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The super fixed point property for asymptotically nonexpansive mappings

by

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Abstract. We show that the super fixed point property for nonexpansive mappings and for asymptotically nonexpansive mappings in the intermediate sense are equivalent. As a consequence, we obtain fixed point theorems for asymptotically nonexpansive mappings in uniformly nonsquare and uniformly noncreasy Banach spaces. The results are generalized to commuting families of asymptotically nonexpansive mappings.

1. Introduction. The classical problem in metric fixed point theory, a branch of fixed point theory which emerged from the Banach contraction principle, is the existence of fixed points of nonexpansive mappings. Recall that a mapping $T : C \to C$ is *nonexpansive* if

$$\|Tx - Ty\| \le \|x - y\|$$

for all $x, y \in C$. A Banach space X is said to have the *fixed point property* (FPP for short) if every nonexpansive self-mapping defined on a nonempty bounded closed and convex set $C \subset X$ has a fixed point (see [2, 10, 11]). One natural and extensively studied generalization of nonexpansive mappings was introduced by Goebel and Kirk [9]. A mapping $T : C \to C$ is said to be *asymptotically nonexpansive* if there exists a sequence (k_n) of real numbers with $\lim_n k_n = 1$ such that

$$\|T^n x - T^n y\| \le k_n \|x - y\|$$

for all $x, y \in C$ and $n \in \mathbb{N}$.

Let B be the closed unit ball in ℓ_2 and set

$$T(x_1, x_2, x_3, \ldots) = (0, x_1^2, a_2 x_2, a_3 x_3, \ldots),$$

where $(x_1, x_2, x_3, \ldots) \in B$ and (a_n) is a sequence of reals in $(0, 1)$ such that

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$\prod_{n=2}^{\infty} a_n = 1/2$. Then

$$\|Tx - Ty\| \le 2\|x - y\|$$

and

$$\|T^n x - T^n y\| \le 2 \prod_{i=2}^{n} a_i \|x - y\|$$

(see [9]). This shows that the class of asymptotically nonexpansive mappings is wider than the class of nonexpansive mappings.

In spite of the common belief that asymptotically nonexpansive mappings share a lot of properties with nonexpansive mappings, there exist relatively few results concerning the existence of fixed points for such mappings. The original result from [9] stating that asymptotically nonexpansive mappings have the fixed point property in uniformly convex spaces was generalized in [22] to the case when X is nearly uniformly convex, in [17] to X satisfying the uniform Opial condition, and in [15] to X having uniform normal structure. It is still unknown whether normal structure implies the fixed point property for asymptotically nonexpansive mappings acting on a convex and weakly compact subset of a Banach space X. Until now, the situation has been even worse in Banach spaces without normal structure.

In 1998, Kirk, Martínez Yáñez and Shin [16] showed that if X has the super fixed point property for nonexpansive mappings (i.e., every Banach space finitely representable in X has FPP), then every asymptotically nonexpansive mapping defined on a bounded closed and convex subset of X has approximate fixed points, i.e., there exists a sequence (x_n) such that $\lim_n \|Tx_n - x_n\| = 0$. In the present paper we strengthen this result by showing, in Theorem 2.4, that the super fixed point property for nonexpansive mappings is equivalent to the super fixed point property for asymptotically nonexpansive mappings in the intermediate sense (see Section 2 for the definition). In particular, we obtain fixed point theorems for asymptotically nonexpansive mappings in both uniformly nonsquare and uniformly noncreasy Banach spaces. In Section 3, the above results are extended to commuting families of asymptotically nonexpansive mappings in the intermediate sense.

It was shown in [6, Th. 10] that every Banach space X which contains an isomorphic copy of c_0 fails the fixed point property for asymptotically nonexpansive mappings. Our results support the conjecture that the fixed point property for nonexpansive mappings and for asymptotically nonexpansive mappings are equivalent, which would imply the failure of the FPP inside isomorphic copies of c_0.

2. Main result. Let X and Y be Banach spaces and let $0 < \varepsilon < 1$. A linear map $T : Y \to X$ is an *ε-isometry* if

$$(1-\varepsilon)\|y\| \le \|Ty\| \le (1+\varepsilon)\|y\|$$

for all $y \in Y$. Recall that Y is said to be *finitely representable* in X if for each $\varepsilon \in (0,1)$ and every finite-dimensional subspace $M \subset Y$ there exists an ε-isometry $T : M \to X$.

