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Categorifications of the polynomial ring

by

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Abstract. We develop a diagrammatic categorification of the polynomial ring $\mathbb{Z}[x]$. Our categorification satisfies a version of Bernstein–Gelfand–Gelfand reciprocity property with the indecomposable projective modules corresponding to x^n and standard modules to $(x-1)^n$ in the Grothendieck ring.

1. Introduction. Inspired by the general idea of categorification, introduced by L. Crane and I. Frenkel, we construct a categorification of the polynomial ring $\mathbb{Z}[x]$, including its elements $(x-1)^n$. This construction can be generalized to orthogonal one-variable polynomials, including Chebyshev polynomials of the second kind and Hermite polynomials [4].

In this paper, we interpret the ring $\mathbb{Z}[x]$ as the Grothendieck ring of a suitable additive monoidal category A-pmod of (finitely-generated) projective modules over an idempotented diagrammatically defined ring A (see Section 2). The monomials x^n become indecomposable projective modules P_n, while the polynomials $(x-1)^m$ turn into the so-called standard modules M_m. The ring A has one more distinguished family of modules, simple modules L_n. A remarkable feature of these three collections of modules is the Bernstein–Gelfand–Gelfand (or BGG) reciprocity property [2]. The projective modules P_n have a filtration by the standard modules M_m, for $m \leq n$, and the respective multiplicities satisfy the relation

$$[P_n : M_m] = [M_m : L_n].$$

The first examples of algebras and modules with this property are due to J. Bernstein, I. Gelfand, and S. Gelfand, and come up in the infinite-dimensional representation theory of simple Lie algebras. Our algebra A, which we call the SLarc algebra, on the other hand, has a purely topological

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definition, yet satisfies the BGG property. Moreover, the standard modules M_n have a clear diagrammatic interpretation. An additional sophistication appears due to the nonunitality of A. Instead of the unit element 1, the SLarc algebra A contains an infinite collection of idempotents 1_n, $n \geq 0$. The projective modules $P_n = A1_n$ and the standard modules M_n are infinite-dimensional, and the multiplicity $[M_m : L_n]$ should be understood in the generalized sense, as $\dim(1_n M_m)$, due to one-dimensionality of the simple modules L_n. We hope that our approach will lead to a topological interpretation of the BGG reciprocity in many other cases, including the ones considered by J. Bernstein, I. Gelfand, and S. Gelfand. In the sequel [4] we will generalize these constructions to categorify the Hermite and Chebyshev polynomials.

2. The algebra of SLarcs and what it categorifies. In this section we define the SLarc algebra A and introduce certain types of A-modules, such as projective and standard modules. Then we compute the Grothendieck group (ring) of an appropriate category and show how it can be identified with the ring $\mathbb{Z}[x]$, via sending indecomposable projective modules to monomials. Finally, we describe various properties of this construction and show that it satisfies the Bernstein–Gelfand–Gelfand reciprocity.

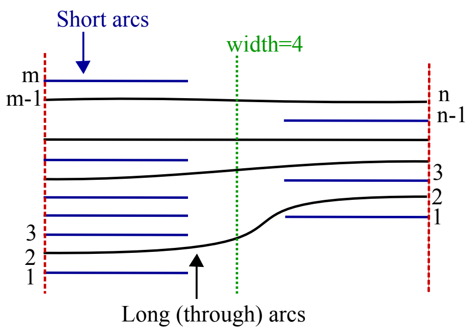


Fig. 1. A diagram in ${}_mB_n$

Definition 2.1. Denote by ${}_mB_n$ the set of isotopy classes of planar diagrams (see Figure 1) which connect k out of m points on the line $x = 0$ to k out of n points on the line $x = 1$ by k disjoint arcs called *larcs* (*long arcs*), $k \leq \min(n, m)$. The remaining $m-k$ left and $n-k$ right points extend to *short arcs* or *sarcs*, with one endpoint on either line $x = 0$ or $x = 1$ and the other in the interior of the strip $0 < x < 1$. We require that the projection of the resulting 1-manifold onto the x-axis has no critical points. The number of larcs k is called the *width* of the diagram. Let ${}_mB_n(k)$ and ${}_mB_n(\leq k)$ denote the subsets of diagrams in ${}_mB_n$ of width k and less than or equal to k, respectively.

The set ${}_mB_n$ has cardinality

$$\sum_{k=0}^{\min(n,m)} \binom{n}{k}\binom{m}{k} = \binom{n+m}{n}.$$

Furthermore, denote by B the set of all diagrams,

$$B := \bigsqcup_{n,m\geq 0} {}_mB_n \quad \text{and} \quad B_n := \bigsqcup_{m\geq 0} {}_mB_n.$$

Definition 2.2. The *SLarc algebra* A over a field $\mathbf{k}$ is a vector space with basis B and multiplication generated by concatenation of elements of B. The product is zero if the resulting diagram has an arc which is not attached to the lines $x=0$ or $x=1$, called a *floating* arc (see Figure 2). Also, if $y \in {}_mB_n$, $z \in {}_kB_l$ and $n \neq k$, so that the concatenation is not defined, then we set $yz = 0$. Thus, for any two elements y, z of B the product yz is either 0 or an element of B.

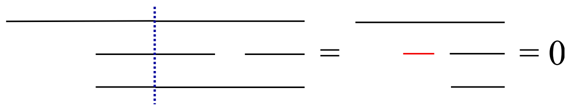


Fig. 2. Concatenation of these two diagrams equals zero since the resulting diagram contains a floating arc.

Remark 2.3. Alternatively, we can avoid drawing sarcs, and instead draw just their endpoints on the vertical lines $x = 0, 1$. Then the product of two diagrams, and their corresponding elements in A, is zero if the composition has an isolated point in the middle of the diagram.

The composition defined above induces an associative $\mathbf{k}$-algebra structure on A. For each n there exists a unique diagram in ${}_nB_n$ without sarcs. We denote this diagram and its image in A by 1_n. These elements are minimal idempotents in A. Therefore, A is a nonunital associative algebra with a system $\{1_n\}_{n\geq 0}$ of mutually orthogonal idempotents.

We consider left modules M over A with the property

$$M = \bigoplus_{n\geq 0} 1_n M.$$

This property is analogous to the unitality condition $1M = M$ for modules over a unital algebra. For a module M, we write M^m for the direct sum of m copies of M.

DEFINITION 2.4. Let $P_n = A1_n$ be the projective A-module with a basis consisting of all diagrams in B_n. Define M_n, called the *standard module*, as the quotient of P_n by the submodule spanned by all diagrams which have right sarcs.

Therefore, a basis of M_n is the set of diagrams in B_n with no right sarcs. In particular, if $1_m M_n \neq 0$ then $m \geq n$. Notice that $b \cdot a = 0$ for any $a \in M_n$ and every diagram $b \in B$ with at least one right sarc (Figure 2).

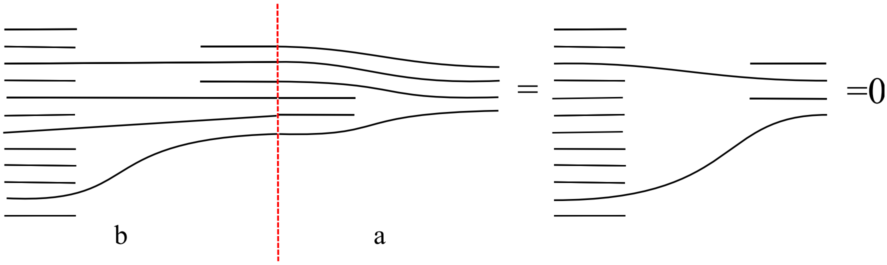


Fig. 3. For any diagram a representing an element of a standard module and every diagram $b \in B$ with right sarcs the product $b \cdot a$ is 0 in M_n.

DEFINITION 2.5. A left A-module M is called *finitely-generated* if for some finite subset $\{m_1, \ldots, m_k\}$ of M we have $M = Am_1 + \cdots + Am_k$.

LEMMA 2.6. *A left A-module M is finitely-generated if and only if it is a quotient of a direct sum $\bigoplus_{n=0}^{N} P_n^{a_n}$ for some $a_n \geq 0$, $N \in \mathbb{N}$.*

Let A-mod be the category of finitely-generated left A-modules and A-pmod the category of finitely-generated projective left A-modules.

PROPOSITION 2.7. *The hom space* $\mathrm{Hom}_A(M', M'')$ *is a finite-dimensional* $\mathbf{k}$*-vector space for any* $M', M'' \in A$-mod.

Proof. It is sufficient to consider the case $M' = P_n$. We have $\mathrm{Hom}(P_n, M'') = 1_n M''$. But $1_n M''$ is finite-dimensional, since M'' is a quotient of a finite direct sum of P_m's, and $1_n P_m$ is finite-dimensional. ∎

COROLLARY 2.8. *The category* A-mod *is Krull–Schmidt.*

DEFINITION 2.9. Let $L_n = \mathbf{k}1_n$ be the one-dimensional module over A on which any element of B other than 1_n acts by zero.

LEMMA 2.10. *Any simple A-module is isomorphic to L_n for some $n \geq 0$.*

Proof. Let L be a simple A-module and I the 2-sided ideal in A spanned by all diagrams with at least one left sarc. Notice that $1_n I^{n+1} = 0$ for all $n \geq 0$. Since IL is a submodule of L, we have either $IL = L$ or $IL = 0$. If $IL = L$ then $I^m L = L$ for every m and $0 = 1_n I^{n+1} L = 1_n L$ for all n, a contradiction. Hence $IL = 0$ and every simple module L is actually an A/I-module. The algebra A/I is directed, in the sense that

$$1_n(A/I)1_m = 0 \quad \text{if } n > m,$$
$$1_n(A/I)1_n \cong \mathbf{k}.$$

Hence, $\bigoplus_{k\le n} 1_k L$ is a submodule of L for every n. With L being simple, $1_n L = L$ for some n, and L is one-dimensional, isomorphic to L_n. ∎

THEOREM 2.11. *Any finitely-generated projective left A-module P is isomorphic to a finite direct sum of indecomposable projective modules P_n,*

$$P \cong \bigoplus_{n=0}^{N} P_n^{a_n}.$$

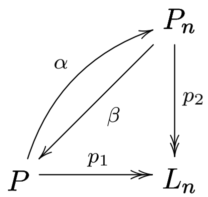
The multiplicities $a_n \in \mathbb{Z}_+$ are invariants of P.

Proof. The module P_n is indecomposable, since its endomorphism ring $R = \mathrm{Hom}_A(P_n, P_n)$ is local. Indeed, the diagrams in ${}_nB_n$ other than 1_n span a two-sided ideal J in R, and $J^N = 0$ for N sufficiently large. Therefore J is the radical of R, $R/J \cong \mathbf{k}$, and R is local.

Take a finitely-generated projective A-module P and any maximal proper submodule Q. The simple module P/Q is isomorphic to L_n for some n. The surjections

$$P \xrightarrow{p_1} L_n \xleftarrow{p_2} P_n$$

lift to homomorphisms $P \xrightarrow{\alpha} P_n \xrightarrow{\beta} P$.



