \Big| \, dx.$$

Since $|(\Delta a_j)/A_j| \le 1$, applying Lemmas 1.6 and 1.10, we get

$$\begin{aligned} \int_0^{\delta} \Big| \sum_{k=n}^{\infty} \Delta a_k D_k^{(r)}(x) \Big| \, dx &\le O(1) \Big[\lim_{N \to \infty} \sum_{k=n}^{N-1} (\Delta A_k)(k+1)^{r+1} + A_n n^{r+1} \Big] \\ &= O(1) \lim_{N \to \infty} \Big[\sum_{k=n}^{N} [(k+1)^{r+1} - k^{r+1}] A_k - (N+1)^{r+1} A_N \Big] + O(n^{r+1} A_n) \\ &= O\Big(\sum_{k=n}^{\infty} k^r A_k \Big) + o(1) = o(1), \qquad n \to \infty. \end{aligned}$$

Next for $r \in \mathbb{N} \cup \{0\}$ we define a new class $\Im_r^2$ as follows: a null sequence $\{a_k\}$ belongs to $\Im_r^2$ if there exists a decreasing null sequence $\{A_k\}$ of nonnegative numbers such that $\sum_{k=1}^{\infty} k^{r+1} (\Delta A_k) < \infty$ and $|\Delta a_k| \le A_k$ for all k.

THEOREM 1.30. *The class $\Im_r$ is equivalent to $\Im_r^2$ for all $r \in \mathbb{N} \cup \{0\}$.*

Proof. Let $\{a_n\} \in \Im_r$. Applying Lemma 1.11, we get $\sum_{n=1}^{\infty} n^{r+1} (\Delta A_n) < \infty$.

Now, if $\{a_n\} \in \Im_r^2$, we have

$$n^{r+1} A_n = n^{r+1} \sum_{k=n}^{\infty} \Delta A_k \leq \sum_{k=n}^{\infty} k^{r+1} (\Delta A_k) = o(1), \qquad n \to \infty,$$

i.e. $n^{r+1} A_n = o(1)$, $n \to \infty$. Then

$$\sum_{k=1}^{n} k^r A_k = \sum_{k=1}^{n-1} (\Delta A_k) \sum_{j=1}^{k} j^r + A_n \sum_{j=1}^{n} j^r = O\Big(\sum_{k=1}^{n-1} k^{r+1} (\Delta A_k) \Big) + O(n^{r+1} A_n).$$

Letting $n \to \infty$, we obtain $\sum_{k=1}^{\infty} k^r A_k < \infty$, i.e. $\{a_n\} \in \Im_r$.

LEMMA 1.12 ([51]). *Let $\{\alpha_j\}_{j=1}^{k}$ be a sequence of real numbers. Then for $1 < p \leq 2$, $v \in \{0, 1, \ldots, r\}$ and $r \in \mathbb{N} \cup \{0\}$,*

$$\begin{aligned} V_k &= \int_{\pi/k}^{\pi} \left| \sum_{j=1}^{k} \alpha_j \frac{(j+1/2)^v \sin[(j+1/2)x + v\pi/2]}{(\sin(x/2))^{r+1-v}} \right| dx \\ &= O_p \left[k^{r+1} \left(\frac{1}{k} \sum_{j=1}^{k} |\alpha_j|^p \right)^{1/p} \right], \end{aligned}$$

where O_p depends only on p.

Proof. Applying first the Hölder inequality yields

$$\begin{aligned} V_k &= \int_{\pi/k}^{\pi} \frac{1}{(\sin(x/2))^{r+1-v}} \left| \sum_{j=1}^{k} \alpha_j (j+1/2)^v \sin[(j+1/2)x + v\pi/2] \right| dx \\ &\leq \left[\int_{\pi/k}^{\pi} \frac{dx}{(\sin(x/2))^{(r+1-v)p}} \right]^{1/p} \\ &\quad \times \left\{ \int_{0}^{\pi} \left| \sum_{j=1}^{k} \alpha_j (j+1/2)^v \sin[(j+1/2)x + v\pi/2] \right|^q dx \right\}^{1/q}. \end{aligned}$$

Since

$$\int_{\pi/k}^{\pi} \frac{dx}{(\sin(x/2))^{(r+1-v)p}} \leq \frac{\pi k^{(r+1-v)p-1}}{(r+1-v)p-1} \leq \frac{\pi}{p-1} k^{(r+1-v)p-1},$$

we have

$$\begin{aligned} V_k &\leq \left(\frac{\pi}{p-1} \right)^{1/p} (k^{(r+1-v)p-1})^{1/p} \\ &\quad \times \left\{ \int_{0}^{\pi} \left| \sum_{j=1}^{k} \alpha_j (j+1/2)^v \sin[(j+1/2)x + v\pi/2] \right|^q dx \right\}^{1/q}. \end{aligned}$$

Then using the Hausdorff–Young inequality we get

$$\left\{ \int_{0}^{\pi} \left| \sum_{j=1}^{k} \alpha_j (j+1/2)^v \sin[(j+1/2)x + v\pi/2] \right|^q dx \right\}^{1/q} = O_p \Big[\Big(\sum_{j=1}^{k} |\alpha_j|^p j^{vp} \Big)^{1/p} \Big].$$

Finally,

$$\begin{aligned}V_k &= O_p\Big[(k^{(r+1-v)p-1})^{1/p}\Big(\sum_{j=1}^{k}|\alpha_j|^p\, j^{vp}\Big)^{1/p}\Big]\\ &= O_p\Big[(k^{(r+1)p-1})^{1/p}\Big(\sum_{j=1}^{k}|\alpha_j|^p\Big)^{1/p}\Big] = O_p\Big[k^{r+1}\Big(\frac{1}{k}\sum_{j=1}^{k}|\alpha_j|^p\Big)^{1/p}\Big],\end{aligned}$$

where O_p depends only on p.

LEMMA 1.13 ([32]). *Let r be a nonnegative integer and $x \in (0,\pi]$. Then*

$$\begin{aligned}D_n^{(r)}(x) = \sum_{k=0}^{r-1}&\frac{(n+1/2)^k\sin[(n+1/2)x+k\pi/2]}{(\sin(x/2))^{r+1-k}}\,\varphi_k(x)\\ &+\frac{(n+1/2)^r\sin[(n+1/2)x+r\pi/2]}{2\sin(x/2)},\end{aligned}$$

where the φ_k are analytic functions of x, independent of n.

LEMMA 1.14. *Let $1<p\le 2$, $r\in\mathbb{N}\cup\{0\}$ and let the coefficients $\{a_j\}_{j=0}^k$ belong to the class S_{pr}. Then*

$$\int_0^{\pi}\Big|\sum_{j=1}^{k}\frac{\Delta a_j}{A_j}\,D_j^{(r)}(x)\Big|\,dx = O_p(k^{r+1}).$$

Proof. We have

$$\int_0^{\pi}\Big|\sum_{j=1}^{k}\frac{\Delta a_j}{A_j}\,D_j^{(r)}(x)\Big|\,dx = \int_0^{\pi/k}+\int_{\pi/k}^{\pi} = I_k+J_k.$$

Applying the inequality $D_n^{(r)}(x)=O(n^{r+1})$, we have

$$\begin{aligned}I_k &\le \alpha\sum_{j=1}^{k}j^r\frac{|\Delta a_j|}{A_j}\le \alpha k^r\sum_{j=1}^{k}\frac{|\Delta a_j|}{A_j}\\ &\le \alpha k^{r+1}\Big(\frac{1}{k}\sum_{j=1}^{k}\frac{|\Delta a_j|^p}{A_j^p}\Big)^{1/p} = O(k^{r+1}),\end{aligned}$$

where α is a positive constant.

Applying Lemma 1.13, we estimate the second integral:

$$\begin{aligned}J_k &= \int_{\pi/k}^{\pi}\Big|\sum_{j=1}^{k}\frac{\Delta a_j}{A_j}D_j^{(r)}(x)\Big|\,dx\\ &\le \int_{\pi/k}^{\pi}\Big|\sum_{j=1}^{k}\frac{\Delta a_j}{A_j}\Big(\sum_{v=0}^{r-1}\frac{(j+1/2)^v\sin[(j+1/2)x+v\pi/2]}{(\sin(x/2))^{r+1-v}}\,\varphi_v(x)\Big)\Big|\,dx\\ &\quad+\int_{\pi/k}^{\pi}\Big|\sum_{j=1}^{k}\frac{\Delta a_j}{A_j}\frac{(j+1/2)^r\sin[(j+1/2)x+r\pi/2]}{2\sin(x/2)}\Big|\,dx = \lambda_k+\mu_k.\end{aligned}$$

Since φ_v are bounded, we have

$$\int_{\pi/k}^{\pi} \left| \sum_{j=1}^{k} \frac{\Delta a_j}{A_j} \frac{(j+1/2)^v \sin[(j+1/2)x + v\pi/2]}{(\sin(x/2))^{r+1-v}} \varphi_v \right| dx \\ \le B \int_{\pi/k}^{\pi} \left| \sum_{j=1}^{k} \alpha_j \frac{(j+1/2)^v \sin[(j+1/2)x + v\pi/2]}{(\sin(x/2))^{r+1-v}} \right| dx,$$

where B is a positive constant and $\alpha_j = (\Delta a_j)/A_j$, $j = 1, \dots, k$. Applying Lemma 1.12 to the last integral, we get

$$\int_{\pi/k}^{\pi} \left| \sum_{j=1}^{k} \frac{\Delta a_j}{A_j} \frac{(j+1/2)^v \sin[(j+1/2)x + v\pi/2]}{(\sin(x/2))^{r+1-v}} \varphi_v(x) \right| dx \\ = O_p\biggl(k^{r+1} \biggl(\frac{1}{k} \sum_{j=1}^{k} \frac{|\Delta a_j|^p}{A_j^p} \biggr)^{1/p} \biggr) = O_p(k^{r+1}).$$

Since r is a finite value, we have $\lambda_k = O_p(k^{r+1})$. Similarly, $\mu_k = O_p(k^{r+1})$. Hence

$$\int_{0}^{\pi} \left| \sum_{j=1}^{k} \frac{\Delta a_j}{A_j} D_j^{(r)}(x) \right| dx = O(k^{r+1}) + O_p(k^{r+1}) = O_p(k^{r+1}).$$

LEMMA 1.15. *Let* $1 < p \le 2$, $r \in \mathbb{N} \cup \{0\}$ *and let the coefficients* $\{a_j\}_{j=0}^{k}$ *belong to the class* S_{pr}. *Then*

$$A_n \int_{0}^{\pi} \left| \sum_{j=1}^{k} \frac{\Delta a_j}{A_j} D_j^{(r)}(x) \right| dx = o(1), \qquad n \to \infty.$$

Proof. Applying first Lemma 1,14, and then Lemma 1.10, we obtain

$$A_n \int_{0}^{\pi} \left| \sum_{j=1}^{k} \frac{\Delta a_j}{A_j} D_j^{(r)}(x) \right| dx = O_p(n^{r+1} A_n) = o(1), \qquad n \to \infty.$$

THEOREM 1.31 ([51]). *For each* $1 < p \le 2$ *and* $r \in \mathbb{N} \cup \{0\}$ *we have the embedding*

$$S_{pr} \subset BV \cap C_r.$$

Proof. We have

$$\begin{aligned} \sum_{k=1}^{n} |\Delta a_k| &\le \sum_{k=1}^{n} k^r |\Delta a_k| = \sum_{k=1}^{n-1} (\Delta A_k) \sum_{j=1}^{k} \frac{|\Delta a_j|}{A_j} j^r + A_n \sum_{j=1}^{n} \frac{|\Delta a_j|}{A_j} j^r \\ &\le \sum_{k=1}^{n-1} k^r (\Delta A_k) \biggl(\sum_{j=1}^{k} \frac{|\Delta a_j|}{A_j} \biggr) + n^r A_n \sum_{j=1}^{n} \frac{|\Delta a_j|}{A_j} \\ &\le \sum_{k=1}^{n-1} k^{r+1} (\Delta A_k) \biggl(\frac{1}{k} \sum_{j=1}^{k} \frac{|\Delta a_j|^p}{A_j^p} \biggr)^{1/p} + n^{r+1} A_n \biggl(\frac{1}{n} \sum_{j=1}^{n} \frac{|\Delta a_j|^p}{A_j^p} \biggr)^{1/p} \\ &= O(1) \Bigl[\sum_{k=1}^{n-1} k^{r+1} (\Delta A_k) + n^{r+1} A_n \Bigr] = O\Bigl(\sum_{k=1}^{n} k^r A_k \Bigr). \end{aligned}$$

Letting $n \to \infty$, we get $\{a_n\} \in BV$.

Then applying the Abel transformation and Lemmas 1.15, 1.14 and 1.11 we obtain

$$\int_0^\delta \Big|\sum_{k=n}^\infty \Delta a_k D_k^{(r)}(x)\Big|\,dx \le \sum_{k=n}^\infty (\Delta A_k)\int_0^\pi \Big|\sum_{j=1}^k \frac{\Delta a_j}{A_j}\,D_j^{(r)}(x)\Big|\,dx + o(1)$$
$$= O_p(1)\Big[\sum_{k=n}^\infty k^{r+1}(\Delta A_k)\Big] = o(1), \qquad n\to\infty.$$

II. Classification of quasi-monotone sequences and its applications to L^1-convergence of trigonometric series

2.1. Remarks on trigonometric series with quasi-monotone coefficients. Quasi-monotone sequences are known to share many properties with decreasing sequences: for example the de la Vallée Poussin theorem [69]: $\sum_{n=1}^\infty a_n < \infty \Rightarrow na_n \to 0$ (see also [39]), the Cauchy condensation test for convergence, and a number of theorems about trigonometric series.

Some proofs of convergence theorems for trigonometric series are based on the use of modified cosine sums defined by Rees–Stanojević [31] as follows:

$$g_n(x) = \frac{1}{2}\sum_{k=0}^n \Delta a_k + \sum_{k=1}^n \Big[\Big(\sum_{i=k}^n \Delta a_i\Big)\cos kx\Big] = S_n(x) - a_{n+1}D_n(x).$$

Marzuq proved the following theorem on L^1-convergence of trigonometric series with quasi-monotone coefficients.

THEOREM 2.1 (Marzuq [27]). *Let $\{a_k\}$ be a nonnegative quasi-monotone sequence tending to zero, with*

$$\sum_{k=1}^\infty \frac{a_k}{k} < \infty, \qquad \sum_{k=1}^\infty (k+1)[|\Delta a_k| - \Delta a_k] < \infty.$$

Then $\lim_{n\to\infty} g_n(x) = g(x) \in L^1[-\pi,\pi]$ *iff* $\sum_{n=1}^\infty a_n < \infty$.

Singh and Sharma [35] defined a class of L^1-convergence as follows. Namely, a sequence $\{a_k\}$ belongs to the class S' if $a_k \to 0$ as $k\to\infty$ and there exists a sequence $\{A_k\}$ such that $\{A_k\}$ is quasi-monotone, $\sum_{k=1}^\infty A_k < \infty$, and $|\Delta a_k| \le A_k$ for all k. They proved the following theorem.

THEOREM 2.2 ([35]). *If $\{a_k\} \in S'$, then g_n converges to g in L^1.*

Let $A_n \Downarrow 0$ mean that $\{A_n\}$ is a quasi-monotone null sequence. For convenience the following notations are used for $\alpha \ge 0$, $p > 1$ and $r \in \{0,1,\dots,[\alpha]\}$:

$$M_\alpha = \Big\{A_n : A_n \downarrow 0 \text{ and } \sum_{n=1}^\infty n^\alpha A_n < \infty\Big\},$$

$$M'_\alpha = \Big\{A_n : A_n \Downarrow 0 \text{ and } \sum_{n=1}^\infty n^\alpha A_n < \infty\Big\},$$

$$S_{p\alpha r}=\Big\{a_n : a_n\to 0 \text{ as } n\to\infty,\ \text{and}$$
$$\frac{1}{n^{p(\alpha-r)+1}}\sum_{k=1}^{n}\frac{|\Delta a_k|^p}{A_k^p}=O(1) \text{ for some } A_n\in M_\alpha\Big\},$$
$$S'_{p\alpha r}=\Big\{a_n : a_n\to 0 \text{ as } n\to\infty,\ \text{and}$$
$$\frac{1}{n^{p(\alpha-r)+1}}\sum_{k=1}^{n}\frac{|\Delta a_k|^p}{A_k^p}=O(1) \text{ for some } A_n\in M'_\alpha\Big\}.$$

We note that the classes M'_α and $S'_{p\alpha r}$ were defined by Sheng [32].

THEOREM 2.3 ([62, 64]). *The classes M_α and M'_α are identical.*

Proof. It is obvious that $M_\alpha\subset M'_\alpha$. To prove $M'_\alpha\subset M_\alpha$, we use an idea of Telyakovskiĭ [47], i.e. we define the sequence

$$B_k=A_k+\beta\sum_{m=k}^{\infty}\frac{A_m}{m}\quad\text{for some } \beta\ge 0,\ \text{where } A_n\in M'_\alpha. \tag{2.1}$$

We have

$$B_k-B_{k+1}=\Delta B_k=\Delta A_k+\beta\,\frac{A_k}{k}\ge 0,$$

i.e. $B_k\downarrow 0$ as $k\to\infty$ and

$$\sum_{k=1}^{\infty}k^\alpha B_k=\sum_{k=1}^{\infty}k^\alpha A_k+\sum_{k=1}^{\infty}\beta k^\alpha\sum_{m=k}^{\infty}\frac{A_m}{m}\le\sum_{k=1}^{\infty}k^\alpha A_k+\beta\sum_{k=1}^{\infty}\sum_{m=k}^{\infty}m^{\alpha-1}A_m$$
$$=\sum_{k=1}^{\infty}k^\alpha A_k+\beta\sum_{m=1}^{\infty}\sum_{n=1}^{m}m^{\alpha-1}A_m=\sum_{k=1}^{\infty}k^\alpha A_k+\beta\sum_{m=1}^{\infty}m^\alpha A_m<\infty.$$

Thus $M_\alpha\equiv M'_\alpha$.

