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Cellular covers of cotorsion-free modules

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Abstract. In this paper we improve recent results dealing with cellular covers of R-modules. Cellular covers (sometimes called colocalizations) come up in the context of homotopical localization of topological spaces. They are related to idempotent cotriples, idempotent comonads or coreflectors in category theory.

Recall that a homomorphism of R-modules $\pi : G \to H$ is called a *cellular cover* over H if π induces an isomorphism $\pi_* : \mathrm{Hom}_R(G,G) \cong \mathrm{Hom}_R(G,H)$, where $\pi_*(\varphi) = \pi\varphi$ for each $\varphi \in \mathrm{Hom}_R(G,G)$ (where maps are acting on the left). On the one hand, we show that every cotorsion-free R-module of rank $\kappa < 2^{\aleph_0}$ is realizable as the kernel of some cellular cover $G \to H$ where the rank of G is $3\kappa + 1$ (or 3, if $\kappa = 1$). The proof is based on Corner's classical idea of how to construct torsion-free abelian groups with prescribed countable endomorphism rings. This complements results by Buckner–Dugas. On the other hand, we prove that every cotorsion-free R-module H that satisfies some rigid conditions admits arbitrarily large cellular covers $G \to H$. This improves results by Fuchs–Göbel and Farjoun–Göbel–Segev–Shelah.

1. Introduction. Cellular covers of groups and modules are the algebraic analogues of the cellular approximations of topological spaces due to J. H. C. Whitehead. These feed into the context of homotopical localization in closed model categories established by Bousfield, Farjoun, Hirschhorn, and others (see e.g. [1], [2], [10], [16], [27], [30]). In some special cases there is even a good interplay between cellularization of spaces and cellularization of groups via the fundamental group [31], as was previously obtained for localizations in [5], [2], [6], [7]. For instance, the universal central extension $0 \to H_2(H;\mathbb{Z}) \to \widetilde{H} \to H \to 1$ of a perfect group H yields a surjective cellular cover, with kernel the Schur multiplier $H_2(H;\mathbb{Z})$. This central extension is the one induced on the lowest homotopy groups of the fiber sequence

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$AX \to X \to X^+$, where $X \to X^+$ is the Quillen plus-construction, $AX \to X$ is the acyclic cellular approximation, and $X = K(H, 1)$ is the Eilenberg–Mac Lane space with fundamental group H; see [31]. Other motivating examples can be found in [31], [29], [17], [19], [32].

Recall that a homomorphism $\pi : G \to H$ of groups is a *cellular cover* over H if every homomorphism $\varphi : G \to H$ lifts uniquely to an endomorphism $\widetilde{\varphi}$ of G such that $\pi\widetilde{\varphi} = \varphi$. In such case $\pi : G \to \mathrm{Im}(\pi)$ is a cellular cover over $\mathrm{Im}(\pi)$, hence one can assume without loss of generality that $\pi : G \to H$ is an epimorphism. We then say that

$$0 \to K \to G \xrightarrow{\pi} H \to 1 \tag{1.1}$$

is a *cellular exact sequence.*

One of the main objectives is to classify (up to isomorphism) all possible cellular exact sequences with either fixed cokernel H or kernel K. It is then crucial to know whether there is a set or a proper class (up to isomorphism) of cellular exact sequences (1.1) for a fixed K or H. Certainly, it is more desirable to find cellular covers of any given cardinality $\lambda \geq |K|$ or $|H|$. Here we are guided by similar results obtained for localizations; see e.g. [6], [13], [14], [15], [23], [25], [28].

First observe that K must be central in G, and conversely every abelian group K is the Schur multiplier $H_2(H; \mathbb{Z})$ of some perfect group H, thus all abelian groups can appear as kernels of cellular covers [17]. It has also been proved in [17] that G is abelian, nilpotent, or an R-module, whenever H is abelian, nilpotent or an R-module, respectively, where R is any commutative ring with 1 (compare with the case of localizations [28], [7]). Of course, other properties like for example being perfect (see [35]) are not transferred in general (cf. [33]). Further results on cellular covers of arbitrary groups have recently been achieved in [11] and [18].

Recall some known results for cellular covers of abelian groups. If H is divisible, then G in (1.1) can be determined explicitly as shown in [12]. A different proof of this result using Maltis duality theory is given in [21]. If H is reduced, then K must be cotorsion-free (see [3], [18], [21]). And if H is torsion and reduced, then the cellular exact sequence collapses and $K = 0$ (see [21]). Furthermore, if K is free, then it is very easy to see ([18]) that $|K| \leq |H|$.

The situation becomes more exciting for cotorsion-free abelian groups. Buckner and Dugas showed in [3] that if K is cotorsion-free, then there exist arbitrarily large cellular covers $G \to H$ with kernel K. Hence (1.1) runs over a proper class in that case. By their construction, $|G| \geq 2^{\aleph_0}$. Here [3] uses a construction based on the combinatorial principle Strong Black Box from [26] which we will replace by the ordinary Black Box, thus filling in missing cardinals $\kappa^{\aleph_0}$ for the size of the kernels (see Corollary 3.1). However, due to the nature of the Black Box this does not say anything about cellular covers of size below the continuum—one problem that we want to attack in the present paper. We would like to point out that these Black Boxes are theorems in ZFC, thus do not depend on additional axioms of set theory (see for instance [24]).

Dually, for every infinite cardinal λ there exists a cotorsion-free abelian group H of cardinal λ which admits arbitrary large cellular covers G (see [18]). The proof is based on [22] concerning the existence of arbitrarily large indecomposable vector spaces with four distinguished subspaces. For instance every rank one group that is not a ring has arbitrarily large cellular covers (see [21]). Note that this result does not fix the group H, but the cardinal λ.

In the present paper we make the following new contributions to the theory of cellular covers: As indicated above we consider the existence of cellular covers of size below the continuum. In this case cotorsion-free is the same as torsion-free and reduced; see [24]. And if K is cotorsion-free of rank $\kappa < 2^{\aleph_0}$, then we prove that there exists a cellular cover G of rank 3, $3\kappa + 1$, or κ, respectively if κ is 1, finite and ≥ 2, or infinite (see Theorem 3.2). This explains our interest in extending the main result of [3] to Corollary 3.1 mentioned above. Dually, if H is cotorsion-free of size $\kappa < 2^{\aleph_0}$ and $\mathrm{End}(H) = \mathbb{Z}$, then there exists a cellular cover G of size κ (see Theorem 4.1). Looking at cokernels of size larger than the continuum we are also able to find arbitrarily large cellular sequences with prescribed cokernel. This is our main result (Theorem 4.4): If H is cotorsion-free of size $\kappa \geq 2^{\aleph_0}$ and satisfies $\mathrm{End}(H) = \mathbb{Z}$ and $\mathrm{Hom}(H, M) = 0$ for all $\aleph_0$-free abelian groups M, then there exist arbitrarily large cellular covers G. To get this result we have to modify the classical Black Box to be suitable for this purpose.

Needless to say, our results hold for R-modules, and are stated in broader generality as indicated here (see Section 2).

We finally remark that cellular covers of groups provide (singly cogenerated) colocalization functors in the category of groups as noticed in [17]. In particular, they can be translated to spaces by simply taking Eilenberg–Mac Lane spaces as in [31] or [23]. That is, if $G \to H$ is a surjective cellular cover of groups then $K(G, n) \to K(H, n)$ is a cellular approximation of spaces (assuming H is abelian for $n \geq 2$). The terminology for abelian groups follows [20].

2. Cellular covers of modules. A homomorphism of R-modules $\pi : G \to H$ is called a *cellular cover* over H if π induces an isomorphism

$$\pi_* : \mathrm{Hom}_R(G, G) \cong \mathrm{Hom}_R(G, H),$$

where $\pi_*(\varphi) = \pi\varphi$ for each $\varphi \in \mathrm{Hom}_R(G,G)$ (where maps are acting on the left). For $R = \mathbb{Z}$ these are precisely cellular covers (or colocalizations) of abelian groups (see e.g. [17], [21]). Recall that $\pi : G \to H$ is a *localization* if it induces an isomorphism $\pi^* : \mathrm{Hom}_R(H,H) \cong \mathrm{Hom}_R(G,H)$, by $\pi^*(\varphi) = \varphi\pi$.

