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How to construct a Hovey triple from two cotorsion pairs

by

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Abstract. Let $\mathcal{A}$ be an abelian category, or more generally a weakly idempotent complete exact category, and suppose we have two complete hereditary cotorsion pairs $(\mathcal{Q}, \widetilde{\mathcal{R}})$ and $(\widetilde{\mathcal{Q}}, \mathcal{R})$ in $\mathcal{A}$ satisfying $\widetilde{\mathcal{R}} \subseteq \mathcal{R}$ and $\mathcal{Q} \cap \widetilde{\mathcal{R}} = \widetilde{\mathcal{Q}} \cap \mathcal{R}$. We show how to construct a (necessarily unique) abelian model structure on $\mathcal{A}$ with $\mathcal{Q}$ (resp. $\widetilde{\mathcal{Q}}$) as the class of cofibrant (resp. trivially cofibrant) objects, and $\mathcal{R}$ (resp. $\widetilde{\mathcal{R}}$) as the class of fibrant (resp. trivially fibrant) objects.

1. Introduction. Let $\mathcal{A}$ be an abelian category. In [8] we can find a one-to-one correspondence between complete cotorsion pairs in $\mathcal{A}$ and abelian model structures on $\mathcal{A}$. The correspondence has proven to be a powerful method for constructing model structures in algebraic settings and for transporting ideas from topology into algebra. The goal of this note is to deepen this correspondence, making it even *easier* to construct abelian model structures in the ubiquitous case that they are hereditary. Before describing this, let us first define the relevant concepts.

A *cotorsion pair* is a pair of classes $(\mathcal{X}, \mathcal{Y})$ of objects in $\mathcal{A}$ satisfying the following two conditions:

- $X \in \mathcal{X}$ iff $\operatorname{Ext}^1_{\mathcal{A}}(X, Y) = 0$ for all $Y \in \mathcal{Y}$.
- $Y \in \mathcal{Y}$ iff $\operatorname{Ext}^1_{\mathcal{A}}(X, Y) = 0$ for all $X \in \mathcal{X}$.

The cotorsion pair is called *complete* if for any $A \in \mathcal{A}$ there exists a short exact sequence $Y \rightarrowtail X \twoheadrightarrow A$ with $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$, and another short exact sequence $A \rightarrowtail Y' \twoheadrightarrow X'$ with $X' \in \mathcal{X}$ and $Y' \in \mathcal{Y}$. The first sequence generalizes the concept of having enough projectives, while the second generalizes the concept of having enough injectives. The canonical nontrivial example of a complete cotorsion pair is the pair $(\mathcal{F}, \mathcal{C})$ in the category of modules over a ring, where $\mathcal{F}$ is the class of flat modules and $\mathcal{C}$ are the cotorsion modules. Finally, a cotorsion pair $(\mathcal{X}, \mathcal{Y})$ is called *hereditary* if $\mathcal{X}$ is closed under taking kernels of

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epimorphisms between objects in $\mathcal{X}$, and $\mathcal{Y}$ is closed under taking cokernels of monomorphisms between objects in $\mathcal{Y}$. The flat cotorsion pair is hereditary. In fact, virtually all the cotorsion pairs we encounter in practice tend to be hereditary. A standard reference for all of this and related concepts is the book [4].

On the other hand, we have the notion of a model category which comes from topology [9], [7]. A model category is a bicomplete category $\mathcal{M}$ along with a model structure: three subclasses of maps called cofibrations, fibrations, and weak equivalences, all of which satisfy several axioms. The axioms allow one to do homotopy theory in the category, and when the underlying category is abelian such a homotopy theory translates to some variety of homological algebra. Hovey [8] introduced and made an important study of abelian model categories. For a model structure on an abelian category to qualify as *abelian*, the model structure and abelian structure ought to be compatible in the following sense:

- A morphism f is a (trivial) cofibration if and only if it is a monomorphism with (trivially) cofibrant cokernel, that is, with $0 \to \operatorname{cok} f$ a (trivial) cofibration.
- A morphism g is a (trivial) fibration if and only if it is an epimorphism with (trivially) fibrant kernel, that is, with $\ker g \to 0$ a (trivial) fibration.