THEOREM 2.4 ([62, 64]). *The classes $S_{p\alpha r}$ and $S'_{p\alpha r}$ are identical.*

Proof. It is obvious that $S_{p\alpha r}\subset S'_{p\alpha r}$. Let $\{a_n\}\in S'_{p\alpha r}$. It suffices to show that the sequence (2.1) satisfies the condition

$$\frac{1}{n^{p(\alpha-r)+1}}\sum_{k=1}^{n}\frac{|\Delta a_k|^p}{B_k^p}=O(1).$$

Clearly,

$$\frac{1}{n^{p(\alpha-r)+1}}\sum_{k=1}^{n}\frac{|\Delta a_k|^p}{B_k^p}\le\frac{1}{n^{p(\alpha-r)+1}}\sum_{k=1}^{n}\frac{|\Delta a_k|^p}{A_k^p}=O(1).$$

Fomin [14], applying the following two theorems, gave a new proof of Theorem 1.25.

THEOREM 2.5 ([13]). *If* (C) *is mean convergent, then*

$$\lim_{n\to\infty}\sum_{k=1}^{m_n}\frac{a_{n+k}}{k}=0$$

for any sequence $\{m_n\}$ of natural numbers such that $m_n\le n$ for all $n\in\mathbb{N}$.

THEOREM 2.6 ([14]). *Let $\{m_n\}$ be a sequence of natural numbers such that* $\lim_{n\to\infty} m_n/n = 0$. *If*

$$\lim_{n\to\infty}\Big[\sum_{k=1}^{m_n-1}|\Delta a_{n+k}|\log(k+1)+|a_{n+m_n}|\log(m_n+1)\Big]=0,$$

then the series (C) *is mean convergent.*

2.2. Trigonometric series with δ-quasi-monotone coefficients. As an extension of quasi-monotone sequences, Boas [4] defined δ-quasi-monotone sequence as follows. A sequence $\{a_n\}$ is called *δ-quasi-monotone* if $a_n \to 0$, $a_n > 0$ ultimately and $\Delta a_n \geq -\delta_n$, where δ_n is sequence of positive numbers. A quasi-monotone sequence with $a_n \to 0$ is one that is δ-quasi-monotone with $\delta_n = \alpha a_n/n$.

Boas [4] proved the following lemmas about δ-quasi-monotone sequences.

LEMMA 2.1. *If $\{a_n\}$ is δ-quasi-monotone with $\sum_{n=1}^{\infty} n\delta_n < \infty$ then the convergence of $\sum_{n=1}^{\infty} a_n$ implies that $na_n = o(1)$, $n\to\infty$.*

REMARK 2.1. This lemma includes the corresponding result for classical quasi-monotone sequences; indeed, if $\{a_n\}$ is quasi-monotone we have $\sum_{n=1}^{\infty} n\delta_n = \sum_{n=1}^{\infty} n\alpha\frac{a_n}{n} = \alpha\sum_{n=1}^{\infty} a_n < \infty$ by hypothesis.

LEMMA 2.2. *Let $\{a_n\}$ be δ-quasi-monotone with $\sum_{n=1}^{\infty} n\delta_n < \infty$. If $\sum_{n=1}^{\infty} a_n < \infty$, then $\sum_{n=1}^{\infty}(n+1)|\Delta a_n| < \infty$.*

Ahmad and Zahid Ali Zaini proved the following theorem.

THEOREM 2.7 ([1]). *Let* (CS) *be a Fourier series with δ-quasi-monotone coefficients with $\sum_{n=1}^{\infty} n\delta_n < \infty$. Then $\|S_n - \sigma_n\| = o(1)$, $n\to\infty$ iff $(a_n+b_n)\log n = o(1)$, $n\to\infty$.*

Applying Theorems 2.5 and 2.6 to the series (C) we shall present a new proof of this theorem, rewritten as follows:

THEOREM 2.8. *Let* (C) *be a Fourier series with δ-quasi-monotone coefficients with $\sum_{n=1}^{\infty} n\delta_n < \infty$. Then* (C) *is mean convergent iff $a_n\log n = o(1)$, $n\to\infty$.*

Proof. Suppose $\|S_n - f\| = o(1)$, $n\to\infty$. We have

$$\frac{a_{2n-1}}{n} \geq \frac{a_{2n}}{n} - \frac{\delta_{2n-1}}{n},$$

$$\frac{a_{2n-2}}{n-1} \geq \frac{a_{2n-1}}{n-1} - \frac{\delta_{2n-2}}{n-1} \geq \frac{a_{2n}}{n-1} - \frac{\delta_{2n-1}}{n-1} - \frac{\delta_{2n-2}}{n-1},$$

$$\frac{a_{2n-3}}{n-2} \geq \frac{a_{2n-2}}{n-2} - \frac{\delta_{2n-3}}{n-2} \geq \frac{a_{2n}}{n-2} - \frac{\delta_{2n-1}}{n-2} - \frac{\delta_{2n-2}}{n-2} - \frac{\delta_{2n-3}}{n-2},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\frac{a_n}{1} \geq \frac{a_{n+1}}{1} - \frac{\delta_n}{1} \geq \frac{a_{2n}}{1} - \frac{\delta_{2n-1}}{1} - \frac{\delta_{2n-2}}{1} - \ldots - \frac{\delta_n}{1}.$$

Adding these inequalities, we obtain

$$\sum_{k=1}^{n}\frac{a_{n+k-1}}{k} \geq \Big(1+\frac{1}{2}+\ldots+\frac{1}{n}\Big)a_{2n} - \Big(1+\frac{1}{2}+\ldots+\frac{1}{n}\Big)\delta_{2n-1}$$

$$-\left(1+\frac{1}{2}+\ldots+\frac{1}{n-1}\right)\delta_{2n-2}-\ldots-\left(1+\frac{1}{2}\right)\delta_{n+1}-\frac{\delta_n}{1}$$
$$>\left(1+\frac{1}{2}+\ldots+\frac{1}{n}\right)a_{2n}-\left(1+\frac{1}{2}+\ldots+\frac{1}{n}\right)\sum_{k=n}^{2n-1}\delta_k.$$

From the inequalities $\log n < 1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n} \leq n$ for $n \in \mathbb{N}$, we obtain

$$a_n+\sum_{k=2}^{n}\frac{a_{n+k}}{k} > a_{2n}\log n-\sum_{k=n}^{2n-1}k\delta_k,$$

i.e.

$$a_{2n}\log n < a_n+\sum_{k=1}^{n}\frac{a_{n+k}}{k}+\sum_{k=n}^{\infty}k\delta_k.$$

Letting $n\to\infty$ and applying Theorem 2.5 we get $a_n\log n = o(1)$, $n\to\infty$.

Conversely, assume $a_n\log n = o(1)$, $n\to\infty$. Applying Theorem 2.6, it suffices to show that

$$A_n=\sum_{k=1}^{m_n-1}|\Delta a_{k+n}|\log(k+1)=o(1),\quad n\to\infty.$$

Indeed,

$$\begin{aligned}
A_n &\leq (\log m_n)\sum_{k=1}^{m_n-1}|\Delta a_{k+n}| \\
&\leq (\log m_n)\Big(\sum_{k=1}^{m_n-1}\Delta a_{k+n}+2\sum_{k=1}^{m_n-1}\delta_{k+n}\Big) \\
&= (\log m_n)(a_{n+1}-a_{n+m_n-1})+2(\log m_n)\sum_{k=1}^{m_n-1}\delta_{k+n} \\
&= O(a_{n+1}\log n)+O\Big(\sum_{i=n+1}^{\infty}i\delta_i\Big)=o(1),\quad n\to\infty.
\end{aligned}$$

This generalizes a theorem of Garrett, Rees and Stanojević [15], where quasi-monotonicity is assumed.

On the other hand, Mazhar [28] defined a class $S(\delta)$ as follows: A null sequence $\{a_n\}$ belongs to $S(\delta)$ if there exists a sequence $\{A_n\}$ such that $\{A_n\}$ is δ-quasi-monotone, $\sum_{n=1}^{\infty}n\delta_n<\infty$, $\sum_{n=1}^{\infty}A_n<\infty$ and $|\Delta a_n|\leq A_n$ for all n.

Later, Bor showed that the condition $\{a_n\}\in S(\delta)$ is sufficient for the integrability of the limit $g(x)=\lim_{n\to\infty}g_n(x)$.

Theorem 2.9 (Bor [7]). *Let* $\{a_n\}\in S(\delta)$. *Then*

$$\frac{1}{x}\sum_{k=1}^{\infty}\Delta a_k\sin[(k+1/2)x]=\frac{h(x)}{x}$$

converges for $x\in(0,\pi]$ *and* $h(x)/x\in L(0,\pi]$.

In [49] we defined a new class of positive sequences. Namely, we say that a null sequence $\{a_k\}$ belongs to the class $S_p(\delta)$, $p > 1$, if there exists a sequence $\{A_k\}$ of numbers such that

(a) $\{A_k\}$ is δ-quasi-monotone and $\displaystyle\sum_{k=1}^{\infty} k\delta_k < \infty$.

(b) $\displaystyle\sum_{k=1}^{\infty} A_k < \infty$,

(c) $\displaystyle\frac{1}{n+1}\sum_{k=0}^{n}\frac{|\Delta a_k|^p}{A_k^p} = O(1)$.

In view of the above definitions it is obvious that $S(\delta) \subset S_p(\delta)$. Applying the Hölder–Hausdorff–Young technique (as in the proof of Theorem 1.14) we can get the following lemma.

Lemma 2.3 ([56]). *Let $1 < p \le 2$ and $\{a_j\} \in S_p(\delta)$. Then*

$$\int_0^{\pi}\left|\sum_{j=0}^{k}\frac{\Delta a_j}{A_j}\,D_j(x)\right| dx = O_p(k+1), \qquad k \to \infty,$$

where O_p depends only on p.

Lemma 2.4 ([56]). *Let $1 < p \le 2$ and $\{a_j\} \in S_p(\delta)$. Then*

$$A_n\int_0^{\pi}\left|\sum_{j=0}^{n}\frac{\Delta a_j}{A_j}\,D_j(x)\right| dx = o(1), \quad n \to \infty.$$

Proof. Applying first Lemma 2.3, and then Lemma 2.1, yields

$$A_n\int_0^{\pi}\left|\sum_{j=0}^{n}\frac{\Delta a_j}{A_j}\,D_j(x)\right| dx = O_p((n+1)A_n) = o(1), \quad n \to \infty.$$

Theorem 2.10 ([56]). *Let $\{a_k\} \in S_p(\delta)$ for some $1 < p \le 2$. Then* (C) *is the Fourier series of some $f \in L^1(0,\pi)$ and $\|S_n - f\| = o(1)$, $n \to \infty$ if and only if $a_n \log n = o(1)$, $n \to \infty$.*

Proof. By summation by parts, and by Hölder's inequality, we have

$$\begin{aligned}\sum_{k=1}^{n}|\Delta a_k| &= \sum_{k=1}^{n} A_k\frac{|\Delta a_k|}{A_k}\\ &\le \sum_{k=1}^{n-1}(k+1)|\Delta A_k|\left(\frac{1}{k+1}\sum_{j=0}^{k}\frac{|\Delta a_j|^p}{A_j^p}\right)^{1/p} + (n+1)A_n\left(\frac{1}{n+1}\sum_{j=0}^{n}\frac{|\Delta a_j|^p}{A_j^p}\right)^{1/p}\\ &= O(1)\Big[\sum_{k=1}^{n-1}(k+1)|\Delta A_k| + (n+1)A_n\Big].\end{aligned}$$

Application of Lemmas 2.1 and 2.2 yields $\sum_{n=1}^{\infty}|\Delta a_n| < \infty$, i.e. $S_n(x)$ converges to $f(x)$

for $x \neq 0$. Using the Abel transformation, we obtain

$$f(x) = \sum_{k=0}^{\infty} \Delta a_k D_k(x),$$

since $\lim_{n\to\infty} a_n D_n(x) = 0$ if $x \neq 0$. Then

$$\|S_n - f\| = \|g_n - f + a_{n+1} D_n\|,$$

where g_n is the Rees–Stanojević sum. Using the Abel transformation, we have

$$g_n(x) = S_n(x) - a_{n+1} D_n(x) = \sum_{k=0}^{n} \Delta a_k D_k(x).$$

Since $\sum_{n=1}^{\infty} |\Delta a_n| < \infty$, the series $\sum_{k=0}^{\infty} \Delta a_k D_k(x)$ converges. Hence $\lim_{n\to\infty} g_n(x)$ exists for $x \neq 0$. Then

$$\|f - g_n\| = \Big\| \sum_{k=n+1}^{\infty} \Delta a_k D_k \Big\| = \frac{1}{\pi} \int_0^{\pi} \Big| \sum_{k=n+1}^{\infty} \Delta a_k D_k(x) \Big| \, dx.$$

Application of the Abel transformation and of Lemmas 2.4, 2.3 and 2.2 yields

$$\int_0^{\pi} \Big| \sum_{k=n+1}^{\infty} A_k \frac{\Delta a_k}{A_k} D_k(x) \Big| \, dx \leq \sum_{k=n+1}^{\infty} |\Delta A_k| \int_0^{\pi} \Big| \sum_{j=0}^{k} \frac{\Delta a_j}{A_j} D_j(x) \Big| dx + o(1) = o(1), \quad n \to \infty.$$

Hence, $\|f - g_n\| = o(1)$, $n \to \infty$.

"If": Assume $\|S_n - f\| = o(1)$, $n \to \infty$. Since $\|D_n\| = O(\log n)$, by the estimate

$$\|a_{n+1} D_n\| = \|S_n - g_n\| \leq \|S_n - f\| + \|f - g_n\| = o(1) + o(1), \quad n \to \infty,$$

we have $a_n \log n = o(1)$, $n \to \infty$.

"Only if": Assume $a_n \log n = o(1)$, $n \to \infty$. Then

$$\|S_n - f\| \leq \|g_n - f\| + \|a_{n+1} D_n(x)\| = o(1) + a_{n+1} O(\log n) = o(1), \quad n \to \infty.$$

Corollary 2.1 ([57]). *Let $\{a_n\} \in S_p(\delta)$ for some $1 < p \leq 2$. Then*

$$\frac{1}{x} \sum_{k=1}^{\infty} \Delta a_k \sin[(k + 1/2)x] = \frac{h(x)}{x}$$

converges for $x \in (0, \pi]$ and $h(x)/x \in L(0, \pi]$.

Proof. Since

$$\begin{aligned}
2\sin(x/2)\, f(x) &= a_0 \sin(x/2) + \sum_{k=1}^{\infty} a_k (2\sin(x/2)\cos kx) \\
&= a_0 \sin(x/2) + \sum_{k=1}^{\infty} a_k \sin[(k + 1/2)]x - (k - 1/2)x] \\
&= (a_0 - a_1)\sin(x/2) + (a_1 - a_2)\sin(3x/2) + (a_2 - a_3)\sin(5x/2) + \dots \\
&= \sum_{k=1}^{\infty} \Delta a_k \sin[(2k+1)x/2] = h(x),
\end{aligned}$$

by Theorem 2.10, the proof is obvious.

2.3. On the equivalence of classes of Fourier coefficients. In [6] Bor considered the following class $S^2(\delta)$. A sequence $\{a_k\}$ belongs to $S^2(\delta)$ if $a_k \to 0$ as $k \to \infty$, and there exists a sequence $\{A_k\}$ of numbers which is δ-quasi-monotone, $\sum_{k=1}^{\infty} k\delta_k < \infty$, $\sum_{k=1}^{\infty} k|\Delta A_k| < \infty$ and $|\Delta a_k| \le A_k$ for all k. Also, he proved Theorems 2.2 and 2.9 with $S^2(\delta)$ in place of S' and $S(\delta)$.

Recently, Telyakovskiĭ [47] and Leindler [25] proved that these classes and the Sidon–Telyakovskiĭ class S are all equivalent. Now, we shall present the proof of S. A. Telyakovskiĭ.

Theorem 2.11. *The classes S, S', $S(\delta)$ and $S^2(\delta)$ are all equivalent.*

Proof. First we prove that $S(\delta)$ and $S^2(\delta)$ are equivalent.

Let $\{a_n\} \in S(\delta)$. It suffices to show that $\sum_{n=1}^{\infty} n|\Delta A_n| < \infty$, but this holds by Lemma 2.2.

If $\{a_n\} \in S^2(\delta)$, then

$$nA_n = n\sum_{k=n}^{\infty} \Delta A_k \le \sum_{k=n}^{\infty} k|\Delta A_k| = o(1), \qquad n \to \infty.$$

But

$$\sum_{k=1}^{n} A_k = \sum_{k=1}^{n-1} k\,\Delta A_k + nA_n \le \sum_{k=1}^{n-1} k|\Delta A_k| + nA_n,$$

and this implies that $\sum_{n=1}^{\infty} A_n < \infty$, i.e. $\{a_n\} \in S(\delta)$.

Next we prove that S and $S(\delta)$ are equivalent. It is obvious that $S \subset S(\delta)$. If $\{a_n\} \in S(\delta)$, we define

$$B_k = A_k + \sum_{m=k}^{\infty} \delta_m.$$

Then $B_k - B_{k+1} = \Delta A_k + \delta_k \ge 0$, i.e. $B_k \downarrow 0$ as $k \to \infty$. On the other hand,

$$\sum_{k=1}^{\infty} B_k = \sum_{k=1}^{\infty} A_k + \sum_{k=1}^{\infty}\sum_{m=k}^{\infty} \delta_m = \sum_{k=1}^{\infty} A_k + \sum_{m=1}^{\infty}\sum_{k=1}^{m} \delta_m = \sum_{k=1}^{\infty} A_k + \sum_{m=1}^{\infty} m\delta_m < \infty,$$

and $|\Delta a_n| \le A_n < B_n$ for all n, i.e. $\{a_n\} \in S$. Now we have

$$S \subset S' \subset S(\delta) \subset S.$$

Consequently, $S \equiv S' \equiv S(\delta) \equiv S^2(\delta)$.

Applying this result, the inequality

$$\frac{1}{n}\sum_{k=1}^{n} \frac{|\Delta a_k|^p}{B_k^p} \le \frac{1}{n}\sum_{k=1}^{n} \frac{|\Delta a_k|^p}{A_k^p} = O(1),$$

and also Theorem 2.4, we obtain the following corollary.

