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Analytic Baire spaces

by

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To Mary Ellen and in memory of Walter Rudin (1921–2010)

Abstract. We generalize to the non-separable context a theorem of Levi characterizing Baire analytic spaces. This allows us to prove a joint-continuity result for non-separable normed groups, previously known only in the separable context.

1. Introduction. This paper is inspired by Sandro Levi's [Levi], similarly titled paper, *On Baire cosmic spaces*, containing an Open Mapping Theorem (a consequence of the Direct Baire Property given below) and a useful corollary on comparison of topologies; all these results are in the separable realm. Here we give non-separable generalizations (see Main Theorem 1.6) and, as an illustration of their usefulness, Main Theorem 1.9 offers a non-separable version of an Ellis-type theorem (see [Ell, Cor. 2], cf. [Bou1], [Bou2], the more recent [SolSri], or the textbook [AT]) with a 'one-sided' continuity condition implying that a right-topological group generated by a right-invariant metric (i.e. a normed group in the terminology of §4) is a topological group. Unlike Ellis we do not assume that the group is abelian, nor that it is locally compact; the non-separable context requires some preservation of σ-discreteness as a side-condition (see below).

Given that the application in mind is metrizable, references to non-separable descriptive theory remain, for transparency, almost exclusively in the metric realm, though we do comment on the regular Hausdorff context in §5 (see Remark 5.4).

Levi's work draws together two notions: BP—the Baire *set* property (i.e. that a set is open modulo a meagre set, so 'almost open'), and BS—the Baire *space* property (i.e. that Baire's theorem holds in the space). Below we keep

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the distinction clear by using the terms 'Baire property' and 'Baire space'. The connection between BP and BS is not altogether surprising, and as we explain in Remark 4.5 the two are 'almost' the same in a precise sense, at least in the context of normed groups (cf. [Ost-S], where this closeness is fully exploited).

We refer to [Eng] for general topological usage (though we prefer 'meagre' as a term). We say that a subspace S of a metric space X has a *Souslin-*$\mathcal{F}(X)$ *representation* if there is a 'determining' system $\langle F(i|n)\rangle := \langle F(i|n) : i \in \mathbb{N}^{\mathbb{N}}\rangle$ of sets in $\mathcal{F}(X)$ (the closed sets) with

$$S = \bigcup_{i\in I} \bigcap_{n\in\mathbb{N}} F(i|n), \qquad \text{where } I = \mathbb{N}^{\mathbb{N}}$$

and $i|n$ denotes $(i_1, \dots, i_n)$. We will say that a topological space is *classically analytic* if it is the continuous image of a Polish space (Levi terms these 'Souslin') and not necessarily metrizable, in distinction to an (absolutely) *analytic* space, i.e. one that *is* metrizable and is embeddable as a Souslin-$\mathcal{F}$ set in its own metric completion; in particular, in a complete metric space $\mathcal{G}_\delta$-subsets (being $\mathcal{F}_{\sigma\delta}$) are analytic. We call a Hausdorff space *almost analytic* if it is analytic modulo a meagre set. Similarly, a space X' is absolutely $\mathcal{G}_\delta$, or an *absolute-*$\mathcal{G}_\delta$, if X' is a $\mathcal{G}_\delta$ in all spaces X containing X' as a subspace. (This is equivalent to complete metrizability in the narrowed realm of metrizable spaces [Eng, Th. 4.3.24], and to topological/Čech completeness in the narrowed realm of completely regular spaces [Eng, §3.9].) So a metrizable absolute-$\mathcal{G}_\delta$ is analytic; we use this fact in Lemma 6.2.

Levi's results follow from the following routine observation.

THEOREM 1.1 (on the Direct Baire Property, [Levi]). *Let X be a classically analytic space and Y Hausdorff. Every continuous map $f : X \to Y$ has the Direct Baire Property: the image of any open set in X has the Baire property in Y.*

The nub of the theorem is that, with X as above, continuity preserves various analyticity properties such as that open, and likewise closed, sets are taken to analytic sets, in brief: a continuous map is *open-analytic* and *closed-analytic* in the terminology of [Han-74], and so preserves the Baire property. (See Remark 1.5.3 below for a reprise of this theme.) Levi deduces the following characterization of Baire spaces in the category of classically analytic spaces.

THEOREM 1.2 (Levi's Open Mapping Theorem, [Levi]). *Let X be a regular classically analytic space. Then X is a Baire space iff $X = f(P)$ for some continuous map f on some Polish space P with the property that there exists a subspace $X' \subseteq X$ which is a dense metrizable absolute-$\mathcal{G}_\delta$ such that the restriction map $f|P' : P' \to X'$ is open for $P' = f^{-1}(X')$.*

The result may be regarded as implying an 'inner regularity' property (compare the capacitability property) of a classically analytic space X: if X is a Baire space, then X contains a dense absolute-$\mathcal{G}_\delta$ subspace, so a Baire space. The existence of a dense completely metrizable subspace—making X *almost complete* in the sense of Frolík ([Frol], but the term is due to Michael [Mich91])—is a result that implicitly goes back to Kuratowski ([Kur-1, IV.2, p. 88], because a classically analytic set has the Baire property in the restricted sense [Kur-1, Cor. 1, p. 482]). Generalizations of the latter result, including the existence of a restriction map that is a homeomorphism between a $\mathcal{G}_\delta$-subset and a dense set, are given by Michael in [Mich86]; but there the continuous map f requires stronger additional properties such as openness on P (unless P is separable), which Levi's result delivers.

Theorem 1.2 has a natural extension characterizing a Baire space (in the same way) when it is almost analytic. Indeed, with X' as above in Th. 1.2, the space X is *almost complete* and so almost analytic. On the other hand, if X is a Baire space and almost analytic, then by supressing a meagre $\mathcal{F}_\sigma$ and passing to an absolutely $\mathcal{G}_\delta$-subspace, we may assume that X is a Baire space which is analytic, so has the open mapping representation of the theorem, and in particular is almost complete (for more background see [Ost-S]; cf. Cor. 1.8 below).

Since an analytic space is a continuous image, Theorem 1.2 may be viewed as an 'almost preservation' result for complete metrizability under continuity in the spirit of the classical theorem of Hausdorff (resp. Vaĭnstein) on the preservation of complete metrizability by open (resp. closed) continuous mappings—see Remark 1.11.4 at the end of this section for the most recent improvements and the literature of preservation. We note that Michael [Mich91, Prop. 6.5] shows that almost completeness is preserved by demi-open maps (i.e. continuous maps under which inverse images of dense open sets are dense). Theorem 1.2 has an interesting corollary on the comparison of refinement topologies. For a discussion of refinements see [Ost-S, §7.1] (for examples of completely metrizable and of analytic refinements see [Kech, Ths. 13.6, 25.18, 25.19]).

Theorem 1.3 (Levi's Comparison Theorem, [Levi]). *For $\mathcal{T},\mathcal{T}'$ two topologies on a set X with $(X,\mathcal{T}')$ classically analytic (e.g. Polish) and $\mathcal{T}'$ refining $\mathcal{T}$ (i.e. $\mathcal{T}\subseteq\mathcal{T}'$), if $(X,\mathcal{T})$ is a regular Baire space, then there is a $\mathcal{T}$-dense $\mathcal{G}(\mathcal{T})_\delta$-set on which $\mathcal{T}$ and $\mathcal{T}'$ agree.*

We offer a generalization in Main Theorem 1.6 below to the broader category of (absolutely) analytic spaces—be they separable or non-separable metric spaces. We will need the following definitions (see below for comments). Recall that a Hausdorff space X is *paracompact* [Eng, Ch. 5] if every open cover of X has a locally-finite open refinement, and further that for an (indexed) family $\mathcal{B} := \{B_t : t \in T\}$:

(i) $\mathcal{B}$ is *index-discrete* in the space X (or just *discrete* when the index set T is understood) if every point in X has a neighbourhood meeting the sets B_t for at most one $t \in T$,

(ii) $\mathcal{B}$ is *σ-discrete* (abbreviated to σ-d) if $\mathcal{B} = \bigcup_n \mathcal{B}_n$ where each set $\mathcal{B}_n$ is discrete as in (i), and

(iii) $\mathcal{B}$ is a *base* for $\mathcal{E}$ if every member of $\mathcal{E}$ may be expressed as the union of a subfamily of $\mathcal{B}$. For $\mathcal{T}$ a topology (the family of all open sets) with $\mathcal{B} \subseteq \mathcal{T}$ a base for $\mathcal{T}$, this reduces to $\mathcal{B}$ being simply a (topological) *base*.

