Contents

Abstract

The Complex Absorbing Potential (CAP) method is widely used to compute resonances in Quantum Chemistry, both for scalar valued and matrix valued Hamiltonians. In the semiclassical limit $\hbar \to 0$ we consider resonances near the real axis and we establish the CAP method rigorously in an abstract matrix valued setting by proving that resonances are perturbed eigenvalues of the nonselfadjoint CAP Hamiltonian, and vice versa. The proof is based on pseudodifferential operator theory and microlocal analysis.

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1. Introduction

The Complex Absorbing Potential (CAP) method has emerged as a very useful method for computing resonances in Quantum Chemistry. Depending upon the definition of what a resonance is, its existence can be justified in several ways. Within the semiclassical limit, i.e., as Planck's "constant" $\hbar$ tends to zero, we analyze resonances in the spectral meaning by studying the meromorphic continuation of the resolvent of the Hamiltonian describing the quantum system. The resonant eigenstate associated with complex poles of the resolvent play a central role in scattering processes of atomic and molecular physics. Numerous techniques have been developed for calculating these poles.

In a typical quantum scattering scenario particles with positive energy arrive from infinity, interact with a localized potential $V(x)$ whereafter they leave to infinity. The absolutely continuous spectrum of the corresponding Hamiltonian $T(\hbar) = -\hbar^2\Delta + V(x)$ coincides with the positive semi-axis. Nevertheless, the resolvent function $(T(\hbar)-z)^{-1}$ admits a meromorphic continuation from the upper half-plane $\{\operatorname{Im} z > 0\}$ to (some part of) the lower half-plane $\{\operatorname{Im} z < 0\}$. Generally, this continuation has poles $z_k = E_k - i\Gamma_k/2$, $\Gamma_k > 0$, which are called *resonances* of the scattering system. The probability density of the corresponding "eigenfunction" $\psi_k(x)$ decays in time like $e^{-t\Gamma_k/\hbar}$, thus physically ψ_k represents a metastable state with a decay rate $\Gamma_k/\hbar$ or, re-phrased, a lifetime $\tau_k = \hbar/\Gamma_k$. In the semiclassical limit $\hbar \to 0$, resonances z_k satisfying $\Gamma_k = \mathcal{O}(\hbar)$ (equivalently, with lifetimes bounded away from zero) are called "long-lived". Physically, the generalized eigenfunction $\psi_k(x)$ only makes sense near the interaction region, whereas its behaviour away from that region is evidently nonphysical; outgoing waves of exponential growth. If one perturbs $T(\hbar)$ by an artificial CAP $-iW(x)$ which is supposed to vanish in the interaction region and to be positive outside ("switched on"), then the resulting Hamiltonian $J(\hbar) := T(\hbar) - iW(x)$ is a nonselfadjoint operator and the effect of the potential $W(x)$ is to *absorb* outgoing waves (up to an $\mathcal{O}(\hbar^\infty)$ error); on the contrary, a real valued positive potential would reflect the waves back into the interaction region. In some neighbourhood of the positive axis, the spectrum of $J(\hbar)$ consists of discrete eigenvalues $\tilde{z}_k$ corresponding to L^2-eigenfunctions $\widetilde{\psi}_k$. The CAP method provides a recipe for computing resonances of width $-\operatorname{Im} z(\hbar) \le c(\hbar) = \mathcal{O}(\hbar^N)$, $N \gg 1$. Studying the CAP method from a mathematical point of view amounts to relating z_k and $\tilde{z}_k$ quantitatively; the so-called approximating resonances context. We limit ourselves to a detailed study of the behaviour of the resonances and resonant states near the real axis. By resonances near the real axis we mean resonances in a "box" $\Omega(\hbar) = [l_0, r_0] + i[-c(\hbar), 0]$ where $0 < c(\hbar) = \mathcal{O}(\hbar^N)$, $N \gg 1$. In particular, we do not need to worry about pseudospectra [68, 9].

It is evident that the CAP method is useful for numerical computations, where one works on bounded domains. Indeed, when a wavepacket gets near to the edge of a numerical grid, artificial reflections occur which rapidly deteriorate the quality of the computed solution. A CAP attenuates the asymptotic part of the wavepacket and thus suppresses the reflection. The idea of using an artificial CAP in resonant scattering process was first conceived by Jolicard and Austin [23]. It was then used by Kosloff and Kosloff [27] in the field of time-dependent wavepacket propagation. Neuhauser and Baer [41, 42] used a CAP to give a time-dependent treatment of reactive scattering and later these authors applied the CAP in a time-independent framework [43]. Seideman and Miller used a CAP to compute cumulative reaction probabilities [53]. Work on approximating resonances, in particular analytic investigations, are found in Jolicard and Austin [23, 24, 25], Child [7], Riss and Meyer [48, 49], Poirier and Carrington [47], Mandelshtam and Neumaier [30], and Manolopoulos [31]. The ease of implementation of the CAP method in the discrete variable representation (DVR)/pseudo-spectral methods [52] explains its increasing use: one just add a complex (diagonal) potential to the Hamiltonian. The CAP is assumed to be zero in the interaction region and "switched on" in the region where there are no interactions. In concrete implementations, however, "switch-on" point is moved inward towards the interaction region as much as possible to minimize the number of grid points used. The results are very good: see e.g., Seideman and Miller [53], Vibók and Balint-Kurti [69], Riss and Meyer [49, 50], Vibók and Halász [70], Neumaier and Mandelshtam [44], and Santra [52]. We refer to Muga et al. [38] for a survey on the level of theoretical physics.

The discussion above, including all references, concerns scalar valued Hamiltonians. Matrix valued Hamiltonians and similar systems appear frequently in quantum mechanics, e.g., coupled electronic states [4, 45, 71], multichannel scattering [15, 32, 33], resonances for diatomic molecules [8, 36, 37] and when one reduces the dimension by means of the Born–Oppenheimer approximation [18] and similar methods that lead to so-called effective Hamiltonians.

We give results in a matrix valued "black box" scattering setting which allow us to include a wide class of (effective) Hamiltonians. Although these (effective) Hamiltonians are not in general exactly equal to matrix valued Schrödinger operators, the latter form important models for more general systems that could also be studied with more or less the same methods. Hence, as a toy model which works as our main "case study" in this introduction and one we return to repeatedly throughout the paper, we consider the semiclassical matrix valued Schrödinger Hamiltonian

$$\boldsymbol{T}(\hbar) = -\hbar^2\Delta \otimes \boldsymbol{I}_2 + \boldsymbol{V}(x) = \hbar^2 \begin{pmatrix} -\Delta & 0 \\ 0 & -\Delta \end{pmatrix} + \begin{pmatrix} V_{aa} & V_{ab} \\ V_{ba} & V_{bb} \end{pmatrix} \tag{1.1}$$

acting on the Hilbert space $\mathcal{H} = \mathfrak{h} \oplus \mathfrak{h} = L^2(\mathbb{R}^n) \oplus L^2(\mathbb{R}^n) = L^2(\mathbb{R}^n)^2 = L^2(\mathbb{R}^n, \mathbb{C}^2)$ equipped with the scalar product

$$\langle \boldsymbol{\psi}, \boldsymbol{\phi} \rangle = \int_{\mathbb{R}^n} (\psi_a \overline{\phi_a} + \psi_b \overline{\phi_b})\, d^n x.$$

of two spinor valued functions $\boldsymbol{\psi} = (\psi_a, \psi_b)^t$, $\boldsymbol{\phi} = (\phi_a, \phi_b)^t \in \mathcal{H}$. The entries of the

Hermitian matrix potential $\boldsymbol{V}$ are taken to be bounded and compactly supported real valued functions. Each entry of $\boldsymbol{V}$ looks as in Figure 1.

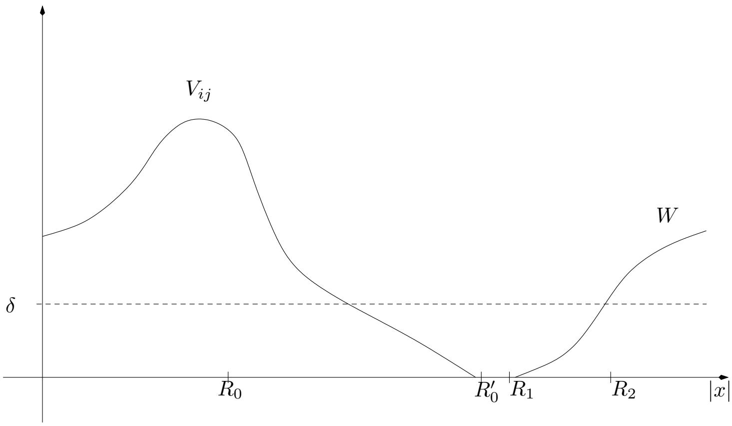

Fig. 1. Entries of matrix potentials $\boldsymbol{V}$ and $\boldsymbol{W}$ when $R_0' < R_1$

Adopting the convention that resonances lie in the lower half-plane, we define resonances of $\boldsymbol{T}(\hbar)$ in a neighbourhood Ω of some energy $E > 0$ as either (1) the poles of the meromorphic extension of the resolvent $(\boldsymbol{T}(\hbar) - z\mathbf{1})^{-1}$ from $\mathbb{C}_+ \cap \Omega$ to Ω (see Appendix A.1), or (2) as the eigenvalues of the complex scaled version $\boldsymbol{T}_\theta(\hbar)$ of $\boldsymbol{T}(\hbar)$; see Section 4.2.

In accordance with the CAP method, the Hamiltonian $\boldsymbol{T}(\hbar)$ is perturbed by a complex valued diagonal potential $-i\boldsymbol{W} = -i\,\mathrm{diag}(W, W)$ with W being a nonnegative function supported outside $\mathrm{supp}\,\boldsymbol{V}$. More precisely, $\mathrm{supp}\,W \subset \mathbb{R}^n \setminus B(0, R_1)$ with $R_0 < R_1$. The resulting Hamiltonian $\boldsymbol{J}(\hbar) := \boldsymbol{T}(\hbar) - i\boldsymbol{W}$ is the CAP Hamiltonian. We shall prove that in a neighbourhood of the real axis of polynomial width $c(\hbar) = \mathcal{O}(\hbar^N)$, the resonances of $\boldsymbol{T}(\hbar)$ are perturbed eigenvalues of the CAP Hamiltonian $\boldsymbol{J}(\hbar)$ and, vice versa the eigenvalues of $\boldsymbol{J}(\hbar)$ are perturbed resonances of $\boldsymbol{T}(\hbar)$. Specifically, Theorem 5.1 estimates the distance between the resonances of $\boldsymbol{T}(\hbar)$ and the spectrum of $\boldsymbol{J}(\hbar)$ (where $\boldsymbol{J}$ is either $\boldsymbol{J}_\infty$ or $\boldsymbol{J}_R$, see Chapter 7 for an explanation) provided we are close to the real axis. The error is $e^{-C/\hbar}$ (up to a fixed polynomial factor). Theorem 5.5 addresses the same question as Theorem 5.1 but we allow the supports of $\boldsymbol{V}$ and $\boldsymbol{W}$ to intersect (see Figure 1 where R_0, R_1, and R_0' are introduced). This yields an $\mathcal{O}(\hbar^\infty)$ error and in order to treat this case, we need that $\boldsymbol{V}$ has diagonal structure for $|x| \geq R_0$. Theorems 5.6 and 5.7, the most substantial results, estimate the number of resonances of $\boldsymbol{T}(\hbar)$ in a box close to the real axis, by the number of eigenvalues of $\boldsymbol{J}(\hbar)$ and assert that the error is $\max\{\sqrt{c(\hbar)}, e^{-\hbar^{-2/3+\epsilon}}\}$ (up to a fixed polynomial factor). We consider both cases ($\mathrm{supp}\,\boldsymbol{V} \cap \mathrm{supp}\,\boldsymbol{W} = \emptyset$ and, respectively, $\mathrm{supp}\,\boldsymbol{V} \cap \mathrm{supp}\,\boldsymbol{W} \neq \emptyset$).

Our results are analogous to the scalar valued ones by Stefanov [63], who gave the first rigorous results within the context of approximating resonances. Needless to say, numerous modifications are necessary to carry over the results to the matrix valued setting, where no prior results exist. The natural framework for the semiclassical limit is the theory of pseudodifferential operators and indeed our proofs rely heavily on pseudodifferential techniques. We give a summary of the necessary facts in Chapter 3. As mentioned above we work within the semiclassical setting of "black box scattering" introduced by Sjöstrand and Zworski [57] and extended by Sjöstrand [54]. In Chapter 4 we show how to carry over this framework to the matrix valued setting and, furthermore, we define the abstract black box Hamiltonian $\boldsymbol{H}(\hbar)$ (generalizing $\boldsymbol{T}(\hbar)$ above) under fairly general conditions given in Assumption 4.1. We formulate our main results in Chapter 5. In Chapter 6 we establish an important a priori cutoff resolvent estimate for the Hamiltonian $\boldsymbol{H}(\hbar)$, resp. $\boldsymbol{H}_R(\hbar)$. The CAP Hamiltonians $\boldsymbol{J}_\infty(\hbar)$, resp. $\boldsymbol{J}_R(\hbar)$, are defined in Chapter 7, their resolvents are analyzed, and estimates of the number of their eigenvalues on rectangles are given in Chapter 8. Using the matrix valued cutoff resolvent estimate and the matrix valued semiclassical maximum principle given in Appendix B, we prove in Chapter 9 that quasimodes of $\boldsymbol{H}(\hbar)$ generate perturbed resonances of $\boldsymbol{H}(\hbar)$. The proofs of the main theorems are given in Chapters 10, 11, and 12. Throughout the paper we have strived to make it self-contained to some reasonable extent.

The strategy of the proof of Theorem 5.1 is to start from a resonance of $\boldsymbol{H}(\hbar)$ and then, by considering a cutoff resonant state of $\boldsymbol{H}(\hbar)$, construct a quasimode which satisfies the assumptions in Proposition 9.1. Theorem 5.5 holds under a nontrapping condition, Assumption 5.4, and the requirement that the principal symbol of $\boldsymbol{H}(\hbar)$ is scalar valued away from the black box; see Assumption 5.2. Its proof is more involved than the proof of Theorem 5.1. We begin by solving Heisenberg's equations of motion semiclassically. Next, by utilizing a standard localization result away from the semiclassical wavefront set, we are able to investigate how singularities propagate. This allows us to propagate microlocally the key estimate in the proof of Theorem 5.1 and hence we complete the proof by arguments similar to the ones in the proof of Theorem 5.1. Theorems 5.6 and 5.7 require a "decomposition" approach to treat clusters of resonances which are too close and to ensure that the multiplicities are kept the same. For this purpose the box $\Omega(\hbar)$ in (5.4) is expressed as a union $\bigcup \Omega_j(\hbar)$ of disjoint boxes having smaller widths. By an application of Proposition 9.1 we show that $m_j(\hbar)$ resonances of $\boldsymbol{H}(\hbar)$ in $\Omega_j(\hbar)$ imply that there exist at least $m_j(\hbar)$ eigenvalues of $\boldsymbol{J}(\hbar)$ in a larger domain $\widetilde{\Omega}_j(\hbar)$, like (9.2). Since the domains $\widetilde{\Omega}_j(\hbar)$ intersect each other, we must ensure that we do not count some resonances more than once. We show how to avoid this and, as a matter of fact, there are at least $m(\hbar) = \sum_j m_j(\hbar)$ eigenvalues in $\Omega(\hbar)$. The latter is shown by demonstrating that the set of all $m(\hbar)$ cutoff resonant states satisfy (9.1). Once again the "propagation of singularities" result is applied, in combination with the above mentioned auxiliary matrix valued results. An interesting open problem is to treat the case $R_1 < R_0'$, when the principal symbol of $\boldsymbol{H}(\hbar)$ is also matrix valued away from the black box.

To the best of our knowledge the CAP method has not previously been investigated rigorously in the matrix valued setting. For abstract block matrix Hamiltonians, Mennicken and Motovilov [35] consider analytic continuation of the transfer function to the nonphysical sheet of its Riemann surface. Nonselfadjoint operators whose spectra include the resonances of the initial operator are constructed. The authors treat resonances as the discrete spectrum of the transfer function situated in the so-called nonphysical sheets of its Riemann surface (Lax–Phillips approach). A factorization theorem for the transfer function is established and basis properties for the "root" vectors are proven. For matrix valued Schrödinger operators Nedelec gives a simple criterion for the potential to produce resonances in [39] and, in [40] the lower bound $C\hbar^{-1}|\ln \hbar|^{-3/2}$ on the number of resonances near a point is established in dimension three. Bolte and Glaser [3] prove a semiclassical version of Egorov's theorem for Pauli type Hamiltonians.

2. Preliminaries

Notation. Throughout the paper we denote by C (with or without indices) various positive constants whose precise value is of no importance. We shall denote by $\mathrm{M}_2(\mathbb{C})$ the set of all 2×2 matrices over $\mathbb{C}$, equipped with the operator norm denoted by $\| \cdot \|_{2\times 2}$. We denote by $\boldsymbol{I}_2$ the corresponding identity matrix. For $n \in \mathbb{N}$ we let $L^2(\mathbb{R}^n, \mathbb{C}^2)$ be the space of $\mathbb{C}^2$-valued L^2 functions on $\mathbb{R}^n$ endowed with its usual norm $\| \cdot \|$ and scalar product $\langle \cdot, \cdot \rangle$. The space $C_0^\infty(\mathbb{R}^n)$ consists of all infinitely differentiable functions on $\mathbb{R}^n$ with compact support, and $C_b^\infty(\mathbb{R}^n)$ consists of all bounded continuous functions, with all derivatives bounded. We let $D_x = -i\partial/\partial x$ and $D^\alpha = D_{x_1}^{\alpha_1} \cdots D_{x_n}^{\alpha_n}$ with standard multi-index notation $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$. For $x \in \mathbb{R}^n$ we denote $\langle x \rangle := (1 + |x|^2)^{1/2}$ and, analogously, $\langle \hbar D \rangle = (1 + (\hbar D)^2)^{1/2}$, where $(\hbar D)^2 = \sum_{j=1}^n (\hbar D_{x_j})^2$.

The standard Sobolev spaces are denoted $\mathbf{H}^k(\mathbb{R}^n, \mathbb{C}^r)$ or just $H^k(\mathbb{R}^n)$ if $r = 1$. The semiclassical variant, denoted $\mathbf{H}_\hbar^k(\mathbb{R}^n, \mathbb{C}^r)$, is endowed with the $\hbar$-Sobolev norm $\|\langle \hbar D \rangle^2 \otimes \boldsymbol{I}_r \cdot \|_{L^2(\mathbb{R}^n)\otimes\mathbb{C}^r}$. The Schwartz space of rapidly decreasing functions and its adjoint space of tempered distributions are denoted by $\mathscr{S}(\mathbb{R}^n)$ and $\mathscr{S}'(\mathbb{R}^n)$, respectively. For $f, g \in C_0^\infty(\mathbb{R}^n)$ we use $f \prec g$ to mean that $g = 1$ in a neighbourhood of $\operatorname{supp} f$ (i.e., the support of f).

Operators. Let $\mathcal{H}$ be a separable complex Hilbert space. We denote its scalar product and norm by $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ and $\| \cdot \|_{\mathcal{H}}$, respectively. Let T be a linear operator on $\mathcal{H}$ with domain $\mathfrak{D}(T)$, range $\mathrm{Ran}(T)$ and kernel $\mathrm{Ker}(T)$. Its adjoint (when it exists) is denoted T^*. The spectrum and resolvent set are denoted by $\mathrm{spec}(T)$ and $\rho(T)$, respectively. The resolvent of a linear operator T is denoted by $R(T, z) = (T - zI)^{-1}$ or merely $R(z)$ if it is clear which operator we mean. If $\mathcal{X}_1$ and $\mathcal{X}_2$ are normed linear vector spaces, then $\mathcal{B}(\mathcal{X}_1, \mathcal{X}_2)$ denotes the space of all bounded operators from $\mathcal{X}_1$ into $\mathcal{X}_2$. If $\mathcal{X} = \mathcal{X}_1 = \mathcal{X}_2$, then we write $\mathcal{B}(\mathcal{X})$. The number of eigenvalues (or resonances) of T on a set $\Omega \subset \mathbb{C}$ will be denoted $\mathrm{Count}(T, \Omega)$, the so-called counting function. Scalar valued, respectively matrix valued, operators are denoted by capitals, respectively boldface capitals.

Meromorphic operator valued functions. Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be two normed linear vector spaces. Holomorphic and meromorphic functions with values in operators acting from $\mathcal{K}_1$ into $\mathcal{K}_2$ are defined as follows. A holomorphic function $A(z)$, defined on an open set $\Omega \subset \mathbb{C}$, with values in $\mathcal{B}(\mathcal{K}_1, \mathcal{K}_2)$, is a function with values in the space of linear operator from $\mathcal{K}_1$ into $\mathcal{K}_2$ such that $\chi_1 A(z) \chi_2$ is holomorphic for all $\chi_j \in C_0^\infty(\mathbb{R}^n)$. Analogously, a meromorphic function is one which is holomorphic on $\Omega \setminus S$, where $S \subset \Omega$ is a discrete set, and such that if $z_0 \in S$ then near z_0 we have

$$A(z) = \sum_{j=1}^{N} \frac{A_j}{(z-z_0)^j} + B(z)$$

with $A_j : \mathcal{K}_1 \to \mathcal{K}_2$ (continuous in the sense that $\chi_1 A_j \chi_2$ is bounded for all $\chi_j \in C_0^\infty(\mathbb{R}^n)$) of finite rank, and $B(z)$ holomorphic with values in $\mathcal{B}(\mathcal{K}_1, \mathcal{K}_2)$ for z in a neighbourhood of z_0.

Elliptic regularity. The following a priori estimate will turn out to be useful [13, Lemma 7.1].

THEOREM 2.1 (Semiclassical elliptic estimate). *Let $W \Subset U$ be open sets. Then for differential operators $A(h) = \sum_{|\alpha| \le m} a_\alpha(x)(hD_x)^\alpha$ which are classically elliptic, i.e. $\sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha \neq 0$ for $\xi \neq 0$, one has*

$$\|u\|_{H_\hbar^m(W)} \le C(\|u\|_{L^2(U)} + \|A(\hbar)u\|_{L^2(U)})$$

for some constant $C > 0$.

3. Matrix valued pseudodifferential operators

We recall elements of the pseudodifferential operator theory which will be used in what follows. The basic motivation for pseudodifferential calculus is to obtain an algebraic correspondence between classical observables and quantum observables. In more mathematical terms this "quantization" involves turning functions on phase space $\mathsf{T}^*(\mathbb{R}_x^n) = \mathbb{R}^{2n} = \mathbb{R}_x^n \times \mathbb{R}_\xi^n$, henceforth referred to as *symbols*, into operators on some function space over $\mathbb{R}_x$, most notably $L^2(\mathbb{R}^n) \otimes \mathbb{C}^2$. The subject is vast and we only pinpoint results and definitions relevant to us. The corresponding theory for scalar objects is well-known [21, 67, 51, 14, 12]. Most of the results carry over to the matrix valued case, if one takes into account some minor modifications.

3.1. Operators and symbols. For any

$$\boldsymbol{a} \in \mathscr{S}(\mathsf{T}^*(\mathbb{R}^n)) \otimes \mathrm{M}_2(\mathbb{C})$$

and $\boldsymbol{u} \in \mathscr{S}(\mathbb{R}^n) \otimes \mathbb{C}^2$, we define the operator $\boldsymbol{A} : \mathscr{S}(\mathbb{R}^n) \otimes \mathbb{C}^2 \to \mathscr{S}(\mathbb{R}^n) \otimes \mathbb{C}^2$ by

$$(\boldsymbol{A}\boldsymbol{u})(x) = (\mathrm{Op}^w(\boldsymbol{b})\boldsymbol{u})(x) = (2\pi\hbar)^{-n} \iint_{\mathsf{T}^*\mathbb{R}^n} e^{\frac{i}{\hbar}\langle x-y,\xi\rangle} \boldsymbol{a}\left(\frac{x+y}{2}, \xi\right) \boldsymbol{u}(y)\, dy\, d\xi, \qquad (3.1)$$

where the uniformly convergent integral is defined in the Bochner sense. These operators $\boldsymbol{A} = \mathrm{Op}^w(\boldsymbol{a})$ are called Weyl operators, and we designate the Weyl symbol of $\boldsymbol{A}$ by $\mathrm{symb}^w(\boldsymbol{A}) = \boldsymbol{a}$. Henceforth we also use the short-hand notation $\mathrm{Op}(\boldsymbol{a})$ and $\boldsymbol{a}^w$ for such an operator $\boldsymbol{A}$. We remark that our choice of taking $\boldsymbol{a}((x+y)/2, \xi)$ rather than the more general $\boldsymbol{a}(tx + (1-t)y, \xi)$ corresponds to what is called Weyl quantization. This is a convenient choice that we shall make throughout. In theory one can jump from one type of quantization to another although for computations it is easiest to settle for one version. Having in mind operators such as $-\hbar^2\Delta \otimes \boldsymbol{I}_2 + \boldsymbol{V}(x)$ that are quantizations of Hermitian symbols $|\xi|^2\boldsymbol{I}_2 + \boldsymbol{V}(x)$ (independently of choice of t-quantization) we would like to allow for symbols that grow at infinity. If we permit symbols belonging to $\mathscr{S}'(\mathsf{T}^*(\mathbb{R}^n)) \otimes \mathrm{M}_2(\mathbb{C})$

it turns out that in general we cannot expect that $\mathrm{Op}(\boldsymbol{a})\,\mathrm{Op}(\boldsymbol{b}) = \mathrm{Op}(\boldsymbol{c})$ for some $\boldsymbol{c} \in \mathscr{S}'(\mathsf{T}^*(\mathbb{R}^n))\otimes\mathrm{M}_2(\mathbb{C})$, i.e. such operators cannot be composed with one another. Therefore we retreat for a moment to discuss certain intermediate classes of symbols for which the associated operators enjoy better properties.

Definition 3.1. A function $m : \mathbb{R}^{2n} \to [0, \infty)$ is said to be an *order function* if there exist $C, N > 0$ such that

$$m(x, \xi) \leq C(1 + (x - y)^2 + (\xi - \eta)^2)^{N/2} m(y, \eta)$$

for all $(x, \xi), (y, \eta) \in \mathsf{T}^*\mathbb{R}^n$.

Notice that the classes we shall work with are better than $\mathscr{S}(\mathbb{R}^n) \otimes \mathrm{M}_2(\mathbb{C})$ in that they do allow for some growth of the symbols and their derivatives at infinity. Examples of order functions we will encounter are $\langle\xi\rangle^2$ and 1. We now make the following definition.

Definition 3.2. Let $m : \mathbb{R}^{2n} \to [0, \infty)$ be an order function. Define $\mathsf{S}^q(m) \subset C^\infty(\mathsf{T}^*\mathbb{R}^n)$ $\otimes\mathrm{M}_2(\mathbb{C})$ to consist of all $\boldsymbol{a} \in C^\infty(\mathsf{T}^*\mathbb{R}^n)\otimes\mathrm{M}_2(\mathbb{C})$ such that for all multi-indices $\alpha, \beta \in \mathbb{N}_0^n$ there are constants $C_{\alpha,\beta} > 0$ with

$$\|\partial_\xi^\alpha \partial_x^\beta \boldsymbol{a}(x, \xi)\|_{n\times n} \leq C_{\alpha,\beta} \hbar^{-q} m(x, \xi) \quad \text{for all } (x, \xi) \in \mathsf{T}^*\mathbb{R}^n.$$

Here we have tacitly assumed $\boldsymbol{a}$ to depend on the semiclassical parameter $\hbar$ and the above estimate to hold uniformly for $\hbar \in (0, 1]$. According to this definition, $\boldsymbol{a}$ and its derivatives vanish the more rapidly the more *negative* q is. We shall abbreviate $\mathsf{S}^0(m) = \mathsf{S}(m)$. In direct analogy with the corresponding result for scalar symbols (see [14, Theorem 2.21] and [12, Lemma 7.8]), the following result holds.

Proposition 3.3. *For* $\boldsymbol{a} \in \mathsf{S}(m)$ *the operator* $\boldsymbol{A}$ *as defined by* (3.1) *is bounded on* $\mathscr{S}(\mathbb{R}^n) \otimes \mathbb{C}^2$.

By duality the same result holds for $\boldsymbol{A} : \mathscr{S}'(\mathbb{R}^n)\otimes\mathbb{C}^2 \to \mathscr{S}'(\mathbb{R}^n)\otimes\mathbb{C}^2$. For symbols that are bounded with all their derivatives we even have the celebrated result by Calderón–Vaillancourt [12, Theorem 7.11].

Proposition 3.4. *Let* $\boldsymbol{a} \in \mathsf{S}(1)$*. Then* $\mathrm{Op}(\boldsymbol{a})$ *defines a continuous operator on* $L^2(\mathbb{R}^n)$ $\otimes\, \mathbb{C}^2$.

Actually it is enough to assume that $\boldsymbol{a}$ and its derivatives up to and including order $2n + 1$ are bounded on $\mathbb{R}^n$. An upper bound for $\|\boldsymbol{A}\|_{\mathcal{B}(L^2(\mathbb{R}^n)\otimes\mathbb{C}^2)}$ is then given by

$$C_n \sup_{|\alpha|\leq 2n+1} \sup_{x\in\mathbb{R}^n} \|(\partial_x^\alpha \boldsymbol{a})(x)\|_{2\times 2}$$

for some constant C_n that only depends on n [12].

Next we state the fundamental product formula $\mathrm{Op}(\boldsymbol{a})\,\mathrm{Op}(\boldsymbol{b}) = \mathrm{Op}(\boldsymbol{a} \# \boldsymbol{b})$ together with a recipe for calculating the new symbol $\boldsymbol{a} \# \boldsymbol{b}$ (see, e.g., [14, Section 2.3] or [12, Theorem 7.9]).

Proposition 3.5. *Let* m_1, m_2 *be order functions and assume* $\boldsymbol{a} \in \mathsf{S}(m_1)$ *and* $\boldsymbol{a} \in \mathsf{S}(m_2)$*. Then there exists* $\boldsymbol{c} \in \mathsf{S}(m_1 m_2)$ *such that*

$$\boldsymbol{AB} = \mathrm{Op}(\boldsymbol{a})\,\mathrm{Op}(\boldsymbol{b}) = \mathrm{Op}(\boldsymbol{c}).$$

Explicitly

$$\boldsymbol{c}(x,\xi) =: \boldsymbol{a} \,\#\, \boldsymbol{b} = \exp\left(\frac{ih}{2}(\nabla_x \cdot \nabla_\eta - \nabla_\xi \cdot \nabla_y)\right)\boldsymbol{a}(x,\xi)\boldsymbol{b}(y,\eta)\Big|_{\substack{y=x\\ \eta=\xi}}.$$

The symbol $\boldsymbol{a} \,\#\, \boldsymbol{b}$ is called the *Weyl*, or *twisted*, *product* of the symbols $\boldsymbol{a}$ and $\boldsymbol{b}$ and is unique modulo $\mathsf{S}^{-\infty}(m) := \bigcap_{q\in\mathbb{R}} \mathsf{S}^q(m)$.

Next we give a semiclassical variant of Beals' characterization of pseudodifferential operators.

LEMMA 3.6. *Let $\boldsymbol{A}(\hbar) : \mathscr{S}(\mathbb{R}^d)\otimes\mathbb{C}^2 \to \mathscr{S}'(\mathbb{R}^d)\otimes\mathbb{C}^2$ be a linear and continuous operator depending on the semiclassical parameter $\hbar \in (0,\hbar_0]$. Then the following statements are equivalent:*

(1) *$\boldsymbol{A}(\hbar) = \mathrm{Op}(\boldsymbol{a})$ is a Weyl operator with symbol $\boldsymbol{a} \in \mathsf{S}^0(m)$.*
(2) *For every sequence $l_1(x,\xi),\dots,l_N(x,\xi)$, $N\in\mathbb{N}$, of linear forms on $\mathsf{T}^*\mathbb{R}^d$ the operator given by the multiple commutator*

$$[\mathrm{Op}(l_N), [\mathrm{Op}(l_{N-1}),\dots,[\mathrm{Op}(l_1),\boldsymbol{A}]\dots]]$$

is bounded as an operator on $L^2(\mathbb{R}^d)\otimes\mathbb{C}^2$ and its norm is of the order $\hbar^N$.

The direction (1)⇒(2) follows immediately from the symbolic calculus above. For the opposite direction we refer to [12, Proposition 8.3].