Notice that $p_1\beta\alpha = p_1$ and $p_2\alpha\beta = p_2$, which gives $p_2(\alpha\beta - 1) = 0$. Hence $1 - \alpha\beta \in J(\mathrm{End}(P_n))$, the Jacobson radical of the endomorphism ring, and there exists an integer N such that $(1-\alpha\beta)^N = 0$. Thus, there exists an endomorphism δ of P_n such that $1 - \alpha\beta\delta = 0$. Hence for $\beta' = \beta\delta$ we get $\alpha\beta' = 1$, which means

$$P \cong \mathrm{Im}\,\beta \oplus \mathrm{Ker}\,\alpha \cong P_n \oplus \mathrm{Ker}\,\alpha,$$

i.e. P_n is a direct summand of P. Proceeding by induction, we get $P \cong \bigoplus_{n=0}^{N} P_n^{a_n}$. The Krull–Schmidt property [1] implies that the multiplicities a_n are invariants of P. ∎

Next, we prove that the nonunital algebra A is Noetherian, hence the category A-mod is abelian.

Proposition 2.12. *A submodule of a finitely-generated left A-module is finitely-generated.*

Proof. Any finitely-generated A-module is a quotient of $\bigoplus_{i=0}^{N} P_i^{n_i}$ for some N and some $n_0, n_1, \ldots, n_N$, hence it suffices to show $\bigoplus_{i=0}^{N} P_i^{n_i}$ is Noetherian. Furthermore it is enough to show that any submodule of P_n is finitely-generated. Since P_n has a finite filtration by standard modules, it suffices to check that any submodule of M_n is finitely-generated. The induction base case $n = 0$ is trivial, since $M_0 = \bigoplus_{m\geq 0} 1_m M_0$, each term $1_m M_0$ is one-dimensional and generates a submodule of finite codimension in M_0.

Basis elements b of M_n can be labeled by length $n+1$ sequences of nonnegative integers $(a_1, \ldots, a_{n+1})$. Here a_1 is the number of sarcs below the bottom larc and a_{n+1} is the number of sarcs above the top larc. Each a_i, $2 \leq i \leq n$, is the number of sarcs between the $(i-1)$st and the ith larc, counting larcs from bottom to top (Figure 4).

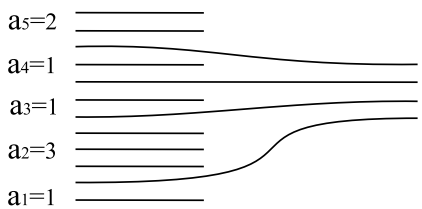


Fig. 4. Basis element for M_n

We call a_{n+1} the *degree* $\deg(b)$ of the basis element $b = (a_1, \ldots, a_{n+1}) \in M_n$. The degree of an arbitrary element $d = \sum_i x_i b_i \in M_n$, $x_i \in \mathbf{k}^*$, is equal to $\deg(d) = \max_i \deg(b_i)$. For $d = \sum_i x_i b_i \in M_n$ define $d' = \sum_{\deg(b_i)=\deg(d)} x_i b_i \in M_n$, which is the sum of the terms of d with the highest degree.

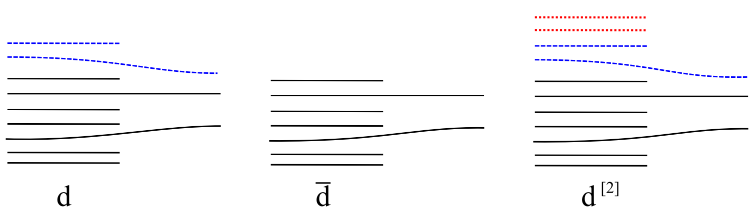


Fig. 5. This figure shows an element $d \in M_3$, the corresponding $\overline{d}$ and the element obtained by degree shift 2 denoted by $d^{[2]}$. The top larc and sarcs above it are denoted by dashed lines. Two added sarcs in $d^{[2]}$ are shown as dotted lines.

Given $d \in M_n$ let $\overline{d} \in M_{n-1}$ be the element obtained by removing the top larc and all of the sarcs above it in each of the diagrams in d. Moreover, we define the element $d^{[p]} \in M_n$ obtained from d by adding p sarcs on top

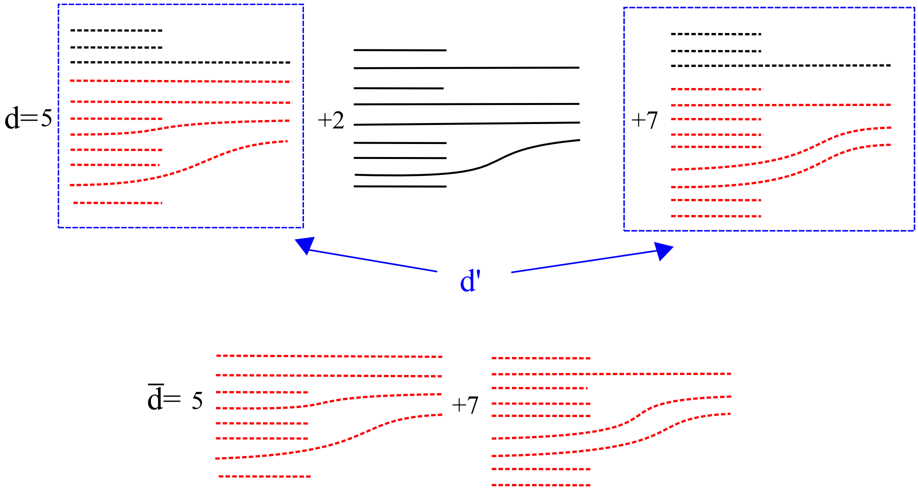


Fig. 6. Highest degree summands of the element $d \in M_4$ are contained in the top left and right rectangles. The bottom picture shows $\overline{d}$.

of each diagram in d. In particular, $\deg(d^{[p]}) = \deg(d) + p$ (Figure 2). To continue with the proof, let M be any submodule of M_n and d_0 be an element of the least degree in M. Assuming that $d_0, \ldots, d_k$ have already been chosen, take $d_{k+1} \in M \setminus (d_0, \ldots, d_k)$ where $(d_0, \ldots, d_k)$ is the submodule generated by the elements $\{d_0, \ldots, d_k\}$. Continuing by induction we obtain a sequence of elements $d_i \in M$.

Let $c_i := \overline{d_i'} \in M_{n-1}$ and let $\overline{M}$ denote the submodule of M_{n-1} generated by the c_i's. According to the induction hypothesis M_{n-1} is Noetherian, hence $\overline{M} = (c_0, c_1, \ldots)$ must be finitely generated. In other words, there exists $N \in \mathbb{N}$ such that $\overline{M} = (c_0, c_1, \ldots, c_N)$.

Assume that $M \neq (d_0, \ldots, d_N)$. Then there exist $d_{N+1} \in M \setminus (d_0, \ldots, d_N)$ and $c_{N+1} = \sum_{k=0}^{N} \alpha_k c_k$ for some $\alpha_k \in A$. Let $d^* = \sum_{k=1}^{N} \alpha_k d_k^{[\deg(d_{N+1}) - \deg(d_k)]}$. Now $d_{N+1} - d^* \notin (d_0, \ldots, d_N)$ and $\deg(d_{N+1} - d^*) < \deg(d_{N+1})$, which contradicts the minimality of $\deg(d_{N+1})$. Therefore $M = (d_0, \ldots, d_N)$ and M_n is Noetherian ([1]). ■

The involution of the set B which reflects a diagram about a vertical axis takes ${}_nB_m$ to ${}_mB_n$ and induces an anti-involution of A. Hence the ring A is right Noetherian as well.

Definition 2.13. The *Grothendieck group* $K_0(A)$ of finitely-generated projective A-modules is the abelian group generated by symbols $[P]$ for all finitely-generated projective left A-modules P, with defining relations $[P] = [P'] + [P'']$ if $P \cong P' \oplus P''$.

([1]) This proof is analogous to the proof that $\mathbf{k}[x_1, \ldots, x_n]$ is Noetherian.

PROPOSITION 2.14. *$K_0(A)$ is a free abelian group with basis $\{[P_n]\}_{n\geq 0}$.*

Proposition 2.14 follows from Theorem 2.11.

Observe that the existence of the filtration (2.1) of the projective module P_n by the standard modules M_m implies that M_m has a finite projective resolution $P(M_m)$ by P_n's, for $n \leq m$. Consequently, we can view M_m as an object of the category $\mathcal{C}(A\text{-pmod})$ of bounded complexes of finitely-generated projective A-modules. Morphisms in this category are homomorphisms of complexes modulo zero-homotopic homomorphisms [3, 5]. The Grothendieck groups of the categories A-pmod and $\mathcal{C}(A\text{-pmod})$ are canonically isomorphic:

$$K_0(\mathcal{C}(A\text{-pmod})) \cong K_0(A\text{-pmod})$$

via the isomorphism taking the symbol of

$$Q = (\cdots \to P^i \to P^{i+1} \to \cdots) \in \mathcal{C}(A\text{-pmod})$$

to

$$[Q] = \sum_{i\in\mathbb{Z}} (-1)^i [P^i] \in K_0(A).$$

Hence, equality (2.2) below can be interpreted within $K_0(A)$.

The projective module P_n has a filtration by the standard modules M_m, over $m \leq n$. Specifically, consider the filtration

$$P_n = P_n(\leq n) \supset P_n(\leq n-1) \supset \cdots \supset P_n(\leq 0) = 0, \tag{2.1}$$

where $P_n(\leq m)$ is spanned by the diagrams in B_n of width at most m (equivalently, with at least $n-m$ right sarcs). Left multiplication by a basis vector cannot increase the width, hence $P_n(\leq m)$ is a submodule of P_n (see Figure 8). The quotient $P_n(\leq m)/P_n(\leq m-1)$ has a basis of diagrams of width exactly m. These diagrams can be partitioned into $\binom{n}{m}$ classes enumerated by positions of the $n-m$ right sarcs. The quotient $P_n(\leq m)/P_n(\leq m-1)$ is isomorphic to the direct sum of $\binom{n}{m}$ copies of the standard module M_m. Consequently, we have an equality in the Grothendieck group of the additive category A-mod:

$$[P_n] = \sum_{m=0}^{n} \binom{n}{m} [M_m]. \tag{2.2}$$

The transformation matrix from the basis of the symbols $[P_n]$ of indecomposable projective modules to the basis of the symbols $[M_m]$ of standard modules is upper-triangular, with ones on the diagonal and nonzero coefficients being the binomials $\binom{n}{m}$. The entries of the inverse matrix are $(-1)^{n+m}\binom{n}{m}$. Thus we have the following equality in $K_0(A)$:

$$[M_n] = \sum_{m=0}^{n} (-1)^{n+m} \binom{n}{m} [P_m]. \tag{2.3}$$

We identify the projective Grothendieck group $K_0(A)$ with $\mathbb{Z}[x]$ by sending the symbols of the projective modules $[P_n]$ to the monomials x^n and define an inner product on the basis $\{x^n\}_{n\geq 0}$ by

$$(x^n, x^m) = \dim \mathrm{Hom}(P_n, P_m) = |{}_nB_m| = \binom{n+m}{m}. \tag{2.4}$$

This identification will be justified in Section 3.1 by introducing a monoidal structure on A-pmod under which $P_n \otimes P_m \cong P_{n+m}$.