DEFINITIONS 1.4 ([Han-74, §3]; cf. [Han-71, §3.1] and [Mich82, Def. 3.3]).

1. Call $f : X \to Y$ *base-σ-discrete* (or *co-σ-discrete*) if the image under f of any discrete family in X has a σ-discrete base in Y. We need two refinements that are more useful and arise in practice: call $f : X \to Y$ an *analytic* (resp. *Baire*) *base-σ-discrete map* (henceforth A-σ-d, resp. B-σ-d map) if *in addition*, for any discrete family $\mathcal{E}$ of *analytic* sets in X, the family $f(\mathcal{E})$ has a σ-d base consisting of analytic sets (resp. sets with the Baire property) in Y. We explain in §2 (Th. 2.6 and thereafter) why A-σ-d maps, though not previously isolated, are really the only base-σ-discrete maps needed in practice in analytic space theory.

2 ([Han-74, §2]). An indexed family $\mathcal{A} := \{A_t : t \in T\}$ is *σ-discretely decomposable* (σ-d decomposable) if there are discrete families $\mathcal{A}_n := \{A_{tn} : t \in T\}$ such that $A_t = \bigcup_n A_{tn}$ for each t. (The open family $\{(-r, r) : r \in \mathbb{R}\}$ on the real line has a σ-d base, but is not σ-d decomposable—see [Han-73b, §3].)

3 ([Mich82, Def. 3.3]). Call $f : X \to Y$ *index-σ-discrete* if the image under f of any discrete family $\mathcal{E}$ in X is σ-d decomposable in Y. (Note $f(\mathcal{E})$ is regarded as indexed by $\mathcal{E}$, so could be discrete without being index-discrete; this explains the prefix 'index-' in the terminology.) An index-σ-discrete function is A-σ-d (analytic base-σ-discrete): see Theorem 2.6 below.

REMARKS 1.5. Recall Bing's Theorem ([Eng, Th. 4.4.8]) that a regular space is metrizable iff it has a σ-discrete base. In a separable space discrete sets are at most countable. So all the notions above generalize various aspects of countability; in particular, in a separable metric setting all maps are (Baire) base-σ-discrete. We comment briefly on their standing. (The paper [Han-74] is the primary source for these.)

1. In Definition 1.4.3, above f has a stronger property than base-σ-discreteness. For a proof see [Han-74, Prop. 3.7(i)]; cf. also [Mich82, Prop. 4.3] which shows that f with closed fibres has the property in 1.4.3 iff it is

base-σ-discrete and has fibres that are $\aleph_1$-compact, i.e. separable (in the metric setting). The stronger property is often easier to work with than Baire base-σ-discreteness; in any case the concepts are close, since for metric spaces and κ an infinite cardinal, X is a base-σ-discrete continuous image of $\kappa^{\mathbb{N}}$ iff X is an index-σ-discrete continuous image of a closed subset of $\kappa^{\mathbb{N}}$, both equivalent to analyticity [Han-74, Th. 4.1]; cf. Proposition 2.9 below. (See also [Han-92, Th. 4.2]. In fact the natural continuous index-σ-discrete representation of an analytic set has separable fibres, for which see §2 below; for a study of fibre conditions see [Han-98].)

2. Base-σ-discrete *continuous* maps (in particular, Baire base-σ-discrete and index-σ-discrete continuous maps) preserve analyticity [Han-74, Cor. 4.2].

3. If X is metric and (absolutely) analytic and $f : X \to Y$ is injective and *closed-analytic* (or *open-analytic*), then f is base-σ-discrete [Han-74, Prop. 3.14]. Base-σ-discreteness is key to this paper just as open-analyticity is key to the separable context of the Levi results above.

4. If $\mathcal{B} = \bigcup_{n\in\mathbb{N}} \mathcal{B}_n$, with each $\mathcal{B}_n$ discrete, is a σ-d base for the metrizable space X and each $f(\mathcal{B}_n)$ is σ-d decomposable, then f is index-σ-discrete, and so base-σ-discrete [Han-74, Prop. 3.9].

5. A discrete collection $\mathcal{A} = \{A_t : t \in T\}$ of analytic sets has the property that any subfamily has analytic union, i.e. is 'completely additive analytic'. It turns out that in an analytic space a disjoint (or a point-finite) collection $\mathcal{A}$ is completely additive analytic iff it is σ-d decomposable (see [KP] generalizing the disjoint case in [Han-71, Th. 2]; see also [FH]). By the proof of Theorem 2.6 the decompositions can be into analytic sets.

Main Theorem 1.6 (Generalized Levi Open Mapping Theorem—nonseparable case). *Let X be an analytic space (more generally, paracompact and $\mathcal{K}$-analytic—as defined in* §2*). Then X is a Baire space iff $X = f(P)$ for some continuous, index-σ-discrete map f on a completely metrizable space P with the property that there exists a dense completely metrizable $\mathcal{G}_\delta$-subset X' of X such that the restriction $f|P' : P' \to X'$ is an open mapping, with $P' = f^{-1}(X')$, again a topologically complete subspace.*

This is proved in §3. For related results on restriction maps of other special maps, see [Mich91, §7]. (Compare also [Han-92, Ths. 6.4 and 6.25].) As immediate corollaries one has:

Corollary 1.7 (Generalized Levi Comparison Theorem). *If $\mathcal{T}, \mathcal{T}'$ are two topologies on a set X with $(X, \mathcal{T})$ a regular Baire space, and $\mathcal{T}'$ an absolutely analytic (e.g. completely metrizable) refinement of $\mathcal{T}$ such that every $\mathcal{T}'$-index-discrete collection is σ-discretely decomposable under $\mathcal{T}$, then there is a $\mathcal{T}$-dense $\mathcal{G}(\mathcal{T})_\delta$-set on which $\mathcal{T}$ and $\mathcal{T}'$ agree.*

Proof. For f take the identity map from $(X, \mathcal{T}')$ to $(X, \mathcal{T})$, which is continuous and index-σ-discrete. ■

COROLLARY 1.8 (Almost Completeness Theorem). *X is almost analytic and a Baire space iff X is almost complete.*

Proof. If X contains a co-meagre analytic subspace A, then by Theorem 1.6, A, being a Baire space, contains a dense completely metrizable subspace X' which is co-meagre in X. So X is almost complete. ■

For a sharper characterization in the case of normed groups see [Ost-LB3, Th. 2]; cf. also Theorem 4.4. Corollary 1.7 will enable us to prove (in §6) the automatic continuity result of Main Theorem 1.9 below for right-topological groups with a right-invariant metric d_R (the normed groups of §4). We write $d_L(x, y) := d_R(x^{-1}, y^{-1})$, which is left-invariant, and $d_S := \max\{d_R, d_L\}$ for the symmetrized ('ambidextrous') metric. The basic open sets under d_S take the form $B^R_\varepsilon(x) \cap B^L_\varepsilon(x)$, i.e. an intersection of balls of ε-radius under d_R and d_L centered at x, giving the join (coarsest common refinement) of the d_R and d_L topologies. Following [Ost-J], for P a topological property it is convenient to say that the metric space (X, d_R) is topologically *symmetrized-P,* or just *semi-P*, if (X, d_S) has property P. In particular (X, d_R) is *semi-complete* if (X, d_S) is topologically complete. As d_R and d_L are isometric under inversion, (X, d_R) is semi-complete iff (X, d_L) is.