If $\pi : G \to H$ is a cellular cover of R-modules, then π induces a morphism $\mathrm{End}_R(H) \to \mathrm{End}_R(G)$, given by $\varphi \mapsto \tilde{\varphi}$, where $\tilde{\varphi} : G \to G$ is the unique lifting of φ, i.e. such that $\pi\tilde{\varphi} = \varphi\pi$. In fact, it is a homomorphism of R-algebras, also by uniqueness of liftings. The first part of the following result can be found in [21].

Proposition 2.1. *Let $\eta : R_0 \to R$ be a homomorphism of rings, H be an R-module and $\pi : G \to H$ a cellular cover as R_0-modules. Then G admits a unique R-module structure for which $\pi : G \to H$ is a morphism of R-modules. Furthermore, π is also a cellular cover viewed as R-modules if $\eta(R_0)$ is central in R.*

Proof. The last assertion is shown as follows. If $\varphi : G \to H$ is an R-homomorphism, then it is an R_0-homomorphism (via η), hence it lifts to a unique R_0-homomorphism $\widetilde{\varphi} : G \to G$ such that $\pi\widetilde{\varphi} = \varphi$. For a fixed $r \in R$, left multiplication by r on G is an R_0-homomorphism since $\eta(R_0)$ is central in R. Therefore, $r\widetilde{\varphi}$ and $\widetilde{\varphi}r$ are two R_0-homomorphisms such that $\pi r\widetilde{\varphi} = r\pi\widetilde{\varphi} = r\varphi = \varphi r = \pi\widetilde{\varphi}r$, since π and φ are R-homomorphisms. Because π is an R_0-cellular cover it follows that $r\widetilde{\varphi} = \widetilde{\varphi}r$, and hence $\widetilde{\varphi}$ is an R-homomorphism. ∎

In particular, for $R_0 = \mathbb{Z}$ and R any ring with 1, cellular covers $G \to H$ as abelian groups over an R-module H are also cellular covers as R-modules (this improves Proposition 2.6 in [21]).

The following easy observation allows us to consider surjective cellular covers:

Proposition 2.2. *A homomorphism of R-modules $\pi : G \to H$ is a cellular cover if and only if $\pi : G \to \mathrm{Im}(\pi)$ is a cellular cover and the induced homomorphism $\mathrm{Hom}(G,H) \to \mathrm{Hom}(G, \mathrm{Coker}\,\pi)$ is trivial.* ∎

In this paper we will construct surjective cellular covers $G \to H$ which are also localizations (cf. [17]). These properties are easy to verify when the modules involved are *rigid* in the sense that $\mathrm{End}_R(G) = R = \mathrm{End}_R(H)$, where R is a ring with 1. The following fact is immediate and will be used in our theorems:

Proposition 2.3. *Let $\pi : G \to H$ be an epimorphism of R-torsion-free modules and suppose that G is rigid. Then π is a cellular cover if and only if $\mathrm{Hom}_R(G,H) = \pi R$. In that case H is rigid as well, and π is a localization.* ∎

Proof. If $\mathrm{End}_R(G) = R$ and G is R-torsion-free then $\mathrm{Hom}_R(G, \mathrm{Ker}\,\pi) = 0$ and therefore π_* is injective. It is also clear that π_* is surjective if and only if $\mathrm{Hom}_R(G, H) = \pi R$. The last statement is immediate. ∎

We now fix some notation and setting from [24] for cotorsion-free modules, from which we will build up our desired rigid modules.

Let R be a commutative ring with 1, and $\mathbb{S} = \{s_n : n \in \omega\}$ a distinguished countable multiplicatively closed subset such that R is $\mathbb{S}$-reduced and $\mathbb{S}$-torsion-free. Thus $\mathbb{S}$ induces a Hausdorff topology on R, taking $q_m R$ ($m \in \mathbb{Z}$) as the neighborhoods of zero where $q_m = \prod_{n<m} s_n$. We let $\widehat{R}$ be the $\mathbb{S}$-adic completion of R. We will also assume that R is cotorsion-free (with respect to $\mathbb{S}$), that is, $\mathrm{Hom}(\widehat{R}, R) = 0$. More generally, an R-module M is $\mathbb{S}$-*cotorsion-free* if $\mathrm{Hom}_R(\widehat{R}, M) = 0$. We must say what it means that M has rank $\kappa \le |R|$. (Note that R may not be a domain.) If $|M| > |R|$ it suffices to let $\mathrm{rk}(M) = |M|$. If $|M| \le |R|$, then $\mathrm{rk}(M) = \kappa$ means that there is a free submodule $E = \bigoplus_{i<\kappa} Re_i$ of M such that M/E is $\mathbb{S}$-torsion. (Note that E also exists if $|M| > |R|$.) Recall that M is $\mathbb{S}$-*torsion* if for all $m \in M$ there is $s \in \mathbb{S}$ such that $sm = 0$. Similarly, a submodule N of M is $\mathbb{S}$-pure if $sM \cap N = sN$ for all $s \in \mathbb{S}$. If M is $\mathbb{S}$-torsion-free and $N \subseteq M$, then we denote by N_* the smallest $\mathbb{S}$-pure submodule of M containing N, i.e. $N_* = \{m \in M : sm \in N \text{ for some } s \in \mathbb{S}\}$.

We will write $\mathrm{Hom}(M, N)$ for $\mathrm{Hom}_R(M, N)$. In what follows, all appearances of torsion, pure, etc. refer to $\mathbb{S}$ and we will therefore not mention the underlying set $\mathbb{S}$.

3. A theorem about kernels of cellular covers. Using the Strong Black Box from [26] it was shown in [3] that any cotorsion-free R-module K can be the kernel of a cellular cover of arbitrarily large cardinality κ. Analyzing the proof by Buckner and Dugas [3] we note that the Strong Black Box can easily be replaced by the (general) Black Box as in [9] or [24]. Thus the existence of cotorsion-free kernels of cellular covers extends to a wider spectrum of cardinals. We skip the proof of the following corollary concerning uncountable cardinals κ since the necessary changes can be deduced from our proof of Theorem 4.4 below.

Corollary 3.1. *Let R be a commutative, cotorsion-free ring with* 1 *and κ be any infinite cardinal with $\kappa^{\aleph_0} > |R|$. If K is a cotorsion-free R-module of size κ, then there is a cellular exact sequence* $0 \to K \to G \to M \to 0$ *and $|G| = \kappa^{\aleph_0}$. If $\kappa = \kappa^{\aleph_0}$, then all members of the cellular exact sequence have the same size κ.*

We note that applications of the (Strong) Black Box in [3] provide cellular covers of size greater than or equal to the continuum. Using a classical idea due to A. L. S. Corner (see [8]) we will be able to derive cellular covers of size $\kappa < 2^{\aleph_0}$ (which complements the results in Buckner and Dugas [3]). On top of this, thanks to the preliminary work in [24, p. 16, Theorem 1.1.20], our construction is much simpler.

THEOREM 3.2. *Let R be a commutative, torsion-free, reduced ring with* 1, *of size* $< 2^{\aleph_0}$. *Let K be any torsion-free and reduced R-module of rank* $\kappa < 2^{\aleph_0}$. *Then there is a cotorsion-free R-module G of rank* 3 *if* $\kappa = 1$, *and of rank* $3\kappa + 1$ *if* $2 \leq \kappa < 2^{\aleph_0}$, *with submodule K such that* $\mathrm{Hom}(G, K) = 0$ *and* $\mathrm{Hom}(G, G/K) = \pi R$ *where* $\pi : G \to G/K$ $(g \mapsto g + K)$ *is the canonical epimorphism. In particular,*

$$0 \to K \to G \to G/K \to 0$$

is a cellular exact sequence.

