FUNDAMENTA
MATHEMATICAE
230 (2015)

Template iterations and maximal cofinitary groups

by

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Abstract. Jörg Brendle (2003) used Hechler's forcing notion for adding a maximal almost disjoint family along an appropriate template forcing construction to show that $\mathfrak{a}$ (the minimal size of a maximal almost disjoint family) can be of countable cofinality. The main result of the present paper is that $\mathfrak{a}_g$, the minimal size of a maximal cofinitary group, can be of countable cofinality. To prove this we define a natural poset for adding a maximal cofinitary group of a given cardinality, which enjoys certain combinatorial properties allowing it to be used within a similar template forcing construction. Additionally we find that $\mathfrak{a}_p$, the minimal size of a maximal family of almost disjoint permutations, and $\mathfrak{a}_e$, the minimal size of a maximal eventually different family, can be of countable cofinality.

1. Introduction. The subject of cardinal characteristics of the real line concerns various combinatorial properties of the reals and the possible cardinalities of sets of reals which are characterized by such properties. An excellent exposition can be found in [2].

The main focus of the present paper is on maximal cofinitary groups. A subgroup of $\mathrm{Sym}(\omega)$ is said to be *cofinitary* if all of its non-identity elements have only finitely many fixed points. A *maximal cofinitary group*, abbreviated mcg, is a cofinitary group which is not properly contained in any other cofinitary group. The symbol $\mathfrak{a}_g$ denotes the minimal cardinality of a maximal cofinitary group. Clearly, if $\mathcal{G}$ is a cofinitary group, then the graphs of its elements form an almost disjoint family in $\omega \times \omega$. Recall that the almost disjointness number $\mathfrak{a}$ is defined as the minimal cardinality of an infinite maximal almost disjoint family of infinite subsets of ω.

Even though cofinitary groups and almost disjoint families are so closely related, for every pair $\kappa < \lambda$ of regular uncountable cardinals, it is consistent that $\mathfrak{a} = \kappa < \mathfrak{a}_g = \lambda$. Indeed, fix $\kappa < \lambda$ and consider the model of $\mathfrak{a} = \mathfrak{b} = \kappa < \mathfrak{s} = \mathfrak{c} = \lambda$ from [5]. By [6, Theorem 2.4], $\mathrm{non}(\mathcal{M}) \leq \mathfrak{a}_g$, while $\mathfrak{s} \leq$

2010 *Mathematics Subject Classification*: Primary 03E17; Secondary 03E35.
Key words and phrases: cardinal characteristics, maximal cofinitary groups, template forcing iterations.

DOI: 10.4064/fm230-3-1

non$(\mathcal{M})$ (see [2, Theorem 5.19]), and so in the same model $\mathfrak{a} = \kappa < \mathfrak{a}_g = \lambda$. Thus $\mathfrak{a}$ and $\mathfrak{a}_g$ can be quite different. But is it consistent that $\mathfrak{a}_g < \mathfrak{a}$? Or is it a ZFC theorem that $\mathfrak{a} \leq \mathfrak{a}_g$? Both of these questions remain open.

Some of the longstanding open questions in the field regard the cofinalities of various combinatorial cardinal characteristics of the reals: for example it is not known if the splitting number can be singular. A major breakthrough in this area is the appearance of Shelah's template iteration technique (see [9]) ([1]).

The method provides in particular the consistency of $\mathfrak{a} = \mathfrak{a}_g$ being singular. However in models obtained by Shelah's original template iteration technique we have $\mathfrak{a} = \mathfrak{a}_g = \mathfrak{c}$, and so cardinalities of countable cofinality remain unattainable. The consistency of $\mathrm{cof}(\mathfrak{a}) = \omega$ is due to Jörg Brendle [4]. He modified Shelah's template iteration construction to obtain a forcing notion $\mathbb{P}$ for which Hechler's poset $\mathbb{Q}$ for adding a mad family of arbitrary size, say $\aleph_\omega$ (see [7]), is a complete suborder. The poset $\mathbb{P}$ not only has all the advantages of Shelah's original template construction, namely it adds a short scale while an isomorphism of names argument eliminates all mad families of cardinalities μ for say $\aleph_2 \leq \mu < \aleph_\omega$, but in addition the mad family added by $\mathbb{Q}$ remains maximal in the $\mathbb{P}$-generic extension. Adding a scale of length λ_0 for some λ_0, say $\aleph_2 \leq \lambda_0 < \aleph_\omega$, implies that $\mathfrak{b} = \mathfrak{d} = \lambda_0$ in the final extension, and since $\mathfrak{b} \leq \mathfrak{a}$ the isomorphism of names argument provides $\mathfrak{a} = \aleph_\omega$ in the final model.

Below we state the main result of our paper.

THEOREM 1.1. *Assume CH. Let λ be a singular cardinal of countable cofinality. Then there is a ccc generic extension in which $\mathfrak{a}_g = \lambda$.*

The problem of finding a poset which adds a maximal cofinitary group of a desired cardinality and which can be embedded into a template forcing construction is non-trivial. One of the pioneers in studying the possible sizes of infinite maximal cofinitary groups, Yi Zhang [10], provides a ccc poset which to a given cofinitary group $\mathcal{G}$ adds a generic permutation g such that $\mathcal{G} \cup \{g\}$ generates a cofinitary group in the resulting generic extension. Thus finite support iterations of Zhang's poset will provide a mcg of desired size. The main problem of using such finite support iterations within a template iteration is the lack of an analogue of the complete embedding property (see Lemma 2.13 and Remark 3.19).

More precisely, suppose $\mathbb{P} = \langle \mathbb{P}_\alpha, \mathbb{Q}_\beta : \alpha \leq \lambda, \beta < \lambda \rangle$ is a finite support iteration of Zhang's poset, where for each α the poset $\mathbb{Q}_\alpha$ adds a generic permutation g_α such that in $V^{\mathbb{P}_{\alpha+1}}$ the group $\mathcal{G}_{\alpha+1}$ generated by $\mathcal{G}_\alpha \cup \{g_\alpha\}$ is cofinitary. If we are to use this poset within a template iteration, we

([1]) The technique was introduced to establish the consistency of $\mathfrak{d} < \mathfrak{a}$.

will need the following: for every $I \subseteq \lambda$ there is a complete suborder $\mathbb{P}(I)$ of $\mathbb{P}$ such that the reals $\{g_i\}_{i\in I}$ are contained in $V^{\mathbb{P}(I)}$, while none of the reals from $\{g_i\}_{i\in\lambda\setminus I}$ belongs to $V^{\mathbb{P}(I)}$. However classical linear finite support iterations do not have this property. Note also that just taking finite support products of Zhang's poset will fail to capture the interactions of different generic permutations, and so the generics will not necessarily generate a cofinitary group in the resulting extension.

The original applications of template iterations seem to be very sensitive to the precise combinatorial properties of the posets used in such constructions: every time a new poset is being iterated along a template, one has to establish all the basic properties of such a construction including the fact that a certain recursively defined set, the intended poset, is a forcing notion (see [3, Lemmas 1.1, 4.4, 4.8] and [4, Main Lemma]).

Taking a slightly more axiomatic approach we define two classes of forcing notions which in a natural way capture the key properties in the context of template iterations of Hechler's poset for adding a mad family and Hechler's forcing notion for adding a dominating real, respectively. We refer to these posets as *finite function posets with the strong embedding property* (see Definitions 3.17 and 3.18) and *good σ-Suslin forcing notions* (see Definitions 3.14 and 3.15), respectively. We generalize the template iteration techniques of [4] so that arbitrary representatives of the above two classes can be iterated along a template (see Definition 3.22 and Lemma 3.23), and establish some basic combinatorial properties of this generalized iteration.

Whenever $\mathcal{T}$ is a template, $\mathbb{Q}$ is a finite function poset with the strong embedding property, and $\mathbb{S}$ is a good σ-Suslin forcing notion, we denote by $\mathbb{P}(\mathcal{T},\mathbb{Q},\mathbb{S})$ the iteration of $\mathbb{Q}$ and $\mathbb{S}$ along $\mathcal{T}$ (see Definition 3.22). For example we show that whenever $\mathbb{Q}$ is Knaster, then the entire iteration $\mathbb{P}(\mathcal{T},\mathbb{Q},\mathbb{S})$ is Knaster (see Lemma 3.28).

Following standard notation, let $\mathfrak{a}_p$ and $\mathfrak{a}_e$ denote the minimal size of a maximal family of almost disjoint permutations on ω and the minimal size of a maximal almost disjoint family of functions from ω to ω, respectively. We not only show that $\mathfrak{a}_g$ can be of countable cofinality, but also obtain (almost) for free the consistency of $\mathfrak{a}_p$ and $\mathfrak{a}_e$ being of countable cofinality. In the more general context of our discussion of cardinal invariants, we want to point out that even though clearly $\mathfrak{a}_p \le \mathfrak{a}_g$, the consistency of $\mathfrak{a}_p \ne \mathfrak{a}_g$ is still open. Let $\mathcal{T}_0$ be the template used by Brendle in [4]. Then our results can be summarized as follows:

Theorem 1.2. *Assume CH. Let λ be a singular cardinal of countable cofinality and let $\bar{\mathfrak{a}} \in \{\mathfrak{a},\mathfrak{a}_p,\mathfrak{a}_g,\mathfrak{a}_e\}$. Then there are a good σ-Suslin poset $\mathbb{S}_{\bar{\mathfrak{a}}}$ and a finite function poset with the strong embedding property $\mathbb{Q}_{\bar{\mathfrak{a}}}$, which is*

Knaster (*and so by Lemma* 3.28, $\mathbb{P}(\mathcal{T}_0, \mathbb{Q}_{\bar{\mathfrak{a}}}, \mathbb{S}_{\bar{\mathfrak{a}}})$ *is Knaster*), *such that*

$$V^{\mathbb{P}(\mathcal{T}_0, \mathbb{Q}_{\bar{\mathfrak{a}}}, \mathbb{S}_{\bar{\mathfrak{a}}})} \models \bar{\mathfrak{a}} = \lambda.$$

Then in particular $V^{\mathbb{P}(\mathcal{T}_0, \mathbb{Q}_{\bar{\mathfrak{a}}}, \mathbb{S}_{\bar{\mathfrak{a}}})} \models \mathrm{cof}(\bar{\mathfrak{a}}) = \omega$.

Thus the answer to the problem of finding an appropriate poset for adding a maximal cofinitary group is a product-like forcing notion (see Definition 2.4), which, even though inspired by Zhang's original poset, might be considered a maximal cofinitary group analogue of Hechler's forcing notion for adding a mad family. The most notable property satisfied by the poset which allows for it to be used within a template iteration construction is the existence of strong reductions (see Definition 3.18 and Remark 2.15) ([2]). We do not claim that our axiomatization is optimal, only that it is general enough to provide a uniform proof of the consistency of $\mathrm{cof}(\bar{\mathfrak{a}}) = \omega$ for each $\mathfrak{a} \in \{\mathfrak{a}, \mathfrak{a}_g, \mathfrak{a}_p, \mathfrak{a}_e\}$ ([3]). While to guarantee that in the final extension there are no mcg's of size $\aleph_1$ it is sufficient to add a short scale (as in Brendle's proof of $\mathrm{con}(\mathrm{cof}(\mathfrak{a}) = \omega)$), we achieve a bit more: we add a short cofinal sequence of slaloms, each of which localizes the corresponding ground model reals, and so obtain a generic extension in which all invariants of the Cichoń diagram have a fixed predetermined value.

Organization of the paper. In §2, we introduce and study a forcing notion $\mathbb{Q}_{A,\rho}$ for adding a maximal cofinitary group with a generating set indexed by some given uncountable set A. In §3, we introduce the classes of good σ-Suslin forcing notions and finite function posets with the strong embedding properties. We define the template iteration $\mathbb{P}(\mathcal{T}, \mathbb{Q}, \mathbb{S})$ of arbitrary representatives $\mathbb{S}$ and $\mathbb{Q}$ of the above two classes respectively, along a given template $\mathcal{T}$, and show that $\mathbb{P}(\mathcal{T}, \mathbb{Q}, \mathbb{S})$ is a forcing notion. In §4, we establish some basic combinatorial properties of this generalized iteration. Theorem 1.1 is proved in §5, and Theorem 1.2 in §6.

2. Maximal cofinitary groups.

In this section we introduce our poset for adding a maximal cofinitary group of arbitrary cardinality. We begin by giving several basic definitions and fixing notation.

DEFINITION 2.1. (1) Let A be a set. We denote by W_A the set of reduced words in the alphabet $\langle a^i : a \in A,\, i \in \{-1, 1\}\rangle$. The *free group* on the generator set A is the group $\mathbb{F}_A$ we obtain by giving W_A the obvious concatenate-and-reduce operation. When $A = \emptyset$ then $\mathbb{F}_A$ is by definition the trivial group. Note that A can be naturally identified with a subset of $\mathbb{F}_A$

([2]) The existence of strong reductions implies the complete embedding property mentioned earlier.

([3]) In fact, most of the classical applications of template iteration can be seen as particular instances in this axiomatization.

which generates $\mathbb{F}_A$, and every function $\rho : B \to G$, where G is any group, extends to a group homomorphism $\hat{\rho} : \mathbb{F}_B \to G$.

(2) A word $w \in W_A$ is said to be *good* if either $w = a^n$ for some $a \in A$ and $n \in \mathbb{Z} \setminus \{0\}$, or w starts and ends with a different letter. In the latter case, this means that there are $u \in W_A$, $a, b \in A$, $a \neq b$, and $i, j \in \{-1, 1\}$ such that $w = a^i u b^j$ *without cancellation*. Let $\widehat{W_A}$ be the set of all good words in W_A. Note that every word $w \in W_A$ can be written as $w = u^{-1} w' u$ for some $w' \in \widehat{W_A}$ and $u \in W_A$ ([4]).

(3) For a (partial) function $f : \omega \to \omega$, let

$$\mathrm{fix}(f) = \{n \in \omega : f(n) = n\}.$$

We denote by $\mathrm{cofin}(S_\infty)$ the set of cofinitary permutations in S_∞, i.e. permutations $\sigma \in S_\infty$ such that $\mathrm{fix}(\sigma)$ is finite.

(4) For a group G, a *cofinitary representation* of G is a homomorphism $\varphi : G \to S_\infty$ such that $\mathrm{im}(\varphi) \subseteq \{I\} \cup \mathrm{cofin}(S_\infty)$, where I denotes the identity permutation. If B is a set and $\rho : B \to S_\infty$, we say that ρ *induces* a cofinitary representation of $\mathbb{F}_B$ if the canonical extension of ρ to a homomorphism $\hat{\rho} : \mathbb{F}_B \to S_\infty$ is a cofinitary representation of $\mathbb{F}_B$.

(5) Let A be a set and let $s \subseteq A \times \omega \times \omega$. For $a \in A$, let

$$s_a = \{(n, m) \in \omega \times \omega : (a, n, m) \in s\}.$$

For a word $w \in W_A$, define the relation $e_w[s] \subseteq \omega \times \omega$ recursively by stipulating that for $a \in A$, if $w = a$ then $(n, m) \in e_w[s]$ iff $(n, m) \in s_a$, if $w = a^{-1}$ then $(n, m) \in e_w[s]$ iff $(m, n) \in s_a$, and if $w = a^i u$ for some word $u \in W_A$ and $i \in \{1, -1\}$ without cancellation then

$$(n, m) \in e_w[s] \;\Leftrightarrow\; (\exists k)\; e_{a^i}[s](k, m) \wedge e_u[s](n, k).$$

If s_a is a partial injection defined on a subset of ω for all $a \in A$, then $e_w[s]$ is always a partial injection defined on some subset of ω, and we call $e_w[s]$ the *evaluation of w given s*. By definition, let $e_\emptyset[s]$ be the identity in S_∞.

(6) If $s \subseteq A \times \omega \times \omega$ is such that s_a is always a partial injection, and $w \in W_A$, then we will write $e_w[s](n)\downarrow$ when $n \in \mathrm{dom}(e_w[s])$, and $e_w[s](n)\uparrow$ when $n \notin \mathrm{dom}(e_w[s])$.

(7) Finally, let A and B be disjoint sets and let $\rho : B \to S_\infty$ be a function. For a word $w \in W_{A \cup B}$ and $s \subseteq A \times \omega \times \omega$, we define

$$(n, m) \in e_w[s, \rho] \;\Leftrightarrow\; (n, m) \in e_w[s \cup \{(b, k, l) : \rho(b)(k) = l\}].$$

If s_a always is a partial injection for $a \in A$, then $e_w[s, \rho]$ is also a partial injection, and we call it the *evaluation of w given s and ρ*. The notations $e_w[s, \rho]\downarrow$ and $e_w[s, \rho]\uparrow$ are defined as before.

([4]) The presentation $w = u^{-1} w' u$ does allow cancellation.

The following lemma is obvious from the definitions. It will be used again and again, often without explicit mention.

Lemma 2.2. *Fix sets A and B such that $A \cap B = \emptyset$, and a function $\rho : B \to S_\infty$. Let $w \in W_{A\cup B}$ and $s \subseteq A \times \omega \times \omega$ be such that s_a is a partial injection for all $a \in A$. Suppose $w = uv$ without cancellation for some $u, v \in W_{A\cup B}$. Then $n \in \operatorname{dom}(e_w[s,\rho])$ if and only if $n \in \operatorname{dom}(e_v[s,\rho])$ and $e_v[s,\rho](n) \in \operatorname{dom}(e_u[s,\rho])$. If moreover $w \in \widehat{W}_{A\cup B}$ then $n \in \operatorname{fix}(e_w[s,\rho])$ if and only $e_v[s,\rho](n) \in \operatorname{fix}(e_{vu}[s,\rho])$. In particular, $\operatorname{fix}(e_w[s,\rho])$ and $\operatorname{fix}(e_{vu}[s,\rho])$ have the same cardinality.*

Remark 2.3. Note that if $w = uv$ *with* cancellation, or $w \notin \widehat{W}_{A\cup B}$, the above lemma may fail.

Definition 2.4. Fix sets A and B such that $A \cap B = \emptyset$, and a function $\rho : B \to S_\infty$ such that ρ induces a cofinitary representation $\hat{\rho} : \mathbb{F}_B \to S_\infty$. We define the forcing notion $\mathbb{Q}_{A,\rho}$ as follows:

(1) Conditions of $\mathbb{Q}_{A,\rho}$ are pairs (s, F) where $s \subseteq A \times \omega \times \omega$ is finite and s_a is a finite injection for every $a \in A$, and $F \subseteq \widehat{W}_{A\cup B}$ is finite.
(2) $(s, F) \leq_{\mathbb{Q}_{A,\rho}} (t, E)$ if and only if $s \supseteq t$, $F \supseteq E$ and for all $n \in \omega$ and $w \in E$, if $e_w[s,\rho](n) = n$ then already $e_w[t,\rho](n)\downarrow$ (and clearly also $e_w[t,\rho](n) = n$).

If $B = \emptyset$ then we write $\mathbb{Q}_A$ for $\mathbb{Q}_{A,\rho}$.

