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On a volume element of a Hitchin component

by

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Abstract. Let Σ be a closed oriented Riemann surface of genus at least 2. By using symplectic chain complex, we construct a volume element for a Hitchin component of $\mathrm{Hom}(\pi_1(\Sigma), \mathrm{PSL}_n(\mathbb{R}))/\mathrm{PSL}_n(\mathbb{R})$ for $n > 2$.

1. Introduction. Reidemeister torsion was introduced by K. Reidemeister in 1935 [19]. This topological but not homotopy invariant enabled him to classify (up to PL equivalence) 3-dimensional lens spaces $\mathbb{S}^3/\Gamma$, where Γ is a finite cyclic group of fixed point free orthogonal transformations. W. Franz [7] extended Reidemeister torsion in 1935 and classified the higher dimensional lens spaces $\mathbb{S}^{2n+1}/\Gamma$, where Γ is a cyclic group acting freely and isometrically on the sphere $\mathbb{S}^{2n+1}$.

In 1964, G. de Rham [6] extended the results of Reidemeister and Franz to spaces of constant curvature $+1$. The topological invariance of Reidemeister torsion for manifolds was proved by R. C. Kirby and L. C. Siebenmann [12]. T. A. Chapman [2, 3] proved the invariance for arbitrary simplicial complexes. Thus, the classification of lens spaces of Reidemeister and Franz was actually topological (i.e. up to homeomorphism).

In 1961, J. Milnor disproved *Hauptvermutung* by using Reidemeister torsion. He constructed two homeomorphic but combinatorially distinct finite simplicial complexes. In 1962, he also identified Reidemeister torsion with the Alexander polynomial which plays an important role in knot theory and links [14, 16].

In [29], E. Witten considered the moduli space $\mathcal{M}$ of gauge equivalence classes of flat connections on a compact Riemann surface Σ. For Σ orientable, $\mathcal{M}$ has a natural symplectic form ω (see [1]), and thus there is a natural volume form $\theta = \omega^n/n!$, where $2n = \dim \mathcal{M}$. Using Reidemeister

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torsion, Witten defined a volume element on $\mathcal{M}$, whether Σ is orientable or not. This volume form coincides with θ for orientable Σ. He also computed the volume of $\mathcal{M}$ by using this volume form.

In [20], using a sympletic chain complex and Thurston's geodesic lamination theory, we constructed a volume element on the moduli space of representations of the fundamental group $\pi_1(\Sigma)$ of a closed oriented Riemann surface Σ of negative Euler characterstic (i.e. genus at least 2) in $\mathrm{PSL}_2(\mathbb{R})$. We also explained in [21] the relation between Reidemeister torsion and the Fubini–Study form ω_{FS} of the complex projective n-space $\mathbb{CP}^n$ by using symplectic chain complex. Moreover, this technique enabled us to prove a connection between the Reidemeister torsion of a closed oriented Riemann surface and its period matrix [22]. Recently, we gave a formula for the Reidemeister torsion of even-dimensional smooth closed oriented manifolds [23].

For a closed oriented Riemann surface Σ of negative Euler characteristic and a semisimple Lie group G, it is well known that the orbit space $\mathrm{Hom}(\pi_1(\Sigma), G)/G$ of all homomorphisms from $\pi_1(\Sigma)$ to G modulo conjugation in G has the structure of a real analytic variety. Recall that it is not necessarily Hausdorff (see e.g. [8]) but the space $\mathrm{Rep}(\pi_1(\Sigma), G) = \mathrm{Hom}^+(\pi_1(\Sigma), G)/G$ of all reductive representations of $\pi_1(\Sigma)$ in G is Hausdorff.

The *Teichmüller space* $\mathrm{Teich}(\Sigma)$ of Σ is the space of isotopy classes of complex structures on Σ. It is a differentiable manifold diffeomorphic to $\mathbb{R}^{3|\chi(\Sigma)|}$, where χ is the Euler characteristic. By the Uniformization Theorem, $\mathrm{Teich}(\Sigma)$ can also be considered as the space $\mathrm{Rep}_{\mathrm{df}}(\pi_1(\Sigma), \mathrm{PSL}_2(\mathbb{R}))$ of discrete faithful representations of $\pi_1(\Sigma)$ in $\mathrm{PSL}_2(\mathbb{R})$. It is well known that this representation space is a connected component of $\mathrm{Rep}(\pi_1(\Sigma), \mathrm{PSL}_2(\mathbb{R}))$.

For a finite cover G of the group $\mathrm{PSL}_2(\mathbb{R})$, the connected components of the space $\mathrm{Rep}(\pi_1(\Sigma), G)$ were investigated by W. Goldman [9]. He proved that $\mathrm{Rep}(\pi_1(\Sigma), G)$ has $2|\chi(\Sigma)| + 1$ connected components, two of which, called Teichmüller spaces, are homeomorphic to $\mathbb{R}^{|\chi(\Sigma)| \dim \mathrm{PSL}_2(\mathbb{R})}$.

In [10], N. Hitchin investigated the connected components of the space $\mathrm{Rep}(\pi_1(\Sigma), G)$ for a split real semisimple Lie group G, and proved the existence of an interesting connected component not detected by characteristic classes. A *Hitchin component* $\mathrm{Rep}_{\mathrm{Hitchin}}(\pi_1(\Sigma), G)$ of $\mathrm{Rep}(\pi_1(\Sigma), G)$ is a connected component containing Fuchsian representations. More precisely, these are representations of the form $\varrho \circ \imath$, where $\varrho : \pi_1(\Sigma) \to \mathrm{PSL}_2(\mathbb{R})$ is Fuchsian and $\imath : \mathrm{PSL}_2(\mathbb{R}) \to G$ is the representation corresponding to the 3-dimensional principal subgroup discovered by B. Kostant [11]. For the case $G = \mathrm{PSL}_n(\mathbb{R})$, the embedding $\imath : \mathrm{PSL}_2(\mathbb{R}) \to \mathrm{PSL}_n(\mathbb{R})$ is an n-dimensional irreducible representation of $\mathrm{PSL}_2(\mathbb{R})$.

This enables one to identify $\mathrm{Teich}(\Sigma)$ with a subset of $\mathrm{Rep}(\pi_1(\Sigma), G)$. In [10], N. Hitchin proved that each Hitchin component is diffeomorphic to $\mathbb{R}^{|\chi(\Sigma)| \dim G}$. Moreover, $\mathrm{Rep}(\pi_1(\Sigma), \mathrm{PSL}_n(\mathbb{R}))$, $n > 2$, has three (respectively, six) connected components when n is odd (respectively, even). There exists only one Hitchin component for odd n, and two isomorphic ones for even n [10]. Recall also that S. Choi and W. M. Goldman proved in [4] that for $n = 3$, the Hitchin component consists of holonomies of convex real projective structures on Σ.

In [13], F. Labourie proved that representations in the Hitchin component $\mathrm{Rep}_{\mathrm{Hitchin}}(\pi_1(\Sigma), \mathrm{PSL}_n(\mathbb{R})), n > 2$ are discrete, faithful, irreducible, and purely loxodromic.

In the present article, the main result is:

Main Theorem 1.1. *Let Σ be a closed oriented Riemann surface of genus $g \geq 2$. Let $\varrho : \pi_1(\Sigma) \to \mathrm{PSL}_n(\mathbb{R})$, $n > 2$, be an irreducible, purely loxodromic representation. Let K be a cell decomposition of Σ, $\mathbf{c}_p$ the geometric basis of $C_p(K; \mathfrak{sl}_n(\mathbb{R})_{\mathrm{Ad}\circ\varrho})$, $p = 0, 1, 2$, and $\mathbf{h}_1$ a basis of $H_1(\Sigma; \mathfrak{sl}_n(\mathbb{R})_{\mathrm{Ad}\circ\varrho})$. Then*

$$\mathbb{T}(C_*(K; \mathfrak{sl}_n(\mathbb{R})_{\mathrm{Ad}\circ\varrho}), \{\mathbf{c}_p\}_{p=0}^2, \{0, \mathbf{h}_1, 0\}) = M_{g,n}\sqrt{\det \Omega_{\omega_B}},$$

where $M_{g,n} = (2(g-1)(n^2-1)/\|H\|^2)^{(g-1)(n^2-1)}$, H is the matrix of the intersection form $(\cdot,\cdot)_{1,1} : H_1(\Sigma; \mathfrak{sl}_n(\mathbb{R})_{\mathrm{Ad}\circ\varrho}) \times H_1(\Sigma; \mathfrak{sl}_n(\mathbb{R})_{\mathrm{Ad}\circ\varrho}) \to \mathbb{R}$ in $\mathbf{h}_1$, Ω_{ω_B} is the matrix of $\omega_B : H^1(\Sigma; \mathfrak{sl}_n(\mathbb{R})_{\mathrm{Ad}\circ\varrho}) \times H^1(\Sigma; \mathfrak{sl}_n(\mathbb{R})_{\mathrm{Ad}\circ\varrho}) \to \mathbb{R}$ in $\mathbf{h}^1$, $\|H\| = \sqrt{HH^t}$ is the norm of H, and $\mathbf{h}^1$ is the Poincaré dual basis of $H^1(\Sigma; \mathfrak{sl}_n(\mathbb{R})_{\mathrm{Ad}\circ\varrho})$ corresponding to $\mathbf{h}_1$ of $H_1(\Sigma; \mathfrak{sl}_n(\mathbb{R})_{\mathrm{Ad}\circ\varrho})$.