Under this identification, (2.3) gives

$$[M_n] = \sum_{m\leq n} (-1)^{n+m} \binom{n}{m} x^m = (x-1)^n, \tag{2.5}$$

so the symbols of standard modules $[M_n]$ correspond to $(x-1)^n$.

Equation (2.3) hints at the existence of a projective resolution of M_n which starts with P_n and has $\binom{n}{m}$ copies of P_m in the $(n-m)$th position:

$$0 \to P_0 \to \cdots \to P_{n-m}^{\binom{n}{m}} \to \cdots \to P_{n-2}^{\binom{n}{2}} \to P_{n-1}^{\binom{n}{1}} \to P_n \to M_n \to 0. \tag{2.6}$$

Let us construct this resolution. Denote the diagram with $n-1$ larcs and one left sarc at the ith position by ${}^ib_{n-1} \in {}_nB_{n-1}$. The diagram obtained from ${}^ib_{n-1}$ by reflection along the vertical axis is denoted by $b_n^i \in {}_{n-1}B_n$ (Figure 7). The product of ${}^ib_{n-1}$ or b_n^i with an arbitrary diagram $a \in B$, when defined and nonzero, differs from the diagram a in the following way (see Figure 8):

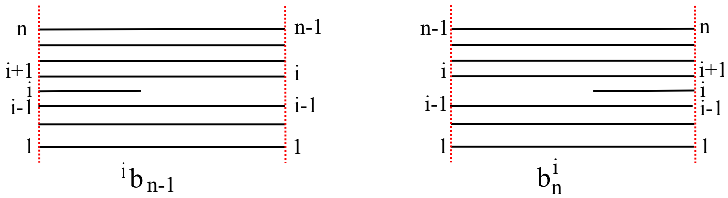


Fig. 7. The diagrams ${}^ib_{n-1}$ and b_n^i used in defining differentials in projective resolution of standard modules and resolution of simple by standard modules.

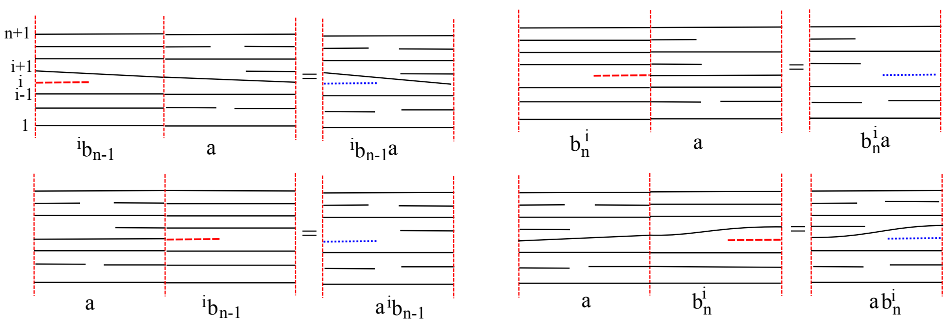


Fig. 8. The diagrams ${}^ib_n$ and b_n^i and their products with a diagram $a \in B$. The dashed line represents the difference between them and the diagram 1_n, and the dotted line in the resulting diagram shows the difference between the diagram a we started with and the product diagram.

(1) $a \cdot {}^{i_j}b_{n-1}$ turns the i_jth larc in the diagram a into a left sarc,
(2) ${}^{i_j}b_{n-1} \cdot a$ adds a left sarc between the ith and $(i+1)$st larcs in a,
(3) $a \cdot b_n^{i_j}$ adds a right sarc between the ith and $(i+1)$st larcs in a,
(4) $b_n^{i_j} \cdot a$ turns the i_jth larc in a into a right sarc.

Let $I_m = \{i_1, \dots, i_m\} \subseteq \{1, \dots, n\}$, $i_1 < \dots < i_m$, be a subset of cardinality $m \leq n$. Label the summands of the mth term $P_{n-m}^{\binom{n}{m}}$ by these subsets I_m, denoting the summand by $P_{n-m}^{I_m}$. Let $I_{m,l} := I_m \setminus \{i_l\}$. Removing an element i_l of I_m can be interpreted as composing a diagram in B_{n-m} on the right with a diagram b_{n-m+1}^p, obtained in the following way. Take a diagram $b_n^{i_l}$ and delete all larcs at positions labeled by elements in $I_{m,l}$, resulting in a diagram b_{n-m+1}^p, where p denotes the position of i_l in the ordered set $\{1, \dots, n\} \setminus I_m \cup \{i_l\}$ (see Figures 7 and 9).

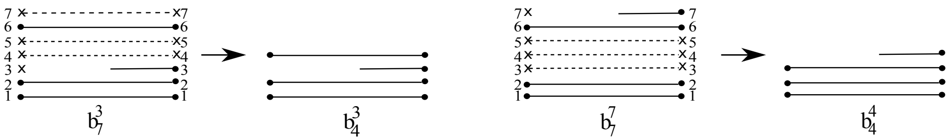


Fig. 9. The differentials $d^{+1}_{\{3,4,5,7\}}$ and $d^{+4}_{\{3,4,5,7\}}$ in the projective resolution of M_7 sending $P_3^{\{3,4,5,7\}}$ to $P_4^{\{4,5,7\}}$ and $P_4^{\{3,4,5\}}$, respectively. They are determined by composing on the right with the diagrams b_4^3 and b_4^4 obtained from b_7^3 and b_7^7 by deleting the dashed larcs corresponding to the label sets of $P_4^{\{4,5,7\}}$ and $P_4^{\{3,4,5\}}$.

Next, define the differential

$$d : P_{n-m}^{\binom{n}{m}} \to P_{n-(m-1)}^{\binom{n}{m-1}}$$

as the sum

$$d = \sum_{I_m} \sum_{l=1}^{m} d_{I_m}^{+l}$$

of the maps $d_{I_m}^{+l} : P_{n-m}^{I_m} \to P_{n-(m-1)}^{I_{m,l}}$ sending $a \in P_{n-m}^{I_m}$ to $d_{I_m}^{+l}(a) = (-1)^{l-1} a \cdot b_{n-m+1}^p$, For example, Figure 9 shows how to define $d^{+1}_{\{3,4,5,7\}}$ and $d^{+4}_{\{3,4,5,7\}}$ in the resolution of M_7 sending $P_3^{\{3,4,5,7\}}$ to $P_4^{\{4,5,7\}}$ and $P_4^{\{3,4,5\}}$, respectively.

PROPOSITION 2.15. *The complex* (2.6) *with the differential defined above is exact.*

Proof. The proof that $d^2 = 0$ follows from the sign convention and the commutative diagram in Figure 10 which shows $d_{I_{m,r}}^{+s-1} \cdot d_{I_m}^{+r} = d_{I_{m,s}}^{+r} \cdot d_{I_m}^{+s}$ for $r < s$. The proof that (2.6) is exact uses a slight generalization of this square. Viewed as a complex of vector spaces, (2.6) splits into the sum of complexes:

$$0 \to 1_k P_0 \to \cdots \to 1_k P_{n-m}^{\binom{n}{m}} \to \cdots \to 1_k P_{n-2}^{\binom{n}{2}} \to 1_k P_{n-1}^{\binom{n}{1}} \\ \to 1_k P_n \to 1_k M_n \to 0,$$

one for each element of ${}_kB_n$, $k \leq n$, with no left sarcs. Each of the complexes in the sum is isomorphic to the total complex of a k-dimensional cube with a copy of the ground field $\mathbf{k}$ at each vertex and each edge an isomorphism. Hence, all complexes are contractible. ∎

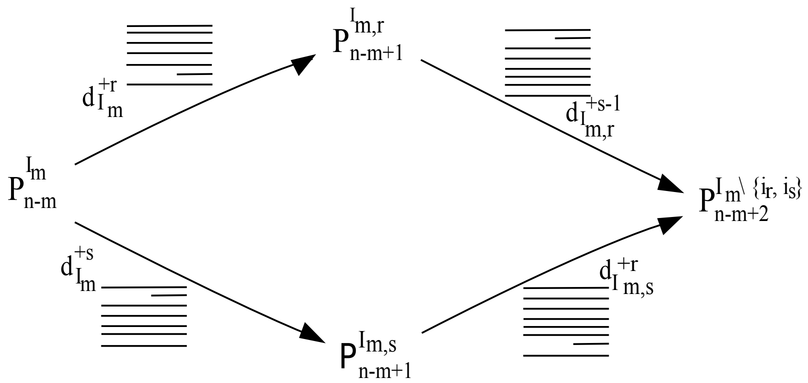


Fig. 10. A commutative diagram for the projective resolution of standard modules

A finite-dimensional A-module M has a finite filtration with simple modules L_n as subquotients. Due to the one-dimensionality of L_n the multiplicity of L_n in M, denoted by $[M : L_n]$, equals $\dim(1_n M)$. A finitely-generated A-module M is not necessarily finite-dimensional but it satisfies

$$\dim(1_n M) < \infty \quad \text{for } n \geq 0,$$

and therefore we call it *locally finite-dimensional.*

For a locally finite-dimensional module M we define the multiplicity of L_n in M as:

$$[M : L_n] := \dim(1_n M).$$

This definition is compatible with the usual notion of multiplicity of L_n in M as the number of times L_n appears in the composition series of M when M is finite-dimensional.

THEOREM 2.16 (SLarc BGG). *The SLarc algebra satisfies the Bernstein–Gelfand–Gelfand* (*BGG*) *reciprocity property:*

$$[P_n : M_m] = [M_m : L_n]. \tag{2.7}$$

The multiplicity on the right side of (2.7) *is understood in the generalized sense, as explained above.*

Proof. Recall that the indecomposable projective module P_n has a filtration by the standard modules M_m for $m \le n$ with $[P_n : M_m] = \binom{n}{m}$. What remains is to compute the multiplicity of a simple module L_n in a standard module M_n:

$$[M_m : L_n] = \dim(1_n M_m) = \begin{cases} \binom{n}{m} & \text{if } n \ge m, \\ 0 & \text{if } n < m. \end{cases} \tag{2.8}$$ ∎

Define the Cartan matrix $C(A)$ by

$$C(A)_{i,j} := \dim \mathrm{Hom}(P_i, P_j) \tag{2.9}$$

and by $m(A)$ the multiplicity matrix $m(A)_{i,j} := [P_i : M_j] = [M_j : L_i]$. Then

$$C(A) = m(A)m(A)^t. \tag{2.10}$$

Indeed,

$$\begin{aligned} C(A)_{i,j} &= \dim \mathrm{Hom}(P_i, P_j) = [P_i : L_j] \\ &= \sum_k [P_i : M_k][M_k : L_j] = \sum_k m(A)_{i,k} m(A)_{j,k} \\ &= \sum_k m(A)_{i,k} m(A)^t_{k,j} = (m(A)m(A)^t)_{i,j}. \end{aligned}$$

Proposition 2.17. $\mathrm{Ext}^i(M_n, M_m) = (1_{n-i} M_m)^{\binom{n}{i}}$.