Remark 2.5. When A, B and $\rho : B \to S_\infty$ are clear from the context, we may write $\leq$ instead of $\leq_{\mathbb{Q}_{A,\rho}}$. For $w \in W_{A\cup B}$, write $\operatorname{oc}(w)$ for the (finite) set of letters occurring in w, and for $F \subseteq W_{A\cup B}$ let $\operatorname{oc}(F) = \bigcup_{w\in F} \operatorname{oc}(w)$. For $C \subseteq A\cup B$ and w and F as before, let $\operatorname{oc}_C(w) = \operatorname{oc}(w)\cap C$ and $\operatorname{oc}_C(F) = \operatorname{oc}(F) \cap C$. For $s \subseteq A \times \omega \times \omega$ let $\operatorname{oc}(s) = \{a : (\exists n, m \in \omega)\ (a, n, m) \in s\}$. For $p \in \mathbb{Q}_{A,\rho}$ let $\operatorname{oc}(p) = \operatorname{oc}(s) \cup \operatorname{oc}(F)$.

Unless otherwise stated, we now always assume that A and B are disjoint sets, $A \neq \emptyset$ and $\rho : B \to S_\infty$ induces a cofinitary representation of $\mathbb{F}_B$.

Lemma 2.6. *The poset $\mathbb{Q}_{A,\rho}$ has the Knaster property.*

Proof. Suppose that $\langle (s^\alpha, F^\alpha) \in \mathbb{Q}_{A,\rho} : \alpha < \omega_1\rangle$ is a sequence of conditions. By applying the Δ-system lemma [8, Theorem 1.5] repeatedly, we may assume that there are $A_0, A_1 \subseteq A$ finite and $t \subseteq A \times \omega \times \omega$ finite such that for all $\alpha \neq \beta$ we have $s^\alpha \cap s^\beta = t$, $\operatorname{oc}_A(F^\alpha) \cap \operatorname{oc}_A(F^\beta) = A_0$ and

$$(\operatorname{oc}(s^\alpha) \cup \operatorname{oc}_A(F^\alpha)) \cap (\operatorname{oc}(s^\beta) \cup \operatorname{oc}_A(F^\beta)) = A_1.$$

Note that $\operatorname{oc}(t)$ and A_0 are subsets of A_1. Further, we may assume that $s^\alpha \cap A_1 \times \omega \times \omega = t$, since this must be true for uncountably many α as A_1

is finite. Note then that $(s^\alpha \cup s^\beta, F^\alpha \cup F^\beta) \in \mathbb{Q}_{A,\rho}$, and that if $\alpha \neq \beta$ then

$$s^\alpha \cap \mathrm{oc}(F^\beta) \times \omega \times \omega \subseteq t. \tag{2.1}$$

We claim that $(s^\alpha \cup s^\beta, F^\alpha \cup F^\beta) \leq_{\mathbb{Q}_{A,\rho}} (s^\beta, F^\beta)$. For this, suppose that $w \in F^\beta$ and $e_w[s^\alpha \cup s^\beta, \rho](n) = n$. Then by 2.1 we have $e_w[t \cup s^\beta, \rho](n) = n$, and so $e_w[s^\beta, \rho](n) = n$. The proof that $(s^\alpha \cup s^\beta, F^\alpha \cup F^\beta) \leq_{\mathbb{Q}_{A,\rho}} (s^\alpha, F^\alpha)$ is similar. ∎

Let G be $\mathbb{Q}_{A,\rho}$-generic (over V, say). We define $\rho_G : A \cup B \to S_\infty$ by

$$\rho_G(x) = \begin{cases} \rho(x) & \text{if } x \in B, \\ \bigcup\{s_x : (\exists F \subseteq \widehat{W}_{A\cup B})\ (s, F) \in G\} & \text{if } x \in A. \end{cases} \tag{2.2}$$

We will see that ρ_G induces a cofinitary representation of $A \cup B$ which extends ρ. Of course, we first need to check that when G is generic then for $x \in A$,

$$\bigcup\{s_x : (\exists F \subseteq \widehat{W}_{A\cup B})\ (s, F) \in G\}$$

is a permutation. This is the content of the next lemma, which is parallel to [10, Lemma 2.2].

Lemma 2.7. *Let A and B be disjoint sets, and $\rho : B \to S_\infty$ a function inducing a cofinitary representation of $\mathbb{F}_B$. Then:*

(1) ("Domain extension") *For any $(s, F) \in \mathbb{Q}_{A,\rho}$, $a \in A$ and $n \in \omega$ such that $n \notin \mathrm{dom}(s_a)$ there are cofinitely many $m \in \omega$ such that $(s \cup \{(a, n, m)\}, F) \leq (s, F)$.*

(2) ("Range extension") *For any $(s, F) \in \mathbb{Q}_{A,\rho}$, $a \in A$ and $m \in \omega$ such that $m \notin \mathrm{ran}(s_a)$ there are cofinitely many $n \in \omega$ such that $(s \cup \{(a, n, m)\}, F) \leq (s, F)$.*

We will first prove a slightly stronger version of this, but at first only for certain special "good" words.

Definition 2.8. Let $a \in A$ and $j \geq 1$. A word $w \in W_{A\cup B}$ is called *a-good* of *rank j* if it has the form

$$w = a^{k_j} u_j a^{k_{j-1}} u_{j-1} \cdots a^{k_1} u_1, \tag{2.3}$$

where $u_i \in W_{A\setminus\{a\}\cup B} \setminus \{\emptyset\}$ and $k_i \in \mathbb{Z} \setminus \{0\}$ for $1 \leq i \leq j$.

Lemma 2.9. *Let $s \subseteq A \times \omega \times \omega$ be finite such that s_a is a partial injection for all $a \in A$. Fix $a \in A$, and let $w \in W_{A\cup B}$ be a-good of rank j for some $j \geq 1$. Then for any $n \in \omega \setminus \mathrm{dom}(s_a)$ and $C \subseteq \omega$ finite there are cofinitely many $m \in \omega$ such that*

$$(\forall l \in \omega)\ e_w[s \cup \{(a, n, m)\}, \rho](l) \in C \ \Leftrightarrow\ e_w[s, \rho](l){\downarrow} \wedge e_w[s, \rho](l) \in C.$$

Proof. Fix n and C as in the statement of the lemma. We proceed by induction on the rank j. Let w be an a-good word of rank 1,

$$w = a^{k_1} u_1.$$

Assume first that $k_1 > 0$. Then pick $m \notin \mathrm{dom}(s_a)$ and $m \notin C$. Suppose $e_w[s \cup \{(a,n,m)\}, \rho](l) \in C$ but $e_w[s,\rho](l)\uparrow$. Then there is some $0 < i < k_1$ such that $e_{a^i u_1}[s,\rho](l) = n$. If $i < k_1 - 1$ then $e_{a^{i+1}u_1}[s \cup \{(a,n,m)\}, \rho](l)\uparrow$, so we must have $i = k_1 - 1$. But then $e_w[s \cup \{(a,n,m)\}, \rho](l) = m \notin C$, a contradiction.

Assume then $k_1 < 0$. Pick $m \notin \mathrm{ran}(e_{a^i u_1}[s,\rho])$ for all $k_1 \leq i < 0$. If $e_w[s \cup \{(a,n,m)\}, \rho](l) \in C$ but $e_w[s,\rho](l)\uparrow$, then there is some $k_1 < i < 0$ such that $e_{a^i u_1}[s,\rho](l)\downarrow$ but $e_{a^{i-1}u_1}[s,\rho](l)\uparrow$. Since $e_{a^i u_1}[s,\rho](l) \neq m$, it follows that $e_{a^{i-1}u_1}[s \cup \{(a,n,m)\}, \rho]\uparrow$, a contradiction.

Now let w be a-good of rank $j > 1$, and write $w = a^{k_j} u_j \bar{w}$, where $\bar{w}$ is a-good of rank $j-1$. Let $C' = e_{u_j^{-1} a^{-k_j}}[s,\rho](C)$. By the inductive assumption there is $I_0 \subseteq \omega$ cofinite such that for all $m \in I_0$,

$$(\forall l \in \omega)\ e_{\bar{w}}[s \cup \{(a,n,m)\}, \rho](l) \in C' \ \Leftrightarrow\ e_{\bar{w}}[s,\rho](l)\downarrow \wedge\ e_{\bar{w}}[s,\rho](l) \in C'.$$

Let $I_1 \subseteq \omega$ be cofinite such that for all $m \in I_1$,

$$\begin{aligned}(\forall l \in \omega)\ e_{a^{k_i} u_j}[s \cup \{(a,n,m)\}, \rho](l) \in C \\ \Leftrightarrow\ e_{a^{k_i} u_j}[s,\rho](l)\downarrow \wedge\ e_{a^{k_i} u_j}[s,\rho](l) \in C.\end{aligned}$$

Then let $m \in I_1 \cap I_0$, and suppose that $e_w[s \cup \{(a,n,m)\}, \rho](l) \in C$. Then $e_{\bar{w}}[s \cup \{(a,n,m)\}, \rho](l) \in C'$, and so $e_{\bar{w}}[s,\rho](l) \in C'$. It follows that

$$e_{a^{k_j} u_j}[s \cup \{(a,n,m)\}, \rho](e_{\bar{w}}[s,\rho](l)) \in C,$$

and so we have $e_{a^{k_j} u_j}[s,\rho](e_{\bar{w}}[s,\rho](l)) = e_w[s,\rho](l) \in C$, as required. ■

Proof of Lemma 2.7. (1) It suffices to prove this when $F = \{w\}$. Further, we may assume that a occurs in w, since otherwise there is nothing to show.

If w is a-good, then the statement follows from Lemma 2.9. If w is not a-good, then write $w = uva^k$ (without cancellation), where $u \in W_{A \setminus \{a\} \cup B}$, v is a-good, and $k \in \mathbb{Z}$. Let $\bar{w} = va^k u$. Then $\bar{w}$ is a-good, and so there is $I \subseteq \omega$ cofinite such that

$$(\forall m \in I)\ (s \cup \{(a,n,m)\}, \{\bar{w}\}) \leq_{\mathbb{Q}_{A,\rho}} (s, \{\bar{w}\}).$$

We claim that $(s \cup \{(a,n,m)\}, \{w\}) \leq (s, \{w\})$ when $m \in I$. Indeed, if $e_w[s \cup \{(a,n,m)\}, \rho](l) = l$ then by Lemma 2.2 it follows that

$$e_{\bar{w}}[s \cup \{(a,n,m)\}, \rho]\big(e_{va^k}[s \cup \{(a,n,m)\}, \rho](l)\big) = e_{va^k}[s \cup \{(a,n,m)\}, \rho](l),$$

and so

$$e_{\bar{w}}[s,\rho]\big(e_{va^k}[s \cup \{(a,n,m)\}, \rho](l)\big) = e_{va^k}[s \cup \{(a,n,m)\}, \rho](l).$$

Applying Lemma 2.2 once more, we get $e_w[s,\rho](l) = l$.

(2) Let $(s,F)\in\mathbb{Q}_{A,\rho}$, $a\in A$, and suppose $m_0\notin\operatorname{ran}(s_a)$. As above, we may assume that $F=\{w\}$. Define $\bar{s}\subseteq A\times\omega\times\omega$ by

$$(x,n,m)\in\bar{s}\ \Leftrightarrow\ (x\neq a\wedge(x,n,m)\in s)\vee(x=a\wedge(x,m,n)\in s).$$

Let $\bar{w}$ be the word in which every occurrence of a is replaced with a^{-1}. Notice that $e_{\bar{w}}[\bar{s},\rho]=e_w[s,\rho]$, and that $m_0\notin\operatorname{dom}(\bar{s})$. By (1) above there are cofinitely many n such that $(\bar{s}\cup\{(a,m_0,n)\},\{\bar{w}\})\leq(\bar{s},\{\bar{w}\})$, and so for cofinitely many n we have $(s\cup\{(a,n,m_0)\},\{w\})\leq(s,\{w\})$. ■

The following easy consequence of Lemma 2.7 will be useful. We leave the proof to the reader.

COROLLARY 2.10. *Let $w\in W_{A\cup B}$, and let $A_0\subseteq A$ be the set of letters from A occurring in w. For any condition $(s,F)\in\mathbb{Q}_{A,\rho}$ and finite sets $C_0,C_1\subseteq\omega$ there is $t\subseteq A_0\times\omega\times\omega$ such that $(t\cup s,F)\leq(s,F)$ and $\operatorname{dom}(e_w[s\cup t,\rho])\supset C_0$ and $\operatorname{ran}(e_w[s\cup t,\rho])\supset C_1$.*

LEMMA 2.11. *Let $w\in\widehat{W}_{A\cup B}$ and suppose $(s,F)\Vdash_{\mathbb{Q}_{A,\rho}} e_w[\rho_G](n)=n$ for some $n\in\omega$. Then $e_w[s,\rho](n){\downarrow}$ and $e_w[s,\rho](n)=n$.*

Proof. Let G be $\mathbb{Q}_{A,\rho}$-generic such that $(s,F)\in G$. Then $V[G]\vDash e_w[\rho_G](n)=n$. The definition of the partial order implies that there is $(t,E)\in G$ such that $e_w[t,\rho](n)=n$. Without loss of generality, $(t,E)\leq(s,F)$. But then $e_w[s,\rho](n)$ is defined and $e_w[s,\rho](n)=n$. ■

PROPOSITION 2.12. *Let G be $\mathbb{Q}_{A,\rho}$-generic. Then ρ_G, defined in 2.2, is a function $A\cup B\to S_\infty$ such that $\rho_G\restriction B=\rho$, and ρ_G induces a cofinitary representation $\hat{\rho}_G:\mathbb{F}_{A\cup B}\to S_\infty$ satisfying $\hat{\rho}_G\restriction\mathbb{F}_B=\hat{\rho}$.*

Proof. For each $a\in A$ and $n\in\omega$, let

$$D_{a,n}=\{(s,F)\in\mathbb{Q}_{A,\rho}:(\exists m)\ (a,n,m)\in s\},$$
$$R_{a,n}=\{(s,F)\in\mathbb{Q}_{A,\rho}:(\exists m)\ (a,m,n)\in s\}.$$

For $w\in\widehat{W}_{A\cup B}$, let

$$D_w=\{(s,F)\in\mathbb{Q}_{A,\rho}:w\in F\}.$$

Then D_w is easily seen to be dense, and $D_{a,n}$ and $R_{a,n}$ are dense by Lemma 2.7. Thus ρ_G is a function $A\cup B\to S_\infty$, as promised.

It remains to prove that ρ_G induces a cofinitary representation. For this let $w\in W_{A\cup B}$. Then we can find $w'\in\widehat{W}_{A\cup B}$ and $u\in W_{A\cup B}$ such that $w=u^{-1}w'u$. Since $D_{w'}$ is dense, there is some condition $(s,F)\in G$ such that $w'\in F$. Suppose then that $e_{w'}[\rho_G](n)=n$ in $V[G]$. Then there is some condition $(t,E)\leq_{\mathbb{Q}_{A,\rho}}(s,F)$ and $(t,E)\in G$ forcing this. It follows by Lemma 2.11 that $e_{w'}[t,\rho](n)=n$. But then by the definition of $\leq_{\mathbb{Q}_{A,\rho}}$ we have $e_{w'}[s,\rho](n)=n$, and so $\operatorname{fix}(e_{w'}[\rho_G])=\operatorname{fix}(e_{w'}[s,\rho])$, which is finite. Finally, $\operatorname{fix}(e_w[\rho_G])=e_u[\rho_G]^{-1}(\operatorname{fix}(e_{w'}[\rho_G]))$, so $\operatorname{fix}(e_w[\rho_G])$ is finite. ■

Notation. For $s \subseteq A \times \omega \times \omega$ and $A_0 \subseteq A$, write $s \restriction A_0$ for $s \cap A_0 \times \omega \times \omega$. For a condition $p = (s, F) \in \mathbb{Q}_{A,\rho}$, we will write $p \restriction A_0$ for $(s \restriction A_0, F)$, and $p \| \restriction A_0$ ("strong restriction") for $(s \restriction A_0, F \cap \widehat{W}_{A_0 \cup B})$. (So $p \| \restriction A_0$ is a condition of $\mathbb{Q}_{A_0,\rho}$ but $p \restriction A_0$ is in general still only a condition of $\mathbb{Q}_{A,\rho}$.)

For the notion of complete containment see Section 3.1.2.

Lemma 2.13. *If $A_0 \subseteq A$ then $\mathbb{Q}_{A_0,\rho}$ is completely contained in $\mathbb{Q}_{A,\rho}$.*

Proof. Let $A_1 = A \setminus A_0$. We may of course assume that $A_0, A_1 \neq \emptyset$, since otherwise there is nothing to show. We first note that all $\mathbb{Q}_{A_0,\rho}$ conditions are also $\mathbb{Q}_{A,\rho}$ conditions, and so $\mathbb{Q}_{A_0,\rho} \subseteq \mathbb{Q}_{A,\rho}$. Clearly $p \leq_{\mathbb{Q}_{A_0,\rho}} q$ implies $p \leq_{\mathbb{Q}_{A,\rho}} q$. Moreover, if $p, q \in \mathbb{Q}_{A,\rho}$ and $p \leq_{\mathbb{Q}_{A,\rho}} q$ then clearly $p \| \restriction A_0 \leq_{\mathbb{Q}_{A_0,\rho}} q \| \restriction A_0$. Hence $p \perp^{\mathbb{Q}_{A_0,\rho}} q$ if and only if $p \perp^{\mathbb{Q}_{A,\rho}} q$. It remains to see that if $q \in \mathbb{Q}_{A,\rho}$, then there is $p_0 \in \mathbb{Q}_{A_0,\rho}$ such that whenever $p \leq_{\mathbb{Q}_{A_0,\rho}} p_0$ then p and q are $\leq_{\mathbb{Q}_{A,\rho}}$-compatible. This follows from the next claim.

Claim 2.14. *For every $(s, F) \in \mathbb{Q}_{A,\rho}$ there is $t_0 \supseteq s \restriction A_0$, $t_0 \subseteq A_0 \times \omega \times \omega$, such that if $(t, E) \leq_{\mathbb{Q}_{A_0,\rho}} (t_0, F \cap \widehat{W}_{A_0 \cup B})$ then $(s \cup t, F) \leq_{\mathbb{Q}_{A,\rho}} (s, F)$. Thus, for any $q \in \mathbb{Q}_{A,\rho}$ there is $p_0 \leq_{\mathbb{Q}_{A_0,\rho}} p \| \restriction A_0$ such that whenever $p \leq_{\mathbb{Q}_{A_0,\rho}} p_0$ then p is $\leq_{\mathbb{Q}_{A,\rho}}$-compatible with q.*

To see this, let $\{w_1, \dots, w_n\} = F \setminus W_{A_0 \cup B}$. Then each word w_i may be written as

$$w_i = u_{i,k_i} v_{i,k_i} \cdots u_{i,1} v_{i,1} u_{i,0}$$

where $u_{i,j} \in W_{A_0}$ and $v_{i,j} \in W_{A_1}$, all words are non-empty except possibly u_{i,k_i} and $u_{i,0}$, and each $v_{i,j}$ starts and ends with a letter from A_1; hence the domain and range of $e_{v_{i,j}}[s, \rho]$ are finite for every i, j. By repeated applications of Corollary 2.10 to (s, F) and the $u_{i,j}$ we can find $t_0 \subseteq A_0 \times \omega \times \omega$ with $t_0 \supseteq s \restriction A_0$ and $\mathrm{dom}(e_{u_{i,j}}[s \cup t_0, \rho]) \supseteq \mathrm{ran}(e_{v_{i,j}}[s, \rho])$ and $\mathrm{ran}(e_{u_{i,j}}[s \cup t_0, \rho] \supseteq \mathrm{dom}(e_{v_{i,j+1}}[s, \rho])$ for all i, j, and satisfying $(s \cup t_0, F) \leq_{\mathbb{Q}_{A,\rho}} (s, F)$.