As shown in [8], this definition is stronger than it needs to be. But the definition makes it clear that we have shifted our focus from morphisms to objects. We can now state Hovey's correspondence between cotorsion pairs and abelian model structures. First, call a class of objects $\mathcal{W}$ *thick* if it is closed under direct summands and has the property that whenever two out of three terms in a short exact sequence are in $\mathcal{W}$, then so is the third.

Theorem 1.1 (Hovey's correspondence). *Let $\mathcal{A}$ be an abelian category. There is a one-to-one correspondence between abelian model structures on $\mathcal{A}$ and triples $(\mathcal{Q}, \mathcal{W}, \mathcal{R})$ having $\mathcal{W}$ thick and admitting two complete cotorsion pairs*

$$(\mathcal{Q}, \mathcal{W} \cap \mathcal{R}) \quad \textit{and} \quad (\mathcal{Q} \cap \mathcal{W}, \mathcal{R}).$$

Given such a triple $(\mathcal{Q}, \mathcal{W}, \mathcal{R})$, the class $\mathcal{Q}$ is precisely the class of cofibrant objects, $\mathcal{R}$ the class of fibrant objects, and $\mathcal{W}$ the class of trivial objects in the abelian model structure.

Now let us turn to the new fact proved in this note. Given such a triple $(\mathcal{Q}, \mathcal{W}, \mathcal{R})$, called a *Hovey triple*, denote the associated cotorsion pairs by

$$(\mathcal{Q}, \widetilde{\mathcal{R}}) = (\mathcal{Q}, \mathcal{W} \cap \mathcal{R}) \quad \text{and} \quad (\widetilde{\mathcal{Q}}, \mathcal{R}) = (\mathcal{Q} \cap \mathcal{W}, \mathcal{R}).$$

Then the following containments and equality are clear:

(1) $\widetilde{\mathcal{R}} \subseteq \mathcal{R}$ and $\widetilde{\mathcal{Q}} \subseteq \mathcal{Q}$.
(2) $\widetilde{\mathcal{Q}} \cap \mathcal{R} = \mathcal{Q} \cap \widetilde{\mathcal{R}}$.

Remarkably, there is a converse when we assume the cotorsion pairs are hereditary. That is, we have the following theorem.

Main Theorem 1.2. *Let* $(\mathcal{Q}, \widetilde{\mathcal{R}})$ *and* $(\widetilde{\mathcal{Q}}, \mathcal{R})$ *be complete hereditary cotorsion pairs satisfying conditions* (1) *and* (2) *above. Then there is a unique thick class* $\mathcal{W}$ *for which* $(\mathcal{Q}, \mathcal{W}, \mathcal{R})$ *is a Hovey triple. Moreover, this thick class* $\mathcal{W}$ *can be described in the following two ways:*

$$\begin{aligned}\mathcal{W} &= \{X \in \mathcal{A} \mid \exists \text{ a short exact sequence } X \rightarrowtail R \twoheadrightarrow Q \text{ with } R \in \widetilde{\mathcal{R}},\ Q \in \widetilde{\mathcal{Q}}\} \\ &= \{X \in \mathcal{A} \mid \exists \text{ a short exact sequence } R' \rightarrowtail Q' \twoheadrightarrow X \text{ with } R' \in \widetilde{\mathcal{R}},\ Q' \in \widetilde{\mathcal{Q}}\}.\end{aligned}$$

The proof we give is both elementary and very general. It holds in the general setting of when $\mathcal{A}$ is a weakly idempotent complete exact category. The author showed in [5] that Hovey's correspondence carries over to this setting. So while we state the theorem in the setting of abelian categories, the proof has purposely been written to hold in the more general setting of weakly idempotent complete exact categories.

The author again wishes to thank Hanno Becker and the referee of his paper [6]. The construction of Hovey triples that we give here is a direct generalization of a construction of Becker from [1]. In particular, when the two given cotorsion pairs are *injective*, then our construction is exactly Becker's right Bousfield localization construction from [1]. On the other hand, when the two given cotorsion pairs are *projective*, then our construction coincides with Becker's left Bousfield localization construction from [1]. The main difference in our proof, when comparing it to Becker's proof, is in how we show that the class of trivial objects $\mathcal{W}$ is thick. Our proof, while longer, is more direct and uses only elementary properties of short exact sequences.