Proof. We first note that the assumptions on R and K imply that R and K are cotorsion-free, by [24, p. 19, Corollary 1.1.25]. Also recall that any ordinal α is the same as the set $\{\beta : \beta < \alpha\}$, in particular any natural number k is $k = \{0, 1, \dots, k-1\}$. Choose a free R-submodule $E = \bigoplus_{\alpha<\kappa} Re_\alpha \subseteq K$ of rank κ such that K/E is torsion, and let $F = \bigoplus_{\alpha<\kappa} Rf_\alpha$ be a free R-module of the same rank. Note that E also exists if $\mathrm{rk}(K) = |K|$. If $C = K \oplus F$, then we define the R-module G as a pure submodule of the completion $\widehat{C}$. We distinguish two cases: If $1 < \kappa$ is finite, then we let $f' = f_0 + \cdots + f_{\kappa-1}$ and define

$$G = \langle K, F, w_\alpha(we_\alpha + f_\alpha), w'f' \mid \alpha < \kappa\rangle_* \subseteq \widehat{C} \tag{3.1}$$

where w, w' and $w_\alpha \in \widehat{R}$ $(\alpha < \kappa)$ is a family of algebraically independent elements over C. Its existence follows from a theorem of Göbel–May; see [24, p. 16, Theorem 1.1.20]. If $\kappa = 1$ we can omit the element $w'f'$ (as the argument (3.8) below will not be needed).

In case κ is infinite, we define

$$G = \langle K, F, w_\alpha(we_\alpha + f_\alpha), w'_\alpha(f_0 + f_\alpha) \mid \alpha < \kappa\rangle_* \subseteq \widehat{C} \tag{3.2}$$

where w, w'_α and w_α $(\alpha < \kappa)$ is again a family of algebraically independent elements over C; for its existence we can apply the same result as above because $\kappa < 2^{\aleph_0}$.

Clearly the rank of G is 3 for $\kappa = 1$, $3\kappa + 1$ if $1 < \kappa$ is finite, and κ if κ is infinite. We must now show that the exact sequence $0 \to K \to G \to M \to 0$ with $M = G/K$ satisfies the conditions stated in the theorem.

Let $\pi : \widehat{C} = \widehat{K} \oplus \widehat{F} \to \widehat{F}$ be the canonical projection with kernel $\widehat{K}$.

We consider the case when $\kappa \neq 1$ is finite. The infinite case is similar and left to the reader. The case $\kappa = 1$ is trivial.

If $x \in G$, then by (3.1) there is $s \in \mathbb{S}$ such that

$$sx = k + f + \sum_\alpha r_\alpha w_\alpha (we_\alpha + f_\alpha) + r'w'f' \quad \text{for some } k \in K, f \in F, r', r, r_\alpha \in R. \tag{3.3}$$

By continuity of π it follows that $\pi(sx) = f + \sum_\alpha r_\alpha w_\alpha f_\alpha + r'w'f'$, thus

$$\pi(G) \subseteq \langle F, w_\alpha f_\alpha, w'f' \mid \alpha < \kappa \rangle_* \subseteq \widehat{F}. \tag{3.4}$$

Next we show that

$$G \cap \widehat{K} = K. \tag{3.5}$$

It will suffice to verify $G \cap \widehat{K} \subseteq K$. If $x \in G \cap \widehat{K}$, by (3.1) there is $s \in \mathbb{S}$ such that $sx = k + f + \sum_\alpha r_\alpha w_\alpha (we_\alpha + f_\alpha) + r'w'f' = k' \in \widehat{K}$. Thus

$$\Big[k + \sum_\alpha r_\alpha w_\alpha we_\alpha - k'\Big] + \Big[f + \sum_\alpha r_\alpha w_\alpha f_\alpha + r'w'f'\Big] = 0.$$

Since the sum $\widehat{K} \oplus \widehat{F}$ is direct we have $k + \sum_\alpha r_\alpha w_\alpha we_\alpha - k' = f + \sum_\alpha r_\alpha w_\alpha f_\alpha + r'w'f' = 0$. Algebraic independence in the second term implies $f = 0$, $r' = 0$ and $r_\alpha = 0$ for all $\alpha < \kappa$. From the first term we get $k - k' = 0$, so $sx = k' = k \in K$. Purity and torsion-freeness of K imply $x \in K$ as required.

We now show that

$$\operatorname{Hom}(G, K) = 0. \tag{3.6}$$

Let $\varphi : G \to K$ be a homomorphism. By the continuity of φ, from $w_\alpha(we_\alpha + f_\alpha) \in G$ we get $\varphi(w_\alpha(we_\alpha + f_\alpha)) = w_\alpha(w\varphi(e_\alpha) + \varphi(f_\alpha)) \in K$, hence $w_\alpha(w\varphi(e_\alpha) + \varphi(f_\alpha)) = k'$ for some $k' \in K$. Again by algebraic independence of w_α we get $\varphi(e_\alpha) = 0$ and $\varphi(f_\alpha) = 0$ for all $\alpha < \kappa$. Thus $\varphi(E) = \varphi(F) = 0$, so that φ induces a map $\varphi' : G/(E \oplus F) \to K$. The torsion part of $G/(E \oplus F)$ is $(E \oplus F)_*/(E \oplus F)$. It must vanish under the induced map φ', because K is torsion-free. Thus φ factors through $G/(E \oplus F)_*$ which is divisible while the image is reduced, so all of G is in the kernel and $\varphi = 0$ as claimed in (3.6).

Finally we show

$$\operatorname{Hom}(G, G/K) = \pi R. \tag{3.7}$$

By (3.4) and (3.5) it follows that $\pi(G) = \langle F, w_\alpha f_\alpha, w'f' \mid \alpha < \kappa \rangle_* = G/K$ canonically. Thus we can view $\varphi : G \to G/K$ as a map

$$\varphi : G \to \langle F, w_\alpha f_\alpha, w'f' \mid \alpha < \kappa \rangle_* \subseteq \widehat{C},$$

and also $G \subseteq \widehat{C}$.

For any $\beta < \kappa$ we have $sw_\beta(w\varphi(e_\beta) + \varphi(f_\beta)) = s\varphi(w_\beta(we_\beta + f_\beta)) = f + \sum_\alpha r_\alpha w_\alpha f_\alpha + r'w'f'$ for suitable coefficients. Algebraic independence and torsion-freeness imply $f = 0$, $r_\alpha = 0$ $(\alpha \neq \beta)$, $r' = 0$ and $sw_\beta(w\varphi(e_\beta) + \varphi(f_\beta)) = r_\beta w_\beta f_\beta$. Thus, $s\varphi(we_\beta) = r_\beta f_\beta - s\varphi(f_\beta)$. Now

$\varphi(f_\beta) \in \langle F, w_\alpha f_\alpha, w'f' \mid \alpha \leq \kappa\rangle_*$, and by algebraic independence of w also $\varphi(e_\beta) = 0$ and $\varphi(sf_\beta) = r_\beta f_\beta$ for all $\beta < \kappa$. In particular $E \subseteq \operatorname{Ker}\varphi$, so $\varphi\restriction K$ induces $\varphi' : K/E \to G/K$. However, K/E is torsion while G/K is torsion-free, hence $\varphi' = 0$. This shows that $K \subseteq \operatorname{Ker}\varphi$, and φ factors through $\varphi : G/K \to G/K$. But we have seen before that $\varphi(f_\beta) = r_\beta f_\beta$ (we may assume $s = 1$ by purity and torsion-freeness). If we apply φ to $w'f'$ we will obtain similarly $\varphi(f') = r'f'$ for some $r' \in R$ and derive

$$(3.8)\qquad \begin{aligned} r'f_0 + \cdots + r'f_{\kappa-1} = r'f' = \varphi(f') &= \varphi(f_0) + \cdots + \varphi(f_{\kappa-1}) \\ &= r_0 f_0 + \cdots + r_{\kappa-1} f_{\kappa-1}. \end{aligned}$$

By linear independence, $r_\beta = r'$ for all $\beta < \kappa$ and $\varphi = \pi r' \in \pi R$ as desired. ■

We remark that the rank of G is chosen minimal (as stated in the theorem).

Example 3.3. For $K = \mathbb{Z}$ this yields a cellular cover $\mathbb{Z} \to G \to M$ with G and M of rank 3 and 2, respectively. Here is an explicit presentation of G: Let $R = \mathbb{Z}$, $\mathbb{S} = \{1, p, p^2, \ldots\}$ for a given prime p, $\widehat{R}$ the ring of p-adic integers. Then $G = \langle e, f, w(w'e + f)\rangle_*$ where w and w' are two linearly independent p-adic numbers, not integers.

This is the simplest example of a cellular cover of abelian groups with kernel $\mathbb{Z}$. In fact, it will be shown in [34] that no group of rank 1 admits cellular covers with free kernel.