The content of the paper is as follows. In §2, the basic definitions and facts about the Reidemeister torsion of a general chain complex are provided. We also describe symplectic chain complexes. §3 concerns the Reidemeister torsion of a representation of a surface group. In §4, the symplectic chain complex associated to a representation is described and the proof of Theorem 1.1 is given. We apply Theorem 1.1 in §5 to exhibit a volume element on a Hitchin component of $\mathrm{Hom}(\pi_1(\Sigma), \mathrm{PSL}_n(\mathbb{R}))/\mathrm{PSL}_n(\mathbb{R})$.

2. Reidemeister torsion and symplectic chain complexes. Let us start with the necessary definitions and basic facts about Reidemeister torsion. See for example [17, 20, 25, 26, 29] and the references therein for detailed proofs and more information.

Let $C_* = (C_n \xrightarrow{\partial_n} C_{n-1} \to \cdots \to C_1 \xrightarrow{\partial_1} C_0 \to 0)$ be a chain complex of finite-dimensional vector spaces over the field $\mathbb{R}$ of real numbers. Let $H_p(C_*) = Z_p(C_*)/B_p(C_*)$, where $B_p(C_*) = \mathrm{Im}\,\partial_{p+1}$ and $Z_p(C_*) = \ker \partial_p$.

Clearly, we have the short exact sequences

$$0 \to Z_p(C_*) \to C_p \to B_{p-1}(C_*) \to 0$$

and

$$0 \to B_p(C_*) \to Z_p(C_*) \to H_p(C_*) \to 0.$$

If $\mathbf{b}_p$, $\mathbf{h}_p$ are bases of $B_p(C_*)$, $H_p(C_*)$, respectively, and if $\ell_p : H_p(C_*) \to Z_p(C_*)$, $s_p : B_{p-1}(C_*) \to C_p$ are sections of $Z_p(C_*) \to H_p(C_*)$, $C_p \to B_{p-1}(C_*)$, respectively, then we obtain a new basis of C_p, namely $\mathbf{b}_p \oplus \ell_p(\mathbf{h}_p) \oplus s_p(\mathbf{b}_{p-1})$.

Definition 2.1. The *Reidemeister torsion* of the chain complex C_* with respect to the bases $\{\mathbf{c}_p\}_{p=0}^n$, $\{\mathbf{h}_p\}_{p=0}^n$ is the alternating product

$$\mathbb{T}(C_*, \{\mathbf{c}_p\}_{p=0}^n, \{\mathbf{h}_p\}_{p=0}^n) = \prod_{p=0}^{n} [\mathbf{b}_p \oplus \ell_p(\mathbf{h}_p) \oplus s_p(\mathbf{b}_{p-1}), \mathbf{c}_p]^{(-1)^{p+1}}.$$

Here, $[\mathbf{e}_p, \mathbf{f}_p]$ is the determinant of the base-change matrix from the basis $\mathbf{f}_p$ to $\mathbf{e}_p$ in C_p.

Remark. The independence of the Reidemeister torsion from the bases $\mathbf{b}_p$ and sections s_p, ℓ_p was proved by J. Milnor [15]. Let $\mathbf{c}'_p, \mathbf{h}'_p$ be other bases of C_p, $H_p(C_*)$ respectively. Then the base-change formula

$$\mathbb{T}(C_*, \{\mathbf{c}'_p\}_{p=0}^n, \{\mathbf{h}'_p\}_{p=0}^n) = \prod_{p=0}^{n} \left(\frac{[\mathbf{c}'_p, \mathbf{c}_p]}{[\mathbf{h}'_p, \mathbf{h}_p]} \right)^{(-1)^p} \mathbb{T}(C_*, \{\mathbf{c}_p\}_{p=0}^n, \{\mathbf{h}_p\}_{p=0}^n) \tag{2.1}$$

can be easily derived from the independence of the Reidemeister torsion from $\mathbf{b}_p$ and sections s_p, ℓ_p.

Clearly, the short exact sequence of chain complexes

$$0 \to A_* \xrightarrow{\imath} B_* \xrightarrow{\pi} D_* \to 0 \tag{2.2}$$

yields the long exact sequence of vector spaces of length $3n+2$

$$\mathcal{H}_* : \cdots \to H_p(A_*) \xrightarrow{\imath_p} H_p(B_*) \xrightarrow{\pi_p} H_p(D_*) \xrightarrow{\delta_p} H_{p-1}(A_*) \to \cdots, \tag{2.3}$$

where $\mathcal{H}_{3p} = H_p(D_*)$, $\mathcal{H}_{3p+1} = H_p(A_*)$, and $\mathcal{H}_{3p+2} = H_p(B_*)$.

Clearly, the bases $\mathbf{h}_p^D$, $\mathbf{h}_p^A$, and $\mathbf{h}_p^B$ serve as bases for $\mathcal{H}_{3p}$, $\mathcal{H}_{3p+1}$, and $\mathcal{H}_{3p+2}$, respectively. The following result of J. Milnor states that the alternating product of the Reidemeister torsions of the chain complexes in (2.2) is equal to the Reidemeister torsion of (2.3).

Theorem 2.2 ([15]). *Suppose that $\mathbf{c}_p^A$, $\mathbf{c}_p^B$, $\mathbf{c}_p^D$, $\mathbf{h}_p^A$, $\mathbf{h}_p^B$, and $\mathbf{h}_p^D$ are bases of A_p, B_p, D_p, $H_p(A_*)$, $H_p(B_*)$, and $H_p(D_*)$, respectively. Furthermore, if $\mathbf{c}_p^A$, $\mathbf{c}_p^B$, and $\mathbf{c}_p^D$ are compatible in the sense that $[\mathbf{c}_p^B, \mathbf{c}_p^A \oplus \widetilde{\mathbf{c}_p^D}] = \pm 1$, where $\pi(\widetilde{\mathbf{c}_p^D}) = \mathbf{c}_p^D$, then*

$$\begin{aligned}\mathbb{T}(B_*, \{\mathbf{c}_p^B\}_{p=0}^n, \{\mathbf{h}_p^B\}_{p=0}^n) &= \mathbb{T}(A_*, \{\mathbf{c}_p^A\}_{p=0}^n, \{\mathbf{h}_p^A\}_{p=0}^n) \\ &\quad \times \mathbb{T}(D_*, \{\mathbf{c}_p^D\}_{p=0}^n, \{\mathbf{h}_p^D\}_{p=0}^n) \mathbb{T}(\mathcal{H}_*, \{\mathbf{c}_{3p}\}_0^{3n+2}, \{0\}_0^{3n+2}).\end{aligned}$$

We have the following sum lemma:

Lemma 2.3 ([23]). *Let A_*, D_* be two chain complexes. Let $\mathbf{c}_p^A$, $\mathbf{c}_p^D$, $\mathbf{h}_p^A$, and $\mathbf{h}_p^D$ be bases of A_p, D_p, $H_p(A_*)$, and $H_p(D_*)$, respectively. Then*

$$\mathbb{T}(A_* \oplus D_*, \{\mathbf{c}_p^A \oplus \mathbf{c}_p^D\}_{p=0}^n, \{\mathbf{h}_p^A \oplus \mathbf{h}_p^D\}_{p=0}^n) = \mathbb{T}(A_*, \{\mathbf{c}_p^A\}_{p=0}^n, \{\mathbf{h}_p^A\}_{p=0}^n) \times \mathbb{T}(D_*, \{\mathbf{c}_p^D\}_{p=0}^n, \{\mathbf{h}_p^D\}_{p=0}^n).$$

The proof of Lemma 2.3 is based on Theorem 2.2. It uses the short exact sequence

$$0 \to A_* \xrightarrow{\imath} A_* \oplus D_* \xrightarrow{\pi} D_* \to 0,$$

where each $\imath_p : A_p \to A_p \oplus D_p$ is the inclusion and $\pi_p : A_p \oplus D_p \to D_p$ is the projection. It also uses the compatibility of the bases $\mathbf{c}_p^A$, $\mathbf{c}_p^A \oplus \mathbf{c}_p^D$, and $\mathbf{c}_p^D$, where one can consider the inclusion as a section of $\pi_p : A_p \oplus D_p \to D_p$.