Proof. Since the map between $\mathrm{Hom}(P_k, M_m)$ and $\mathrm{Hom}(P_{k-1}, M_m)$ induced by the differential in the projective resolution of M_n is trivial, the proof follows from the fact that $\mathrm{Hom}(P_k, M_m) = \mathrm{Hom}(A1_k, M_m) = 1_k M_m$. ∎

Proposition 2.18.

$$\mathrm{Ext}^i(M_n, L_m) \cong \begin{cases} \mathbf{k}^{\binom{n}{n-m}} & \textit{if } m \le n,\ i = n-m, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. Obviously, $\mathrm{Ext}^i(M_n, L_m) = 0$ for $m > n$. To compute $\mathrm{Ext}^i(M_n, L_m)$ we use the projective resolution (2.6) and get the complex

$$0 \leftarrow \mathrm{Hom}(P_0, L_m) \leftarrow \cdots \leftarrow \mathrm{Hom}(P_{n-1}, L_m)^{\oplus n} \leftarrow \mathrm{Hom}(P_n, L_m) \leftarrow 0. \tag{2.11}$$

Notice that

$$\mathrm{Hom}(P_{n-k}, L_m) = \begin{cases} \mathbf{k} & \text{if } m = n-k, \\ 0 & \text{otherwise.} \end{cases}$$

In the case $m = n-k$, $k \in \mathbb{Z}_+$, the complex (2.11) will be nontrivial only in degree $n-m$, and the $(n-m)$th homology is isomorphic to $\mathbf{k}^{\binom{n}{n-m}} = \mathrm{Ext}^{n-m}(M_n, L_m)$. All other Ext's are zero. ∎

Proposition 2.19. *The homological dimension of the standard module M_n is n.*

Proof. The projective dimension of M_n is at most n, as we have constructed a projective resolution (2.6) of that length. For $m = 0$, Proposition 2.18 says that $\mathrm{Ext}^n(M_n, L_0) = \mathbf{k}$, hence the projective dimension is equal to n. ■

Next we construct a resolution of each simple module L_k by the standard modules M_m for $m \geq k$:

$$\xrightarrow{d} M_{k+m}^{\binom{k+m}{m}} \xrightarrow{d} \cdots \xrightarrow{d} M_{k+2}^{\binom{k+2}{2}} \xrightarrow{d} M_{k+1}^{\binom{k+1}{1}} \xrightarrow{d} M_k \xrightarrow{d} L_k \to 0. \tag{2.12}$$

Let $I_m = \{i_1, \ldots, i_m\}$ be a subset of $\{1, \ldots, n\}$, $m \leq n$, $i_1 < \cdots < i_m$. Let $I_{m,-p}$ denote the set obtained from I_m by removing the pth element and subtracting 1 from all subsequent elements:

$$I_{m,-p} = \{i_1, \ldots, i_{p-1}, i_{p+1} - 1, \ldots, i_m - 1\} = I_m \setminus \{i_p\}. \tag{2.13}$$

The mth term of the resolution is the direct sum $M_{k+m}^{\binom{k+m}{m}}$ of the standard modules M_{k+m}. On the level of diagrams, the multiplicity $\binom{k+m}{m}$ represents the number of ways to add m larcs to the identity diagram 1_k in M_k to obtain a diagram in M_{k+m}. Let $I_m = \{i_1, \ldots, i_m\} \subseteq \{1, \ldots, k+m\}$ be the set describing the positions of the added larcs. Each summand $M_{k+m}^{I_m}$ is labeled by one of these subsets, and the differential will take the summand labeled by I_m into summands labeled by $I_{m,-l}$, for $0 < l \leq m$, by composing on the right with diagrams containing a single short right arc and no left sarcs (see Figure 7).

More precisely, let

$$d_{I_m}^{-l} : M_{k+m}^{I_m} \xrightarrow{\ {}^{l}b_{k+m-1}\ } M_{k+m-1}^{I_{m,-l}}$$

send $a \in M_{k+m}^{I_m}$ to

$$d_{I_m}^{-l}(a) = (-1)^l\, a \cdot {}^{l}b_{k+m-1}$$

where the diagram ${}^{l}b_k$ is shown in Figure 7. The differential

$$d : M_{k+m}^{\binom{k+m}{m}} \to M_{k+m-1}^{\binom{k+m-1}{m-1}}$$

is an alternating sum of these maps,

$$d = \sum_{I_m} \sum_{l=1}^{m} (-1)^l d_{I_m}^{-l}.$$

For example, the diagrams in Figure 11 show how to define $d_{\{3,6,8\}}^{-1}$, $d_{\{3,6,8\}}^{-2}$ and $d_{\{3,6,8\}}^{-3}$ in the resolution of L_5 sending $M_8^{\{3,6,8\}}$ into $M_7^{\{5,7\}}$, $M_7^{\{3,7\}}$, and $M_7^{\{3,6\}}$. In general, for a map $d_{I_m}^{-l}$, $0 < l \leq m$, sending $M_{n+1}^{I_m} \to M_n$ in the resolution of L_{n+1-m}, start with a diagram 1_{n+1}, turn the arc i_l

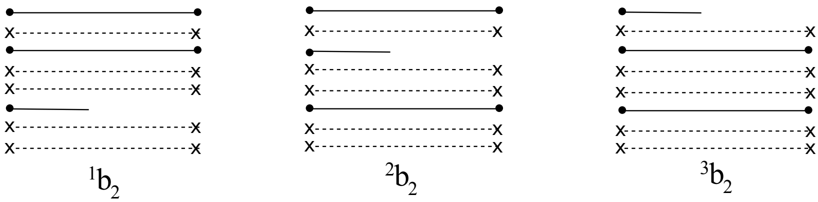


Fig. 11. Examples of diagrams used in defining the differentials $d^{-1}_{\{3,6,8\}}$, $d^{-2}_{\{3,6,8\}}$ and $d^{-3}_{\{3,6,8\}}$ in the resolution of a simple module L_5 by standard modules: dashed lines are the ones to be removed to obtain the appropriate diagram.

in a short left arc, then remove all long arcs labeled by numbers which are not in $I_m = \{i_1, \dots, i_m\}$, shown as dotted lines in Figure 11.

Proposition 2.20. *The complex* (2.12) *with the differential defined above is exact.*

Proof. The proof that $d^2 = 0$ is the same as in Proposition 2.15, except that the differential is defined using diagrams that lower the number of larcs (see Figures 7 and 10).

To prove exactness, notice that the complex (2.12) splits into the sum of complexes of vector spaces

$$1_n M_n^{\binom{n}{n-k}} \to 1_n M_{n-1}^{\binom{n-1}{(n-1)-k}} \to \cdots \to 1_n M_{k+1}^{\binom{k+1}{1}} \to 1_n M_k$$

for each $n > 0$. In turn, each of these complexes splits into the sum of $(n-k)$-dimensional cubes, corresponding to diagrams in ${}_nB_{n-k}$ with k larcs, $n-k$ left sarcs and no right sarcs, containing a copy of the field $\mathbf{k}$ at each vertex. For example, the resolution of L_2 contains a summand corresponding to $M_5^{\{2,3,4\}}$ represented by the total complex of a 3-dimensional cube shown in Figure 12. Sets labeling the vertices denote positions of short arcs in the corresponding diagrams shown to the left of the module symbol. Arrows are labeled with positions of elements which are being removed. ∎

Informally, at the level of Grothendieck groups we have the relation

$$\begin{aligned}[L_n] &= \sum_{k=0}^{\infty} (-1)^k \binom{n+k}{k} [M_{n+k}] \\ &= \sum_{k=0}^{\infty} (-1)^k \binom{n+k}{k} (x-1)^{n+k} = \frac{(x-1)^n}{x^{n+1}}.\end{aligned}$$

We will not try to make sense of this infinite sum.

In order to obtain a projective resolution of a simple module L_n we construct a bicomplex (see Figure 13), with a projective resolution (2.6) of M_{n+k}, $k \geq 0$, lying above each copy of a standard module in the resolution (2.12) of L_n by the standard modules M_m, $m \geq n$.

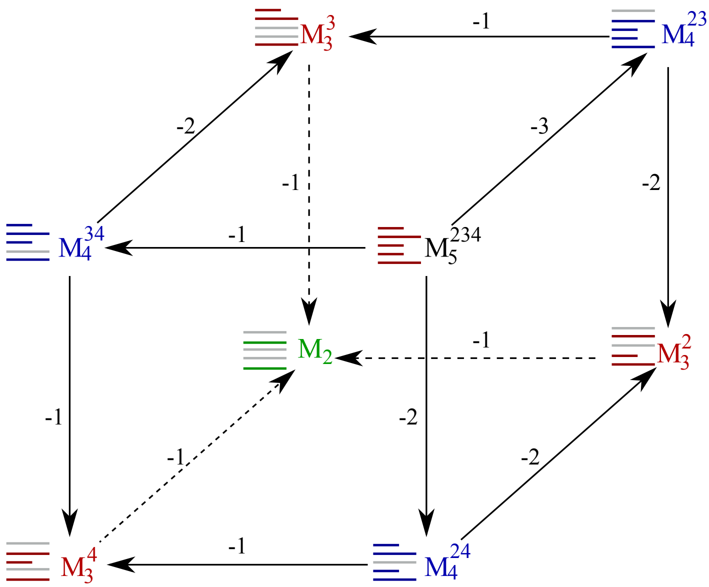


Fig. 12. A 3-dimensional cube in the resolution of the simple module L_2, corresponding to $M_5^{\{2,3,4\}}$, where label $\{2,3,4\}$ describes a diagram in B_2 with three left sarcs and the remaining two larcs shown to the left of the symbol M_5. Negative labels on the arrows specify the order of the element of the set in the superscript that is removed. For example, $M_5^{\{2,3,4\}}$ is mapped to $M_4^{\{3,4\}}$ by an arrow labeled by -1, which means that 2 is removed from $\{2,3,4\}$.

To complete the construction of the bicomplex, we define the horizontal differential denoted by d_H. Each copy of the projective module P_{n+m-k} in the bicomplex shown in Figure 13 comes with a pair of labels $P_{n+m-k}^{I_{m+n},J_k}$. The first label I_{n+m} is equal to the label of the standard module M_{n+m} in the resolution of L_n, and J_k is the label of P_{n+m-k} in the projective resolution of M_{n+m}.

The horizontal differential $d_H : P_{n+m-k}^{\binom{n+m}{m}\binom{n+m}{k}} \to P_{n+(m-1)+k}^{\binom{n+m-1}{m-1}\binom{n+m-1}{k}}$ is a signed sum of maps $d_{I_{m+n}}^{J_k}$ sending $a \in P_{n+m-k}^{I_{m+n},J_k}$ to

$$d_{I_{m+n}}^{J_k}(a) = \sum_{\substack{p=0 \\ i_p \notin J_k}}^{n+m} (-1)^{i_p - 1}\, a\, {}^{i_p}b \in \bigoplus_{\substack{p=0 \\ i_p \notin J_k}}^{n+m} P_{n+m-1-k}^{I_{m+n,-p},J_{k,-p}} \qquad (2.14)$$

where $I_{m+n,-p}$ and $J_{k,-p}$ are defined in (2.13).