We say that X is *superreflexive* if every Banach space Y which is finitely representable in X is reflexive. A Banach space X has the *super fixed point property for nonexpansive mappings* (SFPP) if every Banach space Y which is finitely representable in X has FPP. It follows from the result of van Dulst and Pach [7, Th. 3.2] that SFPP implies superreflexivity.

The notion of finite representability is closely related to the construction of the Banach space ultrapower. Let $\mathcal{U}$ be an ultrafilter defined on a set I. The ultrapower $\widetilde{X}$ (or $(X)_{\mathcal{U}}$) of a Banach space X is the quotient space of

$$l_\infty(X) = \Big\{(x_n) : x_n \in X \text{ for all } n \in I \text{ and } \|(x_n)\| = \sup_n \|x_n\| < \infty\Big\}$$

by

$$\Big\{(x_n) \in l_\infty(X) : \lim_{n\to\mathcal{U}} \|x_n\| = 0\Big\}.$$

Here $\lim_{n\to\mathcal{U}}$ denotes the ultralimit over $\mathcal{U}$. One can prove that the quotient norm on $(X)_{\mathcal{U}}$ is given by

$$\|(x_n)_{\mathcal{U}}\| = \lim_{n\to\mathcal{U}} \|x_n\|,$$

where $(x_n)_{\mathcal{U}}$ is the equivalence class of (x_n). It is also clear that X is isometric to a subspace of $(X)_{\mathcal{U}}$ by the mapping $x \mapsto (x)_{\mathcal{U}}$.

The connection between ultrapowers and finite representability was observed independently by Henson and Moore [13] and Stern [21] (see also [1, 12, 20]).

THEOREM 2.1. *A Banach space Y is finitely representable in X if and only if there exists an ultrafilter $\mathcal{U}$ such that Y is isometric to a subspace of $(X)_{\mathcal{U}}$.*

It follows from the above theorem that X has SFPP iff every ultrapower $(X)_{\mathcal{U}}$ has FPP.

In 1980, the Banach space ultrapower construction was applied in fixed point theory by Maurey [18] who proved the fixed point property for all reflexive subspaces of $L_1[0,1]$ and the weak fixed point property for c_0 and H^1. Inspired by [16], we apply this construction to asymptotically nonexpansive mappings in a slightly more general setting. Recall that a mapping $T : C \to C$ is said to be *asymptotically nonexpansive in the intermediate* *sense* if T is continuous and

$$\limsup_{n\to\infty} \sup_{x,y\in C} (\|T^n x - T^n y\| - \|x-y\|) \le 0 \tag{2.1}$$

(in the original definition, in [4], T was assumed to be uniformly continuous). In particular, the condition (2.1) is satisfied if $\limsup_{n\to\infty} |T^n| \le 1$, where $|T^n|$ denotes the (exact) Lipschitz constant of T^n (and C is bounded).

Let C be a nonempty bounded closed and convex subset of a Banach space X and $T : C \to C$ be asymptotically nonexpansive in the intermediate sense. Take a free ultrafilter p on $\mathbb{N}$ and denote by $\widetilde{C} \subset (X)_p$ the set

$$\widetilde{C} = \{(x_n)_p \in (X)_p : x_n \in C \text{ for all } n \in \mathbb{N}\}.$$

Let

$$\mathcal{N} = \{(\alpha_n) \in \mathbb{N}^{\mathbb{N}} : \alpha_0 < \alpha_1 < \cdots\}$$

be the family of all increasing sequences of natural numbers directed by the relation $(\alpha_n) \preceq (\beta_n)$ iff $\alpha_n \le \beta_n$ for every $n \in \mathbb{N}$. Notice that if $(x_n)_p, (y_n)_p \in \widetilde{C}$ and $(\alpha_n) \in \mathcal{N}$, then

$$\lim_{n\to p} (\|T^{\alpha_n} x_n - T^{\alpha_n} y_n\| - \|x_n - y_n\|) \le \limsup_{n\to\infty} \sup_{x,y\in C} (\|T^n x - T^n y\| - \|x-y\|) \le 0.$$

Therefore, we may extend the mapping T by setting, unambiguously,

$$\widehat{T}_{(\alpha_n)}(x_n)_p = (T^{\alpha_n} x_n)_p. \tag{2.2}$$

It is not difficult to see that $\widehat{T}_{(\alpha_n)} : \widetilde{C} \to \widetilde{C}$ is nonexpansive for every (α_n) in $\mathcal{N}$. For $x \in C$, we shall write $\dot{x} = (x)_p = (x, x, \ldots)_p$.