A general chain complex C_* can (unnaturally) be split as a direct sum of an exact complex and a ∂-zero complex. The Reidemeister torsion $\mathbb{T}(C_*)$ is interpreted as an element of $\bigotimes_{p=0}^n (\det(H_p(C_*)))^{(-1)^{p+1}}$, where $\det(H_p(C_*)) = \bigwedge^{\dim_{\mathbb{R}} H_p(C_*)} H_p(C_*)$, the top exterior power of $H_p(C_*)$, and $\det(H_p(C_*))^{-1}$ is the dual of $\det(H_p(C_*))$. See [20, 29] for details.

Definition 2.4. Let $q \equiv 2 \pmod 4$. A *symplectic chain complex* of length q is $(C_*, \partial_*, \{\omega_{*,q-*}\})$, where

$$C_* : 0 \to C_q \xrightarrow{\partial_q} C_{q-1} \to \cdots \to C_{q/2} \to \cdots \to C_1 \xrightarrow{\partial_1} C_0 \to 0$$

is a chain complex and for $p = 0, \dots, q$, $\omega_{p,q-p} : C_p \times C_{q-p} \to \mathbb{R}$ is a ∂-compatible anti-symmetric non-degenerate bilinear form. More explicitly,

$$\omega_{p,q-p}(\partial_{p+1} a, b) = (-1)^{p+1} \omega_{p+1,q-(p+1)}(a, \partial_{q-p} b),$$
$$\omega_{p,q-p}(a, b) = (-1)^{p(q-p)} \omega_{q-p,p}(b, a).$$

Remark. As $q \equiv 2 \pmod 4$, we have $\omega_{p,q-p}(a,b) = (-1)^p \omega_{q-p,p}(b,a)$. From the ∂-compatibility of the non-degenerate anti-symmetric bilinear maps $\omega_{p,q-p}$ it follows that they can be extended to homologies [20].

Definition 2.5. Let C_* be a symplectic chain complex. Let $\mathbf{c}_p$ and $\mathbf{c}_{q-p}$ be bases of C_p and C_{q-p}, respectively. These bases are said to be *ω-compatible* if the matrix of $\omega_{p,q-p}$ in bases $\mathbf{c}_p$, $\mathbf{c}_{q-p}$ equals the $k \times k$ identity matrix $\mathrm{I}_{k\times k}$ when $p \neq q/2$ and $\left(\begin{smallmatrix} 0_{l\times l} & \mathrm{I}_{l\times l} \\ -\mathrm{I}_{l\times l} & 0_{l\times l} \end{smallmatrix}\right)$ when $p = q/2$, where $k = \dim C_p = \dim C_{q-p}$ and $2l = \dim C_{q/2}$.

By using the existence of ω-compatible bases, we were able to prove in [20] that a symplectic chain complex C_* can be split ω-orthogonally as a direct sum of an exact complex and a ∂-zero symplectic complex. We also proved Theorem 2.6 below, which is one of the main results of [20].

Theorem 2.6 ([20]). *Let $(C_*, \partial_*, \{\omega_{*,q-*}\})$ be a symplectic chain complex with ω-compatible bases. For $p = 0, \ldots, q$, let $\mathbf{c}_p$, $\mathbf{h}_p$ be any bases of C_p, $H_p(C_*)$, respectively. Then*

$$\mathbb{T}(C_*, \{\mathbf{c}_p\}_{p=0}^q, \{\mathbf{h}_p\}_{p=0}^q) = \prod_{p=0}^{(q/2)-1} (\det[\omega_{p,q-p}])^{(-1)^p} \sqrt{\det[\omega_{q/2,q/2}]}^{(-1)^{q/2}},$$

where $\det[\omega_{p,q-p}]$ *denotes the determinant of the matrix of the non-degenerate pairing* $[\omega_{p,q-p}] : H_p(C_*) \times H_{q-p}(C_*) \to \mathbb{R}$ *in the bases* $\mathbf{h}_p$, $\mathbf{h}_{q-p}$.

The proof and unexplained issues can be found in [20]. For further applications of Theorem 2.6, we refer the reader to [21, 22, 23].

3. Reidemeister torsion of a surface group representation. Let Σ be a closed oriented Riemann surface with genus $g \geq 2$ and $\widetilde{\Sigma}$ be the universal covering of Σ. Consider the Lie group $\mathrm{PSL}_n(\mathbb{R})$, $n > 2$, and its Lie algebra $\mathfrak{sl}_n(\mathbb{R})$ with the non-degenerate Killing form B.

Let $\varrho : \pi_1(\Sigma) \to \mathrm{PSL}_n(\mathbb{R})$ be a homomorphism. There is the associated adjoint bundle $E_\varrho = \widetilde{\Sigma} \times \mathfrak{sl}_n(\mathbb{R})/\!\sim$ over Σ. Here, all the elements in the orbit $\{(\gamma \cdot x, \gamma \cdot t);\ \gamma \in \pi_1(S)\}$ of (x, t) are identified, γ acts in the first component as a deck transformation and in the second component as conjugation by $\varrho(\gamma)$, i.e. the adjoint action.

Let K be a cell decomposition of Σ so that the adjoint bundle E_ϱ over Σ is trivial over each cell. Let $\widetilde{K}$ be the lift of K to the universal covering $\widetilde{\Sigma}$ of Σ. Let $\mathbb{Z}[\pi_1(\Sigma)] = \{\sum_{i=1}^p m_i\gamma_i;\ m_i \in \mathbb{Z},\ \gamma_i \in \pi_1(\Sigma),\ p \in \mathbb{N}\}$ be the integral group ring. By the deck transformation action of $\pi_1(\Sigma)$ on $\widetilde{\Sigma}$, $C_*(\widetilde{K}; \mathbb{Z})$ becomes a right $\mathbb{Z}[\pi_1(S)]$-module and by the adjoint action of $\pi_1(\Sigma)$ on $\mathfrak{sl}_n(\mathbb{R})$, $\mathfrak{sl}_n(\mathbb{R})$ is a left $\mathbb{Z}[\pi_1(S)]$-module. More precisely, if $\gamma \in \pi_1(\Sigma)$, $t \in \mathfrak{sl}_n(\mathbb{R})$, and $\sigma \in C_*(\widetilde{\Sigma}; \mathbb{Z})$, then $\gamma \cdot t = \varrho(\gamma)t\varrho(\gamma)^{-1} = \mathrm{Ad}_{\varrho(\gamma)}\, t$ and $\sigma \cdot \gamma = \gamma^{-1} \cdot \sigma$, where γ^{-1} acts as a deck transformation.

Clearly, the tensor relation $\sigma \cdot \gamma \otimes t = \sigma \otimes \gamma \cdot t$ results in $\gamma^{-1} \cdot \sigma \otimes t = \sigma \otimes \gamma \cdot t$, or equivalently $\sigma' \otimes t = \gamma \cdot \sigma' \otimes \gamma \cdot t$, where $\sigma' = \gamma^{-1} \cdot \sigma$. Thus, tensoring with $\mathbb{Z}[\pi_1(\Sigma)]$ has the same effect as factoring with $\pi_1(\Sigma)$. Hence, $C_*(K; \mathfrak{sl}_n(\mathbb{R})_{\mathrm{Ad}\circ\varrho}) = C_*(\widetilde{K}; \mathbb{Z}) \otimes_\varrho \mathfrak{sl}_n(\mathbb{R})$ is defined as $C_*(\widetilde{K}; \mathbb{Z}) \otimes \mathfrak{sl}_n(\mathbb{R})/\sim$, where $\sigma \otimes t$ and all the elements in the orbit $\{\gamma \cdot \sigma \otimes \gamma \cdot t;\ \gamma \in \pi_1(\Sigma)\}$ are identified.

Thus, there is the following chain complex:

$$\begin{aligned}(3.1)\qquad 0 \to C_2(K; \mathfrak{sl}_n(\mathbb{R})_{\mathrm{Ad}\circ\varrho}) &\xrightarrow{\partial_2\otimes\mathrm{id}} C_1(K; \mathfrak{sl}_n(\mathbb{R})_{\mathrm{Ad}\circ\varrho}) \\ &\xrightarrow{\partial_1\otimes\mathrm{id}} C_0(K; \mathfrak{sl}_n(\mathbb{R})_{\mathrm{Ad}\circ\varrho}) \to 0,\end{aligned}$$

where ∂_p is the usual boundary operator. Let $H_*(K; \mathfrak{sl}_n(\mathbb{R})_{\mathrm{Ad}\circ\varrho})$ be the homologies of (3.1). Considering the cochains $C^*(K; \mathfrak{sl}_n(\mathbb{R})_{\mathrm{Ad}\circ\varrho})$, we get

$H^*(K;\mathfrak{sl}_n(\mathbb{R})_{\mathrm{Ad}\circ\varrho})$, where $C^*(K;\mathfrak{sl}_n(\mathbb{R})_{\mathrm{Ad}\circ\varrho})$ is the set of $\mathbb{Z}[\pi_1(S)]$-module homomorphisms from $C_*(\widetilde{K};\mathbb{Z})$ to $\mathfrak{sl}_n(\mathbb{R})$. See [17, 20, 29] for more information.