PROPOSITION 2.21. *The diagram in Figure 13 is a bicomplex—all squares are anticommutative.*

Proof. Direct computation, see Figure 14. ∎

$$
\begin{array}{ccccccccccccc}
 & & \downarrow{\scriptstyle d_M} & & & & & & & & & & \\
\cdots & \xrightarrow{d_H} & P_{m-2}^{\binom{n+m}{m}\binom{n+m}{n-2}} & & & & & & & & & & \\
 & & \downarrow{\scriptstyle d_M} & & \downarrow{\scriptstyle d_M} & & & & & & & & \\
\cdots & \xrightarrow{d_H} & P_{m-1}^{\binom{n+m}{m}\binom{n+m}{n-1}} & \xrightarrow{d_H} & \cdots & \xrightarrow{d_H} & P_0^{n+1} & & & & & & \\
 & & \downarrow{\scriptstyle d_M} & & \downarrow{\scriptstyle d_M} & & \downarrow{\scriptstyle d_M} & & & & & & \\
 & \xrightarrow{d_H} & \cdots & \xrightarrow{d_H} & \cdots & \xrightarrow{d_H} & P_1^{(n+1)^2} & \xrightarrow{d_H} & P_0 & & & & \\
 & & \downarrow{\scriptstyle d_M} & & \downarrow{\scriptstyle d_M} & & \downarrow{\scriptstyle d_M} & & \downarrow{\scriptstyle d_M} & & & & \\
\cdots & \xrightarrow{d_H} & \cdots & \xrightarrow{d_H} & \cdots & \xrightarrow{d_H} & \cdots & \xrightarrow{d_H} & \cdots & & & & \\
 & & \downarrow{\scriptstyle d_M} & & \downarrow{\scriptstyle d_M} & & \downarrow{\scriptstyle d_M} & & \downarrow{\scriptstyle d_M} & & & & \\
\cdots & \xrightarrow{d_H} & P_{n+m-1}^{\binom{n+m}{m}\binom{n+m}{1}} & \xrightarrow{d_H} & \cdots & \xrightarrow{d_H} & P_n^{(n+1)^2} & \xrightarrow{d_H} & P_{n-1}^n & & & & \\
 & & \downarrow{\scriptstyle d_M} & & \downarrow{\scriptstyle d_M} & & \downarrow{\scriptstyle d_M} & & \downarrow{\scriptstyle d_M} & & & & \\
\cdots & \xrightarrow{d_H} & P_{n+m}^{\binom{n+m}{m}} & \xrightarrow{d_H} & \cdots & \xrightarrow{d_H} & P_{n+1}^{n+1} & \xrightarrow{d_H} & P_n & & & & \\
 & & \downarrow{\scriptstyle d_M} & & \downarrow{\scriptstyle d_M} & & \downarrow{\scriptstyle d_M} & & \downarrow{\scriptstyle d_M} & & & & \\
\cdots & \xrightarrow{d_L} & M_{n+m}^{\binom{n+m}{m}} & \xrightarrow{d_L} & \cdots & \xrightarrow{d_L} & M_{n+1}^{n+1} & \xrightarrow{d_L} & M_n & \xrightarrow{d_L} & L_n & \xrightarrow{d_L} & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle d_M} & & & & \\
0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & & & &
\end{array}
$$

Fig. 13. A bicomplex whose total complex is a projective resolution of L_n

$$
\begin{array}{ccc}
P_{n+m-(k+1)}^{\oplus\binom{n+m}{m}\binom{n+m}{k+1}} & \xrightarrow{d_H} & P_{n+(m-1)-(k+1)}^{\oplus\binom{n+m-1}{k+1}\binom{n+m-1}{k+1}} \\
\downarrow{\scriptstyle d_M} & & \downarrow{\scriptstyle d_M} \\
P_{n+m-k}^{\oplus\binom{n+m}{m}\binom{n+m}{k}} & \xrightarrow{d_H} & P_{n+(m-1)-k}^{\oplus\binom{n+m-1}{m-1}\binom{n+m-1}{k}}
\end{array}
$$

Fig. 14. An anticommutative square in the bicomplex of Figure 13

The projective resolution

$$P(L_n) : \cdots \to C_{n,t} \to C_{n,t-1} \to \cdots \to C_{n,0} \to L_n \to 0 \tag{2.15}$$

of the simple module L_n is defined in the following way:

$$C_{n,t} = \bigoplus_{\substack{m+k=t \\ n+m\geq k}} P_{n+m-k}^{\binom{n+m}{m}\binom{n+m}{k}} \tag{2.16}$$

The total differential d_t is the sum of the horizontal differential d_H, and the vertical differential d_M in the projective resolution of standard modules:

$$d_t = d_H + d_M.$$

In other words, the resolution (2.16) is the total complex of the bicomplex in Figure 13. Since each column in the bicomplex is exact, the following proposition holds:

PROPOSITION 2.22. *The chain complex* (2.16) *is exact.*

PROPOSITION 2.23. *The simple modules* L_n *have infinite homological dimension.*

Proof. By the resolution (2.16), it is sufficient to show that $\mathrm{Ext}^i(L_n, M)$ is nontrivial for arbitrarily large $i \in \mathbb{N}$ and some A-module M. Recall that

$$\mathrm{Hom}(P_i, L_m) = \begin{cases} \mathbf{k} & \text{if } m = i, \\ 0 & \text{otherwise.} \end{cases}$$

$C_{n,t}$ contains all P_i for $\max(0, n-t) \leq i < n+t$ such that $n+t-i \equiv 0 \pmod 2$. Let $M = L_0$ and notice that $P_0 \in C_{n,t}$ for every $t \geq n$ such that $n+t$ is even. Hence, the chain complex built out of the hom spaces $\mathrm{Hom}(C_{n,t}, L_0)$ (with the differential induced from the resolution) reduces to the infinite chain complex having trivial groups in odd degrees and nontrivial groups in even degrees for $t \geq n$:

$$\mathrm{Ext}^{n+t}(L_n, L_0) \cong \mathrm{Hom}(C_{n,t}, L_0) \cong \begin{cases} \mathbf{k} & \text{if } t = n, \\ \mathbf{k}^{\binom{(t+n)/2}{(t-n)/2}} & \text{if } t+n \text{ even, } t > n. \end{cases}$$

Therefore, $\mathrm{Ext}^{n+t}(L_n, L_0)$ is nontrivial for arbitrarily large $t > n$ such that $n+t$ is even. ∎

The SLarc algebra A can be viewed as a graded algebra with the grading defined by the total number of sarcs in a diagram. In particular, if we regard (2.16) as a graded resolution, the differential increases the degree by 1.

COROLLARY 2.24. *The algebra* A *is Koszul.*

3. Functors. In this section we describe a monoidal structure on A-pmod, justifying the identification

$$K_0(A\text{-pmod}) \cong \mathbb{Z}[x].$$

Next, we explain how the identity functor on A-mod can be approximated. On the pre-categorified level, given a basis $\{v_i\}_{i=1}^N$ of a separable Hilbert space $\mathcal{H}$, the identity operator acting on $\mathcal{H}$ can be viewed as the limit of finite sums $\sum_{i=1}^N v_i \otimes v_i^*$. In Section 3.2 we explain a categorified analogue of this construction for the case of A-modules. Notice that we are not categorifying a Hilbert space but its small subspace $\bigoplus_{i=1}^\infty \mathbb{Z} v_i \otimes v_i^*$, and the operator $v_i \otimes v_i^*$ should be thought of as acting on this space.

The most obvious inclusion $A \hookrightarrow A$ is given by adding a through (long) line either at the top or at the bottom of each diagram in A. In Section 3.3 we investigate restriction and induction functors for this inclusion and induced maps on Grothendieck groups. Converting each line to k parallel lines leads to a cabling functor, considered in Section 3.4. In Section 3.5 we compute the derived tensor product of standard modules.

3.1. Monoidal structure. We define the tensor product bifunctor

$$A\text{-pmod} \times A\text{-pmod} \to A\text{-pmod}$$

on indecomposable projective modules by $P_n \otimes P_m = P_{n+m}$ and extend it to all objects using Theorem 2.11. Next, define the tensor functor on basic morphisms of projective modules $\alpha : P_n \to P_{n'}$ and $\beta : P_m \to P_{m'}$, where $\alpha \in {}_nB_{n'}$, $\beta \in {}_mB_{m'}$ by placing α on top of β, and extend to all morphisms and objects using bilinearity (see Figure 15).

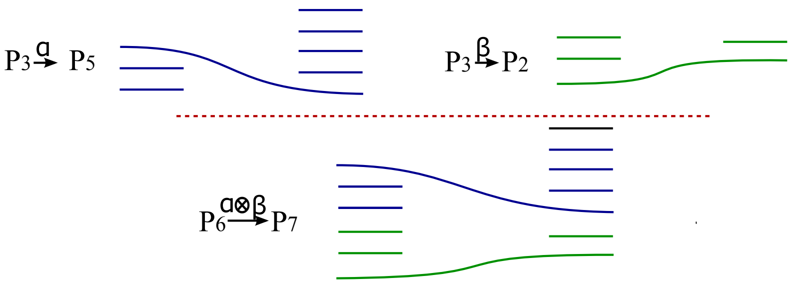


Fig. 15. Tensor product defined on basic morphisms of projective modules

The tensor product extends to a bifunctor $\mathcal{C}(A\text{-pmod}) \times \mathcal{C}(A\text{-pmod}) \to \mathcal{C}(A\text{-pmod})$. Hence, A-pmod and $\mathcal{C}(A\text{-pmod})$ are monoidal categories. Since standard modules have finite projective resolutions, they can be viewed as objects of $\mathcal{C}(A\text{-pmod})$. Let $P(M_n)$ be the projective resolution (2.6) of the standard module M_n.

Note that in the Grothendieck group, $[M_n] = (x-1)^n$ and

$$[M_n] \cdot [M_m] = (x-1)^{n+m} = [M_{n+m}].$$

This equality lifts to the category A-mod or $\mathcal{C}(A\text{-pmod})$.

Lemma 3.1. *In $\mathcal{C}(A\text{-pmod})$, $P(M_n) \otimes P(M_m) \cong P(M_{m+n})$ for $m, n \geq 0$.*

Proof. The pth term in the product of the projective resolutions $P(M_m)$ and $P(M_n)$ is

$$\bigoplus_{k+l=p} P_k^{\binom{n}{k}} \otimes P_l^{\binom{m}{l}} \cong P_p^{\binom{n+m}{p}}.$$

This module isomorphism respects differentials and gives an isomorphism of complexes. Notice that the isomorphism also holds in the category of complexes before modding out by null-homotopic morphisms. ∎

Corollary 3.2. *The following relation holds between standard modules viewed as objects of* $\mathcal{C}(A\text{-pmod})$: $M_n \otimes M_m \cong M_{m+n}$.