On the other hand, the construction of Theorem 3.2 can be easily modified so that the rank of G becomes larger. For this, one can take F free of rank κ', with $\kappa < \kappa' < 2^{\aleph_0}$, and

$$G = \langle K, F, w_\beta(we_\alpha + f_\beta), w'f' \mid \alpha < \kappa\rangle_* \mid \alpha < \kappa,\ \beta < \kappa'\rangle_*$$

where $\alpha = \psi(\beta)$ for any fixed surjective function $\psi : \kappa' \to \kappa$ (cf. Theorem 4.1).

We end this section with an interesting case of Theorem 4.1 not obtained in Buckner and Dugas's paper [4].

Corollary 3.4. *If K is a torsion-free, reduced R-module of infinite size $\kappa < 2^{\aleph_0}$, then there is a cellular exact sequence $0 \to K \to G \to M \to 0$ with $\operatorname{rk}(G) = \operatorname{rk}(M) = \kappa$.* ■

4. Two theorems about cokernels of cellular covers. In this section we take the opposite point of view and want to prescribe certain cotorsion-free modules H such that $0 \to K \to G \to H \to 0$ is a cellular exact sequence for suitable $0 \to K \to G$'s. Our method will work for particular rings R and $\mathbb{S}$-topologies. Recall that an R-module M is called $\aleph_0$-*free* if every finite rank submodule of M is contained in a pure free submodule of M. For a cardinal κ let $E = \bigoplus_{\alpha<\kappa} Re_\alpha$ be a free module of rank κ. It will be clear from the context to which cardinal κ the module E refers to.

As in Section 3, we have a borderline $2^{\aleph_0}$ and must distinguish between cokernels of size below or above and equal to $2^{\aleph_0}$. Moreover, due to technical reasons in the construction we will have to assume that H is rigid and that H has no non-trivial homomorphism into any $\aleph_0$-free R-module (in the Black Box construction). By Proposition 2.1 this is a reasonable restriction on H.

4.1. Cokernels of size below the continuum. The main result of this section reads as follows

THEOREM 4.1. *Let $\kappa' < 2^{\aleph_0}$ be a cardinal, R a commutative, torsion-free ring with $1 \neq 0$, of size $< 2^{\aleph_0}$. Moreover, let H be a cotorsion-free R-module which is κ'-generated such that* $\mathrm{End}(H) = R$*. Then for any cardinal κ with $\kappa' \leq \kappa < 2^{\aleph_0}$ there is a cotorsion-free R-module G with the following properties:*

(i) *G is κ-generated and there is $K \subseteq G$ with $G/K = H$.*
(ii) *If $\varphi \in \mathrm{End}(G)$, then there is a unique element $r \in R$ such that $(\varphi - r \cdot \mathrm{id}_G)(K) = 0$, so there is an induced homomorphism $\varphi_r : H \to G$ $((g + K) \mapsto (\varphi - r)g)$.*
(iii) *If* $\mathrm{Hom}(H, G) = 0$*, then* $\mathrm{End}(G) = R$*.*
(iv) $\mathrm{Hom}(G, H) = \pi R$ *where $\pi : G \to H$ $(g \mapsto g + K)$ is the canonical epimorphism.*

In particular, if $\mathrm{Hom}(H, G) = 0$*, then $0 \to K \to G \to H \to 0$ is a cellular exact sequence.*

Note that for R-modules of size $< 2^{\aleph_0}$ cotorsion-freeness is equivalent to the R-module being torsion-free and reduced (see [24]).

Proof. We enumerate a generating system $\{h_\beta : \beta < \kappa'\}$ of H and choose a surjection $\psi : \kappa \to \kappa'$. If $C = E \oplus H$ (with E as above), then we define the desired module G as a pure submodule of the completion $\widehat{C}$, as follows:

$$G = \langle E, w_\alpha e_\alpha, w'_\alpha(e_1 + e_\alpha), (\tilde{w}_\alpha e_\alpha + h_\beta) \mid \alpha < \kappa,\ \psi(\alpha) = \beta \rangle_* \subseteq \widehat{C}, \tag{4.1}$$

where w_α, w'_α, $\tilde{w}_\alpha \in \hat{R}$ $(\alpha < \kappa)$ is a family of algebraically independent elements over C. Using $|C| < 2^{\aleph_0}$ its existence follows again from a theorem of Göbel–May; see [24, p. 16, Theorem 1.1.20].

Clearly G is κ-generated as required. We also consider the projection induced by the decomposition $\widehat{C} = \widehat{E} \oplus \widehat{H}$, which is $\pi : \widehat{C} \to \widehat{H}$ with kernel $\widehat{E}$.

We will not distinguish between π and its natural restrictions to submodules of $\widehat{C}$. If $x \in G$, then there is $s \in \mathbb{S}$ such that

$$sx = f + \sum_{\alpha<\kappa} r_\alpha w_\alpha e_\alpha + \sum_{\alpha<\kappa} r'_\alpha w'_\alpha (e_1 + e_\alpha) + \sum_{\alpha<\kappa,\, \psi(\alpha)=\beta} \tilde{r}_\alpha (\tilde{w}_\alpha e_\alpha + h_\beta) \tag{4.2}$$

for some $f \in E$ and $r_\alpha, r'_\alpha, \tilde{r}_\alpha \in R$.

By definition and continuity of the map π it now follows that

$$\pi(sx) = \sum_{\alpha<\kappa,\, \psi(\alpha)=\beta} \tilde{r}_\alpha h_\beta,$$

thus

$$\pi(G) = H \tag{4.3}$$

since H is pure in its completion $\widehat{H}$, and we let $K := \widehat{E} \cap G$. Obviously $K = \operatorname{Ker}(\pi \restriction G)$ and thus

$$E \subseteq K = \Big\{ x \in G : \sum_{\alpha<\kappa,\, \psi(\alpha)=\beta} \tilde{r}_\alpha h_\beta = 0 \text{ in } (4.2) \Big\}.$$

This implies that K/E is divisible and the quotient $(G/E)/(K/E) \cong G/K \cong H$ is reduced. There is a decomposition into a divisible summand and a reduced part H:

$$G/E = (K/E) \oplus H \quad \text{with } K/E \text{ the maximal divisible summand.} \tag{4.4}$$

Next we establish (ii) and study $\varphi \in \operatorname{End}(G)$. If $\delta < \kappa$ and $\gamma = \psi(\delta)$, then there is $s = s_\delta \in S$ such that

$$\begin{aligned} s\varphi(w_\delta e_\delta) &= s w_\delta \varphi(e_\delta) \\ &= f + \sum_{\alpha<\kappa} r_\alpha w_\alpha e_\alpha + \sum_{\alpha<\kappa} r'_\alpha w'_\alpha (e_1 + e_\alpha) + \sum_{\alpha<\kappa,\, \psi(\alpha)=\beta} \tilde{r}_\alpha (\tilde{w}_\alpha e_\alpha + h_\beta) \end{aligned}$$

for some $f \in E$ and $r_\alpha, r'_\alpha, \tilde{r}_\alpha \in R$.

Again, the sum $\widehat{E} \oplus \widehat{H}$ is direct and $w_\alpha, w'_\alpha, \tilde{w}_\alpha$ $(\alpha < \kappa)$ are algebraically independent elements over C. Therefore, equating coefficients we obtain $s\varphi(e_\delta) = r_\delta e_\delta$ and φ acts on e_δ as multiplication by r_δ. A similar argument shows that φ also acts on $e_1 + e_\delta$ as multiplication by some $r'_\delta \in R$, for all $\delta < \kappa$. Therefore,

$$r'_\delta (e_1 + e_\delta) = \varphi(e_1 + e_\delta) = \varphi(e_1) + \varphi(e_\delta) = r_1 e_1 + r_\delta e_\delta.$$

And comparing components we get $r_\delta = r$ for all $\delta < \kappa$ (which does not depend on f). It follows that $\varphi \restriction E = r \cdot \mathrm{id}$. Using that G is reduced and (4.4) we see that $\varphi - r \cdot \mathrm{id}$ induces a unique homomorphism $\varphi_r : H = G/K \to G$, which shows (ii).

By the assumption of (iii) we note that the induced map φ_r from (ii) vanishes, thus $\mathrm{End}(G) = R$ in this case and also $\mathrm{Hom}(G, K) = 0$.