3.2. Asymptotic series. It is often convenient to think of a semiclassical symbol as a formal power series in $\hbar$ so that an element $\boldsymbol{a}\in\mathsf{S}^q(m)$ corresponds to a series of the form $\hbar^{-q}\boldsymbol{a}_0 + \hbar^{-q+1}\boldsymbol{a}_1 + \cdots$ with $\boldsymbol{a}_j \in \mathsf{S}(m)$ for every j, abbreviated

$$\boldsymbol{a} \sim \sum_{j=0}^{\infty} \hbar^{-q+j}\boldsymbol{a}_j.$$

We will reserve the notation $\mathsf{S}^q_{\mathrm{cl}}(m)$ for elements of $\mathsf{S}^q(m)$ that have such asymptotic expansions in integer powers of $\hbar$. We call $\boldsymbol{a}_0$ the *principal symbol* of $\boldsymbol{a}$ and the subsequent coefficient $\boldsymbol{a}_1$ the *subprincipal symbol* of $\boldsymbol{a}$. Regarding Proposition 3.5 at the level of power series one can show that if

$$\boldsymbol{a} \sim \sum_{j\ge 0} \hbar^j \boldsymbol{a}_j \in \mathsf{S}_{\mathrm{cl}}(m_1) \quad \text{and} \quad \boldsymbol{b} \sim \sum_{j\ge 0} \hbar^j \boldsymbol{b}_j \in \mathsf{S}_{\mathrm{cl}}(m_2),$$

then $\boldsymbol{a}\,\#\,\boldsymbol{b} \in \mathsf{S}_{\mathrm{cl}}(m_1m_2)$ with $\boldsymbol{a}\,\#\,\boldsymbol{b} \sim \sum_{k\ge 0}\hbar^k(\boldsymbol{a}\,\#\,\boldsymbol{b})_k$, where

$$(\boldsymbol{a}\,\#\,\boldsymbol{b})_k(x,\xi) = (2i)^{-k} \sum_{|\alpha|+|\beta|+j+l=k} \frac{(-1)^{|\alpha|}}{|\alpha|!|\beta|!}((\partial_x^\alpha\partial_\xi^\beta \boldsymbol{a}_j)(\partial_\xi^\alpha\partial_x^\beta \boldsymbol{b}_l))(x,\xi),$$

where $k,j,l\in\mathbb{N}_0$ and $\alpha,\beta\in\mathbb{N}_0^n$. In this context the product is referred to as the *Moyal product*. In particular, we see that

$$(\boldsymbol{a}\,\#\,\boldsymbol{b})_0 = \boldsymbol{a}_0\boldsymbol{b}_0 \quad \text{and} \quad (\boldsymbol{a}\,\#\,\boldsymbol{b})_1 = \boldsymbol{a}_0\boldsymbol{b}_1 + \boldsymbol{a}_1\boldsymbol{b}_0 - \frac{i}{2}\{\boldsymbol{a}_0,\boldsymbol{b}_0\},$$

where $\{\cdot,\cdot\}$ denotes the Poisson bracket defined through

$$\{\boldsymbol{a},\boldsymbol{b}\} = \sum_{j=1}^{n} \left(\frac{\partial \boldsymbol{a}}{\partial \xi_j} \frac{\partial \boldsymbol{b}}{\partial x_j} - \frac{\partial \boldsymbol{a}}{\partial x_j} \frac{\partial \boldsymbol{b}}{\partial \xi_j} \right).$$

As opposed to the scalar case, we have in general $\{\boldsymbol{a},\boldsymbol{b}\} \neq -\{\boldsymbol{b},\boldsymbol{a}\}$ and in particular $\{\boldsymbol{a},\boldsymbol{a}\} \neq 0$.

For the record we mention the following result (see, e.g., [51, Theorem II.53]):

PROPOSITION 3.7. *Let $m : \mathbb{R}^{2n} \to [0,\infty)$ be an order function. If*

$$\iint_{\mathbb{R}^{2n}} m(x,\xi)\, dx\, d\xi$$

is finite then every $\boldsymbol{A} \in \mathrm{Op}(\mathsf{S}_{\mathrm{cl}}(m))$ is of trace class with $\|\boldsymbol{A}\|_{\mathrm{tr}} = \mathcal{O}(\hbar^{-n})$.

3.3. Inverses. An operator $\boldsymbol{A} = \mathrm{Op}(\boldsymbol{a})$ is called *elliptic* if its symbol $\boldsymbol{a} \in \mathsf{S}(m)$ is invertible, i.e., if the matrix inverse $\boldsymbol{a}^{-1}$ exists in $\mathsf{S}(m^{-1})$. One can then construct a parametrix $\boldsymbol{q} \in \mathsf{S}(m^{-1})$ which is an asymptotic inverse of $\boldsymbol{a}$ in the sense of symbol products:

LEMMA 3.8. *Suppose $\boldsymbol{a} \in \mathsf{S}(m)$ is elliptic in the sense that $\boldsymbol{a}^{-1}(x,\xi)$ exists for all $(x,\xi) \in \mathsf{T}^*\mathbb{R}^d$ and belongs to the class $\mathsf{S}(m^{-1})$. Then there exists a parametrix $\boldsymbol{q} \in \mathsf{S}(m^{-1})$ with an asymptotic expansion of the form*

$$\boldsymbol{q} \sim \boldsymbol{a}^{-1} + \hbar(\boldsymbol{a}^{-1} \,\#\, \boldsymbol{r}) + \hbar^2(\boldsymbol{a}^{-1} \,\#\, \boldsymbol{r} \,\#\, \boldsymbol{r}) + \cdots \tag{3.2}$$

such that

$$\boldsymbol{a} \,\#\, \boldsymbol{q} \sim \boldsymbol{q} \,\#\, \boldsymbol{a} \sim \boldsymbol{I}_2.$$

Proof. Consider

$$\mathrm{Op}(\boldsymbol{a})\,\mathrm{Op}(\boldsymbol{a})^{-1} = \mathbf{1} - \hbar\,\mathrm{Op}(\boldsymbol{r}),$$

where $\boldsymbol{r} \in \mathsf{S}(m)$. For sufficiently small $\hbar$, the operator $\mathbf{1} - \hbar\,\mathrm{Op}(\boldsymbol{r})$ possesses a bounded inverse and one can define a (left and right) inverse $\mathrm{Op}(\boldsymbol{a})^{-1}(\mathbf{1} - \hbar\,\mathrm{Op}(\boldsymbol{r}))^{-1}$ for $\mathrm{Op}(\boldsymbol{a})$. Moreover, Lemma 3.6 implies that that this inverse is again a bounded pseudodifferential operator. To derive an asymptotic expansion for the parametrix $\boldsymbol{q}$, one proceeds by defining the operator $\boldsymbol{Q}_N := \mathrm{Op}(\boldsymbol{a})^{-1}(\mathbf{1} + \hbar\boldsymbol{R} + \cdots + \hbar^N \boldsymbol{R}^N)$, with $\boldsymbol{R} = \mathrm{Op}(\boldsymbol{r})$, which is equivalent to $\boldsymbol{Q} = \mathrm{Op}(\boldsymbol{q})$ modulo terms of order $\hbar^{N+1}$. Hence one can write

$$\boldsymbol{q} \sim \boldsymbol{a}^{-1} + \hbar(\boldsymbol{a}^{-1} \,\#\, \boldsymbol{r}) + \hbar^2(\boldsymbol{a}^{-1} \,\#\, \boldsymbol{r} \,\#\, \boldsymbol{r}) + \cdots$$

and this yields the result. ■

3.4. Semiclassical wavefront set. In the semiclassical matrix valued setting we introduce the wavefront set as for a family of $\hbar$-tempered smooth functions $\{\boldsymbol{u}(\hbar)\}$ (cf. [51, 58, 13]).

DEFINITION 3.9 (Wavefront set). We say that $(x_0,\xi_0) \notin \mathrm{WF}_\hbar^s\, \boldsymbol{u}(\hbar)$ if and only if there exists $\boldsymbol{a} \in \mathsf{S}(1)$, invertible near (x_0,ξ_0), such that

$$\|\boldsymbol{a}^w(x,\hbar D)\boldsymbol{u}(\hbar)\|_{L^2} \leq C\hbar^s.$$

The definition can be illustrated by the following fact.

LEMMA 3.10. *If* $(x_0, \xi_0) \notin \mathrm{WF}^s_\hbar \boldsymbol{u}$ *then*

$$\|\boldsymbol{b}^w(x, \hbar D)\boldsymbol{u}(\hbar)\|_{L^2} = \mathcal{O}(\hbar^s)$$

for any $\boldsymbol{b} \in C_0^\infty(\mathbb{R}^{2n})$ *with support in a sufficiently small neighbourhood of* (x_0, ξ_0).

Proof. Let $\boldsymbol{a} \in \mathsf{S}(1)$ be elliptic near (x_0, ξ_0) with $\|\boldsymbol{a}^w(x, \hbar D)\boldsymbol{u}(\hbar)\|_{L^2} = \mathcal{O}(\hbar^s)$ and select $\boldsymbol{\chi} \in C_0^\infty(\mathbb{R}^{2n})$ with $\boldsymbol{\chi} = \boldsymbol{a}(x_0, \xi_0)^{-1}$ near (x_0, ξ_0) such that

$$\boldsymbol{\chi}(x, \xi)(\boldsymbol{a}(x, \xi) - \boldsymbol{a}(x_0, \xi_0)) + \boldsymbol{I}_2$$

is invertible on $\mathbb{R}^{2n}$. In view of Lemma 3.8 there exists a parametrix $\boldsymbol{c}^w(x, \hbar D)$ so that $\boldsymbol{b}^w(x, \hbar D)$ can be decomposed according to

$$\begin{aligned}\boldsymbol{b}^w(x, \hbar D)\boldsymbol{u}(\hbar) &= \boldsymbol{b}^w(x, \hbar D)\boldsymbol{c}^w(x, \hbar D)\boldsymbol{\chi}^w(x, \hbar D)\boldsymbol{a}^w(x, \hbar D)\boldsymbol{u}(\hbar)\\ &\quad + \boldsymbol{b}^w(x, \hbar D)\boldsymbol{c}^w(x, \hbar D)(\mathbf{1} - \boldsymbol{\chi}^w(x, \hbar D)\boldsymbol{a}(x_0, \xi_0))\boldsymbol{u}(\hbar) + \mathcal{O}(\hbar^\infty).\end{aligned}$$

Hence if $\operatorname{supp} \boldsymbol{b} \cap \operatorname{supp}(\mathbf{1} - \boldsymbol{\chi}\boldsymbol{a}(x_0, \xi_0)) = \emptyset$, then we conclude that $\|\boldsymbol{b}(x, \hbar D)^w \boldsymbol{u}(\hbar)\|_{L^2} = \mathcal{O}(\hbar^s)$. ∎

3.5. Helffer–Robert–Sjöstrand calculus. We shall sometimes make use of the Helffer–Robert–Sjöstrand functional calculus for $\hbar$-pseudodifferential operators, namely the one that originates with the Cauchy–Green–Riemann–Stokes formula

$$f(\boldsymbol{A}) = -\frac{1}{\pi}\int_{\mathbb{C}} \frac{\partial \tilde{f}}{\partial \bar{z}}(\boldsymbol{A} - z\mathbf{1})^{-1}\, dz \tag{3.3}$$

where $f \in C_0^\infty(\mathbb{R}^n)$ and $\boldsymbol{A} = \mathrm{Op}(\boldsymbol{a})$ is an essentially selfadjoint operator with symbol $\boldsymbol{a} \in \mathsf{S}(m)$. Here $\tilde{f} \in C_0^\infty(\mathbb{C})$ denotes an extension of f such that $|\overline{\partial}\tilde{f}(z)| \le C_N |\mathrm{Im}\, z|^N$ for all $N \in \mathbb{N}_0$. Therefore we call $\tilde{f}$ an almost analytic extension of f. The resulting functional calculus was developed in [19, 20] (see also [12]) for the scalar setting, and the results were carried over to the matrix valued setting by Dimassi [10, 11].

4. Framework and assumptions

In this section we introduce the abstract Hamiltonian under fairly general assumptions. Moreover, we summarize some of its basic properties.

4.1. Matrix valued black box framework. We carry over the semiclassical framework of "black box scattering" from the scalar valued setting in [54]. Let $\mathcal{H}$ be a complex separable Hilbert space admitting the orthogonal decomposition

$$\mathcal{H} = \mathcal{H}_{R_0} \oplus (L^2(\mathbb{R}^n \setminus B(0, R_0)) \otimes \mathbb{C}^2),$$

where $R_0 > 0$ is fixed and $B(0, R) = \{x \in \mathbb{R}^n : |x| < R\}$ for any $R > 0$. For the corresponding orthogonal projections we will employ the notation $\mathbf{1}_{B(0,R_0)} = 1_{|x|\le R_0} \otimes \boldsymbol{I}_2$ and $\mathbf{1}_{\mathbb{R}^n \setminus B(0,R_0)} = 1_{\mathbb{R}^n \setminus B(0,R_0)} \otimes \boldsymbol{I}_2$, respectively. Moreover, we will use the notation $\mathcal{H}_R$ for the space $\mathcal{H}_{R_0} \oplus (L^2(B(0, R) \setminus B(0, R_0)) \otimes \mathbb{C}^2)$, where $R > R_0$.

Hamiltonian. Our operators shall mostly depend on a semiclassical parameter $\hbar \in (0, 1]$ although we will not always write this dependence explicitly, especially in proofs. We consider a family of selfadjoint unbounded operators $\boldsymbol{H}(\hbar)$ in $\mathcal{H}$ with a common domain $\mathcal{D}$ (independent of $\hbar$) equipped with the graph norm

$$\|\boldsymbol{u}\|_{\mathcal{D}} := \|(\boldsymbol{H}(\hbar) + i\mathbf{1})\boldsymbol{u}\|_{\mathcal{H}}.$$

Specifically, we need that $\boldsymbol{H}(\hbar)$ is an essentially selfadjoint operator on $\mathscr{S}(\mathbb{R}^n) \otimes \mathbb{C}^2$ with symbol $\boldsymbol{h}(\hbar) \in \mathsf{S}^0_{\mathrm{cl}}(m)$; its principal symbol will be denoted $\boldsymbol{h}_0(\hbar)$. We impose the following conditions:

Assumption 4.1.

(i) Let $\boldsymbol{h} \in \mathsf{S}^0_{\mathrm{cl}}(m)$ be Hermitian.
(ii) Let $\boldsymbol{h}_0$ be elliptic in the sense that

$$\|(\boldsymbol{h}_0 + i\boldsymbol{I}_2)^{-1}\|_{2\times 2} \leq cm(x,\xi)^{-1}.$$

(iii) Suppose $\mathbf{1}_{\mathbb{R}^n\setminus B(0,R_0)}\mathcal{D} = H^2(\mathbb{R}^n \setminus B(0,R_0), \mathbb{C}^2)$
(iv) Let

$$\mathbf{1}_{B(0,R_0)}(\boldsymbol{H}(\hbar) - i\mathbf{1})^{-1} : \mathcal{H} \longrightarrow \mathcal{H} \tag{4.1}$$

be compact.
(v) Suppose

$$(\boldsymbol{H}(\hbar)\boldsymbol{u})|_{\mathbb{R}^n\setminus\overline{B(0,R_0)}} = \boldsymbol{H}_0(\hbar)(\boldsymbol{u}|_{\mathbb{R}^n\setminus\overline{B(0,R_0)}}) \tag{4.2}$$

where

$$\boldsymbol{H}_0(\hbar)\boldsymbol{u} = \Big(\sum_{|\alpha|\leq 2} \boldsymbol{a}_\alpha(x)(\hbar D)^\alpha\Big)\boldsymbol{u} \quad \text{and } \boldsymbol{a}_\alpha \text{ Hermitian.} \tag{4.3}$$

(vi) Suppose $\boldsymbol{H}_0(\hbar) = (-\hbar^2\Delta) \otimes \boldsymbol{I}_2$ for $|x| > R_0' > R_0$

Essential selfadjointness is ensured by (i)–(ii). Condition (iii) is understood uniformly in $\hbar$ in the sense that if $H^2(\mathbb{R}^n \setminus B(0,R_0)) \otimes \mathbb{C}^2$ comes equipped with the $\hbar$-Sobolev norm $\|(\langle\hbar D\rangle^2 \otimes \boldsymbol{I}_2)\boldsymbol{u}\|_{L^2(\mathbb{R}^n\setminus B(0,R_0))\otimes\mathbb{C}^2}$, where $\mathcal{D}$ carries the graph norm $\|(\boldsymbol{H} + i\mathbf{1})\boldsymbol{u}\|_{\mathcal{H}}$ then $\mathbf{1}_{\mathbb{R}^n\setminus B(0,R_0)} : \mathcal{D} \to H^2(\mathbb{R}^n \setminus B(0,R_0)) \otimes \mathbb{C}^2$ is uniformly bounded in $\hbar$ with a uniformly bounded right inverse. Conversely we assume that if $\boldsymbol{u} \in H^2(\mathbb{R}^n \setminus B(0,R_0)) \otimes \mathbb{C}^2$ vanishes near $\{|x| = R_0\}$ then $\boldsymbol{u} \in \mathcal{D}$ with $\mathbf{1}_{B(0,R_0)}\boldsymbol{u} = 0$. Condition (v) says that outside $\mathcal{H}_{0,R_0}$, $\boldsymbol{H}(\hbar)$ is supposed to coincide with $\boldsymbol{H}_0(\hbar)$.

Example 4.2. Under the conditions mentioned in the Introduction, it is clear that the matrix valued Schrödinger operator in (1.1) satisfies Assumption 4.1.

4.2. Complex scaling. We briefly describe how to define a complex scaled version $\boldsymbol{H}_\theta(\hbar)$ of $\boldsymbol{H}(\hbar)$. The interested reader may consult [57] for further details. Given $\varepsilon_0 \in (0,\pi/2)$, $0 \leq \theta_0 \leq \pi$ and $\widetilde{R} > R_0'$ we let $[0,\theta_0] \times [0,\infty) \ni (\theta,t) \mapsto f_\theta(t) \in \mathbb{C}$ be smooth with $f_\theta(\cdot)$ injective and such that

(i) $f_\theta(t) = t$ for $0 \leq t \leq \widetilde{R}$;
(ii) $0 \leq \arg f_\theta(t) \leq \theta$, $0 \leq \arg \partial_t f_\theta(t) \leq \theta + \varepsilon_0$, $\partial_t f_\theta \neq 0$;
(iii) $\arg f_\theta(t) \leq \arg \partial_t f_\theta(t) \leq \arg f_\theta(t) + \varepsilon_0$;
(iv) $f_\theta(t) = e^{i\theta_0}t$ for $t \geq \widetilde{R} + \delta/2$ for some $\delta > 0$.

We will also add the additional assumption that

$$\theta(r) = e^{-(r-\widetilde{R})^{-k}} \quad \text{for } \widetilde{R} \le r \le \widetilde{R} + 1/C \quad \text{for some } k > 0 \text{ and } C \gg 1.$$

Denote by $r \in \mathbb{R}_+$ and $\omega \in \mathbb{S}^{n-1}$ the radial and angular part of $x \in \mathbb{R}^n \setminus \{0\}$ respectively, i.e. $r = |x|$ and $\omega = x/|x|$. A map κ_θ from $\mathbb{R}_+ \times \mathbb{S}^{n-1}$ to $\mathbb{C}^n$ is defined by

$$\kappa_\theta : (r, \omega) \mapsto f_\theta(r)\omega.$$

Upon identifying $\mathbb{R}^n \setminus \{0\}$ with $\mathbb{R}_+ \times \mathbb{S}^{n-1}$ we may regard κ_θ as a map from $\mathbb{R}^n$ to $\mathbb{C}^n$. Let $\Gamma_\theta = \kappa_\theta(\mathbb{R}^n)$. By locally identifying $\mathbb{C}^n$ near a point of $\Gamma_\theta \setminus \{0\}$ with $\{(s, \omega) \in \mathbb{C} \times \mathbb{C}^n : \sum \omega_j^2 = 1\}$ (using the substitution $x = s\omega$), we see that Γ_θ is a totally real submanifold (meaning $\mathsf{T}_x\Gamma_\theta \cap i\mathsf{T}_x\Gamma_\theta = \{0\}$, see e.g. [57]) of (real) maximal dimension n.

We define the dilated operator $\boldsymbol{H}_\theta$ as follows: we work in the Hilbert space

$$\mathcal{H}_\theta = \mathcal{H}_{R_0} \oplus L^2(\Gamma_\theta \setminus B(0, R_0'), \mathbb{C}^2),$$

where $B(0, R_0')$ denotes the real ball as above. If $\chi \in C_0^\infty(B(0, R_0' + \epsilon))$ equals one in a neighbourhood of $\overline{B(0, R_0')}$, we define

$$\mathcal{D}_\theta = \{\boldsymbol{u} \in \mathcal{H}_\theta : \boldsymbol{\chi u} \in \mathcal{D},\ (\boldsymbol{1} - \boldsymbol{\chi})\boldsymbol{u} \in H^2(\Gamma_\theta \setminus B(0, R_0')) \otimes \mathbb{C}^2\}.$$

This definition does not depend on the choice of $\boldsymbol{\chi}$. For $\boldsymbol{v} \in \mathcal{D}_\theta$ denote by $\tilde{\boldsymbol{v}}$ an almost analytic extension of $\boldsymbol{v}$ (i.e., a smooth extension of $\boldsymbol{v}$ to a neighbourhood of Γ_θ such that $\overline{\partial}\boldsymbol{v}$ vanishes to infinite order on Γ_θ). The unbounded operator $\boldsymbol{H}_\theta : \mathcal{H}_\theta \to \mathcal{H}_\theta$ with domain $\mathcal{D}_\theta$ is defined as

$$\begin{aligned} \boldsymbol{H}_\theta \boldsymbol{u}|_{B(0,R_0')} &= \boldsymbol{H}(\boldsymbol{\chi u})|_{B(0,R_0')}, \\ \boldsymbol{H}_\theta \boldsymbol{u}|_{\Gamma_\theta \setminus B(0,R_0')} &= -\Delta_{\Gamma_\theta} \otimes \boldsymbol{I}_2(\boldsymbol{u}|_{\Gamma_\theta \setminus B(0,R_0')}) := -\Delta_z \otimes \boldsymbol{I}_2((\boldsymbol{u}|_{\Gamma_\theta \setminus B(0,R_0')})^{\sim})|_{\Gamma_\theta}, \end{aligned}$$

which does not depend on the choice of $\boldsymbol{\chi}$.

Proposition 4.3. *Let Assumptions* 4.1 *and* 4.6 *be satisfied.*

(1) *If* $z \in \mathbb{C} \setminus \{0\}$, $\arg z \neq -2\theta$, *then* $\boldsymbol{H}_\theta - z\boldsymbol{1} : \mathcal{D}_\theta \to \mathcal{H}_\theta$ *is a Fredholm operator with index zero.*
(2) *A point* $z \in \mathbb{C} \setminus e^{-2i\theta}[0, +\infty)$ *belongs to the spectrum of* $\boldsymbol{H}_\theta$ *if and only if* $\mathrm{Ker}(\boldsymbol{H}_\theta - z\boldsymbol{1}) \neq \{0\}$.

The proof is similar to the one in the scalar valued setting [57, Lemma 3.2 and Lemma 3.3]. Under different assumptions, a matrix valued version is found in Nedelec [39, Proposition 3.1].

4.3. Resonances and resonant states. In the following we use

$$\begin{aligned} \mathcal{H}_{\mathrm{comp}} &= \{\boldsymbol{u} \in \mathcal{H} : \mathrm{supp}(\boldsymbol{1}_{\mathbb{R}^n \setminus B(0,R_0)}\boldsymbol{u}) \text{ is bounded}\}, \\ \mathcal{H}_{\mathrm{loc}} &= \mathcal{H}_{R_0} \oplus L^2_{\mathrm{loc}}(\mathbb{R}^n \setminus B(0, R_0), \mathbb{C}^2), \\ \mathcal{D}_{\mathrm{loc}} &= \{\boldsymbol{u} \in \mathcal{H}_{\mathrm{loc}} : \boldsymbol{\chi u} \in \mathcal{D} \text{ for } \chi \in C_0^\infty(\mathbb{R}^n),\ \chi \text{ constant near } B(0, R_0)\}. \end{aligned}$$

In light of the following result, quantum resonances in a neighbourhood Ω of some energy $E > 0$ can be defined as the poles of the meromorphic extension of the resolvent $(\boldsymbol{H}(\hbar) - z\boldsymbol{1})^{-1}$ from $\mathbb{C}_+ \cap \Omega$ to Ω.

Proposition 4.4. *Let Assumption* 4.1 *hold. The operator* $\boldsymbol{H}(\hbar)$ *has only discrete spectrum in* $\mathbb{R}_-$ *and* $\boldsymbol{R}(z,\hbar):\mathcal{H}_{\rm comp}\to\mathcal{D}_{\rm loc}$ *admits a meromorphic continuation from* $\mathbb{C}_+$ *to*

(1) $\mathbb{C}^*=\mathbb{C}\setminus\{0\}$ *when* $n=1$*;*
(2) *the Riemann surface associated with* $z\mapsto\sqrt{z}$ *when* $n\geq 3$ *is odd;*
(3) *the Riemann surface associated with* $z\mapsto\log z$ *when* n *is even.*

For a proof we refer to Appendix A.1. Alternatively, resonances can be characterized as the eigenvalues of $\boldsymbol{H}_\theta(\hbar)$ in $e^{-2i[0,\theta)}\mathbb{R}_+$ for some $\theta\in(0,\theta_0]$. We will denote this set by $\operatorname{Res}\boldsymbol{H}(\hbar)$ and include them with their multiplicity (see below).

Next we introduce the corresponding resonant states (see, e.g., [56, pp. 12–13]).

Definition 4.5. Let $z_0(\hbar)$ be a resonance of $\boldsymbol{H}(\hbar)$. An element

$$\boldsymbol{u}\in\operatorname{Ran}\left(\frac{1}{2\pi i}\int_{|z-z_0|\ll\varepsilon}(\boldsymbol{H}(\hbar)-z\mathbf{1})^{-1}\,dz\right) \tag{4.4}$$

such that $(\boldsymbol{H}(\hbar)-z_0(\hbar)\mathbf{1})\boldsymbol{u}=0$ will be called a *resonant state* corresponding to the resonance $z_0(\hbar)$.

If $z_0(\hbar)$ is a resonance then the dimension of the range appearing in (4.4) is finite (i.e. the spectral projection is of finite rank) and this number is what we take to be the multiplicity of $z_0(\hbar)$ [56].

4.4. Reference operator. As for the scalar valued setting in Sjöstrand and Zworski [57] and Sjöstrand [54], we construct a selfadjoint reference operator $\boldsymbol{H}^\sharp(\hbar)$ from the matrix valued operator $\boldsymbol{H}(\hbar)$. Let $R>R_0'$ and introduce the flat n-torus $\mathbb{T}^n:=(\mathbb{R}/R^\sharp\mathbb{Z})^n$, where $R^\sharp>2R$. Define

$$\mathcal{H}^\sharp=\mathcal{H}_{R_0'}\oplus(L^2(\mathbb{T}^n\setminus B(0,R_0'))\otimes\mathbb{C}^2),$$

where the decomposition is orthogonal. Put, for $1_{B(0,R_0')}\prec\chi\prec 1_{B(0,R)}$,

$$\mathcal{D}^\sharp=\{\boldsymbol{u}\in\mathcal{H}^\sharp:\boldsymbol{\chi}\boldsymbol{u}\in\mathcal{D},\,(\mathbf{1}-\boldsymbol{\chi})\boldsymbol{u}\in H^2(\mathbb{T}^n)\otimes\mathbb{C}^2\} \tag{4.5}$$

and

$$\boldsymbol{H}^\sharp(\hbar)\boldsymbol{u}=\boldsymbol{H}(\hbar)\boldsymbol{\chi}\boldsymbol{u}+\boldsymbol{Q}^\sharp(\hbar)(\mathbf{1}-\boldsymbol{\chi})\boldsymbol{u},$$

where

$$\boldsymbol{Q}^\sharp\boldsymbol{u}=\sum_{|\alpha|\leq 2}\boldsymbol{a}_\alpha^\sharp(x,\hbar)(\hbar D)^\alpha$$

is a formally selfadjoint operator on $\mathbb{T}^n$. Viewing $B(0,R)\subset\mathbb{T}^n$ we assume $\boldsymbol{a}_\alpha^\sharp(x,\hbar)=\boldsymbol{a}_\alpha(x,\hbar)$ for $|x|<R$. Moreover we make the assumption that $\boldsymbol{a}_\alpha$ is independent of $\hbar$ for $|\alpha|=2$, $\boldsymbol{a}_\alpha^\sharp\in C_b^\infty(\mathbb{T})$, uniformly in $\hbar$ and satisfies the uniform ellipticity condition for matrix valued operators (see, e.g., Agranovich [2, Section 3.2]):

$$\left|\det\sum_{|\alpha|=2}\boldsymbol{a}_\alpha^\sharp(x)\xi^\alpha\right|\geq C>0,\qquad x,\xi\in\mathbb{R}^n,\ |\xi|=1. \tag{4.6}$$

In Proposition A.1 of the Appendix, we prove that the operator $\boldsymbol{H}^\sharp(\hbar):\mathcal{H}^\sharp\to\mathcal{H}^\sharp$ is selfadjoint and its spectrum is purely discrete.

We impose the following condition:

Assumption 4.6. Let the reference operator $\boldsymbol{H}^\sharp$ be defined under the above-mentioned requirements. Suppose

$$\#\{\, z \in \operatorname{spec}(\boldsymbol{H}^\sharp(\hbar)) : |z| \leq \lambda\} \leq C(\lambda/\hbar^2)^{n^\sharp/2} \tag{4.7}$$

for some $C > 0$ and $n^\sharp \geq n$.

Below (see (6.9)) we show that Assumption 4.6 implies that

$$\#\{z \in \operatorname{Res} \boldsymbol{H}(\hbar) : 0 < l_0 \leq \operatorname{Re} z \leq r_0, 0 \leq -\operatorname{Im} z \leq c_0\} \leq C(l_0, r_0, c_0)\hbar^{-n^{\#}}. \tag{4.8}$$

Example 4.7. For the matrix valued Schrödinger operator $\boldsymbol{T}(\hbar)$ in (1.1), Assumption 4.6 is fulfilled with $n^\sharp = n$.

5. Results

Below we always require that Assumptions 4.1 and 4.6 are satisfied. Moreover, $\boldsymbol{J}(\hbar)$ denotes either $\boldsymbol{J}_\infty(\hbar)$ or $\boldsymbol{J}_R(\hbar)$; rigorous definitions are found in Chapter 7.

5.1. Individual resonances. The case $R_0' < R_1$. A sketch is given in Figure 1; recall that $\operatorname{supp} W \subset \mathbb{R}^n \setminus B(0, R_1)$, $R_0 < R_1$. We obtain the following result.

Theorem 5.1. *Let Assumptions 4.1 and 4.6 hold.*

(1) *If $R_0' < R_1$ and $z_0(\hbar)$ is a resonance of $\boldsymbol{H}(\hbar)$ in*

$$[l_0, r_0] + i\left[-\left(\frac{\hbar^{n^\sharp+1}}{C \log \frac{1}{\hbar}} \right)^2, 0 \right], \tag{5.1}$$

then there is an $\hbar_0 \in (0, 1]$ such that, for $0 < \hbar \leq \hbar_0$, $\boldsymbol{J}(\hbar)$ has an eigenvalue in

$$\left[\operatorname{Re} z_0(h) - \varepsilon(\hbar) \log \frac{1}{\hbar}, \operatorname{Re} z_0(\hbar) + \varepsilon(\hbar) \log \frac{1}{\hbar} \right] + i[-\varepsilon(\hbar), 0],$$

where $\varepsilon(\hbar) = C\hbar^{-n^\sharp-1/2}\sqrt{-\operatorname{Im} z_0(\hbar)} + e^{-\gamma(R_1)/\hbar}$. Here the constant $\gamma(R_1) > 0$ satisfies $\lim_{R_1 \to \infty} \gamma(R_1)/R_1 = C_0^{-1}$ for some constant $C_0 > 0$.

(2) *If $w_0(\hbar)$ is an eigenvalue of $\boldsymbol{J}(\hbar)$ in (5.1), then there is an $\hbar_0 \in (0, 1]$ such that for $0 < \hbar \leq \hbar_0$ and $B > 0$ fixed, $\boldsymbol{H}(\hbar)$ has a resonance in*

$$\left[\operatorname{Re} w_0(\hbar) - \delta(\hbar) \log \frac{1}{\hbar}, \operatorname{Re} w_0(\hbar) + \delta(\hbar) \log \frac{1}{\hbar} \right] + i[-\delta(\hbar), 0],$$

where $\delta(\hbar) = CB\hbar^{-n^\sharp-1}\sqrt{-\operatorname{Im} w_0(\hbar)} + e^{-B/\hbar}$.

5.2. Individual resonances. The case $R_1 < R_0'$. For this case we shall henceforth impose the following assumption.

Assumption 5.2. Suppose that the principal symbol $\boldsymbol{h}_0$ of $\boldsymbol{H}$ is scalar valued away from the black box, i.e.,

$$\boldsymbol{h}_0 = h_0 \otimes \boldsymbol{I}_2 \quad \text{for } |x| \geq R_0, \tag{5.2}$$

Under Assumption 5.2, $\boldsymbol{h}_0$ generates classical dynamics. The Hamiltonian vector field $X_{h_0} := (\partial_\xi h_0, -\partial_x h_0)$ of h_0 generates a flow via the solution of the Hamilton–Jacobi equations

$$\begin{cases} \frac{d}{dt}(x(t),\xi(t)) = X_{h_0}(x(t),\xi(t)), \\ (x(0),\xi(0)) = (x_0,\xi_0), \end{cases}$$

that we denote by $\Phi^t(x_0,\xi_0)$.