Let $\varrho,\varrho':\pi_1(\Sigma)\to \mathrm{PSL}_n(\mathbb{R})$ be conjugate, i.e. $\varrho'(\cdot)=A\varrho(\cdot)A^{-1}=\mathrm{Ad}_A\circ\varrho(\cdot)$ for some $A\in\mathrm{PSL}_n(\mathbb{R})$. Then $C_*(K;\mathfrak{sl}_n(\mathbb{R})_{\mathrm{Ad}\circ\varrho})$ and $C_*(K;\mathfrak{sl}_n(\mathbb{R})_{\mathrm{Ad}\circ\varrho'})$ are isomorphic. Likewise, $C^*(K;\mathfrak{sl}_n(\mathbb{R})_{\mathrm{Ad}\circ\varrho})$ and $C^*(K;\mathfrak{sl}_n(\mathbb{R})_{\mathrm{Ad}\circ\varrho'})$ are isomorphic. Moreover, $H_*(K;\mathfrak{sl}_n(\mathbb{R})_{\mathrm{Ad}\circ\varrho})$ does not depend on the cell decomposition of Σ. These can be proved by similar arguments to those in [20, Lemma 1.2.1].

Definition 3.1. $\varrho:\pi_1(\Sigma)\to\mathrm{PSL}_n(\mathbb{R})$ is *purely loxodromic* if for every non-trivial $\gamma\in\pi_1(\Sigma)$, the eigenvalues of $\varrho(\gamma)$ are real with multiplicity 1.

Let us assume that $\varrho:\pi_1(\Sigma)\to\mathrm{PSL}_n(\mathbb{R})$ is purely loxodromic. Let us consider the chain complex (3.1). Let e_j^p be the p-cells of K which gives us a $\mathbb{Z}$-basis for $C_p(K;\mathbb{Z})$. Let us fix a lift $\widetilde{e}_j^p$ of e_j^p, $j=1,\dots,m_p$. Then $c_p=\{\widetilde{e}_j^p\}_{j=1}^{m_p}$ becomes a $\mathbb{Z}[\pi_1(\Sigma)]$-basis for $C_p(\widetilde{K};\mathbb{Z})$. Let $\mathcal{A}=\{\mathfrak{a}_k\}_{k=1}^{\dim\mathfrak{sl}_n(\mathbb{R})}$ be an $\mathbb{R}$-basis of the semisimple Lie algebra $\mathfrak{sl}_n(\mathbb{R})$ so that the matrix of the Killing form B is the diagonal matrix $\mathrm{Diag}(1,\overset{(p)}{\dots},1,-1,\overset{(r)}{\dots},-1)$, where $p+r=\dim\mathfrak{sl}_n(\mathbb{R})$. We call such a basis a *B-orthonormal basis.* Then $\mathbf{c}_p=c_p\otimes_\varrho\mathcal{A}$ is an $\mathbb{R}$-basis for $C_p(K;\mathfrak{sl}_n(\mathbb{R})_{\mathrm{Ad}\circ\varrho})$ and we call it a *geometric basis* for $C_p(K;\mathfrak{sl}_n(\mathbb{R})_{\mathrm{Ad}\circ\varrho})$.

Definition 3.2. Let $\mathbf{h_p}$ be an $\mathbb{R}$-basis for $H_p(K;\mathfrak{sl}_n(\mathbb{R})_{\mathrm{Ad}\circ\varrho})$. Then we call $\mathbb{T}(C_*(K;\mathfrak{sl}_n(\mathbb{R})_{\mathrm{Ad}\circ\varrho}),\{c_p\otimes_\varrho\mathcal{A}\}_{p=0}^2,\{\mathbf{h}_p\}_{p=0}^2)$ the *Reidemeister torsion* of the triple K, $\mathrm{Ad}\circ\varrho$, and $\{\mathbf{h}_p\}_{p=0}^2$.

Following similar arguments to [15], [20, Lemmas 1.4.2 and 2.0.5], one can conclude that the above definition is independent of $\mathcal{A}$, of the lifts $\widetilde{e}_j^p$, and of the cell decomposition; also this value is constant on the conjugacy class of ϱ. For completeness, we prove the independence from $\mathcal{A}$ and $\widetilde{e}_j^p$, and constance on the conjugacy class of ϱ below; for the independence from the cell decomposition, we refer to [20, Lemma 2.0.5].

Lemma 3.3. *$\mathbb{T}(C_*(K;\mathfrak{sl}_n(\mathbb{R})_{\mathrm{Ad}\circ\varrho}),\{c_p\otimes_\varrho\mathcal{A}\}_{p=0}^2,\{\mathbf{h}_p\}_{p=0}^2)$ does not depend on $\mathcal{A}$, and is constants on the lifts $\widetilde{e}_j^p$, and the cell decomposition K conjugacy class of ϱ.*

Proof. If $\mathcal{A}'$ is another B-orthonormal basis of $\mathfrak{sl}_n(\mathbb{R})$, then by the base-change formula (2.1) for Reidemeister torsion, we get

$$\frac{\mathbb{T}(C_*(K;\mathfrak{sl}_n(\mathbb{R})_{\mathrm{Ad}\circ\varrho}),\{\mathbf{c}'_p\}_{p=0}^2,\{\mathbf{h}_p\}_{p=0}^2)}{\mathbb{T}(C_*(K;\mathfrak{sl}_n(\mathbb{R})_{\mathrm{Ad}\circ\varrho}),\{\mathbf{c}_p\}_{p=0}^2,\{\mathbf{h}_p\}_{p=0}^2)}=\det(T)^{-\chi(\Sigma)},$$

where $\mathbf{c}'_p=c_p\otimes_\varrho\mathcal{A}'$ and T is the base-change matrix from $\mathcal{A}'$ to $\mathcal{A}$.

Clearly, $\det T$ is ± 1 for $\mathcal{A}$, $\mathcal{A}'$ being B-orthonormal bases of $\mathfrak{sl}_n(\mathbb{R})$. As the Euler characteristic $\chi(\Sigma)$ is even, we conclude that the Reidemeister torsion is independent of the B-orthonormal basis $\mathcal{A}$.

Let us now explain the independence from the lifts $\widetilde{e}^p_j$. Fix $\gamma \in \pi_1(\Sigma)$ and let $c'_p = \{\widetilde{e}^p_1 \cdot \gamma, \widetilde{e}^p_2, \dots, \widetilde{e}^p_{m_p}\}$ be another lift of $\{e^p_1, \dots, e^p_{m_p}\}$, where we consider another lift of e^p_1 only and leave the others the same. Note that by the adjoint action of $\pi_1(\Sigma)$ on $\mathfrak{sl}_n(\mathbb{R})$, the tensor product property $\widetilde{e}^p_1 \cdot \gamma \otimes t = \widetilde{e}^p_1 \otimes \gamma \cdot t$ becomes $\widetilde{e}^p_1 \cdot \gamma \otimes t = \widetilde{e}^p_1 \otimes \mathrm{Ad}_{\varrho(\gamma)}(t)$. From (2.1) it follows that

$$\frac{\mathbb{T}(C_*(K; \mathfrak{sl}_n(\mathbb{R})_{\mathrm{Ad}\circ\varrho}), \{\mathbf{c}'_p\}_{p=0}^2, \{\mathbf{h}_p\}_{p=0}^2)}{\mathbb{T}(C_*(K; \mathfrak{sl}_n(\mathbb{R})_{\mathrm{Ad}\circ\varrho}), \{\mathbf{c}_p\}_{p=0}^2, \{\mathbf{h}_p\}_{p=0}^2)} = \det(A),$$

where $\mathbf{c}_p = c_p \otimes_\varrho \mathcal{A}$, $\mathbf{c}'_p = c'_p \otimes_\varrho \mathcal{A}$, and A is the matrix of the map $\mathrm{Ad}_{\varrho(\gamma)} : \mathfrak{sl}_n(\mathbb{R}) \to \mathfrak{sl}_n(\mathbb{R})$ with respect to the basis $\mathcal{A}$.

The determinant of the matrix of $\mathrm{Ad}_{\varrho(\gamma)}$ does not depend on the basis of $\mathfrak{sl}_n(\mathbb{R})$. Let us consider the basis $\mathcal{B} = \{E_{ij}, i \neq j,\ H_k, k = 1, \dots, n-1\}$, where E_{ij} is the matrix with 1 in the ij entry and 0 elsewhere, and $H_i = E_{ii} - E_{i+1,i+1}$. Note that $\mathcal{B}$ is not B-orthonormal. By the assumption that ϱ is purely loxodromic, for $\gamma \in \pi_1(\Sigma)$ there exists $Q = Q(\gamma) \in \mathrm{PSL}_n(\mathbb{R})$ such that $Q\varrho(\gamma)Q^{-1} = D = \mathrm{Diag}(\lambda_1, \dots, \lambda_n)$.

Hence, it suffices to find the determinant of the matrix of Ad_D in the basis $\mathcal{B}$. One can easily see that this matrix is

$$\mathrm{Diag}(\underbrace{\lambda_i/\lambda_j}_{1\leq i\neq j\leq n}, \underbrace{1, \dots, 1}_{n-1})$$

and has determinant 1. Thus, the Reidemeister torsion is also independent of the lifts.