LEMMA 2.2. *Let $T : C \to C$ be asymptotically nonexpansive in the intermediate sense and suppose that there exists $\widetilde{y} \in \widetilde{C}$ such that*

$$\widehat{T}_{(\alpha_n)}\widetilde{y} = \widetilde{y} \tag{2.3}$$

for all $(\alpha_n) \in \mathcal{N}$. Let $\|\widetilde{y} - \dot{x}_0\| < \delta$ for some $x_0 \in C$ and $\delta > 0$. Then for every $\varepsilon > 0$ there exist $x \in C$ and $n_0 \in \mathbb{N}$ such that $\|x - x_0\| < \delta$ and $\|T^n x - x\| < \varepsilon$ for every $n \ge n_0$.

Proof. Since $\|\widetilde{y} - \dot{x}_0\| < \delta$, there exists a sequence (y_n) in C such that $\|y_n - x_0\| < \delta$ for all $n \in \mathbb{N}$ and $\widetilde{y} = (y_n)_p$. Assume, contrary to our claim, that there exists $\varepsilon_0 > 0$ such that for every $x \in C$ and $n_0 \in \mathbb{N}$ there exists $n \ge n_0$ such that $\|x - x_0\| \ge \delta$ or $\|T^n x - x\| \ge \varepsilon_0$. We shall define a sequence (β_n) by induction. For $n = 0$ and $y_0 \in C$, there exists β_0 such that $\|T^{\beta_0} y_0 - y_0\| \ge \varepsilon_0$. Suppose that we have chosen $\beta_0 < \beta_1 < \cdots < \beta_n$ such that $\|T^{\beta_i} y_i - y_i\| \ge \varepsilon_0$ for $i = 0, 1, \ldots, n$. By assumption, since $\|y_{n+1} - x_0\| < \delta$, there exists $\beta_{n+1} > \beta_n$ such that $\|T^{\beta_{n+1}} y_{n+1} - y_{n+1}\| \ge \varepsilon_0$. (To be more precise, we can define, for example, β_{n+1} as the minimum of $\{\beta > \beta_n : \|T^{\beta} y_{n+1} - y_{n+1}\| \ge \varepsilon_0\}$.) Thus we obtain a sequence $(\beta_n) \in \mathcal{N}$ such that

$\|T^{\beta_n}y_n - y_n\| \geq \varepsilon_0$ for all $n \in \mathbb{N}$. Hence $\|\widehat{T}_{(\beta_n)}\widetilde{y} - \widetilde{y}\| \geq \varepsilon_0$, a contradiction with (2.3). ∎

A Banach space X is said to have the *fixed point property for asymptotically nonexpansive mappings* (*in the intermediate sense*) if every asymptotically nonexpansive (in the intermediate sense) self-mapping acting on a nonempty bounded closed and convex set $C \subset X$ has a fixed point.

THEOREM 2.3. *Assume that X has the super fixed point property for nonexpansive mappings. Then X has the fixed point property for asymptotically nonexpansive mappings in the intermediate sense.*

Proof. Assume that X has the super fixed point property for nonexpansive mappings. Let $T : C \to C$ be an asymptotically nonexpansive mapping in the intermediate sense acting on a nonempty bounded closed and convex set $C \subset X$. By [7, Th. 3.2], X is superreflexive and hence C is weakly compact. Without loss of generality we can assume that $\operatorname{diam} C = 1$. Take a free ultrafilter p on $\mathbb{N}$, $(\alpha_n) \in \mathcal{N}$, and define $\widehat{T}_{(\alpha_n)}$ by (2.2). Notice that for every $(\alpha_n), (\beta_n) \in \mathcal{N}$ and any $(z_n)_p \in \widetilde{C}$,

$$(\widehat{T}_{(\alpha_n)} \circ \widehat{T}_{(\beta_n)})(z_n)_p = (T^{\alpha_n}T^{\beta_n}z_n)_p = (\widehat{T}_{(\beta_n)} \circ \widehat{T}_{(\alpha_n)})(z_n)_p.$$

It follows from Bruck's theorem (see [3, Th. 1]) that there exists $\widetilde{y}_0 \in \widetilde{C}$ such that $\widehat{T}_{(\alpha_n)}\widetilde{y}_0 = \widetilde{y}_0$ for all $(\alpha_n) \in \mathcal{N}$ (a similar argument but for two mappings was used in [16, Th. 4.1]). Fix $\varepsilon < 1$ and $x_0 \in C$. We shall define by induction a sequence (n_j) of natural numbers and a sequence (x_j) of elements in C such that

$$\text{(2.4)} \quad \|x_j - x_{j-1}\| < 3\varepsilon^{j-1} \text{ and } \|T^n x_j - x_j\| < \varepsilon^j \text{ for every } n \geq n_j, j \geq 1.$$