Finally, the author wishes to thank Mark Hovey, who the author has been fortunate to work with over the years. It was during a recent exchange that the author found Theorem 1.2. It solves the problem of finding the Gorenstein AC-flat model structure on the category of chain complexes of modules over an arbitrary ring. This is the flat analog to the projective and injective models that recently appeared in [2]. This application, and others, will appear elsewhere. But at the end of the paper we give an example indicating how Theorem 1.2 immediately yields new and interesting model structures. In any case, the author finds the theorem interesting in its own right.

2. Proof of the theorem. Assume $(\mathcal{Q}, \widetilde{\mathcal{R}})$ and $(\widetilde{\mathcal{Q}}, \mathcal{R})$ are two complete hereditary cotorsion pairs satisfying

(1) $\widetilde{\mathcal{R}} \subseteq \mathcal{R}$ and $\widetilde{\mathcal{Q}} \subseteq \mathcal{Q}$.
(2) $\widetilde{\mathcal{Q}} \cap \mathcal{R} = \mathcal{Q} \cap \widetilde{\mathcal{R}}$.

We wish to construct a Hovey triple $(\mathcal{Q}, \mathcal{W}, \mathcal{R})$ with the properties in Theorem 1.2.

Proof of Theorem 1.2. We start by showing that the two classes below that define $\mathcal{W}$ do coincide:

$$\begin{aligned}&\{X \in \mathcal{A} \mid \exists \text{ a short exact sequence } X \rightarrowtail R \twoheadrightarrow Q \text{ with } R \in \widetilde{\mathcal{R}},\ Q \in \widetilde{\mathcal{Q}}\}\\ =\ &\{X \in \mathcal{A} \mid \exists \text{ a short exact sequence } R' \rightarrowtail Q' \twoheadrightarrow X \text{ with } R' \in \widetilde{\mathcal{R}},\ Q' \in \widetilde{\mathcal{Q}}\}.\end{aligned}$$

So say X is in the top class, that is, that there is a short exact sequence $X \rightarrowtail R \twoheadrightarrow Q$ where $R \in \widetilde{\mathcal{R}}$ and $Q \in \widetilde{\mathcal{Q}}$. Since $(\mathcal{Q}, \widetilde{\mathcal{R}})$ has enough projectives, we can find a short exact sequence $R' \rightarrowtail Q' \twoheadrightarrow R$ where $R' \in \widetilde{\mathcal{R}}$ and $Q' \in \mathcal{Q}$. We take a pullback, and from [3, Proposition 2.12] we get the commutative diagram below with exact rows and columns and whose lower left corner is a bicartesian (pushpull) square.

$$\begin{tikzcd} R' \arrow[r, equal] \arrow[d, tail] & R' \arrow[d, tail] & \\ P \arrow[r, tail] \arrow[d, two heads] & Q' \arrow[r, two heads] \arrow[d, two heads] & Q \arrow[d, equal] \\ X \arrow[r, tail] & R \arrow[r, two heads] & Q \end{tikzcd}$$

Since $\widetilde{\mathcal{R}}$ is closed under extensions, and by (2) we have $\widetilde{\mathcal{Q}} \cap \mathcal{R} = \mathcal{Q} \cap \widetilde{\mathcal{R}}$, we deduce that $Q' \in \widetilde{\mathcal{Q}} \cap \mathcal{R}$. Now since $Q', Q \in \widetilde{\mathcal{Q}}$ and the cotorsion pairs are hereditary, we conclude that $P \in \widetilde{\mathcal{Q}}$. Now the left vertical column shows that X is in the bottom class describing $\mathcal{W}$. A similar argument will show that any X in the bottom class must be in the top class. So the two descriptions of $\mathcal{W}$ coincide.

$\mathcal{W}$ is thick. We must show $\mathcal{W}$ is closed under retracts and that whenever two out of three terms in a short exact sequence $X \rightarrowtail Y \twoheadrightarrow Z$ are in $\mathcal{W}$ then so is the third.