Finally, the constance on the conjugacy class of ϱ follows from the fact that the twisted chains and cochains for conjugate representations are isomorphic. ∎

4. Proof of Main Theorem. Let $\varrho : \pi_1(\Sigma) \to \mathrm{PSL}_n(\mathbb{R})$ be a homomorphism and K be a cell decomposition of the closed oriented Riemann surface Σ of genus at least 2 so that the adjoint bundle E_ϱ over Σ is trivial over each cell. The twisted complex $C_*(K; \mathfrak{sl}_n(\mathbb{R})_{\mathrm{Ad}\circ\varrho})$ and the twisted co-complex $C^*(K; \mathfrak{sl}_n(\mathbb{R})_{\mathrm{Ad}\circ\varrho}) = \mathrm{Hom}_{\mathbb{Z}[\pi_1(\Sigma)]}(C_*(\widetilde{K}; \mathbb{Z}), \mathfrak{sl}_n(\mathbb{R}))$ are well-defined. Here $\widetilde{K}$ is the lift of K to the universal covering $\widetilde{\Sigma}$ of Σ.

From the invariance of the Cartan–Killing form B of $\mathfrak{sl}_n(\mathbb{R})$ under conjugation it follows that for $k = 0, 1, 2$ there is a non-degenerate form, called the *Kronecker pairing*,

$$\langle \cdot, \cdot \rangle : C^k(K; \mathfrak{sl}_n(\mathbb{R})_{\mathrm{Ad}\circ\varrho}) \times C_k(K; \mathfrak{sl}_n(\mathbb{R})_{\mathrm{Ad}\circ\varrho}) \to \mathbb{R}$$

defined by $B(t,\theta(\sigma))$, $\theta \in C^k(K;\mathfrak{sl}_n(\mathbb{R})_{\mathrm{Ad}\circ\varrho})$, $\sigma \otimes_\varrho t \in C_k(K;\mathfrak{sl}_n(\mathbb{R})_{\mathrm{Ad}\circ\varrho})$. Clearly, the Kronecker pairing can be extended to

$$\langle\cdot,\cdot\rangle : H^k(\Sigma;\mathfrak{sl}_n(\mathbb{R})_{\mathrm{Ad}\circ\varrho}) \times H_k(\Sigma;\mathfrak{sl}_n(\mathbb{R})_{\mathrm{Ad}\circ\varrho}) \to \mathbb{R}.$$

Considering the chain complex $C_*(K;\mathfrak{sl}_n(\mathbb{R})_{\mathrm{Ad}\circ\varrho})$, the cup product

$$\widetilde{\cup} : C^k(K;\mathfrak{sl}_n(\mathbb{R})_{\mathrm{Ad}\circ\varrho}) \times C^\ell(K;\mathfrak{sl}_n(\mathbb{R})_{\mathrm{Ad}\circ\varrho}) \to C^{k+\ell}(\widetilde{\Sigma};\mathfrak{sl}_n(\mathbb{R})\otimes\mathfrak{sl}_n(\mathbb{R}))$$

is defined by the formula $(\theta_k \,\widetilde{\cup}\, \theta_\ell)(\sigma_{k+\ell}) = \theta_k((\sigma_{k+\ell})_{\mathrm{front}}) \otimes \theta_\ell((\sigma_{k+\ell})_{\mathrm{back}})$, $\sigma_{k+\ell} \in C_{k+\ell}(\widetilde{K};\mathbb{Z})$. By the non-degeneracy of B, $\theta_k \in C^k(K;\mathfrak{sl}_n(\mathbb{R})_{\mathrm{Ad}\circ\varrho})$, and $\theta_\ell \in C^\ell(K;\mathfrak{sl}_n(\mathbb{R})_{\mathrm{Ad}\circ\varrho})$, we have

$$\cup' : C^k(K;\mathfrak{sl}_n(\mathbb{R})_{\mathrm{Ad}\circ\varrho}) \times C^\ell(K;\mathfrak{sl}_n(\mathbb{R})_{\mathrm{Ad}\circ\varrho}) \to C^{k+\ell}(\widetilde{\Sigma};\mathbb{R})$$

defined by $(\theta_k \cup' \theta_\ell)(\sigma_{k+\ell}) = B(\theta_k((\sigma_{k+\ell})_{\mathrm{front}})),\theta_\ell((\sigma_{k+\ell})_{\mathrm{back}})$. From the invariance of B under conjugation, $\theta_k \cup' \theta_\ell$ is invariant under the action of $\pi_1(\Sigma)$, and thus we get the cup product

$$\smile_B : C^k(K;\mathfrak{sl}_n(\mathbb{R})_{\mathrm{Ad}\circ\varrho}) \times C^\ell(K;\mathfrak{sl}_n(\mathbb{R})_{\mathrm{Ad}\circ\varrho}) \to C^{k+\ell}(K;\mathbb{R}).$$

Clearly, $\smile_B$ can be extended to twisted cohomologies. By the independence of twisted cohomologies from the cell decomposition, we obtain

$$\smile_B : H^k(\Sigma;\mathfrak{sl}_n(\mathbb{R})_{\mathrm{Ad}\circ\varrho}) \times H^\ell(\Sigma;\mathfrak{sl}_n(\mathbb{R})_{\mathrm{Ad}\circ\varrho}) \to H^{k+\ell}(\Sigma;\mathbb{R}).$$

To define the intersection form, let us consider the dual cell decomposition K' of Σ associated to K [14]. Let $\widetilde{K}, \widetilde{K'}$ be the lifts of K, K', respectively. Clearly, they are also dual in $\widetilde{\Sigma}$. Let $\alpha \in C_k(\widetilde{K};\mathbb{Z})$, $\beta \in C_{2-k}(\widetilde{K'};\mathbb{Z})$. Recall that for $\alpha\cap\beta = \emptyset$, the intersection number $\alpha.\beta$ is 0. For $\alpha\cap\beta = \{x\}$, $\alpha.\beta$ is 1 if the orientation of $T_x\alpha \oplus T_x\beta$ coincides with that of $T_x\widetilde{\Sigma}$, otherwise it is -1.

The intersection form

$$(\cdot,\cdot)_{k,2-k} : C_k(K;\mathfrak{sl}_n(\mathbb{R})_{\mathrm{Ad}\circ\varrho}) \times C_{2-k}(K';\mathfrak{sl}_n(\mathbb{R})_{\mathrm{Ad}\circ\varrho}) \to \mathbb{R}$$

is defined by

$$(\sigma_1 \otimes t_1, \sigma_2 \otimes t_2)_{k,2-k} = \sum_{\gamma\in\pi_1(\Sigma)} \sigma_1.(\gamma\cdot\sigma_2)\, B(t_1,\gamma\cdot t_2)$$

where "." is the above intersection pairing, the action of γ on σ_2 is by deck transformation and on t_2 by conjugation by $\varrho(\gamma)$.

Since the action of $\pi_1(\Sigma)$ on $\widetilde{\Sigma}$ is properly discontinuous and free and σ_1, σ_2 are compact, the sum above is over a finite set. From the fact that the intersection number is anti-symmetric and B is invariant under conjugation it follows that the intersection form is also anti-symmetric.

The intersection form can be extended to twisted homologies. By the independence of the twisted homologies from the cell decomposition, we get a non-degenerate anti-symmetric form

$$(\cdot,\cdot)_{k,2-k} : H_k(\Sigma;\mathfrak{sl}_n(\mathbb{R})_{\mathrm{Ad}\circ\varrho}) \times H_{2-k}(\Sigma;\mathfrak{sl}_n(\mathbb{R})_{\mathrm{Ad}\circ\varrho}) \to \mathbb{R}.$$

Combining the isomorphisms induced by the Kronecker pairing and the intersection form, we get the Poincaré duality isomorphisms

$$\mathrm{PD} : H_k(\Sigma;\mathfrak{sl}_n(\mathbb{R})_{\mathrm{Ad}\circ\varrho}) \cong H_{2-k}(\Sigma;\mathfrak{sl}_n(\mathbb{R})_{\mathrm{Ad}\circ\varrho})^* \cong H^{2-k}(\Sigma;\mathfrak{sl}_n(\mathbb{R})_{\mathrm{Ad}\circ\varrho}).$$

Hence, the following commutative diagram exists for $k = 0,1,2$:

$$\begin{array}{ccccc} H^{2-k}(\Sigma;\mathfrak{sl}_n(\mathbb{R})_{\mathrm{Ad}\circ\varrho}) & \times & H^{k}(\Sigma;\mathfrak{sl}_n(\mathbb{R})_{\mathrm{Ad}\circ\varrho}) & \xrightarrow{\smile_B} & H^2(\Sigma;\mathbb{R}) \\ \uparrow \mathrm{PD} & & \uparrow \mathrm{PD} & \circlearrowleft & \uparrow \\ H_k(\Sigma;\mathfrak{sl}_n(\mathbb{R})_{\mathrm{Ad}\circ\varrho}) & \times & H_{2-k}(\Sigma;\mathfrak{sl}_n(\mathbb{R})_{\mathrm{Ad}\circ\varrho}) & \xrightarrow{(,)_{k,2-k}} & \mathbb{R} \end{array}$$

Here, the isomorphism $\mathbb{R} \to H^2(\Sigma;\mathbb{R})$ sends 1 to the fundamental class of $H^2(\Sigma;\mathbb{R})$.