MAIN THEOREM 1.9 (Semi-Completeness Theorem, cf. [Ost-J]). *For a normed group X, if (X, d_R) is semi-complete and a Baire space, and the continuous embedding map $j : (X, d_S) \to (X, d_R)$ is Baire base-σ-discrete (e.g. index-σ-discrete), in particular this is so if X is separable, then the right and left uniformities of d_R and d_L coincide and so (X, d_R) is a topologically complete topological group.*

The following result will be needed later in conjunction with Theorem 5.2.

LEMMA 1.10. *For a normed group X, if the continuous embedding map $j : (X, d_S) \to (X, d_R)$ is index-σ-discrete (resp. base-σ-discrete), then so also is the inversion mapping from (X, d_R) to (X, d_R), i.e. $i : x \mapsto x^{-1}$.*

Proof. Suppose $\mathcal{V}$ is a family of sets that is discrete in (X, d_R). Then $\mathcal{V}^{-1} := \{V^{-1} : V \in \mathcal{V}\}$ is a family of sets that is discrete in (X, d_L). As the d_S topology refines the d_L topology, $\mathcal{V}^{-1}$ is discrete in (X, d_S). Assuming that j is index-σ-discrete (resp. base-σ-discrete), the family $j(\mathcal{V}^{-1}) = \mathcal{V}^{-1}$ is σ-d decomposable (has a σ-d base) in (X, d_R). So inversion maps $\mathcal{V}$ to a family $\mathcal{V}^{-1}$ that is σ-d decomposable (has a σ-d base) in (X, d_R). ■

REMARKS 1.11. 1. Theorem 1.9 generalizes a classical result for *abelian* locally compact groups due to Ellis [Ell].

2. Remarks 1.5 noted that the 'index-σ-discreteness condition' imposed in the theorem is a natural one from the perspective of the non-separable theory of analytic sets, and Lemma 1.10 interprets this in terms of inversion; compare point 4 below.

3. For separable spaces, where discrete families are countable and so the embedding j above automatically preserves σ-discreteness, the result here was proved in [Ost-J] (to which we refer for the literature) in the form that a semi-Polish, normed group X, Baire in the right norm-topology, is a topologically-complete topological group. Rephrased in the language of uniformities generated by the norm ([Kel, Ch. 6, Pb. O]), this says that a normed group, Polish in the ambidextrous uniformity and Baire in either of the right or left uniformities, has coincident right and left uniformities, and so is a topological group. Key to the proof is that a continuous image of a complete separable metric space is a classically analytic space. So the 'index σ-discreteness condition' is exactly the condition that secures preservation of analyticity. In the non-separable context continuity is not enough to preserve analyticity, and an additional property is needed, involving σ-d as above: see [Han-98, Example 4.2] for a non-analytic metric space that is a one-to-one continuous image of $\kappa^{\mathbb{N}}$ for some uncountable cardinal κ (so a continuous image of a countable product of discrete, hence absolutely analytic, spaces of cardinal κ).

4. Recent work by Holický and Pol [HP], in response to Ostrovsky's insights and based on [Mich86, esp. §6] (which itself goes back to [GM]), connects preservation of (topological) completeness under continuous maps between metric spaces to the classical notion of *resolvable* sets. (The latter notion provides the natural generalization to Ostrovsky's setting; see also Holický [H] for non-metrizable spaces.) They find that a map f preserves completeness if it 'resolves countable discrete sets', i.e. for every countable metrically-discrete set C and open neighbourhood V of C there is L with $C \subseteq L \subseteq V$ such that $f(L)$ is a resolvable set.

Consider the implications for a group X with right-invariant metric d_R (see above), when for f one takes j the identity embedding $j : (X, d_S) \to (X, d_R)$, and $C = \{c_n : n \in \mathbb{N}\}$ is a d_S-discrete set (so that C and C^{-1} are d_R-discrete). To obtain the desired resolvability for j, it is necessary and sufficient, for each C as above and each assignment $r : \mathbb{N} \to \mathbb{R}_+$ with $r_n \to 0$, that there exist d_R-resolvable sets $L_n \subseteq B^R_{r_n}(c_n) \cap B^L_{r_n}(c_n)$. Since $B^L_r(c) = \{x : d_R(c^{-1}, x^{-1}) < r\} = \{x : d_R(c^{-1}, y) < r \text{ and } y = x^{-1}\} = \{y^{-1} : d_R(c^{-1}, y) < r\}$, this is yet another condition relating inversion to the d_R-topology, via the sets $B^R_r(c^{-1})^{-1}$.

2. Analyticity: upper-semicontinuity and the Baire property. This section prepares three tools for later use. The first two lift to the non-separable context results standard in the separable case, namely Hansell's Characterization Theorem (Th. 2.1), yielding representation of analytic sets in the form $\bigcup_{j\in\kappa^{\mathbb{N}}} H(j)$ with H upper-semicontinuous and compact-valued, and Nikodym's Theorem (Th. 2.2), implying their Baire property. The third is the conclusion that analytic base-σ-discrete maps (A-σ-d maps, for short) are the only ones that matter (compare Prop. 2.9).

We content ourselves mostly with a metric context, though a wider one is feasible (consult [Han-92]). Recall that a metric space S is said to be absolutely analytic, or just *analytic,* if it is Souslin-$\mathcal{F}(S^*)$, i.e. is Souslin in its (metric) completion S^*. A Hausdorff space S is $\mathcal{K}$-*analytic* if $S = \bigcup_{i\in I} K(i)$ for some upper-semicontinuous map K from I to $\mathcal{K}(S)$, the compact subsets of S. In a separable metric space, an absolutely analytic subset is $\mathcal{K}$-analytic [Rog-Jay, Cor. 2.4.3 plus Th. 2.5.3]. In a non-separable complete metric space X, it is not possible to represent a Souslin-$\mathcal{F}(X)$ subset S of X as a $\mathcal{K}$-analytic set relative to $I = \mathbb{N}^{\mathbb{N}}$. Various generalizations of countability enter the picture here, as we now recall, referring to two survey papers: [St] and the more recent [Han-92].

Denoting by $\mathrm{wt}(X)$ the weight of the space X (i.e. the smallest cardinality of a base for the topology), and replacing $I = \mathbb{N}^{\mathbb{N}}$ by $J = \kappa^{\mathbb{N}}$ for $\kappa = \mathrm{wt}(X)$, with basic open sets $J(j|n) := \{j' \in J : j'|n = j|n\}$, consider sets S represented by the following *extended κ-Souslin operation* (briefly: the *extended Souslin operation*):

$$S = \bigcup_{j\in J} H(j), \quad \text{where} \quad H(j) := \bigcap_{n\in\mathbb{N}} H(j|n),$$

applied to a *determining system* $\langle H(j|n)\rangle := \langle H(j|n) : j \in \kappa^{\mathbb{N}}\rangle$ of sets from a family $\mathcal{H}$ subject to the requirement that:

(i) $\{H(j|n) : j|n \in \kappa^n\}$ is σ-discrete for each n.

For $\mathcal{H} = \mathcal{F}$ the corresponding extended Souslin-$\mathcal{F}$ sets reduce to the κ-*Souslin* sets of [Han-92]. (This slightly refines Hansell's terminology, and abandons Stone's term 'κ-restricted Souslin' of [St].) Say that the determining system is *shrinking* if

(ii) $\mathrm{diam}_X H(j|n) < 2^{-n}$, so that $H(j)$ is empty or single-valued, and so compact.

With X above complete (e.g. $X = S^*$) and for $\mathcal{H} = \mathcal{F}(X)$, the mapping $H : J \to \mathcal{K}(X)$ evidently yields a natural upper-semicontinuous representation of S. We refer to it below, in relation to the Analytic Cantor Theorem, and also in Proposition 2.9; there the fact that $C := \{j : H(j) \neq \emptyset\}$ is closed in $\kappa^{\mathbb{N}}$ yields a *natural representation* of S as the image of C under a map h defined by $H(j) = \{h(j)\}$. The map h is continuous and index-σ-discrete with countable fibres (by (i) above), as noted in Remark 1.5.1.