Definition 5.3. A Hamiltonian $\boldsymbol{H}(\hbar)$ is said to be *nontrapping* for energies $\lambda \in [l_0, r_0]$ if the corresponding classical trajectories $\Phi^t(x,\xi) = (x(t),\xi(t))$ at energy λ, i.e. such that $h_0(x(t),\xi(t)) = \lambda$, fulfill

$$\lim_{|t|\to\infty} |x(t)| = \infty \quad \text{and} \quad |x(t)| > R_0 \quad \text{for all } t \in \mathbb{R}.$$

In addition to Assumption 5.2 we need to impose the following condition:

Assumption 5.4. Suppose that

$$\boldsymbol{H}_0 = \boldsymbol{H}|_{|x|>R_0} \quad \text{is nontrapping for energies in } [l_0, r_0]. \tag{5.3}$$

Theorem 5.5. *Let* $R_1 < R_0'$ *and suppose Assumptions* 5.2–5.4 *hold. Suppose, moreover, that the uniform estimates in* (11.2) *are satisfied. Then the conclusion of Theorem* 5.1.1 *holds true with*

$$\varepsilon(\hbar) = C\hbar^{-n^\sharp-3/2}\sqrt{-\operatorname{Im} z_0(\hbar)} + \mathcal{O}(\hbar^\infty).$$

5.3. Clusters of resonances. The case $R_0' < R_1$. Bear in mind that the notation $\operatorname{Count}(\boldsymbol{H}(\hbar),\Omega(\hbar))$ is used for the number of resonances in $\Omega(\hbar)$ (and, similarly, $\operatorname{Count}(\boldsymbol{J}(\hbar),\Omega(\hbar))$ denotes the number of eigenvalues of $\boldsymbol{J}(\hbar)$ in $\Omega(\hbar)$), counting multiplicities.

Theorem 5.6. *Suppose* $R_0' < R_1$. *Fix* $0 < l_0 < r_0 < \infty$ *and let* $\boldsymbol{J}(\hbar)$ *denote either* $\boldsymbol{J}_\infty(\hbar)$, *or* $\boldsymbol{J}_R(\hbar)$. *Let*

$$\Omega(\hbar) = [l(\hbar), r(\hbar)] + i[-c(\hbar), 0], \tag{5.4}$$

where $l_0 \le l(\hbar) < r(\hbar) \le r_0$,

$$e^{-\hbar^{-2/3+\varepsilon_0}} \le c(\hbar) \le \hbar^M \quad \text{and} \quad 2c(\hbar) \le r(\hbar) - l(\hbar)$$

for some $\varepsilon_0 \in (0, 2/3)$ *and a positive constant* M. *Then there exists* $N > 0$ *such that*

$$\operatorname{Count}(\boldsymbol{J}(\hbar),\Omega_-(\hbar)) \le \operatorname{Count}(\boldsymbol{H}(\hbar),\Omega(\hbar)) \le \operatorname{Count}(\boldsymbol{J}(\hbar),\Omega_+(\hbar)) \tag{5.5}$$

where

$$\begin{aligned} \Omega_-(\hbar) &= [l(\hbar) + c(\hbar), r(\hbar) - c(\hbar)] + i[-\hbar^N c(\hbar)^2, 0], \\ \Omega_+(\hbar) &= [l(\hbar) - \hbar^{-N} c(\hbar)^{1/2}, r(\hbar) + \hbar^{-N} c(\hbar)^{1/2}] + i[-\hbar^{-N} c(\hbar)^{1/2}, 0]. \end{aligned}$$

If one weakens the lower bound for $c(\hbar)$ such that $e^{-C/\hbar} \le c(\hbar)$, then the first inequality in (5.5) is still valid.

5.4. Clusters of resonances. The case $R_1 < R_0'$. We obtain the following result.

THEOREM 5.7. *Let $R_1 < R_0'$ and suppose Assumptions* 5.2–5.4 *hold. Suppose, moreover, that the uniform estimates in* (11.2) *are satisfied. Then the assertions of Theorem* 5.6 *are valid with $\Omega_+(\hbar)$ replaced by*

$$\begin{aligned}\widetilde{\Omega}_+(\hbar) = [l(\hbar) - \hbar^{-N}c(\hbar)^{1/2} - \mathcal{O}(\hbar^\infty), r(\hbar) + \hbar^{-N}c(\hbar)^{1/2} + \mathcal{O}(\hbar^\infty)]\\ + i[-\hbar^{-N}c(\hbar)^{1/2} - \mathcal{O}(\hbar^\infty), 0].\end{aligned}$$

6. Cutoff resolvent estimate

We will need the following important a priori cutoff resolvent estimate which comes from estimates for the resolvent of the matrix valued scaled operator $\boldsymbol{H}_\theta(\hbar)$.

PROPOSITION 6.1. *Let Assumptions* 4.1 *and* 4.6 *hold. For any simply connected*

$$\Omega \Subset S_\theta = \{z : \max(-\pi, 2\theta - 2\pi < -\arg z < 2\theta)\}$$

and $g : (0, \hbar_0) \to (0, 1)$ for some $\hbar_0 > 0$ there is a constant $A = A(\Omega) > 0$ and $\hbar_1 \in (0, \hbar_0)$ such that

$$\|\boldsymbol{\chi}(\boldsymbol{H}(\hbar) - z\mathbf{1})^{-1}\boldsymbol{\chi}\|_{\mathcal{B}(\mathcal{H})} \leq A\exp(-A\hbar^{-n^\sharp}\log g(\hbar))\\ \forall z \in \Omega \setminus \bigcup_{z_j \in \mathrm{Res}(\boldsymbol{H})\cap\Omega} D(z_j, g(\hbar)), \qquad (6.1)$$

for all $\hbar \in (0, \hbar_1)$ and $1_{B(0,R_0)} \prec \chi \in C_0^\infty(\mathbb{R}^n)$. Here $D(z, r) = \{w : |w - z| \leq r\}$.

Before we give the proof, we mention that the estimate goes back to Stefanov and Vodev [64, 65], who established a global scalar valued version which allowed them to prove that for scattering by compactly supported perturbations in odd dimensional Euclidean space, existence of localized quasimodes implies existence of resonances converging to the real axis. Within the scalar valued setting, a local version, like the one in (6.1), was first proved by Tang and Zworski [66, Lemma 1] but it essentially comes from Sjöstrand's work on the local trace formula for resonances [54]. Our proof in the matrix valued setting borrows ideas from Sjöstrand and Zworski [57], Sjöstrand [54], and [66]; see also Sjöstrand [56, Lemma 11.3].

LEMMA 6.2. *Let $f \in C_0^\infty(\mathbb{R})$ and $1_{B(0,R_0)} \prec \psi_0 \prec \widetilde{\chi}_0$. Then*

$$\|(\mathbf{1} - \widetilde{\boldsymbol{\chi}}_0)f(\boldsymbol{H}^\sharp)\boldsymbol{\psi}_0\|_{\mathrm{tr}} = \mathcal{O}(\hbar^\infty),$$

where $f(\boldsymbol{H}^\sharp)$ is defined as in (3.3).

Proof. Let $\psi_0 \prec \psi_1 \prec \cdots \prec \psi_N \prec \widetilde{\chi}_0$ and iterate the identity

$$\begin{aligned}(\boldsymbol{H}^\sharp - z\mathbf{1})^{-1}\boldsymbol{\psi}_j = \boldsymbol{\psi}_{j+1}(\boldsymbol{Q}^\sharp - z\mathbf{1})^{-1}\boldsymbol{\psi}_j\\ + (\boldsymbol{H}^\sharp - z\mathbf{1})^{-1}[\boldsymbol{Q}^\sharp, \boldsymbol{\psi}_{j+1}](\boldsymbol{Q}^\sharp - z\mathbf{1})^{-1}\boldsymbol{\psi}_j\end{aligned}$$

N times to obtain

$$(\boldsymbol{H}^\sharp - z\mathbf{1})^{-1}\boldsymbol{\psi}_0 = \sum_{j=1}^{N} \boldsymbol{\psi}_j(\boldsymbol{Q}^\sharp - z\mathbf{1})^{-1}[\boldsymbol{Q}^\sharp, \boldsymbol{\psi}_{j-1}](\boldsymbol{Q}^\sharp - z\mathbf{1})^{-1} \times [\boldsymbol{Q}^\sharp, \boldsymbol{\psi}_{j-2}]\cdots[\boldsymbol{Q}^\sharp, \boldsymbol{\psi}_1](\boldsymbol{Q}^\sharp - z\mathbf{1})^{-1}\boldsymbol{\psi}_0$$
$$+ (\boldsymbol{H}^\sharp - z\mathbf{1})^{-1}[\boldsymbol{Q}^\sharp, \boldsymbol{\psi}_N](\boldsymbol{Q}^\sharp - z\mathbf{1})^{-1}[\boldsymbol{Q}^\sharp, \boldsymbol{\psi}_{N-1}]\cdots[\boldsymbol{Q}^\sharp, \boldsymbol{\psi}_1](\boldsymbol{Q}^\sharp - z\mathbf{1})^{-1}\boldsymbol{\psi}_0.$$

Since $(1-\widetilde{\chi}_0)\psi_j = 0$ for all $1 \le j \le N$ we get

$$(\mathbf{1} - \widetilde{\boldsymbol{\chi}}_0)(\boldsymbol{H}^\sharp - z\mathbf{1})^{-1}\boldsymbol{\psi}_0$$
$$= (\mathbf{1} - \widetilde{\boldsymbol{\chi}}_0)(\boldsymbol{H}^\sharp - z\mathbf{1})^{-1}[\boldsymbol{Q}^\sharp, \boldsymbol{\psi}_N](\boldsymbol{Q}^\sharp - z\mathbf{1})^{-1}[\boldsymbol{Q}^\sharp, \boldsymbol{\psi}_{N-1}]\cdots[\boldsymbol{Q}^\sharp, \boldsymbol{\psi}_1](\boldsymbol{Q}^\sharp - z\mathbf{1})^{-1}\boldsymbol{\psi}_0.$$

Using $(\boldsymbol{H}^\sharp - z\mathbf{1})^{-1} = \mathcal{O}(|\operatorname{Im} z|^{-1})$ and $[\boldsymbol{H}^\sharp, \boldsymbol{\psi}_j] = \mathcal{O}(\hbar)$ we see that $(\mathbf{1}-\widetilde{\boldsymbol{\chi}}_0)(\boldsymbol{H}^\sharp - z\mathbf{1})^{-1}\boldsymbol{\psi}_0$ is negligible in the sense that for all $N \in \mathbb{N}$ there exists $M(N) > 0$ such that its norm is $\mathcal{O}_N(1)\hbar^N|\operatorname{Im} z|^{-M(N)}$. With $N > n$ we have

$$\|(\mathbf{1} - \widetilde{\boldsymbol{\chi}}_0)(\boldsymbol{H}^\sharp - z\mathbf{1})^{-1}\boldsymbol{\psi}_0\|_{\mathrm{tr}} = \mathcal{O}_N(1)\frac{\hbar^{N-n}}{|\operatorname{Im} z|^{-N}}$$

and it follows that $\|(\mathbf{1} - \widetilde{\boldsymbol{\chi}}_0)f(\boldsymbol{H}^\sharp)\boldsymbol{\psi}_0\|_{\mathrm{tr}} = \mathcal{O}(\hbar^\infty)$. ∎

For $|x| \ge R_0'$ the operator $\boldsymbol{H}_\theta = H_\theta \otimes \mathbf{1}$ is a scalar elliptic differential operator with principal symbol $\boldsymbol{h}_\theta = h_\theta \otimes \boldsymbol{I}_2$ where h_θ globally takes values in the sector $e^{i[-2(\theta+\varepsilon_0),\varepsilon_0]}\mathbb{R}_{\ge 0}$. Let F be a smooth map from a neighbourhood of this sector into itself with the property that $F = \mathrm{Id}$ for $|z| \gg 1$ as well as in a neighbourhood of the ray $e^{-2i\theta}\mathbb{R}_{\ge 0}$ and $\overline{\Omega} \cap \operatorname{Ran} F = \emptyset$. We observe that $F \circ h_\theta$ is well-defined with values away from $\overline{\Omega}$. Moreover, with $f := F|_{\mathbb{R}} =: x + g(x)$ for some $g \in C_0^\infty(\mathbb{R}^n)$ we deduce from the functional calculus (see Chapter 3) that

$$f(\boldsymbol{H}^\sharp) = \boldsymbol{H}^\sharp + g(\boldsymbol{H}^\sharp)$$

with $g(\boldsymbol{H}^\sharp) : \mathcal{H}^\sharp \to \mathcal{H}^\sharp$ of rank $\mathcal{O}(\hbar^{-n^\sharp})$. Let $\chi_0 + \chi_1 + \chi_2 = 1$ be such that $1_{\overline{B(0,R_0')}} \prec \chi_0 \prec 1_{B(0,\widetilde{R})}$ and $\chi_1 \in C_0^\infty(\Gamma_\theta)$ with $\chi_0 + \chi_1 = 1$ near $\overline{B(0,R_2)}$ where $R_2 > \widetilde{R}$ is such that $F \circ h_\theta = h_\theta$ for $|x| \ge R_2$. For $\chi_j \prec \widetilde{\chi}_j \in C^\infty(\Gamma_\theta)$, where the $\widetilde{\chi}_j$ have the same support properties, we define

$$\widetilde{\boldsymbol{H}}_\theta = \widetilde{\boldsymbol{\chi}}_0 f(\boldsymbol{H}^\sharp)\boldsymbol{\chi}_0 + \widetilde{\boldsymbol{\chi}}_1\boldsymbol{R}_F\boldsymbol{\chi}_1 + \widetilde{\boldsymbol{\chi}}_2\boldsymbol{H}_\theta\boldsymbol{\chi}_2, \tag{6.2}$$

where $\boldsymbol{R}_F$ is an $\hbar$-pseudodifferential operator with the leading symbol $F(h_\theta)\boldsymbol{I}_2$ such that the total Weyl symbol of $\boldsymbol{R}_F - \boldsymbol{H}_\theta$ for x near $\operatorname{supp}\chi_1$ has compact support in ξ.

LEMMA 6.3. *Provided $\hbar > 0$ is sufficiently small, the operator $\widetilde{\boldsymbol{H}}_\theta - z\mathbf{1}$ is invertible for $z \in \Omega$ with*

$$(\widetilde{\boldsymbol{H}}_\theta - z\mathbf{1})^{-1} = \mathcal{O}(1) : \mathcal{H} \to \mathcal{D},$$

uniformly for $z \in \Omega$.

Proof. Rewrite (6.2) as

$$\widetilde{\boldsymbol{H}}_\theta = \boldsymbol{H}_\theta + \widetilde{\boldsymbol{\chi}}_0 g(\boldsymbol{H}^\sharp)\boldsymbol{\chi}_0 + \widetilde{\boldsymbol{\chi}}_1(\boldsymbol{R}_F - \boldsymbol{H}_\theta)\boldsymbol{\chi}_1 \tag{6.3}$$

to see that $\widetilde{\boldsymbol{H}}_\theta$ is a perturbation of $\boldsymbol{H}_\theta$ by compact operators [51, 12]. It follows (see Proposition 4.3) that $\widetilde{\boldsymbol{H}}_\theta - z\mathbf{1}$ is Fredholm of index zero. Therefore the claim follows provided we can show the a priori estimate

$$\|\boldsymbol{u}\| \leq C\|(\widetilde{\boldsymbol{H}}_\theta - z\mathbf{1})\boldsymbol{u}\| \quad \text{for all } \boldsymbol{u} \in \mathcal{D}. \tag{6.4}$$

Indeed, this estimate implies that $\operatorname{Ker}(\widetilde{\boldsymbol{H}}_\theta - z\mathbf{1}) = \{\mathbf{0}\}$ so that codim $\operatorname{Ran}(\widetilde{\boldsymbol{H}}_\theta - z\mathbf{1}) = \dim \operatorname{coker}(\widetilde{\boldsymbol{H}}_\theta - z\mathbf{1}) = 0$ and, consequently, $\widetilde{\boldsymbol{H}}_\theta - z\mathbf{1}$ is bijective. In order to prove (6.4), let $\{\psi_j\}_{j=0}^2 \subset C_b^\infty(\Gamma_\theta)$ have the same support properties as the χ_j with $\psi_0 \prec \chi_0$ and

$$\psi_0^2 + \psi_1^2 + \psi_2^2 = 1.$$

Using (6.2), $1 = \chi_0 + (1 - \chi_0)$ and $\psi_0 \prec \chi_0$, in conjunction with Lemma 6.2, we deduce that

$$\|(\widetilde{\boldsymbol{H}}_\theta - z\mathbf{1})\boldsymbol{\psi}_0\boldsymbol{u}\|^2 = \|(f(\boldsymbol{H}^\sharp) - z\mathbf{1})\boldsymbol{\psi}_0\boldsymbol{u}\|^2 + \mathcal{O}(\hbar^\infty)\|\boldsymbol{u}\|^2. \tag{6.5}$$

Moreover, for some constants C_j, $j = 1, 2$, we have the elliptic estimates

$$C_j^2\|\boldsymbol{\psi}_j\boldsymbol{u}\|^2 \leq \|(\widetilde{\boldsymbol{H}}_\theta - z\mathbf{1})\boldsymbol{\psi}_j\boldsymbol{u}\|^2.$$

As a consequence,

$$\begin{aligned}\|(\widetilde{\boldsymbol{H}}_\theta - z\mathbf{1})\boldsymbol{u}\|^2 &= \sum_{j=0}^{2} \|\boldsymbol{\psi}_j(\widetilde{\boldsymbol{H}}_\theta - z\mathbf{1})\boldsymbol{u}\|^2 \\ &\geq \sum_{j=0}^{2} (\|(\widetilde{\boldsymbol{H}}_\theta - z\mathbf{1})\boldsymbol{\psi}_j\boldsymbol{u}\| - \|[\boldsymbol{\psi}_j, \widetilde{\boldsymbol{H}}_\theta]\boldsymbol{u}\|)^2 \geq C_0^2\|\boldsymbol{u}\|^2 - \mathcal{O}(\hbar)\|\boldsymbol{u}\|_{\mathcal{D}}^2,\end{aligned}$$

which completes the proof of the invertibility of $\widetilde{\boldsymbol{H}}_\theta - z\mathbf{1}$ provided $\hbar$ is small enough because

$$\|\boldsymbol{u}\|_{\mathcal{D}}^2 \leq C(\|(\widetilde{\boldsymbol{H}}_\theta - z\mathbf{1})\boldsymbol{u}\|^2 + \|\boldsymbol{u}\|^2). \ \blacksquare$$

The next lemma is an improvement of Lemma 6.3:

LEMMA 6.4. *There exists an operator $\boldsymbol{S} : \mathcal{H} \to \mathcal{H}$ of rank $\mathcal{O}(\hbar^{-n^\sharp})$, compactly supported in the sense that $\boldsymbol{S} = \boldsymbol{\chi}\boldsymbol{S}\boldsymbol{\chi}$ for $\chi \succ 1_{B(0,R)}$ if R is sufficiently large, such that*

$$(\boldsymbol{H}_\theta + \boldsymbol{S} - z\mathbf{1})^{-1} = \mathcal{O}(1) : \mathcal{H} \to \mathcal{D},$$

uniformly for $z \in \overline{\Omega}$.

Proof. The support property of the symbol of $\boldsymbol{R}_F - \boldsymbol{H}_\theta$ implies that we can find $\boldsymbol{T}_F$ of rank $\mathcal{O}(\hbar^{-n^\sharp})$ such that

$$\widetilde{\boldsymbol{\chi}}_1((\boldsymbol{R}_F - \boldsymbol{H}_\theta) - \boldsymbol{T}_F)\boldsymbol{\chi}_1 = \mathcal{O}(\hbar^\infty).$$

Using this $\boldsymbol{T}_F$ to replace the latter term in (6.3) we define

$$\boldsymbol{S} := \widetilde{\boldsymbol{\chi}}_0 g(\boldsymbol{H}^\sharp)\boldsymbol{\chi}_0 + \widetilde{\boldsymbol{\chi}}_1\boldsymbol{T}_F\boldsymbol{\chi}_1,$$

which in view of Lemma 6.3 satisfies the desired conclusion. $\blacksquare$

We now pose the Grushin problem and prove the cut-off resolvent estimate (6.1):

Proof of Proposition 6.1. Denote $N = \operatorname{rank}\boldsymbol{S}$ so $N = \mathcal{O}(\hbar^{-n^\sharp})$ and let $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_N\}$ be an orthonormal basis of $\operatorname{Ran}\langle\boldsymbol{H}\rangle^{-2}\boldsymbol{S}^*$, where $\boldsymbol{S}^*$ denotes the adjoint of $\boldsymbol{S} : \mathcal{H} \to \mathcal{H}$.

Define

$$\widetilde{X}_b : (u_{jb})_{j\in\{1,\dots,N\}} \mapsto \sum_{j=1}^{N} u_{jb} e_j,$$

$$X_b(z) : (u_{jb})_{j\in\{1,\dots,N\}} \mapsto \sum_{j=1}^{N} u_{jb}(\boldsymbol{H}_\theta + \boldsymbol{S} - z)\boldsymbol{e}_j, \quad z \in \Omega,$$

$$X_a : \boldsymbol{u} \mapsto (\langle \boldsymbol{u}, \boldsymbol{e}_j\rangle_{\mathcal{D}})_{j\in\{1,\dots,N\}},$$

$$\mathcal{T}(z) = \begin{pmatrix} \boldsymbol{H}_\theta - z & X_b(z) \\ X_a & 0 \end{pmatrix} : \mathcal{D} \oplus \mathbb{C}^N \to \mathcal{H} \oplus \mathbb{C}^N.$$

Denote the inverse of $\mathcal{T}(z)$ by

$$\mathcal{Y}(z) = \begin{pmatrix} \boldsymbol{Y}(z) & Y_a(z) \\ Y_b(z) & Y_{ba}(z) \end{pmatrix} : \mathcal{H} \oplus \mathbb{C}^N \to \mathcal{D} \oplus \mathbb{C}^N.$$

Here the entries satisfy

$$\begin{aligned} \boldsymbol{Y}(z) &= (1 - \widetilde{X}_b X_a)(\boldsymbol{H}_\theta + \boldsymbol{S} - z\boldsymbol{1})^{-1}, \\ Y_a(z) &= -(1 - \widetilde{X}_b X_a)(\boldsymbol{H}_\theta + \boldsymbol{S} - z\boldsymbol{1})^{-1}(\boldsymbol{H}_\theta - z\boldsymbol{1})\widetilde{X}_b + \widetilde{X}_b, \\ Y_b(z) &= X_a(\boldsymbol{H}_\theta + \boldsymbol{S} - z\boldsymbol{1})^{-1}, \\ Y_{ba}(z) &= -X_a(\boldsymbol{H}_\theta + \boldsymbol{S} - z\boldsymbol{1})^{-1}(\boldsymbol{H}_\theta - z\boldsymbol{1})\tilde{X}_b. \end{aligned}$$

Some further identities are (with prime standing for ∂_z):

$$\begin{aligned} (\boldsymbol{H}_\theta - z\boldsymbol{1})^{-1} &= \boldsymbol{Y}(z) - Y_a(z)Y_{ba}^{-1}(z)Y_b(z), \\ (\boldsymbol{H}_\theta - z\boldsymbol{1})Y_a(z) &= -X_b(z)Y_{ba}(z), \\ Y_b(z)(\boldsymbol{H}_\theta - z\boldsymbol{1}) &= Y_{ba}(z)X_a, \\ Y_b(z)Y_a(z) &= Y_{ba}'(z) + Y_b(z)\boldsymbol{R}_b'(z)Y_a(z). \end{aligned} \tag{6.6}$$

The functions $Y_a(z), Y_b(z), \boldsymbol{Y}(z)$ and $Y_{ab}(z)$ are analytic for $z \in \Omega$.

If we introduce the notation $D(z;\hbar) := \det Y_{ba}(z)$ we have by Cramer's rule

$$Y_{ba}^{-1}(z) = \frac{1}{D(z;\hbar)}\widetilde{Y}_{ba}(z), \tag{6.7}$$

where $\widetilde{Y}_{ba}$ is the adjugate of Y_{ba}, having cofactor elements C_{ij}. We estimate $\widetilde{Y}_{ba}$ from above according to

$$\|\widetilde{Y}_{ba}(z)\| \le N \sup_{1\le i,j\le N} |C_{ij}| \le NC^N \le C\hbar^{-n^\sharp} e^{C\hbar^{-n^\sharp}} \qquad \text{for some } C > 0, \tag{6.8}$$

where we have also used the fact that $\|Y_{ba}\| = \mathcal{O}(1)$ so that all minors are bounded.

Next we estimate the denominator in (6.7) from below. Start with the factorization

$$D(z;\hbar) = G(z;\hbar)D_w(z;\hbar) \quad \text{where} \quad D_w(z;\hbar) = \prod_{z_j \in \mathrm{Res}(\boldsymbol{H})\cap\Omega} (z - z_j).$$

As always, the z_j are counted according to their multiplicity. By a classical result on the Grushin problem, z is an eigenvalue of $\boldsymbol{H}_\theta$ if and only if zero is an eigenvalue of the effective Hamiltonian $Y_{ba}(z)$ and multiplicities agree.

It follows that G and its inverse are both holomorphic on Ω. Notice also how the bound $|D(z;\hbar)| \leq e^{C\hbar^{-n^\sharp}}$ in conjunction with Jensen's inequality for the number of zeros of an analytic function immediately implies that

$$\#(\operatorname{Res} \boldsymbol{H}(\hbar) \cap \Omega) \leq C\hbar^{-n^\sharp}. \tag{6.9}$$

This in turn implies $|D_w| \leq Ce^{C\hbar^{-n^\sharp}}$ for all $z \in \Omega$. Since $\|Y_{ba}^{-1}(z)\|$ is uniformly bounded if we stay uniformly away from $\operatorname{Res} \boldsymbol{H}$ we obtain similarly for any $\delta > 0$ the lower bound

$$|D_w(z;\hbar)| \geq e^{-C\hbar^{-n^\sharp}}, \quad z \in \Omega_\delta := \{z \in \Omega : \operatorname{Im} z \geq \delta\},$$

because the resonances in Ω are confined to $\Omega \cap \mathbb{C}_-$. Hence $|G(z;\hbar)| \geq e^{-C\hbar^{-n^\sharp}}$ for all $z \in \Omega_\delta$. With $\widetilde{\Omega} \Subset \Omega$ any simply connected relatively open $\hbar$-independent subset of Ω, consider the nonnegative harmonic function

$$0 \leq \ell(z;\hbar) = C\hbar^{-n^\sharp} - \log|G(z;\hbar)|$$

on $\widetilde{\Omega}$. Then $\ell(z;\hbar) \leq C\hbar^{-n^\sharp}$ on Ω_δ and by Harnack's inequality for nonnegative harmonic functions, ℓ is of uniformly constant order of magnitude throughout all of $\widetilde{\Omega}$. Consequently, $\log|G(z;\hbar)| \geq -C\hbar^{-n^\sharp}$, i.e. $|G(z;\hbar)| \geq e^{-C\hbar^{-n^\sharp}}$ on $\widetilde{\Omega}$. Thus, assuming

$$z \in \widetilde{\Omega} \setminus \bigcup_{z_j \in \operatorname{Res}(\boldsymbol{H}) \cap \widetilde{\Omega}} D(z_j, g(\hbar)),$$

we get

$$|D(z;\hbar)| = |G(z;\hbar)|\,|D_w(z;\hbar)| \geq e^{-C\hbar^{-n^\sharp}} (g(\hbar))^{C\hbar^{-n^\sharp}} \geq Ce^{C\hbar^{-n^\sharp}\log g},$$

where in the last step the exponential $e^{-C\hbar^{-n^\sharp}}$ has been absorbed by the exponential factor containing g, possibly at the expense of "worse" constants. Combining this with (6.7) and (6.8) we see that there is an $A > 0$ such that $\|Y_{ba}^{-1}\| \leq Ae^{-A\hbar^{-n^\sharp}\log g}$. The proposition now follows from (6.6) and

$$\boldsymbol{\chi}(\boldsymbol{H} - z\mathbf{1})^{-1}\boldsymbol{\chi} = \boldsymbol{\chi}(\boldsymbol{H}_\theta - z\mathbf{1})^{-1}\boldsymbol{\chi}, \quad z \in \Omega, \quad 1_{B(0,R_0)} \prec \chi \in C_0^\infty(\mathbb{R}^n),$$

together with the fact that $\|\boldsymbol{Y}(z)\|$, $\|Y_a(z)\|$ and $\|Y_b(z)\|$ are all $\mathcal{O}(1)$ for $z \in \Omega$. ∎

Remark 6.5. The estimate (6.9) on the number of resonances is also derived for the matrix valued Schrödinger operator (1.1) within the simpler setting in Nedelec [39, p. 219]; for this case $n^\sharp = n$.

7. Hamiltonians with complex absorbing potentials

Let $W \in L^\infty(\mathbb{R}^n)$ be a complex valued potential such that

$$\operatorname{Re} W(x) \geq 0, \quad \operatorname{supp} W \subset \mathbb{R}^n \setminus B(0,R_1), \quad R_0 < R_1.$$

We also assume that for some $\delta_0 > 0$ and $R_2 > R_1$,

$$\operatorname{Re} W \geq \delta_0 \quad \text{for } |x| > R_2.$$

Furthermore, we assume that

$$|\operatorname{Im} W| \leq C\sqrt{\operatorname{Re} W}. \tag{7.1}$$

We point out that this condition is obviously fulfilled in the special case of a real W. Then we define

$$\boldsymbol{W} = \begin{pmatrix} W & 0 \\ 0 & W \end{pmatrix}$$

We now define two CAP operators. First,

$$\boldsymbol{J}_\infty(\hbar) = \boldsymbol{H}(\hbar) - i\boldsymbol{W} \quad \text{ on } \mathcal{H}.$$

Second, given $R > R_2$, let $\mathcal{H}_R(\hbar)$ be as above (roughly speaking it is the restriction of $\mathcal{H}$ to the ball $B(0,R)$) and let $\boldsymbol{H}_R(\hbar)$ be the Dirichlet realization of $\boldsymbol{H}(\hbar)$ there. Put

$$\boldsymbol{J}_R(\hbar) = \boldsymbol{H}_R(\hbar) - i\boldsymbol{W} \quad \text{ on } \mathcal{H}_R. \tag{7.2}$$

We see that both $\boldsymbol{J}_\infty(\hbar)$ and $\boldsymbol{J}_R(\hbar)$ are closed unbounded operators with

$$\mathcal{D}(\boldsymbol{J}_\infty(\hbar)) = \mathcal{D}(\boldsymbol{H}(\hbar)) =: \mathcal{D} \quad \text{and} \quad \mathcal{D}(\boldsymbol{J}_R(\hbar)) = \mathcal{D}(\boldsymbol{H}_R(\hbar)).$$

Furthermore, since $\operatorname{Re} W \geq 0$, we see that $\mathbb{C}_+$ is contained in their resolvent sets.

Next we prove that for any $\hbar > 0$, $\operatorname{spec}(\boldsymbol{J}_\infty(\hbar)) \cap \{z \in \mathbb{C} : \operatorname{Im} z > -\delta_0\}$ consists only of eigenvalues of finite multiplicity. The same is true for $\boldsymbol{J}_R(\hbar)$ in all of $\mathbb{C}$ (see below). The underlying ideas are close to the ones in the proof of Proposition 4.3 (see also [56, Theorem 2.2]). Compared to the scalar valued case in [63], we avoid semiclassical elliptic estimates and the Fourier transform.