Suppose now $(t, E) \leq_{\mathbb{Q}_{A_0,\rho}} (t_0, F \cap \widehat{W}_{A_0 \cup B})$. If $e_{w_i}[s \cup t, \rho](n)\downarrow$ for some $n \in \omega$, then by definition of t_0 it must be the case that $e_{w_i}[s \cup t_0, \rho](n)\downarrow$. Therefore if $e_{w_i}[s \cup t, \rho](n) = n$, we have $e_{w_i}[s \cup t_0, \rho](n) = n$, and so since $(s \cup t_0, F) \leq_{\mathbb{Q}_{A,\rho}} (s, F)$ it follows that $e_{w_i}[s, \rho](n) = n$. Thus $(s \cup t, F) \leq_{\mathbb{Q}_{A,\rho}} (s, F)$ as required. ∎

Remark 2.15. Note that in Claim 2.14 we in fact obtained a slightly stronger property than stated, namely the following. Let $A \subseteq \mathrm{dom}(\mathbb{Q})$ and $p = (s, F) \in \mathbb{Q}$. Then there is $t_0 \subseteq \mathrm{oc}(p) \cap A \times \omega \times \omega$ such that $s \restriction A \subseteq t_0$, $(t_0, F \cap \widehat{W}_A) \leq_{\mathbb{Q}_{\mathrm{oc}(p) \cap A}} p \| \restriction A$, and whenever $(t, E) \leq_{\mathbb{Q}} (t_0, F \cap \widehat{W}_A)$ is such that $\mathrm{oc}(t) \cap (\mathrm{oc}(p) \setminus A) = \mathrm{oc}(E) \cap (\mathrm{oc}(p) \setminus A) = \emptyset$, then $(t \cup s, F) \leq (s, F)$, $(t \cup s, E) \leq (t, E)$, and so $(t \cup s, E \cup F)$ is a common extension of (s, F) and (t, E).

LEMMA 2.16. *Let $A = A_0 \cup A_1$. If $(t, E) \in \mathbb{Q}_{A_0,\rho}$ and*

$$(t, E) \Vdash_{\mathbb{Q}_{A_0,\rho}} (s_0, F_0) \leq_{\mathbb{Q}_{A_1,\rho_{\dot{G}}}} (s_1, F_1)$$

then $(t \cup s_0, F_0) \leq_{\mathbb{Q}_{A,\rho}} (t \cup s_1, F_1)$.

Proof. Let $w \in F_1$ and suppose $e_w[t \cup s_0, \rho](n) = n$. If G is $\mathbb{Q}_{A_0,\rho}$-generic such that $(t, E) \in G$, then in $V[G]$ we have $e_w[s_0, \rho_G](n) = n$, and so in $V[G]$ we have $e_w[s_1, \rho_G](n) = n$, from which it follows that $e_w[t \cup s_1, \rho](n) = n$. ∎

Let $A_0 \subseteq A$ and $A_1 = A \setminus A_0$. By Lemma 2.13 we have $\mathbb{Q}_{A_0,\rho} \lessdot \mathbb{Q}_{A,\rho}$. Let H be $\mathbb{Q}_{A_0,\rho}$-generic. Lemma 2.17 below implies in particular that in $V[H]$ the quotient $\mathbb{Q}_{A,\rho}/H$ is equal to Q_{A_1,ρ_H}, where ρ_H is the generic extension of ρ given by H. This nice combinatorial representation of the quotients of $\mathbb{Q}_{A,\rho}$ will be of importance for establishing the maximality of the cofinitary group added by $\mathbb{Q}_{A,\rho}$ in Theorem 2.18, as well as for establishing the maximality of the cofinitary group added by our mcg poset within a template iteration (see Lemma 4.2).

LEMMA 2.17. *Suppose G is $\mathbb{Q}_{A,\rho}$-generic over V and $A = A_0 \cup A_1$ where $A_0, A_1 \neq \emptyset$ and $A_0 \cap A_1 = \emptyset$. Then $H = G \cap \mathbb{Q}_{A_0,\rho}$ is $\mathbb{Q}_{A_0,\rho}$-generic over V and $K = \{p{\restriction}A_1 : p \in G\} = \{(s{\restriction}A_1, F) : (s, F) \in G\}$ is $\mathbb{Q}_{A_1,\rho_H}$-generic over $V[H]$. Moreover, $\rho_G = (\rho_H)_K$.*

Proof. That H is $\mathbb{Q}_{A_0,\rho}$-generic over V follows from Lemma 2.13. To see that K is $\mathbb{Q}_{A_1,\rho_H}$-generic over $V[H]$, suppose $D \subseteq \mathbb{Q}_{A_1,\rho_H}$ is dense and $D \in V[H]$. Define

$$D' = \{p \in \mathbb{Q}_{A,\rho} : p|{\restriction}A_0 \Vdash_{\mathbb{Q}_{A_0,\rho}} p{\restriction}A_1 \in \dot{D}\}$$

and let $p_0 \in H$ be a condition such that $p_0 \Vdash_{\mathbb{Q}_{A_0,\rho}}$ "D is dense". We claim that D' is dense below p_0 (in $\mathbb{Q}_{A,\rho}$.) For this, let $(s, F) = p \leq_{\mathbb{Q}_{A,\rho}} p_0$. Then by Claim 2.14 we can find $p_0 \leq_{\mathbb{Q}_{A_0,\rho}} p|{\restriction}A_0$ such that for any $p_1 \leq_{\mathbb{Q}_{A_0,\rho}} p_0$, p_1 is compatible with p. Thus we can find $q = (s_0, F_0) \in \mathbb{Q}_{A_1,\rho_H}$ and (t, E) $\leq_{\mathbb{Q}_{A_0,\rho}} p_0$ such that

$$(t, E) \Vdash_{\mathbb{Q}_{A_0,\rho}} \dot{q} \in \dot{D} \wedge \dot{q} \leq_{\mathbb{Q}_{A_1,\rho_{\dot{H}}}} \dot{p}{\restriction}A_1.$$

By Lemma 2.16 we have $(s_0 \cup t, F_0) \leq_{\mathbb{Q}_{A,\rho}} (s{\restriction}A_1 \cup t, F)$, and therefore

$$(s_0 \cup t, F_0 \cup E) \leq_{\mathbb{Q}_{A,\rho}} (s, F).$$

Since clearly $(s_0 \cup t, F_0 \cup E) \in D'$, this shows that D' is dense below p_0.

Now, since $p_0 \in G$ it follows that there is $q' \in D' \cap G$. In $V[H]$ we then have $q'{\restriction}A_1 \in D$, which shows that $K \cap D \neq \emptyset$.

That $(\rho_H)_K = \rho_G$ follows directly from the definition of H and K. ∎

Our next goal is to prove the following.

THEOREM 2.18. *Suppose $\rho : B \to S_\infty$ induces a cofinitary representation of $\mathbb{F}_B$. If* $\mathrm{card}(A) > \aleph_0$ *and G is $\mathbb{Q}_{A,\rho}$-generic over V, then* $\mathrm{im}(\rho_G)$ *is a maximal cofinitary group in $V[G]$.*

The theorem is a consequence of the following lemma, which is parallel to [10, Lemma 3.3].

LEMMA 2.19. *Suppose that $\rho : B \to S_\infty$ induces a cofinitary representation $\hat\rho : \mathbb{F}_B \to S_\infty$ and that there is $b_0 \in B$ such that $\rho(b_0) \neq I$. Let $(s, F) \in \mathbb{Q}_{A,\rho\restriction B\setminus\{b_0\}}$ and $a_0 \in A$. Then there is $N \in \omega$ such that for all $n \geq N$,*

$$\bigl(s \cup \{(a_0, n, \rho(b_0)(n))\}, F\bigr) \leq_{\mathbb{Q}_{A,\rho\,\restriction\, B\setminus\{b_0\}}} (s, F).$$

Proof. Let $w_1, \ldots, w_l \in F$ enumerate the words in F in which a_0 occurs. Then we may write each word w_i in the form

$$w_i = u_{i,j_i} a_0^{k(i,j_i)} u_{i,j_i-1} a_0^{k(i,j_i-1)} \cdots u_{i,1} a_0^{k(i,1)} u_{i,0}$$

where the $u_{i,m}$ are in $W_{A\setminus\{a_0\}\cup B\setminus\{b_0\}}$ and are non-empty whenever $m \notin \{j_i, 0\}$. By Lemma 2.7 we may assume that for all $u_{i,m}$ with $\mathrm{dom}(e_{u_{i,m}}[s,\rho])$ and $\mathrm{ran}(e_{u_{i,m}}[s,\rho])$ finite,

$$\mathrm{dom}(e_{a_0^{k(i,m+1)}}[s,\rho]) \supseteq \mathrm{ran}(e_{u_{i,m}}[s,\rho]),$$
$$\mathrm{ran}(e_{a_0^{k(i,m)}}[s,\rho]) \supseteq \mathrm{dom}(e_{u_{i,m}}[s,\rho]).$$

Let $\bar w_i$ be the word in which every occurrence of a_0 in w_i has been replaced by b_0. If $e_{\bar w_i}[\rho]$ is totally defined, then since ρ induces a cofinitary representation there are at most finitely many n such that $e_{\bar w_i}[\rho](n) = n$. For each $\bar w_i$ with $e_{\bar w_i}[\rho]$ totally defined and $1 \leq m \leq j_i$ let

$$\bar w_{i,m} = u_{i,m} b_0^{k(i,m)} \cdots u_{i,1} b_0^{k(i,1)} u_{i,0},$$

and let

$$N_i = \max\bigl\{e_v[\rho](k) : e_{\bar w_i}[\rho](k) = k \wedge v = b_0^{\mathrm{sign}(k(i,m))p}\bar w_{i,m} \wedge 0 \leq p \leq \mathrm{sign}(k(i,m))k(i,m) \wedge 0 \leq m \leq j_i\bigr\}.$$

Then let $N \in \omega$ be such that $N \geq \max\{N_i : i \leq l\}$ and $n \notin \mathrm{dom}(s_{a_0})$ and $\rho(b_0)(n) \notin \mathrm{ran}(s_{a_0})$ whenever $n \geq N$. Then for any $n \geq N$, on the one hand, if $e_{\bar w_i}[\rho]$ is not everywhere defined then

$$\mathrm{dom}(e_{w_i}[s,\rho]) = \mathrm{dom}\bigl(e_{w_i}[s \cup \{(a_0, n, \rho(b_0)(n))\}, \rho]\bigr),$$

while if $e_{\bar w_i}[\rho]$ is everywhere defined then necessarily

$$e_{w_i}[s \cup \{(a_0, n, \rho(b_0)(n))\}, \rho](k) = k$$

only when $e_{w_i}[s,\rho](k) = k$. ■

Proof of Theorem 2.18. Let $b_0 \notin B \cup A$. Suppose $\mathrm{card}(A) > \aleph_0$ and that G is $\mathbb{Q}_{A,\rho}$-generic, and suppose further that there is a permutation $\sigma \in \mathrm{cofin}(S_\infty)^{V[G]} \setminus \mathrm{im}(\rho_G)$ such that $\rho'_G : B \cup \{b_0\} \to S_\infty$ defined by

$\rho'_G{\restriction}B = \rho_G$, and $\rho'_G(b_0) = \sigma$ induces a cofinitary representation of $\mathbb{F}_{B\cup\{b_0\}}$. Let $\dot\sigma$ be a name for σ. Then there is $A_0 \subseteq A$ countable so that $\dot\sigma$ is a $\mathbb{Q}_{A_0,\rho}$-name, and so we already have $\sigma \in V[H]$, where $H = G \cap \mathbb{Q}_{A_0,\rho}$. Let $A_1 = A \setminus A_0$, and let K be as in Lemma 2.17. Define

$$D_{\sigma,N} = \{(s,F) \in \mathbb{Q}_{A_1,\rho_H} : (\exists n \geq N)\ s(n) = \sigma(n)\}.$$

By Lemma 2.19 this set is dense. Thus in $V[H][K]$, for any $a_0 \in A \setminus A_0$ we have $(\rho_H)_K(a_0)(n) = \sigma(n)$ for infinitely many n. Since $(\rho_H)_K = \rho_G$ by Lemma 2.17, this contradicts the fact that ρ'_G induces a cofinitary representation. ∎

3. Iteration along a two-sided template

3.1. Preliminaries. We now recall various definitions and introduce several notions that are needed to set up the framework in which we will treat the iteration along a two-sided template.

3.1.1. *Localization.* As indicated, we are aiming to give an iterated forcing construction which will provide a generic extension in which the minimal size of a maximal cofinitary group is of countable cofinality. In order to provide a lower bound for $\mathfrak{a}_g$, along this iteration construction cofinally often we will force with the following partial order $\mathbb{L}$, known as *localization*.

Definition 3.1. The forcing notion $\mathbb{L}$ consists of pairs (σ,ϕ) such that $\sigma \in {}^{<\omega}({}^{<\omega}[\omega])$, $\phi \in {}^{\omega}({}^{<\omega}[\omega])$, $\sigma \subseteq \phi$, $|\sigma(i)| = i$ for all $i < |\sigma|$, and $|\phi(i)| \leq |\sigma|$ for all $i \in \omega$. The extension relation is defined as follows: $(\sigma,\phi) \leq (\tau,\psi)$ if and only if σ end-extends τ and $\psi(i) \subseteq \phi(i)$ for all $i \in \omega$.

Recall that a *slalom* is a function $\phi : \omega \to [\omega]^{<\omega}$ such that for all $n \in \omega$ we have $|\phi(n)| \leq n$. We say that a slalom *localizes* a real $f \in {}^{\omega}\omega$ if there is $m \in \omega$ such that for all $n \geq m$ we have $f(n) \in \phi(n)$. The following is well-known and follows easily from the definition of $\mathbb{L}$.

Lemma 3.2. *The poset $\mathbb{L}$ adds a slalom which localizes all ground model reals.*

Let $\mathrm{add}(\mathcal{N})$ denote the additivity of the (Lebesgue) null ideal, and let $\mathrm{cof}(\mathcal{N})$ denote the cofinality of the null ideal. We will use the following well-known combinatorial characterizations of $\mathrm{add}(\mathcal{N})$ and $\mathrm{cof}(\mathcal{N})$ which are due to Bartoszyński and Judah.

Theorem 3.3 (Bartoszyński, Judah [1, Ch. 2]).

(1) $\mathrm{add}(\mathcal{N})$ *is the least cardinality of a family $F \subseteq \omega^\omega$ such that no slalom localizes all members of F.*
(2) $\mathrm{cof}(\mathcal{N})$ *is the least cardinality of a family Φ of slaloms such that every member of ω^ω is localized by some $\phi \in \Phi$.*

Finally, we will need the following result due to Brendle, Spinas and Zhang:

THEOREM 3.4 ([6]). *$\mathfrak{a}_g \geq \mathrm{non}(\mathcal{M})$.*

In our intended forcing construction cofinally often we will force with the partial order $\mathbb{L}$, which using the above characterizations will provide a lower bound for $\mathfrak{a}_g$.

3.1.2. *Complete embeddings.* Recall that if $\mathbb{P}$ and $\mathbb{Q}$ are posets such that $\mathbb{P} \subseteq \mathbb{Q}$, then we say that $\mathbb{P}$ is *completely contained* in $\mathbb{Q}$, written $\mathbb{P} \lessdot \mathbb{Q}$, if $\mathbb{P} \subseteq \mathbb{Q}$ and

(1) if $p, p' \in \mathbb{P}$ and $p \leq_{\mathbb{P}} p'$ then $p \leq_{\mathbb{Q}} p'$;
(2) if $p, p' \in \mathbb{P}$ and $p \perp_{\mathbb{P}} p'$ then $p \perp_{\mathbb{Q}} p'$;
(3) if $q \in \mathbb{Q}$ then there is $r \in \mathbb{P}$ (called a *reduction* of q) such that for all $p \in \mathbb{P}$ with $p \leq_{\mathbb{P}} r$, the conditions p and q are compatible.

We note that (3.1.2) above may be seen to be equivalent to

(3′) all maximal antichains in $\mathbb{P}$ are maximal in $\mathbb{Q}$.

LEMMA 3.5. *Let $\mathbb{P}$ and $\mathbb{Q}$ be posets, and suppose $\mathbb{P} \lessdot \mathbb{Q}$. Let $q \in \mathbb{Q}$, $p \in \mathbb{P}$ and $q \leq_{\mathbb{Q}} p$. Then any reduction of q to $\mathbb{P}$ is compatible in $\mathbb{P}$ with p, and so q has a reduction extending p.*

Proof. Suppose $r \in \mathbb{P}$ is a reduction of q and $r \perp_{\mathbb{P}} p$. Let $x \in \mathbb{P}$, $x \leq_{\mathbb{P}} r$. Then since r is a reduction of q, we see that x is compatible with q in $\mathbb{Q}$ and so there is $x' \in \mathbb{Q}$ which is their common extension. But then $x' \leq_{\mathbb{Q}} x \leq_{\mathbb{P}} r$ and so $x' \leq_{\mathbb{Q}} r$. Also $x' \leq_{\mathbb{Q}} q \leq_{\mathbb{Q}} p$ and hence $x' \leq_{\mathbb{Q}} p$. Therefore r is compatible with p in $\mathbb{Q}$. But by assumption $\mathbb{P} \lessdot \mathbb{Q}$, and so $r \perp_{\mathbb{P}} p \to r \perp_{\mathbb{Q}} p$ must be true. Therefore r and p are compatible in $\mathbb{P}$, which is a contradiction.

To complete the proof, consider any reduction r of q to $\mathbb{P}$. Then r is compatible in $\mathbb{P}$ with p, and so they have a common extension r_0. However, any extension of a reduction is a reduction, and so r_0 is a reduction of q with $r_0 \leq_{\mathbb{P}} p$. ∎

3.1.3. *Canonical projection of a name for a real*

DEFINITION 3.6. Let $\mathbb{B}$ be a partial order and $y \in \mathbb{B}$. For each $n \geq 1$ let $\mathcal{B}_n$ be a maximal antichain below y. We will say that the set $\{(b, s(b))\}_{b \in \mathcal{B}_n, n \geq 1}$ is a *nice name for a real below y* if

(1) whenever $n \geq 1$, $b \in \mathcal{B}_n$ then $s(b) \in {}^n\omega$;
(2) whenever $m > n \geq 1$, $b \in \mathcal{B}_n$, $b' \in \mathcal{B}_m$ and b, b' are compatible, then $s(b)$ is an initial segment of $s(b')$.