THEOREM 2.1 (Characterization of analytic sets, [Han-73a]). *In a metric space X, the Souslin-$\mathcal{F}(X)$ subsets of X are precisely the sets S represented by a shrinking determining system of closed sets through the extended κ-Souslin representation above with $\kappa = \mathrm{wt}(X)$.*

For other equivalent representations, including a weakening of σ-discreteness in X above to σ-d relative to its union, as well as to σ-d decompositions, see [Han-73b] and [Han-73a]. Thus, working relative to J, the corresponding extended Souslin sets exhibit properties similar to the $\mathcal{K}$-*analytic* sets *relative* to I. In particular, of interest here is:

THEOREM 2.2 (Nikodym Theorem for analytic sets). *In a metric space S, analytic sets have the Baire property.*

Proof. Since $S \cap \mathcal{F}(S^*) = \mathcal{F}(S)$, the theorem follows immediately from the definition of analytic sets as Souslin-$\mathcal{F}(S^*)$ and from Nikodym's classical theorem ([Rog-Jay, §2.9], or [Kech, Th. 29.14, cf. Th. 21.6]) asserting that the Baire property is preserved by the usual Souslin operation, with the consequence that Souslin-$\mathcal{F}$ sets have the Baire property (since a closed set differs from its interior by a nowhere dense set). ■

Using Hansell's characterization theorem and again Nikodym's classical theorem, one also has the equally thematic result:

THEOREM 2.3 (Nikodym Theorem for extended Souslin sets). *In a metric space, sets with a shrinking extended Souslin-$\mathcal{F}$ representation have the Baire property.*

Actually, this is a direct consequence of the following result, apparently unrecorded in the literature, so for completeness and in view of its brevity we include a proof (despite not needing it).

THEOREM 2.4 (Nikodym Stability Theorem for the extended Souslin operation). *In a topological space, the extended Souslin operation applied to a determining system of sets with the Baire property yields a set with the Baire property.*

Proof. One need only check that the classical 'separable' proof given for the usual Souslin operation in [Rog-Jay, Th. 2.9.2, pp. 43–44] continues to hold mutatis mutandis for the choice $\mathcal{M}$ of the family of sets with the Baire property and $\mathcal{N}$ of the meagre subsets of the metric space. In particular, we must interpret $\mathbb{N}^{\mathbb{N}}$ there as $\kappa^{\mathbb{N}}$ throughout, with $\kappa^{(\mathbb{N})}$ denoting finite sequences with terms in κ. So, consider the extended Souslin operation above applied to a determining system of sets $\langle B(\sigma|n)\rangle$ in $\mathcal{M}$. Then $\{B(\sigma|n) : \sigma|n \in \kappa^n\}$ is a σ-d family for each n. By Banach's localization principle, or Category Theorem ([Oxt, Ch. 16], [Kel, Th. 6.35], [Rog-Jay, pp. 44–45], or [Kur-1, §10.III, Union Theorem]), $\mathcal{N}$ is closed under σ-d unions, and hence so is $\mathcal{M}$ (open sets being closed under arbitrary unions). As the system $\langle D(\sigma|n)\rangle$ is a refinement of the $\langle B(\sigma|n)\rangle$ system, $\{D(\sigma|n) : \sigma|n \in \kappa^n\}$ is also σ-d for each n, and so the union $\bigcup\{D(\sigma|n,t) : t \in \kappa\}$ is in $\mathcal{M}$. (Note that the sets $D(\sigma|n)$ are defined as finite intersections of sets in $\mathcal{M}$.) Each set $N(\sigma|n) := D(\sigma|n) \setminus \bigcup\{D(\sigma|n,t) : t \in \kappa\}$ is in $\mathcal{N}$, as $\mathcal{N}$ is closed under subset formation, and again the family $\{N(\sigma|n) : \sigma|n \in \kappa^n\}$ is σ-d for each n, as before by refinement: $N(\sigma|n) \subseteq D(\sigma|n)$. Hence

$$L := \bigcup\{N(\sigma) : \sigma \in \kappa^{(\mathbb{N})}\} = \bigcup_{n\in\mathbb{N}} \bigcup\{N(\sigma|n) : \sigma|n \in \kappa^n\}$$

is in $\mathcal{N}$, again by Banach's Category Theorem. The remainder of the proof in [Rog-Jay, Th. 2.9.2, pp. 43–44] now applies verbatim. ■

A similar, but simpler, argument with $\mathcal{M}$ the *Radon measurable* sets shows these to be stable under the extended Souslin operation (using measure completeness and local determination, for which see [Fre4, 412J, cf. 431A], and measurable envelopes [Fre2, 213L]).

Evidently, the standard separable category arguments may also be applied to σ-d decompositions of a set, in view of Banach's Category Theorem just cited.

Finally, since $H : J \to \mathcal{K}(X)$ above is upper-semicontinuous (for X complete and $\mathcal{H} = \mathcal{F}$), the following theorem, used in the separable context of [Ost-AH, §2] and [Ost-LB3, Th. AC], continues to hold in the non-separable context (by the same proof), which permits us to quote freely some of its consequences in §4 in such a context.

Theorem 2.5 (Analytic Cantor Theorem). *Let X be a Hausdorff space and $A = K(J)$, with $K : J \to \mathcal{K}(X)$ compact-valued and upper-semicontinuous. If F_n is a decreasing sequence of (non-empty) closed sets in X such that $F_n \cap K(J(j_1,\ldots,j_n)) \neq \emptyset$ for some $j = (j_1,\ldots) \in J$ and each n, then $K(j) \cap \bigcap_n F_n \neq \emptyset$.*

Here, beyond upper-semicontinuity, we do not need properties related to the notion of σ-d possessed by the mapping H (for which see [HJR]).

We return to a discussion of analytic base-σ-discrete maps, promised in §1. Recall that their definition requires in addition to base-σ-discreteness that, *for any discrete family $\mathcal{E}$ of analytic sets* in X, the family $f(\mathcal{E})$ has a σ-d base consisting of sets with the Baire property. The remaining results in this section are gleaned from a close reading of the main results in [Han-74] in respect of base-σ-discrete maps, i.e. Hansell's sequence of results 3.6–3.10 and Th. 4.1, all of which derive the required base-σ-discrete property by arguments that combine σ-d decompositions with discrete collections of singletons. We shall see below that all these results may be refined to the A-σ-d context.

THEOREM 2.6. *An index-σ-discrete function is A-σ-d (analytic base-σ-discrete).*

Proof. Suppose that an indexed family $\mathcal{A} := \{A_t : t \in T\}$ has a σ-d decomposition using discrete families $\mathcal{A}_n := \{A_{tn} : t \in T\}$, with $A_t = \bigcup_n A_{tn}$ for each t, and all the sets A_t have the Baire property (so, in particular, if the sets A_t are analytic—by Nikodym's Theorem above (Th. 2.2)). Putting $\tilde{A}_{tn} := A_t \cap \bar{A}_{tn}$ and $\widetilde{\mathcal{A}}_n := \{\tilde{A}_{tn} : t \in T\}$, which is discrete, we obtain a B-σ-d decomposition for $\mathcal{A}$ (or an A-σ-d one in the case of an analytic σ-d decomposition), and so a fortiori a B-σ-d base for $\mathcal{A}$ (or an A-σ-d one).