THEOREM 7.1. *Let Assumption* 4.1 *be satisfied. For any $\hbar > 0$ the resolvent $(\boldsymbol{J}_\infty(\hbar) - z\mathbf{1})^{-1} : \mathcal{H} \to \mathcal{H}$ has a meromorphic extension from $\mathbb{C}_+$ into $\{z \in \mathbb{C} : \operatorname{Im} z > -\delta_0\}$. The poles of $(\boldsymbol{J}_\infty(\hbar) - z\mathbf{1})^{-1}$ are eigenvalues of $\boldsymbol{J}_\infty(\hbar)$ of finite multiplicity.*

Proof. Since the nature of this result is not semiclassical, we will simply take $\hbar = 1$. We will construct an approximate right parametrix of $\boldsymbol{J}_\infty - z\mathbf{1}$. We let $\sum_{i=1}^3 \chi_i = 1$ be a smooth partition of unity such that $\chi_1 = 1$ in a neighbourhood of $B(0,R_0)$, $\operatorname{supp}\chi_1 \subset B(0,(R_0+R_1)/2)$, $\operatorname{supp}\chi_3 \subset \mathbb{R}^n \backslash B(0,R_2)$ and $\chi_3 = 1$ for $|x| \gg 1$. Let $\chi_i \prec \widetilde{\chi}_i$, $i = 1,2,3$, have the same support properties (but $\sum_{i=1}^3 \widetilde{\chi}_i \neq 1$). We introduce a modified version of $\boldsymbol{W}$ by $\widetilde{\boldsymbol{W}} = \operatorname{diag}(\widetilde{W}, \widetilde{W})$, where

$$\widetilde{W} = \begin{cases} \delta_0 & \text{for } |x| < R_2, \\ W & \text{otherwise.} \end{cases}$$

For $z_0 \in \mathbb{C}_+$, we introduce

$$\begin{aligned} \boldsymbol{E}(z) = {} & \widetilde{\boldsymbol{\chi}}_1(\boldsymbol{H} - z_0\mathbf{1})^{-1}\boldsymbol{\chi}_1 + \widetilde{\boldsymbol{\chi}}_2(\boldsymbol{H}_0 - i\boldsymbol{W} - z_0\mathbf{1})^{-1}\boldsymbol{\chi}_2 \\ & + \widetilde{\boldsymbol{\chi}}_3(\boldsymbol{H}_0 - i\widetilde{\boldsymbol{W}} - z\mathbf{1})^{-1}\boldsymbol{\chi}_3. \end{aligned} \tag{7.3}$$

In particular $\operatorname{Re}\widetilde{W} \geq \delta_0$ and this implies $(\boldsymbol{H}_0 - i\widetilde{\boldsymbol{W}} - z\mathbf{1})^{-1}$ is analytic for $\operatorname{Im} z > -\delta_0$. Since $\boldsymbol{H}$ is selfadjoint, this makes $\boldsymbol{E}(z)$ analytic in $\{\operatorname{Im} z > -\delta_0\}$. We now apply $\boldsymbol{J}_\infty - z\mathbf{1}$

to the terms of $\boldsymbol{E}(z)$:

$$\begin{aligned}
&(\boldsymbol{J}_\infty - z\mathbf{1})\widetilde{\boldsymbol{\chi}}_1(\boldsymbol{H} - z_0\mathbf{1})^{-1}\boldsymbol{\chi}_1 \\
&\qquad = (\boldsymbol{H} - i\boldsymbol{W} - z\mathbf{1})\widetilde{\boldsymbol{\chi}}_1(\boldsymbol{H} - z_0\mathbf{1})^{-1}\boldsymbol{\chi}_1 \\
&\qquad = \boldsymbol{H}\widetilde{\boldsymbol{\chi}}_1(\boldsymbol{H} - z_0\mathbf{1})^{-1}\boldsymbol{\chi}_1 - i\boldsymbol{W}\widetilde{\boldsymbol{\chi}}_1(\boldsymbol{H} - z_0\mathbf{1})^{-1}\boldsymbol{\chi}_1 - \widetilde{\boldsymbol{\chi}}_1 z(\boldsymbol{H} - z_0\mathbf{1})^{-1}\boldsymbol{\chi}_1 \\
&\qquad = \widetilde{\boldsymbol{\chi}}_1\boldsymbol{H}(\boldsymbol{H} - z_0\mathbf{1})^{-1}\boldsymbol{\chi}_1 + [\boldsymbol{H}_0, \widetilde{\boldsymbol{\chi}}_1](\boldsymbol{H} - z_0\mathbf{1})^{-1}\boldsymbol{\chi}_1 - \widetilde{\boldsymbol{\chi}}_1 z(\boldsymbol{H} - z_0\mathbf{1})^{-1}\boldsymbol{\chi}_1 \\
&\qquad = [\boldsymbol{H}_0, \widetilde{\boldsymbol{\chi}}_1](\boldsymbol{H} - z_0\mathbf{1})^{-1}\boldsymbol{\chi}_1 + \widetilde{\boldsymbol{\chi}}_1(\boldsymbol{H} - z\mathbf{1})(\boldsymbol{H} - z_0\mathbf{1})^{-1}\boldsymbol{\chi}_1 \\
&\qquad = [\boldsymbol{H}_0, \widetilde{\boldsymbol{\chi}}_1](\boldsymbol{H} - z_0\mathbf{1})^{-1}\boldsymbol{\chi}_1 + \widetilde{\boldsymbol{\chi}}_1(\boldsymbol{H} - z_0\mathbf{1} + (z_0 - z)\mathbf{1})(\boldsymbol{H} - z_0\mathbf{1})^{-1}\boldsymbol{\chi}_1 \\
&\qquad = [\boldsymbol{H}_0, \widetilde{\boldsymbol{\chi}}_1](\boldsymbol{H} - z_0\mathbf{1})^{-1}\boldsymbol{\chi}_1 + \widetilde{\boldsymbol{\chi}}_1(\boldsymbol{I} + (z - z_0)(\boldsymbol{H} - z_0\mathbf{1})^{-1})\boldsymbol{\chi}_1.
\end{aligned}$$

Here we have used the fact that $\boldsymbol{W}\widetilde{\boldsymbol{\chi}}_1\boldsymbol{u} = 0$ for all $\boldsymbol{u} \in \mathcal{H}$, since $\operatorname{supp}\widetilde{\chi}_1 \subset B(0, R_1)$ whereas $\operatorname{supp}\boldsymbol{W} \subset \mathbb{R}^n \setminus B(0, R_1)$ as well as $\boldsymbol{H}\widetilde{\boldsymbol{\chi}}_1\boldsymbol{u} = \widetilde{\boldsymbol{\chi}}_1\boldsymbol{H}\boldsymbol{u} + [\boldsymbol{H}_0, \widetilde{\boldsymbol{\chi}}_1]\boldsymbol{u}$. Similarly we obtain, observing the fact that $\widetilde{\boldsymbol{\chi}}_2\boldsymbol{H}\boldsymbol{u} = \widetilde{\boldsymbol{\chi}}_2\boldsymbol{H}_0\boldsymbol{u}$ as $\operatorname{supp}\widetilde{\chi}_2 \subset \mathbb{R}^n \setminus B(0, R_0)$:

$$\begin{aligned}
(\boldsymbol{J}_\infty - z\mathbf{1})\widetilde{\boldsymbol{\chi}}_2(\boldsymbol{H}_0 - i\boldsymbol{W} - z_0\mathbf{1})\boldsymbol{\chi}_2 &= [\boldsymbol{H}_0, \widetilde{\boldsymbol{\chi}}_2](\boldsymbol{H}_0 - i\boldsymbol{W} - z_0\mathbf{1})^{-1}\boldsymbol{\chi}_2 \\
&\quad + \widetilde{\boldsymbol{\chi}}_2(\boldsymbol{I} + (z_0 - z)(\boldsymbol{H}_0 - i\boldsymbol{W} - z_0\mathbf{1})^{-1})\boldsymbol{\chi}_2.
\end{aligned}$$

In the same manner, using $\widetilde{\boldsymbol{\chi}}_3\boldsymbol{\chi}_3 = \boldsymbol{\chi}_3$, we obtain

$$(\boldsymbol{J}_\infty - z\mathbf{1})\widetilde{\boldsymbol{\chi}}_3(\boldsymbol{H}_0 - i\widetilde{\boldsymbol{W}} - z\mathbf{1})^{-1}\boldsymbol{\chi}_3 = [\boldsymbol{H}_0, \widetilde{\boldsymbol{\chi}}_3](\boldsymbol{H}_0 - i\widetilde{\boldsymbol{W}} - z\mathbf{1})^{-1}\boldsymbol{\chi}_3 + \boldsymbol{\chi}_3.$$

Therefore, as $\sum_{i=1}^3 \widetilde{\boldsymbol{\chi}}_i\boldsymbol{\chi}_i = \sum_{i=1}^3 \boldsymbol{\chi}_i = \mathbf{1}$,

$$(\boldsymbol{J}_\infty - z\mathbf{1})\boldsymbol{E}(z) = \mathbf{1} + \boldsymbol{K}(z), \tag{7.4}$$

where

$$\begin{aligned}
\boldsymbol{K}(z) &= [\boldsymbol{H}_0, \widetilde{\boldsymbol{\chi}}_1](\boldsymbol{H} - z_0\mathbf{1})^{-1}\boldsymbol{\chi}_1 + (z_0 - z)\widetilde{\boldsymbol{\chi}}_1(\boldsymbol{H} - z_0\mathbf{1})^{-1}\boldsymbol{\chi}_1 \\
&\quad + [\boldsymbol{H}_0, \widetilde{\boldsymbol{\chi}}_2](\boldsymbol{H}_0 - i\boldsymbol{W} - z_0\mathbf{1})^{-1}\boldsymbol{\chi}_2 \\
&\quad + (z_0 - z)\widetilde{\boldsymbol{\chi}}_2(\boldsymbol{H}_0 - i\boldsymbol{W} - z_0\mathbf{1})^{-1}\boldsymbol{\chi}_2 \\
&\quad + [\boldsymbol{H}_0, \widetilde{\boldsymbol{\chi}}_3](\boldsymbol{H}_0 - i\widetilde{\boldsymbol{W}} - z\mathbf{1})^{-1}\boldsymbol{\chi}_3 \\
&= \boldsymbol{K}_1 + \boldsymbol{K}_2(z) + \boldsymbol{K}_3 + \boldsymbol{K}_4(z) + \boldsymbol{K}_5(z),
\end{aligned} \tag{7.5}$$

with the obvious interpretation of the $\boldsymbol{K}_i$. Arguing as in the proof of Proposition 4.4 we see that $\boldsymbol{K}_2(z)$ is compact on $\mathcal{H}$ and we can easily see that the same holds for $\boldsymbol{K}_4(z)$. Since $\boldsymbol{H}_0$ and $\widetilde{\boldsymbol{\chi}}_j$ commute at the level of principal symbols, the commutators will be first order operators and clearly with compactly supported smooth coefficients. As a consequence of the Rellich–Kondrashov theorem, the image of a bounded sequence in $H^2(\mathbb{R}^n \setminus B(0, R_0)) \otimes \mathbb{C}^2$ will be bounded in $H^1(\mathbb{R}^n \setminus B(0, R_0)) \otimes \mathbb{C}^2$ and therefore have a convergent subsequence in $L^2(\mathbb{R}^n \setminus B(0, R_0)) \otimes \mathbb{C}^2$. It follows from these considerations that $\boldsymbol{K}(z)$ is a compact operator on $\mathcal{H}$. By inspection we see that $\boldsymbol{K}(z)$ depends analytically on $z \in \{z \in \mathbb{C} : \operatorname{Im} z > -\delta_0\}$.

Next we claim that for $\operatorname{Im} z_0 \gg 1$ and z close to z_0 we have $\|\boldsymbol{K}(z)\| \leq 1/2$ and therefore $\mathbf{1} + \boldsymbol{K}(z)$ is invertible (with inverse given by a Neumann series). Since $\boldsymbol{H}$ is selfadjoint it follows from the basic estimate $\|(\boldsymbol{H} - z\mathbf{1})^{-1}\| \leq |\operatorname{Im} z|^{-1}$ for $\operatorname{Im} z \neq 0$ that $\boldsymbol{K}_2(z)$ and $\boldsymbol{K}_4(z)$ are $\mathcal{O}((\operatorname{Im} z_0)^{-1})$ for $\operatorname{Im} z_0 \gg 1$ and $|z - z_0|$ small. Writing

$$\boldsymbol{K}_1 = [\boldsymbol{H}_0, \widetilde{\boldsymbol{\chi}}_1](\mathbf{1} - \boldsymbol{\chi}_1)(\boldsymbol{H} - z_0\mathbf{1})^{-1}\boldsymbol{\chi}_1$$

where

$$(\mathbf{1}-\boldsymbol{\chi}_1)(\boldsymbol{H}-z_0\mathbf{1})^{-1}\boldsymbol{\chi}_1 = \begin{cases}\mathcal{O}((\operatorname{Im} z_0)^{-1}) : \mathcal{H} \to L^2(\mathbb{R}^n \setminus B(0,R_0)) \otimes \mathbb{C}^2 \\ \mathcal{O}(1) : \mathcal{H} \to H^2(\mathbb{R}^n \setminus B(0,R_0)) \otimes \mathbb{C}^2\end{cases}$$

we find by interpolation that $(\mathbf{1}-\boldsymbol{\chi}_1)(\boldsymbol{H}-z_0\mathbf{1})^{-1}\boldsymbol{\chi}_1 = \mathcal{O}((\operatorname{Im} z_0)^{-1/2})$ as an operator $\mathcal{H} \to H^1(\mathbb{R}^n \setminus B(0,R_0)) \otimes \mathbb{C}^2$. Since $[\boldsymbol{H}_0, \widetilde{\boldsymbol{\chi}}_1] : H^1(\mathbb{R}^n \setminus B(0,R_0)) \otimes \mathbb{C}^2 \to L^2(\mathbb{R}^n \setminus B(0,R_0)) \otimes \mathbb{C}^2$ is bounded it follows that $\boldsymbol{K}_1 = \mathcal{O}((\operatorname{Im} z_0)^{-1/2})$ for $\operatorname{Im} z_0 \gg 1$ on $\mathcal{H}$. The same type of estimate holds for $\boldsymbol{K}_3$ and $\boldsymbol{K}_5(z)$, so for $\operatorname{Im} z_0$ and $\operatorname{Im} z$ sufficiently large we have $\|\boldsymbol{K}(z)\| \le 1/2$ and as a consequence $(\mathbf{1}+\boldsymbol{K}(z))^{-1}$ exists in $\mathcal{B}(\mathcal{H})$ for such z and z_0.

By the construction of $\widetilde{\boldsymbol{W}}$, $\{\boldsymbol{K}(z)\}_{z\in\{\operatorname{Im} z > -\delta_0\}}$ depends analytically on z. By analytic Fredholm theory we conclude that $z \mapsto (\mathbf{1}+\boldsymbol{K}(z))^{-1}$ is a meromorphic family of operators for $\operatorname{Im} z > -\delta_0$ so that $\boldsymbol{E}(z)(\mathbf{1}+\boldsymbol{K}(z))^{-1}$ is a meromorphic right inverse of $\boldsymbol{J}_\infty - z\mathbf{1}$. Similarly

$$\boldsymbol{F}(z) = \boldsymbol{\chi}_1(\boldsymbol{H}-z_0\mathbf{1})^{-1}\widetilde{\boldsymbol{\chi}}_1 + \boldsymbol{\chi}_2(\boldsymbol{H}_0 - i\boldsymbol{W} - z_0\mathbf{1})\widetilde{\boldsymbol{\chi}}_2 + \boldsymbol{\chi}_3(\boldsymbol{H}_0 - i\boldsymbol{W}_1 - z\mathbf{1})^{-1}\widetilde{\boldsymbol{\chi}}_3$$

satisfies $\boldsymbol{F}(z)(\boldsymbol{J}_\infty - z\mathbf{1}) = \mathbf{1} + \boldsymbol{L}(z)$, where

$$\begin{aligned}\boldsymbol{L}(z) = {}& (z_0 - z)\boldsymbol{\chi}_1(\boldsymbol{H}-z_0\mathbf{1})^{-1}\widetilde{\boldsymbol{\chi}}_1 - \boldsymbol{\chi}_1(\boldsymbol{H}-z_0\mathbf{1})^{-1}[\boldsymbol{H}_0, \widetilde{\boldsymbol{\chi}}_1] \\ & + (z_0 - z)\boldsymbol{\chi}_2(\boldsymbol{H}_0 - i\boldsymbol{W} - z\mathbf{1})^{-1}\widetilde{\boldsymbol{\chi}}_2 - \boldsymbol{\chi}_2(\boldsymbol{H}_0 - i\boldsymbol{W} - z_0\mathbf{1})^{-1}[\boldsymbol{H}_0, \widetilde{\boldsymbol{\chi}}_2] \\ & - \boldsymbol{\chi}_3(\boldsymbol{H}_0 - i\boldsymbol{W}_1 - z\mathbf{1})^{-1}[\boldsymbol{H}_0, \widetilde{\boldsymbol{\chi}}_3]\end{aligned}$$

and we conclude as before that $(\mathbf{1}+\boldsymbol{L})^{-1}\boldsymbol{F}$ is a meromorphic left inverse of $\boldsymbol{J}_\infty - z\mathbf{1}$. Outside the poles,

$$(\mathbf{1}+\boldsymbol{L})^{-1}\boldsymbol{F} = (\mathbf{1}+\boldsymbol{L})^{-1}\boldsymbol{F}(\boldsymbol{J}_\infty - z\mathbf{1})\boldsymbol{E}(\mathbf{1}+\boldsymbol{K})^{-1} = \boldsymbol{E}(\mathbf{1}+\boldsymbol{K})^{-1},$$

so the one-sided “inverses” share the same poles and are equal elsewhere, meaning they are equal as meromorphic functions. We are thus justified in writing

$$(\boldsymbol{J}_\infty - z\mathbf{1})^{-1} = \boldsymbol{E}(z)(\mathbf{1}+\boldsymbol{K}(z))^{-1}$$

and so the meromorphic extension of $(\boldsymbol{J}_\infty - z\mathbf{1})^{-1}$ follows from that of $(\mathbf{1}+\boldsymbol{K}(z))^{-1}$.

We see that $(\boldsymbol{J}_\infty - z\mathbf{1})^{-1}$ is meromorphic in $\{z \in \mathbb{C} : \operatorname{Im} z > -\delta_0\}$ with poles among those of $(\mathbf{1}+\boldsymbol{K}(z))^{-1}$, as $\boldsymbol{E}(z)$ is analytic. Moreover, the residue of the resolvent $(\boldsymbol{J}_\infty - z\mathbf{1})^{-1}$ at each pole is of finite order as well as finite rank, because the same is true for each residue of $(\mathbf{1}+\boldsymbol{K}(z))^{-1}$ by analytic Fredholm theory.

The second statement of the proposition follows from the general theory of nonselfadjoint operators (see for instance [17]). ∎

Remark 7.2. For the matrix valued Schrödinger operator a more direct (but somewhat more tedious) proof can be given by using the Feshbach formula (used essentially to reduce the problem to the scalar setting), the spectral theorem, semiclassical elliptic estimates and Green’s formula.

It is straightforward to establish the analogue of Theorem 7.1 for the resolvent of the CAP Hamiltonian $\boldsymbol{J}_R(\hbar)$.

Theorem 7.3. *Let Assumption* 4.1 *be satisfied. For any* $\hbar > 0$ *the resolvent* $(\boldsymbol{J}_R(\hbar) - z\mathbf{1})^{-1} : \mathcal{H} \to \mathcal{H}$ *has a meromorphic extension from* $\mathbb{C}_+$ *into* $\mathbb{C}$. *The poles of* $(\boldsymbol{J}_R(\hbar) - z\mathbf{1})^{-1}$ *are eigenvalues of* $\boldsymbol{J}_R(\hbar)$ *of finite multiplicity.*

8. Eigenvalue estimate on a rectangle

Here we give an estimate of the number of eigenvalues of the CAP Hamiltonian $\boldsymbol{J}_\infty(\hbar)$ on a rectangle. The result is an analogue of the estimate (4.8) for $\boldsymbol{H}(\hbar)$ and it provides us with the same upper bound, but this time for the number of eigenvalues of $\boldsymbol{J}_\infty(\hbar)$ rather than the resonances of $\boldsymbol{H}(\hbar)$.

The scalar valued case is treated in Stefanov [63, Proposition 2]. The proof in the matrix valued setting is quite different because we utilize pseudodifferential operator theory whereas Stefanov uses various estimates of characteristic values (Weyl type asymptotics etc.) close to the account in [56, Section 6]. Furthermore, in contrast to Stefanov, we do not use the reference operator $\boldsymbol{H}^\sharp(\hbar)$ in the proof.

Proposition 8.1. *Let Assumption* 4.1 *hold. Suppose* $l < r$, $0 < c < \delta_0$ *and* $\Omega = [l, r] + i[-c, 0]$. *Then the number of eigenvalues of* $\boldsymbol{J}_\infty(\hbar)$ *in* Ω *satisfies*

$$\mathrm{Count}(\boldsymbol{J}_\infty(\hbar), \Omega) = \mathcal{O}(\hbar^{-n^\sharp}).$$

Proof. Bear in mind the representation (7.4), i.e.,

$$(\boldsymbol{J}_\infty - z\mathbf{1})\boldsymbol{E}(z) = \mathbf{1} + \boldsymbol{K}(z),$$

where

$$\boldsymbol{K}(z) = \boldsymbol{K}_1 + \boldsymbol{K}_2(z) + \boldsymbol{K}_3 + \boldsymbol{K}_4(z) + \boldsymbol{K}_5(z)$$

and the z_0 appearing in $\boldsymbol{K}(z)$ satisfies $\operatorname{Im} z_0 \gg 0$, and is chosen as before. Let $r > 0$ be such that $\Omega \subset D(z_0, r) = \{z \in \mathbb{C} : |z - z_0| < r\}$ and $\overline{D(z_0, r)} \subset \{z \in \mathbb{C} : \operatorname{Im} z > -\delta_0\}$.

We recall from the proof of Theorem 7.1 that for all $\hbar \in (0, 1]$ we can make $\|\boldsymbol{K}_1\| + \|\boldsymbol{K}_3\|$ arbitrarily small by choosing $\operatorname{Im} z_0 > 0$ large and that

$$\boldsymbol{K}_5(z) = [\boldsymbol{H}_0, \widetilde{\boldsymbol{\chi}}_3](\boldsymbol{H}_0 - i\widetilde{\boldsymbol{W}} - z\mathbf{1})^{-1}\boldsymbol{\chi}_3.$$

Our choice of r implies, by the construction of $\widetilde{\boldsymbol{W}}$, that

$$(\boldsymbol{H}_0 - i\widetilde{\boldsymbol{W}} - z\mathbf{1})^{-1} : \mathcal{H} \to \mathcal{D}$$

is a bounded operator for any $z \in D(z_0, r)$ (not necessarily close to z_0). Since $\boldsymbol{H}_0$ and $\widetilde{\boldsymbol{\chi}}_3$ commute at the level of principal symbols, $[\boldsymbol{H}_0, \widetilde{\boldsymbol{\chi}}_3]$ is of first order and therefore

$$\|[\boldsymbol{H}_0, \widetilde{\boldsymbol{\chi}}_3]\boldsymbol{u}\|_{\mathcal{H}} \le C\hbar\|\boldsymbol{u}\|_{H^1_\hbar(\operatorname{supp}\nabla\widetilde{\chi}_3)\otimes\mathbb{C}^2} \le C\hbar\|\boldsymbol{u}\|_{\mathcal{D}}.$$

It follows that $\boldsymbol{K}_5(z)$ is $\mathcal{O}(\hbar)$ on $\mathcal{H}$. Therefore $(\mathbf{1} + \boldsymbol{K}_1 + \boldsymbol{K}_3 + \boldsymbol{K}_5(z))^{-1}$ exists provided $\operatorname{Im} z_0$ is chosen large enough and $\hbar$ is sufficiently small, and we may write

$$\mathbf{1} + \boldsymbol{K}(z) = (\mathbf{1} - \widetilde{\boldsymbol{K}}(z))(\mathbf{1} + \boldsymbol{K}_1 + \boldsymbol{K}_3 + \boldsymbol{K}_5(z)), \tag{8.1}$$

where

$$\widetilde{\boldsymbol{K}}(z) = -(\boldsymbol{K}_2(z) + \boldsymbol{K}_4(z))(\mathbf{1} + \boldsymbol{K}_1 + \boldsymbol{K}_3 + \boldsymbol{K}_5(z))^{-1}.$$

From the meromorphic identity

$$(\mathbf{1} - \widetilde{\boldsymbol{K}}^{n^\sharp}(z))^{-1} = (\mathbf{1} + \widetilde{\boldsymbol{K}}(z) + \cdots + \widetilde{\boldsymbol{K}}^{n^\sharp - 1}(z))^{-1}(\mathbf{1} - \widetilde{\boldsymbol{K}}(z))^{-1}$$

we infer that the poles of $(\mathbf{1} - \widetilde{\boldsymbol{K}}^{n^\sharp}(z))^{-1}$ include those of $(\mathbf{1} - \widetilde{\boldsymbol{K}}(z))^{-1}$, which, by (8.1), in turn include the poles of $(\mathbf{1} + \boldsymbol{K}(z))^{-1}$, i.e. the eigenvalues of $\boldsymbol{J}_\infty$. Since $\widetilde{\boldsymbol{K}}^{n^\sharp}(z)$ is of trace class it suffices to estimate the number of zeros of $f(z) = \det(\mathbf{1} - \widetilde{\boldsymbol{K}}^{n^\sharp}(z))$ in Ω. We have

$$\mathrm{symb}^w([\boldsymbol{H}_0, \boldsymbol{\chi}]) = -i\hbar\{\boldsymbol{h}_0, \boldsymbol{\chi}\} + \mathcal{O}(\hbar^2) \in \mathsf{S}(\langle\xi\rangle).$$

Since $(\boldsymbol{h}_0 - z\mathbf{1})^{-1} \in \mathsf{S}(\langle\xi\rangle^{-2})$ it follows from Proposition 3.5 that

$$\mathrm{symb}^w(\boldsymbol{K}_1) \in \mathsf{S}(\langle\xi\rangle^{-1}).$$

Similar arguments work for $\boldsymbol{K}_3$ and $\boldsymbol{K}_5(z)$ and we get

$$\mathrm{symb}^w((\mathbf{1} + \boldsymbol{K}_1 + \boldsymbol{K}_3 + \boldsymbol{K}_5(z))^{-1}) \in \mathsf{S}(1).$$

Since $\boldsymbol{K}_2(z), \boldsymbol{K}_4(z) \in \mathsf{S}(\langle\xi\rangle^{-2})$ a repeated application of Proposition 3.5 yields

$$\mathrm{symb}^w(\widetilde{\boldsymbol{K}}^{n^\sharp}(z)) \in \mathsf{S}(\langle\xi\rangle^{-2n^\sharp}).$$

From Proposition 3.7 it then follows that $\widetilde{\boldsymbol{K}}^{n^\sharp}(z)$ is of trace class with $\|\widetilde{\boldsymbol{K}}^{n^\sharp}(z)\|_{\mathrm{tr}} = \mathcal{O}(\hbar^{-n^\sharp})$. Using

$$|\det(\mathbf{1} + \widetilde{\boldsymbol{K}}^{n^\sharp}(z))| \le \exp(\|\widetilde{\boldsymbol{K}}^{n^\sharp}(z)\|_{\mathrm{tr}})$$

(see [56, p. 57]), we get $|f(z)| \le e^{C\hbar^{-n^\sharp}}$. Applying Jensen's formula for the number of zeros of an analytic function in a disk $D(z_0, r+\varepsilon)$ for ε sufficiently small it becomes evident that f has no more than $\mathcal{O}(\hbar^{-n^\sharp})$ zeros in Ω. ∎

We continue to carry over properties of $\boldsymbol{H}$ to the analogous ones for $\boldsymbol{J}_\infty$ by showing how the matrix valued cutoff resolvent estimate in Proposition 6.1 remains true also for the CAP Hamiltonian; the proof follows essentially the scalar valued reasoning in Petkov and Zworski [46, Proposition 4.2] which is closely related to Sjöstrand [56, p. 104].

Proposition 8.2. *Let Assumptions* 4.1 *and* 4.6 *hold. If* $\Omega := [l, r] + i[-c, 0]$ *where* $0 < l < r$ *and* $0 < c < \delta_0$, *then there is an* $A = A(\Omega)$ *such that*

$$\|(\boldsymbol{J}_\infty(\hbar) - z\mathbf{1})^{-1}\| \le A e^{A\hbar^{-n^\sharp}\log(1/g(\hbar))}, \quad \forall z \in \Omega \setminus \bigcup_{w_j \in \mathrm{spec}(\boldsymbol{J}_\infty)\cap\Omega} D(w_j, g(\hbar))$$

for any $0 < g(\hbar) \ll 1$.

Proof. With the notation of the proofs of Propositions 6.1 and 7.1 we get, with $|\boldsymbol{K}| := \sqrt{\boldsymbol{K}^*\boldsymbol{K}}$ (see [17]),

$$\|(\boldsymbol{J}_\infty(\hbar) - z\mathbf{1})^{-1}\| \le \|\boldsymbol{E}(z)\|\,\|(\mathbf{1} + \boldsymbol{K}(z))^{-1}\| \le \frac{\det(\mathbf{1} + |\boldsymbol{K}|^{n^\sharp})}{|f(z)|} \le Ce^{C\hbar^{-n^\sharp}}|f(z)|^{-1},$$

so it suffices to obtain lower bounds on $|f(z)|$. Let $\{w_j(\hbar)\}_{j=1}^{N(\hbar)}$ be the eigenvalues of $\boldsymbol{J}_\infty$

in $D(z_0, r+\varepsilon)$ so that $N(\hbar) = \mathcal{O}(\hbar^{-n^\sharp})$ according to Proposition 8.1, and write

$$f(z,\hbar) = e^{k(z,\hbar)} \prod_{j=1}^{N(\hbar)} (z - w_j(\hbar)), \quad z \in D(z_0, r),$$

with $k(z,\hbar)$ analytic in z on $D(z_0,r)$. Using the upper bound on $|f|$ from the proof of Proposition 8.1 and an estimate of Cartan for the product [29, Ch. I] we obtain, for $|z - z_0| = r$,

$$e^{C\hbar^{-n^\sharp}} \geq |f(z)| = e^{\operatorname{Re} k(z)} \prod_{j=1}^{N(\hbar)} |z - w_j| > e^{\operatorname{Re} k(z)} \eta_0^{N(\hbar)}$$

for some $\eta_0 > 0$. Using the maximum principle for the harmonic function $\operatorname{Re} k$, this implies $\operatorname{Re} k(z) < C\hbar^{-n^\sharp}$ for all $z \in D(z_0, r)$. Writing

$$\frac{1}{f(z_0)} = \det((\mathbf{1} + \widetilde{\boldsymbol{K}}^{n^\sharp}(z_0))^{-1}) = \det(\mathbf{1} - \widetilde{\boldsymbol{K}}^{n^\sharp}(z_0)(\mathbf{1} + \widetilde{\boldsymbol{K}}^{n^\sharp}(z_0))^{-1})$$

we can obtain the same upper bound as before so that $\log|f(z_0)| \geq -C\hbar^{-n^\sharp}$. Moreover for $z_0 \notin \bigcup_{j=1}^{N(\hbar)} D(w_j, g(\hbar))$ we have

$$\log\Big(\prod_{j=1}^{N(\hbar)} |z_0 - w_j|\Big) \geq N(\hbar)\log(g) \geq -C\hbar^{-n^\sharp} \log\frac{1}{g},$$

because $N(\hbar) = \mathcal{O}(\hbar^{-n^\sharp})$. Thus

$$-C\hbar^{-n^\sharp} \leq \log|f(z_0)| = \operatorname{Re} k(z_0) + \log\Big(\prod_{j=1}^{N(\hbar)} |z_0 - w_j|\Big) \leq \operatorname{Re} k(z_0) + C\hbar^{-n^\sharp} \log\frac{1}{g},$$

so all in all $|\operatorname{Re}(k(z_0))| \leq C\hbar^{-n^\sharp} \log(1/g)$. Considering instead $e^{i\varphi} f(z)$ we may assume $\operatorname{Im} k$ is such that $|k(z_0)| \leq C\hbar^{-n^\sharp}\log(1/g)$. To say something about $\operatorname{Re} k(z)$ for a general point $z \in D(z_0,\rho)$ with $\rho < r$ we use the Borel–Carathéodory theorem [28, Ch. XII, §3] which tells us that

$$\max_{|z-z_0|=\rho} |k(z)| \leq \frac{2\rho}{r-\rho} \max_{|z-z_0|=r} \operatorname{Re} k(z) + \frac{r+\rho}{r-\rho}|k(z_0)|, \quad \rho < r.$$

This implies

$$|\operatorname{Re} k(z)| \leq C\hbar^{-n^\sharp} \log(1/g) \quad \text{for all } z \in D(z_0,\rho)$$

by the maximum modulus principle and hence

$$\log|f(z)| \geq -C\hbar^{-n^\sharp} \log\frac{1}{g} \quad \text{for all } z \in D(z_0,\rho) \setminus \bigcup_{j=1}^{N(\hbar)} D(w_j, g(\hbar)).$$

The proposition now follows by covering Ω by finitely many disks and repeating the argument. ∎

We close this chapter with an easy lemma.

LEMMA 8.3. *Let Assumption* 4.1 *be satisfied. Then*

$$\|(\boldsymbol{J}_\infty - z)^{-1}\| \leq \frac{1}{\operatorname{Im} z} \quad \textit{for } z \in \mathbb{C}_+.$$

Proof. We begin by noticing that, for any $\boldsymbol{f} \in \mathcal{D}$,

$$\begin{aligned}
-\operatorname{Im}\langle(\boldsymbol{J}_\infty - z\mathbf{1})\boldsymbol{f}, \boldsymbol{f}\rangle &= -\operatorname{Im}\langle \boldsymbol{H}\boldsymbol{f} - i(\operatorname{Re}\boldsymbol{W})\boldsymbol{f} + (\operatorname{Im}\boldsymbol{W})\boldsymbol{f} - z\boldsymbol{f}, \boldsymbol{f}\rangle \\
&= \langle(\operatorname{Re}\boldsymbol{W})\boldsymbol{f}, \boldsymbol{f}\rangle + (\operatorname{Im} z)\|\boldsymbol{f}\|^2 \\
&\geq (\operatorname{Im} z)\|\boldsymbol{f}\|^2.
\end{aligned}$$

Therefore, by the Cauchy–Schwarz inequality,

$$\|(\boldsymbol{J}_\infty - z\mathbf{1})\boldsymbol{f}\|\|\boldsymbol{f}\| \geq (\operatorname{Im} z)\|\boldsymbol{f}\|^2.$$

On substituting $\boldsymbol{g} = (\boldsymbol{J}_\infty - z\mathbf{1})\boldsymbol{f}$ this reads

$$\|\boldsymbol{g}\|\|(\boldsymbol{J}_\infty - z\mathbf{1})^{-1}\boldsymbol{g}\| \geq (\operatorname{Im} z)\|(\boldsymbol{J}_\infty - z\mathbf{1})^{-1}\boldsymbol{g}\|^2,$$

or

$$\|(\boldsymbol{J}_\infty - z\mathbf{1})^{-1}\boldsymbol{g}\| \leq \frac{1}{\operatorname{Im} z}\|\boldsymbol{g}\|,$$

i.e. $\|(\boldsymbol{J}_\infty - z\mathbf{1})^{-1}\| \leq 1/\operatorname{Im} z$. ■

Results similar to the ones for $\boldsymbol{J}_\infty(\hbar)$ and its resolvent given above are valid for $\boldsymbol{J}_R(\hbar)$ and its resolvent.