REMARK 3.7. Whenever $\dot{f}$ is a $\mathbb{B}$-name for a real, we can associate with $\dot{f}$ a family of maximal antichains $\{\mathcal{B}_n\}_{n \geq 1}$ and initial approximations $s(b) \in {}^n\omega$ of $\dot{f}$ for $b \in \mathcal{B}_n$ such that for all n and b, we have $b \Vdash_{\mathbb{B}} \dot{f} \restriction n = \check{s}(b)$ and the collection $\{(b, s(b))\}_{b\in\mathcal{B}_n,\, n\in\omega}$ has the above properties. Thus we can assume that all names for reals are nice, and abusing notation we will write $\dot{f} = \{(b, s(b))\}_{b\in\mathcal{B}_n,\, n\in\omega}$.

LEMMA 3.8. *Let $\mathbb{A}$ be a complete suborder of $\mathbb{B}$, $y \in \mathbb{B}$ and x a reduction of y to $\mathbb{A}$. Let $\dot{f} = \{(b, s(b))\}_{b\in\mathcal{B}_n,\, n\geq 1}$ be a nice name for a real below y. Then there is $\dot{g} = \{(a, s(a))\}_{a\in\mathcal{A}_n,\, n\geq 1}$, an $\mathbb{A}$-nice name for a real below x, such that for all $n \geq 1$, for all $a \in \mathcal{A}_n$, there is $b \in \mathcal{B}_n$ such that a is a reduction of b and $s(a) = s(b)$.*

REMARK 3.9. Whenever $\dot{f}, \dot{g}$ are as above, we will say that $\dot{g}$ is a *canonical projection* of $\dot{f}$ below x.

Proof of Lemma 3.8. Recursively we will construct the antichains $\mathcal{A}_n$. Along this construction we will guarantee that for all $a \in \mathcal{A}_n$, $a' \in \mathcal{A}_{n+1}$ either $a' \leq a$ or $a \perp a'$, and that if $a' \leq a$ then $s(a')$ end-extends $s(a)$.

First we will define $\mathcal{A}_1$. Let $t \in \mathbb{A}$ be an arbitrary extension of x. Since x is a reduction of y, there is $\hat{t} \in \mathbb{B}$ such that $\hat{t} \leq_{\mathbb{B}} t, y$. Therefore there is $b \in \mathcal{B}_1$ such that $\hat{t}$ and b are compatible with a common extension $\bar{t}$. Then in particular $\bar{t} \leq_{\mathbb{B}} t$, and so we can find a reduction a of $\bar{t}$ extending t. Since $\bar{t} \leq b$, a is also a reduction of b. Define $s(a) = s(b)$, $a(t) = a$. Let $\mathcal{A}_1$ be a maximal antichain in the dense below x set $D_1 = \{a(t) : t \leq x\}$.

Suppose $\mathcal{A}_n$ has been defined. Let $a \in \mathcal{A}_n$ and $t \leq_{\mathbb{A}} a$. By the inductive hypothesis, there is $b \in \mathcal{B}_n$ such that a is a reduction of b and $s(a) = s(b)$. Then t is compatible in $\mathbb{B}$ with b, with common extension $\hat{t}$. Hence in particular $\hat{t} \leq_{\mathbb{B}} y$, and so there is $\bar{b} \in \mathcal{B}_{n+1}$ such that $\hat{t}$ is compatible with $\bar{b}$ in $\mathbb{B}$ with common extension $\tilde{t}$. Then in particular $\tilde{t} \leq \bar{b}, b$, and so $s(b)$ is an initial segment of $s(\bar{b})$. Since $\tilde{t} \leq t$, it has a reduction $\bar{a} \leq_{\mathbb{A}} t$. Define $a(t) = \bar{a}$, $s(\bar{a}) = s(\bar{b})$. Again since $\tilde{t} \leq \bar{b}$, $\bar{a}$ is also a reduction of $\bar{b}$. Let $\mathcal{A}_{n+1,a}$ be a maximal antichain in the dense below a set $\{\bar{a}(t) : t \leq_{\mathbb{A}} a\}$ and let $\mathcal{A}_{n+1} = \bigcup_{a\in\mathcal{A}_n} \mathcal{A}_{n+1,a}$. ■

3.1.4. *Canonical projection of a name for a slalom*

DEFINITION 3.10. Let $\mathbb{B}$ be a partial order and $y \in \mathbb{B}$. Let $\sigma \in {}^{<\omega}({}^{<\omega}[\omega])$ be such that $(\forall i < |\sigma|)\ |\sigma(i)| = i$, and for each $n \geq 1$ let $\mathcal{B}_n$ be a maximal antichain below y. We will say that the pair $(\check{\sigma}, \dot{\phi})$ is a *nice name for an element of* $\mathbb{L}$ *below* y, where $\dot{\phi} = \{(b, \sigma(b))\}_{b\in\mathcal{B}_n,\, n\geq 1}$, if the following conditions hold:

(1) whenever $n \geq 1$ and $b \in \mathcal{B}_n$, then $\sigma(b) \in {}^{n}({}^{<\omega}[\omega])$;
(2) whenever $1 \leq n \leq |\sigma|$ and $b \in \mathcal{B}_n$, then $\sigma(b) = \sigma \restriction n$;
(3) whenever $n > |\sigma|$, then $\sigma \subset \sigma(b)$ and $(\forall i : |\sigma| \leq i < n)\ |\sigma(b)(i)| \leq |\sigma|$;
(4) whenever $m > n \geq |\sigma|$, $b \in \mathcal{B}_n$, $b' \in \mathcal{B}_m$ and b, b' are compatible, then $\sigma(b)$ is an initial segment of $\sigma(b')$.

Remark 3.11. If $(\check{\sigma}, \dot{\phi})$, where $\dot{\phi} = \{(b, \sigma(b))\}_{b \in \mathcal{B}_n, n \geq 1}$, is a nice name for an element of $\mathbb{L}$ below y, then $y \Vdash (\check{\sigma}, \dot{\phi}) \in \mathbb{L}$ and for all $n \in \omega$ and $b \in \mathcal{B}_n$ we have $b \Vdash \dot{\phi}\restriction n = \check{\sigma}(b)$.

Lemma 3.12. *Let $\mathbb{A}$ be a complete suborder of $\mathbb{B}$, $y \in \mathbb{B}$ and x a reduction of y to $\mathbb{A}$. Let $(\check{\sigma}, \dot{\phi})$, where $\dot{\phi} = \{(b, \sigma(b))\}_{b \in \mathcal{B}_n, n \geq 1}$, be a nice name for an element of $\mathbb{L}$ below y. Then there is an $\mathbb{A}$-nice name $(\check{\sigma}, \dot{\psi})$, where $\dot{\psi} = \{(a, \sigma(a))\}_{a \in \mathcal{A}_n, n \geq 1}$, for an element in $\mathbb{L}$ below x such that for all $n \geq 1$, for all $a \in \mathcal{A}_n$, there is $b \in \mathcal{B}_n$ such that a is a reduction of b and $\sigma(a) = \sigma(b)$.*

Proof. Similar to the proof of Lemma 3.8. ■

Another forcing notion of interest for us is Hechler forcing $\mathbb{D}$. Recall that it consists of pairs $(s, f) \in {}^{<\omega}\omega \times {}^{\omega}\omega$ such that $s \subseteq f$ and of the extension relation given by $(s, f) \leq (t, g)$ iff s end-extends t and $(\forall i \in \omega)\ g(i) \leq f(i)$. Clearly, if $\mathbb{A}$, $\mathbb{B}$, x and y are as in the statement of Lemma 3.8, and y forces that $(\check{s}, \dot{f})$ is a condition in $\mathbb{D}$ where $\dot{f}$ is a nice name for a real below y, then $\dot{f}$ has a canonical projection $\dot{f}'$ below x such that x forces that $(\check{s}, \dot{f}')$ is a Hechler condition.

3.1.5. *Suslin, σ-Suslin and good σ-Suslin posets.* Recall that a *Suslin poset* is a poset $(\mathbb{S}, \leq_{\mathbb{S}})$ such that $\mathbb{S}(\subseteq \omega^\omega)$, $\leq_{\mathbb{S}}$ and $\perp_{\mathbb{S}}$ have $\mathbf{\Sigma}^1_1$ definitions (with parameters in the ground model). For a Suslin forcing $\mathbb{S}$, the ordering $\leq_{\mathbb{S}}$ will be defined by the $\mathbf{\Sigma}^1_1$ predicate in whatever model we work in (that has a code for $\leq_{\mathbb{S}}$). The key property of Suslin forcings that we need is the following well-known fact. A detailed proof of it can be found in [5] (in Lemma 3.13 below take $\dot{\mathbb{A}}$ to be a $\mathbb{P}$-name for $\mathbb{S}$, and $\dot{\mathbb{B}}$ to be a $\mathbb{Q}$-name for $\mathbb{S}$).

Lemma 3.13. *Let $\mathbb{P}$ and $\mathbb{Q}$ be posets and let $\mathbb{S}$ be a ccc Suslin poset. If $\mathbb{P} \lessdot \mathbb{Q}$ then $\mathbb{P} * \dot{\mathbb{S}} \lessdot \mathbb{Q} * \dot{\mathbb{S}}$ (where $\dot{\mathbb{S}}$ denotes the name of $\mathbb{S}$ for the relevant poset).*

We will work with the following strengthening of the notion of Suslin forcing:

Definition 3.14. Let $(\mathbb{S}, \leq_{\mathbb{S}})$ be a Suslin forcing notion, whose conditions can be written in the form (s, f) where $s \in {}^{<\omega}\omega$ and $f \in {}^{\omega}\omega$. We will say that $\mathbb{S}$ is *n-Suslin* if whenever $(s, f) \leq_{\mathbb{S}} (t, g)$ and (t, h) is a condition in $\mathbb{S}$ such that $h\restriction n \cdot |s| = g\restriction n \cdot |s|$, then (s, f) and (t, h) are compatible. A forcing notion is called *σ-Suslin* if it is n-Suslin for some n.

Clearly, if $\mathbb{S}$ is n-Suslin and $m \geq n$, then $\mathbb{S}$ is also m-Suslin. If $\mathbb{S}$ is n-Suslin and (s, f) and (s, g) are conditions in $\mathbb{S}$ such that $f\restriction n \cdot |s| = g\restriction n \cdot |s|$, then (s, f) and (s, g) are compatible. Thus every σ-Suslin forcing notion is σ-linked and so has the Knaster property. Hechler forcing $\mathbb{H}$ is 1-Suslin, and localization $\mathbb{L}$ is 2-Suslin.

Definition 3.15. Let $(\mathbb{S}, \leq_{\mathbb{S}})$ be a Suslin forcing notion, whose conditions can be written in the form (s, f) where $s \in^{<\omega} \omega$ and $f \in {}^\omega\omega$. Let $\mathbb{B}$ be a partial order. The pair $(\check{s}, \dot{f})$ is a *nice name for a condition* in $\mathbb{S}$ below $y \in \mathbb{B}$ if $\dot{f}$ is a nice name for a real below y and $y \Vdash_{\mathbb{B}} (\check{s}, \dot{f}) \in \dot{\mathbb{S}}$.

Suppose $(\check{s}, \dot{f})$ is a nice $\mathbb{B}$-name for a condition in $\mathbb{S}$ below $y \in \mathbb{B}$, where $\mathbb{B}$ is an arbitrary partial order. Let $\mathbb{A}$ be a complete suborder of $\mathbb{B}$ and let $x \in \mathbb{A}$ be a reduction in y. By Lemma 3.8 there is a projection $\dot{g}$ of $\dot{f}$ below x. Say $\dot{f} = \{(b, s(b))\}_{b \in \mathcal{B}_n, n \geq 1}$ and $\dot{g} = \{(a, s(a))\}_{a \in \mathcal{A}_n, n \geq 1}$. For every $a \in \mathcal{A}_1$ we can choose $b \in \mathcal{B}_1$, denote it $\psi(b)$, such that a is a reduction of b, and so we can define a mapping $\psi : \mathcal{A}_1 \to \mathcal{B}_1$. However ψ might not be surjective, which implies that for a $\mathbb{B}$-generic filter G it very well might be the case that $\dot{f}[G] \neq \dot{g}[G \cap \mathbb{A}]$. This gives rise to the following definition.

Definition 3.16. Let $(\mathbb{S}, \leq_{\mathbb{S}})$ be a Suslin forcing notion, whose conditions can be written in the form (s, f) where $s \in^{<\omega} \omega$ and $f \in {}^\omega\omega$.

(1) Let $\mathbb{A}$, $\mathbb{B}$ be partial orders such that $\mathbb{A} \lessdot \mathbb{B}$. Let $x \in \mathbb{A}$ be a reduction of $y \in \mathbb{B}$ and let $(\check{s}, \dot{f})$ be a nice name for a condition in $\mathbb{S}$ below $y \in \mathbb{B}$. If $\dot{g}$ is a canonical projection of $\dot{f}$ below x such that $x \Vdash_{\mathbb{B}} (\check{s}, \dot{g}) \in \dot{\mathbb{S}}$, we will say that $(\check{s}, \dot{g})$ is a *canonical projection of the nice name* $(\check{s}, \dot{f})$ below x.

(2) $\mathbb{S}$ is called *good* if whenever $\mathbb{A}$, $\mathbb{B}$ are partial orders, $\mathbb{A} \lessdot \mathbb{B}$, $x \in \mathbb{A}$, $y \in \mathbb{B}$ and x is a reduction of y, then every nice name for a condition in $\mathbb{S}$ below y has a canonical projection below x.

An immediate corollary of Lemma 3.12 is that the localization poset $\mathbb{L}$ is a good σ-Suslin forcing notion. It is straightforward to verify that the Hechler poset $\mathbb{D}$ is also good σ-Suslin.

3.1.6. *Finite function posets*

Definition 3.17. Let A be a fixed set and let $\mathbb{Q}$ be a poset of pairs $p = (s^p, F^p)$ where $s^p \subseteq A \times \omega \times \omega$ is finite, for every $a \in A$, $s^p_a = \{(n, m) : (a, n, m) \in s\}$ is a finite partial function and $F \in [\widehat{W_A}]^{<\omega}$. For $p \in \mathbb{Q}$ let $\mathrm{oc}(s^p) = \{a : (\exists n, m)\ (a, n, m) \in s^p\}$ and let $\mathrm{oc}(p) = \mathrm{oc}(s^p) \cup \{a :$ a is a letter from a word in $F^p\}$. For $B \subseteq A$ let $p{\restriction}B = (s^p \cap B \times \omega \times \omega, F^p)$, let $p{\parallel}{\restriction}B = (s^p \cap B \times \omega \times \omega, F^p \cap \widehat{W_B})$ and let $\mathrm{dom}(\mathbb{Q}) = A$. Then $\mathbb{Q}$ is a *finite function poset* (*with side conditions*) if:

(i) ("Restrictions"). Whenever p and q are conditions in $\mathbb{Q}$, $B \subseteq A$ then

- $p{\restriction}B$, $p{\parallel}{\restriction}B$ are conditions in $\mathbb{Q}$, and $p{\restriction}B \leq p{\parallel}{\restriction}B$;
- if $p \leq q$ then $p{\parallel}{\restriction}B \leq q{\parallel}{\restriction}B$.

(ii) ("Extensions"). Whenever $p = (s, F) \in \mathbb{Q}$

- and $t \subseteq A \times \omega \times \omega$ is finite such that $\mathrm{oc}(p) \cap \mathrm{oc}(t) = \emptyset$, then $(s \cup t, F) \leq p$;
- and $E \in [\widehat{W_A}]^{<\omega}$ contains F, then $(s, E) \leq (s, F)$.

Whenever $B \subseteq \mathrm{dom}(\mathbb{Q})$, we denote by $\mathbb{Q}_B$ the suborder $\{p \| \restriction B : p \in \mathbb{Q}\}$. Thus in particular if $L_0 := \mathrm{dom}(\mathbb{Q})$ then $\mathbb{Q}_{L_0} = \mathbb{Q}$.

Definition 3.18. Let $\mathbb{Q}$ be a finite function poset. We say that $\mathbb{Q}$ has the *strong embedding property* if whenever $A_0 \subseteq \mathrm{dom}(\mathbb{Q})$ and $p = (s, F) \in \mathbb{Q}$, then there is $t_0 \subseteq (\mathrm{oc}(p) \cap A_0) \times \omega \times \omega$ such that $s \restriction A_0 \subseteq t_0$, $(t_0, F \cap \widehat{W_{A_0}}) \leq_{\mathbb{Q}_{\mathrm{oc}(p) \cap A_0}} p \| \restriction A_0$, and whenever $(t, E) \leq_{\mathbb{Q}} (t_0, F \cap \widehat{W_{A_0}})$ is such that $\mathrm{oc}(t)$ and $\mathrm{oc}(E)$ are disjoint from $\mathrm{oc}(p) \setminus A_0$, then $(t \cup s, F) \leq (s, F)$ and $(t \cup s, E) \leq (t, E)$. We say that $(t_0, F \cap \widehat{W_{A_0}})$ is a *strong reduction* of p and $(s \cup t, E \cup F)$ a *canonical extension* of (s, F) and (t, E).

Remark 3.19. Note that if $\mathbb{Q}$ is a finite function poset with the strong embedding property, then whenever $A \subseteq B \subseteq \mathrm{dom}(\mathbb{Q})$, $C \subseteq \mathrm{dom}(\mathbb{Q})$ are such that $C \cap B = A$, for every condition $p \in \mathbb{Q}_B$ there is $p_0 \leq_{\mathbb{Q}_A} p \| \restriction A$ such that $\mathrm{oc}(p_0) \subseteq \mathrm{oc}(p) \cap A$ and if q_0 is a $\mathbb{Q}_C$-extension of p_0, then q_0 is compatible with p. We will say that p_0 is a *strong $\mathbb{Q}_A$-reduction* of p. The existence of strong reductions implies in particular that for every $A \subseteq \mathrm{dom}(\mathbb{Q})$ the poset $\mathbb{Q}_A$ is a complete suborder of $\mathbb{Q}$. We will refer to this property as the *complete embedding property*.

Lemma 3.20. *$\mathbb{Q}_{A,\rho}$ is a finite function poset with the strong embedding property.*

Another example of a finite function poset with the strong embedding property is the following forcing notion $\mathbb{D}_A$. Let A be a non-empty set and let $\mathbb{D}_A$ be the poset of all pairs (s^p, F^p) where $s^p \subseteq A \times \omega \times 2$ is a finite set such that for all $a \in A$, $s^p_a = \{(n, m) : (a, n, m) \in s\}$ is a finite partial function and $F \in [A]^{<\omega}$. The condition q is said to *extend* p if $s^q \supset s^p$, $F^q \supset F^p$ and for all distinct $a, b \in F^p$ we have $s^a_q \cap s^b_q \subseteq s^a_p \cap s^b_p$. If $|A| > \omega$, then $\mathbb{D}_A$ adds a maximal almost disjoint family of size $|A|$.