Thus, if $\mathcal{E}$ is a discrete family of analytic sets, and f is index-σ-discrete, then $\mathcal{A} := f(\mathcal{E})$, which consists of analytic sets (see Remark 1.5.2 above), has for its σ-d base the family $\bigcup_n \widetilde{\mathcal{A}}_n$ of analytic sets, so with the Baire property, where $\widetilde{\mathcal{A}}_n$ are as just given above. ■

PROPOSITION 2.7 (Portmanteau Proposition, [Han-74: (i) Prop. 3.4; (ii) & (iii) Prop. 3.7; (iv) Cor. 3.8]). *The map $f : X \to Y$ is A-σ-d in each of the following circumstances:*

- (i) *it is a composition of A-σ-d maps;*
- (ii) *$f(\mathcal{E})$ is σ-d decomposable for discrete $\mathcal{E} \subseteq \mathcal{F}$;*
- (iii) *$f(\mathcal{E})$ is σ-d decomposable for discrete $\mathcal{E} \subseteq \mathcal{G}$, for metric X;*
- (iv) *f is injective open/closed;*
- (v) *each $f(\mathcal{B}_n)$ is σ-d decomposable, for some σ-d base $\bigcup_n \mathcal{B}_n$, for metric X.*

The proofs are as in [Han-74] using the construction of $\widetilde{\mathcal{A}}_n$ in Theorem 2.6. In similar vein are the next two refinements of results due to Hansell—together verifying the adequacy of A-σ-d maps for analytic-sets theory.

PROPOSITION 2.8 ([Han-74, Prop. 3.10]). *A closed surjective map onto a metrizable space is A-σ-d.*

Proof. Since singletons are analytic, a base that is a discrete family of singletons is an A-σ-d base. This combined with the construction in Theorem 2.6 above refines the argument for Hansell's Prop. 3.10 proving that a closed surjective map onto a metrizable space is an A-σ-d map. ■

PROPOSITION 2.9 ([Han-74, Th. 4.1]). *Analytic metric spaces are the A-σ-d continuous images of $\kappa^{\mathbb{N}}$.*

Proof. By Remark 1.5.2 again, there is only one direction to consider. So let S be analytic; we refine Hansell's argument. Observe first that the argument for Hansell's Prop. 3.5(ii) proves more: if Y is σ-discrete, then any map into Y is A-σ-d (as in Th. 2.6 above). Next, using the notation H for analytic sets established above (in a complete context, with $\mathcal{H} = \mathcal{F}$), work in the closed subspace $C \subseteq \kappa^{\mathbb{N}}$ comprising those j with $H(j) \neq \emptyset$, and define, as above, the (continuous) map h on C via $H(j) = \{h(j)\}$. Observe that h takes, for each n, the discrete family of basic open sets $J(j|n)$, relativized to C, to the σ-d family of analytic sets $h(J(j|n) \cap C)$, and so h is an A-σ-d map (by [Han-74, Cor. 3.9], reported in Remark 1.5.4 above). Stone's canonical retraction r of $\kappa^{\mathbb{N}}$ onto any closed subspace as applied to the closed subspace C has σ-discrete range on $\kappa^{\mathbb{N}} \setminus C$ and is the identity homeomorphism on C—for details see [Eng-ret] (where r is also shown to be a closed map). So, in view of the preceding two observations, r is an A-σ-d map. Hence $h \circ r$ is A-σ-d, since composition of A-σ-d maps is A-σ-d, and provides the required characterization. ∎

3. Generalized Levi Theorem. The generalized Levi characterization of Baire spaces in Main Theorem 1.6 is a consequence of the following result, which we also apply in §6.

LEMMA 3.1 (Generalized Levi Lemma). *If $f : X \to Y$ is surjective, continuous and Baire base-σ-discrete (in particular, index-σ-discrete) from X metric and analytic to Y a paracompact space, then there is a dense metrizable $\mathcal{G}_\delta$-subspace $Y' \subseteq Y$ such that for $X' := f^{-1}(Y')$ the restriction map $f|X' : X' \to Y'$ is open.*

Proof. Let $\mathcal{A} = \bigcup_n \mathcal{A}_n = \{A_t : t \in T\}$ be an open base for X with $\mathcal{A}_n = \{A_{tn} : t \in T_n\}$ discrete. Then $E_t := f(A_t)$ is analytic (see Remark 1.5.2), so has the Baire property (Th. 2.2, Nikodym's Theorem for analytic sets). Let $\mathcal{E}_n = \{f(A) : A \in \mathcal{A}_n\}$ and let $\mathcal{B}_n$ be a σ-d base for $\mathcal{E}_n$ consisting of sets with the Baire property. Put $\mathcal{B}_n = \bigcup_m \mathcal{B}_{nm}$ with each $\mathcal{B}_{nm}$ discrete. Thus for each $t \in T$ and $E_t \in \mathcal{E}_n$ one has

$$E_t = \bigcup_m \bigcup \{B : B \subseteq E_t \text{ and } B \in \mathcal{B}_{nm}\}.$$

Put $\mathcal{B} = \bigcup_{nm} \mathcal{B}_{nm}$. For each $B \in \mathcal{B}$ pick an open set U_B and meagre sets N_B and M_B such that

$$B = (U_B \setminus N_B) \cup M_B$$

with M_B disjoint from U_B and with $N_B \subseteq U_B$. As $\{B : B \in \mathcal{B}_{nm}\}$ is discrete, the set

$$M := \bigcup_{n,m} \bigcup \{M_B : B \in \mathcal{B}_{nm}\}$$

is meagre. By paracompactness of Y (cf. [Eng, Th. 5.1.18]), since $\{U_B \setminus N_B : B \in \mathcal{B}_{nm}\}$ is discrete, for $B \in \mathcal{B}_{nm}$ we may select open sets W_B with $U_B \setminus N_B \subseteq W_B$ with $\{W_B : B \in \mathcal{B}_{nm}\}$ discrete. Without loss of generality $W_B \subseteq U_B$ (otherwise replace W_B by $W_B \cap U_B$). So $U_B \setminus N_B \subseteq W_B \subseteq U_B$ and hence $U_B \setminus N_B = W_B \setminus N_B$. Then

$$N := \bigcup_{n,m} \bigcup \{W_B \cap N_B : B \in \mathcal{B}_{nm}\}$$

is also meagre. Now put $Y' := Y\backslash(M\cup N)$ and $W := \bigcup_{m,n}\bigcup\{W_B : B\in\mathcal{B}_{nm}\}$, which is open. Then for $B \in \mathcal{B}_{nm}$ one has

$$B \cap Y' = W_B \cap Y',$$

so that B is open relative to Y' and also $\{Y' \cap W_B : B \in \mathcal{B}_{nm}\}$ is open and discrete in Y'. Now for $t \in T$, since $E_t \in \mathcal{E}_n$ for some n and $\mathcal{B}_n$ is a base for $\mathcal{E}_n$,

$$\begin{aligned} E_t \cap Y' &= \bigcup_{n,m}\bigcup\{B \cap Y' : B \subseteq E_t \ \&\ B \in \mathcal{B}_{nm}\} \\ &= \bigcup_{n,m}\bigcup\{W_B \cap Y' : B \subseteq E_t \ \&\ B \in \mathcal{B}_{nm}\} \end{aligned}$$

is open in Y'.

For $G \subseteq X$ open, since $\mathcal{A}$ is a (topological) base, we may write

$$G := \bigcup_n \mathcal{A}_n^G \quad \text{with} \quad \mathcal{A}_n^G := \{A : A \subseteq G \ \&\ A \in \mathcal{A}_n\}.$$

Then

$$f(G) := \bigcup_n f(\mathcal{A}_n^G) \quad \text{with} \quad f(\mathcal{A}_n^G) := \{E : E = f(A) \ \&\ A \subseteq G \ \&\ A \in \mathcal{A}_n\}.$$

So for $X' := f^{-1}(Y')$,

$$f(G \cap X') = Y' \cap \bigcup_{n,m} \mathcal{W}_{nm}^G \quad \text{with}$$

$$\mathcal{W}_{nm}^G := \{W_B : B \subseteq f(A) \ \&\ B \in \mathcal{B}_{nm} \ \&\ A \subseteq G \ \&\ A \in \mathcal{A}_n\},$$

which is open in Y'.