We start by showing that $\mathcal{W}$ is closed under retracts. So let $W \in \mathcal{W}$ and $X \xrightarrow{i} W \xrightarrow{p} X$ be such that $pi = 1_X$. We wish to show $X \in \mathcal{W}$. Start by writing $W \rightarrowtail \widetilde{R} \twoheadrightarrow \widetilde{Q}$ and also apply the fact that $(\widetilde{\mathcal{Q}}, \mathcal{R})$ has enough injectives to get a short exact sequence $X \rightarrowtail R \twoheadrightarrow \widetilde{Q}'$. Now construct a commutative diagram as shown below:

$$\begin{tikzcd} X \arrow[r] \arrow[d, "i"'] & R \arrow[r] \arrow[d, "j"] & \widetilde{Q}' \arrow[d] \\ W \arrow[r] \arrow[d, "p"'] & \widetilde{R} \arrow[r] \arrow[d, "q"] & \widetilde{Q} \arrow[d] \\ X \arrow[r] & R \arrow[r] & \widetilde{Q}' \end{tikzcd}$$

The map j exists because $\mathrm{Ext}^1_{\mathcal{A}}(\widetilde{Q}', \widetilde{R}) = 0$, and similarly q exists because $\mathrm{Ext}^1_{\mathcal{A}}(\widetilde{Q}, R) = 0$. Next, the two right vertical maps exist simply by the universal property of $\widetilde{Q}'$ and $\widetilde{Q}$ being cokernels. Next denote the map $X \rightarrowtail R$ by k, and its cokernel $R \twoheadrightarrow \widetilde{Q}'$ by h. Then we see $(1_R - qj)k = k - qjk = k - k = 0$. So again the universal property of $\widetilde{Q}'$ being the cokernel of k gives us a map $\widetilde{Q}' \xrightarrow{t} R$ such that $th = 1_R - qj$. This proves that $1_R - qj$ factors through $\widetilde{Q}'$, but now we argue that this implies $1_R - qj$ actually factors through an object of $\widetilde{\mathcal{Q}} \cap \mathcal{R} = \mathcal{Q} \cap \widetilde{\mathcal{R}}$. Indeed, using the fact that $(\widetilde{\mathcal{Q}}, \mathcal{R})$ has enough injectives we find $\widetilde{Q}' \rightarrowtail R' \twoheadrightarrow \widetilde{Q}''$, but this time it follows that $R' \in \widetilde{\mathcal{Q}} \cap \mathcal{R}$. Now since $\mathrm{Ext}^1_{\mathcal{A}}(\widetilde{Q}'', R) = 0$, we see that $\widetilde{Q}' \xrightarrow{t} R$ extends over $\widetilde{Q}' \rightarrowtail R'$. So we see that we can find maps $R \xrightarrow{\alpha} R'$ and $R' \xrightarrow{\beta} R$ such that $1_R - qj = \beta\alpha$ and $R' \in \widetilde{\mathcal{Q}} \cap \mathcal{R} = \mathcal{Q} \cap \widetilde{\mathcal{R}}$. Thus the composition

$$R \xrightarrow{(j\ \alpha)} \widetilde{R} \oplus R' \xrightarrow{q+\beta} R$$

is the identity 1_R. But this just means R is a retract of $\widetilde{R} \oplus R'$. Since $\widetilde{\mathcal{R}}$ is closed under direct sums and retracts, we see that $R \in \widetilde{\mathcal{R}}$. This proves $X \in \mathcal{W}$ and we are done.

We now turn to the two-out-of-three property, and our next immediate goal is to show closure of $\mathcal{W}$ under extensions. Note now that $\mathcal{W}$ clearly contains both $\widetilde{\mathcal{Q}}$ and $\widetilde{\mathcal{R}}$; we will use this ahead. We start by making the following claim.