3.2. Two-sided templates. If $(L, \leq)$ is a linearly ordered set and $x \in L$, we let $L_x = \{y \in L : y < x\}$ and $L^{=}_x = \{y \in L : y \leq x\}$. If $L_0 \subseteq L$ is a distinguished subset of L and $A \subseteq L$, then the L_0-*closure* of A is defined as

$$\mathrm{cl}_{L_0}(A) = A \cup \bigcup_{x \in A} (L_x \cap L_0),$$

and we will say that A is L_0-*closed* if $A = \mathrm{cl}_{L_0}(A)$. Note that $\mathrm{cl}_{L_0}(A)$ is the smallest set $B \supseteq A$ with the property that if $x \in B$ then $L_x \cap L_0 \subseteq B$. We will usually drop mention of L_0 when it is clear from the context, and write "closed" instead of "L_0-closed" and cl instead of cl_{L_0}.

Definition 3.21 (J. Brendle [4]). A *two-sided template* is a 4-tuple $\mathcal{T} = ((L, \leq), \mathcal{I}, L_0, L_1)$ consisting of a linear ordering $(L, \leq)$, a family $\mathcal{I} \subseteq \mathcal{P}(L)$, and a decomposition $L = L_0 \cup L_1$ into two disjoint pieces such that:

(1) $\mathcal{I}$ is closed under finite intersections and unions, and $\emptyset, L \in \mathcal{I}$.
(2) If $x, y \in L$, $y \in L_1$ and $x < y$ then there is $A \in \mathcal{I}$ such that $A \subseteq L_y$ and $x \in A$.
(3) If $A \in \mathcal{I}$, $x \in L_1 \setminus A$, then $A \cap L_x \in \mathcal{I}$.
(4) The family $\{A \cap L_1 : A \in \mathcal{I}\}$ is well-founded when ordered by inclusion.
(5) All $A \in \mathcal{I}$ are L_0-closed.

Given a two-sided template $\mathcal{T}$ as above, $x \in L$ and $A \in \mathcal{I}$, we define

$$\mathcal{I}_A = \{B \in \mathcal{I} : B \subset A\}, \quad \mathcal{I}_x = \{B \in \mathcal{I} : B \subseteq L_x\}$$

and $\mathcal{I}_{A,x} = \mathcal{I}_A \cap \mathcal{I}_x$. Finally we define the rank function $\mathrm{Dp} : \mathcal{I} \to \mathbb{ON}$ by letting $\mathrm{Dp}(A) = 0$ for $A \subseteq L_0$ and $\mathrm{Dp}(A) = \sup\{\mathrm{Dp}(B) + 1 : B \in \mathcal{I} \wedge B \cap L_1 \subset A \cap L_1\}$. We define $\mathrm{Rk}(\mathcal{T})$, the *rank* of $\mathcal{T}$, to be $\mathrm{Rk}(\mathcal{T}) = \mathrm{Dp}(L)$.

If $A \subseteq L$ then $\mathcal{T}_A$ is the template $((A, \leq), \mathcal{I}{\restriction}A, L_0 \cap A, L_1 \cap A)$, where

$$\mathcal{I}{\restriction}A = \{A \cap B : B \in \mathcal{I}\}.$$

Note that if $A \in \mathcal{I}$ then $\mathrm{Rk}(\mathcal{T}_A) = \mathrm{Dp}(A)$. Moreover, if $A \subseteq L$ is arbitrary, then $\mathrm{Rk}(\mathcal{T}_A) \leq \mathrm{Rk}(\mathcal{T})$.

3.3. Iteration along a two-sided template. We are now ready to define the iteration along a two-sided template. This definition is a generalization of the definition of iterating "Hechler forcing and adding a mad family along a template" given in [4].

Definition 3.22. Let $\mathcal{T} = ((L, \leq), \mathcal{I}, L_0, L_1)$ be a two-sided template, $\mathbb{Q}$ a finite function forcing with the strong embedding property such that $L_0 = \mathrm{dom}(\mathbb{Q})$ and $\mathbb{S}$ a good σ-Suslin forcing notion. The poset $\mathbb{P}(\mathcal{T}, \mathbb{Q}, \mathbb{S})$ is defined recursively according to the following clauses:

(1) If $\mathrm{Rk}(\mathcal{T}) = 0$, then $\mathbb{P}(\mathcal{T}, \mathbb{Q}, \mathbb{S}) = \mathbb{Q}_{L_0}$ $(= \mathbb{Q})$.
(2) Assume that for all $\mathcal{T}$ with $\mathrm{Rk}(\mathcal{T}) < \kappa$, $\mathbb{P}(\mathcal{T}, \mathbb{Q}, \mathbb{S})$ has been defined (and *is* a poset, see comment below). Let $\mathcal{T}$ be a two-sided template of rank κ, and for $B \in \mathcal{I}$ of $\mathrm{Dp}(B) < \kappa$ let $\mathbb{P}_B = \mathbb{P}(\mathcal{T}_B, \mathbb{Q}, \mathbb{S})$. We define $\mathbb{P} = \mathbb{P}(\mathcal{T}, \mathbb{Q}, \mathbb{S})$ as follows:
    (i) $\mathbb{P}$ consists of all pairs $P = (p, F^p)$ where p is a finite partial function with $\mathrm{dom}(p) \subseteq L$, $P{\restriction}L_0 := (p{\restriction}L_0, F^p) \in \mathbb{Q}$, and if $x_p := \max\{\mathrm{dom}(p) \cap L_1\}$ is defined then there is $B \in \mathcal{I}_{x_p}$ (called a *witness* to $P \in \mathbb{P}$) such that $P{\|}{\restriction}L_{x_p} := (p{\restriction}L_{x_p}, F^p \cap \widehat{W_B}) \in \mathbb{P}_B$,

$p(x_p) = (\check{s}^p_x, \dot{f}^p_x)$, where $s^p_x \in {}^{<\omega}\omega$, $\dot{f}^p_x$ is a $\mathbb{P}_B$-name for a real and $(P|\restriction L_{x_p}, p(x_p)) \in \mathbb{P}_B * \dot{\mathbb{S}}$.

(ii) For $P, Q \in \mathbb{P}$, let $Q \leq_{\mathbb{P}} P$ iff $\mathrm{dom}(p) \subseteq \mathrm{dom}(q)$, $(q\restriction L_0, F^q) \leq_{\mathbb{Q}} (p\restriction L_0, F^p)$, and if x_p is defined then either

(ii.a) $x_p < x_q$ and there exists $B \in \mathcal{I}_{x_q}$ such that $P|\restriction L_{x_q}, Q|\restriction L_{x_q} \in \mathbb{P}_B$ and $Q|\restriction L_{x_q} \leq_{\mathbb{P}_B} P|\restriction L_{x_q}$, or

(ii.b) $x_p = x_q$ and there exists $B \in \mathcal{I}_{x_q}$ witnessing $P, Q \in \mathbb{P}$ and such that $(Q|\restriction L_{x_q}, q(x_q)) \leq_{\mathbb{P}_B * \dot{\mathbb{S}}} (P|\restriction L_{x_p}, p(x_p))$.

Below we will call B as in (ii.a) or (ii.b) a *witness* to $Q \leq_{\mathbb{P}} P$.

Whenever the side condition F^p is clear from the context, we will denote the condition $P = (p, F^p)$ simply by the finite partial function p. Furthermore, for $A \subseteq L$, let $P\restriction A = (p\restriction A, F^p)$ and $P|\restriction A = (p\restriction A, F^p \cap \widehat{W_A})$. Note also that if $P = (p, F^p)$ then $p\restriction L_0 \subseteq (\mathrm{dom}(p) \cap L_0) \times \omega \times \omega$. The definition is recursive and it is not clear to what extent it succeeds in defining a poset. However this will follow from Lemma 3.23 below, which establishes not only transitivity but also a strong version of the complete embedding property, which is necessary for this definition to succeed. This lemma is a generalization of the Main Lemma of [4]. We note that if $A \in \mathcal{I}$ then it is clear from the definition that $\mathbb{P}_A := \mathbb{P}(\mathcal{T}_A, \mathbb{Q}, \mathbb{S})$ is a subset of $\mathbb{P}(\mathcal{T}, \mathbb{Q}, \mathbb{S})$ and that the relation $\leq_{\mathbb{P}_A}$ is contained in $\leq_{\mathbb{P}}$. The above definition also defines $\mathbb{P}_A = \mathbb{P}(\mathcal{T}_A, \mathbb{Q}, \mathbb{S})$ for arbitrary $A \subseteq L$.

Lemma 3.23 (Completeness of embeddings). *Let $\mathcal{T} = ((L, \leq), \mathcal{I}, L_0, L_1)$ be a template, let $\mathbb{Q}$ be a finite function poset with $L_0 = \mathrm{dom}(\mathbb{Q})$ which satisfies the strong embedding property, and let $\mathbb{S}$ be a good σ-Suslin poset. Let $B \in \mathcal{I}$, and let $A \subset B$ be closed. Then $\mathbb{P}_B$ is a partial order, $\mathbb{P}_A \subset \mathbb{P}_B$ and even $\mathbb{P}_A \lessdot \mathbb{P}_B$. Furthermore, any $P = (p, F^p) \in \mathbb{P}_B$ has a reduction $P_0 = (p_0, F^{p_0}) \in \mathbb{P}_A$ such that*

(i) *$\mathrm{dom}(p_0) = \mathrm{dom}(p) \cap A$, $F^{p_0} = F^p \cap \widehat{W_A}$,*

(ii) *$s^{p_0}_x = s^p_x$ for all $x \in \mathrm{dom}(p_0) \cap L_1$,*

(iii) *$P_0\restriction L_0 = (p_0\restriction L_0, F^{p_0})$ is a strong $\mathbb{Q}_{A\cap L_0}$-reduction of $P\restriction L_0 = (p\restriction L_0, F^p)$,*

and such that whenever $D \in \mathcal{I}$, $C \subseteq L$, C is L_0-closed, $B, C \subseteq D$ and $C \cap B = A$, then for every $Q_0 \in \mathbb{P}_C$ which extends P_0 there is $Q \in \mathbb{P}_D$ which is a common extension of Q_0 and P.

We refer to the reduction P_0 of the condition P from Lemma 3.23 as *a canonical reduction from $\mathbb{P}_B$ to $\mathbb{P}_A$*. For $P \in \mathbb{P}(\mathcal{T}, \mathbb{Q}, \mathbb{S})$ let $\mathrm{oc}(P) = \mathrm{oc}(P\restriction L_0)$ and let $\mathrm{dom}(P) = \mathrm{dom}(p) \cup \mathrm{oc}(P)$. Also, for $B \subseteq L$ we will write $\mathbb{Q}_B$ for the set $\mathbb{Q}_{B\cap L_0}$. The lemma is proved by induction on the rank of $\mathcal{T}$. It uses the following lemmas, which are helpful for making simple manipulations with the conditions of $\mathbb{P}(\mathcal{T},\mathbb{Q},\mathbb{S})$. In Lemmas 3.24 through 3.27 assume that $\mathcal{T}$, $\mathbb{Q}$ and $\mathbb{S}$ are as in Definition 3.22 and that the Completeness of Embeddings Lemma 3.23 has been established for all templates of rank $<\mathrm{Rk}(\mathcal{T})$. Let $\mathbb{P}=\mathbb{P}(\mathcal{T},\mathbb{Q},\mathbb{S})$.

LEMMA 3.24. *If $P=(p,F^P)$ and $Q=(q,F^q)$ are conditions in $\mathbb{P}$ such that* $\mathrm{dom}(P)$ *and* $\mathrm{dom}(Q)$ *are contained in L_x for some $x\in L_1$ and $Q\le_{\mathbb{P}} P$, then there is $B\in\mathcal{I}_x$ such that $Q\le_{\mathbb{P}_B} P$.*

Proof. If x_p is defined and $x_p=x_q$ (resp. $x_p<x_q$), let $B'\in\mathcal{I}_{x_p}$ (resp. $B'\in\mathcal{I}_{x_q}$) be a witness to $Q\le_P P$. Using Definition 3.21, find $B\in\mathcal{I}_x$ such that $B'\subseteq B$ and $\mathrm{dom}(P)\cup\mathrm{dom}(Q)\subseteq B$. Then $B'\in\mathcal{I}_{B,x_p}$ (resp. $B'\in\mathcal{I}_{B,x_q}$) is a witness to $Q\le_{\mathbb{P}_B} P$. If x_p is not defined and $B\in\mathcal{I}_x$ is such that $\mathrm{dom}(P)\cup\mathrm{dom}(Q)\subseteq B$, then since $Q{\restriction}L_0\le_{\mathbb{Q}_B} P{\restriction}L_0$ we obtain $Q\le_{\mathbb{P}_B} P$. ∎

LEMMA 3.25. *Let $P=(p,F^p)$ and $Q=(q,F^q)$ be conditions in $\mathbb{P}$ and let $x_0\in L$. Then $Q\|{\restriction}L_{x_0}\in\mathbb{P}$, $Q\|{\restriction}L^{=}_{x_0}\in\mathbb{P}$, and if $Q\le_{\mathbb{P}} P$ then $Q\|{\restriction}L_{x_0}\le_{\mathbb{P}} P\|{\restriction}L_{x_0}$ and $Q\|{\restriction}L^{=}_{x_0}\le_{\mathbb{P}} P\|{\restriction}L^{=}_{x_0}$.*

Proof. The proofs of $Q\|{\restriction}L_{x_0}\in\mathbb{P}$ and $Q\|{\restriction}L_{x_0}\le P\|{\restriction}L_{x_0}$ proceed by induction on $n_q=|\mathrm{dom}(q)\cap L_1|$. The case $n_q=0$ follows by Definition 3.17. Thus suppose each of those is true whenever $n_q<n$ and let $n_q=n$. To see that $Q\|{\restriction}L_{x_0}\in\mathbb{P}$ note that if $x_q<x_0$ and B is a witness to $Q\in\mathbb{P}$, then B also witnesses $Q\|{\restriction}L_{x_0}\in\mathbb{P}$. If $x_0\le x_q$, then $n_{q{\restriction}L_{x_0}}<n$, and so we can use the inductive hypothesis.

If $\mathrm{dom}(p{\restriction}L_{x_0})\subseteq L_0$, then $Q\|{\restriction}L_{x_0}\le_{\mathbb{P}} P\|{\restriction}L_{x_0}$ follows from Definition 3.17. Suppose $n_{p{\restriction}L_{x_0}}\ne 0$ and let B be a witness to $Q\le P$. If $x_q<x_0$, then B also witnesses $Q\|{\restriction}L_{x_0}\le P\|{\restriction}L_{x_0}$. If $x_0<x_q$, then $Q\|{\restriction}L_{x_q}\le_{\mathbb{P}_B} P\|{\restriction}L_{x_q}$, and since $\le_{\mathbb{P}_B}\subseteq\le_{\mathbb{P}}$ we find that $Q\|{\restriction}L_{x_q}\le_{\mathbb{P}} P\|{\restriction}L_{x_q}$. If $x_0=x_q$ we are done, and if $x_0<x_q$ then $n_{q{\restriction}L_{x_q}}<n$ and so $Q\|{\restriction}L_{x_0}\le_{\mathbb{P}} P\|{\restriction}L_{x_0}$ by the inductive hypothesis.

The $L^{=}_x$ case is proved analogously. ∎

LEMMA 3.26. *Let $P=(p,F^p)$ and $Q=(q,F^q)$ be conditions in $\mathbb{P}$. If $\mathrm{dom}(p)\subseteq\mathrm{dom}(q)$, $Q{\restriction}L_0\le_{\mathbb{Q}} P{\restriction}L_0$ and $Q\|{\restriction}L^{=}_{x_p}\le_{\mathbb{P}} P\|{\restriction}L^{=}_{x_p}$, then $Q\le_{\mathbb{P}} P$.*

Proof. If x_p is not defined, the claim is straightforward by Definition 3.22. Thus assume x_p is defined. Note that $x_q\ge x_p$. If $x_q=x_p$, then if B is a witness to $Q\|{\restriction}L^{=}_{x_p}\le_{\mathbb{P}} P\|{\restriction}L^{=}_{x_p}$, then B is also a witness to $Q\le_{\mathbb{P}} P$. Thus suppose $x_q>x_p=x$. Let $\{x_j\}_{j=1}^n$ be an increasing (in the linear order L) enumeration of $(\mathrm{dom}(Q)\cap L_1)\setminus L^{=}_{x_p}=(\mathrm{dom}(q)\cap L_1)\setminus L^{=}_{x_p}$, and let $H\in\mathcal{I}_{x_p}$ be a witness to $Q\|{\restriction}L^{=}_{x_p}\le_{\mathbb{P}} P\|{\restriction}L^{=}_{x_p}$. In particular $x_n=x_q$. Since $(\mathrm{dom}(\mathbb{Q})\cap L_{x_1})\setminus L_{x_p}$ and $(\mathrm{dom}(P)\cap L_{x_1})\setminus L_{x_p}$ are finite, by Definition 3.21

there is a set $H_1 \in \mathcal{I}_{x_1}$ such that H, $\mathrm{dom}(Q|{\restriction}L_{x_1})$ and $\mathrm{dom}(P|{\restriction}L_{x_1})$ are contained in H_1. Then $H \in \mathcal{I}_{H_1}$ is a witness to $\mathbb{Q}|{\restriction}L_{x_1} \leq_{\mathbb{P}_{H_1}} P|{\restriction}L_{x_1}$. Similarly we can find $H_2 \in \mathcal{I}_{x_2}$ such that H_1, $\mathrm{dom}(Q|{\restriction}L_{x_2})$ and $\mathrm{dom}(P|{\restriction}L_{x_2})$ are contained in H_2. Then $H_1 \in \mathcal{I}_{H_2,x_1}$ is a witness to $Q|{\restriction}L_{x_2} \leq_{\mathbb{P}_{H_2}} P|{\restriction}L_{x_2}$. Thus in finitely many steps we can find an increasing sequence $\{H_j\}_{j=1}^n$ of elements of $\mathcal{I}$ such that for all j, $H_j \subseteq L_{x_j}$, $\mathrm{dom}(Q|{\restriction}L_{x_j})$, $\mathrm{dom}(P|{\restriction}L_{x_j})$ are contained in H_j, and H_{j-1} is a witness to $Q|{\restriction}L_{x_j} \leq_{\mathbb{P}_{H_j}} P|{\restriction}L_{x_j}$. Then H_{n-1} is a witness to $Q|{\restriction}L_{x_n} \leq_{\mathbb{P}_{H_n}} P|{\restriction}L_{x_n}$, and so $H_n \in \mathcal{I}_{x_n}$ is a witness to $Q \leq_{\mathbb{P}} P$. ∎

Lemma 3.27. *Let $Q = (q, F^q) = Q|{\restriction}L_x^=$ with $x = \max\{\mathrm{dom}(q) \cap L_1\}$ be a condition in $\mathbb{P}$. Let $P = (p, F^p)$ be a condition such that $(P|{\restriction}L_x){\restriction}L_0$ is a strong $\mathbb{Q}_{L_0 \cap L_x}$-reduction of $P{\restriction}L_0$ and $Q \leq_{\mathbb{P}} P|{\restriction}L_x^=$. Then $Q \mathbin{\bar{\ltimes}_x} P = (q \mathbin{\bar{\ltimes}_x} p, F^q \cup F^p)$ is a common extension of Q and P where $q \mathbin{\bar{\ltimes}_x} p = q \cup p{\restriction}L \setminus L_x^=$.*

Proof. Since $q \mathbin{\bar{\ltimes}_x} p{\restriction}L_0 = q{\restriction}L_0 \cup p{\restriction}L_0 \setminus L_x^=$ and $(P|{\restriction}L_x){\restriction}L_0$ is a strong $\mathbb{Q}_{L_0 \cap L_x}$-reduction of $P{\restriction}L_0$, we see that $(Q \mathbin{\bar{\ltimes}_x} P){\restriction}L_0 \leq_{\mathbb{Q}} P{\restriction}L_0$. On the other hand $\mathrm{dom}(p{\restriction}L_0 \setminus L_x^=) \cap \mathrm{dom}(Q) = \emptyset$ and so $(Q \mathbin{\bar{\ltimes}_x} P){\restriction}L_0 \leq Q{\restriction}L_0$.