Corollary 2.2 ([62]). *For all $p > 1$, the classes S_p, S_p' (case $\alpha = r = 0$) and $S_p(\delta)$ are equivalent.*

Remark 2.2. If $c_n \equiv a_n$ is a real even sequence ($c_n = c_{-n} = a_n$, $n = 0, 1, 2, \ldots$) then Theorem 1.8 of Č. V. Stanojević and V. B. Stanojević, the Sheng theorem (see Chapter III, 3.3, Theorem 3.17) and Theorem 2.10 are equivalent.

2.4. Trigonometric series with regularly quasi-monotone coefficients. A positive measurable function $L(u)$ is said to be *slowly varying* in the sense of Karamata [20] if $\lim_{u\to\infty} L(\lambda u)/L(u) = 1$ for every $\lambda > 0$. A basic property of slowly varying functions is the asymptotic relation [20]:

$$u^{\alpha} \max_{u\le s<\infty} s^{-\alpha} L(s) \sim L(u), \qquad u \to \infty, \qquad \text{for any} \quad \alpha > 0.$$

Slowly varying sequences are defined analogously: a positive sequence $\{l_n\}$ is said to be *slowly varying* if $\lim_{n\to\infty} l_{[\lambda n]}/l_n = 1$ for every $\lambda > 0$. The class of *slowly varying sequences* is denoted by $SV(\mathbb{N})$.

A nondecreasing sequence $\{r_n\}$ of positive numbers is *regularly varying*, i.e. $\{r_n\} \in (RV)(\mathbb{N})$ in the sense of J. Karamata [21], if for some $\alpha \ge 0$,

$$\lim_{n\to\infty} \frac{r_{[\lambda n]}}{r_n} = \lambda^{\alpha}, \qquad \lambda > 1.$$

Regularly varying sequences are characterized [20] as follows: $\{r_n\} \in (RV)(\mathbb{N})$ if and only if $r_n = n^{\alpha} l_n$ for some $\alpha > 0$ and some $\{l_n\} \in (SV)(\mathbb{N})$. On the other hand, a sequence $\{a_n\}$ is called *regularly quasi-monotone*, written $\{a_n\} \in RQM$, if $a_n/r_n \downarrow 0$ for some $\{r_n\} \in (RV)(\mathbb{N})$. It is obvious that the class of quasi-monotone sequences is a subclass of RQM. The next theorem is a generalization of the de la Vallée Poussin theorem (see Chapter II, 2.1).

THEOREM 2.12 ([65]). *If* $\{a_n\} \in RQM$ *and* $\sum_{n=1}^{\infty} a_n < \infty$ *then* $na_n = o(1)$, $n \to \infty$.

Proof. We have

$$a_{2n-1} \ge \left(1 + \frac{\alpha}{2n-1}\right)^{-1} a_{2n} \frac{l_{2n-1}}{l_{2n}},$$

$$a_{2n-2} \ge \left(1 + \frac{\alpha}{2n-2}\right)^{-1} a_{2n-1} \frac{l_{2n-2}}{l_{2n-1}} \ge \left(1 + \frac{\alpha}{2n-2}\right)^{-2} a_{2n} \frac{l_{2n-2}}{l_{2n}},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$a_n \ge \left(1 + \frac{\alpha}{n}\right)^{-n} a_{2n} \frac{l_n}{l_{2n}}.$$

Adding these inequalities, we obtain

$$\sum_{v=n}^{2n-1} a_v \ge \frac{a_{2n}}{l_{2n}} \sum_{v=n}^{2n-1} l_v \left(1 + \frac{\alpha}{v}\right)^{-(2n-v)}.$$

But

$$\left(1 + \frac{\alpha}{v}\right)^{2n-v} \le \left(1 + \frac{\alpha}{n}\right)^{2n-v} \le \left(1 + \frac{\alpha}{n}\right)^{n}$$

implies that

$$\sum_{v=n}^{2n-1} a_v \ge \frac{a_{2n}}{l_{2n}} \sum_{v=n}^{2n-1} l_v \left(1 + \frac{\alpha}{n}\right)^{-n}.$$

Thus

$$a_{2n} \frac{\sum_{v=n}^{2n-1} l_v}{l_{2n}} \le \left(1 + \frac{\alpha}{n}\right)^{n} \sum_{v=n}^{2n-1} a_v \le e^{\alpha} \sum_{v=n}^{2n-1} a_v.$$

The asymptotic relation

$$l_k \sim k^\beta \sup_{n\ge k} n^{-\beta} l_n, \quad k\to\infty, \tag{2.2}$$

gives for large n,

$$\begin{aligned}
\sum_{v=n}^{2n-1} l_v &\approx \sum_{v=n}^{2n-1} v^\beta \sup_{m\ge v} m^{-\beta} l_m \ge \sup_{m\ge 2n-1} m^{-\beta} l_m \ge \sum_{v=n}^{2n-1} v^\beta \\
&\ge n^\beta [\sup_{m\ge 2n-1} m^{-\beta} l_m] \sum_{v=n}^{2n-1} 1 \\
&= \left(\frac{n}{2n-1}\right)^\beta (2n-1)^\beta [\sup_{m\ge 2n-1} m^{-\beta} l_m] n \sim \frac{1}{2^\beta}\, n l_{2n-1}
\end{aligned}$$

for some $\beta>0$. Consequently,

$$n a_{2n}\,\frac{l_{2n-1}}{l_{2n}} \le e^\alpha 2^\beta \sum_{v=n}^{\infty} a_v.$$

Letting $n\to\infty$, we obtain $na_n = o(1)$, $n\to\infty$.

Sheng proved the following results on L^1-approximation of trigonometric series with regularly quasi-monotone coefficients.

THEOREM 2.13 ([33]). *Let* (CS) *be a Fourier series with* $\{a_n\},\{b_n\}\in RQM$. *Then there exist positive constants* C_1 *and* C_2 *such that*

$$C_1 \sum_{v=n+1}^{2n-1} \frac{a_v+b_v}{v-n} \le \|S_n-\tau_n\|,$$

$$\begin{aligned}
\|S_n-\tau_n\| \le C_2\biggl\{ &\sum_{v=n+1}^{2n-1} (a_v+b_v)\left(\frac{l_{v+1}}{(v-n)l_v}+\varepsilon_v\right) \\
&+ \frac{1}{n}\sum_{v=n+1}^{2n-1} (a_v+b_v)\log(v-n+1)\biggr\}
\end{aligned}$$

where $\tau_n(x) = n^{-1}\sum_{k=n}^{2n-1} S_k(x)$ *and* $\varepsilon_v = l_{v+1}/l_v - 1$.

The next theorem generalizes Theorem 1.25.

THEOREM 2.14 ([65]). *Let* (C) *be a Fourier series with* $\{a_n\}\in RQM$. *Then* (C) *is mean convergent iff* $a_n\log n = o(1)$, $n\to\infty$.

Proof. For the necessity we apply Theorem 2.5, i.e. $\sum_{k=1}^n a_{n+k}/k = o(1)$, $n\to\infty$. Since $\{a_n\}\in RQM$ we obtain the inequalities

$$\begin{aligned}
\sum_{k=1}^n \frac{a_{n+k}}{k} &= \sum_{k=1}^n \frac{a_{n+k}}{(n+k)^\alpha l_{n+k}}\,\frac{(n+k)^\alpha l_{n+k}}{k} \\
&\ge \frac{a_{2n}}{(2n)^\alpha l_{2n}} \sum_{k=1}^n \frac{(n+k)^\alpha l_{n+k}}{k} \\
&\ge \left(\frac{n+1}{2n}\right)^\alpha \frac{a_{2n}}{l_{2n}} \sum_{k=1}^n \frac{l_{n+k}}{k}.
\end{aligned}$$

Applying the asymptotic relation (2.2) for large n, we have

$$\begin{aligned}\sum_{k=1}^{n}\frac{l_{n+k}}{k}&=\sum_{k=n+1}^{2n}\frac{l_k}{k-n}\approx\sum_{k=n+1}^{2n}\frac{k^\beta\sup_{m\ge k}m^{-\beta}l_m}{k-n}\\&\ge(\sup_{m\ge 2n}m^{-\beta}l_m)\sum_{k=n+1}^{2n}\frac{k^\beta}{k-n}\\&\ge(n+1)^\beta(\sup_{m\ge 2n}m^{-\beta}l_m)\sum_{k=1}^{n}\frac{1}{k}\\&=\left(\frac{n+1}{2n}\right)^\beta[(2n)^\beta\sup_{m\ge 2n}m^{-\beta}l_m]\sum_{k=1}^{n}\frac{1}{k}\approx\frac{1}{2^\beta}\,l_{2n}\log n.\end{aligned}$$

Letting $n\to\infty$ in the inequality

$$a_{2n}\log n<2^\beta\left(\frac{2n}{n+1}\right)^\alpha\sum_{k=1}^{n}\frac{a_{n+k}}{k}$$

completes the proof of the necessity.

For the sufficiency, we apply Theorem 2.6. From the monotonicity of the sequence a_n/r_n, we get

$$\begin{aligned}A_n=&\sum_{k=1}^{m_n-1}|\Delta a_{k+n}|\log(k+1)=\sum_{i=n+1}^{n+m_n-1}|\Delta a_i|\log(i-n+1)\\&\sum_{i=n+1}^{n+m_n-1}\left|r_i\Delta\left(\frac{a_i}{r_i}\right)+\frac{a_{i+1}}{r_{i+1}}(r_i-r_{i+1})\right|\log(i-n+1)\\&\le r_{n+m_n-1}\log m_n\sum_{i=n+1}^{n+m_n-1}\Delta\left(\frac{a_i}{r_i}\right)+\sum_{i=n+1}^{n+m_n-1}\frac{a_{i+1}}{r_{i+1}}(r_{i+1}-r_i)\log(i+1)\\&=r_{n+m_n-1}\log m_n\left(\frac{a_{n+1}}{r_{n+1}}-\frac{a_{n+m_n-1}}{r_{n+m_n-1}}\right)\\&\quad+\max_{n+1\le i\le n+m_n-1}(a_{i+1}\log(i+1))\sum_{i=n+1}^{n+m_n-1}\left(1-\frac{r_i}{r_{i+1}}\right).\end{aligned}$$

Since

$$\sum_{i=n+1}^{n+m_n-1}\left(1-\frac{r_i}{r_{i+1}}\right)\le\log\prod_{i=n+1}^{n+m_n-1}\frac{r_{i+1}}{r_i}=\log\frac{r_{n+m_n}}{r_{n+1}}\le\frac{r_{n+m_n}}{r_n},$$

we obtain

$$A_n\le\frac{r_{n+m_n-1}}{r_n}\frac{\log m_n}{\log n}(a_{n+1}\log(n+1))+\frac{r_{n+m_n}}{r_n}\max_{n+1\le i\le n+m_n-1}[a_{i+1}\log(i+1)].$$

The hypothesis $a_n\log n=o(1)$, $n\to\infty$ and $\{r_n\}\in(RV)(\mathbb{N})$ imply that the first and second terms on the right side are $o(1)$, $n\to\infty$. Finally, $A_n=o(1)$, $n\to\infty$, i.e. the series (C) is mean convergent.

Remark 2.3. The proof of the necessity of this theorem can be simplified by using the monotonicity of the sequence $\{r_n\}$ and the fact that $\{a_n/r_n\}\downarrow$. We have

$$\sum_{k=1}^{n}\frac{a_{n+k}}{k}=\sum_{k=1}^{n}\frac{a_{n+k}}{r_{n+k}}\,\frac{r_{n+k}}{k}\ge\frac{a_{2n}}{r_{2n}}\,r_n\sum_{k=1}^{n}\frac{1}{k}\approx(a_{2n}\log n)\frac{r_n}{r_{2n}}.$$

Taking $n\to\infty$ in the inequality

$$a_{2n}\log n\le\frac{r_{2n}}{r_n}\sum_{k=1}^{n}\frac{a_{n+k}}{k}$$

completes the proof.

III. Estimates of trigonometric series, useful in problems of approximation theory

3.1. Some L^1-estimates for trigonometric series with the Fomin coefficient condition. Let $f(x)$ and $g(x)$ be the sums of the series (C) and (S) respectively. It is well known (see [2], [22], [73]) that if $\{a_n\}$ is a quasi-convex null sequence of real numbers, then the series (C) is the Fourier series of some $f\in L^1$ and

$$\int_0^{\pi}|f(x)|\,dx\le\frac{\pi}{2}\sum_{k=1}^{\infty}k|\Delta^2a_{k-1}|. \tag{3.1}$$

The following two theorems were proved by Telyakovskiĭ [42], [43].

Theorem 3.1 ([42]). *Let $\{a_n\}\in BV$, $a_n\to 0$ and*

$$\sum_{i=2}^{\infty}\biggl|\sum_{k=1}^{[i/2]}\frac{\Delta a_{i-k}-\Delta a_{i+k}}{k}\biggr|<\infty.$$

Then

$$\int_0^{\pi}|f(x)|\,dx\le C\biggl(\sum_{k=0}^{\infty}|\Delta a_k|+\sum_{i=2}^{\infty}\biggl|\sum_{k=1}^{[i/2]}\frac{\Delta a_{i-k}-\Delta a_{i+k}}{k}\biggr|\biggr),$$

where C is some absolute constant.

Theorem 3.2 ([43]). *Let $\{a_n\}\in BV$, $a_n\to 0$, $a_0=0$, and*

$$\sum_{i=2}^{\infty}\biggl|\sum_{k=1}^{[i/2]}\frac{\Delta a_{i-k}-\Delta a_{i+k}}{k}\biggr|<\infty.$$

Then the following estimate holds uniformly with respect to $s\in\mathbb{N}$:

$$\biggl|\int_{\pi/(2s+1)}^{\pi}|g(x)|\,dx-\sum_{k=1}^{s}\frac{|a_k|}{k}\biggr|\le C\biggl(\sum_{k=0}^{\infty}|\Delta a_k|+\sum_{i=2}^{\infty}\biggl|\sum_{k=1}^{[i/2]}\frac{\Delta a_{i-k}-\Delta a_{i+k}}{k}\biggr|\biggr),$$

where C is some absolute constant.

Also, Telyakovskiĭ [42], [44] proved the following inequality:

$$\sum_{i=2}^{\infty}\left|\sum_{k=1}^{[i/2]}\frac{\Delta a_{i-k}-\Delta a_{i+k}}{k}\right|\le C\sum_{k=1}^{\infty}k\,|\Delta^2 a_{k-1}|. \tag{3.2}$$

REMARK 3.1. If $\{a_k\}$ is a quasi-convex null sequence then $\sum_{k=0}^{\infty}|\Delta a_k|\le\sum_{k=1}^{\infty}k|\Delta^2 a_k|$. Thus the estimate (3.1) follows from Theorem 3.1 and the estimate (3.2) with some absolute constant C instead of $\pi/2$.

THEOREM 3.3. *If $\{a_k\}$ is a quasi-convex null sequence with $a_0=0$, then* (S) *is a Fourier series iff $\sum_{n=1}^{\infty}|a_n|/n<\infty$. Moreover, if $\sum_{n=1}^{\infty}|a_n|/n<\infty$, then*

$$\int_0^{\pi}\left|\sum_{k=1}^{\infty}a_k\sin kx\right|dx\le\sum_{k=1}^{\infty}\frac{|a_k|}{k}+C\sum_{k=1}^{\infty}k|\Delta^2 a_{k-1}|.$$

REMARK 3.2. If $\{a_n\}$ is a quasi-convex null sequence then the estimate of Theorem 3.3 is a consequence of Theorem 3.2 and of the estimate (3.2).

Also, Telyakovskiĭ [41] gave a direct proof of Theorem 3.3, by proving the following estimate:

$$\left|\int_{1/(s+1)}^{\pi}\left|\sum_{k=1}^{\infty}a_k\sin kx\right|dx-\sum_{k=1}^{s}\frac{|a_k|}{k}\right|\le C\sum_{k=1}^{\infty}k|\Delta^2 a_{k-1}|,\quad C>0.$$

In [45] the indicated results on series with quasi-convex coefficients were extended to the more general case when the coefficients $\{a_k\}$ satisfy the Sidon–Telyakovskiĭ class S. Namely, Telyakovskiĭ proved the following theorems.

THEOREM 3.4 ([45]). *Let the coefficients of the series* (C) *belong to the class S. Then* (C) *is the Fourier series of some $f\in L^1(0,\pi)$ and*

$$\int_0^{\pi}|f(x)|\,dx\le M\sum_{n=0}^{\infty}A_n,\quad M>0.$$

THEOREM 3.5 ([45]). *Let the coefficients of the series* (S) *belong to the class S. Then*

$$\int_{\pi/(p+1)}^{\pi}|g(x)|\,dx=\sum_{n=1}^{p}\frac{|a_n|}{n}+O\Big(\sum_{n=1}^{\infty}A_n\Big),$$

for $p\in\mathbb{N}$. In particular $g(x)$ is a Fourier series iff $\sum_{n=1}^{\infty}|a_n|/n<\infty$.

COROLLARY 3.1 ([66]). *Let the coefficients of the series* (C) *belong to the class $S(\delta)$. Then* (C) *is the Fourier series of some $f\in L^1(0,\pi)$ and*

$$\int_0^{\pi}|f(x)|\,dx\le M\Big(\sum_{n=0}^{\infty}A_n+\sum_{n=1}^{\infty}n\delta_n\Big),\quad M>0.$$

Proof. Applying Theorems 2.11 and 3.4, we obtain

$$\int_0^{\pi}|f(x)|\,dx\le M\sum_{n=0}^{\infty}B_n=M\Big(\sum_{n=0}^{\infty}A_n+\sum_{n=0}^{\infty}\sum_{m=n}^{\infty}\delta_m\Big)$$

$$=M\Big(\sum_{n=0}^{\infty}A_n+\sum_{n=1}^{\infty}n\delta_n\Big),\qquad M>0.$$

Analogously, applying Theorems 2.11 and 3.5, we obtain

Corollary 3.2 ([66]). *Let the coefficients of the series* (S) *belong to the class* $S(\delta)$. *Then*

$$\int\limits_{\pi/(p+1)}^{\pi} |g(x)|\,dx = \sum_{n=1}^{p} \frac{|a_n|}{n} + O\Big(\sum_{n=1}^{\infty} A_n\Big) + O\Big(\sum_{n=1}^{\infty} n\delta_n\Big)$$

for $p \in \mathbb{N}$.