Claim 1. *Suppose $R \rightarrowtail Y \twoheadrightarrow W$ is exact with $R \in \widetilde{\mathcal{R}}$ and $W \in \mathcal{W}$. Then there exists a commutative diagram as below where $\widetilde{Q}, \widetilde{Q}', \widetilde{Q}'' \in \widetilde{\mathcal{Q}}$ and $\widetilde{R}, \widetilde{R}', \widetilde{R}'' \in \widetilde{\mathcal{R}}$:*

$$\begin{tikzcd}
\widetilde{R} \arrow[r, tail] \arrow[d, tail] & \widetilde{R}'' \arrow[r, two heads] \arrow[d, tail] & \widetilde{R}' \arrow[d, tail] \\
\widetilde{Q} \arrow[r, tail] \arrow[d, two heads] & \widetilde{Q}'' \arrow[r, two heads] \arrow[d, two heads] & \widetilde{Q}' \arrow[d, two heads] \\
R \arrow[r, tail] & Y \arrow[r, two heads] & W
\end{tikzcd}$$

Indeed, since $R, W \in \mathcal{W}$, there are short exact sequences $\widetilde{R} \rightarrowtail \widetilde{Q} \twoheadrightarrow R$ and $\widetilde{R}' \rightarrowtail \widetilde{Q}' \twoheadrightarrow W$ with $\widetilde{R}, \widetilde{R}' \in \widetilde{\mathcal{R}}$ and $\widetilde{Q}, \widetilde{Q}' \in \widetilde{\mathcal{Q}}$. But in this case we also have $R \in \widetilde{\mathcal{R}} \subseteq \mathcal{R}$ and so $\mathrm{Ext}^1_{\mathcal{A}}(\widetilde{Q}', R) = 0$. This means there exists a lift as shown:

$$\begin{tikzcd}
 & & \widetilde{Q}' \arrow[dl, dashed] \arrow[d, two heads] \\
R \arrow[r, tail] & Y \arrow[r, two heads] & W
\end{tikzcd}$$

This lift allows for the construction, analogous to the usual Horseshoe Lemma of homological algebra, of a commutative diagram as below:

$$\begin{tikzcd}
\widetilde{R} \arrow[r, tail] \arrow[d, tail] & \widetilde{R}'' \arrow[r, two heads] \arrow[d, tail] & \widetilde{R}' \arrow[d, tail] \\
\widetilde{Q} \arrow[r, tail] \arrow[d, two heads] & \widetilde{Q} \oplus \widetilde{Q}' \arrow[r, two heads] \arrow[d, two heads] & \widetilde{Q}' \arrow[d, two heads] \\
R \arrow[r, tail] & Y \arrow[r, two heads] & W
\end{tikzcd}$$

Since any class that is part of a cotorsion pair is closed under direct sums and extensions, we now have $\widetilde{Q} \oplus \widetilde{Q}' \in \widetilde{\mathcal{Q}}$ and $\widetilde{R}'' \in \widetilde{\mathcal{R}}$, and so we have proved our first claim.

Claim 2. *$\mathcal{W}$ is closed under extensions.*

Now suppose $W \rightarrowtail Y \twoheadrightarrow W'$ is exact with $W, W' \in \mathcal{W}$. We need to prove that $Y \in \mathcal{W}$ too. Since $W \in \mathcal{W}$, we may now find an exact sequence $W \rightarrowtail R \twoheadrightarrow Q$ where $R \in \widetilde{\mathcal{R}}$ and $Q \in \widetilde{\mathcal{Q}}$. Now form the pushout diagram

$$\begin{tikzcd}
W \arrow[r, tail] \arrow[d, tail] & Y \arrow[r, two heads] \arrow[d, tail] & W' \arrow[d, equal] \\
R \arrow[r, tail] \arrow[d, two heads] & P \arrow[r, two heads] \arrow[d, two heads] & W' \\
Q \arrow[r, equal] & Q &
\end{tikzcd}$$

Note the second row is the type of row from Claim 1. So the Horseshoe argument provides a diagram as shown below where $\widetilde{Q}, \widetilde{Q}', \widetilde{Q}'' \in \widetilde{\mathcal{Q}}$ and $\widetilde{R}, \widetilde{R}', \widetilde{R}'' \in \widetilde{\mathcal{R}}$:

$$\begin{tikzcd}
 & & & & & & \widetilde{R}' \arrow[dd, tail] \\
 & & & & \widetilde{R}'' \arrow[urr, two heads] \arrow[dd, tail] & & \\
 & & \widetilde{R} \arrow[urr, tail] \arrow[dd, tail] & & & & \widetilde{Q}' \arrow[dd, two heads] \\
 & & & & \widetilde{Q}'' \arrow[urr, two heads] \arrow[dd, two heads] & W' \arrow[dr, equal] & \\
 & & \widetilde{Q} \arrow[urr, tail] \arrow[dd, two heads] & Y \arrow[urr, two heads] \arrow[dr, tail] & & & W' \\
 & W \arrow[urr, tail] \arrow[dr, tail] & & & P \arrow[urr, two heads] & & \\
 & & R \arrow[urr, tail] & & & &
\end{tikzcd}$$