Let $n_p := |\mathrm{dom}(p) \cap L_1 \setminus L_x|$. Suppose $n_p = 0$. Then $x_p < x = x_q = x_{q \bar{\ltimes}_x p}$. Since $(q \mathbin{\bar{\ltimes}_x} p){\restriction}L_x = q{\restriction}L_x$ and $F^q \supseteq F^p \cap \widehat{W}_{L_x \cap L_0}$ (because $Q|{\restriction}L_x \leq P|{\restriction}L_x$), we have $Q \mathbin{\bar{\ltimes}_x} P|{\restriction}L_x = Q|{\restriction}L_x$. This easily implies that $Q \mathbin{\bar{\ltimes}_x} P$ is a common extension of Q and P. Now suppose that the claim is true whenever $0 \leq n_p < n$ and let P be a condition with $n_p = n$. Then (without loss of generality) $x_p > x$ and $Q \leq (P|{\restriction}L_{x_p})|{\restriction}L_x^=$. By the inductive hypothesis $Q \mathbin{\bar{\ltimes}_x} (P|{\restriction}L_{x_p})$ is a condition in $\mathbb{P}$ extending both Q and $P|{\restriction}L_{x_p}$. By Lemma 3.24 there is $B_0 \in \mathcal{I}_{x_p}$ such that $Q \mathbin{\bar{\ltimes}_x} (P|{\restriction}L_{x_p}) \leq_{\mathbb{P}_{B_0}} Q, P|{\restriction}L_{x_p}$. Let B_1 be a witness to $P \in \mathbb{P}$. Thus $P|{\restriction}L_{x_p} \in \mathbb{P}_{B_1}$ and $P|{\restriction}L_{x_p} \Vdash_{\mathbb{P}_{B_1}} p(x_p) \in \dot{\mathbb{S}}$. Then $B = B_0 \cup B_1$ is in $\mathcal{I}_{x_p}$. Since x_p does not belong to any of B_0, B_1 and B, all of those sets are of smaller rank than $\mathrm{Rk}(\mathcal{T})$ (see discussion right after the statement of Lemma 3.23), and so by the inductive hypothesis $\mathbb{P}_{B_0}, \mathbb{P}_{B_1}$ completely embed into $\mathbb{P}_B$. This implies that $Q \mathbin{\bar{\ltimes}_x} (P|{\restriction}L_{x_p}) \leq_{\mathbb{P}_B} Q, P|{\restriction}L_{x_p}$, and so in particular $Q \mathbin{\bar{\ltimes}_x} (P|{\restriction}L_{x_p}) \Vdash_{\mathbb{P}_B} p(x_p) \in \dot{\mathbb{S}}$. Then B is also a witness to $Q \mathbin{\bar{\ltimes}_x} P \leq_{\mathbb{P}} P$. Since $x_q < x_{q \bar{\ltimes}_x p} = x_p$, the set B_0 is a witness to $Q \mathbin{\bar{\ltimes}_x} P \leq_{\mathbb{P}} Q$. ∎

Proof of Lemma 3.23. We use recursion on the rank of the underlying template. The $\mathrm{Rk}(\mathcal{T}_B) = 0$ case is clear. So assume that the lemma holds for all templates of rank $< \alpha$, and let $\mathrm{Rk}(\mathcal{T}_B) = \alpha$. Let $\mathbb{P} = \mathbb{P}_B$.

Transitivity: To see that $\leq_{\mathbb{P}}$ is transitive, fix $P_0, P_1, P_2 \in \mathbb{P}$ such that $P_1 \leq_{\mathbb{P}} P_0$ and $P_2 \leq_{\mathbb{P}} P_1$, and assume that x_{p_0} is defined (since otherwise there is nothing to show). Fix witnesses $B_1 \in \mathcal{I}_{x_{p_1}}$ and $B_2 \in \mathcal{I}_{x_{p_2}}$ to $P_1 \leq_{\mathbb{P}} P_0$ and $P_2 \leq_{\mathbb{P}} P_1$. Since $\mathrm{Dp}(B_1 \cup B_2) < \alpha$, the inductive hypothesis implies that $\mathbb{P}_{B_1}, \mathbb{P}_{B_2} \lessdot \mathbb{P}_{B_1 \cup B_2}$, and so $P_i|{\restriction}L_{x_{p_2}} = P_i|{\restriction}B_1 \cup B_2 \in \mathbb{P}_{B_1 \cup B_2}$ for $0 \leq i \leq 2$, and

$$P_2 \| {\restriction} L_{x_{p_2}} \leq_{\mathbb{P}_{B_1 \cup B_2}} P_1 \| {\restriction} L_{x_{p_2}} \leq_{\mathbb{P}_{B_1 \cup B_2}} P_0 \| {\restriction} L_{x_{p_2}}.$$

Thus by the inductive hypothesis we have $P_2\|{\restriction}L_{x_{p_2}} \leq_{\mathbb{P}_{B_1\cup B_2}} P_0\|{\restriction}L_{x_{p_2}}$. If $x_{p_0} < x_{p_2}$ then the definition of $\leq_{\mathbb{P}}$ yields $P_2 \leq_{\mathbb{P}} P_0$. So assume that $x_{p_0} = x_{p_2}$. It is clear that $p_i(x_{p_2})$ is a $\mathbb{P}_{B_1\cup B_2}$-name for $0 \leq i \leq 2$. Since $\mathbb{P}_{B_1}, \mathbb{P}_{B_2} \lessdot \mathbb{P}_{B_1\cup B_2}$, we must have $P_1\|{\restriction}L_{x_{p_2}} \Vdash_{\mathbb{P}_{B_1\cup B_2}} p_1(x_{p_2}) \leq_{\dot{\mathbb{S}}} p_0(x_{p_2})$ and $P_2\|{\restriction}L_{x_{p_2}} \Vdash_{\mathbb{P}_{B_1\cup B_2}} p_2(x_{p_2}) \leq_{\dot{\mathbb{S}}} p_1(x_{p_2})$. But then $P_2{\restriction}L_{x_{p_2}} \Vdash_{\mathbb{P}_{B_1\cup B_2}} p_1(x_{p_2}) \leq_{\dot{\mathbb{S}}} p_0(x_{p_2})$, and so $P_2\|{\restriction}L_{x_{p_2}} \Vdash_{\mathbb{P}_{B_1\cup B_2}} p_2(x_{p_2}) \leq_{\dot{\mathbb{S}}} p_0(x_{p_2})$. Thus

$$(P_2\|{\restriction}L_{x_{p_2}}, p_2(x_{p_2})) \leq_{\mathbb{P}_{B_1\cup B_2} * \dot{\mathbb{S}}} (P_0\|{\restriction}L_{x_{p_2}}, p_0(x_{p_2})),$$

as required.

Suborders: Let $A \subset B$ be closed, $B \in \mathcal{I}$ be given. We will show that $\mathbb{P}_A \subset \mathbb{P}_B$. Assume $R = (r, F^r) \in \mathbb{P}_A$ ([5]). Let $x = x_r$. By the definition of the iteration there is $\bar{A} \in (\mathcal{I}{\restriction}A)_x$ such that $R\|{\restriction}(A \cap L_x) \in \mathbb{P}_{\bar{A}}$ and $\dot{f}^r_x$ is a $\mathbb{P}_{\bar{A}}$-name.

Note that $\bar{A} \in \mathcal{I}{\restriction}A$ means that there is $B_0 \in \mathcal{I}$ such that $\bar{A} = B_0 \cap A$. On the other hand $A \subset B$, so $\bar{A} \subset B$ and hence $B_0 \cap A = B_0 \cap B \cap A$. But $\mathcal{I}$ is closed under finite intersections and so $B_0 \cap B \in \mathcal{I}$, even $B_0 \cap B \in \mathcal{I}_B$. So without loss of generality there is $\bar{B} \in \mathcal{I}_B$ (just take $\bar{B} = B_0 \cap B$) such that $\bar{A} = A \cap \bar{B}$. Since $\bar{A} \subseteq L_x$, we have $x \notin \bar{B}$. Then by Definition 3.21(3), $\bar{B} \cap L_x \in \mathcal{I}_B$. Therefore we can assume that $\bar{B} \subseteq L_x$. Thus $\bar{B} \subset B$ and $\mathrm{Dp}(\bar{B}) < \mathrm{Dp}(B) = \alpha$. By the inductive hypothesis, $\mathbb{P}_{\bar{A}} \subseteq \mathbb{P}_{\bar{B}}$ and $\mathbb{P}_{\bar{A}} \lessdot \mathbb{P}_{\bar{B}}$. Therefore $\dot{f}^r_x$ is a $\mathbb{P}_{\bar{B}}$-name as well. Thus $R\|{\restriction}L_x \in \mathbb{P}_{\bar{B}}$ and $\dot{f}^r_x$ is a $\mathbb{P}_{\bar{B}}$-name. That is, $R \in \mathbb{P}_B$.

Complete embeddings: Assume $P = (p, F^p) \in \mathbb{P}_B$. We will construct a canonical reduction P_0 of P from $\mathbb{P}_B$ to $\mathbb{P}_A$. Let $x = x_p$. By definition of the iteration, there is $\bar{B} \in \mathcal{I}_{B,x}$ such that $P\|{\restriction}L_x = \bar{P} \in \mathbb{P}_{\bar{B}}$ and $\dot{f}^p_x$ is a $\mathbb{P}_{\bar{B}}$-name, where $p(x) = (\check{s}^p_x, \dot{f}^p_x)$. Let $\bar{A} = A \cap \bar{B}$. Then $\bar{A} \in \mathcal{I}{\restriction}A$, $\bar{A} \subset \bar{B}$ and $\bar{A} \in \mathcal{P}(L_x)$. Since x does not belong to $\bar{A}$ and $\bar{B}$, the sets $\bar{A}$ and $\bar{B}$ are of rank smaller than $\mathrm{Rk}(\mathcal{T})$, and so by the inductive hypothesis $\mathbb{P}_{\bar{A}} \lessdot \mathbb{P}_{\bar{B}}$. Therefore $\bar{P} = (p{\restriction}L_x, F^p \cap \widehat{W}_{L_x \cap L_0})$ has a canonical reduction $\bar{P}_0 = (\bar{p}_0, F^p \cap \widehat{W}_{\bar{A}\cap L_0})$ from $\mathbb{P}_{\bar{B}}$ to $\mathbb{P}_{\bar{A}}$. By definition $\bar{P}_0{\restriction}L_0 \leq_{\mathbb{Q}_{\bar{A}}} (\bar{P}{\restriction}L_0)\|{\restriction}\bar{A}$, $\bar{P}_0{\restriction}L_0 = (\bar{p}_0{\restriction}L_0, F^p \cap \widehat{W}_{\bar{A}\cap L_0})$ where $\bar{p}_0{\restriction}L_0 \subseteq (\mathrm{oc}(p{\restriction}L_0) \cap \bar{A}) \times \omega \times \omega$ and $p{\restriction}(L_0 \cap \bar{A}) \subseteq \bar{p}_0{\restriction}L_0$. Now consider $(\bar{p}_0{\restriction}L_0 \cup p{\restriction}L_0 \setminus \bar{A}, F^p)$. By the definition of strong reduction we have $(\bar{p}_0{\restriction}L_0 \cup p{\restriction}L_0 \setminus \bar{A}, F^p) \leq_{\mathbb{Q}_B} P{\restriction}L_0$. Now take a strong $\mathbb{Q}_A$-reduction $(p_0{\restriction}L_0, F^p \cap \widehat{W}_{A\cap L_0})$ of $(\bar{p}_0{\restriction}L_0 \cup p{\restriction}L_0 \setminus \bar{A}, F^p)$. Then $(p_0{\restriction}L_0, F^{p_0})$ where $F^{p_0} := F^p \cap \widehat{W}_A$ is a strong $\mathbb{Q}_A$-reduction of $P{\restriction}L_0$ such

[5] Recall $\mathbb{P}_A = \mathbb{P}(\mathcal{T}_A, \mathbb{Q}, \mathbb{S})$ where $\mathcal{T}_A$ is the template $((A, \leq), \mathcal{I}{\restriction}A, L_0 \cap A, L_1 \cap A)$, and so in particular r is a finite partial function with domain contained in A.

that $p_0{\restriction}L_0 \cap \bar{A} \supseteq \bar{p}_0{\restriction}L_0$. Let $p_0{\restriction}L_1 \cap L_x = \bar{p}_0{\restriction}L_1 \cap L_x$ and let $P_0||{\restriction}L_x := (p_0{\restriction}L_x, F^{p_0})$. Then $P_0||{\restriction}L_x \le_{\mathbb{P}_{\bar{A}}} \bar{P}_0$, and so $P_0||{\restriction}L_x$ is a canonical reduction of $\bar{P}$. We can assume that $p(x)$ is a nice name for a condition in $\mathbb{S}$ below $\bar{P}$. If $x \in A$, let $p_0(x)$ be a canonical projection of $p(x)$ below $P_0||{\restriction}L_x$. With this the construction of $P_0 = (p_0, F^{p_0})$ is complete.

We will show that P_0 is a canonical projection of P from $\mathbb{P}_B$ to $\mathbb{P}_A$. For this assume $D \in \mathcal{I}$, $C \subseteq D$, C is L_0-closed, $B \cup C \subseteq D$, $A = B \cap C$ and $\mathrm{Dp}(D) = \alpha$. Let $Q_0 = (q_0, F^{q_0}) \le_{\mathbb{P}_C} P_0$. We will construct a common extension of Q_0 and P.

Case 1: $x \notin A$. Then clearly $x \notin C$. Let $y = \max(\mathrm{dom}(q_0) \cap L_x \cap L_1)$ (6). Then $y < x$. By Lemma 3.25, $Q_0||{\restriction}L_y^{=} \le_{\mathbb{P}_C} P_0||{\restriction}L_y^{=}$, and so there is $\bar{E}$ in $(\mathcal{I}{\restriction}C)_y$ witnessing this fact. Using 3.21 find $\bar{F} \in \mathcal{I}_{D,y}$ such that $\bar{E} = \bar{F} \cap C$. Now, let $\{y_i\}_{i\in k}$ enumerate $\mathrm{dom}(q_0) \cap L_x \setminus L_y$. By Definition 3.21(2), for all $i \in k$ there exists $G_i \in \mathcal{I}$ such that $y_i \in G_i$. Then $\bar{G} := D \cap \bigcup_{i\in k} G_i \in \mathcal{I}_{D,x}$ and $\mathrm{dom}(q_0) \cap L_x \setminus L_y \subseteq \bar{G}$. Let $\bar{D} = \bar{B} \cup \bar{F} \cup \bar{G}$, $\bar{C} = (\bar{G} \cap C) \cup \bar{E} \cup \bar{A}$ and note that $\bar{D} \in \mathcal{I}_{D,x}$, $\bar{C} \in (\mathcal{I}{\restriction}C)_x$. Clearly $\bar{C} \subseteq \bar{D}$, $\bar{C} \cap \bar{B} = \bar{A}$.

Note that $\bar{Q}_0 := Q_0||{\restriction}L_x \le_{\mathbb{P}_{\bar{C}}} P_0||{\restriction}L_x$ with witness $\bar{E}$ (observe that $\bar{E}$ also belongs to $(\mathcal{I}{\restriction}\bar{C})_y$). Passing to an extension if necessary, we can assume that $\bar{Q}_0{\restriction}L_0$ is a strong $\mathbb{Q}_{\bar{C}}$-reduction of $Q_0{\restriction}L_0$. Since $\mathrm{Dp}_{\mathcal{I}\restriction C}(\bar{C}) \le \mathrm{Dp}_{\mathcal{I}}(\bar{D}) < \mathrm{Dp}_{\mathcal{I}}(D) = \alpha$, we can apply the inductive hypothesis to $\bar{A}, \bar{B}, \bar{C}, \bar{D}$. Thus there is a common extension $\bar{Q} = (\bar{q}, F^{\bar{q}}) \le_{\mathbb{P}_{\bar{D}}} \bar{Q}_0, P||{\restriction}L_x$. With this we are ready to define a common extension $Q = (q, F^q)$ of Q_0 and P as follows:

Let $q' = \bar{q} \cup \{(x, p(x))\}$, $F^{q'} = F^{\bar{q}}$ and let $Q' = (q', F^{q'})$. Then $\bar{D}$ witnesses not only $Q' \in \mathbb{P}_D$, but also $Q' \le_{\mathbb{P}_D} Q_0||{\restriction}L_x^{=}$. By Lemma 3.27, $Q'' := Q' \bar{\ltimes}_x Q_0$ is a common extension in $\mathbb{P}_D$ of Q' and Q_0. Denote $Q'' = (q'', F^{q'} \cup F^{q_0})$ and let $\hat{p} = p{\restriction}L_0 \setminus \mathrm{dom}(q'')$. Let $q = q'' \cup \hat{p}$, $F^q = F^{q'} \cup F^{q_0}$ and let $Q = (q, F^q)$. Since $\mathrm{oc}(Q'') \cap \mathrm{dom}(\hat{p}) = \emptyset$, we deduce that $Q = (q'' \cup \hat{p}, F^{q_0} \cup F^p)$ is a condition in $\mathbb{P}$, extending Q''. Thus in particular $Q \le Q_0$.

To see that $Q \le P$, first observe that $Q''{\restriction}L_0 \le Q_0{\restriction}L_0 \le P_0{\restriction}L_0$, and since by definition $P_0{\restriction}L_0$ is a strong $\mathbb{Q}_A$-reduction of $P{\restriction}L_0$, we obtain $(q''{\restriction}L_0 \cup \hat{p}{\restriction}L_0, F^p) \le P{\restriction}L_0$. But then $Q{\restriction}L_0 \le P{\restriction}L_0$, $Q||{\restriction}L_{x_p}^{=} \le P||{\restriction}L_{x_p}^{=}$ and $\mathrm{dom}(p) \subseteq \mathrm{dom}(q)$, which by Lemma 3.26 gives $Q \le P$.