PROPOSITION 8.4. *Let Assumptions* 4.1 *and* 4.6 *hold. Then the statements of Propositions* 8.1*,* 8.2 *and Lemma* 8.3 *hold for* $\boldsymbol{J}_R(\hbar)$ *as well.*

9. Quasimodes

Quasimodes are defined as a sequence of approximate real "resonances" with associated approximate solutions supported in a fixed compact set. In this chapter we prove that the existence of quasimodes yields the existence of many resonances exponentially close to the quasimodes (not only the imaginary, but the real part as well). The first scalar valued result, for well-separated quasimodes, of this type goes back to the remarkable paper by Stefanov–Vodev [65] (using the Phragmén–Lindelöf principle and a global cutoff resolvent estimate) in the context of Rayleigh surface waves in linear elasticity, where it was shown, for general compactly supported perturbations in odd-dimensional Euclidean spaces, that existence of real quasimodes with polynomially small error implies existence of resonances converging to the real axis at the same rate. Their method, however, was not sensitive enough to yield information on the density of those resonances. An essential step ahead was taken by Tang and Zworski [66] who realized that one can localize resonances not only close to the real axis but even near a quasimode so that, if the quasimode is large enough, then there is always a resonance close to it; confirming that quasimodes are perturbed resonances. The main ingredients are the local cutoff resolvent estimate and a semiclassical maximum principle. They managed to obtain lower bounds on the number of resonances near the real axis; at least linear bounds and, if the quasimodes are "well distributed" in some sense, one could also achieve finer bounds. If quasimodes are distributed in an "irregular way", specifically, if there can be multiple quasimodes or clusters of quasimodes too close to each other, then the method in [66] only enables one to show that these multiple quasimodes or clusters generate only a single resonance.

Stefanov [59] managed to treat multiplicities in the case when quasimodes are very close to each other. He showed that such clusters of quasimodes generate (asymptotically) at least the same number of resonances. In [63] he improved the latter result in several ways by modifying the reasoning in [59, Theorem 1]. The underlying ideas, however, are the same as in Tang and Zworski [66] (see also [56, Theorem 11.2]).

We use the matrix valued local cutoff resolvent estimate in Proposition 6.1 and the matrix valued semiclassical maximum principle in Lemma B.1 to prove a result similar to [63, Theorem 3].

Proposition 9.1. *Let $N \geq 0$, $M > 0$, A as in Proposition 8.2 and $0 < l_0 < l(\hbar) \leq r(\hbar) \leq r_0 < \infty$ where $\hbar \in (0, \hbar_0]$ for some $\hbar_0 \in (0, 1]$.*

(1) *Assume that for any $\hbar \in (0, \hbar_0]$ there exists $m(\hbar) \in \mathbb{Z}_+$, $E_j(\hbar) \in [l(\hbar), r(\hbar)]$ and $\{\boldsymbol{u}_j(\hbar)\}_{j=1}^{m(\hbar)} \subset \mathcal{D}$ with $\|\boldsymbol{u}_j\| = 1$ and $\operatorname{supp} \boldsymbol{u}_j(\hbar) \subset K \Subset \mathbb{R}^n$ where K is independent of $\hbar$ such that*

$$\|(\boldsymbol{H}(\hbar) - E_j(\hbar)\mathbf{1})\boldsymbol{u}_j(\hbar)\| \leq R(\hbar), \tag{9.1}$$

where $R(\hbar) \leq \hbar^{n^\sharp+N+1}/(C\log(1/\hbar))$, and any $\{\tilde{\boldsymbol{u}}_j(\hbar)\}_{j=1}^{m(\hbar)} \subset \mathcal{H}$ with $\|\tilde{\boldsymbol{u}}_j(\hbar) - \boldsymbol{u}_j(\hbar)\| = \hbar^N/M$ is linearly independent. Then there is an $\hbar_1(A, B, M, N) \in (0, \hbar_0]$ such that for all $\hbar \in (0, \hbar_1]$ and any $B > 0$, $\boldsymbol{H}(\hbar)$ has at least $m(\hbar)$ resonances in

$$\left[l(\hbar) - c(\hbar)\log\frac{1}{\hbar}, r(\hbar) + c(\hbar)\log\frac{1}{\hbar}\right] + i[-c(\hbar), 0], \tag{9.2}$$

where $c(\hbar) = \max\{CAMR(\hbar)\hbar^{-n^\sharp-N-1}, e^{-B/\hbar}\}$.

(2) *The first assertion remains true if we replace $\boldsymbol{H}(\hbar)$ by $\boldsymbol{J}(\hbar)$ and "resonances" by "eigenvalues" in its statement.*

Proof. 1. Let $z_1(\hbar), \dots, z_J(\hbar)$ be all distinct resonances in the larger domain

$$\Omega_2(\hbar) := [l(\hbar) - 2w(\hbar), r(\hbar) + 2w(\hbar)] + i\left[-2A\hbar^{-n^\sharp}\log\frac{1}{\widetilde{r}(\hbar)}, \widetilde{r}(\hbar)\right], \tag{9.3}$$

where

$$w(\hbar) = 4n^\sharp A\hbar^{-n^\sharp}\widetilde{r}(\hbar)\log\frac{1}{\widetilde{r}(\hbar)}\log\frac{1}{\hbar}$$

and $\widetilde{r}(\hbar)$ is to be specified below. Fix $\chi \in C_0^\infty(\mathbb{R}^n)$ with $\chi \succ 1_K$ and expand $\boldsymbol{\chi}\boldsymbol{R}(z)\boldsymbol{\chi}$ as in (B.7). The multiplicity of $z_j(\hbar)$ is then given by the dimension of (4.4). We must prove

$$\widetilde{m} := \sum_{j=1}^{J} \operatorname{rank} \boldsymbol{A}_{-1}^{(j)} \geq m.$$

Let Π be the orthogonal projection onto $\bigcup_{j=1}^J \boldsymbol{A}_{-1}^{(j)}\mathcal{H}$, and let $\Pi' = \mathbf{1} - \Pi$. Then clearly $\operatorname{rank}\Pi \leq \widetilde{m}$ so it is enough to show that

$$\operatorname{rank}\Pi \geq m. \tag{9.4}$$

According to Lemma B.3,

$$\boldsymbol{F}(z) := \Pi'\boldsymbol{\chi}\boldsymbol{R}(z)\boldsymbol{\chi}$$

is analytic in a neighbourhood of $\Omega_2(\hbar)$. Moreover, as $\boldsymbol{H}$ is selfadjoint, we have for $z \in \mathbb{C}_+$ the estimate

$$\|\boldsymbol{F}(z)\| \leq \|(\boldsymbol{H} - z\mathbf{1})^{-1}\| \leq \frac{1}{\operatorname{Im} z}.$$

As a consequence of Proposition 6.1 we get

$$\|\boldsymbol{F}(z)\| \leq Ae^{A\hbar^{-n^\sharp}\log(1/\widetilde{r})} \qquad \text{for all } z \in \Omega_2 \setminus \bigcup_{j=1}^{J} D(z_j, \widetilde{r}(\hbar)). \tag{9.5}$$

Using the fact that $\boldsymbol{F}(z)$ is analytic in Ω_2 we next prove that this estimate is in fact valid in the whole of the smaller domain Ω. This follows from the standard maximum modulus principle (which holds true also for operator valued functions since $\|\boldsymbol{F}(z)\| = \max_{\|\boldsymbol{u}\|=\|\boldsymbol{v}\|=1} |\langle \boldsymbol{F}(z)\boldsymbol{u}, \boldsymbol{v}\rangle|$) provided we can show that any connected union of disks $D(z_{j_k}, \widetilde{r})$ having a point in common with Ω has no point in common with the complement of Ω_2. To prove this it is enough, since resonances live in $\mathbb{C}_-$, to show that any such connected union has length no more than

$$\min_{\hbar \ll 1} \left\{ w(\hbar), A\hbar^{-n^\sharp} \left(\log \frac{1}{\widetilde{r}} \right) \widetilde{r} \right\} = A\hbar^{-n^\sharp} \left(\log \frac{1}{\widetilde{r}} \right) \widetilde{r}.$$

In view of Assumption 4.6 there are no more than $\mathcal{O}(\hbar^{-n^\sharp})$ resonances in Ω_2, and therefore any connected union of disks $D(z_{j_k}, \widetilde{r})$ has diameter at most $\mathcal{O}(\hbar^{-n^\sharp}\widetilde{r})$. Clearly, since $\widetilde{r}(\hbar) = o(1)$, there is $\hbar_2 \in (0,1]$ such that

$$C\hbar^{-n^\sharp}\widetilde{r} < A\hbar^{-n^\sharp} \left(\log \frac{1}{\widetilde{r}} \right) \widetilde{r}$$

for $0 < \hbar < \hbar_2$ and the claim follows. Thus, since (9.5) is true on the boundary of any connected union of disks $D(z_{j_k}, \widetilde{r})$ intersecting Ω and since such unions never intersect the complement of Ω_2 (where $\boldsymbol{F}(z)$ may not be analytic), it follows from the maximum modulus principle that (9.5) is also true in the interior of all such connected unions of disks. Therefore

$$\|\boldsymbol{F}(z)\| \leq Ae^{A\hbar^{-n^\sharp}\log(1/\widetilde{r})} \qquad \text{for all } z \in \Omega.$$

Thus the requirements of Corollary B.2 are satisfied. The hypothesis in (9.1) in conjunction with Corollary B.2 yields

$$\|\Pi'\boldsymbol{u}_j\| = \|\Pi'\boldsymbol{\chi}\boldsymbol{R}(z)\boldsymbol{\chi}(\boldsymbol{H} - z\mathbf{1})\boldsymbol{u}_j\| \leq \|\Pi'\boldsymbol{\chi}\boldsymbol{R}(z)\boldsymbol{\chi}\| R \leq \frac{e^3 R}{\widetilde{r}(\hbar)}$$

and consequently $\|\tilde{\boldsymbol{u}}_j - \boldsymbol{u}_j\| \leq e^3 R/\widetilde{r}$ for $\tilde{\boldsymbol{u}}_j := \Pi\boldsymbol{u}_j$. By our hypothesis $\{\tilde{\boldsymbol{u}}_j\}_{j=1}^m$ will be linearly independent if we take

$$\widetilde{r}(\hbar) = \max\{e^3 M\hbar^{-N} R(\hbar), e^{-B/\hbar}\}$$

and from this (9.4) follows.

(2) Using Lemma 8.1, Proposition 8.2 and Lemma 8.3 we can argue as in the proof of (1). ■

Remark 9.2. If we apply the theorem for only one quasimode $E(\hbar)$, then the theorem implies the existence of a resonance $z(\hbar)$ with

$$|\mathrm{Re}\, z_0(\hbar) - E(\hbar)| \leq CR(\hbar)\hbar^{-n^\sharp-1}\log\frac{1}{\hbar}$$

and

$$0 \leq -\,\mathrm{Im}\, z_0(\hbar) \leq CR(\hbar)\hbar^{-n^\sharp-1}.$$

10. Individual resonances. The case $R_0' < R_1$

Below we always require that Assumptions 4.1 and 4.6 are satisfied. To establish Theorem 5.1 we prove that cutoff resonant states of $\boldsymbol{H}(\hbar)$ are quasimodes of $\boldsymbol{J}(\hbar)$, and cutoff eigenfunctions of $\boldsymbol{J}(\hbar)$ are quasimodes of $\boldsymbol{H}(\hbar)$. Specifically, starting from a resonance of $\boldsymbol{H}(\hbar)$ belonging to the rectangle (5.1), the strategy of the proof of Theorem 5.1 is to construct a quasimode which satisfies the assumptions in Proposition 9.1. An application of the latter proposition then yields the desired assertion. The first ingredient in the proof is Burq's absorption estimate [5, Proposition 2.2] in its improved version (10.1); see below and notice γ, found in Stefanov [61, Proposition 2]. The difference between the two versions is that one can choose γ large provided ρ is made large enough; this is even implicit in Burq's original proof. Moreover, Stefanov proved that γ is bounded from below; a fact which is essential for the proof below.

Proof of Theorem 5.1. (1) Let $z_0(\hbar)$ be a resonance of $\boldsymbol{H}(\hbar)$ in the rectangle defined in (5.1), and let $\boldsymbol{u} = (u_a, u_b)^t$ be the corresponding resonant state (cf. Definition 4.5). For $|x| > R_0'$ we see that

$$(\boldsymbol{H}(\hbar) - z_0(\hbar)\mathbf{1})\boldsymbol{u} = \mathbf{0} \;\Rightarrow\; (-\hbar^2\Delta \otimes \boldsymbol{I}_2 - z_0(\hbar)\boldsymbol{I}_2)\boldsymbol{u} = \mathbf{0},$$

which implies that the components of $\boldsymbol{u}$ are outgoing resonant states of $-\hbar^2\Delta$ for $|x| \geq R_0'$. In particular, $u_\#$, $\# = a, b$, admits Burq's absorption estimate [5] in its improved version (see also Lemma 12.1)

$$\int_{|x|=\rho} (|u_\#|^2 + |\hbar\nabla_x u_\#|^2)\, dS_x \leq C(-\hbar^{-1}\,\mathrm{Im}\, z_0(\hbar) + e^{-\gamma(\rho)/\hbar}) \int_{B(0,\rho)} |u_\#|^2\, dx \qquad (10.1)$$

for any $\rho > R_0'$ and some $\gamma(\rho)$ which is bounded from below. More specifically, $\gamma(\rho) \geq C_0(\rho - R_0')$ for some $C_0 > 0$ provided $\hbar$ is sufficiently small [61]. In particular, $\gamma(\rho) \to \infty$ as $\rho \to \infty$. By integrating (10.1) with respect to $\rho \in [R_0', R_1]$ and estimating the first integrand by $C\int_{B(0,R_1)} |u_\#|^2\, dx$ and the second by $Ce^{-C_0(R_1)/\hbar}\int_{B(0,R_1)} |u_\#|^2\, dx$, we obtain, after multiplying by $\hbar^2$,

$$\begin{aligned}\|\hbar u_\#\|^2_{L^2(B(0,R_0',R_1))} &+ \|\hbar^2\nabla_x u_\#\|^2_{L^2(B(0,R_0',R_1))} \\ &\leq C(-\hbar\,\mathrm{Im}\, z_0(\hbar) + e^{-C_0(R_1)/\hbar})\|u_\#\|^2_{L^2(B(0,R_1))}, \qquad (10.2)\end{aligned}$$

where we have kept the notation γ since it possesses the same properties as before and the exponential term has absorbed a polynomial factor $\hbar^2$.

Letting $\boldsymbol{v}(\hbar) = \boldsymbol{\chi}\boldsymbol{u}(\hbar)$, we find that

$$(\boldsymbol{H}(\hbar) - z_0(\hbar)\mathbf{1})\boldsymbol{v}(\hbar) = [\boldsymbol{H}_0(\hbar), \boldsymbol{\chi}]\boldsymbol{u}(\hbar), \tag{10.3}$$

because $\boldsymbol{u}(\hbar)$ is a resonant state corresponding to the resonance $z_0(\hbar)$. Since $\chi = \mathrm{const}$ on $B(0, R_0')$, we notice that

$$[\boldsymbol{H}_0(\hbar), \boldsymbol{\chi}]\boldsymbol{u}(\hbar) = ([-\hbar^2\Delta, \chi] \otimes I_2)\boldsymbol{u}(\hbar),$$

Then, by straightforward calculations,

$$[-\hbar^2\Delta, \chi]u_\#(\hbar) = -\hbar^2(2\nabla\chi \cdot \nabla u_\#(\hbar) + (\Delta\chi)u_\#(\hbar)) \tag{10.4}$$

implies that

$$[\boldsymbol{H}_0, \boldsymbol{\chi}]\boldsymbol{u}(\hbar) = -\hbar^2((\Delta\chi) + 2(\nabla\chi)\cdot\nabla) \otimes \boldsymbol{I}_2\, \boldsymbol{u}(\hbar),$$

and therefore, by (10.3)–(10.4) and the simple inequality $2ab \le a^2 + b^2$,

$$\begin{aligned}&\|(\boldsymbol{H}(\hbar) - z_0(\hbar)\mathbf{1})\boldsymbol{v}(\hbar)\|^2_{L^2(\mathbb{R}^n)\otimes\mathbb{C}^2}\\ &\quad\le C \sum_{\#=a,b} \big(\hbar^2\|\nabla\chi\cdot\nabla u_\#(\hbar)\|^2_{L^2(B(0,R_1)\setminus B(0,R_0'))} + \hbar^2\|(\Delta\chi)u_\#(\hbar)\|^2_{L^2(B(0,R_1)\setminus B(0,R_0'))}\big),\end{aligned}$$

because $\mathrm{supp}(\nabla\chi) \subset B(0,R_1) \setminus B(0,R_0')$. Since χ is smooth on $\mathbb{R}^n$, we therefore infer that

$$\begin{aligned}&\|(\boldsymbol{H}(\hbar) - z_0(\hbar)\mathbf{1})\boldsymbol{v}(\hbar)\|^2_{L^2(\mathbb{R}^n)\otimes\mathbb{C}^2}\\ &\qquad\le C \sum_{\#=a,b} \big(\hbar^2\|\nabla u_\#(\hbar)\|^2_{L^2(B(0,R_0',R_1))} + \hbar^2\|u_\#(\hbar)\|^2_{L^2(B(0,R_0',R_1))}\big),\end{aligned}$$

which, together with (10.2) and the inequality $\sqrt{a+b} \le \sqrt{a} + \sqrt{b}$ for $a, b \ge 0$, yields

$$\begin{aligned}&\|(\boldsymbol{H}(\hbar) - z_0(\hbar)\mathbf{1})\boldsymbol{v}(\hbar)\|_{L^2(\mathbb{R}^n)\otimes\mathbb{C}^2}\\ &\qquad\le C(\hbar^{1/2}\sqrt{-\operatorname{Im} z_0(\hbar)} + e^{-C_0(R_1)/2\hbar})\|\boldsymbol{u}(\hbar)\|_{L^2(B(0,R_1))\otimes\mathbb{C}^2}.\end{aligned} \tag{10.5}$$

Next we claim that

$$\|\boldsymbol{u}\|_{L^2(B(0,R_1))\otimes\mathbb{C}^2} \le C\|\boldsymbol{v}\|_{L^2(\mathbb{R}^n)\otimes\mathbb{C}^2}. \tag{10.6}$$

Indeed, by the definition of $\boldsymbol{v}(\hbar)$,

$$\begin{aligned}&\|\boldsymbol{v}(\hbar) - \boldsymbol{u}(\hbar)\|^2_{\mathbf{H}^1(B(0,R_0',R_1))}\\ &\qquad= \sum_{\#=a,b} \int_{B(0,R_0',R_1)} \big((\chi-1)^2|u_\#(\hbar)|^2 + |\hbar\nabla((\chi-1)u_\#)|^2\big)\,dx.\end{aligned} \tag{10.7}$$

Splitting this integral into two in the natural way one easily sees the first term can be estimated by $C\|u_\#\|_{L^2(B(0,R_0',R_1))}$, whereas for the second term we have

$$\begin{aligned}&\int B(0,R_0',R_1)|\hbar\nabla((\chi-1)u_\#)|^2\,dx\\ &\qquad\le \hbar^2\int_{B(0,R_0',R_1)} \big(|(\nabla(\chi-1))u_\#| + |(\chi-1)\nabla u_\#|\big)^2\,dx.\end{aligned}$$

Since both $\nabla(\chi - 1)$ and $\chi - 1$ are bounded on the annulus $B(0, R_0', R_1)$, the above integral may be estimated by

$$C\hbar^2 \int_{B(0,R_0',R_1)} (|u_\#| + |\nabla u_\#|)^2 \, dx,$$

or, as $2|u_\#||\nabla u_\#| \leq |u_\#|^2 + |\nabla u_\#|^2$, it can be estimated by

$$C\hbar^2 \int_{B(0,R_0',R_1)} (|u_\#|^2 + |\nabla u_\#|^2) \, dx = C\hbar^2 (\|u_\#\|^2_{L^2(B(0,R_0',R_1))} + \|\nabla u_\#\|^2_{L^2(B(0,R_0',R_1))}). \tag{10.8}$$

Denoting $-\hbar^{-1} \operatorname{Im} z_0(\hbar) + e^{-C_0(R_1)/\hbar}$ by $\varepsilon(\hbar)$ we infer from the above discussion, together with (10.2), that

$$\|\boldsymbol{v}(\hbar) - \boldsymbol{u}(\hbar)\|_{\mathbf{H}^1(B(0,R_0',R_1))} \leq C\sqrt{\varepsilon(\hbar)} \|\boldsymbol{u}\|_{L^2(B(0,R_1)) \otimes \mathbb{C}^2}.$$

In particular,

$$\|\boldsymbol{v}(\hbar) - \boldsymbol{u}(\hbar)\|_{L^2(B(0,R_0',R_1)) \otimes \mathbb{C}^2} \leq C\sqrt{\varepsilon(\hbar)} \|\boldsymbol{u}\|_{L^2(B(0,R_1)) \otimes \mathbb{C}^2}. \tag{10.9}$$

Since $\boldsymbol{v}(\hbar) = \boldsymbol{u}(\hbar)$ on $B(0, R_0')$ we have

$$\|\boldsymbol{u}(\hbar)\|^2_{L^2(B(0,R_1)) \otimes \mathbb{C}^2} = \|\boldsymbol{v}(\hbar)\|^2_{L^2(B(0,R_1)) \otimes \mathbb{C}^2} + \|\boldsymbol{u}(\hbar)\|^2_{L^2(B(0,R_0',R_1)) \otimes \mathbb{C}^2}. \tag{10.10}$$

Using

$$\|\boldsymbol{u}(\hbar)\|_{L^2(B(0,R_0',R_1)) \otimes \mathbb{C}^2} \leq \|\boldsymbol{v}(\hbar) - \boldsymbol{u}(\hbar)\|_{L^2(B(0,R_0',R_1)) \otimes \mathbb{C}^2} + \|\boldsymbol{v}(\hbar)\|_{L^2(B(0,R_0',R_1)) \otimes \mathbb{C}^2},$$

the inequality $2ab \leq a^2 + b^2$, (10.9)–(10.10) and $\lim_{\hbar \to 0} \varepsilon(\hbar) = 0$, we find

$$\|\boldsymbol{u}(\hbar)\|^2_{L^2(B(0,R_0',R_1)) \otimes \mathbb{C}^2} \leq \frac{C_1}{1 - C_2 \varepsilon(\hbar)} \|\boldsymbol{v}(\hbar)\|^2_{L^2(B(0,R_1)) \otimes \mathbb{C}^2} \leq C_3 \|\boldsymbol{v}(\hbar)\|^2_{L^2(\mathbb{R}^n) \otimes \mathbb{C}^2}$$

for some constants $C_1, C_2, C_3 > 0$. Again using $\boldsymbol{v}(\hbar) = \boldsymbol{u}(\hbar)$ on $B(0, R_0')$ we arrive at (10.6). Using the latter in (10.5), we have

$$\|(\boldsymbol{H}(\hbar) - z_0(\hbar)\mathbf{1})\boldsymbol{v}(\hbar)\|_{L^2(\mathbb{R}^n) \otimes \mathbb{C}^2} \leq C(\hbar^{1/2}\sqrt{-\operatorname{Im} z_0(\hbar)} + e^{-C_0(R_1)/\hbar}) \|\boldsymbol{v}(\hbar)\|_{L^2(\mathbb{R}^2) \otimes \mathbb{C}^2}.$$

Here we may in fact replace $z_0(\hbar)$ by $\operatorname{Re} z_0(\hbar)$. Indeed, using the triangle inequality, we have

$$\begin{aligned} \|(\boldsymbol{H}(\hbar) - \operatorname{Re} z_0(\hbar)\mathbf{1})\boldsymbol{v}(\hbar)\|_{L^2(\mathbb{R}^n) \otimes \mathbb{C}^2} \leq{}& \|(\boldsymbol{H}(\hbar) - z_0(\hbar)\mathbf{1})\boldsymbol{v}(\hbar)\|_{L^2(\mathbb{R}^n) \otimes \mathbb{C}^2} \\ &+ |\operatorname{Im} z_0(\hbar)| \, \|\boldsymbol{v}(\hbar)\|_{L^2(\mathbb{R}^n) \otimes \mathbb{C}^2}, \end{aligned}$$

so it suffices to show

$$|\operatorname{Im} z_0(\hbar)| = -\operatorname{Im} z_0(\hbar) \leq C\hbar^{1/2}\sqrt{-\operatorname{Im} z_0(\hbar)},$$

or, equivalently, $-\operatorname{Im} z_0(\hbar) \leq C\hbar$, which is true for small $\hbar$ by the assumption on $z_0(\hbar)$. Since $\operatorname{supp} \chi \subset B(0, R_1)$ whereas $\operatorname{supp} \boldsymbol{W} \subset \mathbb{R}^n \setminus B(0, R_1)$, we see that $\boldsymbol{W}\boldsymbol{v} = \boldsymbol{W}\boldsymbol{\chi}\boldsymbol{u} = \mathbf{0}$ so that $\boldsymbol{H}(\hbar)\boldsymbol{v}(\hbar) = \boldsymbol{J}(\hbar)\boldsymbol{v}(\hbar)$. We thus even have

$$\|(\boldsymbol{J}(\hbar) - \operatorname{Re} z_0(\hbar)\mathbf{1})\boldsymbol{v}(\hbar)\|_{L^2(\mathbb{R}^n) \otimes \mathbb{C}^2} \leq C(\hbar^{1/2}\sqrt{-\operatorname{Im} z_0(\hbar)} + e^{-C_0(R_1)/\hbar}) \|\boldsymbol{v}(\hbar)\|_{L^2(\mathbb{R}^n) \otimes \mathbb{C}^2}.$$

By interpreting $\boldsymbol{v}(\hbar)$ as a quasimode for $\boldsymbol{J}(\hbar)$, an application of Proposition 9.1(2) in conjunction with Remark 9.2 ensures the existence of an eigenvalue w_0 of $\boldsymbol{J}(\hbar)$ with

$$|\operatorname{Re} w_0 - \operatorname{Re} z_0| \le \varepsilon(\hbar) \log \frac{1}{\hbar} \quad \text{and} \quad -\operatorname{Im} w_0 \le \varepsilon(\hbar)$$

where $\varepsilon(\hbar)$ is as in the theorem.

(2) Let $\boldsymbol{u} \in \mathcal{D}$ be a normalized eigenfunction corresponding to w_0. Since $\boldsymbol{H}(\hbar)$ is symmetric, it follows that

$$0 = \operatorname{Im} \langle (\boldsymbol{J}(\hbar) - w_0)\boldsymbol{u}, \boldsymbol{u}\rangle = -\langle (\operatorname{Re} \boldsymbol{W})\boldsymbol{u}, \boldsymbol{u}\rangle - \operatorname{Im} w_0 \|\boldsymbol{u}\|^2,$$

or, equivalently,

$$\|(\operatorname{Re} \boldsymbol{W})^{1/2}\boldsymbol{u}\| = (-\operatorname{Im} w_0)^{1/2}.$$

With $1_{B(0,R_2)} \prec \chi \in C_0^\infty(\mathbb{R}^n)$ we have

$$(\boldsymbol{H}(\hbar) - w_0\mathbf{1})\boldsymbol{\chi}\boldsymbol{u} = \boldsymbol{\chi}(\boldsymbol{H}(\hbar) - w_0\mathbf{1})\boldsymbol{u} + [\boldsymbol{H}_0, \boldsymbol{\chi}]\boldsymbol{u} = i\boldsymbol{\chi}\boldsymbol{W}\boldsymbol{u} + [\boldsymbol{H}_0, \boldsymbol{\chi}]\boldsymbol{u}.$$

Since by assumption $\operatorname{Im} \boldsymbol{W} \le C(\operatorname{Re} \boldsymbol{W})^{1/2}$ and $\boldsymbol{W}$ is essentially bounded we see that $\|i\boldsymbol{\chi}\boldsymbol{W}\boldsymbol{u}\| \le C(-\operatorname{Im} w_0)^{1/2}$. By the semiclassical elliptic estimates in Theorem 2.1 we get, componentwise,

$$\begin{aligned}\|[-\hbar^2\Delta, \chi]u_\#\| &= \hbar\|\hbar(\Delta\chi)u_\# + 2\nabla\chi \cdot (\hbar\nabla f_\#)\| \le C\hbar\|u_\#\|_{H^1_\hbar(B(0,R_2,R_2+\mu))} \\ &\le C\hbar(\|(-\hbar^2\Delta - iW - w_0)u_\#\|_{B(0,R_2,R_2+\mu)} + \|u_\#\|_{B(0,R_2,R_2+\mu)}) \\ &\le C\hbar\|u_\#\|_{B(0,R_2,R_2+\mu)},\end{aligned}$$

where $\mu > 0$ is such that $\operatorname{supp}(\nabla\chi) \subset B(0, R_2, R_2 + \mu)$. Since for $|x| \ge R_2$ we can write

$$\int_{R_2<|x|<R_2+\mu} |u_\#|^2\, dx \le \delta_0^{-1} \int \operatorname{Re} W |u_\#|^2\, dx = C\|(\operatorname{Re} W)^{1/2}u_\#\|^2 \tag{10.11}$$

we conclude that $\|(\boldsymbol{H}(\hbar) - w_0\mathbf{1})\boldsymbol{\chi}\boldsymbol{u}\| \le C(-\operatorname{Im} w_0)^{1/2}$. Since

$$1 = \|\boldsymbol{u}\| = \|\boldsymbol{\chi}\boldsymbol{u} + (\mathbf{1} - \boldsymbol{\chi})\boldsymbol{u}\| \le \|\boldsymbol{\chi}\boldsymbol{u}\| + (-\delta_0^{-1} \operatorname{Im} w_0)^{1/2},$$

where the last inequality can be seen as in (10.7), we see that $\|\boldsymbol{\chi}\boldsymbol{u}\|$ is uniformly bounded away from 0 for small enough $\hbar$ and therefore we can regard $\boldsymbol{\chi}\boldsymbol{u}/\|\boldsymbol{\chi}\boldsymbol{u}\|$ as a quasimode for $\boldsymbol{H}(\hbar)$. An application of Proposition 9.1 in conjunction with Remark 9.2 yields the conclusion. ■

11. Individual resonances. The case $R_1 < R_0'$

In this chapter we establish Theorem 5.5 and we begin by outlining the strategy of the proof. Let $R_0' < R_1'$ and $\mathbf{1}_{B(0,R_0')} \prec \chi \prec \mathbf{1}_{B(0,R_1')}$. An application of the same arguments as in the proof of Theorem 5.1 gives us the estimate

$$\|(\boldsymbol{H}(\hbar) - \operatorname{Re} z_0(\hbar)\mathbf{1})\boldsymbol{v}(\hbar)\| \le C\tilde{\varepsilon}(\hbar), \tag{11.1}$$

where $\tilde{\varepsilon}(\hbar) = (\sqrt{-\hbar \operatorname{Im} z_0(\hbar)} + e^{-\gamma(\rho)/\hbar})\|\boldsymbol{u}(\hbar)\|_{L^2(B(0,R_1'))}$. Since, by hypothesis, the supports of $\boldsymbol{v}(\hbar)$ and $\boldsymbol{W}$ intersect, we cannot just replace $\boldsymbol{H}(\hbar)$ by $\boldsymbol{J}(\hbar)$ in (11.1) (as we did for its analogue in the proof of Theorem 5.1). Under the nontrapping condition in Assumption 5.4, we will prove that $\boldsymbol{v}(\hbar)$ is "small" away from $B(0, R_0)$; not merely outside

$B(0, R_0')$ as (10.1) and (10.5) suggest. For this purpose we employ microlocal analysis. We solve Heisenberg's equation of motion semiclassically by adapting ideas from Ivrii [22, Section 2.3] and Bolte–Glaser [3, Proposition 3.1]. Next, by a standard localization result away from the semiclassical wavefront set, we investigate how singularities propagate. The resulting auxiliary result, Proposition 11.1, is inspired by the discussion in Ivrii [22, Section 2.3] (see also [62, Lemma 3.1]). The Egorov type statement, which is part of its proof, differs from the scalar case in that one also needs to propagate the matrix degrees of freedom. These are contained in the dynamics of $\boldsymbol{a}(t)$ in the form of unitary transport matrices $\boldsymbol{d}(t)$; see below for details.

Proposition 11.1 (Propagation of singularities). *Let Assumptions 5.2 and 5.4 be satisfied. Suppose, moreover, that*

$$\|\partial_x^\alpha \partial_\xi^\beta \boldsymbol{h}_j\| \le C \quad \text{for all } (x,\xi) \in \mathsf{T}^*\mathbb{R}^n \text{ and } |\alpha| + |\beta| + j \ge 2, \tag{11.2}$$

where $\boldsymbol{h} \sim \sum \hbar^l \boldsymbol{h}_l$. Furthermore, suppose that for some $z_0(\hbar) \in [l_0, r_0]$, one has $(\boldsymbol{H}(\hbar) - z_0(\hbar))\boldsymbol{u}(\hbar) = \boldsymbol{g}(\hbar)$ with $\|\boldsymbol{u}(\hbar)\| \le C$, uniformly in $\hbar$, and where $\|\boldsymbol{g}(\hbar)\| = \mathcal{O}(\hbar^{s+1})$. Put, for any fixed $(x_0, \xi_0) \in \mathsf{T}^*\mathbb{R}^n$ with $|x_0| \ge R_0$ and $0 < T < \infty$, $(x_1, \xi_1) = \Phi^T(x_0, \xi_0)$; see Figure 2. Then, provided $\hbar$ is sufficiently small, $(x_1, \xi_1) \notin \mathrm{WF}^s\, \boldsymbol{u}$ implies that the same holds true for (x_0, ξ_0).