On the other hand, Fomin [13] proved the following estimate:

$$\sum_{i=2}^{\infty} \bigg| \sum_{k=1}^{[i/2]} \frac{\Delta a_{i-k} - \Delta a_{i+k}}{k} \bigg| \le C_p \sum_{s=0}^{\infty} 2^s \Delta_s^{(p)} \tag{3.3}$$

for any $1 < p \le 2$, where the positive constant C_p depends only on p.

Lemma 3.1 (Elliot [18]). *If* $0 < q < 1$, $b_n \ge 0$ *and* $\sum_{n=1}^{\infty} b_n^q < \infty$ *then*

$$\Big(\frac{q}{1-q}\Big)^q \sum_{n=1}^{\infty} b_n^q < \sum_{n=1}^{\infty} \Big(\frac{b_n + b_{n+1} + \ldots}{n}\Big)^q,$$

unless all the b_n *are zero.*

Theorem 3.6 ([59], [60]). *Let* $\{a_n\} \in F_p$ *for some* $1 < p \le 2$. *Then the series* (C) *is the Fourier series of some* $f \in L^1(0,\pi)$ *and*

$$\int\limits_0^{\pi} |f(x)|\,dx \le C_p \sum_{n=1}^{\infty} \Big(\frac{\sum_{k=n}^{\infty} |\Delta a_k|^p}{n}\Big)^{1/p}.$$

Proof. Putting $b_n = |\Delta a_n|^p$ in Lemma 3.1, where $q = 1/p$, we get

$$\Big(\frac{1}{p-1}\Big)^{1/p} \sum_{n=1}^{\infty} |\Delta a_n| < \sum_{n=1}^{\infty} \Big(\frac{|\Delta a_n|^p + |\Delta a_{n+1}|^p + \ldots}{n}\Big)^{1/p},$$

i.e.

$$\sum_{n=1}^{\infty} |\Delta a_n| < (p-1)^{1/p} \sum_{n=1}^{\infty} \Big(\frac{\sum_{k=n}^{\infty} |\Delta a_k|^p}{n}\Big)^{1/p}.$$

On the other hand, since $U_s = s^{-1}\sum_{k=s}^{\infty} |\Delta a_k|^p$ is a decreasing sequence,

$$\begin{aligned}\sum_{s=1}^{n} 2^s \Delta_s^{(p)} &\le 2\sum_{s=1}^{n} \Big[2^{(s-1)(p-1)} \sum_{k=2^{s-1}+1}^{2^s} |\Delta a_k|^p\Big]^{1/p} \\ &\le 2\sum_{s=1}^{n} 2^{s-1}\Big(\frac{1}{2^{s-1}} \sum_{k=2^{s-1}}^{\infty} |\Delta a_k|^p\Big)^{1/p} = O\Big(\sum_{s=1}^{2^{n-1}} (U_s)^{1/p}\Big).\end{aligned}$$

Letting $n \to \infty$, we have

$$\sum_{s=1}^{\infty} 2^s \Delta_s^{(p)} = O\bigg(\sum_{s=1}^{\infty} \Big(\frac{1}{s}\sum_{k=s}^{\infty} |\Delta a_k|^p\Big)^{1/p}\bigg).$$

Then applying Theorem 3.1 and inequality (3.3) completes the proof.

Similarly, applying Theorem 3.2, we can get

THEOREM 3.7 ([59], [60]). *Let* $1 < p \le 2$, $\{a_n\} \in F_p$ *and* $a_0 = 0$. *Then* (S) *is a Fourier series iff* $\sum_{n=1}^{\infty} |a_n|/n < \infty$. *Moreover if* $\sum_{n=1}^{\infty} |a_n|/n < \infty$, *then*

$$\int_0^{\pi} \Big|\sum_{k=1}^{\infty} a_k \sin kx\Big|\, dx \le \sum_{k=1}^{\infty} \frac{|a_k|}{k} + C_p \sum_{n=1}^{\infty} \left(\frac{\sum_{k=n}^{\infty} |\Delta a_k|^p}{n}\right)^{1/p}.$$

3.2. Some results on L^1-approximation of the rth derivative of Fourier series. In this section we obtain L^1-inequalities for rth derivatives of the series (C) and (S).

Generalizations of the Telyakovskiĭ inequalities [52], [53], [55] are obtained by considering the condition $\Im_r$, $r \in \mathbb{N} \cup \{0\}$, and $S_{p\alpha r}$, $1 < p \le 2$, $\alpha \ge 0$, $r \in \{0, 1, \dots, [\alpha]\}$, instead of S. An equivalent form of the condition $\Im_r$, $r \in \mathbb{N} \cup \{0\}$, and an extension of Sidon's Theorem 1.3 are given.

THEOREM 3.8 ([52]). *Let* $r \in \mathbb{N} \cup \{0\}$ *and let the coefficients of the series* (C) *belong to the class* $\Im_r$. *Then the* r*th derivative of* (C) *is the Fourier series of some* $f^{(r)} \in L^1(0, \pi)$ *and*

$$\int_0^{\pi} |f^{(r)}(x)|\, dx \le M \sum_{n=1}^{\infty} n^r A_n, \quad \textit{where} \quad 0 < M = M(r) < \infty.$$

Proof 1. We have

$$\begin{aligned}
\sum_{k=1}^{n} |\Delta(k^r a_k)| &\le \sum_{k=1}^{n} |(k+1)^{r+1} a_{k+1} - k^r a_{k+1}| + \sum_{k=1}^{n} |k^r a_{k+1} - k^r a_k| \\
&= \sum_{k=1}^{n} |\Delta(k^r) a_{k+1}| + \sum_{k=1}^{n} k^r |\Delta a_k| \\
&= O_r\Big(\sum_{k=1}^{n} k^{r-1} |a_{k+1}|\Big) + O\Big(\sum_{k=1}^{n} k^r A_k\Big).
\end{aligned}$$

Applying the Abel transformation, we have

$$\begin{aligned}
\sum_{k=1}^{n} k^{r-1} |a_{k+1}| &= \sum_{k=1}^{n-1} \Delta |a_{k+1}| \sum_{j=1}^{k} j^{r-1} + |a_{n+1}| \sum_{j=1}^{n} j^{r-1} \\
&\le \sum_{k=1}^{n-1} |\Delta a_{k+1}| k^r + |a_{n+1}| n^r \\
&\le \sum_{k=1}^{n-1} |\Delta a_{k+1}| k^r + \sum_{k=n+1}^{\infty} k^r |\Delta a_k| \\
&\le \sum_{k=1}^{n-1} k^r A_k + \sum_{k=n+1}^{\infty} k^r A_k.
\end{aligned}$$

Letting $n \to \infty$, we get $\sum_{k=1}^{\infty} |\Delta(k^r a_k)| < \infty$, i.e. $\lim_{n\to\infty} S_n^{(r)}(x) = f^{(r)}(x)$.

Since $|D_n^{(r)}(x)| = O(n^r/x)$ (see [32]), the series $\sum_{k=1}^{\infty} \Delta a_k D_k^{(r)}(x)$ is uniformly convergent on any compact subset of $(0, \pi)$. Thus the representation $f(x) = \sum_{k=0}^{\infty} \Delta a_k D_k(x)$

implies that

$$f^{(r)}(x) = \sum_{k=1}^{\infty} \Delta a_k D_k^{(r)}(x).$$

From Lemmas 1.6 and 1.10, we obtain

$$A_N \int_0^{\pi} \left| \sum_{j=0}^{N} \frac{\Delta a_j}{A_j} D_j^{(r)}(x) \right| dx = O((N+1)^{r+1} A_N) = o(1), \quad N \to \infty. \tag{3.4}$$

Again applying the Abel transformation, (3.4) and Lemma 1.6, we get

$$\begin{aligned}
\int_0^{\pi} |f^{(r)}(x)|\, dx &\le \lim_{N\to\infty} \sum_{k=0}^{N-1} (\Delta A_k) \int_0^{\pi} \left| \sum_{j=0}^{k} \frac{\Delta a_j}{A_j} D_j^{(r)}(x) \right| dx \\
&= O(1) \lim_{N\to\infty} \sum_{k=0}^{N-1} (\Delta A_k)(k+1)^{r+1} \\
&= O(1) \lim_{N\to\infty} \left\{ \sum_{k=0}^{N} [(k+1)^{r+1} - k^{r+1}] A_k - (N+1)^{r+1} A_N \right\} \\
&= O_r\Big(\sum_{k=0}^{\infty} k^r A_k \Big),
\end{aligned}$$

where O_r depends on r.

Proof 2 ([63]). First we prove that if $\{a_n\} \in \Im_r$, then $\{n^r a_n\} \in S$. We define the sequence $\{B_k\}$ as follows:

$$B_k = k^r A_k + \sum_{i=k+1}^{\infty} [i^r - (i-1)^r] A_i.$$

We have

$$B_k - B_{k+1} = k^r A_k - (k+1)^r A_{k+1} + (k+1)^r A_{k+1} - k^r A_{k+1} = k^r \Delta A_k \ge 0.$$

Then

$$\begin{aligned}
\sum_{k=1}^{\infty} B_k &= \sum_{k=1}^{\infty} k^r A_k + \sum_{k=1}^{\infty} \sum_{i=k}^{\infty} [(i+1)^r - i^r] A_{i+1} \\
&= \sum_{k=1}^{\infty} k^r A_k + \sum_{i=1}^{\infty} \sum_{k=1}^{i} [(i+1)^r - i^r] A_{i+1} \\
&= \sum_{k=1}^{\infty} k^r A_k + \sum_{i=1}^{\infty} i[(i+1)^r - i^r] A_{i+1} \\
&< \sum_{k=1}^{\infty} k^r A_k + \sum_{i=1}^{\infty} [(i+1)^{r+1} - i^{r+1}] A_{i+1} \\
&= \sum_{k=1}^{\infty} k^r A_k + O_r\Big(\sum_{i=1}^{\infty} i^r A_i \Big) < \infty.
\end{aligned}$$

Then $\Delta(k^r a_k) = k^r a_k - (k+1)^r a_{k+1} = k^r \Delta a_k - ((k+1)^r - k^r)a_{k+1}$. The function $h(x) = (x+1)^r - x^r$ is increasing on $[0,\infty)$, since $h'(x) = r[(x+1)^{r-1} - x^{r-1}] \geq 0$ for $x \geq 0$. This implies that

$$\begin{aligned} |\Delta(k^r a_k)| &\leq k^r |\Delta a_k| + ((k+1)^r - k^r)|a_{k+1}| \\ &\leq k^r A_k + ((k+1)^r - k^r) \sum_{i=k+1}^{\infty} |\Delta a_i| \\ &\leq k^r A_k + \sum_{i=k+1}^{\infty} (i^r - (i-1)^r)|\Delta a_i| \\ &\leq k^r A_k + \sum_{i=k+1}^{\infty} (i^r - (i-1)^r) A_i = B_k. \end{aligned}$$

Thus $\{n^r a_n\} \in S$. Now, applying Theorem 3.4, we obtain

$$\begin{aligned} \int_0^\pi |f^{(r)}(x)|\,dx \leq M \sum_{n=0}^{\infty} B_n &< M\Big[\sum_{k=1}^{\infty} k^r A_k + O_r\Big(\sum_{i=1}^{\infty} i^r A_i\Big)\Big] \\ &= O_r\Big(\sum_{k=1}^{\infty} k^r A_k\Big), \end{aligned}$$

where O_r depends on r.

Theorem 3.9 ([55]). *Let $r \in \mathbb{N} \cup \{0\}$. A null sequence $\{a_n\}$ belongs to the class $\Im_r$ if and only if it can be represented as*

$$a_n = \sum_{k=n}^{\infty} \frac{p_k}{k} \sum_{l=n}^{k} \alpha_l, \qquad n \in \mathbb{N},$$

where $\{\alpha_n\}_{n=1}^{\infty}$ and $\{p_n\}_{n=1}^{\infty}$ are sequences such that $|\alpha_n| \leq 1$ for all n and

$$\sum_{n=1}^{\infty} n^r |p_n| < \infty. \tag{3.5}$$

Proof. Let (3.5) hold. Then

$$\Delta a_k = \alpha_k \sum_{m=k}^{\infty} \frac{p_m}{m}$$

and we define

$$A_k = \sum_{m=k}^{\infty} \frac{|p_m|}{m}.$$

Since $|\alpha_k| \leq 1$, we get

$$|\Delta a_k| \leq |\alpha_k| \sum_{m=k}^{\infty} \frac{|p_m|}{m} \leq A_k \quad \text{for all } k.$$

However,

$$\sum_{k=1}^{\infty} k^r A_k = \sum_{k=1}^{\infty} k^r \sum_{m=k}^{\infty} \frac{|p_m|}{m} = \sum_{m=1}^{\infty} \frac{|p_m|}{m} \sum_{k=1}^{m} k^r \leq \sum_{m=1}^{\infty} m^r |p_m| < \infty,$$

and $A_k \downarrow 0$, i.e. $\{a_k\} \in \Im_r$.

Conversely, if $\{a_k\} \in \Im_r$, we put $\alpha_k = (\Delta a_k)/A_k$ and $p_k = k(A_k - A_{k+1})$. Hence $|\alpha_k| \le 1$, and by Lemma 1.10 we get

$$\sum_{k=1}^{\infty} k^r |p_k| = \sum_{k=1}^{\infty} k^{r+1}(A_k - A_{k+1}) = O\Big(\sum_{k=1}^{\infty} k^r A_k\Big) < \infty.$$

Finally,

$$a_k = \sum_{i=k}^{\infty} \Delta a_i = \sum_{i=k}^{\infty} \alpha_i A_i = \sum_{i=k}^{\infty} \alpha_i \sum_{m=i}^{\infty} \Delta A_m = \sum_{i=k}^{\infty} \alpha_i \sum_{m=i}^{\infty} \frac{p_m}{m} = \sum_{m=k}^{\infty} \frac{p_m}{m} \sum_{i=k}^{m} \alpha_i,$$

i.e. (3.5) holds.

Corollary 3.3 ([55]). *Let $r \in \mathbb{N} \cup \{0\}$ and let $\{\alpha_n\}_{n=1}^{\infty}$ and $\{p_n\}_{n=1}^{\infty}$ be sequences such that $|\alpha_n| \le 1$ for every n and let $\sum_{n=1}^{\infty} n^r |p_n| < \infty$. If*

$$a_n = \sum_{k=n}^{\infty} \frac{p_k}{k} \sum_{l=n}^{k} \alpha_l, \qquad n \in \mathbb{N},$$

then the rth derivative of the series (C) *is the Fourier series of some $f^{(r)} \in L^1$.*

Proof. This follows from Theorems 3.8 and 3.9.

Lemma 3.2 ([52]). *Let $\{\alpha_j\}_{j=0}^{k}$ be a sequence of real numbers. Then*

$$U_k = \int_{\pi/(k+1)}^{\pi} \left| \sum_{j=0}^{k} \alpha_j \frac{(j+1/2)^v \sin[(j+1/2)x + (v+3)\pi/2]}{(\sin(x/2))^{r+1-v}} \right| dx$$
$$= O\Big((k+1)^{r-v+1/2}\Big(\sum_{j=0}^{k} \alpha_j^2 (j+1)^{2v}\Big)^{1/2}\Big),$$

for $v \in \{0, 1, \ldots, r\}$ and $r \in \mathbb{N} \cup \{0\}$.

Proof. Applying first the Cauchy–Bunyakovskiĭ inequality yields

$$U_k \le \Bigg[\int_{\pi/(k+1)}^{\pi} \frac{dx}{(\sin(x/2))^{2(r+1-v)}} \Bigg]^{1/2}$$
$$\times \Bigg\{ \int_{\pi/(k+1)}^{\pi} \Big[\sum_{j=0}^{k} \alpha_j (j+1/2)^v \sin[(j+1/2)x + (v+3)\pi/2] \Big]^2 dx \Bigg\}^{1/2}.$$

Since

$$\int_{\pi/(k+1)}^{\pi} \frac{dx}{(\sin(x/2))^{2(r+1-v)}} \le \pi^{2(r+1-v)} \int_{\pi/(k+1)}^{\pi} \frac{dx}{x^{2(r+1-v)}}$$
$$\le \frac{\pi(k+1)^{2(r+1-v)-1}}{2(r+1-v)-1} \le \pi(k+1)^{2(r+1-v)-1},$$

we have

$$U_k \le \sqrt{\pi}[(k+1)^{2(r+1-v)-1}]^{1/2} \times \Big\{\int_0^{\pi}\Big[\sum_{j=0}^{k}\alpha_j(j+1/2)^v\sin[(j+1/2)x+(v+3)\pi/2]\Big]^2\,dx\Big\}^{1/2}$$

$$\le \sqrt{2\pi}[(k+1)^{2(r+1-v)-1}]^{1/2}\Big\{\int_0^{2\pi}\Big[\sum_{j=0}^{k}\alpha_j(j+1/2)^v\sin[(2j+1)t+(v+3)\pi/2]\Big]^2\,dt\Big\}^{1/2}.$$

Then applying the Parseval equality, we get

$$U_k \le \sqrt{2\pi}[(k+1)^{2(r+1-v)-1}]^{1/2}\Big[\sum_{j=0}^{k}\alpha_j^2(j+1)^{2v}\Big]^{1/2}.$$

Finally,

$$U_k = O\Big((k+1)^{r-v+1/2}\Big(\sum_{j=0}^{k}\alpha_j^2(j+1)^{2v}\Big)^{1/2}\Big).$$

LEMMA 3.3 ([55]). *Let $r\in\mathbb{N}\cup\{0\}$ and let $\{\alpha_k\}$ be a sequence of real numbers such that $|\alpha_k|\le 1$ for all k. Then there exists a finite constant $M=M(r)>0$ such that*

$$\int_{\pi/(n+1)}^{\pi}\Big|\sum_{k=0}^{n}\alpha_k\overline{D}_k^{(r)}(x)\Big|\,dx \le M(n+1)^{r+1}$$

for any $n\ge 0$.