In the Grothendieck group the tensor product descends to multiplication in the ring $\mathbb{Z}[x]$, under the isomorphism of abelian groups $K_0(A) \cong \mathbb{Z}[x]$.

To define the tensor product for arbitrary modules we need to construct and tensor their projective resolutions. If modules M, N have finite filtrations with successive quotients isomorphic to standard modules M_n for various n, then the derived tensor product $M \widehat{\otimes} N$ has cohomology only in degree zero, and $H^0(M \widehat{\otimes} N) \cong_{D^b} M \widehat{\otimes} N$ has a filtration by standard modules. The derived tensor product restricts to a bifunctor on the category of modules admitting a finite filtration by standard modules.

3.2. Approximations of the identity. Recall that $B(\leq k) = \bigsqcup_{i=0}^{k} B(i)$ denotes the set of diagrams in B of width less than or equal to k. Let $A(\leq k)$, $k \geq 0$, denote the subspace of A spanned by diagrams in $B(\leq k)$. This subspace is an A-subbimodule of A. Let $A(k)$ be the quotient subbimodule $A(\leq k)/A(\leq k-1)$. Let ${}_nP$ denote the right projective module $1_n A$ and, analogously to the standard modules M_n, let ${}_nM$ be the quotient of ${}_nP$ by the submodule spanned by all diagrams with a left sarc. One can think of diagrams of ${}_nM$ as reflections along the vertical axis of diagrams in M_n.

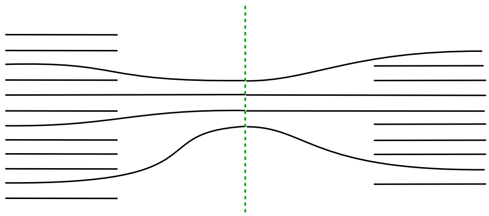


Fig. 16. A diagram in $B(4)$ viewed as a product of elements in M_4 and ${}_4M$

Proposition 3.3. $A(\leq k)/A(\leq k-1) \cong M_k \otimes_{\mathbf{k}} {}_kM$ *as* A*-bimodules* (*Figure* 16).

For a given $k \geq 0$, define a right exact functor $F_k : A\text{-mod} \to A\text{-mod}$ by

$$F_k(M) = A(\leq k) \otimes_A M$$

for an A-module M. The image of the standard module M_m under F_k is

$$A(\leq k) \otimes_A M_m = \begin{cases} M_m & \text{if } k \geq m, \\ 0 & \text{otherwise.} \end{cases} \tag{3.1}$$

By definition $P_m = A1_m$, hence $A(\leq k) \otimes_A P_m = A(\leq k) \otimes_A A1_m = A(\leq k)1_m$, and this is a submodule of P_m spanned by diagrams of width less than or equal to k:

$$F_k(P_m) = A(\leq k) \otimes_A P_m = \begin{cases} P_m & \text{if } k \geq m, \\ P_m(\leq k) & \text{if } k < m. \end{cases} \tag{3.2}$$

Recall that in the Grothendieck group, the projective modules P_n correspond to x^n and the standard modules M_n to $(x-1)^n$. The modules $P_n(\leq k)$ have finite homological dimension, since they admit finite filtrations with successive quotients isomorphic to standard modules. Therefore, the functor F_k descends to an operator on the Grothendieck group $K_0(A)$, denoted by $[F_k]$. The action of $[F_k]$ on $[P_n] = \sum_{m=0}^{n} \binom{n}{m}[M_m]$ is equal to

$$[F_k][P_n] = \begin{cases} [P_n] = x^n & \text{if } k \geq n, \\ \displaystyle\sum_{m=0}^{k} \binom{n}{m}[M_m] = \sum_{m=0}^{k} \binom{n}{m}(x-1)^m & \text{if } k < n. \end{cases} \tag{3.3}$$

In other words, for $k \geq n$ the operator $[F_k]$ acts via the identity on $[P_n]$, and for $k < n$ it approximates the identity and can be viewed as taking the first $k+1$ terms $\sum_{m=0}^{k} \binom{n}{m}[M_m]$ in the expansion of $[P_n]$ in the basis $\{(x-1)^m\}_{m\geq 0}$.

Proposition 3.4. *Higher derived functors of the functor F_k applied to a standard module are zero:*

$$L^i F_k(M_n) = \begin{cases} M_n & \text{if } i = 0, k \geq n, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The projective resolution $P(M_n)$ has the form (2.6):

$$0 \to P_0 \to \cdots \to P_{n-m}^{\binom{n}{m}} \to \cdots \to P_{n-2}^{\binom{n}{2}} \to P_{n-1}^{\binom{n}{1}} \to P_n \to 0. \tag{3.4}$$

Terms in this resolution are multiples of the projective modules P_m for $m \leq n$. By (3.2), if $k \geq n$, F_k acts as the identity on the resolution, implying the proposition in this case. Assume now that $k < n$. The differential in (2.6) applied to a diagram in any P_{n-m} preserves the width of the diagram, and (2.6) splits, as a complex of vector spaces, into a direct sum of complexes over all widths from 0 to n. These complexes are exact unless the width is exactly n, in which case the summand is isomorphic to $0 \to M_n \to 0$.

Applying F_k to the resolution (3.4) produces the complex

$$0 \to P_0 \to \cdots \to P_{n-m}^{\binom{n}{m}}(\leq k) \to \cdots \to P_{n-1}^{\binom{n}{1}}(\leq k) \to P_n(\leq k) \to 0 \tag{3.5}$$

which is exact for $k \leq n$, being a direct sum of exact complexes over all widths from 0 to k. ∎

3.3. Restriction and induction functors and what they categorify. In this section we consider the restriction and induction functors coming from the specific inclusion map on the SLarc algebra and their decategorification.

For a unital inclusion $\iota : B \hookrightarrow A$ of arbitrary rings the induction functor

$$\mathrm{Ind} : B\text{-mod} \to A\text{-mod}$$

given by $\mathrm{Ind}(M) = A \otimes_B M$ is left adjoint to the restriction functor,

$$\mathrm{Hom}_A(\mathrm{Ind}(M), N) \cong \mathrm{Hom}_B(M, \mathrm{Res}(N)).$$

If the inclusion is nonunital, i.e., ι takes the unit element of B to an idempotent $e \neq 1$ of A, the restriction functor has to be redefined: to an A-module N assign the eAe-module eN and then restrict the action to B. The induction functor is defined as before, but now

$$\mathrm{Ind}(M) = A \otimes_B M \cong (Ae \otimes_B M) \oplus (A(1-e) \otimes_B M) = Ae \otimes_B M,$$

and induction is still left adjoint to restriction. A similar construction works for nonunital B and A equipped with systems of idempotents.

We now specialize to the SLarc algebra A and the inclusion $\iota : A \hookrightarrow A$ induced by adding a straight through line on top of every diagram, so that a diagram $d \in {}_mB_n$ goes to $\iota(d) \in {}_{m+1}B_{n+1}$. In particular, the system $\{1_n\}_{n\geq 0}$ of idempotents goes to $\{1_{n+1}\}_{n\geq 0}$ missing 1_0. This inclusion ι gives rise to both induction and restriction functors, with

$$\mathrm{Ind}(N) \cong A \otimes_{\iota(A)} N,$$

$$\mathrm{Res}(N) \cong N/1_0N \cong \bigoplus_{k>0} 1_k\, N \text{ with the algebra } A \text{ acting on the left via } \iota.$$

In particular, $1_{n-1}\,\mathrm{Res}(M) \cong 1_n M$.

Notice that for simple modules

$$\mathrm{Res}(L_n) = \begin{cases} L_{n-1} & \text{if } n > 0, \\ 0 & \text{if } n = 0, \end{cases}$$

while $\mathrm{Ind}(L_n)$ is an infinite-dimensional module such that

$$1_m(\mathrm{Ind}(L_n)) = \begin{cases} \mathbf{k} & \text{if } m > n, \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 3.5. *$\mathrm{Res}(M_n) \cong M_n \oplus M_{n-1}$ for $n > 0$, and $\mathrm{Res}(M_0) \cong M_0$.*

Proof. Let M_n^L and $M_n^\emptyset$ denote the spans of diagrams in M_n with the top left point being a part of a left sarc or a larc, respectively (the diagrams in Figure 17 can be treated as elements of standard modules if we delete right returns). Then $\mathrm{Res}(M_n) \cong M_n^L \oplus M_n^\emptyset$ as left A-modules. Furthermore, $M_n^\emptyset \cong M_n$ and $M_n^L \cong M_{n-1}$. ∎

Proposition 3.6. *$\mathrm{Res}(P_n) \cong \bigoplus_{k=0}^n P_k$ for all $n \geq 0$.*

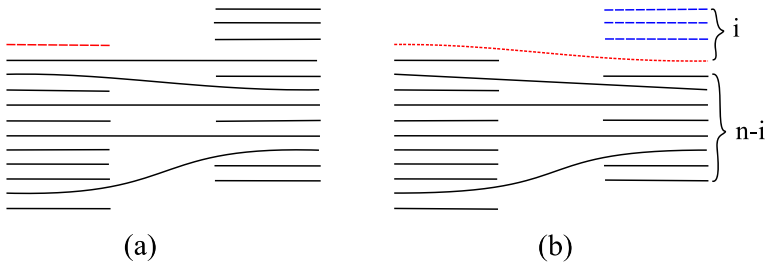


Fig. 17. Decomposition of P_n as a sum of vector spaces spanned by diagrams of type (a) where a left sarc is attached to the top left point and type (b) where the top left point is connected by a larc to the ith point on the right. In particular, the diagram in (a) is an element of $P_{12}^{\emptyset}$, and (b) belongs to $P_{12}^{(i)}$.

Proof. For each $i \geq 1$, let $P_n^{(i)}$ denote the spans of diagrams in P_n with top left point connected by a larc to the ith point on the right, and $P_n^{\emptyset}$ the span of diagrams such that at the top we have a left sarc (Figure 17). Each of these spans is a direct summand of $\mathrm{Res}(P_n)$.

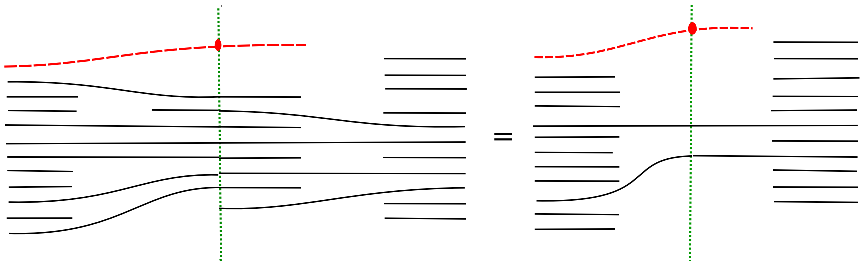


Fig. 18. $P_n^{\emptyset}$ is isomorphic to the projective module P_n.