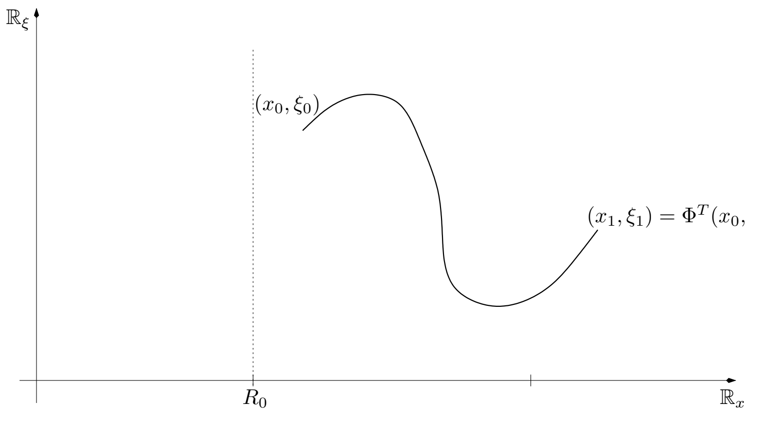

Fig. 2. Sketch of nontrapping scenario in phase space

Proof. By assumption we can find $\boldsymbol{a} \in \mathsf{S}(1)$ such that $\boldsymbol{a}(x, \hbar D)\boldsymbol{u} = \mathcal{O}(\hbar^s)$ and $\boldsymbol{a}_0(x,\xi) = I_2$ near (x_1, ξ_1) where $\boldsymbol{a} \sim \sum \hbar^l \boldsymbol{a}_l$. Our aim is to construct a symbol $\boldsymbol{a}(t) \in \mathsf{S}(1)$, $t \in [0, T]$, which is invertible, supported near $\Phi^{T-t}(x_0, \xi_0)$, and $\|\boldsymbol{a}(T)(x, \hbar D)\boldsymbol{u}\| \le C\hbar^s$. Introduce the strongly continuous one-parameter group of unitary operators $\boldsymbol{U}(t) = e^{-\frac{i}{\hbar}\boldsymbol{H}t}$. These are well-defined for all $t \in \mathbb{R}$ since $\boldsymbol{H}$ is selfadjoint. The time evolution $\boldsymbol{A}(t)$ of $\boldsymbol{A} = \mathrm{Op}^w(\boldsymbol{a})$ is given by the bounded operator

$$\boldsymbol{A}(t) = \boldsymbol{U}^*(t)\boldsymbol{A}\boldsymbol{U}(t),$$

and, as a consequence, it satisfies Heisenberg's equation of motion

$$\frac{\partial}{\partial t}\boldsymbol{A}(t) = \frac{i}{\hbar}[\boldsymbol{H}, \boldsymbol{A}(t)], \quad \boldsymbol{A}(0) = \boldsymbol{A}. \tag{11.3}$$

We will solve this equation semiclassically at the level of symbols, i.e. we shall construct coefficients $\boldsymbol{a}_l(t)$ so that the Weyl symbol $\boldsymbol{a}(t)$ of $\boldsymbol{A}(t)$ has the formal asymptotic expansion

$$\boldsymbol{a}(t) \sim \sum_{l=0}^{\infty} \hbar^l \boldsymbol{a}_l(t).$$

As we will see, $\boldsymbol{a}_l(t) \in \mathsf{S}(1)$ for all $t \in [0, T]$, and consequently we can find (with a slight abuse of notation) $\boldsymbol{A}(t)$ so that for all $0 \le t \le T$ we have

$$\frac{\partial}{\partial t}\boldsymbol{A}(t) = \frac{i}{\hbar}[\boldsymbol{H}, \boldsymbol{A}(t)] + \boldsymbol{R}(t), \quad \boldsymbol{A}(0) = \boldsymbol{A}, \tag{11.4}$$

where $\|\boldsymbol{R}(t)\| = \mathcal{O}(\hbar^{s+1})$. To calculate the coefficients $\boldsymbol{a}_l(t)$ we use (11.3) together with pseudodifferential calculus (see Section 3.2) to obtain the recursive Cauchy problem

$$\begin{aligned}\frac{\partial}{\partial t}&\boldsymbol{a}_l(t) - \{\boldsymbol{h}_0, \boldsymbol{a}_l(t)\} - i[\boldsymbol{h}_1, \boldsymbol{a}_l(t)] \\ &= \sum_{\substack{0 \le k \le l-1 \\ j+|\alpha|+|\beta|=l-k+1}} \frac{i^{|\alpha|-|\beta|}}{2^{|\alpha|+|\beta|}|\alpha|!|\beta|!}\big((\partial_\xi^\beta \partial_x^\alpha \boldsymbol{a}_k(t))(\partial_\xi^\alpha \partial_x^\beta \boldsymbol{h}_j) \\ &\qquad - (-1)^{|\alpha|-|\beta|}(\partial_\xi^\alpha \partial_x^\beta \boldsymbol{h}_j)(\partial_\xi^\beta \partial_x^\alpha \boldsymbol{a}_k(t))\big),\end{aligned} \tag{11.5}$$

with $\boldsymbol{a}_l(0) = \boldsymbol{a}_l$ (cf. also [22, Section 3.2]). For $l = 0$ the sum on the right is empty so at leading order (11.5) can be rewritten as

$$\frac{d}{dt}[\boldsymbol{d}^{-1}(x, \xi, -t)\boldsymbol{a}_0(t)(\Phi^{-t}(x, \xi))\boldsymbol{d}(x, \xi, -t)] = 0,$$

where $\boldsymbol{d}$ is the solution of the Cauchy problem

$$\frac{\partial}{\partial t}\boldsymbol{d}(x, \xi, t) + i\boldsymbol{h}_1(\Phi^t(x, \xi))\boldsymbol{d}(x, \xi, t) = 0, \quad \boldsymbol{d}(x, \xi, 0) = \boldsymbol{I}_2.$$

We remark that the time derivative is to be understood along the trajectory $\Phi^t(x, \xi)$. By differentiating the quantity $\|\boldsymbol{d}(x, \xi, t)\boldsymbol{v}\|^2$ for $\boldsymbol{v} \in \mathbb{C}^2$ with respect to t, using the defining equation for $\boldsymbol{d}$ and the fact that $\boldsymbol{h}_1$ is Hermitian it follows that $\|\boldsymbol{d}(x, \xi, t)\boldsymbol{v}\|^2$ is constant and therefore equal to $\|\boldsymbol{v}\|^2$ so that $\boldsymbol{d}(x, \xi, t)$ is unitary. Moreover, as a consequence of the fact that both $\boldsymbol{f}(t) = \boldsymbol{d}(\Phi^t(x, \xi), -t)$ and $\boldsymbol{g}(t) = \boldsymbol{d}(\Phi^t(x, \xi), T - t)\boldsymbol{d}(\Phi^T(x, \xi), -T)$ satisfy

$$\boldsymbol{f}'(t) + i\boldsymbol{h}_1(\Phi^t(x, \xi))\boldsymbol{f}(t) = 0$$

as well as $\boldsymbol{f}(T) = \boldsymbol{g}(T)$ it follows by uniqueness that also $\boldsymbol{f}(0) = \boldsymbol{g}(0)$ or, in other words,

$$\boldsymbol{d}(\Phi^t(x, \xi), -t) = \boldsymbol{d}^{-1}(x, \xi, t).$$

Using these properties we can write

$$\boldsymbol{a}_0(t)(x, \xi) = \boldsymbol{d}^*(x, \xi, t)\boldsymbol{a}_0(\Phi^t(x, \xi))\boldsymbol{d}(x, \xi, t).$$

We observe that $\operatorname{supp} \boldsymbol{a}_0(T) = \Phi^{-T}(\operatorname{supp} \boldsymbol{a}_0)$ is confined to a neighbourhood of (x_0, ξ_0). For the higher order coefficients we similarly need to solve

$$\begin{aligned}
\frac{d}{dt}[\boldsymbol{d}^{-1}(x,\xi,-t)\boldsymbol{a}_l(t)(\Phi^{-t}(x,\xi))\boldsymbol{d}(x,\xi,-t)]& \\
= \sum_{\substack{0\le k\le l-1\\ j+|\alpha|+|\beta|=l-k+1}} & \frac{i^{|\alpha|-|\beta|}}{2^{|\alpha|+|\beta|}|\alpha|!|\beta|!}\big((\partial_\xi^\beta\partial_x^\alpha \boldsymbol{a}_k(t))(\partial_\xi^\alpha\partial_x^\beta \boldsymbol{h}_j) \\
& - (-1)^{|\alpha|-|\beta|}(\partial_\xi^\alpha\partial_x^\beta \boldsymbol{h}_j)(\partial_\xi^\beta\partial_x^\alpha \boldsymbol{a}_k(t))\big), \qquad (11.6)
\end{aligned}$$

however, this time with an inhomogeneity depending on the previous terms on the right-hand side. The Jacobian $\boldsymbol{D}\Phi^t$ satisfies the variation equation

$$\frac{d}{dt}\boldsymbol{D}\Phi^t = \boldsymbol{D}\big(\partial_\xi h_0(x(t),\xi(t)), -\partial_x h_0(x(t),\xi(t))\big)\boldsymbol{D}\Phi^t$$

with $\boldsymbol{D}\Phi^t|_{t=0} = \boldsymbol{I}_2$. From this it follows that $\|\boldsymbol{D}\Phi^t - \boldsymbol{I}_2\| \le C|t|e^{Ct}$ for all $t\in[0,T]$. By [22, Section 3.2] our assumption on the Hamiltonian implies that $|\partial_x^\alpha\partial_\xi^\beta\Phi^t(x,\xi)| \le C$ for all $\alpha,\beta\in\mathbb{N}_0^n$ and $t\in[0,T]$. Consequently, $\boldsymbol{a}\circ\Phi^t \in \mathsf{S}(1)$ because already $\boldsymbol{a}\in\mathsf{S}(1)$. Since $\boldsymbol{d}(x,\xi,t)$ is unitary, it is also bounded. Moreover, the estimate (11.2) for $j=1$ implies, again by [22, Section 3.2], that $\|\partial_x^\alpha\partial_\xi^\beta \boldsymbol{d}(x,\xi,t)\| \le C$ for all $t\in[0,T]$ and $|\alpha|+|\beta|\ge 1$. Since the right-hand side of (11.6) only contains terms with $|\alpha|+|\beta|+j\ge 2$ it follows that it is bounded in matrix norm for all $(x,\xi)\in\mathsf{T}^*\mathbb{R}^n$. It follows by means of Duhamel's principle that $\boldsymbol{a}_l(t)\in\mathsf{S}(1)$ for all $l\ge 0$. From these considerations (11.4) follows. Consider now

$$\alpha(t) = \|\boldsymbol{a}(t)^w(x,\hbar D)\boldsymbol{u}\|$$

so that $\alpha(0) = \mathcal{O}(\hbar^s)$. With $\boldsymbol{A}(t)$ as in (11.4) we obtain, as a consequence of the fact that $\boldsymbol{H}$ is selfadjoint,

$$\begin{aligned}
\frac{d}{dt}\frac{\alpha(t)^2}{2} &= \operatorname{Re}\left\langle \frac{d}{dt}\boldsymbol{A}(t)\boldsymbol{u}, \boldsymbol{A}(t)\boldsymbol{u}\right\rangle \\
&= \hbar^{-1}\operatorname{Im}\langle([\boldsymbol{H}-z_0,\boldsymbol{A}(t)]+\boldsymbol{R}(t))\boldsymbol{u},\boldsymbol{A}(t)\boldsymbol{u}\rangle \\
&= -\hbar^{-1}\operatorname{Im}\langle\boldsymbol{A}(t)(\boldsymbol{H}-z_0)\boldsymbol{u},\boldsymbol{A}(t)\boldsymbol{u}\rangle + \hbar^{-1}\operatorname{Im}(\boldsymbol{R}(t)\boldsymbol{u},\boldsymbol{A}(t)\boldsymbol{u}) \\
&= -\hbar^{-1}\operatorname{Im}\langle\boldsymbol{A}(t)\boldsymbol{g},\boldsymbol{A}(t)\boldsymbol{u}\rangle + \hbar^{-1}\operatorname{Im}\langle\boldsymbol{R}(t)\boldsymbol{u},\boldsymbol{A}(t)\boldsymbol{u}),
\end{aligned}$$

where $\|\boldsymbol{g}\| = \mathcal{O}(\hbar^{s+1})$, from which it follows that

$$\alpha(t)\frac{d}{dt}\alpha(t) \le C\hbar^s\alpha(t) + \hbar^{-1}\|\boldsymbol{R}(t)\boldsymbol{u}\|\alpha(t).$$

Consequently, since $\alpha(0) = \mathcal{O}(\hbar^s)$ and $\|\boldsymbol{R}(t)\boldsymbol{u}\| = \mathcal{O}(\hbar^{s+1})$,

$$\alpha(t) = \mathcal{O}(\hbar^s) \qquad \text{for } t\in[0,T]. \tag{11.7}$$

Since $\boldsymbol{a}_0(T)$ is close to the identity matrix near (x_0,ξ_0) it follows that $\boldsymbol{a}(T)$ is invertible near (x_0,ξ_0) provided $\hbar$ is sufficiently small. It follows that $(x_0,\xi_0)\notin \mathrm{WF}^s\,\boldsymbol{u}$. ∎

Proof of Theorem 5.5. Arguing as in the proof of Theorem 5.1 where again $\boldsymbol{v} = \boldsymbol{\chi}\boldsymbol{u}$ but this time $\mathbf{1}_{B(0,R_0')} \prec \chi \prec \mathbf{1}_{B(0,R_1')}$ for some $R_1' > R_0'$ we obtain

$$\|(\boldsymbol{H}(\hbar) - \operatorname{Re} z_0\mathbf{1})\boldsymbol{v}\| \le C\tilde{\varepsilon}(\hbar)\|\boldsymbol{u}\|_{L^2(B(0,R_1'))} \tag{11.8}$$

with $\tilde\varepsilon(\hbar)$ as before. Let $\mu > 0$ and pick a point $(x_0, \xi_0) \in h_0^{-1}[l_0, r_0]$ with $|x_0| > R_0 + \mu$. By Assumption 5.4 there exists $T > 0$ such that (x_0, ξ_0) can be connected to some $(x_1, \xi_1) = \Phi^T(x_0, \xi_0)$ with $|x_1| > R_1'$ where $\boldsymbol{v} = 0$. Applying Proposition 11.1 (with $\boldsymbol{H}$ replaced by $\boldsymbol{H}_0$ suffices) near this bicharacteristic implies (see (11.7)) that

$$\|\boldsymbol{v}\| = \mathcal{O}(\hbar^{-1}\tilde\varepsilon(\hbar) + \hbar^\infty) \tag{11.9}$$

microlocally near (x_0, ξ_0). From (11.8) we get

$$\begin{aligned}\|(\boldsymbol{J}(\hbar) - \operatorname{Re} z_0\mathbf{1})\boldsymbol{v}\| &\le \|(\boldsymbol{H}(\hbar) - \operatorname{Re} z_0\mathbf{1})\boldsymbol{v}\| + C\|\boldsymbol{v}\|_{L^2(\{|x|>R_1\})}\\ &= \mathcal{O}(\tilde\varepsilon(\hbar)) + \mathcal{O}(\hbar^{-1}\tilde\varepsilon(\hbar) + \hbar^\infty) = \mathcal{O}(\hbar^{-1}\tilde\varepsilon(\hbar) + \hbar^\infty).\end{aligned}$$

An application of Proposition 9.1 yields Theorem 5.5 because the exponential factor is absorbed by the polynomial one for $\hbar \ll 1$. ■

12. Clusters of resonances

In this chapter we always impose Assumptions 4.1 and 4.6. The matrix valued results in Propositions 6.1, 8.1, 8.2, 8.4 and 9.1 and Lemma B.1 underpin the proof of Theorems 5.6 and 5.7. Moreover, the propagation of singularities result in Proposition 11.1 in combination with the reasoning from the proof of Theorem 5.5 will be used to prove Theorem 5.7.

Decomposition into clusters. We consider the box $\Omega(\hbar)$, defined in (5.4). After possibly altering the box slightly without changing its properties we may assume $\partial\Omega(\hbar)$ is free from resonances. We gather all resonances in $\Omega(\hbar)$ into the interior of mutually disjoint subdomains

$$\Omega_j(\hbar) = [l_j(\hbar), r_j(\hbar)] + i[-c(\hbar), 0], \quad 1 \le j \le J(\hbar),$$

and denote by m_j the number of resonances, counting multiplicities, in $\Omega_j(\hbar)$. Clearly $J(\hbar) = \mathcal{O}(\hbar^{-n^\sharp})$ because, as a consequence of Assumption 4.6, $\operatorname{Count}(\boldsymbol{H}, \Omega(\hbar)) = \mathcal{O}(\hbar^{-n^\sharp})$; see (6.9). Another easy consequence of the latter bound is that we can group the resonances so that for $j, k \in \{1, \dots, J(\hbar)\}$, $j \ne k$,

$$\operatorname{dist}(\Omega_j(\hbar), \Omega_k(\hbar)) \ge 4w(\hbar), \quad 0 < r_j(\hbar) - l_j(\hbar) \le C\hbar^{-n^\sharp}w(\hbar) \tag{12.1}$$

where $0 < w(\hbar) = \mathcal{O}(\hbar^N)$, $N \gg 1$, is some pre-fixed quantity which we take to be

$$w(\hbar) = \hbar^{-(5n^\sharp+1)/2}c(\hbar). \tag{12.2}$$

We notice that with this choice of $w(\hbar)$ the bigger domains (9.3) introduced in the proof of Proposition 9.1 will not intersect and therefore, provided the theorem applies, we can use it for each subdomain separately without overcounting any resonances (eigenvalues). Define

$$\Pi_{\Omega_j(\hbar)} = \frac{1}{2\pi i}\oint_{\partial\Omega_j(\hbar)} (\boldsymbol{H} - z\mathbf{1})^{-1}\,dz, \qquad \Pi_{\Omega(\hbar)} = \sum \Pi_{\Omega_j(\hbar)}, \qquad \mathcal{H}_{\Omega_j} = \Pi_{\Omega_j(\hbar)}\mathcal{H}.$$

Clearly, by Lemma 8.1, the same decomposition can be made for the eigenvalues of $\boldsymbol{J}(\hbar)$. Then $\mathcal{H}_{\Omega_j}$ is the span of the generalized eigenvectors of $\boldsymbol{H}_\theta$ corresponding to the eigenvalues in $\Omega_j(\hbar)$ (see e.g. [26]) and $\boldsymbol{H}_\theta$ is invariant on this set. We define $\boldsymbol{H}_{\Omega_j} =$

$\boldsymbol{H}_\theta|_{\mathcal{H}_{\Omega_j}}$, which is of finite rank since the dimension m_j of $\mathcal{H}_{\Omega_j}$ is finite (bounded by $C\hbar^{-n^\sharp}$).

Dissipation estimate. Fix $\hat{R} < \widetilde{R}$ with $R_0' + \delta < \hat{R} < \widetilde{R} - \delta$ and let

$$\rho = \begin{cases} 1 & \text{for } r < \hat{R} - \delta/2, \\ r^{(n-1)/2} & \text{for } r > \hat{R}. \end{cases}$$

Set $\widetilde{\boldsymbol{H}} = \rho \boldsymbol{H} \rho^{-1}$ so that $\widetilde{\boldsymbol{H}}$ becomes selfadjoint on $\widetilde{\mathcal{H}} := \rho\mathcal{H}$ with the measure $d\mu := \rho^{-2} r^{n-1}\, dr\, d\omega$. With this change of variables we have

$$\widetilde{\boldsymbol{H}}|_{\{r>\hat{R}\}} = \hbar^2 \left(-\partial_r^2 - \frac{\Delta_\omega}{r^2} + \frac{(n-1)(n-3)}{4r^2} \right) \otimes \boldsymbol{I}_2$$

and $d\mu|_{\{r>\hat{R}\}} = dr d\omega$. We denote by $\widetilde{\boldsymbol{H}}_\theta$ the operator obtained from $\widetilde{\boldsymbol{H}}$ by complex scaling for $r \geq \widetilde{R}$. In the following we shall use the notation $|\boldsymbol{u}|^2 = |u_1|^2 + |u_2|^2$.

LEMMA 12.1. *Let $z \in \mathbb{C}$ with $\operatorname{Re} z \geq l_0$ and $\operatorname{Im} z \leq 0$. Then, for $\hbar, \theta_0 > 0$ sufficiently small, we have for any $\boldsymbol{u} \in \mathcal{D}_\theta$, $0 \leq \theta \leq \theta_0$, the estimate*

$$C \int \big((\theta + r\theta')|\hbar \partial_r \boldsymbol{u}|^2 + \theta(|\hbar r^{-1} \nabla_\omega \boldsymbol{u}|^2 + |\boldsymbol{u}|^2) \big)\, dr\, d\omega \\ \leq -\operatorname{Im} \langle e^{i\theta}(\widetilde{\boldsymbol{H}}_\theta - z\mathbf{1})\boldsymbol{u}, \boldsymbol{u} \rangle_{\widetilde{\mathcal{H}}} + (-\operatorname{Im} z + e^{-\hbar^{-2/3+\varepsilon}}) \|\boldsymbol{u}\|^2_{\widetilde{\mathcal{H}}},$$

where $C = \min(l_0, 1)/2$.

Proof. The idea of this proof goes back to Burq [6] (see also [62, 63]). There is no loss of generality to assume $\operatorname{supp} \boldsymbol{u} \subset \{|x| > \hat{R}\}$. Indeed, choose a smooth cutoff

$$\chi = \begin{cases} 0, & r \leq \hat{R}, \\ 1, & r \geq \hat{R} + \delta. \end{cases}$$

Decomposing $\boldsymbol{u}$ according to $\boldsymbol{u} = \boldsymbol{\chi}\boldsymbol{u} + (\mathbf{1} - \boldsymbol{\chi})\boldsymbol{u}$ we have

$$\begin{aligned} \operatorname{Im} \langle e^{i\theta}(\widetilde{\boldsymbol{H}}_\theta - \operatorname{Re} z\mathbf{1})\boldsymbol{u}, \boldsymbol{u} \rangle_{\widetilde{\mathcal{H}}} &= \operatorname{Im} \langle e^{i\theta}(\widetilde{\boldsymbol{H}}_\theta - \operatorname{Re} z\mathbf{1})\boldsymbol{\chi}\boldsymbol{u}, \boldsymbol{\chi}\boldsymbol{u} \rangle_{\widetilde{\mathcal{H}}} \\ &\quad + \operatorname{Im} \langle (\widetilde{\boldsymbol{H}}_\theta - \operatorname{Re} z\mathbf{1})(\mathbf{1} - \boldsymbol{\chi})\boldsymbol{u}, \boldsymbol{\chi}\boldsymbol{u} \rangle_{\widetilde{\mathcal{H}}} \\ &\quad + \operatorname{Im} \langle (\widetilde{\boldsymbol{H}}_\theta - \operatorname{Re} z\mathbf{1})(\mathbf{1} - \boldsymbol{\chi})\boldsymbol{u}, (\mathbf{1} - \boldsymbol{\chi})\boldsymbol{u} \rangle_{\widetilde{\mathcal{H}}} \\ &\quad + \operatorname{Im} \langle (\widetilde{\boldsymbol{H}}_\theta - \operatorname{Re} z\mathbf{1})\boldsymbol{\chi}\boldsymbol{u}, (\mathbf{1} - \boldsymbol{\chi})\boldsymbol{u} \rangle_{\widetilde{\mathcal{H}}} \\ &= \operatorname{Im} \langle e^{i\theta}(\widetilde{\boldsymbol{H}}_\theta - \operatorname{Re} z\mathbf{1})\boldsymbol{\chi}\boldsymbol{u}, \boldsymbol{\chi}\boldsymbol{u} \rangle_{\widetilde{\mathcal{H}}}, \end{aligned}$$

since $\theta = 0$ on $\operatorname{supp}(\mathbf{1} - \boldsymbol{\chi})$ and $\widetilde{\boldsymbol{H}}_\theta$ is symmetric. Using

$$e^{i\theta}\widetilde{\boldsymbol{H}}_\theta = \left(-\frac{1}{1 + ir\theta'} \hbar\partial_r \frac{e^{-i\theta}}{1 + ir\theta'} \hbar\partial_r - e^{-i\theta} \frac{\hbar^2 \Delta_\omega}{r^2} + e^{i\theta}\hbar^2 \frac{(n-1)(n-3)}{4r^2} \right) \otimes \boldsymbol{I}_2 \quad (12.3)$$

for $r > \hat{R}$, integration by parts results in

$$
\begin{aligned}
-\operatorname{Im}\langle e^{i\theta}(\widetilde{\boldsymbol{H}}_\theta - \operatorname{Re} z\mathbf{1})\boldsymbol{u}, \boldsymbol{u}\rangle_{\widetilde{\mathcal{H}}} &= \int \left(\operatorname{Im}\left(-\frac{e^{-i\theta}}{(1+ir\theta')^2}\right)|\hbar\partial_r \boldsymbol{u}|^2 + \sin\theta |\hbar r^{-1}\nabla_\omega \boldsymbol{u}|^2\right) dr\, d\omega \\
&\quad + \int \operatorname{Im}\left(e^{i\theta}\operatorname{Re} z - e^{-i\theta}\hbar^2 \frac{(n-1)(n-3)}{4r^2}\right)|\boldsymbol{u}|^2\, dr\, d\omega \\
&\quad - h \operatorname{Im}\langle g(r)\hbar(\partial_r)\otimes \boldsymbol{I}\boldsymbol{u}, \boldsymbol{u}\rangle_{\widetilde{\mathcal{H}}} \\
&=: (\mathrm{I}) + (\mathrm{II}) + (\mathrm{III}),
\end{aligned}
$$

where

$$
g(r) = \frac{d}{dr}\left(\frac{1}{1+ir\theta'}\right)\frac{e^{-i\theta}}{1+ir\theta'} = -\frac{i(r\theta'' + \theta')e^{-i\theta}}{(1+ir\theta')^3}. \tag{12.4}
$$

It is easy to see that if $\theta_0 > 0$ is sufficiently small then

$$
(\mathrm{I}) \geq \frac{3}{4}\int ((\theta + 2r\theta')|\hbar\partial_r \boldsymbol{u}|^2 + \theta|\hbar r^{-1}\nabla_\omega \boldsymbol{u}|^2)\, dr\, d\omega.
$$

Moreover, for $\hbar$ small enough,

$$
(\mathrm{II}) \geq \int \left(\operatorname{Re} z \sin\theta - \frac{l_0}{4}\theta\right)|\boldsymbol{u}|^2\, dr\, d\omega \geq \frac{3}{4} l_0 \int \theta |\boldsymbol{u}|^2\, dr\, d\omega,
$$

provided θ_0 is sufficiently small. Writing

$$
(\mathrm{III}) = -\frac{\hbar^2}{2i}\left(\int g(\partial_r \boldsymbol{u})\overline{\boldsymbol{u}}\, dr\, d\omega - \int \overline{g}(\partial_r \overline{\boldsymbol{u}})\boldsymbol{u}\, dr\, d\omega\right)
$$

and integrating the second integral by parts we obtain

$$
(\mathrm{III}) = -\frac{\hbar^2}{i}\langle(\operatorname{Re} g)\partial_r \boldsymbol{u}, \boldsymbol{u}\rangle_{\widetilde{\mathcal{H}}} - \frac{\hbar^2}{2i}\langle(\operatorname{Re} g')\boldsymbol{u}, \boldsymbol{u}\rangle_{\widetilde{\mathcal{H}}} + \frac{\hbar^2}{2}\langle(\operatorname{Im} g')\boldsymbol{u}, \boldsymbol{u}\rangle_{\widetilde{\mathcal{H}}}.
$$

Taking real parts of both sides results in

$$
(\mathrm{III}) = -\operatorname{Im}\hbar\langle(\operatorname{Re} g)\hbar\partial_r \boldsymbol{u}, \boldsymbol{u}\rangle_{\widetilde{\mathcal{H}}} + \frac{\hbar^2}{2}\langle(\operatorname{Im} g')\boldsymbol{u}, \boldsymbol{u}\rangle_{\widetilde{\mathcal{H}}} =: (\mathrm{III})^{(1)} + (\mathrm{III})^{(2)}.
$$

Using (12.4) and properties of θ we see that

$$
|\operatorname{Re} g| \leq C(\theta' + |\theta''|)(\theta + \theta') \leq C\theta,
$$

implying

$$
|(\mathrm{III})^{(1)}| \leq C\hbar \int \theta |\hbar\partial_r \boldsymbol{u}|\, |\boldsymbol{u}| \leq C\hbar \int \theta(|\hbar\partial_r \boldsymbol{u}|^2 + |\boldsymbol{u}|^2)
$$

from which it follows that $(\mathrm{III})^{(1)}$ can be absorbed by the estimates for (I) and (II). To obtain a bound on $(\mathrm{III})^{(2)}$ use (12.4) again to see that

$$
|g'| \leq C(\theta' + |\theta''| + |\theta^{(3)}|).
$$

Therefore, with $t := r - \widetilde{R}$, for any $\varepsilon > 0$ we have

$$
|g'| \leq \begin{cases} e^{-\hbar^{-2/3+\varepsilon}} & \text{for } 0 \leq t \leq \hbar^{2/(k+3)}, \\ C\hbar^{-2}\theta & \text{for } t \geq \hbar^{2/(k+3)}, \end{cases}
$$

provided $\hbar \ll 1$ and $k \gg 1$. Thus

$$
|g'(r)| \leq e^{-\hbar^{-2/3+\varepsilon}} + C\hbar^{-2}\theta \quad \text{for } \hbar \text{ sufficiently small and } r \geq \widetilde{R}.
$$

We conclude that

$$(\mathrm{III})^{(2)} \le e^{-\hbar^{-2/3+\varepsilon}} \|\boldsymbol{u}\|^2_{\widetilde{\mathcal{H}}} + \hbar^{1/2} \int \theta |\boldsymbol{u}|^2 \, dr \, d\omega,$$

where the latter integral is absorbed by the estimate for (II). Putting everything together we get, since $\operatorname{Im} z \le 0$,

$$\begin{aligned} -\operatorname{Im} \langle e^{i\theta}(\widetilde{\boldsymbol{H}}_\theta - z\mathbf{1})\boldsymbol{u}, \boldsymbol{u}\rangle_{\widetilde{\mathcal{H}}} &= -\operatorname{Im} \langle e^{i\theta}(\widetilde{\boldsymbol{H}}_\theta - \operatorname{Re} z\mathbf{1})\boldsymbol{u}, \boldsymbol{u}\rangle_{\widetilde{\mathcal{H}}} + \operatorname{Im} z \langle \cos\theta \boldsymbol{u}, \boldsymbol{u}\rangle_{\widetilde{\mathcal{H}}} \\ &\ge -\operatorname{Im} \langle e^{i\theta}(\widetilde{\boldsymbol{H}}_\theta - \operatorname{Re} z\mathbf{1})\boldsymbol{u}, \boldsymbol{u}\rangle_{\widetilde{\mathcal{H}}} + \operatorname{Im} z \|\boldsymbol{u}\|^2_{\widetilde{\mathcal{H}}} \\ &\ge \frac{\min(1,l_0)}{2} \int \big((\theta + r\theta')|\hbar\partial_r \boldsymbol{u}|^2 + \theta(|\hbar r^{-1}\nabla_\omega \boldsymbol{u}|^2 + |\boldsymbol{u}|^2)\big) \, dr\, d\omega \qquad (12.5) \\ &\quad - e^{-\hbar^{-2/3+\varepsilon}} \|\boldsymbol{u}\|^2_{\widetilde{\mathcal{H}}} + \operatorname{Im} z \|\boldsymbol{u}\|^2_{\widetilde{\mathcal{H}}}. \ \blacksquare \end{aligned}$$

Corollary 12.2. *Under the assumptions of Lemma* 12.1 *and* $\operatorname{Im} z > e^{-\hbar^{-2/3+\varepsilon}}/2$, *one has*

$$\|(\boldsymbol{H}_\theta - z\mathbf{1})^{-1}\|_{\mathcal{B}(\mathcal{H})} \le \frac{2}{\operatorname{Im} z - \frac{1}{2} e^{-\hbar^{-2/3+\varepsilon}}}.$$

Proof. Following the proof of Lemma 12.1 until (12.5) we can write, since now $\operatorname{Im} z > 0$,

$$\begin{aligned} -\operatorname{Im} \langle e^{i\theta}(\widetilde{\boldsymbol{H}}_\theta - z\mathbf{1})\boldsymbol{u}, \boldsymbol{u}\rangle_{\widetilde{\mathcal{H}}} &= -\operatorname{Im} \langle e^{i\theta}(\widetilde{\boldsymbol{H}}_\theta - \operatorname{Re} z\mathbf{1})\boldsymbol{u}, \boldsymbol{u}\rangle_{\widetilde{\mathcal{H}}} + \operatorname{Im} z \langle \cos\theta \boldsymbol{u}, \boldsymbol{u}\rangle_{\widetilde{\mathcal{H}}} \\ &\ge -\operatorname{Im} \langle e^{i\theta}(\widetilde{\boldsymbol{H}}_\theta - \operatorname{Re} z\mathbf{1})\boldsymbol{u}, \boldsymbol{u}\rangle_{\widetilde{\mathcal{H}}} + \operatorname{Im} z \langle \cos\theta_0 \boldsymbol{u}, \boldsymbol{u}\rangle_{\widetilde{\mathcal{H}}} \\ &\ge \frac{\min(1,l_0)}{2} \int \big((\theta + r\theta')|\hbar\partial_r \boldsymbol{u}|^2 + \theta(|\hbar r^{-1}\nabla_\omega \boldsymbol{u}|^2 + |\boldsymbol{u}|^2)\big) \, dr\, d\omega \\ &\quad - \tfrac{1}{4} e^{-\hbar^{-2/3+\varepsilon}} \|\boldsymbol{u}\|^2_{\widetilde{\mathcal{H}}} + \tfrac{1}{2} \operatorname{Im} z \|\boldsymbol{u}\|^2_{\widetilde{\mathcal{H}}}, \end{aligned}$$

so that

$$\frac{1}{2}(\operatorname{Im} z - \tfrac{1}{2} e^{-\hbar^{-2/3+\varepsilon}}) \|\boldsymbol{u}\|_{\widetilde{\mathcal{H}}} \le \|(\widetilde{\boldsymbol{H}}_\theta - z\mathbf{1})\boldsymbol{u}\|_{\widetilde{\mathcal{H}}}$$

and finally, after replacing $\boldsymbol{u}$ by $\rho \boldsymbol{u}$,

$$\|(\boldsymbol{H}_\theta - z\mathbf{1})^{-1}\|_{\mathcal{B}(\mathcal{H})} \le \frac{2}{\operatorname{Im} z - \frac{1}{2} e^{-\hbar^{-2/3+\varepsilon}}} \quad \text{for } \operatorname{Im} z > \tfrac{1}{2} e^{-\hbar^{-2/3+\varepsilon}}. \ \blacksquare$$

Resonant state estimates. For the next few lemmas we work with a fixed subdomain Ω_j and shall therefore avoid corresponding subscripts. We apply the matrix valued cutoff resolvent estimate in Proposition 6.1 and the semiclassical maximum principle in Lemma B.1. Using (12.1) we prove that $(\boldsymbol{H}_\Omega(\hbar) - z_0\mathbf{1})\boldsymbol{u} = \mathcal{O}(\hbar^{-(7n^\sharp+1)})$ for any linear combination $\boldsymbol{u}$ of resonant states associated to resonances belonging to a cluster. The strategy of the proof follows closely Stefanov [60, Lemma 2] (see also [62, Proposition 3.2]). It is necessary to work with the complex scaled Hamiltonian $\boldsymbol{H}_\theta(\hbar)$ for technical reasons.