Proof. Since $-\cos[(n+1/2)x]=\sin[(n+1/2)x+3\pi/2]$, by Lemma 1.13, we get

$$\overline{D}_n^{(r)}(x) = \sum_{k=0}^{r-1}\frac{(n+1/2)^k\sin[(n+1/2)x+(k+3)\pi/2]}{(\sin(x/2))^{r+1-k}}\,\varphi_k(x) + \frac{(n+1/2)^r\sin[(n+1/2)x+(r+3)\pi/2]}{2\sin(x/2)},$$

where the φ_k are analytic functions of x, independent of n. Then

$$\int_{\pi/(n+1)}^{\pi}\Big|\sum_{k=0}^{n}\alpha_k\overline{D}_k^{(r)}(x)\Big|\,dx$$

$$\le \int_{\pi/(n+1)}^{\pi}\Big|\sum_{j=0}^{n}\alpha_j\Big(\sum_{v=0}^{r-1}\frac{(j+1/2)^v\sin[(j+1/2)x+(v+3)\pi/2]}{(\sin(x/2))^{r+1-v}}\,\varphi_v(x)\Big)\Big|\,dx$$

$$+ \int_{\pi/(n+1)}^{\pi}\Big|\sum_{j=0}^{n}\alpha_j\frac{(j+1/2)^r\sin[(j+1/2)x+(r+3)\pi/2]}{2\sin(x/2)}\Big|\,dx = \lambda_n+\mu_n.$$

Since φ_v are bounded functions, we have

$$\int_{\pi/(n+1)}^{\pi}\Big|\sum_{j=0}^{n}\alpha_j\frac{(j+1/2)^v\sin[(j+1/2)x+(v+3)\pi/2]}{(\sin(x/2))^{r+1-v}}\,\varphi_v\Big|\,dx \le K\,U_n,$$

where U_n is the integral as in Lemma 3.2 and K is a positive constant. Applying Lemma 3.2 to the last integral, we get

$$\int\limits_{\pi/(n+1)}^{\pi} \Big| \sum_{j=0}^{n} \alpha_j \frac{(j+1/2)^v \sin[(j+1/2)x + (v+3)\pi/2]}{(\sin(x/2))^{r+1-v}} \varphi_v(x) \Big| \, dx$$
$$= O\Big((n+1)^{r-v+1/2} \Big(\sum_{j=0}^{n} \alpha_j^2 (j+1)^{2v}\Big)^{1/2}\Big)$$
$$= O((n+1)^{r-v+1/2}(n+1)^{v+1/2}) = O((n+1)^{r+1}).$$

Since r is a finite value, we have $\lambda_n = O((n+1)^{r+1})$. Similarly, $\mu_n = O((n+1)^{r+1})$. Thus, the inequality is proved.

Remark 3.3. For $r = 0$, we get the Telyakovskiĭ inequality, proved in [45].

Theorem 3.10 ([55]). *Let $r \in \mathbb{N} \cup \{0\}$ and let the coefficients of the series $g(x)$ belong to the class $\Im_r$. Then the rth derivative of the series* (S) *converges to a function and*

$$(*) \qquad \int\limits_{\pi/(m+1)}^{\pi} |g^{(r)}(x)| \, dx \le M\Big(\sum_{n=1}^{m} |a_n| n^{r-1} + \sum_{n=1}^{\infty} n^r A_n\Big),$$

for $m \in \mathbb{N}$, where $0 < M = M(r) < \infty$. Moreover, if $\sum_{n=1}^{\infty} n^{r-1}|a_n| < \infty$, then the rth derivative of (S) *is the Fourier series of some $g^{(r)} \in L^1(0,\pi)$ and*

$$\int\limits_{0}^{\pi} |g^{(r)}(x)| \, dx \le M\Big(\sum_{n=1}^{\infty} |a_n| n^{r-1} + \sum_{n=1}^{\infty} n^r A_n\Big).$$

Proof. We suppose that $a_0 = 0$ and $A_0 = \max(|a_1|, A_1)$. Applying the Abel transformation, we have

$$(3.6) \qquad g(x) = \sum_{k=0}^{\infty} \Delta a_k \overline{D}_k(x), \qquad x \in (0,\pi].$$

Applying the inequality of Lemma 4.3(iii), we see that the series $\sum_{k=0}^{\infty} \Delta a_k \overline{D}_k^{(r)}(x)$ is uniformly convergent on any compact subset of $[\varepsilon, \pi]$, where $\varepsilon > 0$. Thus representation (3.6) implies that

$$g^{(r)}(x) = \sum_{k=0}^{\infty} \Delta a_k \overline{D}_k^{(r)}(x), \quad x \in (0,\pi].$$

Then

$$\int\limits_{\pi/(m+1)}^{\pi} |g^{(r)}(x)| \, dx = \int\limits_{\pi/(m+1)}^{\pi} \Big| \sum_{k=0}^{\infty} \Delta a_k \overline{D}_k^{(r)}(x) \Big| \, dx$$
$$\le \sum_{j=1}^{m} \int\limits_{\pi/(j+1)}^{\pi/j} \Big| \sum_{k=0}^{j-1} \Delta a_k \overline{D}_k^{(r)}(x) \Big| \, dx$$
$$+ O\Big(\sum_{j=1}^{m} \int\limits_{\pi/(j+1)}^{\pi/j} \Big| \sum_{k=j}^{\infty} \Delta a_k \overline{D}_k^{(r)}(x) \Big| \, dx\Big).$$

Let

$$I_1=\sum_{j=1}^{m}\int\limits_{\pi/(j+1)}^{\pi/j}\Big|\sum_{k=0}^{j-1}\Delta a_k\overline{D}_k^{(r)}(x)\Big|\,dx,\quad I_2=\sum_{j=1}^{m}\int\limits_{\pi/(j+1)}^{\pi/j}\Big|\sum_{k=j}^{\infty}\Delta a_k\overline{D}_k^{(r)}(x)\Big|\,dx.$$

Applying the well known expansion

$$\operatorname{ctg}\Big(\frac{x}{2}\Big)=\frac{2}{x}+\sum_{n=1}^{\infty}\frac{4x}{x^2-4n^2\pi^2}$$

it is not difficult to prove the following estimate:

$$\Big(\operatorname{ctg}\Big(\frac{x}{2}\Big)\Big)^{(r)}=\frac{2(-1)^r r!}{x^{r+1}}+O(1),\quad x\in(0,\pi].$$

Thus

$$\overline{D}_n^{(r)}(x)=\frac{(-1)^{r+1}r!}{x^{r+1}}+O((n+1)^{r+1}),\quad x\in(0,\pi].$$

Hence,

$$\begin{aligned}I_1&=r!\sum_{j=1}^{m}\Big|\sum_{k=0}^{j-1}\Delta a_k\Big|\int\limits_{\pi/(j+1)}^{\pi/j}\frac{dx}{x^{r+1}}+O\Big(\sum_{j=1}^{m}\Big[\sum_{k=0}^{j-1}|\Delta a_k|(k+1)^{r+1}\Big]\int\limits_{\pi/(j+1)}^{\pi/j}dx\Big)\\&=O_r\Big(\sum_{j=1}^{m}|a_j|j^{r-1}\Big)+O\Big(\sum_{j=1}^{m}\sum_{k=0}^{j-1}\frac{(k+1)^{r+1}|\Delta a_k|}{j(j+1)}\Big),\end{aligned}$$

where O_r depends on r. But

$$\begin{aligned}\sum_{j=1}^{m}\sum_{k=0}^{j-1}\frac{(k+1)^{r+1}|\Delta a_k|}{j(j+1)}&=\sum_{j=1}^{m}\frac{1}{j(j+1)}\sum_{k=0}^{j-1}(k+1)^{r+1}|\Delta a_k|\\&\le\sum_{k=0}^{\infty}(k+1)^{r+1}|\Delta a_k|\sum_{j=k+1}^{\infty}\frac{1}{j(j+1)}=\sum_{k=0}^{\infty}(k+1)^r|\Delta a_k|\\&=|\Delta a_0|+\sum_{k=1}^{\infty}(k+1)^r|\Delta a_k|\le|a_1|+2^r\sum_{k=1}^{\infty}k^r|\Delta a_k|\\&\le\sum_{k=1}^{\infty}|\Delta a_k|+2^r\sum_{k=1}^{\infty}k^rA_k\le(1+2^r)\sum_{k=1}^{\infty}k^rA_k.\end{aligned}$$

Thus,

$$\sum_{j=1}^{m}\sum_{k=0}^{j-1}\frac{|\Delta a_k|(k+1)^{r+1}}{j(j+1)}=O_r\Big(\sum_{k=1}^{\infty}k^rA_k\Big),$$

where O_r depends on r. Therefore,

$$I_1=O_r\Big(\sum_{j=1}^{m}|a_j|j^{r-1}\Big)+O_r\Big(\sum_{k=1}^{\infty}k^rA_k\Big),$$

where O_r depends on r. Applying the Abel transformation yields

$$\sum_{k=j}^{\infty} \Delta a_k \overline{D}_k^{(r)}(x) = \sum_{k=j}^{\infty} \Delta A_k \sum_{i=0}^{k} \frac{\Delta a_i}{A_i}\, \overline{D}_i^{(r)}(x) - A_j \sum_{i=0}^{j-1} \frac{\Delta a_i}{A_i}\, \overline{D}_i^{(r)}(x).$$

Let us estimate the second integral:

$$I_2 \le \sum_{j=1}^{m} \Big[\sum_{k=j}^{\infty} (\Delta A_k) \int_{\pi/(j+1)}^{\pi} \Big| \sum_{i=0}^{k} \frac{\Delta a_i}{A_i}\, \overline{D}_k^{(r)}(x) \Big| + A_j \int_{\pi/(j+1)}^{\pi/j} \Big| \sum_{i=0}^{j-1} \frac{\Delta a_i}{A_i}\, \overline{D}_i^{(r)}(x) \Big|\, dx \Big].$$

Applying Lemma 3.3, we have

$$J_k = \int_{\pi/(j+1)}^{\pi} \Big| \sum_{i=0}^{k} \frac{|\Delta a_i|}{A_i}\, \overline{D}_i^{(r)}(x) \Big|\, dx = O_r((k+1)^{r+1}), \tag{3.7}$$

where O_r depends on r. Then by Lemma 4.3(iii),

$$\begin{aligned} \int_{\pi/(j+1)}^{\pi/j} \Big| \sum_{i=0}^{j-1} \frac{\Delta a_i}{A_i}\, \overline{D}_i^{(r)}(x) \Big|\, dx \\ &= O\Big(\int_{\pi/(j+1)}^{\pi/j} j^r \Big(\sum_{i=0}^{j-1} \frac{|\Delta a_i|}{A_i} \Big) \frac{dx}{x} \Big) + O\Big(\sum_{i=0}^{j-1} \frac{|\Delta a_i|}{A_i} \int_{\pi/(j+1)}^{\pi/j} \frac{dx}{x^{r+1}} \Big) \\ &= O(j^r) + O_r(j^r) = O_r(j^r), \end{aligned} \tag{3.8}$$

where O_r depends on r.

However, by (3.7), (3.8), Lemma 1.10, we have

$$\begin{aligned} I_2 &\le \sum_{k=1}^{\infty} (\Delta A_k)\, J_k + O_r\Big(\sum_{j=1}^{\infty} j^r A_j \Big) \\ &= O_r(1) \sum_{k=1}^{\infty} (\Delta A_k)(k+1)^{r+1} + O_r\Big(\sum_{j=1}^{\infty} j^r A_j \Big) = O_r\Big(\sum_{j=1}^{\infty} j^r A_j \Big). \end{aligned}$$

COROLLARY 3.4. *Let $r \in \mathbb{N} \cup \{0\}$ and let the coefficients of the series $g(x)$ satisfy the condition $\Im_r$. Then*

$$\int_0^{\pi} |g^{(r)}(x)|\, dx = O_r\Big(\sum_{n=1}^{\infty} n^r A_n \Big),$$

where O_r depends on r.

Proof. By the inequalities

$$\begin{aligned} \sum_{n=1}^{m} |a_n| n^{r-1} &\le \sum_{n=1}^{\infty} n^{r-1} \sum_{k=n}^{\infty} |\Delta a_k| \le \sum_{n=1}^{\infty} n^{r-1} \sum_{k=n}^{\infty} A_k \\ &= \sum_{k=1}^{\infty} A_k \sum_{n=1}^{k} n^{r-1} \le \sum_{k=1}^{\infty} k^r A_k, \end{aligned}$$

and by Theorem 3.10, we obtain

$$\int\limits_{\pi/(m+1)}^{\pi} |g^{(r)}(x)|\,dx = O_r\Big(\sum_{n=1}^{\infty} n^r A_n\Big).$$

Letting $m \to \infty$ completes the proof.

LEMMA 3.4. *Let $\{\alpha_j\}_{j=0}^{k}$ be a sequence of real numbers. Then*

$$\begin{aligned} V_k &= \int\limits_{\pi/(k+1)}^{\pi} \Bigg| \sum_{j=0}^{k} \alpha_j \frac{(j+1/2)^v \sin[(j+1/2)x + v\pi/2]}{(\sin(x/2))^{r+1-v}} \Bigg|\,dx \\ &= O_p\Big((k+1)^{1+\alpha}\Big[(k+1)^{p(r-\alpha)-1}\sum_{j=0}^{k}|\alpha_j|^p\Big]^{1/p}\Big) \end{aligned}$$

for $v \in \{0,1,\dots,r\}$, $\alpha \ge 0$ and $r \in \{0,1,\dots,[\alpha]\}$, where O_p depends only on p.

Proof. Applying Lemma 1.12, we get

$$\begin{aligned} V_k &= O_p\Big[(k+1)^{r+1}\Big(\frac{1}{k+1}\sum_{j=0}^{k}|\alpha_j|^p\Big)^{1/p}\Big] \\ &= O_p\Big((k+1)^{1+\alpha}\Big[(k+1)^{p(r-\alpha)-1}\sum_{j=0}^{k}|\alpha_j|^p\Big]^{1/p}\Big), \end{aligned}$$

where O_p depends only on p.

LEMMA 3.5. *Let $1 < p \le 2$, $\alpha \ge 0$, $r \in \{0,1,\dots,[\alpha]\}$, and let the sequence $\{a_n\}$ of real numbers belong to the class $S_{p\alpha r}$. Then*

$$\int\limits_{0}^{\pi} \Big| \sum_{j=0}^{k} \frac{\Delta a_j}{A_j}\, D_j^{(r)}(x)\Big|\,dx = O_p((k+1)^{\alpha+1}),$$

where O_p depends only on p.

Proof. We have

$$\int\limits_{0}^{\pi} \Big| \sum_{j=0}^{k} \frac{\Delta a_j}{A_j}\, D_j^{(r)}(x)\Big|\,dx = \int\limits_{0}^{\pi/(k+1)} + \int\limits_{\pi/(k+1)}^{\pi} = I_k + J_k.$$

Applying the inequality $D_n^{(r)}(x) = O(n^{r+1})$, we obtain

$$\begin{aligned} I_k &\le \gamma k^r \sum_{j=0}^{k} \frac{|\Delta a_j|}{A_j} \le \gamma (k+1)^{r+1}\Big(\frac{1}{k+1}\sum_{j=0}^{k}\frac{|\Delta a_j|^p}{A_j^p}\Big)^{1/p} \\ &= \gamma (k+1)^{1+\alpha}\Big[(k+1)^{p(r-\alpha)-1}\sum_{j=0}^{k}\frac{|\Delta a_j|^p}{A_j^p}\Big] = O((k+1)^{\alpha+1}). \end{aligned}$$

For the second integral we apply Lemmas 1.13 and 3.4 to get

$$J_k = O_p((k+1)^{1+\alpha}).$$

Thus, the inequality is satisfied.

LEMMA 3.6. *Let* $1 < p \le 2$, $\alpha \ge 0$, $r \in \{0, 1, \ldots, [\alpha]\}$ *and* $\{a_n\} \in S_{p\alpha r}$. *Then*

$$A_N \int_0^\pi \Big| \sum_{j=0}^N \frac{\Delta a_j}{A_j} D_j^{(r)}(x) \Big|\, dx = o(1), \quad N \to \infty.$$

Proof. Applying first Lemma 3.5, then Lemma 1.10, we obtain

$$A_N \int_0^\pi \Big| \sum_{j=0}^N \frac{\Delta a_j}{A_j}\, D_j^{(r)}(x) \Big|\, dx = O_p(A_N(N+1)^{1+\alpha}) = o(1), \quad N \to \infty.$$

THEOREM 3.11 ([53]). *Let* $1 < p \le 2$, $\alpha \ge 0$, $r \in \{0, 1, \ldots, [\alpha]\}$, *and let the coefficients of the series* (C) *belong to the class* $S_{p\alpha r}$. *Then the* r*th derivative of* (C) *is the Fourier series of some* $f^{(r)} \in L^1(0, \pi)$ *and*

$$\int_0^\pi |f^{(r)}(x)|\, dx \le M_{p,\alpha} \sum_{n=0}^\infty n^\alpha A_n,$$

where $M_{p,\alpha}$ *is a positive constant depending on* p *and* α.