For an irreducible representation $\varrho : \pi_1(\Sigma) \to \mathrm{PSL}_n(\mathbb{R})$, both the groups $H_k(\Sigma;\mathfrak{sl}_n(\mathbb{R})_{\mathrm{Ad}\circ\varrho})$ and $H^k(\Sigma;\mathfrak{sl}_n(\mathbb{R})_{\mathrm{Ad}\circ\varrho})$ vanish for $k = 0,2$. Thus, there is only the following commutative diagram:

$$\begin{array}{ccccc} H^{1}(\Sigma;\mathfrak{sl}_n(\mathbb{R})_{\mathrm{Ad}\circ\varrho}) & \times & H^{1}(\Sigma;\mathfrak{sl}_n(\mathbb{R})_{\mathrm{Ad}\circ\varrho}) & \xrightarrow{\smile_B} & H^2(\Sigma;\mathbb{R}) \\ \uparrow \mathrm{PD} & & \uparrow \mathrm{PD} & \circlearrowleft & \uparrow \\ H_1(\Sigma;\mathfrak{sl}_n(\mathbb{R})_{\mathrm{Ad}\circ\varrho}) & \times & H_{1}(\Sigma;\mathfrak{sl}_n(\mathbb{R})_{\mathrm{Ad}\circ\varrho}) & \xrightarrow{(,)_{1,1}} & \mathbb{R} \end{array} \tag{4.1}$$

For the rest of the paper, let

$$\omega_B : H^1(\Sigma;\mathfrak{sl}_n(\mathbb{R})_{\mathrm{Ad}\circ\varrho}) \times H^1(\Sigma;\mathfrak{sl}_n(\mathbb{R})_{\mathrm{Ad}\circ\varrho}) \xrightarrow{\smile_B} H^2(\Sigma;\mathbb{R}) \xrightarrow{\int_\Sigma} \mathbb{R}. \tag{4.2}$$

For $G = \mathrm{PSL}_2(\mathbb{R})$, the form ω_B is related to the Weil–Petersson form and the Thurston symplectic form [24].

Now, let us describe the symplectic chain complex associated to ϱ. Let $\{e_j^p\}_{j=1}^{m_p}$ be the set of p-cells of K, where $m_p = \dim C_p(K;\mathbb{Z})$. For each p and j, if we fix a lift $\widetilde{e}_j^p$ of e_j^p, $j = 1,\ldots,m_p$, then $c_p = \{\widetilde{e}_j^p\}_{j=1}^{m_p}$ is a $\mathbb{Z}[\pi_1(\Sigma)]$-basis for $C_p(\widetilde{K};\mathbb{Z})$. Let $\mathcal{A} = \{\mathfrak{a}_k\}_{k=1}^{\dim \mathfrak{sl}_n(\mathbb{R})}$ be a B-orthonormal basis of $\mathfrak{sl}_n(\mathbb{R})$. Then $\mathbf{c}_p = c_p \otimes_\varrho \mathcal{A}$ is an $\mathbb{R}$-basis for $C_p(K;\mathfrak{sl}_n(\mathbb{R})_{\mathrm{Ad}\circ\varrho})$, i.e. a geometric basis for $C_p(K;\mathfrak{sl}_n(\mathbb{R})_{\mathrm{Ad}\circ\varrho})$.

Let K' be the dual cell decomposition of Σ corresponding to the cell decomposition K. Let us assume that cells $\sigma \in K$, $\sigma' \in K'$ can meet at most in one point and also the diameter of each cell has diameter less than, say, half the injectivity radius of Σ. This is no loss of generality because the Reidemeister torsion is invariant under subdivision. Let c'_p be the basis of $C_p(\widetilde{K'};\mathbb{Z})$ corresponding to the basis c_p of $C_p(\widetilde{K};\mathbb{Z})$. Let $\mathbf{c}'_p = c'_p \otimes_\varrho \mathcal{A}$ be the corresponding basis for $C_p(K';\mathfrak{sl}_n(\mathbb{R})_{\mathrm{Ad}\circ\varrho})$.

For the rest of the paper, we will denote $C_p = C_p(K; \mathfrak{sl}_n(\mathbb{R})_{\mathrm{Ad}\circ\varrho})$, $C'_p = C_p(K'; \mathfrak{sl}_n(\mathbb{R})_{\mathrm{Ad}\circ\varrho})$, and $D_p = C_* \oplus C'_*$.

THEOREM 4.1. *D_* is a symplectic chain complex with ω-compatible bases, which are obtained from the geometric bases.*

Proof. Recall that the intersection form $(\cdot,\cdot)_{p,2-p} : C_p \times C'_{2-p} \to \mathbb{R}$ is defined by $(\sigma_1 \otimes t_1, \sigma_2 \otimes t_2)_{p,2-p} = \sum_{\gamma\in\pi_1(\Sigma)} \sigma_1.(\gamma \cdot \sigma_2)\, B(t_1, \gamma \cdot t_2)$, where the action of γ on t_2 is by conjugation with $\varrho(\gamma)$, and on σ_2 by deck transformation, "." is the intersection form and B is the Cartan–Killing form of $\mathfrak{sl}_n(\mathbb{R})$.

Clearly, the intersection pairing "." is compatible with the usual boundary operator in the sense that $(\partial\alpha).\beta = (-1)^{|\alpha|}\alpha.(\partial\beta)$, where $|\alpha|$ is the dimension of the cell α. By the properly discontinuous free action of $\pi_1(\Sigma)$ on $\widetilde{\Sigma}$ and the compactness of σ_1, σ_2, the cardinality of the set $\{\gamma \in \pi_1(\Sigma); \sigma_1 \cap (\gamma \cdot \sigma_2)\}$ is finite. From the anti-symmetry of the intersection form and the invariance of B under conjugation, $(\cdot,\cdot)_{p,2-p}$ is anti-symmetric. Furthermore, since "." is boundary compatible, so is each $(\cdot,\cdot)_{p,2-p}$. Let us define $(\cdot,\cdot)_{p,2-p}$ to be 0 on $C_p \times C_{2-p}$ and on $C'_p \times C'_{2-p}$. Let $\omega_{p,2-p} : D_p \times D_{2-p} \to \mathbb{R}$ be defined by using $(\cdot,\cdot)_{p,2-p}$. Then D_* is a symplectic complex.

Moreover, D_* has ω-compatible bases obtained from the geometric bases. Recall that the cells of K and K' meet at most in one point. Let $\{e^p_1, \ldots, e^p_{m_p}\}$ be a basis for p-dimensional cells in K. Then the duals $\{(e^p_1)', \ldots, (e^p_{m_p})'\}$ generate the $(2-p)$-dimensional cells in K'. The cell e^p_i meets $(e^p_i)'$ exactly in one point and never meets the other $(e^p_j)'$. Fix the lifts $\{\widetilde{e^p_1}, \ldots, \widetilde{e^p_{m_p}}\}$ of $\{e^p_1, \ldots, e^p_{m_p}\}$ so that the corresponding dual $\{\widetilde{(e^p_1)'}, \ldots, \widetilde{(e^p_{m_p})'}\}$ is already fixed. Recall also that $\mathcal{A} = \{\mathbf{a}_k\}_{k=1}^{\dim \mathfrak{sl}_n(\mathbb{R})}$ is a B-orthonormal basis of $\mathfrak{sl}_n(\mathbb{R})$.

Since the size of the cells is less than half the injectivity radius, it follows that $(\widetilde{(e^p_i)} \otimes x, \widetilde{(e^p_j)'} \otimes y)_{p,2-p} = B(x,y)\widetilde{(e^p_i)}.\widetilde{(e^p_j)'} = B(x,y)\delta_{ij}$.