Since f is continuous and $\mathcal{W}_{nm} := \{W_B : B \in \mathcal{B}_{nm}\}$ is discrete for each n, m, this also shows that the family $\bigcup_{n,m}\{Y' \cap W_B : B \in \mathcal{B}_{nm}\}$ is a σ-d base for Y'. Being paracompact, Y is regular [Eng, Th. 5.1.5], so the subspace Y' is regular [Eng, Th. 2.1.6], and so Y' is metrizable by Bing's Characterization Theorem (see [Eng, Th. 4.4.8], cf. Remarks 1.5). Finally, by replacing the meagre sets M, N by larger sets that are unions of closed nowhere dense sets, we obtain in place of Y' a smaller, metrizable, dense $\mathcal{G}_\delta$-subspace. ∎

REMARKS 3.2. 1. For Lemma 3.1 we replaced each system $\{U_B : B \in \mathcal{B}_{nm}\}$ to obtain discrete systems $\{W_B : B \in \mathcal{B}_{nm}\}$ to reduce the sets N_B to the sets $N_B \cap W_B$ and only then did we take unions. This circumvents the suggested approach (in a parenthetical remark) to the proof of Theorem 6.4(c) in [Han-92] (by way of representing a Baire set as $E = (G_E \setminus P_E) \cup Q_E$ with $P_E \subseteq E$ (sic) for G_E open and P_E, Q_E meagre).

2. Given an arbitrary base $\mathcal{B}$ one may replace each $B \in \mathcal{B}$ with a Baire envelope B^+ such that $B \subseteq B^+ \subseteq \bar{B}$. Then $\mathcal{B}^+ = \bigcup_n \mathcal{B}_n^+$ is σ-d and $E_t \subseteq$

$E_t^+ := \bigcup_n \bigcup\{B^+ : B \subseteq E_t \text{ and } B \in \mathcal{B}_n\}$, with $E_t^+ \setminus E_t$ meagre. However, it is not clear that the union of these meagre sets is meagre.

Proof of Main Theorem 1.6 (Non-separable Levi Open Mapping Theorem). Let X be analytic of weight κ. Then for some closed subset P of $\kappa^{\mathbb{N}}$ there is a continuous index-σ-discrete map $f : P \to X$. Form P' and X' analogously to X' and Y' in the preceding lemma. As X is a Baire space and X' is co-meagre, without loss of generality X' is a dense $\mathcal{G}_\delta$ and is metrizable. Also P' is a $\mathcal{G}_\delta$-subspace of the complete space $\kappa^{\mathbb{N}}$, hence is also topologically complete. So P' has the desired properties. As X' is metrizable, the result now follows from Hausdorff's Theorem that the image under an open continuous mapping of the completely metrizable space P' onto a metrizable space X' is also completely metrizable (for a proof see e.g. [Anc], or for a recent account e.g. [HP]).

For the converse, as X' is metrizable, the result again follows from Hausdorff's Theorem. Thus X' is completely metrizable. But its complement in X is meagre. So X is a Baire space—in fact an almost complete space. ■

4. Normed-group preliminaries. We recall the definition of a normed group from [BOst-N] and cite from [Ost-LB3] four results that we need in the next two sections. The first (Th. 4.3) is quite general, but we need to observe here that in view of §2 the other three (Ths. 4.4, 4.6, 4.8) continue to hold in the new non-separable context here.

DEFINITION 4.1. For T an *algebraic* group (i.e. with no topology) with neutral element e, say that $\|\cdot\| : T \to \mathbb{R}_+$ is a *group-norm* ([BOst-N]) if the following properties hold:

(i) *Subadditivity* (Triangle inequality): $\|st\| \le \|s\| + \|t\|$.
(ii) *Positivity*: $\|t\| > 0$ for $t \neq e$ and $\|e\| = 0$.
(iii) *Inversion* (Symmetry): $\|t^{-1}\| = \|t\|$.

REMARKS 4.2. 1. The group-norm generates a right and a left *norm topology* (equivalently, right and left uniformity—cf. [Kel, Ch. 6, Pr. O]) via the right-invariant and left-invariant metrics $d_R^T(s,t) := \|st^{-1}\|$ and $d_L^T(s,t) := \|s^{-1}t\| = d_R^T(s^{-1},t^{-1})$. We omit the superscript if context allows and identify the two topologies by reference to the metric. Since $d_L(t,e) = d_L(e,t^{-1}) = d_R(e,t)$, convergence at e is identical under either topology. Evidently (T,d_R) is homeomorphic to (T,d_L) under inversion. As a result, topological properties (e.g. density character) of the two norm topologies d_R, d_L are in fact norm properties, although properties of a specific subset of T (e.g. Baire property) may depend on the choice of norm topology. Preference of d_R over d_L is motivated by the right-invariance of the supremum metric on the space of bounded homeomorphisms—see [Ost-S]; in the absence of a qualifier, the 'right' norm topology is to be understood.

2. Under either norm topology, there is continuity of operations at e. At further distances the topology may force the group operations to be increasingly 'less' continuous.

3. Note that in the right norm-topology the right shift $\rho_t(s) := st$ is a uniformly continuous homeomorphism, since $d_R(sy, ty) = d_R(s, t)$; thus d_R makes T a right-topological group.

4. Under the d_R topology, $B_r(x) = \{t : d_R(t, x) < r\} = B_r(e_T)x$.

5. If d^T is a one-sidedly invariant metric on T, then $\|t\| := d^T(t, e_T)$ is a norm.

NOTATION. We use the subscripts R, L, S as in $x_n \to_R x$ etc. to indicate convergence in the corresponding metrics d_R, d_L, d_S derived from the norm (so that e.g. $d_R(x, y) := \|xy^{-1}\|$—see §1). Note that a metrizable topological group is a normed group, by the Birkhoff–Kakutani Theorem ([Bir], [Kak]; in fact this is a normability theorem for certain right-topological groups—see [Ost-LB3], or [Ost-S]).

THEOREM 4.3 (Equivalence Theorem, [BOst-N, Th. 3.4]). *A normed group X is a topological group under the right (resp. left) norm-topology iff each conjugacy*

$$\gamma_g(x) := gxg^{-1}$$

is right-to-right (resp. left-to-left) continuous at $x = e_X$ (and so everywhere), i.e. for $z_n \to_R e_X$ and any g,

$$gz_n g^{-1} \to_R e_X.$$

Equivalently, it is a topological group iff left/right-shifts are continuous for the right/left norm topology, or iff the two norm topologies are themselves equivalent, i.e. the left and right uniformities generated by the norm coincide.

The following results were proved in [Ost-LB3] for classically analytic spaces. Their proofs continue to hold for the more general non-separable definition of analytic space given and reviewed in §2, since those proofs in fact rely only on the Analytic Cantor Theorem as stated in 2.5 above, and σ-d decomposition.

THEOREM 4.4 ([Ost-LB3, Th. 1]). *In a normed group X under d_R, if X contains a non-meagre analytic set, then X is a Baire space.*

REMARK 4.5. In the present normed-group context, in an almost-complete space (cf. §1), 'Baire set', 'set with the Baire property' and 'Baire space' are almost-synonyms in the sense that: for B non-meagre, B has the Baire property iff B is a Baire space iff B is almost-complete (cf. [Ost-S, Th. 7.4]).

Below 'quasi all' means 'all but for a meagre set of exceptions'.

Theorem 4.6 (Analytic Shift Theorem, [Ost-LB3, Th. 3]). *In a normed group under the topology d_R, with $z_n \to e_X$ and A analytic and non-meagre: for a non-meagre set of $a \in A$ with co-meagre Baire envelope, there is an infinite set $\mathbb{M}_a$ and points $a_n \in A$ converging to a such that*

$$\{aa_m^{-1}z_m a_m : m \in \mathbb{M}_a\} \subseteq A.$$

In particular, if the normed group is topological, for quasi all $a \in A$, there is an infinite set $\mathbb{M}_a$ such that

$$\{az_m : m \in \mathbb{M}_a\} \subseteq A.$$

Remark 4.7. When $z_n \to e_X$ one says that z_n is a *null* sequence. Note that $aa_m^{-1}z_m a_m$ above also converges under d_R to a as

$$d_R(aa_m^{-1}z_m a_m, a) = \|aa_m^{-1}z_m a_m a^{-1}\| \le \|aa_m^{-1}\| + \|z_m\| + \|a_m a^{-1}\|.$$

The theorem uses transconjugacies to embed a subsequence of the null sequence into A; it is natural, borrowing from [Par], to term this 'shift-compactness'—see [Ost-LB3] for background and connections with allied notions of generic automorphisms.