Proof. Since

$$(\triangle) \qquad \begin{aligned} \sum_{k=1}^n k^r |\Delta a_k| &= \sum_{k=1}^{n-1} (\Delta A_k) \sum_{j=1}^k \frac{|\Delta a_j|}{A_j}\, j^r + A_n \sum_{j=1}^n \frac{|\Delta a_j|}{A_j}\, j^r \\ &\le \sum_{k=1}^{n-1} (\Delta A_k) k^{1+\alpha} \bigg[k^{p(r-\alpha)-1} \sum_{j=1}^k \frac{|\Delta a_j|^p}{A_j^p} \bigg]^{1/p} \\ &\quad + n^{1+\alpha} A_n \bigg[n^{p(r-\alpha)-1} \sum_{j=1}^n \frac{|\Delta a_j|^p}{A_j^p} \bigg] \\ &= O(1) \Big[\sum_{k=1}^{n-1} (\Delta A_k) k^{1+\alpha} + n^{1+\alpha} A_n \Big] = O\Big(\sum_{k=1}^n k^\alpha A_k \Big). \end{aligned}$$

Applying the same estimates of the proof of Theorem 3.8, we obtain

$$\sum_{k=1}^n |\Delta(k^r a_k)| \le O_r\Big(\sum_{k=1}^n k^{r-1} |a_{k+1}| \Big) + O\Big(\sum_{k=1}^n k^r |\Delta a_k| \Big).$$

But

$$\sum_{k=1}^n k^{r-1} |a_{k+1}| \le \sum_{k=1}^{n-1} |\Delta a_{k+1}| k^r + \sum_{k=n+1}^\infty k^r |\Delta a_k|$$

implies that

$$\begin{aligned} \sum_{k=1}^n |\Delta(k^r a_k)| &\le O_r\Big(\sum_{k=1}^{n-1} |\Delta a_{k+1}| k^r \Big) + O_r\Big(\sum_{k=n+1}^\infty k^r |\Delta a_k| \Big) + O\Big(\sum_{k=1}^n k^r |\Delta a_k| \Big) \\ &= O_r\Big(\sum_{k=1}^n k^\alpha A_k \Big) + o(1), \quad n \to \infty. \end{aligned}$$

Thus

$$\sum_{k=1}^{\infty}|\Delta(k^r a_k)| \le O_r\Big(\sum_{k=1}^{\infty} k^\alpha A_k\Big) < \infty, \quad \text{i.e.} \quad \lim_{n\to\infty} S_n^{(r)}(x) = f^{(r)}(x).$$

Since $f^{(r)}(x) = \sum_{k=0}^{\infty} \Delta a_k D_k^{(r)}(x)$, applying the Abel transformation and Lemmas 3.6 and 3.5 we obtain

$$\begin{aligned}\int_0^\pi |f^{(r)}(x)|\,dx &= \int_0^\pi \Big|\sum_{k=0}^{\infty}\Delta a_k D_k^{(r)}(x)\Big|\,dx \le \lim_{N\to\infty}\sum_{k=0}^{N-1}(\Delta A_k)\int_0^\pi\Big|\sum_{j=0}^{k}\frac{\Delta a_j}{A_j}\,D_j^{(r)}(x)\Big|\,dx\\
&= O_p(1)\lim_{N\to\infty}\sum_{k=0}^{N-1}(\Delta A_k)(k+1)^{\alpha+1}\\
&= O_p(1)\lim_{N\to\infty}\Big\{\sum_{k=0}^{N}[(k+1)^{\alpha+1}-k^{\alpha+1}]A_k-(N+1)^{\alpha+1}A_n\Big\}\\
&= O_{p,\alpha}\Big(\sum_{k=0}^{\infty}k^\alpha A_k\Big).\end{aligned}$$

Finally,

$$\int_0^\pi |f^{(r)}(x)|\,dx \le M_{p,\alpha}\sum_{n=0}^{\infty} n^\alpha A_n,$$

where $M_{p,\alpha}$ depends on p and α.

THEOREM 3.12 ([53]). *Let $1 < p \le 2$, $\alpha \ge 0$, $r \in \{0,1,\dots,[\alpha]\}$, and let the coefficients of the series* (S) *belong to the class $S_{p\alpha r}$. Then the rth derivative of* (S) *converges to a function and*

$$(*) \qquad \int_{\pi/(m+1)}^{\pi} |g^{(r)}(x)|\,dx \le M\sum_{j=1}^{m}|a_j| j^{r-1} + O_{p,\alpha,r}\Big(\sum_{k=1}^{\infty}k^\alpha A_k\Big),$$

for all $m\in\mathbb{N}$, where $0 < M = M(r) < \infty$ and $O_{p,\alpha,r}$ depends on p, r and α. Moreover, if $\sum_{n=1}^{\infty} n^{r-1}|a_n| < \infty$, then the rth derivative of (S) *is the Fourier series of some $g^{(r)} \in L^1(0,\pi)$ and*

$$(**) \qquad \int_0^\pi |g^{(r)}(x)|\,dx = O_r\Big(\sum_{j=1}^{\infty}|a_j| j^{r-1}\Big) + O_{p,\alpha,r}\Big(\sum_{j=1}^{\infty} j^\alpha A_j\Big),$$

Proof. We suppose that $a_0 = 0$ and $A_0 = \max(|a_1|, A_1)$. Applying Lemma 4.3(iii) and the inequality ($\triangle$) (proved in Th. 3.11):

$$\sum_{k=1}^{n} k^r|\Delta a_k| = O\Big(\sum_{k=1}^{n}k^\alpha A_k\Big),$$

we see that the series $\sum_{k=0}^{\infty}\Delta a_k \overline{D}_k^{(r)}(x)$ is uniformly convergent on any compact subset of $[\varepsilon,\pi]$, where $\varepsilon > 0$. Thus representation (3.6) implies that

$$g^{(r)}(x) = \sum_{k=0}^{\infty}\Delta a_k \overline{D}_k^{(r)}(x).$$

Then

$$\int_{\pi/(m+1)}^{\pi} |g^{(r)}(x)|\,dx \le \sum_{j=1}^{m} \int_{\pi/(j+1)}^{\pi/j} \Big|\sum_{k=0}^{j-1} \Delta a_k \overline{D}_k^{(r)}(x)\Big|\,dx + O\Big(\sum_{j=1}^{m} \int_{\pi/(j+1)}^{\pi/j} \Big|\sum_{k=j}^{\infty} \Delta a_k \overline{D}_k^{(r)}(x)\Big|\,dx\Big). \tag{3.9}$$

Applying the same technique as in the proof of Theorem 3.10, we obtain

$$\begin{aligned} T &= \sum_{j=1}^{m} \int_{\pi/(j+1)}^{\pi/j} \Big|\sum_{k=0}^{j-1} \Delta a_k \overline{D}_k^{(r)}(x)\Big|\,dx \\ &= O_r\Big(\sum_{j=1}^{m} |a_j| j^{r-1}\Big) + O\Big(\sum_{k=0}^{\infty} (k+1)^r |\Delta a_k|\Big). \end{aligned}$$

Then

$$\begin{aligned} \sum_{k=0}^{\infty} (k+1)^r |\Delta a_k| &\le |a_1| + 2^r \sum_{k=1}^{\infty} k^r |\Delta a_k| \\ &\le (1+2^r) \sum_{k=1}^{\infty} k^r |\Delta a_k| = O_r\Big(\sum_{k=1}^{\infty} k^{\alpha} A_k\Big), \end{aligned}$$

i.e.

$$T = O_r\Big(\sum_{j=1}^{m} |a_j| j^{r-1}\Big) + O_r\Big(\sum_{k=1}^{\infty} k^{\alpha} A_k\Big). \tag{3.10}$$

Let

$$U = \sum_{j=1}^{m} \int_{\pi/(j+1)}^{\pi/j} \Big|\sum_{k=j}^{\infty} \Delta a_k \overline{D}_k^{(r)}(x)\Big|\,dx.$$

Applying the Abel transformation, we have

$$U \le \sum_{j=1}^{m} \Big[\sum_{k=j}^{\infty} (\Delta A_k)\, J_k + A_j I_j\Big],$$

where

$$J_k = \int_{\pi/(j+1)}^{\pi} \Big|\sum_{i=0}^{k} \frac{\Delta a_i}{A_i}\, \overline{D}_i^{(r)}(x)\Big|\,dx, \qquad I_j = \int_{\pi/(j+1)}^{\pi/j} \Big|\sum_{i=0}^{j-1} \frac{\Delta a_i}{A_i}\, \overline{D}_i^{(r)}(x)\Big|\,dx.$$

Applying the Hölder–Hausdorff–Young technique (see the proof of Lemma 3.5), we obtain $J_k = O_{p,r}((k+1)^{\alpha+1})$, where $O_{p,r}$ depends on r and p. Then by Lemma 4(iii),

$$\begin{aligned} I_j &= O\Big(j^r \ln\Big(1+\frac{1}{j}\Big)\Big(\sum_{i=0}^{j-1} \frac{|\Delta a_i|}{A_i}\Big)\Big) + O\Big(\sum_{i=0}^{j-1} \frac{|\Delta a_i|}{A_i} \int_{\pi/(j+1)}^{\pi/j} \frac{dx}{x^{r+1}}\Big) \\ &= O\Big(j^{\alpha}\Big(j^{p(r-\alpha)-1} \sum_{i=0}^{j-1} \frac{|\Delta a_i|^p}{A_i^p}\Big)^{1/p}\Big) + O_r\Big(j^{r-1} \sum_{i=0}^{j-1} \frac{|\Delta a_i|}{A_i}\Big) \end{aligned}$$

$$= O(j^{\alpha}) + O_r\Bigg(j^{\alpha}\Bigg(j^{p(r-\alpha)-1}\sum_{i=0}^{j-1}\frac{|\Delta a_i|^p}{A_i^p}\Bigg)^{1/p}\Bigg)$$
$$= O(j^{\alpha}) + O_r(j^{\alpha}) = O_r(j^{\alpha}).$$

Thus

$$U \le O_{p,r}(1)\sum_{k=1}^{\infty}(k+1)^{\alpha+1}(\Delta A_k) + O_r(1)\sum_{j=1}^{\infty} j^{\alpha}A_j \tag{3.11}$$
$$= O_{p,\alpha,r}(1)\sum_{k=1}^{\infty} k^{\alpha}A_k + O_r(1)\sum_{j=1}^{\infty} j^{\alpha}A_j$$
$$= O_{p,\alpha,r}\Big(\sum_{k=1}^{\infty} k^{\alpha}A_k\Big),$$

since $n^{\alpha+1}A_n = o(1)$, $n\to\infty$.

Combining the inequalities (3.9), (3.10) and (3.11) yields the inequality $(*)$.

If $\sum_{n=1}^{\infty} n^{r-1}|a_n| < \infty$, then by letting $m\to\infty$ in inequality $(*)$, we find that the rth derivative of the series (S) is the Fourier series of some $g^{(r)} \in L^1(0,\pi)$ and the inequality $(**)$ is satisfied.

Now we consider the case $r=\alpha=0$. Since S_p and $S_p(\delta)$, $p>1$, are identical classes of Fourier coefficients we obtain the following corollaries.

Corollary 3.5 ([49], [62]). *Let $1<p\le 2$ and let the coefficients of the series* (C) *belong to the class $S_p(\delta)$. Then* (C) *is the Fourier series of some $f\in L^1(0,\pi)$ and*

$$\int_0^{\pi} |f(x)|\,dx \le M_p\Big(\sum_{n=0}^{\infty} A_n + \sum_{n=1}^{\infty} n\delta_n\Big),$$

where M_p is a positive constant depending only on p.

Corollary 3.6 ([57]). *Let $1<p\le 2$ and let the coefficients of the series* (S) *belong to the class $S_p(\delta)$. Then the series converges to a function $g(x)$ and*

$$\int_{\pi/(m+1)}^{\pi} |g(x)|\,dx \le \sum_{n=1}^{m}\frac{|a_n|}{n} + O_p\Big(\sum_{n=1}^{\infty} A_n + \sum_{n=1}^{\infty} n\delta_n\Big)$$

for $m\in\mathbb{N}$, where O_p depends only on p.

3.3. Necessary and sufficient conditions for L^1-convergence of the rth derivative of Fourier series. Van and Telyakovskiĭ [70] considered the following class of sequences. A null sequence $\{a_k\}$ belongs to the class $(BV)_r^{\sigma}$, $r\in\mathbb{N}\cup\{0\}$, $\sigma\ge 0$, if $\sum_{k=1}^{\infty} k^r|\Delta^{\sigma}a_k| < \infty$. In the same paper, they proved the following theorem.

Theorem 3.13 ([70]). *Let $\varrho,\sigma\ge 0$. Then for all $\gamma>\sigma$ we have the embedding*

$$(BV)_{\varrho}^{\sigma} \subset (BV)_{\varrho}^{\gamma}.$$

For $r=0$ and $\sigma=m\in\mathbb{N}\cup\{0\}$ we have the well known class $(BV)^m$.

Corollary 3.7. *Let $\{a_n\}\in (BV)^{\sigma}$ for some $\sigma\ge 0$ and $a_n\log n = o(1)$, $n\to\infty$. Then $\|S_n - f\| = o(1)$, $n\to\infty$ iff $\{a_n\}\in C$.*

Proof. Let m be an integer such that $m \ge \sigma$. Then by Theorem 3.13 (case $\varrho = 0$), we have $\{a_n\} \in (BV)^m$. Applying Theorem 1.16 completes the proof.

If $\sigma = 1$, we define $(BV)_r = (BV)_r^\sigma$.

Van and Telyakovskiĭ [70] by considering the complex form of trigonometric series

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k e^{ikx}$$

proved the following theorem.

THEOREM 3.14 ([70]). *If $r \in \mathbb{N} \cup \{0\}$, $\sigma \ge 0$ and $\{a_n\} \in (BV)_r^\sigma$, then the series* (C) *and* (S) *have continuous derivatives of order r on $(0, \pi]$.*

LEMMA 3.7 ([32]). $\|D_n^{(r)}\|_1 = (4/\pi)n^r \log n + O(n^r)$ *for all $r \in \mathbb{N} \cup \{0\}$.*

Next we shall give necessary and sufficient conditions for L^1-convergence of the rth derivative of the series (C).

THEOREM 3.15 ([54]). *Let $r \in \mathbb{N} \cup \{0\}$, $\{a_n\} \in (BV)_r$, and $a_n n^r \log n = o(1)$, $n \to \infty$. Then $\|S_n^{(r)} - f^{(r)}\| = o(1)$, $n \to \infty$ iff $\{a_n\} \in C_r$.*

Proof. Since $\{a_n\} \in (BV)_r$, the series $\sum_{k=0}^{\infty} \Delta a_k D_k^{(r)}(x)$ is uniformly convergent on any segment $[\xi, \pi]$ where $\xi > 0$. Thus, $f^{(r)}(x) = \sum_{k=0}^{\infty} \Delta a_k D_k^{(r)}(x)$.

For the "if" part let $\varepsilon > 0$. Then there exists $\delta > 0$ such that

$$\int_0^\delta \Big| \sum_{k=n}^{\infty} \Delta a_k D_k^{(r)}(x) \Big| < \frac{\varepsilon}{3} \quad \text{for all } n.$$

Then

$$\begin{aligned}
\int_0^\pi |f^{(r)}(x) - S_n^{(r)}(x)|\,dx &= \int_0^\pi \Big| \sum_{k=n}^{\infty} \Delta a_k D_k^{(r)}(x) - a_{n+1} D_n^{(r)}(x) \Big|\,dx \\
&\le \int_0^\pi \Big| \sum_{k=n}^{\infty} \Delta a_k D_k^{(r)}(x) \Big|\,dx + |a_{n+1}| \int_0^\pi |D_n^{(r)}(x)|\,dx \\
&= \Big(\int_0^\delta + \int_\delta^\pi \Big) \Big| \sum_{k=n}^{\infty} \Delta a_k D_k^{(r)}(x) \Big|\,dx + |a_{n+1}|\,\|D_n^{(r)}\| \\
&< \frac{\varepsilon}{3} + \int_\delta^\pi \Big| \sum_{k=n}^{\infty} \Delta a_k D_k^{(r)}(x) \Big|\,dx + \frac{\varepsilon}{3}.
\end{aligned}$$

Applying the estimate for the rth derivative of the Dirichlet kernel (see [32]), we obtain

$$\int_\delta^\pi \Big| \sum_{k=n}^{\infty} \Delta a_k D_k^{(r)}(x) \Big|\,dx = O\Big(\sum_{k=n}^{\infty} k^r |\Delta a_k| \Big).$$

Hence,

$$\int_\delta^\pi \Big| \sum_{k=n}^{\infty} \Delta a_k D_k^{(r)}(x) \Big|\,dx < \frac{\varepsilon}{3}.$$

Thus for sufficiently large n, $\|S_n^{(r)} - f^{(r)}\| < \varepsilon$.

For the "only if" part let $\varepsilon > 0$. Then there exists an integer N such that

$$\int_0^\pi |f^{(r)} - S_n^{(r)}|\, dx < \frac{\varepsilon}{4} \quad \text{if } n \geq N.$$

That is,

$$\int_0^\pi \Big| \sum_{k=n}^\infty \Delta a_k D_k^{(r)}(x) - a_{n+1} D_n^{(r)}(x) \Big|\, dx < \frac{\varepsilon}{4} \quad \text{if } n \geq N.$$

Since $a_n n^r \log n \to 0$, by Lemma 3.7 there exists an integer M such that

$$\int_0^\pi \Big| \sum_{k=n}^\infty \Delta a_k D_k^{(r)}(x) \Big|\, dx < \frac{\varepsilon}{2} \quad \text{if } n \geq M.$$

Now if $\sum_{k=0}^M k^r |\Delta a_k| = 0$, then for $n < M$,

$$\int_0^\pi \Big| \sum_{k=n}^\infty \Delta a_k D_k^{(r)}(x) \Big|\, dx = \int_0^\pi \Big| \sum_{k=M+1}^\infty \Delta a_k D_k^{(r)}(x) \Big|\, dx < \frac{\varepsilon}{2} < \varepsilon.$$

If $\sum_{k=0}^M k^r |\Delta a_k| \neq 0$, let

$$\delta = \frac{\varepsilon}{2M \sum_{k=0}^M k^r |\Delta a_k|}.$$

For $n \geq M$, we have

$$\int_0^\delta \Big| \sum_{k=n+1}^\infty \Delta a_k D_k^{(r)}(x) \Big|\, dx \leq \int_0^\pi \Big| \sum_{k=n+1}^\infty \Delta a_k D_k^{(r)}(x) \Big|\, dx < \frac{\varepsilon}{2} < \varepsilon.$$

For $0 \leq n < M$, we get

$$\begin{aligned}
\int_0^\delta \Big| \sum_{k=n+1}^\infty \Delta a_k D_k^{(r)}(x) \Big|\, dx &\leq \int_0^\delta \Big| \sum_{k=n+1}^M \Delta a_k D_k^{(r)}(x) \Big|\, dx + \int_0^\delta \Big| \sum_{k=M+1}^\infty \Delta a_k D_k^{(r)}(x) \Big|\, dx \\
&\leq \int_0^\delta \Big(\sum_{k=n+1}^M k^{r+1} |\Delta a_k| \Big)\, dx + \int_0^\pi \Big| \sum_{k=M+1}^\infty \Delta a_k D_k^{(r)}(x) \Big|\, dx \\
&< \delta M \sum_{k=0}^M k^r |\Delta a_k| + \frac{\varepsilon}{2} = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Finally, $\{a_n\} \in C_r$.