For $p = 0,1,2$, let $\{\widetilde{e^p_i} \otimes \mathbf{a}_k\}_{i,k}$ be the geometric basis for C_p and $\{\widetilde{(e^p_j)'} \otimes \mathrm{b}_\ell\}_{j,\ell}$ be the one for C'_{2-p}, where $\mathrm{b}_\ell = B(\mathrm{a}_\ell, \mathrm{a}_\ell)\mathrm{a}_\ell$. Considering these bases for $C_p \oplus C'_p$, we obtain ω-compatible bases for D_*. ∎

THEOREM 4.2. *If $\mathbf{c}_p$, $\mathbf{c}'_p$ are the geometric bases of C_p, C'_p respectively, and if $\mathbf{h}_1$ is a basis for $H_1(\Sigma; \mathfrak{sl}_n(\mathbb{R})_{\mathrm{Ad}\circ\varrho})$, then*

$$\mathbb{T}(D_*, \{\mathbf{c}_p \oplus \mathbf{c}'_p\}_{p=0}^2, \{0 \oplus 0, \mathbf{h}_1 \oplus \mathbf{h}_1, 0 \oplus 0\}) = (\mathbb{T}(C_*, \{\mathbf{c}_p\}_{p=0}^2, \{0, \mathbf{h}_1, 0\}))^2.$$

The proof of Theorem 4.2 is based on Lemma 2.3 and the fact that the Reidemeister torsion does not depend on the cell decomposition of Σ. It uses the short exact sequence

$$0 \to C_* \hookrightarrow D_* = C_* \oplus C'_* \twoheadrightarrow C'_* \to 0$$

where $C_* \hookrightarrow D_*$ is the inclusion and $D_* \twoheadrightarrow C'_*$ is the projection. It also uses the compatibility of the bases $\mathbf{c}_p$ of C_p, $\mathbf{c}_p \oplus \mathbf{c}'_p$ of D_*, and $\mathbf{c}'_p$ of C'_*.

Actually, by Theorems 2.6 and 4.1, we have

THEOREM 4.3. *Let $\varrho : \pi_1(\Sigma) \to \mathrm{PSL}_n(\mathbb{R})$ be irreducible and purely loxodromic. Let $\mathbf{c}_p$, $\mathbf{c}'_p$ be the geometric bases of C_p, C'_p respectively. Let $\mathbf{h}_1$ be a basis for $H_1(\Sigma; \mathfrak{sl}_n(\mathbb{R})_{\mathrm{Ad}\circ\varrho})$. Then*

$$(4.3)\qquad \mathbb{T}(D_*, \{\mathbf{c}_p \oplus \mathbf{c}'_p\}_{p=0}^2, \{0 \oplus 0, \mathbf{h}_1 \oplus \mathbf{h}_1, 0 \oplus 0\}) = \Big(\sqrt{\det([\omega]_{1,1})}\Big)^{-1},$$

where $[\omega]_{1,1} : H_1(D_*) \times H_1(D_*) \to \mathbb{R}$ is

$$\begin{pmatrix} 0 & (\cdot,\cdot)_{1,1} \\ -(\cdot,\cdot)_{1,1} & 0 \end{pmatrix},$$

$(\cdot,\cdot)_{1,1} : H_1(C_*) \times H_1(C'_*) \to \mathbb{R}$ being the extension of the intersection form $(\cdot,\cdot)_{1,1} : C_1(K; \mathfrak{sl}_n(\mathbb{R})_{\mathrm{Ad}\circ\varrho}) \times C_1(K'; \mathfrak{sl}_n(\mathbb{R})_{\mathrm{Ad}\circ\varrho}) \to \mathbb{R}$, and the matrix of $[\omega]_{1,1}$ is with respect to the basis $\mathbf{h}_1 \oplus \mathbf{h}_1$.

See [20] for $G = \mathrm{PSL}_2(\mathbb{R})$. The proof of Theorem 4.3 is based on the fact that $D_* = C_* \oplus C'_*$ is a symplectic complex, and the existence of ω-compatible bases obtained from the geometric bases. It also uses the fact that for an irreducible representation $\varrho : \pi_1(\Sigma) \to \mathrm{PSL}_n(\mathbb{R})$, we have $H_k(\Sigma; \mathfrak{sl}_n(\mathbb{R})_{\mathrm{Ad}\circ\varrho}) = H^k(\Sigma; \mathfrak{sl}_n(\mathbb{R})_{\mathrm{Ad}\circ\varrho}) = 0$ for $k = 0, 2$.

By the independence of the twisted homologies from cell decompositions and non-degeneracy of $(\cdot,\cdot)_{1,1} : H_1(\Sigma; \mathfrak{sl}_n(\mathbb{R})_{\mathrm{Ad}\circ\varrho}) \times H_1(\Sigma; \mathfrak{sl}_n(\mathbb{R})_{\mathrm{Ad}\circ\varrho}) \to \mathbb{R}$, equality (4.3) becomes

$$(4.4)\qquad \mathbb{T}(D_*, \{\mathbf{c}_p \oplus \mathbf{c}'_p\}_{p=0}^2, \{0 \oplus 0, \mathbf{h}_1 \oplus \mathbf{h}_1, 0 \oplus 0\}) = \det((\cdot,\cdot)_{1,1})^{-1}.$$

with the matrix of $(\cdot,\cdot)_{1,1}$ in the basis $\mathbf{h}_1$.

From Theorems 2.6 and 4.2, the existence of ω-compatible bases of D_*, and (4.4) it follows that

$$(4.5)\qquad \mathbb{T}(C_*, \{\mathbf{c}_p\}_{p=0}^2, \{0, \mathbf{h}_1, 0\}) = \Big(\sqrt{\det(\cdot,\cdot)_{1,1}}\Big)^{-1}.$$

THEOREM 4.4. *Let $\varrho : \pi_1(\Sigma) \to \mathrm{PSL}_n(\mathbb{R})$ be irreducible and purely loxodromic. Let $\mathbf{c}_p$ be the geometric bases of C_p, $p = 0, 1, 2$, and let $\mathbf{h}_1$ be a basis of $H_1(\Sigma; \mathfrak{sl}_n(\mathbb{R})_{\mathrm{Ad}\circ\varrho})$. Then*

$$(4.6)\qquad \mathbb{T}(C_*, \{\mathbf{c}_p\}_{p=0}^2, \{0, \mathbf{h}_1, 0\}) = M_{g,n}\sqrt{\det \Omega_{\omega_B}},$$

where $M_{g,n} = (2(g-1)(n^2-1)/\|H\|^2)^{(g-1)(n^2-1)}$, H is the matrix of the intersection form $(\cdot,\cdot)_{1,1} : H_1(\Sigma; \mathfrak{sl}_n(\mathbb{R})_{\mathrm{Ad}\circ\varrho}) \times H_1(\Sigma; \mathfrak{sl}_n(\mathbb{R})_{\mathrm{Ad}\circ\varrho}) \to \mathbb{R}$ in $\mathbf{h}_1$, Ω_{ω_B} is the matrix of $\omega_B : H^1(\Sigma; \mathfrak{sl}_n(\mathbb{R})_{\mathrm{Ad}\circ\varrho}) \times H^1(\Sigma; \mathfrak{sl}_n(\mathbb{R})_{\mathrm{Ad}\circ\varrho}) \to \mathbb{R}$ in $\mathbf{h}^1$, $\|H\| = \sqrt{HH^t}$ is the norm of H, and $\mathbf{h}^1$ is the Poincaré dual basis of $H^1(\Sigma; \mathfrak{sl}_n(\mathbb{R})_{\mathrm{Ad}\circ\varrho})$ corresponding to the basis $\mathbf{h}_1$ of $H_1(\Sigma; \mathfrak{sl}_n(\mathbb{R})_{\mathrm{Ad}\circ\varrho})$.

For $G = \mathrm{PSL}_2(\mathbb{R})$, see [20].

Proof. From the diagram (4.1), H is also the matrix of ω_B in the basis $\mathbf{h}^1$ of $H^1(\Sigma; \mathfrak{sl}_n(\mathbb{R})_{\mathrm{Ad}\circ\varrho})$, which is the Poincaré dual of the basis $\mathbf{h}_1$ of $H_1(\Sigma; \mathfrak{sl}_n(\mathbb{R})_{\mathrm{Ad}\circ\varrho})$. Let $A = [a_{ij}]$ be the skew-symmetric matrix $(H^t)^{-1}$. Let $\omega_A = \sum_{i<j} a_{ij}(\mathbf{h}^1)_i \wedge (\mathbf{h}^1)_j$ be the 2-form associated to A. Clearly,

$$\frac{1}{m!}\overbrace{\omega_A \wedge \cdots \wedge \omega_A}^{m} = \sqrt{\det(A)}(\mathbf{h}^1)_1 \wedge \cdots \wedge (\mathbf{h}^1)_{2m}$$

where $m = (g-1)(n^2-1)$. By (4.5), we have

$$\mathbb{T}(C_*, \{\mathbf{c}_p\}_{p=0}^2, \{0, \mathbf{h}_1, 0\}) = \sqrt{\det(H)^{-1}} = \sqrt{\det(A)}.$$

Let $\omega_H = \sum_{i<j} h_{ij}(\mathbf{h}^1)_i \wedge (\mathbf{h}^1)_j$ be the 2-form associated to H. Then

$$\omega_A = c\,\omega_H \tag{4.7}$$

for some $c \in \mathbb{R}$. Taking the integral of both sides of (4.7) over Σ yields

$$\sum_{i=1}^{2m}(AH^t)_{ii} = c\sum_{i=1}^{2m}(HH^t)_{ii}$$

where $c = 2m/\|H\|^2$. Hence, we get

$$\sqrt{\det A} = \left(\frac{2m}{\|H\|^2}\right)^m \sqrt{\det H}.$$

This finishes the proof of Theorem 4.4. ∎

5. Application. Let Σ be a closed oriented Riemann surface of genus at least 2 and G be a semisimple Lie group. Let $\mathrm{Hom}(\pi_1(\Sigma), G)$ be the set of all homomorphisms from the fundamental group $\pi_1(\Sigma)$ to G. Let us consider the presentation of $\pi_1(\Sigma)$ by the usual $2g$ generators $a_1, b_1, \ldots, a_{2g}, b_{2g}$,

$$\pi_1(\Sigma) = \Big\langle a_1, b_1, \ldots, a_{2g}, b_{2g} \;\Big|\; \prod_{i=1}^{2g}[a_i, b_i] = 1\Big\rangle,$$

where $[a_i, b_i] = a_i b_i a_i^{-1} b_i^{-1}$.