By Lemma 2.2, there exist $x_1 \in C$ and $n_1 \in \mathbb{N}$ such that $\|T^n x_1 - x_1\| < \varepsilon$ for every $n \geq n_1$ and $\|x_1 - x_0\| \leq \operatorname{diam} C < 3$.

Suppose that we have chosen natural numbers $n_1, \ldots, n_j$ and $x_1, \ldots, x_j \in C$ $(j \geq 1)$ such that

$$\|x_i - x_{i-1}\| < 3\varepsilon^{i-1} \text{ and } \|T^n x_i - x_i\| < \varepsilon^i \text{ for every } n \geq n_i, 1 \leq i \leq j.$$

Let

$$D_j = \Big\{\widetilde{y} = (y_n)_p \in \widetilde{C} : \limsup_{(\alpha_n)\in\mathcal{N}} \|\widehat{T}_{(\alpha_n)}\dot{x}_j - \widetilde{y}\| \leq \varepsilon^j\Big\},$$

where $\dot{x}_j = (x_j, x_j, \ldots)_p$ and

$$\limsup_{(\alpha_n)\in\mathcal{N}} \|\widehat{T}_{(\alpha_n)}\dot{x}_j - \widetilde{y}\| = \inf_{(\alpha_n)\in\mathcal{N}} \sup_{(\beta_n)\succeq(\alpha_n)} \lim_{n\to p} \|T^{\beta_n}x_j - y_n\|$$

denotes the upper limit of the net $(\|\widehat{T}_{(\alpha_n)}\dot{x}_j - \widetilde{y}\|)_{(\alpha_n)\in\mathcal{N}}$. It is not difficult to see that D_j is a nonempty closed and convex subset of $\widetilde{C}$ (notice that

$\dot{x}_j \in D_j$). Furthermore, for fixed $(\beta_n) \in \mathcal{N}$ and $\widetilde{y} \in D_j$,

$$\begin{aligned}\limsup_{(\alpha_n)\in\mathcal{N}} \|\widehat{T}_{(\alpha_n)}\dot{x}_j - \widehat{T}_{(\beta_n)}\widetilde{y}\| &= \limsup_{(\alpha_n)\in\mathcal{N}} \|\widehat{T}_{(\alpha_n+\beta_n)}\dot{x}_j - \widehat{T}_{(\beta_n)}\widetilde{y}\| \\ &\le \limsup_{(\alpha_n)\in\mathcal{N}} \|\widehat{T}_{(\alpha_n)}\dot{x}_j - \widetilde{y}\| \le \varepsilon^j,\end{aligned}$$

and hence $\widehat{T}_{(\beta_n)}(D_j) \subset D_j$ for every $(\beta_n) \in \mathcal{N}$. Again, by Bruck's theorem, there exists $\widetilde{y}_j \in D_j$ such that $\widehat{T}_{(\alpha_n)}\widetilde{y}_j = \widetilde{y}_j$ for all $(\alpha_n) \in \mathcal{N}$. Notice that $\|\widetilde{y}_j - \dot{x}_j\| \le 2\varepsilon^j < 3\varepsilon^j$ and by Lemma 2.2, there exist $x_{j+1} \in C$ and $n_{j+1} \in \mathbb{N}$ such that $\|x_{j+1} - x_j\| < 3\varepsilon^j$ and $\|T^n x_{j+1} - x_{j+1}\| < \varepsilon^{j+1}$ for every $n \ge n_{j+1}$.

Thus we obtain by induction a sequence (n_j) of natural numbers and a sequence (x_j) of elements in C such that (2.4) is satisfied. It follows that (x_j) is a Cauchy sequence converging to some $x \in C$. Hence

$$\begin{aligned}\|T^n x - x\| &\le \|T^n x - T^n x_j\| + \|T^n x_j - x_j\| + \|x_j - x\| \\ &\le (\|T^n x - T^n x_j\| - \|x_j - x\|) + \varepsilon^j + 2\|x_j - x\|\end{aligned}$$

for every $n \ge n_j$, $j \ge 1$, and consequently $\lim_{n\to\infty} \|T^n x - x\| = 0$. Since T is continuous, $Tx = x$. ∎

A Banach space X is said to have the *super fixed point property for asymptotically nonexpansive mappings* (*in the intermediate sense*) if every Banach space Y which is finitely representable in X has the fixed point property for asymptotically nonexpansive mappings (in the intermediate sense). We can strengthen Theorem 2.3 in the following way.