Then $\mathrm{Res}(P_n) \cong P_n^{\emptyset} \oplus \bigoplus_{i=1}^{n} P_n^{(i)}$ as left A-modules. It is easy to see that $P_n^{\emptyset} \cong P_n$ (Figure 18) since the top left sarc is fixed. Similarly, $P_n^{(i)} \cong P_{n-i}$ since the $i-1$ top right sarcs are fixed (Figure 19). ∎

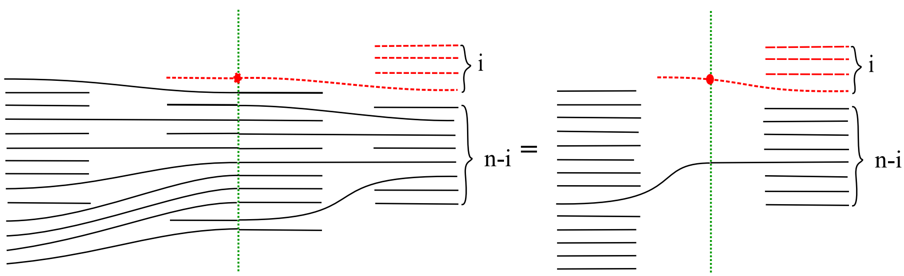


Fig. 19. $P_n^{(i)}$ is isomorphic to the projective module P_{n-i}.

PROPOSITION 3.7. $\operatorname{Ind}(P_n) \cong P_{n+1}$ *for* $n \geq 0$.

Proof. This follows from the definition of the induction functor (see Figure 20). ∎

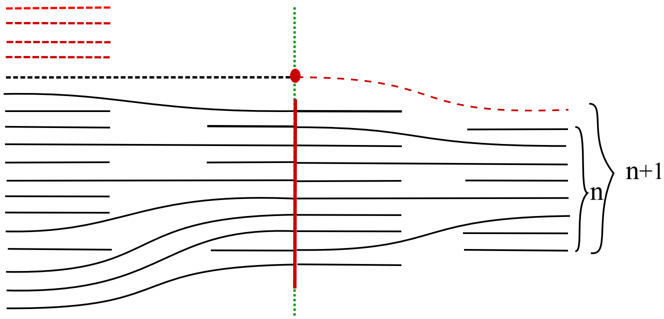


Fig. 20. Induction on projective modules: an element of $A \otimes_{\iota(A)} P_n$ is presented diagrammatically by composing basis elements of A and P_n. Elements of $\iota(A)$ can be exchanged through the vertical line.

PROPOSITION 3.8. *For* $n \geq 0$ *there exists a short exact sequence*

$$0 \to M_n \to \operatorname{Ind}(M_n) \to M_{n+1} \to 0. \tag{3.6}$$

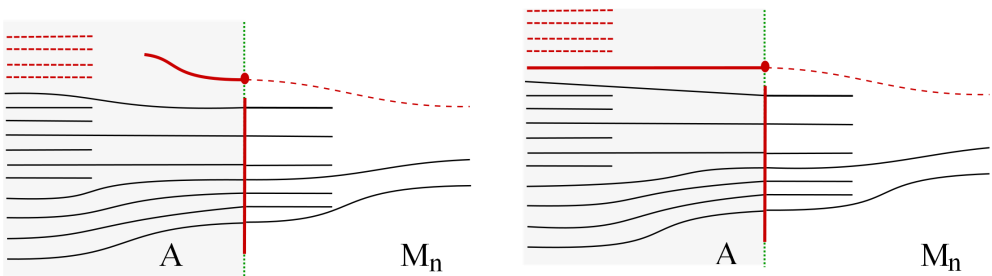


Fig. 21. Induction on standard modules

Proof. Notice that the right action of $\iota(A)$ fixes the top right point of a diagram in A. Depending on whether this point has a right sarc or larc attached to it (see Figure 21), we get a copy of M_n or M_{n+1} as a submodule or a quotient of $\operatorname{Ind}(M_n)$, respectively. ∎

PROPOSITION 3.9. *Higher derived functors of the induction functor applied to a standard module are zero:*

$$L^i \operatorname{Ind}(M_n) = 0 \quad \textit{for every } i > 0.$$

Proof. The induction functor applied to the projective resolution (2.6) gives

$$0 \to P_1 \to P_2^{\binom{n}{1}} \to \cdots \to P_m^{\binom{n}{m-1}} \to \cdots \to P_n^{\binom{n}{n-1}} \to P_{n+1} \to 0$$

where the differential corresponds to the one from (2.6) with a long arc added on top of each diagram. This complex splits, as a complex of vector spaces, into the sum of two copies of the original complex depending on whether the top arc is a larc or right sarc. ∎

Propositions 3.3 to 3.7 imply that at the level of the Grothendieck group, induction sends $[P_n] = x^n$ to $[P_{n+1}] = x^{n+1}$, $[M_n] = (x-1)^n$ to $[M_n] + [M_{n+1}] = (x-1)^n + (x-1)^{n+1}$, and restriction (always exact) acts in the following way:

$$[P_n] = x^n \mapsto \sum_{i=0}^{n} [P_i] = \sum_{i=0}^{n} x^i,$$

$$[M_n] = (x-1)^n \mapsto \sum_{i=0}^{n} [M_i] + [M_{i-1}] = \sum_{i=0}^{n} (x-1)^i + (x-1)^{i-1}.$$

Corollary 3.10. *In the Grothendieck group, induction corresponds to multiplication by x, and restriction* [Res] *acts by sending*

$$f(x) \mapsto \tfrac{xf(x)-f(1)}{x-1}.$$

3.4. Cabling functors. For every A-module M and a positive integer k we construct the corresponding cabled module ${}^{[k]}M$ in the following way:

$$1_n{}^{[k]}M = 1_{nk}M, \quad \text{hence} \quad {}^{[k]}M = \bigoplus_{n \geq 0} 1_{nk}M. \tag{3.7}$$

Given a diagram $y \in {}_sB_l$, construct a diagram ${}^{[k]}y \in {}_{sk}B_{lk}$, called the *k-cabling* of y, by taking k parallel copies of each arc (Figure 22). For example, ${}^{[k]}1_n = 1_{nk}$. By definition, the action of an element $\alpha \in A$ on ${}^{[k]}M_n$ is the regular action of its k-cabling α^k.

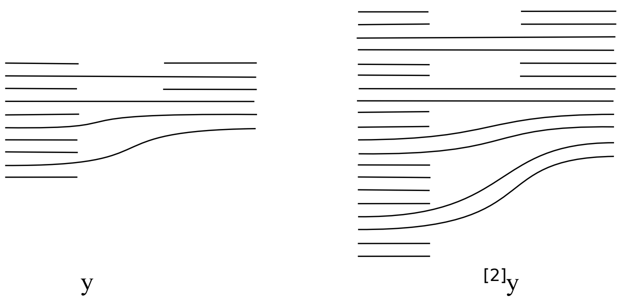


Fig. 22. A diagram $y \in {}_{11}B_6$ and 2-cable ${}^{[2]}y \in {}_{22}B_{12}$

What is the result of k-cabling simple, standard and projective modules? It is easy to see that if k divides n, the k-cabling of L_n is $L_{n/k}$:

$$1_m{}^{[k]}L_n = 1_{km}L_n = \begin{cases} \mathbf{k} & \text{if } km = n, \\ 0 & \text{otherwise.} \end{cases} \tag{3.8}$$

If k does not divide n, then ${}^{[k]}L_n = 0$.

Recall that basis elements of the standard A-modules M_n correspond to diagrams in B_n with n through arcs and an arbitrary number of left sarcs. Let $S(n,k,i)$ denote the number of ways to select n numbers between 1 and ki such that each of the sets $\{kj+1,\dots,k(j+1)\}_{0\le j<i}$ contains at least one of the selected numbers.

Proposition 3.11. ${}^{[k]}M_n \cong \bigoplus_{i=\lceil n/k\rceil}^{n} M_i^{S(n,k,i)}$.

Proof. The proof is left to the reader following the examples shown in Figure 23. $S(n,k,i)$ is the sum of products $\prod_{j=1}^{i}\binom{k}{\lambda_j}$ over all possible partitions $\lambda=(\lambda_1,\dots,\lambda_i)$ of n into i blocks of length at most k. ■

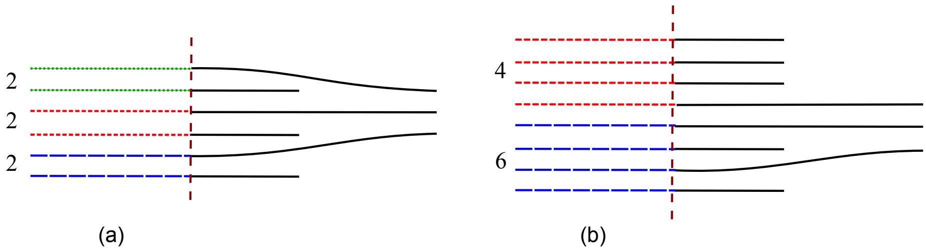


Fig. 23. (a) 2-cabling of M_3; (b) 4-cabling of M_3 corresponding to the partition $(2,1)$: two arcs in the same part contribute 6, hence the total contribution is 24.

We compute cabling modules of M_n for small values of n: ${}^{[k]}M_0 = M_0$, ${}^{[k]}M_1 = M_1^k$, ${}^{[k]}M_2 = M_2^{k^2} \oplus M_1^{\binom{k}{2}}$, ${}^{[k]}M_3 = M_3^{k^3} \oplus M_2^{2\binom{k}{1}\binom{k}{2}} \oplus M_1^{\binom{k}{3}}$.

Studying cablings of projective modules reduces to the case of standard modules: ${}^{[k]}P_n$ has a filtration with the ith term consisting of $\binom{n}{i}{}^{[k]}M_i$, based on the filtration (2.1) of P_n by $P_n(i)$, $i \le n$.

The cabling functor ${}^{[k]}$, sending an A-module M to its k-cabled module ${}^{[k]}M$, is exact, and categorifies the following operator on the Grothendieck group:

$$[M_n] = (x-1)^n \mapsto [{}^{[k]}M_n] = \sum_{i=\lceil n/k\rceil}^{n} S(n,k,i)(x-1)^i.$$

Notice that ${}^{[s][k]}M \cong {}^{[ks]}M$ functorially in M.

Proposition 3.12. *The cabling functor* ${}^{[k]}$ *preserves finitely-generated* A*-modules.*

Proof. The module ${}^{[k]}M_n$ is finitely-generated. Since P_m has a finite filtration by standard modules (2.1), ${}^{[k]}P_m$ is finitely-generated. A finitely-generated module M is a quotient of a finite sum of indecomposable projective modules P_m, thus ${}^{[k]}M$ is finitely-generated, and the functor ${}^{[k]}$ preserves the category A-mod. ∎

Another cabling functor, denoted by $\mathfrak{L}_k$, on the category A-pmod can be defined on objects by $\mathfrak{L}_k(P_n) = P_{nk}$ and on morphisms in the same way as above (Figure 22), i.e. $\mathfrak{L}_k(\alpha) = {}^{[k]}\alpha$ for $\alpha \in {}_mB_n$.