This is an extension theorem of the Garrett–Stanojević Theorem 1.12. Applying Theorem 1.29 and this theorem we obtain

Corollary 3.8. *Let $r \in \mathbb{N} \cup \{0\}$ and $\{a_n\} \in \Im_r$. Then $\|S_n^{(r)} - f^{(r)}\| = o(1)$, $n \to \infty$ iff $a_n n^r \log n = o(1)$, $n \to \infty$.*

On the other hand by Theorem 1.31, we get $S_{pr} \subset (BV)_r \cap C_r$ for any $1 < p \leq 2$ and $r \in \mathbb{N} \cup \{0\}$. Again, applying the Theorem 3.15, we obtain:

Corollary 3.9. *Let $1 < p \leq 2$ and $r \in \mathbb{N} \cup \{0\}$. If $\{a_n\} \in S_{pr}$, then $\|S_n^{(r)} - f^{(r)}\| = o(1)$, $n \to \infty$ iff $a_n n^r \log n = o(1)$, $n \to \infty$.*

Now using Lemmas 3.5, 3.6 and applying the same technique as in the proof of Theorem 1.31, we obtain

THEOREM 3.16 ([64]). *For any $1 < p \le 2$, $\alpha \ge 0$ and $r \in \{0, 1, \dots, [\alpha]\}$ we have the embeddings*

$$S_{p\alpha r} \subset (BV)_r \cap C_r \subset BV \cap C_r.$$

Combining this with Theorem 3.15 we obtain

THEOREM 3.17. *Let $1 < p \le 2$, $\alpha \ge 0$ and $r \in \{0, 1, \dots, [\alpha]\}$. If $\{a_n\} \in S_{p\alpha r}$, then $\|S_n^{(r)} - f^{(r)}\| = o(1)$, $n \to \infty$ iff $a_n n^r \log n = o(1)$, $n \to \infty$.*

REMARK 3.4. This theorem was obtained by Sheng [32], but we have given a new proof.

Denote by I_m the dyadic interval $[2^{m-1}, 2^m)$ for $m \ge 1$. A null sequence $\{a_n\}$ belongs to the class F_{pr}, $p > 1$, $r \in \mathbb{N} \cup \{0\}$, if

$$\sum_{m=1}^{\infty} 2^{m(1/q+r)} \Big(\sum_{k \in I_m} |\Delta a_k|^p \Big)^{1/p} < \infty, \quad \text{where} \quad \frac{1}{p} + \frac{1}{q} = 1.$$

It is obvious that for $r = 0$, we obtain the Fomin class F_p.

THEOREM 3.18. *For any $p > 1$, $\alpha \ge 0$ and $r \in \{0, 1, \dots, [\alpha]\}$ we have the embedding $S_{p\alpha r} \subset F_{pr}$.*

Proof. From the condition

$$\frac{1}{n^{p(\alpha-r)+1}} \sum_{k=1}^{n} \frac{|\Delta a_k|^p}{A_k^p} = O(1)$$

and the monotonicity of $\{A_k\}$, we obtain

$$\begin{aligned} \Big(\sum_{k \in I_m} |\Delta a_k|^p \Big)^{1/p} &\le 2^{(m-1)/p} \Big(\frac{1}{2^{m-1}} \sum_{k \in I_m} \frac{|\Delta a_k|^p}{A_k^p} \Big)^{1/p} A_{2^{m-1}} \\ &\le K \cdot 2^{m/p} 2^{m(\alpha-r)} A_{2^{m-1}}, \end{aligned}$$

where K is an absolute constant. Hence,

$$\begin{aligned} \sum_{m=1}^{\infty} 2^{m(1/q+r)} \Big(\sum_{k \in I_m} |\Delta a_k|^p \Big)^{1/p} &\le K \sum_{m=1}^{\infty} 2^{m(1/p+1/q)} 2^{m\alpha} A_{2^{m-1}} \\ &= K \sum_{m=1}^{\infty} 2^{m(1+\alpha)} A_{2^{m-1}} \\ &= K \cdot 2^{1+\alpha} \sum_{m=1}^{\infty} 2^{(m-1)(1+\alpha)} A_{2^{m-1}} < \infty. \end{aligned}$$

THEOREM 3.19. *Let $1 < p \le 2$ and $r \in \mathbb{N} \cup \{0\}$. If $\{a_n\} \in F_{pr}$, then $\|S_n^{(r)} - f^{(r)}\| = o(1)$ iff $a_n n^r \log n = o(1)$, $n \to \infty$.*

Proof. Using the Abel transformation we obtain

$$S_n^{(r)}(x) = \sum_{k=0}^{n} \Delta a_k D_k^{(r)}(x) + a_{n+1} D_n^{(r)}(x).$$

Since

$$\begin{aligned}\sum_{k=1}^{\infty}|\varDelta a_k D_k^{(r)}(x)| &\le \lim_{n\to\infty}\frac{M}{x}\sum_{k=1}^{n}k^r|\varDelta a_k|\\ &= \lim_{n\to\infty}\frac{M}{x}\sum_{j=1}^{m}\Big(\sum_{k=2^{j-1}}^{2^j-1}k^r|\varDelta a_k|\Big)\quad (\text{for } n=2^m-1 \text{ and } M>0)\\ &\le \lim_{n\to\infty}\frac{M}{x}\sum_{j=1}^{m}2^{j(1/q+r)}\Big(\sum_{k\in I_j}|\varDelta a_k|^p\Big)^{1/p}<\infty\end{aligned}$$

and

$$|a_{n+1}D_n^{(r)}(x)| \le M|a_{n+1}|\,\frac{n^r}{x} \le \frac{M}{x}\sum_{k=n+1}^{\infty}k^r|\varDelta a_k| \to 0,\quad n\to\infty,$$

we get

$$\lim_{n\to\infty}S_n^{(r)}(x) = f^{(r)}(x) = \sum_{k=0}^{\infty}\varDelta a_k D_k^{(r)}(x).$$

This implies that

$$\|f^{(r)} - g_n^{(r)}\| = \Big\|\sum_{k=n+1}^{\infty}\varDelta a_k D_k^{(r)}(x)\Big\|.$$

By Lemma 1.9, we obtain

$$\|f^{(r)} - g_n^{(r)}\| \le A_p\sum_{m=j}^{\infty}2^{m(1/q+r)}\Big(\sum_{k\in I_j}|\varDelta a_k|^p\Big)^{1/p} = o(1),\quad n\to\infty,$$

by the hypothesis of the theorem; here $j=j(n)$ denotes the integer for which $2^{j-1}\le n\le 2^j$. Since $g_n^{(r)}$ is a polynomial, it follows that $f^{(r)}\in L^1$.

Since

$$|\,\|f^{(r)} - S_n^{(r)}\| - \|a_{n+1}D_n^{(r)}\|\,| \le \|f^{(r)} - g_n^{(r)}\| = o(1),\quad n\to\infty,$$

by Lemma 3.7, we obtain $\|S_n^{(r)} - f^{(r)}\| = o(1)$, $n\to\infty$ iff $a_{n+1}n^r\log n = o(1)$, $n\to\infty$.

Similarly, we can get an analogous theorem for sine series (S).

THEOREM 3.20. *Let $1<p\le 2$, $r\in\mathbb{N}\cup\{0\}$ and $\{a_n\}\in F_{pr}$. If $\sum_{n=1}^{\infty}n^{r-1}|a_n|<\infty$ then the rth derivative of the series* (S) *is the Fourier series of some $g^{(r)}\in L^1$ and $\|\widetilde{S}_n^{(r)} - g^{(r)}\| = o(1)$, $n\to\infty$ iff $a_{n+1}n^r\log n = o(1)$, $n\to\infty$.*

IV. Convergence and integrability of the rth derivative of complex trigonometric series

4.1. On a theorem of Bhatia and Ram. Let $\{c_k : k=0,\pm1,\pm2,\dots\}$ be a sequence of complex numbers and denote the partial sums of the complex trigonometric series $\sum_{k=-\infty}^{\infty}c_k e^{ikt}$ by

$$S_n(c,t) = \sum_{k=-n}^{n}c_k e^{ikt},\quad t\in T=\mathbb{R}/2\pi\mathbb{Z}. \tag{4.1}$$

If the trigonometric series is the Fourier series of some $f \in L^1$, we shall write $c_n = \widehat{f}(n)$ for all n and $S_n(c,t) = S_n(f,t) = S_n(f)$.

Bhatia and Ram [3] introduced the following class $\Re^*$ of complex sequences: a null sequence $\{c_n\}$ of complex numbers belongs to $\Re^*$ if

$$\sum_{k=1}^{\infty} \left| \Delta\left(\frac{c_{-k} - c_k}{k}\right)\right| k \log k < \infty, \quad \sum_{k=1}^{\infty} k^2 \left| \Delta^2\left(\frac{c_k}{k}\right)\right| < \infty.$$

Let

$$E_n(t) = \frac{1}{2} + \sum_{k=1}^{n} e^{ikt}, \quad E_{-n}(t) = \frac{1}{2} + \sum_{k=1}^{n} e^{-ikt}.$$

Then the rth derivatives $D_n^{(r)}(t)$ and $\widetilde{D}_n^{(r)}(t)$ can be written as

$$\begin{aligned} 2D_n^{(r)}(t) &= E_n^{(r)}(t) + E_{-n}^{(r)}(t), \\ 2i\widetilde{D}_n^{(r)}(t) &= E_n^{(r)}(t) - E_{-n}^{(r)}(t), \end{aligned} \tag{4.2}$$

where $E_n^{(r)}(t)$ denotes the rth derivative of $E_n(t)$.

Bhatia and Ram [3] introduced the following modified sums:

$$g_n(c,t) = S_n(c,t) + \frac{i}{n+1}\left[c_{n+1}E_n'(t) - c_{-(n+1)}E_{-n}'(t)\right]$$

and proved the following result.

Theorem 4.1 ([3]). *Let $\{c_n\} \in \Re^*$. Then there exists $f(t)$ such that*

(i) $\lim_{n\to\infty} g_n(c,t) = f(t)$ *for all* $0 < |t| \le \pi$.
(ii) $f(t) \in L^1(T)$ *and* $\|g_n(c,t) - f(t)\|_1 = o(1)$, $n \to \infty$.
(iii) $\|S_n(f,t) - f(t)\|_1 = o(1)$ *iff* $\widehat{f}(n) \log |n| = o(1)$, $|n| \to \infty$.

Now we define a new class $\Re^*(r)$, $r \in \mathbb{N} \cup \{0\}$, of complex sequences as follows: a null sequence $\{c_k\}$ of complex numbers belongs to the class $\Re^*(r)$ if

$$\sum_{k=1}^{\infty} \left| \Delta\left(\frac{c_{-k} - c_k}{k}\right)\right| k^{r+1} \log k < \infty, \quad \sum_{k=1}^{\infty} k^{r+2} \left| \Delta^2\left(\frac{c_k}{k}\right)\right| < \infty.$$

We write $\Re^*(0) = \Re^*$.

Č. V. Stanojević and V. B. Stanojević [38] introduced the following modified complex trigonometric sums:

$$U_n(c,t) = S_n(c,t) - (c_n E_n(t) + c_{-n} E_{-n}(t)).$$

The complex form of the rth derivative of these sums, obtained by Sheng [32], is

$$U_n^{(r)}(c,t) = S_n^{(r)}(c,t) - (c_n E_n^{(r)}(t) + c_{-n} E_{-n}^{(r)}(t)).$$

Ram and Kumari [30] introduced another set of modified cosine and sine sums

$$f_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n}\sum_{j=k}^{n} \Delta\left(\frac{a_j}{j}\right) k \cos kx,$$

$$h_n(x) = \sum_{k=1}^{n}\sum_{j=k}^{n} \Delta\left(\frac{a_j}{j}\right) k \sin kx.$$

The complex form of the rth derivative of these modified sums is

$$G_n^{(r)}(c,t) = S_n^{(r)}(c,t) + \frac{i}{n+1}\,[c_{n+1}E_n^{(r+1)}(t) - c_{-(n+1)}E_{-n}^{(r+1)}(t)].$$

Remark 4.1. If $|n|^r c_n \to 0$, $|n| \to \infty$, then $\|G_n^{(r)} - U_n^{(r)}\| \to 0$. Observe that by partial summation, we have

$$E_n^{(r+1)}(t) = -i\sum_{k=1}^{n} E_k^{(r)}(t) + i(n+1)E_n^{(r)}(t)$$

and similarly for $E_{-n}^{(r+1)}(t)$. Then by the formulae

$$U_{n+1}^{(r)}(c,t) = S_n^{(r)}(c,t) - c_{n+1}E_n^{(r)}(t) - c_{-(n+1)}E_{-n}^{(r)}(t)$$

we obtain

$$U_{n+1}^{(r)}(c,t) - G_n^{(r)}(c,t) = -c_{n+1}\,\frac{1}{n+1}\sum_{k=1}^{n} E_k^{(r)}(t) - c_{-(n+1)}\,\frac{1}{n+1}\sum_{k=1}^{n} E_{-k}^{(r)}(t).$$

Then by the well known properties of Fejér kernels, it follows that

$$\|G_n^{(r)} - U_n^{(r)}\| \to 0, \quad n \to \infty.$$

Using the modified complex sums $G_n^{(r)}$ we shall prove the following theorem:

Theorem 4.2. *Let* $r \in \mathbb{N} \cup \{0\}$ *and* $\{c_n\} \in \Re^*(r)$. *Then*

(i) $\lim_{n\to\infty} G_n^{(r)}(c,t) = f^{(r)}(t)$ *for all* $0 < |t| \le \pi$.
(ii) $f^{(r)} \in L^1(T)$ *and* $\|G_n^{(r)}(c,t) - f^{(r)}(t)\|_1 = o(1)$, $n \to \infty$.
(iii) $\|S_n^{(r)}(f,t) - f^{(r)}(t)\|_1 = o(1)$, $n \to \infty$ *iff* $|n|^r \widehat{f}(n)\log|n| = o(1)$, $|n| \to \infty$.

Lemma 4.1. $\|\widetilde{D}_n^{(r)}\|_1 = O(n^r \log n)$ *for all* $r \in \mathbb{N} \cup \{0\}$.

Lemma 4.2 ([32]). *For each nonnegative integer* n *and each complex sequence* $\{c_n\}$, $\|c_n E_n^{(r)} + c_{-n}E_{-n}^{(r)}\|_1 = o(1)$, $|n| \to \infty$ *iff* $|n|^r c_n \log|n| = o(1)$, $|n| \to \infty$.

We note that for $r = 0$ this lemma was obtained by Bray and Stanojević in [9].

Lemma 4.3 ([55]). *Let* r *be a nonnegative integer. Then for all* $0 < |t| \le \pi$ *and all* $n \ge 1$ *the following estimates hold*:

(i) $|E_{-n}^{(r)}(t)| \le 4n^r\pi/|t|$.
(ii) $|\widetilde{D}_n^{(r)}(t)| \le 4n^r\pi/|t|$.
(iii) $|\overline{D}_n^{(r)}(t)| \le 4n^r\pi/|t| + O(1/|t|^{r+1})$.

Proof. (i) The case $r = 0$ is trivial. Indeed, since $E_n(t) = D_n(t) + i\widetilde{D}_n(t)$, we have

$$|E_n(t)| \le |D_n(t)| + |\widetilde{D}_n(t)| \le \frac{\pi}{2|t|} + \frac{\pi}{|t|} = \frac{3\pi}{2|t|} < \frac{4\pi}{|t|},$$
$$|E_{-n}(t)| = |E_n(-t)| < \frac{4\pi}{|t|}.$$

Let $r \ge 1$. Applying the Abel transformation, we have

$$E_n^{(r)}(t) = i^r\sum_{k=1}^{n} k^r e^{ikt} = i^r\left[\sum_{k=1}^{n-1} \Delta(k^r)\left(E_k(t) - \frac{1}{2}\right) + n^r\left(E_n(t) - \frac{1}{2}\right)\right],$$

and so

$$
(4.3) \qquad |E_n^{(r)}(t)| \le \sum_{k=1}^{n-1}[(k+1)^r - k^r]\left(\frac{1}{2} + |E_k(t)|\right) + n^r\left(|E_n(t)| + \frac{1}{2}\right)
$$

$$
\le \left(\frac{\pi}{2|t|} + \frac{3\pi}{2|t|}\right)\left\{\sum_{k=1}^{n-1}[(k+1)^r - k^r] + n^r\right\} = \frac{4\pi n^r}{|t|}.
$$

Since $E_{-n}^{(r)}(t) = E_n^{(r)}(-t)$, we obtain $|E_{-n}^{(r)}(t)| \le 4\pi n^r/|t|$.

(ii) Applying (i) and (4.2) we obtain

$$
|\widetilde{D}_n^{(r)}(t)| = |i\widetilde{D}_n^{(r)}(t)| \le \frac{1}{2}|E_n^{(r)}(t)| + \frac{1}{2}|E_{-n}^{(r)}(t)| \le \frac{4n^r\pi}{|t|}.
$$

(iii) We note that $|(\operatorname{ctg}(t/2))^{(r)}| = O(1/|t|^{r+1})$. Applying (ii), we obtain

$$
|\overline{D}_n^{(r)}(t)| \le |\widetilde{D}_n^{(r)}(t)| + \frac{1}{2}\left|\left(\operatorname{ctg}\frac{t}{2}\right)^{(r)}\right| \le \frac{4n^r\pi}{|t|} + O\left(\frac{1}{|t|^{r+1}}\right).
$$

Lemma 4.4 ([3]). *$\|\widetilde{K}_n'(t)\|_1 = O(n)$.*

Lemma 4.5. *$\|\widetilde{K}_n^{(r)}\|_1 = O(n^r)$ for all $r \in \mathbb{N} \cup \{0\}$.*

Proof. Since

$$
\widetilde{K}_n(x) = \sum_{k=1}^{n} \frac{n+1-k}{n+1}\sin kx,
$$

we see that

$$
T_n(x) = \widetilde{K}_n'(x) = \sum_{k=1}^{n} \frac{k(n+1-k)}{n+1}\cos kx
$$

is a cosine trigonometric polynomial of order n. Applying first the Bernstein inequality, and then Lemma 4.4, yields

$$
\|\widetilde{K}_n^{(r)}\|_1 = \|T_n^{(r-1)}\|_1 \le n^{r-1}\|T_n\|_1 = O(n^r).
$$

Proof of Theorem 4.2. Applying the Abel transformation, we have

$$
G_n^{(r)}(c,t) = S_n^{(r)}(c,t) + \frac{i}{n+1}[c_{n+1}E_n^{(r+1)}(t) - c_{-(n+1)}E_{-n}^{(r+1)}(t)]
$$

$$
= 2\sum_{k=1}^{n} \Delta\left(\frac{c_k}{k}\right)\widetilde{D}_k^{(r+1)}(t) + \sum_{k=1}^{n}\Delta\left(\frac{c_{-k}-c_k}{k}\right)iE_{-k}^{(r+1)}(t).
$$

By Lemma 4.3, we get

$$
\sum_{k=1}^{\infty}\left|\Delta\left(\frac{c_k}{k}\right)\widetilde{D}_k^{(r+1)}\right| \le \frac{4\pi}{|t|}\sum_{k=1}^{\infty}k^{r+1}\left|\Delta\left(\frac{c_k}{k}\right)\right| \le \frac{4\pi}{|t|}\left\{\sum_{k=1}^{\infty}\sum_{j=k}^{\infty}k^{r+1}\left|\Delta^2\left(\frac{c_j}{j}\right)\right|\right\}
$$

$$
= \frac{4\pi}{|t|}\left\{\sum_{j=1}^{\infty}\left(\sum_{k=1}^{j}k^{r+1}\right)\left|\Delta^2\left(\frac{c_j}{j}\right)\right|\right\}
$$

$$
= O\left(\frac{1}{|t|}\sum_{j=1}^{\infty}j^{r+2}\left|\Delta^2\left(\frac{c_j}{j}\right)\right|\right) < \infty
$$

and

$$\sum_{k=3}^{\infty}\left|\Delta\left(\frac{c_{-k}-c_k}{k}\right)E_{-k}^{(r+1)}(t)\right| \le \frac{4\pi}{|t|}\left\{\sum_{k=3}^{\infty}k^{r+1}\left|\Delta\left(\frac{c_{-k}-c_k}{k}\right)\right|\right\}$$
$$= O\left(\frac{1}{|t|}\sum_{k=3}^{\infty}k^{r+1}\log k\left|\Delta\left(\frac{c_{-k}-c_k}{k}\right)\right|\right) < \infty.$$

Consequently,

$$f^{(r)}(t) = 2\sum_{k=1}^{\infty}\Delta\left(\frac{c_k}{k}\right)\widetilde{D}_k^{(r+1)}(t) + \sum_{k=1}^{\infty}\Delta\left(\frac{c_{-k}-c_k}{k}\right)iE_{-k}^{(r+1)}(t)$$

exists and thus (i) follows.