For (iv) we consider any $\psi \in \mathrm{Hom}(G, H)$. Clearly

$$\psi(w_\alpha e_\alpha) = w_\alpha \psi(e_\alpha) \in H \cap w_\alpha H,$$

which is zero by the algebraically independent element w_α. Hence ψ induces a homomorphism $\tilde{\psi} : G/E \to H$. Since H is reduced it follows from (4.4) that $\tilde{\psi}(K/E) = 0$ and we can write $\psi = \pi r$ for some $r \in R$ and $\psi \in \pi R$. The reverse inclusion for (iv) is trivial. Finally note that $0 \to K \to G \to H \to 0$ is a cellular exact sequence by Proposition 2.3. ∎

Note that there are many examples of rings R and $\mathbb{S}$-topologies such that the constructed module G in the above theorem satisfies $\mathrm{Hom}(H, G) = 0$ for given H. For instance, if H is a rank one group that is not a ring, then one could choose $R = \mathbb{Z}$ and the $\mathbb{Z}$-adic topology on R. Consequently, G will be $\mathbb{Z}$-homogeneous and hence $\mathrm{Hom}(H, G) = 0$. This shows Corollary 5.5 from [21].

4.2. Cokernels of size greater than or equal to the continuum. We now consider cellular covers of cotorsion-free modules of size $\geq 2^{\aleph_0}$ and utilize the Black Box from [9] to extend Theorem 4.1 to larger cardinals.

Let $\lambda > |R|$ be an infinite cardinal such that $\lambda = \lambda^{\aleph_0}$. If μ is any infinite cardinal (and $\mu^{\aleph_0} > |R|$), then $\mu^{\aleph_0}$ is a candidate for λ. The cardinal condition ensures that the set of all countable subsets of λ has size λ as well. This will be used to deduce that the completion of our canonical free base module B of size λ has size λ as well. The heart of the Black Box construction is to build the desired R-module on a tree $T = {}^{\omega>}\lambda$ as its underlying set of 'supports' using the additional geometric structure.

Therefore let T be the set of all finite sequences $\tau = \lambda_0{}^\wedge \cdots {}^\wedge\lambda_{n-1}$ in λ, hence

$$T = \{\tau : n \to \lambda : n \in \omega\}.$$

Recall that τ above is a finite branch of length n, thus $\mathrm{Dom}(\tau) = n = \{0, \dots, n-1\}$. Similarly we define the set $\mathrm{Br}(T)$ of all infinite branches of length ω, which is

$$\mathrm{Br}(T) = {}^\omega\lambda = \{v : \omega \to \lambda\}.$$

This set has cardinality λ by the assumption on λ. Finite and infinite branches v have a canonical *support*, which is the subset

$$[v] = \{v{\restriction}m \in T : m < \text{length of } v\}$$

of T. Note that $[v], [w]$ are almost disjoint, that is, $[v] \cap [w]$ is finite, if and only if v, w are distinct branches. Trees also have a natural ordering by extensions as follows: For any $\tau, \nu \in T$,

$$\tau < \nu \;\Leftrightarrow\; \tau \subseteq \nu \;\Leftrightarrow\; [\tau] \subseteq [\nu] \text{ and } \nu{\restriction}\mathrm{Dom}\tau = \tau.$$

The *norm* of a branch $\tau \in \mathrm{Br}(T)$ is defined as

$$\|\tau\| = \sup(\mathrm{Im}(\tau)) \in \lambda.$$

We transport these supports and norms to an R-module, taking

$$E = \bigoplus_{\tau \in T} R\tau$$

to be the free R-module generated by $T \subset E$ (where $\tau \in T$ is identified with $1\tau \in E$). Any element $g \in \widehat{E}$ can be expressed as a countable sum $g = \sum_{n\in\omega} g_n\tau_n$ for some $g_n \in \widehat{R}$, such that for all $m \in \omega$, we have $g_n \in q_m\widehat{R}$ for almost all $n \in \omega$. We denote by $[g] = \{\tau_n : g_n \neq 0, n \in \omega\}$ the *support* of g. If v is an infinite branch, then we also write $v = \sum_{n\in\omega} q_n(v\restriction n) \in \widehat{E}$ and call this element a *branch-element*, which obviously has the same support as the branch v, namely $[v]$. These branch-elements will be useful tools to recognize elements of the module G under construction.

Let H be a cotorsion-free R-module of size less than λ and put $B := E \oplus H$. As before let $\pi : \widehat{B} \to \widehat{H}$ be the canonical projection onto $\widehat{H}$. As in the previous section, our goal is to construct $G \subset_* \widehat{B}$ such that $\pi(G) = H$, $\mathrm{Hom}(G, H) = \pi R$ and $\mathrm{Hom}(G, K) = 0$, where $K = \hat{E} \cap G = \mathrm{Ker}(\pi\restriction G)$. Note that in this case $0 \to K \to G \to H \to 0$ is a cellular cover over H. In order to ensure these properties we need to satisfy two requirements during the construction: $G \cap \widehat{H} = 0$ and $\mathrm{End}(G) = R$. Thus we will have

$$E \subseteq_* G \subseteq_* \widehat{B} \tag{4.5}$$

but H must not be contained in G. This forces us to adjust Shelah's Black Box to (4.5). Thus, if $g+h \in \widehat{B} = \widehat{E} \oplus \widehat{H}$, we let $[g+h] := [g]$ be the support of $g + h$ just looking at the $\widehat{E}$-component of elements.

The notions of support and norm extend naturally to subsets X of $\widehat{B}$ as follows:

$$[X] = \bigcup\{[x] : x \in X\} \quad \text{and} \quad \|X\| = \sup\{\|\tau\| : \tau \in [X]\}.$$

Note that $\|X\|$ is an ordinal which is strictly less than λ if $[X]$ is countable, because the cofinality $\mathrm{cf}(\lambda)$ of λ is greater than $\aleph_0$.

The classical Black Box also needs the notion of traps which are partial approximations to endomorphisms $\widehat{B} \to \widehat{B}$. In our case, we will approximate homomorphisms of the form $\varphi : \widehat{E} \oplus H \to \widehat{B}$.

As before let ${}^{\omega>}\omega$ be the countable tree of finite sequences in ω and let $f : {}^{\omega>}\omega \to T$ be a tree embedding. Let $\varphi : \mathrm{Dom}(\varphi) \to \widehat{B}$ denote a partial homomorphism with countable domain $\mathrm{Dom}(\varphi) \subseteq_* \widehat{B}$ a pure submodule of $\widehat{E} \oplus H$ and suppose that $[\mathrm{Dom}(\varphi)]$ is a countable subtree of T. We will require

$$(f,\varphi) \qquad\qquad \operatorname{Im} f \subseteq [\operatorname{Dom}(\varphi)] \subseteq T$$

and call (f,φ) a *trap*.

By our assumptions on the cardinal λ it is clear that the number of traps is λ, hence the following theorem is an easy modification of Shelah's classical Black Box (see the appendix of [9]). Recall that for an ordinal ρ we let $\rho^o = \{\delta < \rho : \operatorname{cf}(\delta) = \aleph_0\}$.