Now pullback the entire diagram over the original exact sequence $W \rightarrowtail Y \twoheadrightarrow W'$ to get what we want. In particular, the pullback in the middle of the diagram leads to the following bicartesian (pushpull) square:

$$\begin{tikzcd}
\widetilde{R}'' \arrow[r, equal] \arrow[d, tail] & \widetilde{R}'' \arrow[d, tail] & \\
L \arrow[r, tail] \arrow[d, two heads] & \widetilde{Q}'' \arrow[r, two heads] \arrow[d, two heads] & Q \arrow[d, equal] \\
Y \arrow[r, tail] & P \arrow[r, two heads] & Q
\end{tikzcd}$$

and the hereditary property of $\widetilde{\mathcal{Q}}$ gives us $L \in \widetilde{\mathcal{Q}}$.

Claim 3. *If $W \rightarrowtail W' \twoheadrightarrow Z$ is exact with $W, W' \in \mathcal{W}$, then $Z \in \mathcal{W}$.*

Start by writing $W \rightarrowtail \widetilde{\mathcal{R}} \twoheadrightarrow \widetilde{\mathcal{Q}}$ and again forming a pushout diagram

$$\begin{tikzcd}
W \arrow[r, tail] \arrow[d, tail] & W' \arrow[r, two heads] \arrow[d, tail] & Z \arrow[d, equal] \\
\widetilde{\mathcal{R}} \arrow[r, tail] \arrow[d, two heads] & P \arrow[r, two heads] \arrow[d, two heads] & Z \\
\widetilde{\mathcal{Q}} \arrow[r, equal] & \widetilde{\mathcal{Q}} &
\end{tikzcd}$$

Since we have shown $\mathcal{W}$ is closed under extensions, we get $P \in \mathcal{W}$. So now we can write $P \rightarrowtail \widetilde{\mathcal{R}}' \twoheadrightarrow \widetilde{\mathcal{Q}}'$. We now take yet another pushout, of $\widetilde{\mathcal{R}}' \leftarrowtail P \twoheadrightarrow Z$, to get yet another diagram

$$\begin{tikzcd}
\widetilde{\mathcal{R}} \arrow[r, equal] \arrow[d, tail] & \widetilde{\mathcal{R}} \arrow[d, tail] & \\
P \arrow[r, tail] \arrow[d, two heads] & \widetilde{\mathcal{R}}' \arrow[r, two heads] \arrow[d, two heads] & \widetilde{\mathcal{Q}}' \arrow[d, equal] \\
Z \arrow[r, tail] & L \arrow[r, two heads] & \widetilde{\mathcal{Q}}'
\end{tikzcd}$$

(By [3, Proposition 2.12] the lower left square is bicartesian.) Now since $(\mathcal{Q}, \widetilde{\mathcal{R}})$ is a hereditary cotorsion pair, we get $L \in \widetilde{\mathcal{R}}$. So the short exact sequence $Z \rightarrowtail L \twoheadrightarrow \widetilde{\mathcal{Q}}'$ finishes the proof of the claim.

A dual argument shows that if $X \rightarrowtail W' \twoheadrightarrow W$ is exact with $W, W' \in \mathcal{W}$ then so is X. (Use the *other* characterization of $\mathcal{W}$ and pullbacks.) This completes the proof that $\mathcal{W}$ is thick.

$(\mathcal{Q}, \mathcal{W}, \mathcal{R})$ *is a Hovey triple.* We just need to show $\mathcal{Q} \cap \mathcal{W} = \widetilde{\mathcal{Q}}$ and $\mathcal{W} \cap \mathcal{R} = \widetilde{\mathcal{R}}$. The two proofs are similar, so we will just show $\mathcal{W} \cap \mathcal{R} = \widetilde{\mathcal{R}}$.