By sending $\varrho \in \mathrm{Hom}(\pi_1(\Sigma), G)$ to $(\varrho(a_1), \varrho(b_1), \ldots, \varrho(a_{2g}), \varrho(b_{2g})) \in G^{2g}$, we can identify $\mathrm{Hom}(\pi_1(\Sigma), G)$ with the following subset of G^{2g}:

$$\Big\{(A_1, B_1, \ldots, A_{2g}, B_{2g}) \in G^{2g};\ \prod_{i=1}^{2g}[A_i, B_i] = I\Big\},$$

where I is the identity element of G.

Considering the action of G on $\mathrm{Hom}(\pi_1(\Sigma), G)$ by conjugation, that is, $g \cdot \varrho(\gamma) = g\varrho(\gamma)g^{-1}$, $g \in G$, $\varrho \in \mathrm{Hom}(\pi_1(\Sigma), G)$, and $\gamma \in \pi_1(\Sigma)$, we have the orbit space $\mathrm{Hom}(\pi_1(\Sigma), G)/G$ which has the structure of a real analytic variety. It is algebraic if so is G.

It is also well known that this space need not be Hausdorff (see, for example, [8]). However, the orbit space $\mathrm{Rep}(\pi_1(\Sigma), G) = \mathrm{Hom}^+(\pi_1(\Sigma), G)/G$ of all reductive representations of $\pi_1(\Sigma)$ in G is Hausdorff. Recall that a *reductive* representation is one that once composed with adjoint representation of G on its Lie algebra $\mathcal{G}$ is a sum of irreducible representations.

The *Teichmüller space* $\mathrm{Teich}(\Sigma)$ of Σ is by definition the space of isotopy classes of complex structures on Σ. A *complex structure* on Σ is the equivalence class of a homeomorphism $f : \Sigma \to S$, where S is a Riemann surface, and two such homeomorphisms $f : \Sigma \to S$, $f' : \Sigma \to S'$ are equivalent if there is a conformal diffeomorphism $g : S \to S'$ such that $(f')^{-1} \circ g \circ f$ is isotopic to the identity map on Σ.

A complex structure on Σ lifts to a complex structure on the universal covering $\widetilde{\Sigma}$ of Σ. From the Uniformization Theorem, $\widetilde{\Sigma}$ is biholomorphic to the upper half-plane $\mathbb{H}^2 \subset \mathbb{C}$. Every biholomorphic homeomorphism of $\mathbb{H}^2$ is of the form $f(z) = (az+b)/(cz+d)$, where $a, b, c, d \in \mathbb{R}$ with $ad - bc = 1$. This gives a homomorphism of $\pi_1(\Sigma)$ to $\mathrm{PSL}_2(\mathbb{R})$ which is discrete, faithful and well defined up to conjugation by orientation preserving isometries of $\mathbb{H}^2$. In this way, $\mathrm{Teich}(\Sigma)$ can be identified with the *Fricke space*, i.e. the set $\mathrm{Rep}_{\mathrm{df}}(\pi_1(\Sigma), \mathrm{PSL}_2(\mathbb{R}))$ of discrete faithful representations of $\pi_1(\Sigma)$ in $\mathrm{PSL}_2(\mathbb{R})$.

The Fricke space is a connected component of $\mathrm{Rep}(\pi_1(\Sigma), \mathrm{PSL}_2(\mathbb{R}))$. Here, openness follows from [27], closedness from [5, 18], and connectedness from the Uniformization Theorem together with the identification of $\mathrm{Teich}(\Sigma)$ as a cell.

In [9], W. Goldman investigated the connected components of the representation space $\mathrm{Hom}(\pi_1(\Sigma), G)/G$, where G is a finite cover of $\mathrm{PSL}_2(\mathbb{R})$. For $G = \mathrm{PSL}_2(\mathbb{R})$, W. Goldman proved that there are $4g-3$ connected components of $\mathrm{Hom}(\pi_1(\Sigma), G)/G$. There are two homeomorphic components, called Teichmüller spaces. It is well known that they are homeomorphic to $\mathbb{R}^{|\chi(\Sigma)| \dim \mathrm{PSL}_2(\mathbb{R})}$.

N. Hitchin investigated in [10] connected components of $\mathrm{Rep}(\pi_1(\Sigma), G)$, where G is a split real form of a semisimple Lie group. By using Higgs bundle techniques, he proved the existence of an interesting connected component not detected by characteristic classes.

A *Hitchin component* $\mathrm{Rep}_{\mathrm{Hitchin}}(\pi_1(\Sigma), G)$ of $\mathrm{Rep}(\pi_1(\Sigma), G)$ is a connected component containing Fuchsian representations, i.e. representations of the form $\varrho \circ \imath$, where $\varrho : \pi_1(\Sigma) \to \mathrm{PSL}_2(\mathbb{R})$ is Fuchsian and where $\imath : \mathrm{PSL}_2(\mathbb{R}) \to G$ is the representation corresponding to the 3-dimensional principal subgroup of B. Kostant [11]. In the case $G = \mathrm{PSL}_n(\mathbb{R})$, $\imath$ is the n-dimensional irreducible representation corresponding to the symmetric power $\mathrm{Sym}^{n-1}(\mathbb{R}^2)$.

In this way, the Fricke space and hence $\mathrm{Teich}(\Sigma)$ is identified with a subset of $\mathrm{Rep}(\pi_1(\Sigma), G)$. In [10], N. Hitchin proved that each Hitchin component is a cell of dimension $|\chi(\Sigma)| \dim G$. For $G = \mathrm{PSL}_n(\mathbb{R})$, $n > 2$, he also proved that there are three (respectively, six) connected components if n is odd (respectively, even). Moreover, for n odd, there is one Hitchin component, and for n even, there are two isomorphic ones. Recall that for $n = 3$, the Hitchin component consists of holonomies of convex real projective structures on Σ [4].

By using dynamical approach, F. Labourie proved in [13] that each representation $\varrho \in \mathrm{Rep}_{\mathrm{Hitchin}}(\pi_1(\Sigma), \mathrm{PSL}_n(\mathbb{R}))$ is discrete, faithful, irreducible, and purely loxodromic. This is already true for $n = 2$.

Since $\varrho \in \mathrm{Rep}_{\mathrm{Hitchin}}(\pi_1(\Sigma), \mathrm{PSL}_n(\mathbb{R}))$ is in particular irreducible, it follows that $H^0(\Sigma, \mathfrak{sl}_n(\mathbb{R})_{\mathrm{Ad}\circ\varrho})$ and $H^2(\Sigma, \mathfrak{sl}_n(\mathbb{R})_{\mathrm{Ad}\circ\varrho})$ vanish. Recall that $H^1(\Sigma, \mathfrak{sl}_n(\mathbb{R})_{\mathrm{Ad}\circ\varrho})$ and $H_1(\Sigma, \mathfrak{sl}_n(\mathbb{R})_{\mathrm{Ad}\circ\varrho})$ are identified respectively with the tangent space $T_\varrho \mathrm{Rep}_{\mathrm{Hitchin}}(\pi_1(\Sigma), \mathrm{PSL}_n(\mathbb{R}))$ and the cotangent space $T^*_\varrho \mathrm{Rep}_{\mathrm{Hitchin}}(\pi_1(\Sigma), \mathrm{PSL}_n(\mathbb{R}))$ (see e.g. [8]). Recall also that the Reidemeister torsion $\mathbb{T}(A_*)$ of a general complex A_* of length n is an element of $\bigotimes_{p=0}^{n} (\det(H_p(A_*)))^{(-1)^{p+1}}$, where $\det(H_p(A_*)) = \bigwedge^{\dim_{\mathbb{R}} H_p(A_*)} H_p(A_*)$ is the top exterior power of $H_p(A_*)$, and $\det(H_p(A_*))^{-1}$ is the dual space of $\det(H_p(A_*))$ ([20, 29]). Therefore, Theorem 1.1 yields a volume element on $\mathrm{Rep}_{\mathrm{Hitchin}}(\pi_1(\Sigma), \mathrm{PSL}_n(\mathbb{R}))$.

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