THEOREM 4.2 (Shelah's Black Box). *Let $\lambda = \lambda^{\aleph_0} \geq |R|$ be an infinite cardinal and $T = {}^{\omega>}\lambda$ be a tree which is the basis of a free R-module $B = \bigoplus_{\tau\in T} R\tau$. Moreover, let H be a cotorsion-free R-module of size less than λ and let $\rho = \operatorname{cf}(\lambda)$. For any choice of disjoint stationary subsets $S_1, S_2 \subseteq \rho$ there exists an ordinal λ^* of cardinality λ and a list of traps*

$$(f_\alpha, \varphi_\alpha), \qquad \alpha \in \lambda^*,$$

with the following properties:

(a) $\|\operatorname{Dom}(\varphi_\alpha)\| \in \rho^o$ *is a limit ordinal with* $\|v\| = \|\operatorname{Dom}(\varphi_\alpha)\|$ *for all* $v \in \operatorname{Br}(\operatorname{Im} f_\alpha)$.
(b) *If* $\beta < \alpha \in \lambda^*$ *then* $\|\operatorname{Dom}(\varphi_\beta)\| \leq \|\operatorname{Dom}(\varphi_\alpha)\|$ *and* $\operatorname{Br}(\operatorname{Im} f_\beta) \cap \operatorname{Br}(\operatorname{Im} f_\alpha) = \emptyset$.
(c) *If* $\beta + 2^{\aleph_0} \leq \alpha$, *then* $\operatorname{Br}(\operatorname{Dom}(\varphi_\beta)) \cap \operatorname{Br}(\operatorname{Im} f_\alpha) = \emptyset$.
(d) (Prediction) *If X is a countable subset of $\widehat{E} \oplus H$ and $\varphi : \widehat{E} \oplus H \to \widehat{B}$ is a homomorphism, then there exist ordinals $\alpha_1, \alpha_2 \in \lambda^*$ such that $X \subseteq \operatorname{Dom}(\varphi_{\alpha_i})$ and $\varphi{\restriction}\operatorname{Dom}(\varphi_{\alpha_i}) = \varphi_{\alpha_i}$ and $\|\operatorname{Dom}(\varphi_{\alpha_i})\| \in S_i$ for $i = 1, 2$.*

In the classical Black Box it is used that mappings from a free R-module B to B have unique extensions to mappings from $\widehat{B}$ to $\widehat{B}$. We still want to argue with unique extensions of mappings, which explains the following observation.

LEMMA 4.3. *Let $\varphi : \operatorname{Dom}(\varphi) \to \widehat{E} \oplus H$ be such that $E' \subseteq_* \operatorname{Dom}(\varphi) \subseteq_* \widehat{E} \oplus H'$ and $\pi(\operatorname{Dom}(\varphi)) = H'$ for some submodule $H' \subseteq_* H$ and direct summand $E' \subseteq E$. Then φ has a unique extension $\widehat{\varphi} : \widehat{E}' \oplus H' \to \widehat{E} \oplus \widehat{H} = \widehat{B}$.*

Proof. Since the completion commutes with finite direct sums we may assume without loss of generality that $E' = E$. Moreover, since $\operatorname{Dom}(\varphi)$ is pure in $\widehat{E} \oplus H'$ it follows that $E \subseteq_* \operatorname{Dom}(\varphi) \cap \widehat{E}$ is pure in $\widehat{E}$. Hence there is a unique map $\widehat{\varphi}_E : \widehat{E} \to \widehat{E} \oplus \widehat{H} = \widehat{B}$ such that $\widehat{\varphi}_E{\restriction}\operatorname{Dom}(\varphi) \cap \widehat{E} = \varphi{\restriction}\operatorname{Dom}(\varphi) \cap \widehat{E}$. Let $h \in H'$. Then $\widehat{b} + h \in \operatorname{Dom}(\varphi)$ for some $\widehat{b} \in \widehat{E}$ since $\pi(\operatorname{Dom}(\varphi)) = H'$. Define

$$\widehat{\varphi}_{H'} : H' \to \widehat{E} \oplus \widehat{H} = \widehat{B} \quad \text{via} \quad h \mapsto \varphi(\widehat{b} + h) - \widehat{\varphi}_E(\widehat{b}).$$

We claim that $\widehat{\varphi}_{H'}$ is a well-defined homomorphism. Assume first that there are $\widehat{b}_1, \widehat{b}_2 \in \widehat{B}$ such that $\widehat{b}_1 + h$ and $\widehat{b}_2 + h$ are in $\mathrm{Dom}(\varphi)$. Then

$$\begin{aligned}\varphi(\widehat{b}_1 + h) - \widehat{\varphi}_E(\widehat{b}_1) - (\varphi(\widehat{b}_2 + h) - \widehat{\varphi}_E(\widehat{b}_2))\\ = \varphi(\widehat{b}_1 + h - (\widehat{b}_2 + h)) - \widehat{\varphi}_E(\widehat{b}_1 - \widehat{b}_2) = \varphi(\widehat{b}_1 - \widehat{b}_2) - \widehat{\varphi}_E(\widehat{b}_1 - \widehat{b}_2) = 0\end{aligned}$$

since $\widehat{b}_1 - \widehat{b}_2 \in \mathrm{Dom}(\varphi) \cap \widehat{E}$. Hence $\widehat{\varphi}_{H'}$ is well-defined.

Now assume that $h_1, h_2 \in H'$ and $\widehat{b}_1, \widehat{b}_2, \widehat{b}_3 \in \widehat{B}$ are such that $\widehat{b}_1 + h_1$, $\widehat{b}_2 + h_2, \widehat{b}_3 + h_1 + h_2 \in \mathrm{Dom}(\varphi)$. It follows that

$$\widehat{\varphi}_{H'}(h_1 + h_2) - \widehat{\varphi}_{H'}(h_1) - \widehat{\varphi}_{H'}(h_2) = \varphi(\widehat{b}_3 - \widehat{b}_2 - \widehat{b}_1) - \widehat{\varphi}_E(\widehat{b}_3 - \widehat{b}_2 - \widehat{b}_1) = 0$$

since again $\widehat{b}_3 - \widehat{b}_2 - \widehat{b}_1 \in \mathrm{Dom}(\varphi) \cap \widehat{E}$. Hence $\widehat{\varphi}_{H'}$ is a homomorphism.

Define $\widehat{\varphi} : \widehat{E} \oplus H' \to \widehat{E} \oplus \widehat{H}$ by $\widehat{\varphi}(\widehat{b} + h) := \widehat{\varphi}_E(\widehat{b}) + \widehat{\varphi}_{H'}(h)$. Then clearly $\widehat{\varphi}$ extends φ.

Finally, assume that $\widehat{\psi}$ extends φ too. Then $\widehat{\varphi}\restriction E = \widehat{\psi}\restriction E = \varphi\restriction E$, and hence uniqueness of $\widehat{\varphi}_E$ implies that $\widehat{\varphi}_E = \widehat{\psi}\restriction\widehat{E} = \widehat{\varphi}\restriction\widehat{E}$. If $h \in H'$, then $\widehat{b} + h \in \mathrm{Dom}(\varphi)$ for some $\widehat{b} \in \widehat{E}$ and hence

$$\widehat{\varphi}(\widehat{b} + h) = \widehat{\varphi}_E(\widehat{b}) + \widehat{\varphi}_{H'}(h) = \varphi(\widehat{b} + h),$$

and similarly

$$\widehat{\psi}(\widehat{b} + h) = \widehat{\varphi}_E(\widehat{b}) + \widehat{\psi}(h) = \varphi(\widehat{b} + h).$$

Therefore $\widehat{\psi}(h) = \widehat{\varphi}_{H'}(h)$ and so $\widehat{\varphi}$ and $\widehat{\psi}$ coincide on $\widehat{E}$ and on H' and are thus identical. Uniqueness of $\widehat{\varphi}$ is shown. ■

We now want to apply the Black Box to show the following

Theorem 4.4. *Let R be a commutative, torsion-free ring with $1 \neq 0$ and κ a cardinal with $\kappa^{\aleph_0} > |R|$. Moreover, let H be a cotorsion-free R-module such that $\mathrm{End}(H) = R$ and $|H| \leq \kappa$. Then there is a cotorsion-free R-module G of size $|G| = \kappa^{\aleph_0}$ with the following properties:*

- (i) *There is a submodule $K \subseteq G$ with $G/K = H$.*
- (ii) *If $\varphi \in \mathrm{End}(G)$, then there is a unique element $r \in R$ such that $(\varphi - r \cdot \mathrm{id}_G)(K) = 0$, so there is an induced homomorphism $\varphi_r : H \to G$ $((g + K) \mapsto (\varphi - r)g)$.*
- (iii) *If $\mathrm{Hom}(H, M) = 0$ for every $\aleph_0$-free module M, then $\mathrm{End}(G) = R$, and $\mathrm{Hom}(G, K) = 0$.*
- (iv) *$\mathrm{Hom}(G, H) = \pi R$ where $\pi : G \to H$ $(g \mapsto g + K)$ is the canonical epimorphism.*

In particular, if $\mathrm{Hom}(H, M) = 0$ for all $\aleph_0$-free modules M then $0 \to K \to G \to H \to 0$ is a cellular exact sequence.