Case 2: $x \in A$. Then $x \in C$. Let $\bar{C} \in (\mathcal{I}{\restriction}C)_x$ be a witness to $Q_0||{\restriction}L_x^{=} \le_{\mathbb{P}_C} P_0||{\restriction}L_x^{=}$. That is, $\bar{Q}_0 = Q_0||{\restriction}L_x \le_{\mathbb{P}_{\bar{C}}} P_0||{\restriction}L_x (\le \bar{P}_0)$ and

$$Q_0||{\restriction}L_x \Vdash \text{“}(\check{s}_x^{q_0}, \dot{f}_x^{q_0}) \le_{\dot{\mathbb{S}}} (s_x^{p_0}, \dot{f}_x^{p_0})\text{”}.$$

By definition $\bar{A} = A \cap \bar{B}$ where $\bar{B} \in \mathcal{I}_{B,x}$. Also by the definition of $\bar{C} \in \mathcal{I}{\restriction}C$ there is $C_0' \in \mathcal{I}$ such that $\bar{C} = C_0' \cap C$. Then $x \notin C_0'$, and so by Definition 3.21,

(6) If $\mathrm{dom}(q_0) \cap L_x \subseteq L_0$, then we do not need the witness $\bar{E}$. The proof proceeds as in Case 1, but take $\bar{E} = \bar{F} = \emptyset$ and pick $\bar{G} \in \mathcal{I}_{D,x}$ such that $\mathrm{dom}(q_0) \cap L_x \subseteq \bar{G}$. Then in particular $\bar{C} = (\bar{G} \cap C) \cup \bar{A}$ and $\bar{D} = \bar{B} \cup \bar{G}$.

$C_0 := C_0' \cap L_x \in \mathcal{I}_x$ and $\bar{C} = C_0 \cap C$. Passing to an extension if necessary, we can assume that $\bar{Q}_0 \restriction L_0$ is a strong $\mathbb{Q}_{\bar{C}}$-reduction of $Q_0 \restriction L_0$.

Then $\bar{A} \cup \bar{C} \in (\mathcal{I}\restriction C)_x$ (7), and since $\mathrm{Rk}(\mathcal{T}_{\bar{A}\cup\bar{C}}) < \mathrm{Rk}(\mathcal{T})$, we have $\mathbb{P}_{\bar{C}} \lessdot \mathbb{P}_{\bar{A}\cup\bar{C}}$. Therefore $\dot{f}_x^{q_0}$ is also a $\mathbb{P}_{\bar{A}\cup\bar{C}}$-name, and so without loss of generality we may assume that $\bar{A} \subseteq \bar{C}$ (otherwise work with $\bar{A}\cup\bar{C}$ instead of $\bar{C}$). This implies $\bar{A} = \bar{C} \cap \bar{B}$ (8). Note also that $\bar{D} := D \cap C_0 \in \mathcal{I}_{D,x}$ and $\bar{C} = \bar{D} \cap C$. Similarly we may also assume that $\bar{B} \subseteq \bar{D}$. Since $\mathrm{Dp}_{\mathcal{I}}(\bar{D}) < \mathrm{Dp}_{\mathcal{I}}(D) = \alpha$, we can use the inductive hypothesis when working with $\bar{A}, \bar{B}, \bar{C}, \bar{D}$.

Let n be such that $\mathbb{S}$ is n-Suslin. Let $m = |s_x^{q_0}|$. Find $\hat{Q}_0 \leq_{\mathbb{P}_{\bar{C}}} \bar{Q}_0$ and $s' \in {}^{n\cdot m}\omega$ such that $\hat{Q}_0 \Vdash$ "$\dot{f}_x^{p_0} \restriction n\cdot m = \check{s}'$". Let G be a $\mathbb{P}_{\bar{C}}$-generic filter such that $\hat{Q}_0 \in G$. Now note that $\dot{f}_x^{p_0}$ is a $\mathbb{P}_{\bar{A}}$-name and $\mathbb{P}_{\bar{A}} \lessdot \mathbb{P}_{\bar{C}}$ by the inductive hypothesis (here we use the fact that $\mathrm{Dp}_{\mathcal{I}\restriction\bar{C}}(\bar{C}) \leq \mathrm{Dp}_{\mathcal{I}}(\bar{D}) < \alpha$). Therefore $G \cap \mathbb{P}_{\bar{A}}$ is $\mathbb{P}_{\bar{A}}$-generic and there is $U \in G \cap \mathbb{P}_{\bar{A}}$ such that $U \Vdash_{\mathbb{P}_{\bar{A}}} \dot{f}_x^{p_0} \restriction n\cdot m = \check{s}'$. Now $U, P_0 | \restriction L_x \in G \cap \bar{A}$, so they have a common extension $E' \in G \cap \bar{A}$ and $E' \Vdash_{\mathbb{P}_{\bar{A}}} \dot{f}_x^{p_0} \restriction n \cdot m = \check{s}'$.

Since E', $\hat{Q}_0$ are in G, they have a common extension $\hat{\hat{Q}}_0 \in G$ (and so in $\mathbb{P}_{\bar{B}}$). Then in particular $\hat{\hat{Q}}_0 \leq E'$, and so $\hat{\hat{Q}}_0$ has a reduction $\tilde{Q}_0$ in $\mathbb{P}_{\bar{A}}$ which extends E'. Thus $\tilde{Q}_0 \Vdash_{\mathbb{P}_{\bar{A}}} \dot{f}_x^{p_0} \restriction n \cdot m = \check{s}'$ and $\tilde{Q}_0 \leq P_0 | \restriction L_x$. But then $\tilde{Q}_0$ is compatible in $\mathbb{P}_{\bar{A}}$ with some $a \in \mathcal{A}_{n\cdot m}$. Here following the notation of Lemma 3.8, we assume that $\dot{f}_x^{p} = \{(b, s(b))\}_{b\in\mathcal{B},\, n\geq 1}$ and $\dot{f}_x^{p_0} = \{(a, s(a))\}_{a\in\mathcal{A}_n,\, n\geq 1}$.

Since $a \Vdash_{\mathbb{P}_{\bar{A}}} \dot{f}_x^{p_0} \restriction n\cdot m = \check{s}(a)$ and a is compatible with $\tilde{Q}_0$, it must be the case that $s(a) = s'$. Let P_0^* be a common $\mathbb{P}_{\bar{A}}$-extension of a and $\tilde{Q}_0$. Then $P_0^* \leq a$ and P_0^* is a reduction of $\hat{\hat{Q}}_0$ (since $\tilde{Q}_0$ is such a reduction; also P_0^* is a reduction of $\bar{Q}_0$). By construction, a is a reduction of some condition $b \in \mathcal{B}_{n\cdot m}$ such that $s(b) = s(a)$, i.e. $b \leq \bar{P}$ and $b \Vdash_{\mathbb{P}_{\bar{B}}} \dot{f}_x^{p} \restriction n \cdot m = \check{s}'$. Then P_0^* is compatible with b, with common extension $\bar{P}^+$. By the inductive hypothesis $\mathbb{P}_{\bar{A}} \lessdot \mathbb{P}_{\bar{B}}$, and so $\bar{P}^+$ has a canonical reduction $\hat{P}^+$ from $\mathbb{P}_{\bar{B}}$ to $\mathbb{P}_{\bar{A}}$. By Lemma 3.5, $\hat{P}^+$ is compatible with P_0^* (since $\bar{P}^+ \leq P_0^*$ and every canonical reduction is clearly also a reduction). Therefore they have a common extension $\bar{P}_0^+$. Note that $\bar{P}_0^+ \leq P_0^*$ and $\bar{P}_0^+$ is a canonical reduction of $\bar{P}^+$. Since P_0^* is a reduction of $\hat{\hat{Q}}_0$ onto $\mathbb{P}_{\bar{A}}$, there is $\bar{Q}_0^+ \in \mathbb{P}_{\bar{C}}$ extending $\bar{P}_0^+$ and $\hat{\hat{Q}}_0$. Now using the fact that $\bar{Q}_0^+ \leq \bar{P}_0^+$ and $\bar{P}_0^+$ being a canonical reduction of $\bar{P}^+$, we obtain a condition $T = (t, F^t) \in \mathbb{P}_{\bar{D}}$ such that $T \leq_{\mathbb{P}_{\bar{D}}} \bar{P}^+$ and $T \leq_{\mathbb{P}_{\bar{D}}} \bar{Q}_0^+$.

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(7) Indeed, $\bar{C} = C_0 \cap C$ where $C_0 \in \mathcal{I}_x$ and $\bar{A} = A \cap \bar{B}$ where $\bar{B} \in \mathcal{I}_{B,x}$. But $B \in \mathcal{I}$ and so $\bar{B} \in \mathcal{I}$. Then $\bar{A} \cup \bar{C} = (A \cap \bar{B}) \cup (C_0 \cap C) = (B \cap C \cap \bar{B}) \cup (C_0 \cap C) = (\bar{B} \cup C_0) \cap C$.

(8) By definition $\bar{A} = \bar{B} \cap A$, and so $\bar{A} \subseteq \bar{B}$, therefore $\bar{A} \subseteq \bar{C} \cap \bar{B}$. On the other hand, if $z \in \bar{C} \cap \bar{B}$ then $z \in C \cap B$, and so $z \in A$. Hence $\bar{C} \cap \bar{B} \subseteq A \cap \bar{B} = \bar{A}$.

Then

$$T \Vdash_{\mathbb{P}_{\bar{D}}} \text{“}(s_x^{q_0}, \dot{f}_x^{q_0}) \leq_{\dot{\mathbb{S}}} (s_x^{p_0}, \dot{f}_x^{p_0}) \text{ and}$$
$$(s_x^p, \dot{f}_x^p) \text{ is such that } s_x^p = s_x^{p_0} \wedge \dot{f}_x^p \restriction n \cdot m = \dot{f}_x^{p_0} \restriction n \cdot m\text{”}.$$

Since $\mathbb{S}$ is n-Suslin by assumption, we have

$$T \Vdash_{\mathbb{P}_{\bar{D}}} (\exists t(x) \in \dot{\mathbb{S}})\ t(x) \leq_{\dot{\mathbb{S}}} q_0(x), p(x).$$

Find $\bar{Q}^+ \leq T$ and a nice name $(\check{s}_x^q, \dot{f}_x^q)$ for a condition in $\mathbb{S}$ below $\bar{Q}^+$ such that $\bar{Q}^+ \Vdash_{\mathbb{P}_{\bar{D}}}$ “$(\check{s}_x^q, \dot{f}_x^q) \leq_{\dot{\mathbb{S}}} (s_x^{q_0}, \dot{f}_x^{q_0}), (s_x^p, \dot{f}_x^p)$”. Denote $\bar{Q}^+ = (\bar{q}^+, F^{\bar{q}^+})$.

With this we are ready to define a common extension $Q = (q, F^q)$ of Q_0 and P. Let $q' = \bar{q}^+ \cup \{(x, q(x))\}$, $F^{q'} = F^{\bar{q}}$ and $Q' = (q', F^{q'})$. Given Q', define Q'', $\hat{p}$ and Q as in Case 1. Then following the proof of Case 1, one concludes that Q is a common extension of Q_0 and P. ∎

3.4. Basic properties of the iteration. Having established our generalized “Main Lemma” (Lemma 3.23), we now proceed to develop the remaining basic tools that we need to work with the iteration along a two-sided template. These steps are parallel to those taken in Brendle [4, pp. 2640–2642], and we provide complete proofs only where it seems needed. For the discussion in this section fix $\mathcal{T}$, $\mathbb{Q}$ and $\mathbb{S}$ as in Lemma 3.23.

LEMMA 3.28. *Suppose $\mathbb{Q}$ is Knaster. Then $\mathbb{P}(\mathcal{T}, \mathbb{Q}, \mathbb{S})$ is Knaster.*

Proof. Let $\langle Q_\alpha : \alpha < \omega_1\rangle$ be an arbitrary sequence of conditions in $\mathbb{P}$. Since $\mathbb{Q}$ is Knaster, we can assume that $\langle Q_\alpha \restriction L_0 : \alpha < \omega_1\rangle$ are pairwise compatible in $\mathbb{Q}$. Applying the Δ-system lemma and the fact that $\mathbb{Q}$ is Knaster, we can assume that for all distinct $\alpha, \beta < \omega_1$, $\mathrm{dom}(q_\alpha) \cap \mathrm{dom}(q_\beta) = F$ for some fixed finite set $F \subseteq L$. Furthermore, we can assume that for all $x \in F \cap L_1$ there are $s_x \in {}^{<\omega}\omega$, $t_x \in {}^{n\cdot|s_x|}\omega$ such that if B is a witness to $Q_\alpha \restriction L_x^= \in \mathbb{P}$, then $Q_\alpha \restriction L_x \Vdash_{\mathbb{P}_B} \pi_0(q_\alpha(x)) = \check{s}_x \wedge \pi_1(q_\alpha(x)) \restriction n \cdot |s_x| = \check{t}_x$, where π_0 and π_1 denote the projections onto the first and second coordinate respectively.

Fix α, β distinct. We will show that Q_α, Q_β are compatible in $\mathbb{P}$. Let $\{x_i\}_{i \in m}$ enumerate $(\mathrm{dom}(q_\alpha) \cup \mathrm{dom}(q_\beta)) \cap L_1$ in $<_L$-increasing order, and let $R = (r, F)$ be a common extension of $Q_\alpha \restriction L_0$ and $Q_\beta \restriction L_0$. Passing to an extension if necessary, we can assume that $R \| \restriction L_{x_0}$ is a strong $\mathbb{Q}_{L_0 \cap L_{x_0}}$-reduction of R. Furthermore, there are $R_0^* \leq_{\mathbb{Q}_{L_{x_0}}} R \| \restriction L_{x_0}$ and $t(x_0)$ such that $R_0^* \Vdash_{\mathbb{Q}_{L_{x_0}}} t(x_0) \leq q_\alpha(x_0), q_\beta(x_0)$. Let $R^* = (r^*, F^*)$ and let $R_0 = (r_0, F_0) = (r_0^* \cup \{(x_0, t(x_0))\} \cup r \restriction L \setminus L_{x_0}, F_0^* \cup F)$. Since $R \| \restriction L_{x_0}$ is a strong $\mathbb{Q}_{L_{x_0}}$-reduction of R, we obtain $R_0 \leq_{\mathbb{P}} R$. Furthermore, $R_0 \| \restriction L_{x_1}$ is a common extension of $Q_\alpha \| \restriction L_{x_1}$ and $Q_\beta \| \restriction L_{x_1}$ (in $\mathbb{P}$).

Suppose for some $i < m - 1$ we have a condition $R_i = (r_i, F_i) \leq_{\mathbb{P}} R$ such that $r_i \restriction L \setminus L_{x_i}^= = r \restriction L \setminus L_{x_i}^=$, $(R_i \| \restriction L_{x_i}) \| \restriction L_0$ is a strong $\mathbb{Q}_{L_{x_i}}$-extension of R_i and

$R_i\|\restriction L_{x_i} \leq_{\mathbb{P}_{L_{x_i}}} Q_\alpha\|\restriction L_{x_i}, Q_\beta\|\restriction L_{x_i}$. Then we can find an extension R_i^* of $R_i\|\restriction L_{x_i}$ in $\mathbb{P}_{L_{x_i}}$ and a name $t(x_{i+1})$ such that $R_i^* \Vdash_{\mathbb{P}_{L_{x_i}}} t(x_{i+1}) \leq q_\alpha(x_{i+1}), q_\beta(x_{i+1})$. Let $R_i^* = (r_i^*, F_i^*)$. Since $(R_i\|\restriction L_{x_i})\restriction L_0$ is a strong $\mathbb{Q}_{L_{x_i}}$-reduction of R_i, we obtain $R_{i+1} = (r_{i+1}, F_{i+1}) = (r_i^* \cup \{(x_{i+1}, t(x_{i+1}))\} \cup r\restriction L\setminus L_{x_{i+1}}, F_i^* \cup F_i) \leq_{\mathbb{P}} R_i$ and $R_{i+1}\|\restriction L_{x_{i+2}} \leq Q_\alpha\|\restriction L_{x_{i+2}}, Q_\beta\|\restriction L_{x_{i+2}}$. Then for $i = m$, we conclude that $R_m \leq_{\mathbb{P}} Q_\alpha, Q_\beta$. ∎

We omit the proofs of the next three lemmas since they follow very closely the proofs of [4, Lemmas 1.3–1.5].

Lemma 3.29. *Let $x \in L_1$, $A \in \mathcal{I}_x$. Then the two-step iteration $\mathbb{P}_A * \mathbb{S}$ completely embeds into $\mathbb{P}$.*

Lemma 3.30. *For any $p \in \mathbb{P}(\mathcal{T}, \mathbb{Q}, \mathbb{S})$ there is a countable set $A \subseteq L$ such that $p \in \mathbb{P}_{\mathrm{cl}(A)}$. Similarly, if τ is a $\mathbb{P}$-name for a real, then there is a countable $A \subseteq L$ such that τ is a $\mathbb{P}_{\mathrm{cl}(A)}$-name.*

Lemma 3.31. *Let $\mathcal{J} \subseteq \mathcal{I}$ be such that $\mathcal{T}_{\mathcal{J}} = ((L, \leq), \mathcal{J}, L_0, L_1)$ is a template. Suppose $\mathcal{J}$ is cofinal in $\mathcal{I}$. Then $\mathbb{P}(\mathcal{T}_{\mathcal{J}}, \mathbb{Q}, \mathbb{S})$ is forcing equivalent to $\mathbb{P}(\mathcal{T}, \mathbb{Q}, \mathbb{S})$.*

4. $\mathfrak{a}_g$ can be $\aleph_\omega$. We now start working towards the main theorem of the paper. The model in which $\mathrm{cof}(\mathfrak{a}_g) = \omega$ is obtained by forcing with a poset of the form $\mathbb{P}(\mathcal{T}, \mathbb{Q}, \mathbb{S})$, where $\mathbb{Q}$ is the poset $\mathbb{Q}_{L_0}$ that adds a cofinitary group with L_0 generators, $\mathbb{S}$ is localization forcing, and $\mathcal{T}$ is the particular template used by Brendle [4].

4.1. Basic estimates for $\mathfrak{a}_g$. Before specifying $\mathcal{T}$, we prove two generally applicable lemmas, which are parallel to [4, Propositions 1.6 and 1.7].

Lemma 4.1. *Let $\mathcal{T}$ be a template, let $\mathbb{Q}$ be a finite function poset with the complete embedding property and $L_0 = \mathrm{dom}(\mathbb{Q})$, let $\mathbb{S} = \mathbb{L}$ be localization forcing, and let μ be a regular uncountable cardinal. Suppose $\mu \subseteq L_1$ (as an order), μ is cofinal in L, and $L_\alpha \in \mathcal{I}$ for all $\alpha < \mu$. Then $\mathbb{P}(\mathcal{T}, \mathbb{Q}, \mathbb{S})$ forces that* $\mathrm{non}(\mathcal{M}) = \mu$ *and $\mathfrak{a}_g \geq \mu$.*

Proof. Similarly to the classical linear iterations (finite support iterations of ccc posets, or countable support iterations of proper posets of size $\aleph_1$ over a model of CH) the proof will heavily rely on the fact that every real, as well as every small family of reals, appears in some initial segment of the iteration which is itself completely embedded into the entire construction.