Clearly $\mathcal{W} \cap \mathcal{R} \supseteq \widetilde{\mathcal{R}}$, so we just need to show $\mathcal{W} \cap \mathcal{R} \subseteq \widetilde{\mathcal{R}}$. So let $X \in \mathcal{W} \cap \mathcal{R}$. As X is in $\mathcal{W}$, we may write a short exact sequence $X \rightarrowtail \widetilde{R} \twoheadrightarrow \widetilde{Q}$. But since $X \in \mathcal{R}$ and $(\widetilde{\mathcal{Q}}, \mathcal{R})$ is a cotorsion pair, this sequence must split. Thus X is a retract of $\widetilde{R}$ and so must be in $\widetilde{\mathcal{R}}$. This completes the proof that $(\mathcal{Q}, \mathcal{W}, \mathcal{R})$ is a Hovey triple. The uniqueness of $\mathcal{W}$ follows from a general fact: The class of trivial objects in a Hovey triple is always unique by [6, Proposition 3.2]. ∎

We end by giving a glimpse of how Theorem 1.2 can be used. A complete hereditary cotorsion pair $(\mathcal{F}, \mathcal{C})$ in an abelian category $\mathcal{A}$ often gives rise to several complete hereditary cotorsion pairs on the associated chain complex category $\mathrm{Ch}(\mathcal{A})$. To illustrate, take for simplicity $\mathcal{A}$ to be the category of modules over a ring R. Then associated to such a pair $(\mathcal{F}, \mathcal{C})$ we typically have the following complete hereditary cotorsion pairs in $\mathrm{Ch}(R)$:

- $(dw\widetilde{\mathcal{F}}, dw\widetilde{\mathcal{F}}^{\perp})$ where $dw\widetilde{\mathcal{F}}$ is the class of all chain complexes X with each X_n in $\mathcal{F}$, and $dw\widetilde{\mathcal{F}}^{\perp}$ is the class of all complexes Y such that $\mathrm{Ext}^1(X, Y) = 0$ for all $X \in dw\widetilde{\mathcal{F}}$.
- $(ex\widetilde{\mathcal{F}}, ex\widetilde{\mathcal{F}}^{\perp})$ where $ex\widetilde{\mathcal{F}}$ is the class of all exact chain complexes X with each X_n in $\mathcal{F}$, and $ex\widetilde{\mathcal{F}}^{\perp}$ is the class of all complexes Y such that $\mathrm{Ext}^1(X, Y) = 0$ for all $X \in ex\widetilde{\mathcal{F}}$.
- $(dg\widetilde{\mathcal{F}}, \widetilde{\mathcal{C}})$ where $\widetilde{\mathcal{C}}$ is the class of all exact chain complexes Y with each Z_nY in $\mathcal{C}$, and $dg\widetilde{\mathcal{F}} = {}^{\perp}\widetilde{\mathcal{C}}$ is the class of all complexes X such that $\mathrm{Ext}^1(X, Y) = 0$ for all $Y \in \widetilde{\mathcal{C}}$.
- $(\widetilde{\mathcal{F}}, dg\widetilde{\mathcal{C}})$ where $\widetilde{\mathcal{F}}$ is the class of all exact chain complexes X with each Z_nX in $\mathcal{F}$, and $dg\widetilde{\mathcal{C}} = \widetilde{\mathcal{F}}^{\perp}$ is the class of all complexes Y such that $\mathrm{Ext}^1(X, Y) = 0$ for all $X \in \widetilde{\mathcal{F}}$.

For any single one of the above cotorsion pairs, the intersection of the two classes is the same. They all equal the class of contractible chain complexes whose components lie in $\mathcal{F} \cap \mathcal{C}$. It turns out that we have class containments as indicated by the diagram below.

$$\begin{tikzcd}[column sep=small]
 & dw\widetilde{\mathcal{F}} \arrow[dl, dash] \arrow[dr, dash] & \\
dg\widetilde{\mathcal{F}} \arrow[dr, dash] & & ex\widetilde{\mathcal{F}} \arrow[dl, dash] \\
 & \widetilde{\mathcal{F}} &
\end{tikzcd}$$

Thus Theorem 1.2 at once yields five model structures on $\mathrm{Ch}(R)$ starting from the single cotorsion pair $(\mathcal{F}, \mathcal{C})$ in R-Mod.

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