Proof. Let $\lambda := \kappa^{\aleph_0}$ and E, T and B be as above. Then $\lambda^{\aleph_0} = \lambda$ is as required for the Black Box. The module will be the union $G = \bigcup_{\alpha\in\lambda} G_\alpha$ of an ascending, continuous chain of cotorsion-free, $\aleph_0$-free modules G_α with

$$E \subseteq_* G_\alpha \subseteq_* \widehat{B} \quad \text{and} \quad \pi(G_\alpha) \subseteq H.$$

Recall that $\pi : \widehat{B} \to \widehat{H}$ is the canonical projection. Let S_1 and S_2 be two disjoint stationary subsets of ρ^o where $\rho = \mathrm{cf}(\lambda)$ and such that the Black Box 4.2 holds for S_1, S_2. We begin with $G_0 \subseteq_* \widehat{E} \oplus H$ such that G_0 is free with $\pi(G_0) = H$. It is easy to construct G_0 by adding elements of the form $v_h + h$ to E where $h \in H$ and the v_h's form a set of almost disjoint infinite branches (see also below for more details). By continuity we only need to consider the inductive step at any $\alpha \in \lambda^*$. We want to find $g_\alpha \in \widehat{E} \oplus H \subseteq \widehat{B}$ such that

$$G_{\alpha+1} = \langle G_\alpha, g_\alpha \rangle_*$$

and the new element g_α must fulfill several tasks.

First we require that the new element g_α is a 'branch-like' element of $\widehat{B}$: Recall that any branch $v \in \mathrm{Br}(\mathrm{Im}\, f_\alpha)$ has norm $\|v\| = \|\mathrm{Dom}(\varphi_\alpha)\|$, by (a) of Black Box 4.2, and gives rise to a non-constant branch-element, which we also denote by v. Any sum $b + v$ with

$$b \in \widehat{\mathrm{Dom}(\varphi_\alpha)} \quad \text{and} \quad \|b\| < \|v\|$$

is called a *branch-like element*. The point is that a branch-like element has a support which at the top looks like a branch from $\mathrm{Br}(\mathrm{Im}\, f_\alpha)$. The choice of generators for G helps to describe the action of homomorphisms on G. We are ready for two preliminary Step Lemmas. The first one will be used to 'kill' non-inner endomorphisms, and the second to 'kill' homomorphisms $\varphi : G \to H$ with no $\widetilde{\varphi} : H \to H$ such that $\varphi = \widetilde{\varphi}\pi$.

Lemma 4.5. *Let G be the module constructed so far with*

(i) $E \subseteq_* G \subseteq_* \widehat{B}$,
(ii) $\pi(G) := H' \subseteq H$.

Let $\varphi \in \mathrm{End}(G)$ be such that $\varphi{\restriction}E \notin R$. For any $0 \neq h \in H'$ there is an $x' \in \widehat{E}$ such that $x = x' + h$ satisfies

$$\varphi(x) \notin \langle G, x \rangle_*.$$

Moreover, the module $G' = \langle G, x \rangle_*$ is cotorsion-free, $\aleph_0$-free, and $\pi(G') \subseteq H$.

Proof. First note that φ has a unique extension (again denoted by φ) to $\varphi : \widehat{E} \oplus H' \to \widehat{B}$ by Lemma 4.3 which is again not in R. Thus $\varphi(x)$ is a well-defined element in $\widehat{B}$ for any choice of $x \in \widehat{E} \oplus H'$. First we use the assumption on φ to show that there exists a countable subset

(4.6) $\quad C \subseteq T$ such that $D = \langle C \rangle$ satisfies $(\varphi - r)\widehat{D} \not\subseteq G$ for all $r \in R$.

Choose a 'constant branch' $v : \omega \to \{\eta\}$ at some $\eta < \lambda$ and let C' be a countable subset of T such that the branch-element v belongs to $\widehat{D'}$ where

$D' = \langle C' \rangle$. If (4.6) fails for $D = D'$, there is $r \in R$ with $(\varphi - r)\widehat{D'} \subseteq G$. This implies that $(\varphi - r)\widehat{D'} = 0$. Indeed, suppose that for some $z \in \widehat{D'}$ we have $(\varphi - r)z \neq 0$; then, since $\widehat{R}\widehat{D'} \subseteq \widehat{D'}$, we would have a non-trivial homomorphism $\widehat{R} \to G$ given by $\hat{r} \mapsto (\varphi - r)(\hat{r}z)$, which is impossible because G is cotorsion-free. Hence $(\varphi - r)\widehat{D'} = 0$ as desired.

But $\varphi{\restriction}E \notin R$, so there is $x \in E$ such that $(\varphi - r)x \neq 0$. Enlarge C' to C such that $x \in D = \langle C \rangle$. As before, using cotorsion-freeness of G we have $(\varphi - r')\widehat{D} = 0$. Hence $(\varphi - r)v = 0$ and $(\varphi - r')v = 0$, which implies $(r - r')v = 0$, and thus $r = r'$ as $\widehat{B}$ is torsion-free. In particular $(\varphi - r)x = 0$, which is a contradiction.

Here is an alternative support argument to prove (4.6). We include this at this point as similar arguments will be needed several times later on. If (4.6) fails for D, then $(\varphi - r')\widehat{D} \subseteq G$ for some $r' \in R$, hence $v(r - r') \in G$. By construction there is q_n and elements g_α $(\alpha \in I)$ for some finite index set I such that

$$q_n v(r - r') = \sum_{\alpha \in I} r_\alpha g_\alpha + e \tag{4.7}$$

for some $r_\alpha \in R$ and $e \in E$. Now recall that by construction all g_α are branch-like elements coming from branches v_α that are not constant. Hence for every $\alpha \in I$ there is some m_α such that $v{\restriction}m_\alpha \neq v_\alpha{\restriction}m_\alpha$. Let m be the maximum of all these m_α. Restricting equation (4.7) to $v{\restriction}m$ it then follows that $q_n(r - r')v{\restriction}m = 0$ and hence $r = r'$. We derive the contradiction $(\varphi - r)x = 0$ and (4.6) follows.

We may assume that

(4.8) $\quad D$ in (4.6) also satisfies $(q_n\varphi - r)\widehat{D} \not\subseteq G$ for all $n > 0$, $r \in R \setminus q_nR$.

Suppose (4.8) fails for $\psi = q_n\varphi - r$. Now we choose elements $\sigma_m \in T$ of length m which constitute an 'anti-branch', that is, two σ_m's are incomparable in T. Moreover we require

$$\sup_{m \leq k} \|\psi(\sigma_m)\| < \|\sigma_{k+1}\| \quad \text{and let} \quad t = t_n := \sum_{m \in \omega} q_m q_n^m \sigma_m.$$

Then, for every non-zero $k < \omega$, we have

$$\psi(t) \equiv \sum_{m \leq k} q_m q_n^m \psi(\sigma_m) \bmod q_k q_n^{k+1}\widehat{B},$$

hence

$$\psi(t){\restriction}\sigma_k \equiv -q_k q_n^k r \bmod q_k q_n^{k+1} R$$

which cannot be 0 because $r \notin q_nR$. Hence $\|\psi(t)\| = \sup \|\sigma_k\|$ and the element $\psi(t)$ cannot be in G by a support argument. Now we enlarge D such that all the σ_m's belong to D and this failure for (r, q_n) is impossible for the enlarged D. Similarly we deal with the other (at most countably many) potential failures of (4.8) and correct D, hence (4.8) holds.

We are now ready to find the desired element $x \in \widehat{E} \oplus H'$. Let $h \in H'$. First we choose a new constant branch w with $\|D\|, \|\varphi(D)\| < \|w\|$. If the branch-like element $x = w + h$ satisfies the lemma, then the proof is finished. Otherwise $\varphi(x) \in \langle G, x\rangle_*$ and there are $n \in \omega$ and $r \in R$ such that

$$q_n\varphi(w+h) - r(w+h) \in G.$$

If $n = 0$ we can apply (4.6) directly, but if $n > 0$, we may assume that $r \in R \setminus q_nR$ and (4.8) applies as well. (Note that G is pure in $\widehat{B}$ and the $\mathbb{S}$-topology is Hausdorff.) There is a $d \in \widehat{D}$ such that $(q_n\varphi - r)d \notin G$. Now it is easy to check by support arguments that $x = w + (d + h)$ meets the requirements of the lemma. ■

LEMMA 4.6. *Let G be the module constructed so far with*

(i) $E \subseteq_* G \subseteq_* \widehat{B}$,
(ii) $\pi(G) := H' \subseteq H$.