Lemma 12.3. *Under the assumptions above, the choice for* $w(\hbar)$ *in* (12.2), *and* $z_0 \in [l(\hbar), r(\hbar)]$ *one has*

$$\|(\boldsymbol{H}_\Omega - z_0\mathbf{1})\boldsymbol{u}\|_{\mathcal{H}_\Omega} \le C\hbar^{-(7n^\sharp+1)} c(\hbar) \|\boldsymbol{u}\|_{\mathcal{H}} \quad \textit{for all } \boldsymbol{u} \in \Pi_\Omega \mathcal{H}.$$

Proof. Define $\tilde{z}_j(\hbar)$ as the reflection of $z_j(\hbar)$ about the line $\{\operatorname{Im} z = c(\hbar)\}$, i.e.

$$\tilde{z}_j(\hbar) = \overline{z}_j(\hbar) + 2ic(\hbar).$$

By Lemma 12.1 and the lower bound on $c(\hbar)$ we have

$$\|(\boldsymbol{H}_\Omega - z\boldsymbol{1})^{-1}\|_{\mathcal{B}(\mathcal{H}_\Omega)} \leq \frac{4}{c(\hbar)}$$

for $\operatorname{Im} z = c(\hbar)$. Define

$$G(z,\hbar) = \prod_{j=1}^{p} \left(\frac{z - z_j}{z - \tilde{z}_j} \right)^{m_j},$$

where p denotes the number of distinct resonances in Ω. Then

$$|G(z,\hbar)| \leq 1 \quad \text{for } \operatorname{Im} z \leq c(\hbar)$$

because the corresponding bound holds for each factor of the product. By the construction of $\boldsymbol{G} := G \otimes I$ the function $z \mapsto \boldsymbol{F} := \boldsymbol{G}(\boldsymbol{H} - z\boldsymbol{1})^{-1}$ is holomorphic for $\operatorname{Im} z \leq c(\hbar)$.

Since we always work with domains included in $\Omega_0 := [l_0/2, 2r_0] + i[-c_0, c_0]$ with fixed $0 < l_0 < r_0$ and $0 < c_0 \ll 1$, the constant A in Proposition 6.1 can be chosen uniformly and consequently

$$\|(\boldsymbol{H}_\Omega - z\boldsymbol{1})^{-1}\| \leq A e^{A\hbar^{-n^\sharp} \log(1/g)}$$

for $z \in \Omega_0$ with $\operatorname{dist}(z, \operatorname{Res} \boldsymbol{H}) \geq g(\hbar)$ where $g(\hbar) \ll 1$. In view of Assumption 4.6 we can extend the larger domain $\Omega_1 := [l(\hbar) - 5w(\hbar), r(\hbar) + 5w(\hbar)] + i[-D(\hbar)\hbar^{-2n^\sharp - 1}, D(\hbar)]$ (albeit staying within some $\hbar$-independent neighbourhood of $\Omega(\hbar)$, e.g. Ω_0) so that no resonance comes within distance $\hbar^{n^\sharp + 1}$ of $\partial\Omega_1$. Thus

$$\|\boldsymbol{F}\| = \mathcal{O}(\exp(C\hbar^{-n^\sharp} \log \hbar^{-1}))$$

on the boundary of the extended version of Ω_1. By the maximum principle the same bound holds in all of Ω_1. Since, by Corollary 12.2, $\|\boldsymbol{F}\| \leq 4/D(\hbar)$ on the upper side of Ω_1 we can apply the matrix valued semiclassical maximum principle to conclude that

$$\|\boldsymbol{G}(z)(\boldsymbol{H}_\Omega - z\boldsymbol{1})^{-1}\|_{\mathcal{B}(\mathcal{H}_\Omega)} \leq \frac{2e^3}{c(\hbar)} \quad \text{for all } z \in \tilde{\Omega}(\hbar),$$

where $\tilde{\Omega}(\hbar) = [l(\hbar) - w(\hbar), r(\hbar) + w(\hbar)] + i[-\hbar^{-n^\sharp} D(\hbar), D(\hbar)]$. It remains to estimate G from below on $\partial\tilde{\Omega}(\hbar)$. To do so we estimate $(z - \tilde{z}_j)/(z - z_j)$ from above on $\partial\tilde{\Omega}(\hbar)$. We notice that

$$|z(\hbar) - \tilde{z}_j(\hbar)| \leq |z_j(\hbar) - \overline{z}_j(\hbar)| + 2c \leq 4c(\hbar).$$

Moreover, the distance from $z_j(\hbar)$ to any of the sides $\operatorname{Im} z = -c(\hbar)\hbar^{-n^\sharp}$, $\operatorname{Re} z = l - w$ and $\operatorname{Re} z = r + w$ of $\tilde{\Omega}$ is always greater than $D(\hbar)\hbar^{-n^\sharp}/2$ for $\hbar \ll 1$. Therefore

$$\left| \frac{z - \tilde{z}_j}{z - z_j} - 1 \right| = \left| \frac{z_j - \tilde{z}_j}{z - z_j} \right| \leq \frac{4c(\hbar)}{c(\hbar)\hbar^{-n^\sharp}/2} = 8\hbar^{n^\sharp}$$

for all $z \in \partial\tilde{\Omega}(\hbar) \setminus \{\operatorname{Im} z = c(\hbar)\}$. Since $x \mapsto (1+x)^{1/x}$ approaches e from below as $x \to 0$ we obtain

$$\frac{1}{|G(z,\hbar)|} \leq (1 + 8\hbar^{n^\sharp})^{m(\hbar)} \leq (1 + 8\hbar^{n^\sharp})^{C\hbar^{-n^\sharp}} \leq e^{8C}.$$

From this we infer the bound

$$\|(\boldsymbol{H}_\Omega - z\boldsymbol{1})^{-1}\|_{\mathcal{B}(\mathcal{H}_\Omega)} \leq C \|\boldsymbol{G}(\boldsymbol{H}_\Omega - z\boldsymbol{1})^{-1}\|_{\mathcal{B}(\mathcal{H}_\Omega)} \leq \frac{C}{c(\hbar)}, \quad z \in \partial\tilde{\Omega}.$$

For $z_0 \in [l(\hbar), r(\hbar)]$ we write

$$\boldsymbol{H}_\Omega - z_0\mathbf{1} = \frac{1}{2\pi i}\oint_{\partial\tilde{\Omega}} (\boldsymbol{H}_\Omega - z_0\mathbf{1})(z\mathbf{1} - \boldsymbol{H}_\Omega)^{-1}\,dz = \frac{1}{2\pi i}\oint_{\partial\tilde{\Omega}} (z - z_0)(z\mathbf{1} - \boldsymbol{H}_\Omega)^{-1}\,dz,$$

to see that

$$\|\boldsymbol{H}_\Omega - z_0\mathbf{1}\|_{\mathcal{H}_\Omega} \le C|\partial\tilde{\Omega}||z - z_0|\|(z\mathbf{1} - \boldsymbol{H}_\Omega)^{-1}\|_{\mathcal{H}_\Omega} \le C\frac{(r - l + w)^2}{c(\hbar)}.$$

With $w(\hbar) = \hbar^{-(5n^\sharp+1)/2}c(\hbar)$ we get, for any $\boldsymbol{u} \in \Pi_\Omega\mathcal{H}$,

$$\|(\boldsymbol{H}_\theta - z_0\mathbf{1})\boldsymbol{u}\| \le C\biggl(\frac{(r(\hbar) - l(\hbar))^2}{c(\hbar)} + \hbar^{-(5n^\sharp+1)}c(\hbar)\biggr)\|\boldsymbol{u}\|,$$

which under the assumption $r(\hbar) - l(\hbar) \le C\hbar^{-n^\sharp}w(\hbar)$ translates into

$$\|(\boldsymbol{H}_\theta - z_0\mathbf{1})\boldsymbol{u}\| \le C\hbar^{-(7n^\sharp+1)}c(\hbar)\|\boldsymbol{u}\| \quad \text{for any } \boldsymbol{u} \in \Pi_\Omega\mathcal{H}. \ \blacksquare$$

Next we address the degree of linear independence of resonant states associated to resonances too close to each other. As a direct consequence of Lemma 12.3 we establish a bound in the next lemma which states that the spectral projector Π_Ω related to appropriately selected clusters of resonance of $\boldsymbol{H}(\hbar)$ contained in "wide" boxes are polynomially bounded provided the Π_Ω is restricted to generalized eigenfunctions corresponding to eigenvalues in the "wide" box. Its scalar valued analogue is given in Stefanov [62, Proposition 3.4]. This bound is the crucial ingredient which ensures that the assumptions in Proposition 9.1(1) hold.

Lemma 12.4. *Under the assumptions above, and the choice for $w(\hbar)$ in* (12.2), *there exists a constant $C > 0$ such that*

$$\|\Pi_\Omega\|_{\mathcal{B}(\mathcal{H}_\Omega)} \le C\hbar^{-(7n^\sharp+1)/2}.$$

Proof. As in the proof of Lemma 12.3 we have

$$\|(\boldsymbol{H}_\Omega - z\mathbf{1})^{-1}\|_{\mathcal{B}(\mathcal{H}_\Omega)} \le \frac{C}{c(\hbar)} \quad \text{on } \partial\tilde{\Omega}.$$

Since there are no eigenvalues of $\boldsymbol{H}_\Omega$ in $\tilde{\Omega} \setminus \Omega$ we may write

$$\Pi_\Omega|_{\mathcal{H}_\Omega} = \frac{1}{2\pi i}\oint_{\partial\tilde{\Omega}} (z\mathbf{1} - \boldsymbol{H}_\Omega)^{-1}\,dz$$

so that

$$\|\Pi_\Omega\| \le C\frac{|\partial\tilde{\Omega}|}{c(\hbar)} \le C\frac{r(\hbar) - l(\hbar) + w(\hbar)}{c(\hbar)} \le C\frac{\hbar^{-n^\sharp}w(\hbar)}{c(\hbar)} = C\hbar^{-(7n^\sharp+1)/2}. \ \blacksquare$$

By imitating the proof of Stefanov [62, Theorem 3.1] we extract the following result, which measures how well generalized cutoff eigenfunctions approximate the equation $(\boldsymbol{H}(\hbar) - z_0(\hbar)\mathbf{1})\boldsymbol{v} = 0$ and how close these are to $\boldsymbol{u}$.

Lemma 12.5. *Fix $z_0(\hbar) \in [l(\hbar), r(\hbar)]$ and let $\chi_{\widetilde{R}} \in C_0^\infty(\mathbb{R}^n)$ be such that $1_{B(0,\widetilde{R}+3\delta/4)} \prec \chi_{\widetilde{R}} \prec 1_{B(0,\widetilde{R}+\delta)}$. Then, for any $\boldsymbol{u}(\hbar) \in \Pi_\Omega\mathcal{H}$ with $\|\boldsymbol{u}\| = 1$, one has*

$$\|(\boldsymbol{H}(\hbar) - z_0(\hbar)\mathbf{1})\boldsymbol{\chi}_{\widetilde{R}}\boldsymbol{u}\|_{\mathcal{H}} + \|\boldsymbol{u}(\hbar) - \boldsymbol{\chi}_{\widetilde{R}}\boldsymbol{u}(\hbar)\|_{\mathcal{H}} \le C\hbar^{-(7n^\sharp+1)/2}\sqrt{c(\hbar)}.$$

Proof. Lemmas 12.1 and 12.3 combined imply

$$\begin{aligned}\int \theta(|\hbar\nabla\boldsymbol{u}|^2 + |\boldsymbol{u}|^2)\,dx &\le C(\hbar^{-(7n^\sharp+1)}c(\hbar) + e^{-\hbar^{-2/3+\varepsilon}})\|\boldsymbol{u}\|^2 \\ &\le C\hbar^{-(7n^\sharp+1)}c(\hbar)\|\boldsymbol{u}\| \end{aligned}\tag{12.6}$$

for $\hbar \ll 1$. Recall how $\theta = \theta_0$ is constant for $|x| \ge \widetilde{R} + \delta/2$. Set $\boldsymbol{v} = \boldsymbol{\chi}_{\widetilde{R}}\boldsymbol{u}$. Since $\operatorname{supp}(\boldsymbol{u}-\boldsymbol{v}) \subset \{|x| \ge \widetilde{R} + 3\delta/4\}$ we get from (12.6) the estimate

$$\|\boldsymbol{u}-\boldsymbol{v}\|_{\mathcal{H}} \le C(\hbar^{-(7n^\sharp+1)}\sqrt{c(\hbar)} + e^{-\hbar^{-2/3+\varepsilon}}) \le C\hbar^{-(7n^\sharp+1)/2}\sqrt{c(\hbar)}$$

provided $\hbar \ll 1$. Next

$$\|[\boldsymbol{H}_\theta, \boldsymbol{\chi}_{\widetilde{R}}]\boldsymbol{u}\|_{\mathcal{H}} \le C\Big(\int_{\widetilde{R}+3\delta/4\le|r|\le\widetilde{R}+\delta} (|\hbar\nabla\boldsymbol{u}|^2 + |\boldsymbol{u}|^2)\,dx\Big)^{1/2} \le C\hbar^{-(7n^\sharp+1)}\theta_0^{-1/2}\sqrt{c(\hbar)}.$$

Since $\|(\boldsymbol{H}_\theta - z_0\mathbf{1})\boldsymbol{u}\|_{\mathcal{H}} \le C\hbar^{-(7n^\sharp+1)}\sqrt{c(\hbar)}$ and

$$(\boldsymbol{H}_\theta - z_0\mathbf{1})\boldsymbol{v} = [\boldsymbol{H}_\theta, \boldsymbol{\chi}_{\widetilde{R}}]\boldsymbol{u} + \boldsymbol{\chi}_{\widetilde{R}}(\boldsymbol{H}_\theta - z_0\mathbf{1})\boldsymbol{u}$$

we therefore have

$$\|(\boldsymbol{H}_\theta - z_0\mathbf{1})\boldsymbol{v}\|_{\mathcal{H}} \le C\hbar^{-(7n^\sharp+1)/2}\sqrt{c(\hbar)}.\tag{12.7}$$

Thus, in order to estimate $\|(\boldsymbol{H} - z_0\mathbf{1})\boldsymbol{v}\|$ it suffices to estimate $\|(\boldsymbol{H}_\theta - \boldsymbol{H})\boldsymbol{v}\|_{\mathcal{H}}$. Using the explicit representation (12.3) we get

$$\begin{aligned}\|(\boldsymbol{H}_\theta - \boldsymbol{H})\boldsymbol{v}\|_{\widetilde{\mathcal{H}}} \le C\bigg(&(\|(\theta+\theta')\hbar^2\partial_r^2\boldsymbol{v}\|_{\widetilde{\mathcal{H}}} + \|(\theta+\theta'+|\theta''|)\hbar\partial_r\boldsymbol{v}\|_{\widetilde{\mathcal{H}}} \\ &+ \Big\|\theta\frac{\hbar^2\nabla_\omega}{r^2}\boldsymbol{v}\Big\|_{\widetilde{\mathcal{H}}} + \|\theta\boldsymbol{v}\|_{\widetilde{\mathcal{H}}}\bigg) \\ =: \mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3 + \mathrm{I}_4.&\end{aligned}$$

(I_1) By the product rule for differentiation and the fact that $\theta', \theta'' \le C\theta^{1/2}$,

$$\begin{aligned}\|\theta\hbar^2\partial_r^2\boldsymbol{v}\|_{\widetilde{\mathcal{H}}} &\le C(\|\hbar^2\partial_r^2(\theta\boldsymbol{v})\|_{\widetilde{\mathcal{H}}} + \hbar\|\theta'\hbar\partial_r\boldsymbol{v}\|_{\widetilde{\mathcal{H}}} + \hbar^2\|\theta''\boldsymbol{v}\|_{\widetilde{\mathcal{H}}}) \\ &\le C(\|\hbar^2\partial_r^2(\theta\boldsymbol{v})\|_{\widetilde{\mathcal{H}}} + \hbar\|\theta^{1/2}\hbar\partial_r\boldsymbol{v}\|_{\widetilde{\mathcal{H}}} + \hbar^2\|\theta^{1/2}\boldsymbol{v}\|_{\widetilde{\mathcal{H}}}) \\ &\le C(\|\hbar^2\partial_r^2(\theta\boldsymbol{v})\|_{\widetilde{\mathcal{H}}} + \hbar^{-(7^\sharp+1)/2}\sqrt{c(\hbar)}),\end{aligned}$$

where the last inequality follows from (12.6). We treat $\|\theta'\hbar^2\partial_r^2\boldsymbol{v}\|_{\widetilde{\mathcal{H}}}$ similarly since $\theta^{(3)} \le C\theta^{1/2}$. It remains to treat the compactly supported $\|\hbar^2\partial_r^2(\theta\boldsymbol{v})\|_{\widetilde{\mathcal{H}}}$ (since $\|\hbar^2\partial_r^2(\theta'\boldsymbol{v})\|_{\widetilde{\mathcal{H}}}$ can be done likewise). The standard semiclassical elliptic estimate in Theorem 2.1 yields

$$\begin{aligned}\|\hbar^2\partial_r^2(\theta\boldsymbol{v})\|_{\widetilde{\mathcal{H}}} &\le C(\|(\boldsymbol{H}_\theta - z_0\mathbf{1})(\theta\boldsymbol{v})\|_{\widetilde{\mathcal{H}}} + \|\theta\boldsymbol{v}\|_{\widetilde{\mathcal{H}}}) \\ &\le C(\|[\boldsymbol{H}_\theta, \theta]\boldsymbol{v}\|_{\widetilde{\mathcal{H}}} + C\hbar^{-(7n^\sharp+1)/2}\sqrt{c(\hbar)}) \\ &\le C(\hbar^2\|\theta''\boldsymbol{v}\|_{\widetilde{\mathcal{H}}} + \hbar\|\theta'\partial_r\boldsymbol{v}\|_{\widetilde{\mathcal{H}}} + C\hbar^{-(7n^\sharp+1)/2}\sqrt{c(\hbar)}) \\ &\le C(\hbar^2\|\theta^{1/2}\|_{\widetilde{\mathcal{H}}} + \hbar\|\theta^{1/2}\partial_r\boldsymbol{v}\|_{\widetilde{\mathcal{H}}} + C\hbar^{-(7^\sharp+1)/2}\sqrt{c(\hbar)}) \\ &\le C\hbar^{-(7n^\sharp+1)/2}\sqrt{c(\hbar)}.\end{aligned}$$

(I_2) Since this term only involves a first order derivative it follows immediately from (12.6) and the estimates $\theta, \theta', \theta'' \leq C\theta^{1/2}$ that

$$\mathrm{I}_2 \leq C\hbar^{-(7n^\sharp+1)/2}\sqrt{c(\hbar)}.$$

(I_3) It can be seen in much the same way as for I_1 that

$$\mathrm{I}_3 \leq C\hbar^{-(7n^\sharp+1)/2}\sqrt{c(\hbar)}.$$

(I_4) $\mathrm{I}_4 \leq C\hbar^{-(7n^\sharp+1)/2}\sqrt{c(\hbar)}$ follows from (12.6) and $\theta \leq C\theta^{1/2}$.

From $\|(\widetilde{\boldsymbol{H}}_\theta - \boldsymbol{H})\boldsymbol{v}\|_{\widetilde{\mathcal{H}}} \leq C\hbar^{-(7n^\sharp+1)/2}\sqrt{c(\hbar)}$ and (12.7) it now follows that

$$\|(\boldsymbol{H}_\theta - z_0\mathbf{1})\boldsymbol{v}\| \leq C\hbar^{-(7n^\sharp+1)/2}\sqrt{c(\hbar)}. \quad \blacksquare$$

Remark 12.6. In view of Propositions 8.1 and 8.2 one can go through the corresponding arguments above for $\boldsymbol{H}(\hbar)$ replaced by $\boldsymbol{J}(\hbar)$ (denoting either $\boldsymbol{J}_\infty(\hbar)$ or $\boldsymbol{J}_R(\hbar)$) and conclude that Lemmas 12.3 and 12.4 remain true also for $\boldsymbol{J}(\hbar)$. Lemma 12.5 remains valid as well. To see this, replace the spectral projection by the corresponding one for $\boldsymbol{J}$, i.e. put

$$\Pi_{\Omega_j(\hbar)} = \frac{1}{2\pi i}\oint_{\partial\Omega_j(\hbar)} (\boldsymbol{J}(\hbar) - z\mathbf{1})^{-1}\,dz, \tag{12.8}$$

where the Ω_j have the same properties as before (again, this can be done in view of Propositions 8.1 and 8.4). Note that the rank of $\Pi_{\Omega_j(\hbar)}$ equals the multiplicity of each eigenvalue. Let $1_{B(0,R_2)} \prec \chi \leq 1$ and $\boldsymbol{u} \in \Pi_{\Omega(\hbar)}\mathcal{H}$ with $\|\boldsymbol{u}\| = 1$. Arguing as in Chapter 10, using

$$\|((\operatorname{Re} W)^{1/2} \otimes I)\boldsymbol{u}\|^2 = -\operatorname{Im}((\boldsymbol{J}(\hbar) - l_j\mathbf{1})\boldsymbol{u}, \boldsymbol{u})$$

and

$$(\boldsymbol{H}(\hbar) - l_j\mathbf{1})\boldsymbol{\chi}\boldsymbol{u} = [\boldsymbol{H}_0, \chi]\boldsymbol{u} + i\boldsymbol{\chi}\boldsymbol{W}\boldsymbol{u} + \mathcal{O}(\hbar^{-(7n^\sharp+1)}c(\hbar))$$

we see that

$$\|(\boldsymbol{H}(\hbar) - l_j\mathbf{1})\boldsymbol{\chi}\boldsymbol{u}\| + \|\boldsymbol{u} - \boldsymbol{\chi}\boldsymbol{u}\| \leq C\hbar^{-(7n^\sharp+1)/2}\sqrt{c(\hbar)}. \tag{12.9}$$

Proofs of Theorems 5.6 and 5.7. With these preparations we are ready to give the proofs of Theorems 5.6 and 5.7.

Proof of Theorem 5.6. 1. First we prove the estimate $\operatorname{Count}(\boldsymbol{H}, \Omega(\hbar)) \leq \operatorname{Count}(\boldsymbol{J}, \Omega_+(\hbar))$. We adopt the strategy in [62, Theorem 3.2]. Choose $\widetilde{R}$ and δ such that $R_0' < \widetilde{R} < R_1$ and $0 < 2\delta < R_1 - R_0'$. We let $\boldsymbol{\chi}_{\widetilde{R}}\boldsymbol{u}_j(\hbar)$. For every $j = 1, \ldots, J(h)$, let $\{\boldsymbol{u}_{jk}\}_{k=1}^{\operatorname{Count}(\boldsymbol{H},\Omega_j)}$ be an orthonormal basis of $\Pi_{\Omega_j}\mathcal{H}$. The functions $\boldsymbol{u}_{jk}$ are linearly independent according to nonselfadjoint spectral theory [17]. More importantly, in view of Lemma 12.4 the linear independence is inherited under small perturbations. Furthermore, Lemma 12.5 shows that $\boldsymbol{\chi}_{\widetilde{R}}\boldsymbol{u}_{jk}(\hbar)$ are also quasimodes for $\boldsymbol{J}(\hbar)$ because $\boldsymbol{H}_\theta(\hbar)\boldsymbol{\chi}_{\widetilde{R}} = \boldsymbol{J}(\hbar)\boldsymbol{\chi}_{\widetilde{R}}$. To verify the assumptions of Proposition 9.1, let $\{\tilde{\boldsymbol{u}}_{jk}\}_k$ be another set of functions such that $\|\tilde{\boldsymbol{u}}_{jk} - \boldsymbol{\chi}_{\widetilde{R}}\boldsymbol{u}_{jk}\| \leq C\hbar^K$. Suppose that, for some fixed j, $\{\tilde{\boldsymbol{u}}_{jk}\}_{k=1}^{\operatorname{Count}(\boldsymbol{H}(\hbar),\Omega_j(\hbar))}$ are linearly dependent. Then for some choice of scalars c_{jk}, not all zero,

$$\sum_{k=1}^{\mathrm{Count}(\boldsymbol{H},\Omega_j)} c_{jk}\tilde{\boldsymbol{u}}_{jk} = 0.$$

We may, by dividing through by $\max_k |c_{jk}|$, assume $\max_k |c_{jk}| = 1$. Using Lemma 12.5 and the assumption on $\tilde{\boldsymbol{u}}_{jk}$ above we get

$$\begin{aligned}
\Big\| \sum_{k=1}^{\mathrm{Count}(\boldsymbol{H},\Omega_j)} c_{jk}\boldsymbol{u}_{jk}\Big\| &= \Big\| \sum_{k=1}^{\mathrm{Count}(\boldsymbol{H},\Omega_j)} c_{jk}(\boldsymbol{u}_{jk} - \chi_{\widetilde{R}}\boldsymbol{u}_{jk} + \chi_{\widetilde{R}}\boldsymbol{u}_{jk} - \tilde{\boldsymbol{u}}_{jk} + \tilde{\boldsymbol{u}}_{jk})\Big\| \\
&\le \Big\| \sum_{k=1}^{\mathrm{Count}(\boldsymbol{H},\Omega_j)} c_{jk}(\boldsymbol{u}_{jk} - \chi_{\widetilde{R}}\boldsymbol{u}_{jk})\Big\| + \Big\| \sum_{k=1}^{\mathrm{Count}(\boldsymbol{H},\Omega_j)} c_{jk}(\chi_{\widetilde{R}}\boldsymbol{u}_{jk} - \tilde{\boldsymbol{u}}_{jk})\Big\| \\
&\quad + \Big\| \sum_{k=1}^{\mathrm{Count}(\boldsymbol{H},\Omega_j)} c_{jk}\tilde{\boldsymbol{u}}_{jk}\Big\| \\
&= \mathcal{O}(\hbar^{-n^\sharp})\hbar^{-(7n^\sharp+1)/2}\sqrt{c(\hbar)} + \mathcal{O}(\hbar^{-n^\sharp})\hbar^K + 0 = \mathcal{O}(\hbar^{-n^\sharp})(\hbar^{-(7n^\sharp+1)/2}\hbar^{M/2} + \hbar^K).
\end{aligned}$$

Let j_0 be the index for which $|c_{j_0k_0}| = 1$ for some k_0. By applying $\Pi_{\Omega_{j_0}(\hbar)}$ to $\sum c_{jk}u_{jk}$ we see that

$$\begin{aligned}
\Big\| \sum_{k=1}^{\mathrm{Count}(\boldsymbol{H},\Omega_{j_0})} c_{j_0k}\boldsymbol{u}_{j_0k}\Big\| &= \Big\|\Pi_{\Omega_{j_0}(\hbar)} \sum_{k=1}^{\mathrm{Count}(\boldsymbol{H},\Omega_j)} c_{jk}\boldsymbol{u}_{jk}\Big\| \\
&\le \|\Pi_{\Omega_{j_0}(\hbar)}\|_{\mathcal{H}_\Omega} \cdot \Big\| \sum_{k=1}^{\mathrm{Count}(\boldsymbol{H},\Omega_j)} c_{jk}\boldsymbol{u}_{jk}\Big\| \\
&\le C\hbar^{-(7n^\sharp+1)/2}\mathcal{O}(\hbar^{-n^\sharp})(\hbar^K + \hbar^{-(7n^\sharp+1)/2}\hbar^{M/2}) \\
&= \mathcal{O}(\hbar^{-(9n^\sharp+1)/2})(\hbar^K + h^{-(7n^\sharp+1)/2}\hbar^{M/2}),
\end{aligned} \tag{12.10}$$

where we have used Lemma 12.4. However our choice of j_0 and k_0 above implies, since $\{\boldsymbol{u}_{jk}\}_k$ constitute an orthonormal set for each fixed j,

$$1 = |c_{j_0k_0}| \le \Big(\sum_{k=1}^{\mathrm{Count}(\boldsymbol{H},\Omega_{j_0})} |c_{j_0k}|^2\Big)^{1/2} = \Big\| \sum_{k=1}^{\mathrm{Count}(\boldsymbol{H},\Omega_{j_0})} c_{j_0k}\boldsymbol{u}_{j_0k}\Big\|.$$

This together with (12.10) provides a contradiction if

$$K > (9n^\sharp + 1)/2 \quad \text{and} \quad M/2 > 8n^\sharp + 1.$$

Thus, for all $1 \le j \le J(\hbar)$, the $\{\tilde{\boldsymbol{u}}_{jk}\}_{k=1}^{\mathrm{Count}(\boldsymbol{H}(\hbar),\Omega_j(\hbar))}$ as above are linearly independent. An application of Proposition 9.1(2) yields the conclusion.

2. For this proof we treat $\Omega = [l,r] + i[-c,0]$ as a large box Ω_+ as above. If the smaller box is designated by $\Omega_- = [\tilde{l},\tilde{r}] + i[-\tilde{c},0]$ we thus have

$$l = \tilde{l} - \hbar^{-N}\sqrt{\tilde{c}}, \quad r = \tilde{r} + \hbar^{-N}\sqrt{\tilde{c}}, \quad c = \hbar^{-N}\sqrt{\tilde{c}},$$

or, in other words, $\Omega_- = [l+c, r-c] + i[-\hbar^{2N}c^2, 0]$ as in the theorem. By Remark 12.6 cutoff eigenfunctions of $\boldsymbol{J}$ are quasimodes of $\boldsymbol{H}$ with the same type of residuals as above. It follows as in part 1 that $\mathrm{Count}(\boldsymbol{J},\Omega_-) \le \mathrm{Count}(\boldsymbol{H},\Omega)$. ■

Proof of Theorem 5.7. We divide the proof into two steps as in the proof of Theorem 5.6, establishing the two estimates. This time around, however, since $R_1 \leq R_0'$ it is necessary to argue as in the proof of Theorem 5.5. Specifically, using Proposition 11.1 we propagate the estimate in Lemma 12.5 to the whole of $\mathbb{R}^n \setminus B(0, R_0)$ to obtain

$$\|(\boldsymbol{H} - l_j \mathbf{1})\boldsymbol{\chi} \boldsymbol{u}_j)\| + \|\boldsymbol{u}_j - \boldsymbol{\chi}\boldsymbol{u}_j\| \leq C\hbar^{-(7n^\sharp+3)/2}\sqrt{c(\hbar)} + \mathcal{O}(\hbar^\infty).$$

Using this we argue as in the proof of Theorem 5.6. ∎

A. Auxiliary results

A.1. Meromorphic extension. We prove Proposition 4.4; a similar reasoning can be found in [56, Theorem 2.2].