Theorem 4.8 (Analytic Squared Pettis Theorem, [BOst-N, Th. 5.8]). *For X a normed group, if A is analytic and non-meagre under d_R, then e_X is an interior point of $(AA^{-1})^2$.*

5. Non-separable automatic continuity of homomorphisms. In the proof of the Semi-Completeness Theorem (Main Theorem 1.9) we will need to know that the inverse of a certain continuous bijective homomorphism is continuous. In the separable case this follows by noting that the graph of the homomorphism is closed and, as a consequence of the Souslin Graph Theorem, the inverse is a Baire homomorphism (meaning that preimages of open sets have the Baire property), and hence continuous. However, in the non-separable case the paradigm falls foul of the technical requirement for σ-discreteness. We will employ a modified approach based on the following.

Theorem 5.1 (Open Homomorphism Theorem). *For normed groups X, Y with X analytic and Y a Baire space, let $f : X \to Y$ be a surjective, continuous homomorphism which is base-σ-discrete. Then f is open. So if also $X = Y$ and f is bijective, then f^{-1} is continuous.*

Proof. Suppose that G and H are arbitrary open balls around e_X with $G^4 \subseteq H$. Let D be a dense set in X (e.g. X itself). Now $\mathcal{U} = \{Gd : d \in D\}$ is an open cover of X. (Indeed, if $x \in X$ and $d \in D \cap Gx$, then $x \in Gd$, by symmetry.) Let $\mathcal{V}$ be a σ-d open refinement of $\mathcal{U}$, and say $\mathcal{V} = \bigcup_n \mathcal{V}_n$, with each $\mathcal{V}_n$ discrete. Then, as $X = \bigcup_n(\bigcup\{V : V \in \mathcal{V}_n\})$, we have $Y =$

$\bigcup_n(\bigcup\{f(V) : V \in \mathcal{V}_n\})$. As f is base-σ-discrete each $\mathcal{W}_n := f(\mathcal{V}_n) = \{f(V) : V \in \mathcal{V}_n\}$ has a σ-d base $\mathcal{B}_n$; write $\mathcal{B}_n := \bigcup_m \mathcal{B}_{nm}$ with each $\mathcal{B}_{nm}$ discrete. So for each $V \in \mathcal{V}_n$ one has $f(V) := \bigcup_m\{B \in \mathcal{B}_{nm} : B \subseteq f(V)\}$ and so

$$Y = \bigcup\nolimits_{nm}\{B \in \mathcal{B}_{nm} : B \subseteq f(V) \text{ for some } V \in \mathcal{V}_n\}.$$

As Y is non-meagre, there are $n, m \in \mathbb{N}, V \in \mathcal{V}_n$ and $B \in \mathcal{B}_{nm}$ such that $B \subseteq f(V)$ and B is non-meagre; for otherwise, since $\mathcal{B}_{nm}$ is discrete, by Banach's Category Theorem $\{B \in \mathcal{B}_{nm} : B \subseteq f(V)$ for some $V \in \mathcal{V}_n\}$ is meagre implying the contradiction that also Y is meagre. Pick such m, n and B and V with $B \subseteq f(V)$. Now $V \subseteq Gd$ for some $d \in D$, as $\mathcal{V}_n$ refines $\mathcal{U}$, and so $B \subseteq f(V) \subseteq f(Gd) = f(G)f(d)$ is non-meagre. So $f(G)$ is non-meagre and analytic (as G is analytic). By the Squared Pettis Theorem (Th. 4.8), $(f(G)f(G))^{-1})^2 = f(G)f(G)^{-1}f(G)f(G)^{-1} = f(GG^{-1}GG^{-1})$ is a neighbourhood of e_Y contained in $f(H)$. ■

The following corollary will be used together with Lemma 1.10.

THEOREM 5.2 (Continuous Inverse Theorem). *If under d_R the normed group X is an analytic Baire space and the inversion map $i : x \to x^{-1}$ is σ-discrete preserving (takes discrete families to σ-discrete families), then the inverse of any continuous conjugacy $\gamma_x(\cdot)$ is also continuous.*

Proof. If $\{V_t : t \in T\}$ is σ-d, then for any x so is $\{V_t x^{-1} : t \in T\}$, as right shifts are homeomorphisms. Applying our assumption about the inversion map, for any x the family $\{xV_t^{-1} : t \in T\}$ is σ-d, hence $\{xV_t^{-1}x^{-1} : t \in T\}$ is σ-discrete, and so again $\{xV_t x^{-1} : t \in T\}$ is σ-*d*. This means γ_x is index-σ-discrete. By the preceding theorem (Th. 5.1), if γ_x is continuous, then its inverse is also continuous. ■

With some minor amendments and from somewhat different hypotheses, the same proof as in the Open Homomorphism Theorem (Th. 5.1) demonstrates the following generalization of a separable result (given in [BOst-N, Th. 11.11]), but unfortunately without any prospect for achieving the Baire property (see Remark 5.4 below). Here again the assumed discreteness preservation is fulfilled in the realm of separable spaces. We give the proof for the sake of comparison and because of its affinity with a result due to Noll [N, Th. 1] concerning topological groups (not necessarily metrizable), in which the map f has the property that $f^{-1}(U)$ is analytic for each open $\mathcal{F}_\sigma$-set U. In our metric setting, when preimages under the homomorphism f of open sets are analytic, f is Baire by Nikodym's Theorem and, since $f^{-1}(\mathcal{A})$ is disjoint and completely additive analytic for $\mathcal{A}$ discrete, the σ-d decomposability condition given below is satisfied by Hansell's result [Han-71, Th. 2] cited in Remark 1.5.5. Noll shows that the σ-d decomposability condition below is satisfied when X is a topological group that is topologically complete (using the [FH] generalization of Hansell's result and of [KP]—cf. again Remark 1.5.5).

THEOREM 5.3 (Baire Homomorphism Theorem). *For normed groups X, Y with X analytic, a surjective Baire homomorphism $f : X \to Y$ is continuous provided $f^{-1}(\mathcal{A})$ is σ-discretely decomposable for each σ-discrete family $\mathcal{A}$ in Y.*

Proof. We proceed as above but now in Y. For $\varepsilon > 0$, with $B = B_{\varepsilon/4}(e_Y)$ open and D any dense set in Y choose a_d with $f(a_d) = d$. Put $T := f^{-1}(B)$, which has the Baire property (as f is Baire). As Y is metrizable, the open cover $\{Bd : d \in D\}$ has a σ-d refinement $\mathcal{A} = \bigcup_n \mathcal{A}_n$ with $\mathcal{A}_n := \{A_{tn} : t \in T_n\}$ discrete. For each n, by assumption, we may write $\{f^{-1}(A_{tn}) : t \in T_n\} = \bigcup_{n,m}\{B_{tnm} : t \in T_n\}$ with $\{B_{tnm} : t \in T_n\}$ discrete in X for each m and n. Now

$$X = f^{-1}(Y) = \bigcup_n \{f^{-1}(A_{tn}) : t \in T_n\} = \bigcup_{n,m} \{B_{tnm} : t \in T_n\}.$$