Indeed, let G be $\mathbb{P}(\mathcal{T}, \mathbb{Q}, \mathbb{L})$-generic over V and work in $V[G]$. Let ϕ_α be the slalom added in coordinate $\alpha < \mu$ (this makes sense by Lemma 3.29). Since μ is regular and uncountable and is cofinal in L, it is clear by Lemma 3.30 that the family $\langle \phi_\alpha : \alpha < \mu \rangle$ localizes all reals $V[G]$ (since any real must appear in some $V[G \cap \mathbb{P}_{L_\alpha}]$ for some $\alpha < \mu$). Thus $\mathrm{cof}(\mathcal{N}) \leq \mu$. On the other hand, if $F \subseteq \omega^\omega$ is a family of size $< \mu$ in $V[G]$, then there must be some $\alpha < \mu$ such that all reals of F already are in $V[G \cap \mathbb{P}_{L_\alpha}]$, and so ϕ_α localizes all reals in F. Thus $\operatorname{add}(\mathcal{N}) \geq \mu$. Therefore $\operatorname{non}(\mathcal{M}) = \mu$, and so by Theorem 3.4 we have $\mathfrak{a}_g \geq \mu$. ∎

Lemma 4.2. *Let $\mathcal{T}$ be a template, and let $\mathbb{Q} = \mathbb{Q}_{L_0}$ be the poset for adding a cofinitary group with L_0 generators. Suppose that L has uncountable cofinality and L_0 is cofinal in L. Then $\mathbb{P}(\mathcal{T}, \mathbb{Q}, \mathbb{S})$ adds a maximal cofinitary group of size $|L_0|$.*

Proof. Let G be $\mathbb{P} = \mathbb{P}(\mathcal{T}, \mathbb{Q}, \mathbb{S})$-generic. Let $\rho_G : L_0 \to S_\infty$ be defined as follows: for every $x \in L_0$ let $\rho_G(x) = \{s^p_x : P \in G \wedge P{\restriction}L_0 = (s^p, F^p)\}$. Note that $\rho_G = \bigcup\{s^p_x : P \in G \cap \mathbb{P}_{L_0}\}$, and so by Proposition 2.12 the function ρ_G induces a cofinitary representation $\hat{\rho}_G$ of $\mathbb{F}_{L_0}$. We will show that $\operatorname{im}(\rho_G)$ is a maximal cofinitary group (which then clearly has size $|L_0|$).

Suppose not. Then there is a permutation $\sigma \in \operatorname{cofin}(S_\infty)$ and $b_0 \notin L_0$ such that $\rho'_G : L_0 \cup \{b_0\} \to S_\infty$, defined by $\rho'_G{\restriction}L_0 = \rho_G$ and $\rho'_G(b_0) = \sigma$, induces a cofinitary representation. Let $\dot{\sigma}$ be a $\mathbb{P}$-name for σ in V. Then by Lemma 3.30 there is a countable set $A \subseteq L$ such that $\dot{\sigma}$ is a $\mathbb{P}_{\operatorname{cl}(A)}$-name. Since L_0 is cofinal in L and L has uncountable cofinality, there is some $x \in L_0$ such that $\operatorname{cl}(A) \subseteq L_x$, and so $\mathbb{P}_{\operatorname{cl}(A)} \lessdot \mathbb{P}_{L_x}$. Let $G_0 = G \cap \mathbb{P}_{L_0}$ and $H = G \cap \mathbb{P}_{L_x}$.

Claim. $V[H] \vDash$ "$D_{\sigma,N} = \{P \in (\mathbb{P}/H) : (\exists n \geq N)\ s^p_x(n) = \sigma(n)$ *where* $P{\restriction}L_0 = (s^p, F^p)\}$ *is dense*".

Proof. Let $P_0 \in (\mathbb{P}/H)$. Thus $P{\restriction}L_0 \cap L_x \in H_0 := G \cap \mathbb{P}_{L_0 \cap L_x}$. By Lemma 2.19, we have

$$V[H_0] \vDash D^0_{\sigma,N,x} = \{p \in (\mathbb{Q}_{L_0}/\mathbb{Q}_{L_x \cap L_0}) : (\exists n \geq N)\ s^p_x(n) = \sigma(n) \text{ is dense}\}.$$

Thus there is $(t, E) \leq (s^{p_0}{\restriction}L_0 \setminus L_x, F^{p_0})$ such that $(t, E) \in D^0_{\sigma,N}$ i.e. $t_x(n) = \sigma(n)$ for some $n \geq N$. Define $P_1 \in \mathbb{P}/H$ as follows: $P_1{\restriction}L_x = P_0{\restriction}L_x$, $P_1{\restriction}(L_0 \setminus L_x) = (t, E)$, $P_1{\restriction}L_1 \setminus L_x = P_0{\restriction}L_1 \setminus L_x$. Then in $V[H]$ we have $P_1 \leq P_0$ and $P_1 \in D_{\sigma,N}$. ∎

By the Claim, in $V[G]$ there are infinitely many n such that $\sigma(n) = \sigma_x(n)$, contradicting the fact that ρ'_G induces a cofinitary representation. ∎

5. The isomorphism of names argument. Until the end of the paper assume CH. We will use the template construction developed by J. Brendle and S. Shelah to show that the minimal size of a maximal almost disjoint family can be of countable cofinality (see [4]). Let λ be a cardinal of countable cofinality, and more precisely, let $\lambda = \bigcup_{n\in\omega} \lambda_n$, where $\{\lambda_n\}_{n\in\omega}$ is a strictly increasing sequence of regular cardinals, $\lambda_0 \geq \aleph_2$, $\lambda_n^{\aleph_0} = \lambda_n$ for all n, and $\kappa^{\aleph_0} < \lambda_n$ for $\kappa < \lambda_n$. In the following, let μ^* denote a disjoint copy of μ, with the reverse ordering. Let $<_\mu$ denote the ordering of μ. We will refer to the elements of μ as *positive* and to those of μ^* as *negative*. If $\alpha \neq \beta \in \lambda^* \cup \lambda$, we will write $\alpha <_{\lambda^* \cup \lambda} \beta$ if either $\alpha \in \lambda^*$ and $\beta \in \lambda$, or both are in λ and $\alpha < \beta$, or both are in λ^* and $\alpha <_{\lambda^*} \beta$. For each n fix a partition $\lambda^*_n = \bigcup_{\alpha < \omega_1} S^\alpha_n$, where the S^α_n's are co-initial in λ^*_n and for $m < n$, $S^\alpha_n \cap \lambda^*_m = S^\alpha_m$. Definitions 5.1, 5.2, 5.4, 5.5 and Lemma 5.6 can be found in [4].

Definition 5.1. Let $L = L(\lambda)$ consist of all finite non-empty sequences x such that

(1) $x(0) \in \lambda_0$,
(2) $x(n) \in \lambda^*_n \cup \lambda_n$ for $0 < n < |x| - 1$,
(3) for $|x| \geq 2$, if $x(|x| - 2)$ is positive, then $x(|x| - 1) \in \lambda^*_{|x|-1} \cup \lambda$, and if $x(|x| - 2)$ is negative, then $x(|x| - 1) \in \lambda^* \cup \lambda_{|x|-1}$.

Whenever $x, y \in L$ let $x < y$ if and only if

(4) either $x \subset y$ and $y(|x|)$ is positive,
(5) or $y \subset x$ and $x(|y|)$ is negative,
(6) or $n = \min\{k : x(k) \neq y(k)\}$ is defined and $x(n) <_{\lambda^* \cup \lambda} y(n)$.

Clearly, $(L, <)$ is a linear order. We identify ordinals with one-element sequences, and so λ_0 is a cofinal subset of L. Whenever $\alpha \leq \lambda_0$, abusing notation we will write L_α for the set $L_{\langle\alpha\rangle} = \{x \in L : x < \langle\alpha\rangle\}$.

Definition 5.2. Let $L_1 = \{x \in L : |x| = 1 \text{ or } x(|x|-1) \in \lambda^*_{|x|-1} \cup \lambda_{|x|-1}\}$ and let $L_0 = L \setminus L_1$.

Remark 5.3. Note that $x \in L_0$ if and only if $|x| \geq 2$, and if $x(|x| - 2)$ is positive then $x(|x| - 1) \in [\lambda_{|x|-1}, \lambda)$, and if $x(|x| - 2)$ is negative then $x(|x| - 1) \in (\lambda^*, \lambda^*_{|x|-1}]$. Note also that both L_0 and L_1 are cofinal in L.

Definition 5.4. Let L_{rel} be the subset of L_1 of all x such that $|x| \geq 3$ is odd, and $x(n) \in \lambda^*_n$ for odd n, $x(n) \in \lambda_n$ for even n, $x(|x| - 1) \in \omega_1$, and whenever $n < m$ are even such that $x(n), x(m)$ are in ω_1, then there are $\beta < \alpha$ such that $x(n - 1) \in S^\alpha_{n-1}$ and $x(m - 1) \in S^\beta_{m-1}$. We refer to the members of L_{rel} as *relevant elements*.

For $x \in L_{\mathrm{rel}}$, let $J_x = \{z \in L : x\restriction(|x| - 1) \leq z < x\}$. If $x < y$ are relevant, then either $J_x \cap J_y = \emptyset$ or $J_x \subseteq J_y$. In the latter case also $|y| \leq |x|$, $x\restriction(|y| - 1) = y\restriction(|y| - 1)$ and $x(|y| - 1) \leq y(|y| - 1)$.

Definition 5.5. Let $\mathcal{I} = \mathcal{I}(\lambda)$ be the collection of all sets of the form

$$L_\alpha \cup \Big(\bigcup_{x \in I_1} \mathrm{cl}(J_x)\Big) \cup \Big(\bigcup_{x \in I_2} \mathrm{cl}(\{x\})\Big) \cup \Big(\bigcup_{x \in I_3} L_x \cap L_0\Big),$$

where $\alpha \in \lambda_0 \cup \{\lambda_0\}$, $I_1 \in [L_{\mathrm{rel}}]^{<\omega}$ and I_2, I_3 are in $[L_1]^{<\omega}$.

Lemma 5.6 ([4, Lemma 2.1]). *$\mathcal{T} = ((L, \leq), \mathcal{I}, L_0, L_1)$ is a two-sided template.*

Until the end of the section, let $\mathcal{T}$ be as in Lemma 5.6, and let $\mathbb{P} = \mathbb{P}(\mathcal{T}, \mathbb{Q}_{L_0}, \mathbb{L})$ where $\mathbb{Q}_{L_0}$ is the poset for adding a cofinitary group with L_0 generators (see Definition 2.4) and $\mathbb{L}$ is localization forcing. A subset B of L is a tree (recall that the elements of L are finite sequences) if it is closed with respect to initial segments, that is, if $x \in B$ then $x \restriction n \in B$ for all n.

Lemma 5.7. *In $V^{\mathbb{P}}$ there is a maximal cofinitary group of size λ and $\lambda_0 \leq \mathfrak{a}_g$.*

Proof. Since L_0 is cofinal in L and L is of uncountable cofinality, by Lemma 4.2 the forcing $\mathbb{P}$ adds a maximal cofinitary group of size $|L_0| = \lambda$. Since $\lambda_0 \subseteq L_1$ is cofinal in L and $L_\alpha \in \mathcal{I}$ for all $\alpha < \lambda_0$, by Lemma 4.1 we have $\lambda_0 \leq \mathfrak{a}_g$. ∎

We say that a $\mathbb{P}$-name $\dot{g}$ is a *good name* for a real if there are predense sets $\{p_{n,i}\}_{i\in\omega}$, where $n \in \omega$, and sets of integers $\{k_{n,i}\}_{i\in\omega}$, $n \in \omega$, such that $p_{n,i} \Vdash \dot{g}(n) = k_{n,i}$ for all n, i. That is, $\{p_{n,i}\}_{i\in\omega}$ is a predense set of conditions deciding the value of $\dot{g}(n)$. Whenever $\dot{g}$ is a good name for a real, we will refer to $\bigcup_{n,i\in\omega} \mathrm{dom}(p_{n,i})$ as the *L-domain* of $\dot{g}$ and denote it $\mathrm{dom}_L(\dot{g})$. We can assume that all $\mathbb{P}$-names for reals are good.

The following lemma is the essence of the isomorphism of names argument, due to Brendle. Its proof follows [4, pp. 2646–2648] almost identically. We work under the cardinal arithmetic assumptions from the beginning of this section.

Lemma 5.8 (Brendle [4]). *Let $\lambda_0 \leq \kappa < \lambda$, and for every $\beta \in \kappa$ let $B^\beta = \mathrm{dom}_L(\dot{g}^\beta)$ be a countable subset of L which is a tree, where $\dot{g}^\beta$ is a good name for a cofinitary permutation. Then there are a countable subset B^κ of L and a good name for a cofinitary permutation $\dot{g}^\kappa$ such that*

(1) *$\Vdash_{\mathbb{P}} \dot{g}^\kappa \neq \dot{g}^\beta$ for all $\beta < \kappa$,*
(2) *$\mathrm{dom}_L(\dot{g}^\kappa) = B^\kappa$,*
(3) *for every $F \in [\kappa]^{<\omega}$ there is $\alpha < \kappa$ and a partial order isomorphism*

$$\chi_{F,\alpha} : \mathbb{P}_{\mathrm{cl}(\bigcup_{\beta\in F} B^\beta \cup B^\alpha)} \to \mathbb{P}_{\mathrm{cl}(\bigcup_{\beta\in F} B^\beta \cup B^\kappa)}$$

which maps $\dot{g}^\alpha$ to $\dot{g}^\kappa$ and fixes $\dot{g}^\beta$ for $\beta \in F$.

Proof of Theorem 1.1. Let $\mathcal{G}$ be a $\mathbb{P}$-name for a cofinitary group of size κ, where $\lambda_0 \leq \kappa < \lambda$, and let $\{\dot{g}^\beta\}_{\beta\in\kappa}$ be an enumeration of $\mathcal{G}$. For $\beta < \kappa$, let $B^\beta = \mathrm{dom}_L(\dot{g}^\beta)$. Then B^β is at most a countable subset of L, and without loss of generality it is a tree. Let B^κ and $\dot{g}^\kappa$ be as in the conclusion of Lemma 5.8, applied to the families $\{B^\beta\}_{\beta\in\kappa}$ and $\{\dot{g}^\beta\}_{\beta\in\kappa}$. We will show that $\mathcal{H} = \langle \mathcal{G} \cup \{\dot{g}^\kappa\}\rangle$ is a cofinitary group.

Let $h \in \mathcal{H} \setminus \mathcal{G}$ and let $F_0 \cup \{\kappa\}$ be the indices of the permutations involved in h, where $F_0 \in [\kappa]^{<\omega}$. Then by Lemma 5.8, there are $\alpha < \kappa$ and a partial order isomorphism

$$\chi = \chi_{F_0,\alpha} : \mathbb{P}_{\mathrm{cl}(\bigcup_{\beta \in F_0} B^\beta \cup B^\alpha)} \to \mathbb{P}_{\mathrm{cl}(\bigcup_{\beta \in F_0} B^\beta \cup B^\kappa)}$$

which maps $\dot{g}^\alpha$ to $\dot{g}^\kappa$ and fixes $\dot{g}^\beta$ for $\beta \in F_0$. But then $\chi^{-1}(\dot{h})$ is a name for an element of $\mathcal{G}$, and so $|\mathrm{fix}(\chi^{-1}(h))| < \aleph_0$. Since both $\mathbb{P}_{\mathrm{cl}(\bigcup_{\beta \in F_0} B^\beta \cup B^\alpha)}$ and $\mathbb{P}_{\mathrm{cl}(\bigcup_{\beta \in F_0} B^\beta \cup B^\kappa)}$ are completely embedded in $\mathbb{P}$, we conclude that $V^{\mathbb{P}} \models |\mathrm{fix}(h)| < \aleph_0$. ∎

6. Concluding remarks. Let $\mathcal{T}_0$ be the template used in the proof of the consistency of $\mathfrak{a}$ being of countable cofinality (see [4]), whose definition is also stated in the previous section.

The construction presented gives also a proof of the fact that the minimal size of a family of almost disjoint permutations, denoted $\mathfrak{a}_p$, can be of countable cofinality. Let A be a generating set, and let $\mathbb{Q}_A$ be the poset for adding a maximal cofinitary group defined in Section 2. Let $\bar{\mathbb{Q}}_A$ be the suborder consisting of all pairs (s, F) where every word in F is of the form ab^{-1} for some $a, b \in A$. Then $\bar{\mathbb{Q}}_A$ is a finite function poset with the strong embedding property which adds a set of almost disjoint permutations of cardinality $|A|$, which is maximal whenever $|A|$ is uncountable. Then $\mathbb{P}(\mathcal{T}_0, \bar{\mathbb{Q}}_{L_0}, \mathbb{L})$ provides the consistency of $\mathrm{cof}(\mathfrak{a}_p) = \omega$. The proof of maximality follows very closely the maximal cofinitary group case, and the same isomorphism of names argument shows that there are no maximal families of almost disjoint permutations of intermediate cardinalities, i.e. cardinalities between λ_0 and λ. Note also that $\mathrm{non}(\mathcal{M}) \leq \mathfrak{a}_p$.

Another relative of the almost disjointness number, which can be approached in the same way, is the minimal size of a maximal almost disjoint family of functions in ${}^\omega\omega$. Let A be a generating set, and let $\tilde{\mathbb{Q}}_A$ be the poset of all pairs (s, F), where $s \subseteq A \times \omega \times \omega$ is finite, s_a defined as above is a finite function, and F is a finite set of words of the form ab^{-1} for $a \neq b$ in the index set A. The extension relation states that (s, F) extends (t, E) if $s \supseteq t$, $F \supseteq E$ and for all $w \in E$ if $e_w[s](n)$ is defined and $e_w[s](n) = n$ then $e_w[t](n) = n$. Then $\mathbb{P}(\mathcal{T}_0, \tilde{\mathbb{Q}}_{L_0}, \mathbb{L})$ provides the consistency of $\mathfrak{a}_e$ being of countable cofinality. Note also that to obtain a lower bound for $\mathfrak{a}_e$ in the final generic extension, we use the fact that $\mathrm{non}(\mathcal{M}) \leq \mathfrak{a}_e$.

The consistency of $\mathrm{cof}(\mathfrak{a}) = \omega$ is due to Brendle [4]. We mention that his proof also fits into our general framework. More precisely, as described in Section 3 above, given an uncountable generating set A, there is a finite function poset with the strong embedding property $\mathbb{D}_A$ which adds a maximal almost disjoint family of cardinality $|A|$. Then if $\mathbb{D}$ denotes the usual

Hechler forcing for adding a dominating function, the iteration $\mathbb{P}(\mathcal{T}_0, \mathbb{D}_{L_0}, \mathbb{D})$ provides the consistency of $\mathrm{cof}(\mathfrak{a}) = \omega$.

Thus we have obtained Theorem 1.2 of the Introduction.

Acknowledgements. The first author would like to thank the Austrian Science Fund FWF for the generous support through grant no. M1365-N13. The second author would like to thank Denmark's Council for Independent Research for generous support through grant no. 10-082689/FNU.

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Received 11 October 2013;
in revised form 19 October 2014 and 21 January 2015