Proof of Proposition 4.4. From the second resolvent identity it follows that, for $z \in \rho(\boldsymbol{H})$,

$$\mathbf{1}_{B(R_0)}\boldsymbol{R}(z) = \mathbf{1}_{B(R_0)}\boldsymbol{R}(i) + \mathbf{1}_{B(R_0)}\boldsymbol{R}(z)(i\boldsymbol{I} - z\boldsymbol{I})\boldsymbol{R}(i).$$

In view of the hypothesis in (4.1), $\mathbf{1}_{B(R_0)}\boldsymbol{R}(i)$ is compact and therefore $\mathbf{1}_{B(R_0)}\boldsymbol{R}(z)$ is compact as an operator on $\mathcal{H}$. Moreover, $\mathbf{1}_{B(R_0)}\boldsymbol{R}(z)$ is compact if and only if its adjoint $\boldsymbol{R}(z)\mathbf{1}_{B(R_0)}$ is compact. As a consequence of the Rellich–Kondrashov embedding theorem [1, Theorem 6.2] the operator $\mathbf{1}_{B(R_0,R)} : H^2(\mathbb{R}^n \setminus B(0,R_0)) \otimes \mathbb{C}^2 \to L^2(\mathbb{R}^n \setminus B(0,R_0)) \otimes \mathbb{C}^2$, with $B(0,R_0,R) = \{x : R_0 < |x| < R\}$, is compact for any $R \in (R_0, \infty)$. It follows that for any such R the operators $\mathbf{1}_{B(0,R)}\boldsymbol{R}(z)$ and $\boldsymbol{R}(z)\mathbf{1}_{B(R)}$ are compact on $\mathcal{H}$ for all $z \in \rho(\boldsymbol{H})$ and thus so are the cutoff resolvents $\boldsymbol{\chi}\boldsymbol{R}(z) = \boldsymbol{\chi}\mathbf{1}_{B(0,R)}\boldsymbol{R}(z)$ and $\boldsymbol{R}(z)\boldsymbol{\chi} = \boldsymbol{R}(z)\mathbf{1}_{B(R)}\boldsymbol{\chi}$ provided R is such that $\operatorname{supp}\chi \subset B(0,R)$. Let $\chi_0, \chi_1, \chi_2 \in C_0^\infty(\mathbb{R}^n)$ with $1_{B(0,R_0')} \prec \chi_0 \prec \chi_1 \prec \chi_2$. For $z, z_0 \in \mathbb{C}_+$, define $\boldsymbol{Q}_1, \boldsymbol{Q}_2 : \mathcal{H} \to \mathcal{D}$ by

$$\boldsymbol{Q}_1(z) = (\mathbf{1} - \boldsymbol{\chi}_0)\boldsymbol{R}_0(z)(\mathbf{1} - \boldsymbol{\chi}_1), \quad \boldsymbol{Q}_2(z_0) = \boldsymbol{\chi}_2\boldsymbol{R}(z_0)\boldsymbol{\chi}_1.$$

By straightforward computations

$$\begin{aligned}
(\boldsymbol{H} - zI)\boldsymbol{Q}_1 &= (\mathbf{1} - \boldsymbol{\chi}_1) + \hbar^2[\Delta \otimes I, \boldsymbol{\chi}_0]\boldsymbol{R}_0(z)(\mathbf{1} - \boldsymbol{\chi}_1) \\
&=: (\mathbf{1} - \boldsymbol{\chi}_1) + \boldsymbol{K}_1(z), \\
(\boldsymbol{H} - zI)\boldsymbol{Q}_2 &= \boldsymbol{\chi}_1 + \boldsymbol{\chi}_2(z_0 I - zI)\boldsymbol{R}(z_0)\boldsymbol{\chi}_1 + [\boldsymbol{H}_0, \boldsymbol{\chi}_2]\boldsymbol{R}(z_0)\boldsymbol{\chi}_1 \\
&=: \boldsymbol{\chi}_1 + \boldsymbol{K}_2(z, z_0).
\end{aligned}$$

It is well-known [34, Section 1.6] that $\boldsymbol{R}_0(z)$ extends analytically to covering surfaces as in the statement of the proposition, so for fixed $z_0 \in i\mathbb{R}^+$ the operators $\boldsymbol{K}_j(z)$ are analytic on the relevant surface. Since $[\Delta \otimes I, \boldsymbol{\chi}_j] = (2(\nabla\chi_j)\cdot\nabla + (\Delta\chi_j)) \otimes I$ are first order differential operators with C_0^∞ coefficients they take bounded sequences $(u_j) \subset H^2(\mathbb{R}^n \setminus B(0,R_0)) \otimes \mathbb{C}^2$ to sequences bounded in $H^1(\mathbb{R}^n \setminus B(0,R_0)) \otimes \mathbb{C}^2$ and any such sequence has a convergent subsequence in $L^2(\mathbb{R}^n \setminus B(0,R_0)) \otimes \mathbb{C}^2$ by the Rellich–Kondrashov embedding theorem. Therefore

$$[\Delta \otimes I, \boldsymbol{\chi}_j] : H^2(\mathbb{R}^n \setminus B(0,R_0)) \otimes \mathbb{C}^2 \to L^2(\mathbb{R}^n \setminus B(0,R_0)) \otimes \mathbb{C}^2, \quad j = 0, 2,$$

are compact operators. This implies that $\boldsymbol{K}_1(z)$ is compact. Moreover, from

$$[\boldsymbol{R}_0, \boldsymbol{\chi}_2]\boldsymbol{R}(z_0)\boldsymbol{\chi}_1 = [\boldsymbol{H}_0, \boldsymbol{\chi}_2](\mathbf{1} - \boldsymbol{\chi}_0)\boldsymbol{R}(z_0)\boldsymbol{\chi}_1$$

where $(\mathbf{1}-\boldsymbol{\chi}_0)\boldsymbol{R}(z_0) \in \mathcal{B}(\mathcal{H}, H^2(\mathbb{R}^n \setminus B(0,R_0')) \otimes \mathbb{C}^2)$ and the compactness of cutoff resolvents we obtain compactness of $\boldsymbol{K}_2(z,z_0)$ as well.

By selfadjointness, the spectral theorem and the definition of $\|\cdot\|_{\mathcal{D}}$ (see Section 4.1) we have

$$\boldsymbol{R}(z_0) = \begin{cases} \mathcal{O}(|z_0|^{-1}) : \mathcal{H} \to \mathcal{H} \\ \mathcal{O}(1) : \mathcal{H} \to \mathcal{D} \end{cases}$$

so that

$$(\mathbf{1}-\boldsymbol{\chi}_0)\boldsymbol{R}(z_0) = \begin{cases} \mathcal{O}(|z_0|^{-1}) : \mathcal{H} \to L^2(\mathbb{R}^n \setminus B(0,R_0)) \otimes \mathbb{C}^2 \\ \mathcal{O}(1) : \mathcal{H} \to H^2(\mathbb{R}^n \setminus B(0,R_0)) \otimes \mathbb{C}^2. \end{cases}$$

By standard interpolation, we obtain the bound

$$(\mathbf{1}-\boldsymbol{\chi}_0)\boldsymbol{R}(z_0) = \mathcal{O}(|z_0|^{-1/2}) : \mathcal{H} \to H^1(\mathbb{R}^n \setminus B(0,R_0)) \otimes \mathbb{C}^2.$$

With $\boldsymbol{K}(z,z_0) := \boldsymbol{K}_1(z) + \boldsymbol{K}_2(z,z_0)$ we see that $\boldsymbol{K}(z_0,z_0) = \mathcal{O}(|z_0|^{-1/2}) : \mathcal{H} \to \mathcal{H}$ so for $\operatorname{Im} z_0 = |z_0|$ sufficiently large $(\mathbf{1}+\boldsymbol{K}(z_0,z_0))^{-1}$ exists in $\mathcal{B}(\mathcal{H})$. With such z_0 we can apply analytic Fredholm theory with respect to z to see that $(\mathbf{1}+\boldsymbol{K}(z,z_0))^{-1}$ exists in the complement of a discrete set on surfaces as in the proposition. Since

$$\boldsymbol{R}(z) = (\boldsymbol{Q}_1(z) + \boldsymbol{Q}_2(z_0))(\mathbf{1}+\boldsymbol{K}(z,z_0))^{-1} \tag{A.1}$$

this inverse only fails to exist when $\boldsymbol{R}(z)$ does, meaning $\operatorname{spec}(\boldsymbol{H}) \cap \mathbb{R}_- = \operatorname{spec}_{\mathrm{d}}(\boldsymbol{H}) \cap \mathbb{R}_-$. This proves the first part of the proposition.

From (A.1) we see that in order to extend $\boldsymbol{R}(z) : \mathcal{H}_{\mathrm{comp}} \to \mathcal{D}_{\mathrm{loc}}$ meromorphically we must extend $(\mathbf{1}+\boldsymbol{K})^{-1} : \mathcal{H}_{\mathrm{comp}} \to \mathcal{H}_{\mathrm{comp}}$. Unfortunately Fredholm theory does not apply since $\mathcal{H}_{\mathrm{comp}}$ is not even a Banach space, so we resort to a little trick as follows: Pick $\chi \succ \chi_2$ so that $\boldsymbol{\chi}\boldsymbol{K} = \boldsymbol{K}$, i.e. $(\mathbf{1}-\boldsymbol{\chi})\boldsymbol{K} = 0$. From this we obtain the identities

$$\begin{aligned} (\mathbf{1}+\boldsymbol{K}(\mathbf{1}-\boldsymbol{\chi}))^{-1} &= \mathbf{1}-\boldsymbol{K}(\mathbf{1}-\boldsymbol{\chi}), \\ \mathbf{1}+\boldsymbol{K} &= (\mathbf{1}+\boldsymbol{K}(\mathbf{1}-\boldsymbol{\chi}))(\mathbf{1}+\boldsymbol{K}\boldsymbol{\chi}), \end{aligned}$$

which together imply

$$(\mathbf{1}+\boldsymbol{K})^{-1} = (\mathbf{1}+\boldsymbol{K}\boldsymbol{\chi})^{-1}(\mathbf{1}-\boldsymbol{K}(\mathbf{1}-\boldsymbol{\chi})).$$

Thus it suffices to extend $(\mathbf{1}+\boldsymbol{K}\boldsymbol{\chi})^{-1} : \mathcal{H} \to \mathcal{H}$. Applying analytic Fredholm theory as before we arrive at the extension

$$\boldsymbol{R}(z) = (\boldsymbol{Q}_1(z) + \boldsymbol{Q}_2(z_0))(\mathbf{1}+\boldsymbol{K}(z,z_0)\boldsymbol{\chi})^{-1}(\mathbf{1}-\boldsymbol{K}(z,z_0)(\mathbf{1}-\boldsymbol{\chi})). \blacksquare$$

A.2. Properties of reference operator

Proposition A.1. *The operator $\boldsymbol{H}^\sharp(\hbar) : \mathcal{H}^\sharp \to \mathcal{H}^\sharp$ is selfadjoint and its spectrum is purely discrete.*

Proof. Let $\boldsymbol{u} \in \mathcal{D}((\boldsymbol{H}^\sharp)^*) \supset \mathcal{D}(\boldsymbol{H}^\sharp)$ and put $(\boldsymbol{H}^\sharp)^*\boldsymbol{u} = \boldsymbol{v}$. For all $\boldsymbol{\phi} \in C_0^\infty(\mathbb{T}^n \setminus \overline{B(0,R_0')}) \otimes \mathbb{C}^2$ we have

$$\langle \boldsymbol{Q}^\sharp \boldsymbol{u}, \boldsymbol{\phi} \rangle = \langle \boldsymbol{u}, \boldsymbol{Q}^\sharp \boldsymbol{\phi} \rangle = \langle \boldsymbol{u}, \boldsymbol{H}^\sharp \boldsymbol{\phi} \rangle = \langle (\boldsymbol{H}^\sharp)^* \boldsymbol{u}, \boldsymbol{\phi} \rangle$$

so that $\boldsymbol{Q}^\sharp \boldsymbol{u} = \boldsymbol{v}$ in $\mathbb{T}^n \setminus \overline{B(0,R_0')}$ in the distributional sense. Since $\boldsymbol{v}|_{\mathbb{T}^n\setminus\overline{B(0,R_0')}} \in L^2(\mathbb{T} \setminus \overline{B(0,R_0')}) \otimes \mathbb{C}^2$ it follows by standard elliptic estimates [2, (3.21)] that

$$\boldsymbol{u}|_{\mathbb{T}^n\setminus\overline{B(0,R_0')}} \in H^2_{\mathrm{loc}}(\mathbb{T}^n \setminus \overline{B(0,R_0')}) \otimes \mathbb{C}^2,$$

and by compactness we see that the exterior part of $\boldsymbol{u}$ belongs to $\mathcal{D}^\sharp$. For the interior part of $\boldsymbol{u}$ it suffices to consider $\boldsymbol{u}$ with $\boldsymbol{u} = \boldsymbol{\chi}\boldsymbol{u}$ (with $\boldsymbol{\chi}$ as in (4.5)) for which

$$\langle \boldsymbol{u}, \boldsymbol{H}\boldsymbol{\phi}\rangle = \langle \boldsymbol{u}, \boldsymbol{H}^\sharp\boldsymbol{\phi}\rangle = \langle \boldsymbol{v}, \boldsymbol{\phi}\rangle, \quad \boldsymbol{\phi} \in \mathcal{D} \text{ arbitrary},$$

since $\boldsymbol{u}$ and $\boldsymbol{v}$ have supports contained in $B(0,R)$. Since we may also view $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{H}$ the selfadjointness of $\boldsymbol{H}$ implies $\boldsymbol{u} \in \mathcal{D}$ and thus $\boldsymbol{u} \in \mathcal{D}^\sharp$. Now $\mathcal{D}((\boldsymbol{H}^\sharp)^*) \subset \mathcal{D}(\boldsymbol{H}^\sharp)$ in conjunction with the fact that $\boldsymbol{H}^\sharp$ is symmetric means that $\boldsymbol{H}^\sharp$ is selfadjoint.

Now, let $(\boldsymbol{v}_j) \subset \mathcal{H}^\sharp$ be any bounded sequence and put $\boldsymbol{u}_j = (\boldsymbol{H}^\sharp + i\mathbf{1})^{-1}\boldsymbol{v}_j \in \mathcal{D}^\sharp$. Then $((\mathbf{1}-\boldsymbol{\chi})\boldsymbol{u}_j) \subset H^2(\mathbb{T}^n)\otimes\mathbb{C}^2$ is also bounded and hence has a convergent subsequence in $\mathcal{H}^\sharp$. Moreover

$$(\boldsymbol{H} + i\mathbf{1})\boldsymbol{\chi}\boldsymbol{u}_j = [(-\hbar^2\Delta)\otimes I, \boldsymbol{\chi}]\boldsymbol{u}_j + \ \boldsymbol{\chi}\boldsymbol{v}_j =: \boldsymbol{w}_j$$

so that $(\boldsymbol{w}_j)$ is bounded. Thus, with $\widetilde{\chi} \succ \chi$, it follows from

$$\boldsymbol{\chi}\boldsymbol{u}_j = (\boldsymbol{H} + i\mathbf{1})^{-1}\widetilde{\boldsymbol{\chi}}\boldsymbol{w}_j$$

that also $(\boldsymbol{\chi}\boldsymbol{u}_j)$ has a convergent subsequence since $(\boldsymbol{H} + i\mathbf{1})^{-1}\widetilde{\boldsymbol{\chi}}$ is compact (see the proof of Proposition 4.4). Thus $(\boldsymbol{u}_j)$ has a convergent subsequence, i.e. $(\boldsymbol{H}^\sharp + i\mathbf{1})^{-1}$ is compact so that $\operatorname{spec}(\boldsymbol{H}^\sharp)$ is purely discrete. ∎

A similar result in the scalar valued setting goes back to [57].

B. Operator valued semiclassical maximum principle

To ensure that the paper is self-contained we include the proof of the following result.

Lemma B.1 (Operator valued maximum principle). *Let $0 < \hbar < 1$ and $0 < \ell_0 < \ell(\hbar) < r(\hbar) < r_0 < \infty$. Suppose $z \mapsto \boldsymbol{A}(z,\hbar) \in \mathcal{B}(\mathcal{H})$ is an analytic operator valued function defined in a neighbourhood of*

$$\Omega(\hbar) := [\ell(\hbar) - 2w(\hbar), r(\hbar) + 2w(\hbar)] + i[-\alpha(\hbar)D_-(\hbar), D_+(\hbar)],$$

where $0 < D_+(\hbar) \le D_-(\hbar)$, $1 \le \alpha(\hbar)$ and $D_-(\hbar)\alpha(\hbar)\log\alpha(\hbar) \le w(\hbar)$. If $\boldsymbol{A}(z,\hbar)$ satisfies

$$\begin{aligned} |\langle \boldsymbol{A}(z,\hbar)\boldsymbol{\phi}, \boldsymbol{\psi}\rangle| &\le e^{\alpha(\hbar)} && \text{on } \Omega(\hbar), \\ |\langle \boldsymbol{A}(z,\hbar)\boldsymbol{\phi}, \boldsymbol{\psi}\rangle| &\le M(\hbar) && \text{on } [\ell(\hbar) - 2w(\hbar), r(\hbar) + 2w(\hbar)] + iD_+(\hbar), \end{aligned} \tag{B.1}$$

for $\|\boldsymbol{\phi}\| = \|\boldsymbol{\psi}\| = 1$ and some $M(\hbar) \ge 1$, then there exists $h_1 = h_1(D_-, D_+, \alpha) > 0$ (independent of $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$) such that

$$\|\boldsymbol{A}(z,\hbar)\|_{\mathcal{B}(\mathcal{H})} \le e^3 M(\hbar) \quad \forall z \in \widetilde{\Omega} := [\ell(\hbar), r(\hbar)] + i[-D_-(\hbar), D_+(\hbar)]$$

for $\hbar \in (0, \hbar_1)$.

Proof. We suppress dependence on $\hbar$ in this proof in order to simplify notation. Define $G(z) = \langle \boldsymbol{A}(z,\hbar)\boldsymbol{\phi}, \boldsymbol{\psi}\rangle$ for normalized $\boldsymbol{\phi}, \boldsymbol{\psi}$. By hypothesis,

$$\begin{aligned} |G(z)| &\le e^{\alpha(\hbar)} && \text{on } \Omega(\hbar), \\ |G(z)| &\le M(\hbar) && \text{on } [\ell(\hbar) - 2w(\hbar), r(\hbar) + 2w(\hbar)] + iD_+(\hbar). \end{aligned} \tag{B.2}$$

If we establish the statement of the lemma for the scalar valued function $G(z)$, then an application of the scalar valued result yields the result for $\boldsymbol{A}(z,\hbar)$. Let

$$g(z) = \log|G(z)| - \log M \frac{\operatorname{Im} z + \alpha D_-}{\alpha D_- + D_+} - \alpha \frac{D_+ - \operatorname{Im} z}{\alpha D_- + D_+}. \tag{B.3}$$

Since $\log|G(z)| = \operatorname{Re}\log G(z)$ we see that $g(z)$ is a subharmonic function in a neighbourhood of Ω; more precisely, $\Delta g(z) = 2\pi \sum_{j=1}^N \delta(z - z_j)$ where δ stands for Dirac's delta and the z_j's are the zeros of $A(z)$ in Ω. So $\Delta g(z) \ge 0$, with equality if and only if $A(z)$ has no zeros in Ω.

By subharmonicity and (B.1) we have, for $z \in \Omega$,

$$g(z) \le \int_{\partial\Omega} \mathcal{P}(z,y) g(y)|_{\partial\Omega}\, dS_y \le \alpha \int_{\partial\Omega} \mathcal{P}(z,y)\, dS_y \tag{B.4}$$

where $\mathcal{P}(z,y)$ is the Poisson kernel for Ω. We will use this to show $g(z) \le 1$ for $\ell \le \operatorname{Re} z \le r$.

On the lower horizontal side $\{z \in \mathbb{C} : \operatorname{Im} z = -\alpha D_-\}$ we have

$$g(z) = \log|G(z)| - \alpha \le \alpha - \alpha = 0,$$

and similarly on the upper horizontal side we get

$$g(z) = \log|G(z)| - \log M \frac{D_+ + \alpha D_-}{D_+ + \alpha D_-} \le \log M - \log M = 0,$$

so $g \le 0$ on both horizontal sides. Hence we see from the integral representation (B.4) that we can restrict our attention to the case where y belongs to the vertical sides of Ω and $\ell \le \operatorname{Re} z \le r$. To do so we apply Lemma 8.2 in [55]. That lemma tells us that for "long domains", e.g. complex regions of the form $\Omega_R = [\ell, \ell + R] + i[r + \beta]$ where $R \gg 1$ and $0 < \beta < \pi$, we have $\mathcal{P}_{\Omega_R}(z,y) \to \operatorname{dist}(z, \partial\Omega_R) e^{-|z-y|}$ as $R \to \infty$ for $y \in \partial\Omega_R$ and $z \in \Omega_R$. To arrive at such a rectangle we make the change of variable $z = (\alpha D_- + D_+)\zeta$ and view f as defined on

$$\widehat{\Omega} = \left[\frac{\ell - w}{\alpha D_- + D_+}, \frac{r + w}{\alpha D_- + D_+}\right] + i\left[-\frac{\alpha D_-}{\alpha D_- + D_+}, \frac{D_+}{\alpha D_- + D_+}\right].$$

It is easy to see that this is a long domain as defined above in the semiclassical limit $\hbar \to 0$. Making the similar change of variable for $y = y(\eta)$ we obtain from (B.4) the estimate

$$g(z) = g(z(\zeta)) \le \alpha \int_{\partial\widehat{\Omega}} \mathcal{P}\big((\alpha D_- + D_+)\zeta, (\alpha D_- + D_+)\eta\big)(\alpha D_- + D_+)\, dS_\eta.$$

Here the integrand is exactly the Poisson kernel ([1]) for $\widehat{\Omega}$ so, by the above mentioned

([1]) Indeed, the argument is the same as for balls; if $\Delta u(z) = 0$ in Ω and $u(z) = v(z)$ on $\partial\Omega$, then for $\hat{u}(z) = u(kz)$, where k is a constant, we have $\Delta\hat{u}(z) = 0$ in $\widehat{\Omega}$ and $\hat{u}(z) = v(kz)$ on $\partial\widehat{\Omega}$ where $\widehat{\Omega} = k\Omega$. So if $\mathcal{P}(z,y)$ is the Poisson kernel for Ω then $u(z) = \int_{\partial\Omega} \mathcal{P}(z,y)v(y)\, dS_y$ whence $\hat{u}(z) = \int_{\partial\Omega} \mathcal{P}(kz,y)v(y)\, dD_y = \int_{\partial\widehat{\Omega}} \mathcal{P}(kz,ky)v(ky)k\, dS_y$. So with $\widehat{\mathcal{P}}(z,y) = k\mathcal{P}(kz,ky)$ we have $\hat{u}(z) = \int_{\partial\widehat{\Omega}} \widehat{\mathcal{P}}(z,y)\hat{u}(y)|_{\partial\widehat{\Omega}}\, dS_y$, which proves the claim.

result, for $\hbar \ll 1$ we have

$$g(z) \leq 2\alpha \operatorname{dist}(\zeta, \partial\widehat{\Omega}) e^{-|\zeta-\eta|} \leq \alpha e^{-|z-y|/(\alpha D_- + D_+)},$$

since the height of $\widehat{\Omega}$ is 1 and $\operatorname{dist}(\zeta, \partial\widehat{\Omega}) \leq 1/2$.

For $\ell \leq \operatorname{Re} z \leq r$ we have $|z-y| \geq 2w$ when y belongs to the vertical sides of Ω. Since $w \geq D_- \alpha \log \alpha$ and $\alpha D_- \geq D_- \geq D_+$ we get

$$e^{-|z-y|/(\alpha D_- + D_+)} \leq e^{-2w/(\alpha D_- + D_+)} \leq \alpha^{-2\alpha D_-/(\alpha D_- + D_+)} \leq \alpha^{-(\alpha D_- + D_+)/(\alpha D_- + D_+)}.$$

It follows that $g(z) \leq 1$. If, in addition, $-\operatorname{Im} z \leq D_-$ then for the last term in (B.3) we also have

$$\alpha \frac{D_+ - \operatorname{Im} z}{\alpha D_- + D_+} \leq \alpha \frac{D_+ + D_-}{\alpha D_- + D_+} \leq \frac{2\alpha D_-}{\alpha D_-} = 2.$$

Since for $\operatorname{Im} z \leq D_+$,

$$\log M \frac{\operatorname{Im} z + \alpha D_-}{\alpha D_- + D_+} \leq \log M,$$

it now follows from (B.3) that

$$\log |G(z)| \leq 1 + \log M + 2,$$

which is to say that $|G(z)| \leq M e^3$. ∎

For our purposes we prefer to state the above lemma in the following version (which essentially comes from the scalar valued formulation in [66]):

Corollary B.2. *Let $0 < \hbar < 1$ and $0 < \ell_0 < \ell(\hbar) < r(\hbar) < r_0 < \infty$. Suppose $\boldsymbol{A}(z,\hbar)$ is an analytic operator valued function defined in a neighbourhood of*

$$\Omega(\hbar) := [\ell(\hbar) - 2w(\hbar), r(\hbar) + 2w(\hbar)] + i\left[-A\hbar^{-n^\sharp} \log \frac{1}{D(\hbar)} D(\hbar), D(\hbar)\right] \tag{B.5}$$

where $e^{-B/\hbar} < D(\hbar) < 1/2$, $B > 0$ and $2An^\sharp \hbar^{-n^\sharp} \log \frac{1}{\hbar} \log \frac{1}{D(\hbar)} D(\hbar) \leq w(\hbar)$. If, for normalized $\boldsymbol{\phi}$, $\boldsymbol{\psi}$, the function $\boldsymbol{A}(z,\hbar)$ satisfies

$$\begin{aligned} |\langle \boldsymbol{A}(z,\hbar)\boldsymbol{\phi}, \boldsymbol{\psi}\rangle| &\leq e^{A\hbar^{-n^\sharp} \log(1/D(\hbar))} \quad \text{on } \Omega(\hbar), \\ |\langle \boldsymbol{A}(z,\hbar)\boldsymbol{\phi}, \boldsymbol{\psi}\rangle| &\leq \frac{1}{\operatorname{Im} z} \quad \text{on } \Omega(\hbar) \cap \{z \in \mathbb{C} : \operatorname{Im} z > 0\}, \end{aligned} \tag{B.6}$$

then there exists $\hbar_1 > 0$ (independent of $\boldsymbol{\phi}$, $\boldsymbol{\psi}$) such that for $\hbar \leq \hbar_1$,

$$\|\boldsymbol{A}(z,\hbar)\|_{\mathcal{B}(\mathcal{H})} \leq \frac{e^3}{D(\hbar)} \quad \text{for all } z \in \widetilde{\Omega} := [\ell(\hbar), r(\hbar)] + i[-D(\hbar), D(\hbar)].$$

Proof. Again we suppress $\hbar$-dependence. Let us apply Lemma B.1 with the choices $\alpha = A\hbar^{-n^\sharp} \log(1/D)$, $D_- = D_+ =: D$ and $M = 1/D$. The only requirement worth a comment is to show $w \geq D_- \alpha \log \alpha$, which clearly follows if

$$2An^\sharp \hbar^{-n^\sharp} \log \frac{1}{\hbar} \log \frac{1}{D(\hbar)} D(\hbar) \geq AD\hbar^{-n^\sharp} \log\left(\frac{1}{D}\right) \log\left(A\hbar^{-n^\sharp} \log\left(\frac{1}{D}\right)\right).$$

Since the latter inequality is equivalent to $1 \geq A\hbar^{n^\sharp} \log(1/D)$ which easily follows from the assumption $D > e^{-B/\hbar}$ provided $\hbar$ is sufficiently small, the corollary is proved. ∎

We also need the following lemma (see [59, Lemma 3]) and its spin-off.

LEMMA B.3. *Let $C_0^\infty(\mathbb{R}^n) \ni \chi$, $\widetilde{\chi} \succ 1_K$, where $K \Subset \mathbb{R}^n$. Let $\boldsymbol{R}_\chi(z,h) = \boldsymbol{\chi}\boldsymbol{R}(z,h)\widetilde{\boldsymbol{\chi}}$ have a pole $z_0(\hbar)$, i.e.,*

$$\boldsymbol{R}_\chi(z,h) = \boldsymbol{A}_0(z) + \sum_{j=1}^{N}(z-z_0)^{-j}\boldsymbol{A}_{-j}(h), \tag{B.7}$$

where $\boldsymbol{A}_0(z)$ is analytic near z_0. Let $\{\chi_j\}$ be cutoff functions such that $\chi_j \in C_0^\infty(\mathbb{R}^n)$, and

$$\mathbf{1}_K \prec \boldsymbol{\chi}_1 \prec \boldsymbol{\chi}_2 \prec \cdots \prec \boldsymbol{\chi}_{N-1} \prec \boldsymbol{\chi}.$$

Then Ran $\boldsymbol{A}_{-j} \subset$ Ran $\boldsymbol{A}_{-1}$ *for $j = 2,\ldots,N$, and*

$$\boldsymbol{A}_{-j}\boldsymbol{\chi}_1 = \boldsymbol{A}_{-1}(\boldsymbol{H}(h)-z_0\mathbf{1})\boldsymbol{\chi}_{j-1}(\boldsymbol{H}(h)-z_0\mathbf{1})\boldsymbol{\chi}_{j-2}\cdots\boldsymbol{\chi}_2(\boldsymbol{H}(h)-z_0\mathbf{1})\boldsymbol{\chi}_1.$$

Proof. We suppress dependence on $\hbar$. Multiply (B.7) by $\boldsymbol{H} - z\mathbf{1}$ from the right on both sides of the expansion. On the left we obtain

$$\begin{aligned}\boldsymbol{R}_\chi(\boldsymbol{H}-z\mathbf{1}) &= \boldsymbol{\chi}(\boldsymbol{H}-z\mathbf{1})^{-1}\widetilde{\boldsymbol{\chi}}\boldsymbol{H} - \boldsymbol{R}_\chi z = \boldsymbol{\chi}(\boldsymbol{H}-z\mathbf{1})^{-1}(\boldsymbol{H}\widetilde{\boldsymbol{\chi}} + [\widetilde{\boldsymbol{\chi}},\boldsymbol{H}]) - \boldsymbol{R}_\chi \\ &= \boldsymbol{\chi}\widetilde{\boldsymbol{\chi}} + \boldsymbol{\chi}\boldsymbol{R}_\chi(z)[\widetilde{\boldsymbol{\chi}},\boldsymbol{H}],\end{aligned}$$

while on the right-hand side we get (add and subtract z_0),

$$\begin{aligned}&\boldsymbol{A}_0(\boldsymbol{H}-z\mathbf{1}) + \sum\{(z-z_0)^j\boldsymbol{A}_{-j}(z-z_0) - (z-z_0)^{-j+1}\boldsymbol{A}_{-j}\} \\ &\qquad = \boldsymbol{A}_0(\boldsymbol{H}-z\mathbf{1}) - \boldsymbol{A}_{-1} + \sum(z-z_0)^{-j}(\boldsymbol{A}_{-j}(\boldsymbol{H}-z_0) - \boldsymbol{A}_{-(j+1)}).\end{aligned}$$

From multiplication by $\boldsymbol{\chi}_l$ on the right and $\boldsymbol{A}_{-(N+1)} = 0$ using the fact that the left-hand side has no singular terms, we deduce $\boldsymbol{A}_{-j}(\boldsymbol{H}-z_0)\boldsymbol{\chi}_l = \boldsymbol{A}_{-(j+1)}\boldsymbol{\chi}_l$. So, by induction,

$$\boldsymbol{A}_{-j}\boldsymbol{\chi}_1 = \boldsymbol{A}_{-1}(\boldsymbol{H}(h)-z_0)\boldsymbol{\chi}_{j-1}(\boldsymbol{H}(h)-z_0)\boldsymbol{\chi}_{j-2}\cdots\boldsymbol{\chi}_2(\boldsymbol{H}(h)-z_0)\boldsymbol{\chi}_1.$$

This proves the claim. ∎

COROLLARY B.4. *Let $\chi \prec \widetilde{\chi}$ be $C_0^\infty(\mathbb{R}^n)$ cutoff functions. Then the range of the singular part in the Laurent expansion of the cutoff resolvent $\boldsymbol{\chi}\boldsymbol{R}(z,\hbar)\boldsymbol{\chi}$ is the same as the range of the residue $\boldsymbol{A}_{-1}$ (cf. Lemma* B.3*).*

Proof. Since $\widetilde{\chi} \prec \chi$ we can choose $\chi_1 = \chi$ in Lemma B.3. Multiply by $\boldsymbol{\chi}$ from the right in (B.7) to see that

$$\boldsymbol{\chi}\boldsymbol{R}(z,\hbar)\boldsymbol{\chi} = \boldsymbol{A}_0(z,\hbar) + \sum_{k=1}^{N}(z-z_0(\hbar))^{-k}\boldsymbol{A}_{-1}(\hbar)\boldsymbol{Q}_k(\hbar),$$

where the $\boldsymbol{Q}_k(\hbar)$ are unbounded operators (however $\boldsymbol{A}_{-1}(\hbar)\boldsymbol{Q}_k(\hbar)$ are bounded operators). This shows that the range of the singular part of the cutoff resolvent is a subset of $\boldsymbol{A}_{-1}(\hbar)\mathcal{H}$. Since the reverse inclusion is obvious the corollary is proved. ∎

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