As X is a Baire space, there are n, m such that $\bigcup\{B_{tnm} : t \in T_n\}$ is non-meagre. Again by Banach's Category Theorem, and since $\{B_{tnm} : t \in T_n\}$ is discrete, there is t with B_{tnm} non-meagre. But $B_{tnm} \subseteq f^{-1}(A_{tn}) \subseteq f^{-1}(Bd) = Ta_d$ for some $d \in D$, as $\mathcal{A}$ refines $\{Bd : d \in D\}$. Thus Ta_d and so T is non-meagre, as the right shift ρ_{a_d} is a homeomorphism. But T has the Baire property and X is analytic, so T contains a non-meagre analytic subset. By the Squared Pettis Theorem (Th. 4.8), $(TT^{-1})^2$ contains a ball $B_\delta(e_X)$. Then

$$B_\delta(e_X) \subseteq f^{-1}[(B_{\varepsilon/4})^4] = f^{-1}[B_\varepsilon(e_Y)],$$

proving continuity at e_X. ∎

REMARK 5.4. In the separable case, by demanding that the graph Γ of a homomorphism be Souslin-$\mathcal{F}(X \times Y)$, one achieves the Baire property of sets $f^{-1}(U)$, for U open in Y, by projection parallel to the Y-axis of $\Gamma \cap (X \times U)$, provided that Y is a $\mathcal{K}$-analytic space. For Y absolutely analytic, one has an extended Souslin representation, and hence a representation of Y as an upper-semicontinuous image of some product space $\kappa^{\mathbb{N}}$. But the proof of the projection theorem in [Rog-Jay, Ths. 2.6.5 and 2.6.6] now implies that the projection of a Souslin-$\mathcal{F}(X \times Y)$ set has only a Souslin-$\mathcal{F}(X)$ representation relative to $\kappa^{\mathbb{N}}$, without guaranteeing the σ-discreteness condition. In the non-separable context the Baire property can be generated by a projection theorem, provided one knows both that the graph is absolutely analytic and that the relevant projection, namely $(x, f(x)) \mapsto x$, is base-σ-discrete (cf. [Han-92, Th. 4.6], [Han-74, §6], [Han-71, §3.5]).

6. From normed to topological groups. In this section the generalized Levi result in Corollary 1.7 is the key ingredient; we will use it and other results of Sections 1–5 to prove the Semi-Completeness Theorem (Main Theorem 1.9). The proof layout (preparatory lemmas followed by proof) and strategy are the same as in [Ost-J], but, as some of the details differ, it is convenient to repeat the short common part (most of the proof of Lemma 6.2).

LEMMA 6.1. *For a normed group X, if (X, d_S) is topologically complete and the continuous embedding map $j : (X, d_S) \to (X, d_R)$ is Baire base-σ-discrete (e.g. A-σ-d, in particular index-σ-discrete), then there is a dense absolute-$\mathcal{G}_\delta$-subset Y' in (X, d_R) such that the restriction map $j : (Y', d_S) \to (Y', d_R)$ is open, and so all the three topological spaces (Y', d_S), (Y', d_R) and (Y', d_L) are homeomorphic (and topologically complete).*

Proof. Take $\mathfrak{X} = (X, d_S)$ which is metric and analytic and $\mathfrak{Y} = (X, d_R)$ which is paracompact. Then $j : \mathfrak{X} \to \mathfrak{Y}$ is continuous, surjective and Baire base-σ-discrete. By Lemma 3.1, there is a dense $\mathcal{G}_\delta$-subspace Y' of (X, d_R) such that $j|Y'$ is an open mapping from (Y', d_S) to (Y', d_R). Now $j^{-1}(Y')$ is a $\mathcal{G}_\delta$-subspace of (X, d_S), so it follows that (Y', d_S) is topologically complete, so an absolute-$\mathcal{G}_\delta$. So $j|Y'$ embeds Y' as a subset of (X, d_S) homeomorphically into Y' as an absolute-$\mathcal{G}_\delta$ subset of (X, d_R). Since $x_n \to_S x$ iff $x_n \to_R x$ and $x_n \to_L x$, all three topologies on Y' agree. ∎

LEMMA 6.2. *If in the setting of Lemma 6.1 the three topologies d_R, d_L, d_S agree on a dense absolute-$\mathcal{G}_\delta$-set Y of (X, d_R), then for any $\tau \in Y$ the conjugacy $\gamma_\tau(x) := \tau x \tau^{-1}$ is continuous.*

Proof. We work in (X, d_R), which is thus analytic (as a base-σ-discrete continuous image—Remark 1.5.2). Let $\tau \in Y$. We will first show that the conjugacy $x \mapsto \tau^{-1} x \tau$ is continuous in X at e, and then deduce that its inverse $x \mapsto \tau x \tau^{-1}$ is continuous. So let $z_n \to e$ be any null sequence in X. Fix $\varepsilon > 0$; then $T := Y \cap B^L_\varepsilon(\tau)$ is analytic and non-meagre, since X is a Baire space (and $Y \cap B^L_\varepsilon(\tau)$ is d_R-open in Y with Y an absolute $\mathcal{G}_\delta$). By the Analytic Shift Theorem (Th. 4.6), there are $t \in T$ and t_n in T with t_n converging to t (in d_R) and an infinite $\mathbb{M}_t$ such that $\{t t_m^{-1} z_m t_m : m \in \mathbb{M}_t\} \subseteq T$. Since the three topologies agree on Y and the subsequence $t t_m^{-1} z_m t_m$ of points of Y converges to t in Y under d_R (see Remark 4.7), the same is true under d_L. Using the identity $d_L(t t_m^{-1} z_m t_m, t) = d_L(t_m^{-1} z_m t_m, e) = d_L(z_m t_m, t_m)$, we note that

$$\begin{aligned}\|t^{-1} z_m t\| = d_L(t, z_m t) &\le d_L(t, t_m) + d_L(t_m, z_m t_m) + d_L(z_m t_m, z_m t)\\ &\le d_L(t, t_m) + d_L(t t_m^{-1} z_m t_m, t) + d_L(t_m, t) \to 0\end{aligned}$$

as $m \to \infty$ through $\mathbb{M}_t$. So $d_L(t, z_m t) < \varepsilon$ for large enough $m \in \mathbb{M}_t$. Then for such m, as $d_L(\tau, t) < \varepsilon$,

$$\begin{aligned}\|\tau^{-1} z_m \tau\| = d_L(z_m\tau, \tau) &\le d_L(z_m\tau, z_m t) + d_L(z_m t, t) + d_L(t, \tau)\\ &\le d_L(\tau, t) + d_L(t, z_m t) + d_L(t, \tau) \le 3\varepsilon.\end{aligned}$$

Thus there are arbitrarily large m with $\|\tau^{-1} z_m \tau\| \le 3\varepsilon$. Inductively, taking successively $\varepsilon = 1/n$ and $k(n) > k(n-1)$ to be such that $\|\tau^{-1} z_{k(n)} \tau\| \le 3/n$, one has $\|\tau^{-1} z_{k(n)} \tau\| \to 0$. By the weak continuity criterion (Lemma 3.5 on p. 37 of [BOst-N]), $\gamma(x) := \tau^{-1} x \tau$ is continuous. Hence, by Lemma 1.10 and Theorem 5.2, $\gamma^{-1}(x)$ is also continuous. ∎

Proof of Main Theorem 1.9 (Semi-Completeness Theorem). Under d_R, the set $Z_\Gamma := \{x : \gamma_x \text{ is continuous}\}$ is a *closed* subsemigroup of X ([BOst-N, Prop. 3.43] but using the Open Homomorphism Theorem (Th. 5.1) in place of the Souslin Graph Theorem). So as Y is dense, $X = \mathrm{cl}_R Y \subseteq Z_\Gamma$, i.e. γ_x is continuous for all x, and so (X, d_R) is a topological group, by Theorem 4.3. Therefore $x_n \to_R x$ iff $x_n^{-1} \to_R x^{-1}$ iff $x_n \to_L x$ iff $x_n \to_S x$. So (X, d_R) is homeomorphic to (X, d_S). Hence the topological group (X, d_R) is topologically complete, being homeomorphic to (X, d_S). ∎

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