Given a full subcategory $\mathcal{A} \subset \mathcal{B}$, we say that endofunctors $F : \mathcal{A} \to \mathcal{A}$ and $G : \mathcal{B} \to \mathcal{B}$ are *weakly adjoint* if

$$\mathrm{Hom}_{\mathcal{B}}(FM_1, M_2) \cong \mathrm{Hom}_{\mathcal{B}}(M_1, GM_2),$$

functorially in $M_1 \in \mathcal{A}$ and $M_2 \in \mathcal{B}$.

PROPOSITION 3.13. *The cabling functors $\mathfrak{L}_k$ and ${}^{[k]}$ on the categories A-pmod and A-mod, respectively, are weakly adjoint.*

Proof. It is sufficient to prove the statement for $P_n \in A$-pmod and any $M \in A$-mod. Indeed,

$$\mathrm{Hom}(\mathfrak{L}_k(P_n), M) \cong \mathrm{Hom}(P_{nk}, M) \cong 1_{nk}M \cong 1_n{}^{[k]}M \cong \mathrm{Hom}(P_n, {}^{[k]}M). \blacksquare$$

3.5. Monoidal structure and standard modules. The full subcategory $\mathcal{C}'$ of A-pmod which consists of the objects P_n, $n \geq 0$, is monoidal and preadditive, with the unit object $\mathbf{1} = P_0$ and a single generating object P_1, since $P_n = P_1^{\otimes n}$. One can think of $\mathcal{C}'$ as a monoidal category with generating object P_1, generating morphisms $a \in \mathrm{Hom}(P_1, P_0)$ and $b \in \mathrm{Hom}(P_0, P_1)$, and defining relation setting the value of the floating arc, viewed as an endomorphism of $\mathbf{1}$, to zero (see Figure 24).

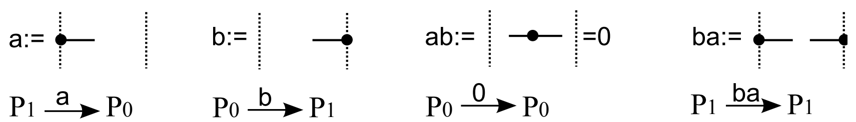


Fig. 24. Generating morphisms in the category $\mathcal{C}'$

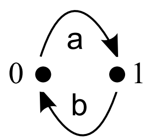


Fig. 25. Quiver description of the algebra $\mathrm{End}(P_0 \oplus P_1)$

The algebra $\mathrm{End}(P_0 \oplus P_1)$ admits a quiver presentation (see Figure 25) as a quiver with two vertices, two edges, and one defining relation $ab = 0$.

This is a five-dimensional algebra, which also describes regular blocks of the category $\mathcal{O}$ for $sl(2)$.

Let $\mathcal{C}'$ be a monoidal $\mathbf{k}$-linear category such that

$$\begin{aligned} \mathrm{Hom}(P_0, P_0) &= \mathbf{k}, & \mathrm{Hom}(P_1, P_0) &= \mathbf{k}b, \\ \mathrm{Hom}(P_0, P_1) &= \mathbf{k}a, & \mathrm{Hom}(P_1, P_1) &= \mathbf{k}1 \oplus \mathbf{k}ba. \end{aligned}$$

From this point of view, the SLarc algebra A can be viewed as the Hom algebra of $\mathcal{C}'$:

$$A = \bigoplus_{n,m\geq 0} \mathrm{Hom}(P_1^{\otimes n}, P_1^{\otimes m}).$$

PROPOSITION 3.14. *The standard module M_n is isomorphic to the nth derived tensor product of M_1: $M_n \simeq M_1^{\widehat{\otimes} n}$.*

Proof. The minimal projective resolution of M_1 is

$$0 \to P_0 \to P_1 \to 0. \tag{3.9}$$

The nth derived tensor power $M_1^{\widehat{\otimes} n}$ can be computed by substituting this resolution for each term in the tensor product $M_1^{\otimes n} \mapsto (0 \to P_0 \to P_1 \to 0)^{\otimes n}$. This tensor power will contain 2^n terms of the form

$$P_{\epsilon_1} \otimes \cdots \otimes P_{\epsilon_n} = P_{\epsilon_1 + \cdots + \epsilon_n}$$

for $\epsilon_i \in \{0, 1\}$.

The projective module P_m will appear $\binom{n}{m}$ times in the complex, and it is easy to match the resulting complex to the projective resolution (2.6) of the standard module M_n. ■

Proposition 3.14 (see also Corollary 3.2) generalizes the observation that

$$[M_n] = (x - 1)^n = [M_1]^n.$$

4. A modification of the SLarc algebra A**.** Assuming that we work over a field $\mathbf{k}$, we have two canonical choices for the value of the floating arc: either 0 or 1. Choosing value zero yields the above-described categorification of the polynomial ring and, interestingly enough, value one leads to yet another categorification of the polynomial ring. Let us denote by A^+ this modification of the SLarc algebra A. The elements 1_n and the projective modules P_n are defined as in the A algebra case.

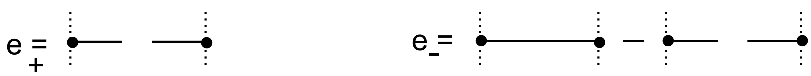


Fig. 26. The idempotents e_+ and e_- in A^+

However, changing the value of the floating arc from 0 to 1 produces additional idempotents, such as the element $e_+ \in {}_1B_1^+$ which is an idempotent according to the calculation shown in Figure 27, and the complementary idempotent $e_- = 1_1 - e_+$ (see Figure 26).

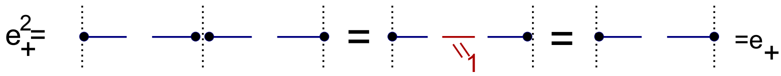


Fig. 27. The element e_+ is an idempotent in the algebra A^+.

Idempotents in $\mathrm{End}(P_n)$ for any $n > 1$ can be obtained from e_+ and e_- by using the monoidal structure of A^+-pmod analogous to the one in A-pmod, for which $P_n \otimes P_m = P_{n+m}$.

Let $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)$, $\varepsilon_i \in \{+, -\}$, denote a sequence of pluses and minuses of length n, and $(-^n)$ the sequence containing exactly n minuses. The corresponding idempotents are denoted by e_ε and $e_{(-^n)}$, respectively. The natural tensor product structure on A^+-pmod satisfies $P_\varepsilon \otimes P_{\varepsilon'} = P_{\varepsilon\varepsilon'}$, where $P_\varepsilon = A^+\varepsilon$. The idempotent $e_\varepsilon = \otimes_{i=1}^{n} e_{\varepsilon_i}$ is just a tensor product of idempotents e_+ and e_-'s, according to the sequence ε (see Figure 28).

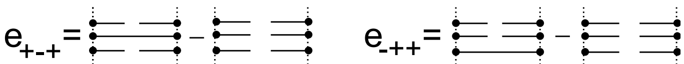


Fig. 28. Additional idempotents in the algebra A^+

Notice that $1_n = \sum_{|\varepsilon|=n} e_\varepsilon$. Moreover, these idempotents are mutually orthogonal, $e_\varepsilon e_{\varepsilon'} = \delta_{\varepsilon,\varepsilon'} e_\varepsilon$. In particular, $e_+e_- = e_-e_+ = 0$.

In general, given a ring R and two idempotents $e, f \in R$, the projective modules Re and Rf are isomorphic iff there exist elements $a = d_{e\to f}, b = d_{f\to e} \in R$ such that $eafbe = e$ and $fbeaf = f$. Moreover, in this case, we say that the elements e, f are *equivalent*, and write $e \simeq f$.

Lemma 4.1. *If a sequence ε contains exactly m minuses then $e_\varepsilon \simeq e_{(-^m)}$.*

Proof. The equivalence is realized by maps corresponding to the following diagrams: $d_{\varepsilon\to m}$ with n left and m right endpoints and m through arcs connecting right endpoints to those left endpoints corresponding to the minus signs in ε, and the remaining points extended to short left arcs. $b = d_{m\to\varepsilon}$ is a reflection of $a = d_{\varepsilon\to m}$ along the vertical axis. We have

$$e_{(-^m)} d_{m\to\varepsilon} e_\varepsilon d_{\varepsilon\to m} e_{(-^m)} = e_{(-^m)}, \qquad e_\varepsilon d_{\varepsilon\to m} e_{(-^m)} d_{m\to\varepsilon} e_\varepsilon = e_\varepsilon.$$

An example is shown in Figure 29.

Fig. 29. The maps $d_{(-,+,-,-,+)\to(-^3)}$ and $d_{(-^3)\to(-,+,-,-,+)}$

LEMMA 4.2. *If sequences ε and ε' contain n and m minuses, respectively, then $e_\varepsilon \simeq e_{\varepsilon'}$ iff $m = n$.*

Proof. By Lemma 4.1 $e_\varepsilon \simeq e_{(-^n)}$ and $e_{\varepsilon'} \simeq e_{(-^m)}$ and $e_{(-^n)}$, $e_{(-^m)}$ are not equivalent unless $m = n$. ∎

COROLLARY 4.3. *The projective modules A^+e_ε and $A^+e_{\varepsilon'}$ are isomorphic iff the sequences ε and ε' contain the same number of minuses.*

To a sequence $(-^n)$ we assign the indecomposable projective A^+-module $P_{(-^n)} = A^+e_{(-^n)}$.

PROPOSITION 4.4. *The projective modules $P_{(-^n)}$ are simple objects satisfying the following properties:*

(i) $\mathrm{Hom}(P_{(-^m)}, P_{(-^n)}) = \begin{cases} \mathbf{k} & \text{if } n = m, \\ 0 & \text{otherwise.} \end{cases}$

(ii) $P_n \cong \bigoplus_{|\varepsilon|=n} P_\varepsilon \cong \bigoplus_{m=0}^{n} \binom{n}{m} P_{(-^m)}$.

Proof. (i) follows from Proposition 4.3 since

$$\mathrm{Hom}(P_{(-^m)}, P_{(-^n)}) = \mathrm{Hom}(A^+e_{(-^m)}, A^+e_{(-^n)}) = e_{(-^m)}A^+e_{(-^n)}.$$

(ii) $P_n = A^+1_n = \bigoplus_{|\varepsilon|=n} A^+e_\varepsilon = \bigoplus_{|\varepsilon|=n} P_\varepsilon$. Each P_ε is equivalent to $P_{(-^m)}$ and there are $\binom{n}{m}$ sequences ε of length n with exactly m minuses. ∎

We see that the category A^+-pmod of projective A^+-modules is semisimple. The idempotented ring A is therefore semisimple and Morita equivalent to an idempotented ring $\mathbf{k} \oplus \mathbf{k} \oplus \cdots$, a countable sum of copies of the field $\mathbf{k}$. Let $K_0(A^+)$ denote the Grothendieck ring of the monoidal category of finitely-generated projective A^+-modules. As before, $[P_n] = x^n$. Based on the decomposition of the projective modules in Proposition 4.4(2) we conclude that $[P_{(-^n)}] = (x-1)^n$.

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