Let $\varphi \in \mathrm{Hom}(G, H)$ be such that $\varphi\restriction E \neq 0$. For any $0 \neq h \in H'$ there is an $x' \in \widehat{E}$ such that $x = x' + h$ satisfies

$$\varphi(x) \notin H.$$

Moreover, the module $G' = \langle G, x\rangle_*$ is cotorsion-free, $\aleph_0$-free and $\pi(G') \subseteq H$.

Proof. As in Lemma 4.5, φ has a unique extension (again denoted by φ) to $\varphi : \widehat{E} \oplus H' \to \widehat{B}$. Since H is cotorsion-free the following is obvious: There exists a countable subset

(4.9) $\qquad C \subseteq T$ such that $D = \langle C\rangle$ satisfies $\varphi(\widehat{D}) \nsubseteq H$.

We now find the desired element $x \in \widehat{E} \oplus H'$. Let $h \in H'$. First we choose a new constant branch w with $\|D\|, \|\varphi(D)\| < \|w\|$. If the branch-like element $x = w + h$ satisfies the lemma, then the proof is finished. Otherwise $\varphi(x) \in H$. But $\varphi\restriction E \neq 0$ and H is cotorsion-free, hence there is $e \in E$ such that $\varphi(\widehat{R}e) \nsubseteq H$. Choose $\gamma \in \widehat{R}$ such that $\varphi(\gamma e) \notin H$. Now it is obvious that $x = w + (\gamma e + h)$ meets the requirements of the lemma. ■

We now continue the construction of G and describe the two tasks depending on the traps $(\varphi_\alpha, f_\alpha)$ of the Black Box in order to control endomorphisms of G and homomorphisms $G \to H$:

(I) First we consider the following 'bad case' for α: *If there is an $x \in \widehat{\mathrm{Dom}(\varphi_\alpha)}$ such that $\|x\| < \|\mathrm{Dom}(\varphi_\alpha)\| \in S_1 \cup S_2$ and either*

$$\varphi_\alpha(x) \notin \langle G_\alpha, x\rangle_* \quad (\textit{if } \|\mathrm{Dom}(\varphi_\alpha)\| \in S_1), \quad \textit{or}$$
$$\varphi_\alpha(x) \notin H \quad (\textit{if } \|\mathrm{Dom}(\varphi_\alpha)\| \in S_2),$$

then choose a branch $v \in \mathrm{Br}(\mathrm{Im}\, f_\alpha)$ *and put* $g_\alpha = v$ *or* $g_\alpha = x+v$*; the choice of* g_α *depends on the requirement that either*

$$\varphi_\alpha(g_\alpha) \notin \langle G_\alpha, g_\alpha\rangle_* = G_{\alpha+1} \quad (\textit{if } \|\mathrm{Dom}(\varphi_\alpha)\| \in S_1), \quad \textit{or}$$
$$\varphi_\alpha(g_\alpha) \notin H \quad (\textit{if } \|\mathrm{Dom}(\varphi_\alpha)\| \in S_2).$$

If α *is not bad, then choose any branch-like* g_α *taking care of* (II).

(II) *If* $\beta < \alpha$ *was bad before and* $\varphi_\beta(g_\beta) \notin G_{\beta+1}$ *then we still want* $\varphi_\beta(g_\beta) \notin G_{\alpha+1}$.

We first show

Lemma 4.7. *There is a choice of* g_α*'s such that the two tasks* (I) *and* (II) *are satisfied.*

Proof. The work on condition (I) has been put into Lemmas 4.5 and 4.6, hence (II) must be verified. We apply a short argument from [9, p. 457] and restrict ourselves to the case of endomorphisms of G (i.e. $\|\mathrm{Dom}(\varphi_\alpha)\| \in S_1$). The case of homomorphisms into H (if $\|\mathrm{Dom}(\varphi_\alpha)\| \in S_2$) is similar but easier and therefore left to the reader. When defining $g_\alpha = g_{\alpha,v} = x + v$ we have a free choice of branches $v \in \mathrm{Br}(\mathrm{Im}\, f_\alpha)$ which we now use. If (II) is violated for some $\beta = \beta_v$, then

$$\varphi_\beta(g_\beta) \in \langle G_\alpha, g_{\alpha,v}\rangle_*, \quad \text{hence} \quad \varphi_{\beta_v}(q_{k_v} g_{\beta_v}) - g_{\alpha,v} a_v \in G_\alpha. \tag{4.10}$$

A support argument and (c) from the Black Box show that

$$\beta_v < \alpha < \beta_v + 2^{\aleph_0},$$

and if β_0 is the least ordinal satisfying this inequality, then, for all $v \in \mathrm{Br}(\mathrm{Im}\, f_\alpha)$,

$$\beta_0 < \beta_v < \beta_0 + 2^{\aleph_0}.$$

By cardinalities, there are two distinct branches $v, w \in \mathrm{Br}(\mathrm{Im}\, f_\alpha)$ such that $\beta_v = \beta_w$. Suppose $k_v \geq k_w$. Subtracting the corresponding expressions (4.10) we get

$$r_v g_{\alpha,v} - \frac{q_{k_v}}{q_{k_w}} g_{\alpha,w} r_w \in G_\alpha$$

and by an easy support argument it can be seen that this is only possible for $v = w$, a contradiction. Hence (II) can be arranged for $g_\alpha = x + v$ and some $v \in \mathrm{Br}(\mathrm{Im}\, f_\alpha)$. ∎

We claim that the two tasks suffice to show the statement of Theorem 4.4 and verify the conditions mentioned there.

First note that $K = \mathrm{Ker}(\pi)$, and G/K is reduced. Consider any $\varphi \in \mathrm{End}\, G$ and assume that $\varphi\restriction E \notin R$.

By Lemma 4.5 there is an $x \in \widehat{E} \oplus H$ such that $\varphi(x) \notin G$ and by the Black Box we can find an $\alpha \in \lambda^*$ such that φ extends φ_α, $x \in \widehat{\mathrm{Dom}\, \varphi_\alpha}$ and $\|x\| < \|\mathrm{Dom}\, \varphi_\alpha\|$. Hence α is a bad case and g_α in the construction must satisfy (I) and (II). It follows by task (I) that $\varphi_\alpha(g_\alpha) = \varphi(g_\alpha) \notin G_{\alpha+1}$. By task (II) we also have $\varphi(g_\alpha) \notin G_\gamma$ for any later ordinal $\alpha < \gamma \in \lambda^*$, hence $\varphi(g_\alpha) \notin G$ and φ is not an endomorphism of G, a contradiction. Thus $\varphi \restriction E = r \cdot \mathrm{id}$ for some $r \in R$. Thus (ii) of the theorem holds, and (iii) follows with the help of (ii): We consider the map $\varphi_r = \varphi - r$. Since G is reduced it follows that $\varphi_r(K) = 0$, so φ_r induces a $\tilde{\varphi} : H = G/K \to G$ which must be zero by the hypothesis in (iii). Thus $\mathrm{End}(G) = R$, which implies $\mathrm{Hom}(G, K) = 0$.

Now we fix $\varphi \in \mathrm{Hom}(G, H)$ and suppose $\varphi \restriction E \neq 0$. Then by Lemma 4.6 there is some $x \in \widehat{E} \oplus H$ such that $\varphi(x) \notin H$. Again, by the Black Box we can find an $\alpha \in \lambda^*$ such that φ extends φ_α, $x \in \widehat{\mathrm{Dom}\,\varphi_\alpha}$ and $\|x\| < \|\mathrm{Dom}\,\varphi_\alpha\|$. Hence α is a bad case and g_α in the construction must satisfy (I). It follows that $\varphi_\alpha(g_\alpha) = \varphi(g_\alpha) \notin H$, a contradiction. Thus $\varphi \restriction E = 0$ and there is an induced homomorphism $\tilde{\varphi} : H = G/K \to H$ which has to be multiplication by some $r \in R$ by the assumptions on H. Thus $\varphi \in \pi R$ as required.

The mapping conditions for the cellular cover $0 \to K \to G \to H \to 0$ are obvious. Thus Theorem 4.4 holds. ∎

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