Now, for $t \neq 0$, we have

$$f^{(r)}(t) - G_n^{(r)}(c,t) = 2\sum_{k=n+1}^{\infty}\Delta\left(\frac{c_k}{k}\right)\widetilde{D}_k^{(r+1)}(t) + i\sum_{k=n+1}^{\infty}\Delta\left(\frac{c_{-k}-c_k}{k}\right)E_{-k}^{(r+1)}(t)$$
$$= 2\sum_{k=n+1}^{\infty}(k+1)\Delta^2\left(\frac{c_k}{k}\right)\widetilde{K}_k^{(r+1)}(t) - 2(n+1)\Delta\left(\frac{c_{n+1}}{n+1}\right)\widetilde{K}_{n+1}^{(r+1)}(t)$$
$$+ i\sum_{k=n+1}^{\infty}\Delta\left(\frac{c_{-k}-c_k}{k}\right)E_{-k}^{(r+1)}(t).$$

Then

$$\|f^{(r)}(t) - G_n^{(r)}(c,t)\|_1 \le 2\sum_{k=n+1}^{\infty}(k+1)\left|\Delta^2\left(\frac{c_k}{k}\right)\right|\int_{-\pi}^{\pi}|\widetilde{K}_k^{(r+1)}(t)|\,dt$$
$$+ 2(n+1)\left|\Delta\left(\frac{c_{n+1}}{n+1}\right)\right|\int_{-\pi}^{\pi}|\widetilde{K}_{n+1}^{(r+1)}(t)|\,dt + \sum_{k=n+1}^{\infty}\left|\Delta\left(\frac{c_{-k}-c_k}{k}\right)\right|\int_{-\pi}^{\pi}|E_{-k}^{(r+1)}(t)|\,dt.$$

Applying Lemmas 4.5, 3,7 and 4.1, we have

$$\|f^{(r)}(t) - G_n^{(r)}(c,t)\|_1 = O\left(\sum_{k=n+1}^{\infty}(k+1)^{r+2}\left|\Delta^2\left(\frac{c_k}{k}\right)\right|\right)$$
$$+ O\left((n+1)^{r+2}\left|\Delta\left(\frac{c_{n+1}}{n+1}\right)\right|\right) + O\left(\sum_{k=n+1}^{\infty}\left|\Delta\left(\frac{c_{-k}-c_k}{k}\right)\right|k^{r+1}\log k\right).$$

But

$$\left|\Delta\left(\frac{c_{n+1}}{n+1}\right)\right| = \left|\sum_{k=n+1}^{\infty}\Delta^2\left(\frac{c_k}{k}\right)\right| \le \sum_{k=n+1}^{\infty}\frac{k^{r+2}}{k^{r+2}}\left|\Delta^2\left(\frac{c_k}{k}\right)\right|$$
$$\le \frac{1}{(n+1)^{r+2}}\sum_{k=n+1}^{\infty}k^{r+2}\left|\Delta^2\left(\frac{c_k}{k}\right)\right| = o\left(\frac{1}{(n+1)^{r+2}}\right), \quad n \to \infty.$$

Hence, $\|f^{(r)}(t) - G_n^{(r)}(c,t)\|_1 = o(1)$, $n \to \infty$ by the hypothesis of the theorem. Since $G_n^{(r)}(c,t)$ is a polynomial, it follows that $f^{(r)} \in L^1(T)$.

The proof of (iii) follows from the estimate

$$\left| \|f^{(r)} - S_n^{(r)}(f)\|_1 - \left\| \frac{i}{n+1}(\widehat{f}(n+1)E_n^{(r+1)} - \widehat{f}(-(n+1))E_{-n}^{(r+1)}) \right\|_1 \right| \le \|f^{(r)} - G_n^{(r)}(c,t)\|_1 = o(1), \quad n \to \infty,$$

and from Lemma 4.2.

Considering the sums $U_n^{(r)}$ instead of $G_n^{(r)}$ and in view of Remark 4.1, statement (ii) in Theorem 4.2 can be replaced by:

(ii′) $f^{(r)} \in L^1(T)$ and $\|U_n^{(r)}(c,t) - f^{(r)}(t)\|_1 = o(1)$, $n \to \infty$.

Thus we have the following result:

THEOREM 4.3. *Under the hypothesis of Theorem* 4.2, *statements* (i), (ii′) *and* (iii) *hold.*

4.2. On a theorem of P. L. Ul'yanov. The function $\varphi(x)$ is called *A-integrable* on $[a,b]$ if

a) $mE\{|\varphi(x)| > n\} = o(1/n)$,
b) the limit $\lim_{n\to\infty} \int_a^b [\varphi(x)]_n \, dx = I$ exists, where

$$[\varphi(x)]_n = \begin{cases} n & \text{if } \varphi(x) > n, \\ \varphi(x) & \text{if } |\varphi(x)| \le n, \\ -n & \text{if } \varphi(x) < -n. \end{cases}$$

The number I is called the *A-integral* of φ on $[a,b]$.

As an application of A-integrals, P. L. Ul'yanov [68] obtained an interesting result concerning the integrability of $|f|^p$ and $|g|^p$, for any $0 < p < 1$, where

$$f(x) = \sum_{k=1}^{\infty} a_k \cos kx, \qquad g(x) = \sum_{k=1}^{\infty} a_k \sin kx,$$

and $\{a_n\}$ is a null sequence of bounded variation:

THEOREM 4.4 ([68]). *Let* $\{a_n\} \in BV$. *Then for any* $0 < p < 1$,

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} |f(x) - S_n(x)|^p \, dx = 0, \qquad \lim_{n\to\infty} \int_{-\pi}^{\pi} |g(x) - \widetilde{S}_n(x)|^p \, dx = 0. \tag{4.4}$$

It is obvious that the assertion of this theorem holds when the coefficients $\{a_n\}$ belong to the classes S, F_q, S_q, $S_{q\alpha}$ (case $r = 0$) for some $q > 1$, $\alpha \ge 0$.

Next, we shall define a new L^p-integrability class $(0 < p < 1)$ as follows. A null sequence $\{a_n\}$ belongs to the class $H_{q\alpha}$, $0 < q \le 1$, $\alpha \ge 0$, if there exists a decreasing sequence $\{A_k\}$ such that

$$\sum_{k=1}^{\infty} k^\alpha A_k < \infty, \qquad \frac{1}{n^{q\alpha+q}} \sum_{k=1}^{n} \frac{|\Delta a_k|^q}{A_k^q} = O(1).$$

THEOREM 4.5. *For any* $0 < q \le 1$ *and any* $\alpha \ge 0$ *the class* $H_{q\alpha}$ *is a subclass of* BV.

Proof. Applying the Abel transformation and the well known inequality

$$\Big(\sum b_i\Big)^q \le \sum b_i^q \quad \text{for } b_i \ge 0 \text{ and } 0 < q \le 1, \tag{4.5}$$

we obtain

$$\begin{aligned}\sum_{k=1}^{n}|\Delta a_k| &= \sum_{k=1}^{n-1}k^{\alpha+1}(\Delta A_k)\left(\frac{1}{k^{\alpha+1}}\sum_{j=1}^{k}\frac{|\Delta a_j|}{A_j}\right)+n^{\alpha+1}A_n\left(\frac{1}{n^{\alpha+1}}\sum_{j=1}^{n}\frac{|\Delta a_j|}{A_j}\right)\\ &\le \sum_{k=1}^{n-1}k^{\alpha+1}(\Delta A_k)\left(\frac{1}{k^{q\alpha+q}}\sum_{j=1}^{k}\frac{|\Delta a_j|^q}{A_j^q}\right)^{1/q}+n^{\alpha+1}A_n\left(\frac{1}{n^{q\alpha+q}}\sum_{j=1}^{n}\frac{|\Delta a_j|^q}{A_j^q}\right)^{1/q}\\ &= O_q(1)\left[\sum_{k=1}^{n-1}k^{\alpha+1}(\Delta A_k)+n^{\alpha+1}A_n\right].\end{aligned}$$

Now, letting $n\to\infty$ and applying Lemmas 1.10 and 1.11, we obtain $\{a_n\}\in BV$.

Combining this theorem and Theorem 4.5, we obtain

COROLLARY 4.1. *Let $\{a_n\}\in H_{q\alpha}$ for some $0<q\le 1$ and $\alpha\ge 0$. Then for any $0<p<1$, the limits (4.4) hold.*

In this section, I shall prove a version of Ul'yanov's theorem and extend it to the rth derivative of the complex series

$$\sum_{|n|<\infty} c_n e^{int},\quad t\in T,$$

where $\{c_n\}$ is a null sequence of complex numbers such that for some $r\in\mathbb{N}\cup\{0\}$,

$$\sum_{|k|<\infty} k^r|\Delta c_k|<\infty. \tag{4.6}$$

The class of null sequences of complex numbers such that (4.6) holds is denoted by $(BV)_r^*$. For $r=0$, we have $(BV)^*=(BV)_r^*$, i.e. it is the class of null sequences of complex numbers of bounded variation.

THEOREM 4.6. *Let $r\in\mathbb{N}\cup\{0\}$ and $\{c_n\}\in(BV)_r^*$. Then the pointwise limit $f^{(r)}$ of the rth derivative of the sums (4.1) exists in $T\setminus\{0\}$ and for any $0<p<1$,*

$$\lim_{n\to\infty}\int_{-\pi}^{\pi}|f^{(r)}(t)-S_n^{(r)}(t)|^p\,dt=0. \tag{4.7}$$

Proof. If $t\ne 0$, we obtain

$$\sum_{k=0}^{n} c_k(e^{ikt})^{(r)}=\sum_{k=1}^{n-1}\Delta c_k E_k^{(r)}(t)+c_nE_n^{(r)}(t).$$

Applying (4.3) and

$$c_n n^r\le\sum_{k=n}^{\infty}k^r|\Delta c_k|\to 0,\quad n\to\infty,$$

we see that $\sum_{k=0}^{\infty}c_k(e^{ikt})^{(r)}$ exists a.e. Similarly $\sum_{k=-\infty}^{-1}c_k(e^{ikt})^{(r)}$ converges a.e. and hence $\lim_{n\to\infty}S_n^{(r)}(t)=f^{(r)}(t)$ exists in $T\setminus\{0\}$. It is obvious that for $t\ne 0$,

$$f(t)-S_n(t)=\sum_{|j|\ge n+1}\Delta c_jE_j(t).$$

By (4.3) the series $\sum_{|j|\ge n+1} \Delta c_j E_j^{(r)}(t)$ is uniformly convergent on any compact subset of $T\setminus\{0\}$. Consequently,

$$f^{(r)}(t) - S_n^{(r)}(t) = \sum_{|j|\ge n+1} \Delta c_j E_j^{(r)}(t).$$

Finally, we obtain

$$\int_{-\pi}^{\pi} |f^{(r)}(t) - S_n^{(r)}(t)|^p\,dt = \int_{-\pi}^{\pi} \Big|\sum_{|j|\ge n+1} \Delta c_j E_j^{(r)}(t)\Big|^p dt$$
$$= O\Big(\Big(\sum_{|j|\ge n+1} j^r|\Delta c_j|\Big)^p\Big)\int_{-\pi}^{\pi}\frac{dt}{|t|^p} \to 0, \qquad n\to\infty.$$

Let us replace the conditions $\Im_r$, S_{pr}, F_{pr}, $S_{p\alpha r}$ by the conditions $\Im_r^*$, S_{pr}^*, F_{pr}^*, $S_{p\alpha r}^*$ when the coefficients are sequences of complex numbers. It is obvious that $\Im_r^* \subset (BV)_r^*$, $S_{pr}^* \subset (BV)_r^*$, $F_{pr}^* \subset (BV)_r^*$, $S_{p\alpha r}^* \subset (BV)_r^*$. Applying these inclusions we obtain the following corollaries of Theorem 4.6.

COROLLARY 4.2. *Let $r\in\mathbb{N}\cup\{0\}$ and $\{c_n\}\in\Im_r^*$. Then the pointwise limit $f^{(r)}$ of the rth derivative of the sums* (4.1) *exists in $T\setminus\{0\}$ and for any $0<p<1$, the limit* (4.7) *holds.*

COROLLARY 4.3. *Let $q>1$, $r\in\mathbb{N}\cup\{0\}$ and $\{c_n\}\in S_{qr}^*$. Then the pointwise limit $f^{(r)}$ of the rth derivative of the sums* (4.1) *exists in $T\setminus\{0\}$ and for any $0<p<1$, the limit* (4.7) *holds.*

COROLLARY 4.4. *Let $q>1$, $r\in\mathbb{N}\cup\{0\}$ and $\{c_n\}\in F_{qr}^*$. Then the pointwise limit $f^{(r)}$ of the rth derivative of the sums* (4.1) *exists in $T\setminus\{0\}$ and for any $0<p<1$, the limit* (4.7) *holds.*

COROLLARY 4.5. *Let $q>1$, $\alpha\ge 0$, $r\in\{0,1,\dots,[\alpha]\}$ and $\{c_n\}\in S_{q\alpha r}^*$. Then the pointwise limit $f^{(r)}$ of the rth derivative of the sums* (4.1) *exists in $T\setminus\{0\}$ and for any $0<p<1$, the limit* (4.7) *holds.*

Now, we shall define a new subclass of $(BV)_r^*$. Namely, a null sequence $\{c_k\}$ of complex numbers belongs to the class $H_{q\alpha r}^*$, $0<q\le 1$, $\alpha\ge 0$, $r\in\{0,1,\dots,[\alpha]\}$, if there exists a decreasing sequence $\{A_k\}$ such that $\sum_{k=1}^{\infty} k^\alpha A_k<\infty$ and

$$\frac{1}{n^{q(\alpha-r)+q}}\sum_{k=1}^{n}\frac{|\Delta c_k|^q}{A_k^q} = O(1).$$

THEOREM 4.7. *For any $0<q\le 1$, $\alpha\ge 0$ and $r\in\{0,1,\dots,[\alpha]\}$ we have the embedding $H_{q\alpha r}^*\subseteq (BV)_r^*$.*

Proof. Let $\{c_n\}\in H_{q\alpha r}^*$. Applying the Abel transformation and (4.5), we obtain

$$\sum_{k=1}^{n} k^r|\Delta c_k| = \sum_{k=1}^{n-1} k^{\alpha+1}(\Delta A_k)\Big(\frac{1}{k^{\alpha+1}}\sum_{j=1}^{k} j^r\frac{|\Delta c_j|}{A_j}\Big) + n^{\alpha+1}A_n\Big(\frac{1}{n^{\alpha+1}}\sum_{j=1}^{n} j^r\frac{|\Delta c_j|}{A_j}\Big)$$
$$\le \sum_{k=1}^{n-1} k^{\alpha+1}(\Delta A_k)\Big(\frac{1}{k^{\alpha-r+1}}\sum_{j=1}^{k}\frac{|\Delta c_j|}{A_j}\Big) + n^{\alpha+1}A_n\Big(\frac{1}{n^{\alpha-r+1}}\sum_{j=1}^{n}\frac{|\Delta c_j|}{A_j}\Big)$$

$$\leq \sum_{k=1}^{n-1} k^{\alpha+1}(\Delta A_k)\left(\frac{1}{k^{q(\alpha-r)+q}}\sum_{j=1}^{k}\frac{|\Delta c_j|^q}{A_j^q}\right)^{1/q}$$
$$+\, n^{\alpha+1}A_n\left(\frac{1}{n^{q(\alpha-r)+q}}\sum_{j=1}^{n}\frac{|\Delta c_j|^q}{A_j^q}\right)^{1/q}$$
$$= O_q(1)\Big[\sum_{k=1}^{n-1} k^{\alpha+1}(\Delta A_k) + n^{\alpha+1}A_n\Big].$$

Letting $n\to\infty$, and applying Lemmas 1.10 and 1.11, we obtain $\{c_n\}\in(BV)_r^*$.

Corollary 4.6. *Let* $0<q\leq 1$, $\alpha\geq 0$, $r\in\{0,1,\ldots,[\alpha]\}$ *and* $\{c_n\}\in H_{q\alpha r}^*$. *Then the pointwise limit* $f^{(r)}$ *of the* r*th derivative of the sums* (4.1) *exists in* $T\setminus\{0\}$ *and for any* $0<p<1$, *the limit* (4.7) *holds.*

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