THEOREM 2.4. *A Banach space X has the super fixed point property for nonexpansive mappings if and only if X has the super fixed point property for asymptotically nonexpansive mappings in the intermediate sense.*

Proof. Assume that X has SFPP (for nonexpansive mappings) and let $\mathcal{U}$ be an ultrafilter defined on a set I. By Theorem 2.1, $(X)_{\mathcal{U}}$ has SFPP too, and it follows from Theorem 2.3 that $(X)_{\mathcal{U}}$ has the fixed point property for asymptotically nonexpansive mappings in the intermediate sense. By Theorem 2.1 again, X has the super fixed point property for asymptotically nonexpansive mappings in the intermediate sense. The reverse implication is obvious. ∎

We conclude this section by giving some consequences of Theorem 2.3. Recall [14] that a Banach space is *uniformly nonsquare* if

$$\sup_{x,y\in S_X} \min\{\|x+y\|, \|x-y\|\} < 2.$$

García Falset, Lloréns Fuster and Mazcuñan Navarro (see [8]) solved a long-standing problem in metric fixed point theory by proving that uniformly nonsquare Banach spaces have FPP and hence SFPP.

COROLLARY 2.5. *Let C be a nonempty bounded closed and convex subset of a uniformly nonsquare Banach space. Then every mapping $T : C \to C$ asymptotically nonexpansive in the intermediate sense has a fixed point.*

In [19], Prus introduced the notion of uniformly noncreasy spaces. A real Banach space X is said to be *uniformly noncreasy* if for every $\varepsilon > 0$ there is $\delta > 0$ such that if $f, g \in S_{X^*}$ and $\|f - g\| \geq \varepsilon$, then $\operatorname{diam} S(f, g, \delta) \leq \varepsilon$, where

$$S(f, g, \delta) = \{x \in B_X : f(x) \geq 1 - \delta \wedge g(x) \geq 1 - \delta\}$$

($\operatorname{diam} \emptyset = 0$). It is known that both uniformly convex and uniformly smooth spaces are uniformly noncreasy. The Bynum space $l^{2,\infty}$, which is l^2 with the norm

$$\|x\|_{2,\infty} = \max\{\|x^+\|_2, \|x^-\|_2\},$$

and the space $X_{\sqrt{2}}$, which is l^2 with the norm

$$\|x\|_{\sqrt{2}} = \max\{\|x\|_2, \sqrt{2}\, \|x\|_\infty\},$$

are examples of uniformly noncreasy spaces without normal structure. It was proved in [19] that all uniformly noncreasy spaces are superreflexive and have SFPP. This yields

COROLLARY 2.6. *Let C be a nonempty bounded closed and convex subset of a uniformly noncreasy Banach space. Then every mapping $T : C \to C$ asymptotically nonexpansive in the intermediate sense has a fixed point.*

Recently, a fixed point theorem in direct sums of two Banach spaces was proved in [23]. Assume that X has SFPP (for nonexpansive mappings) and Y is uniformly convex, uniformly smooth or finite-dimensional. Since uniformly convex, uniformly smooth as well as finite-dimensional spaces are stable under passing to the Banach space ultrapowers and have uniform normal structure, it follows from [23, Th. 3.4], that $X \oplus Y$ with a strictly monotone norm has SFPP. Thus we obtain the following theorem.

COROLLARY 2.7. *Assume that X has SFPP and Y is uniformly convex, uniformly smooth or finite-dimensional. Then $X \oplus Y$ with a strictly monotone norm has the fixed point property for asymptotically nonexpansive mappings in the intermediate sense.*

3. Common fixed points. In this section we generalize Theorem 2.3 to a commuting family of mappings. Let $\{T_t : t \in I\}$ be a commuting family of asymptotically nonexpansive self-mappings in the intermediate sense acting on a nonempty bounded closed and convex subset C of a Banach space X.

Consider the set

$$\mathcal{A} = \{\{(t_1,\alpha_1),(t_2,\alpha_2),\dots,(t_k,\alpha_k)\} : t_1,\dots,t_k \in I,\ t_i \neq t_j \text{ for } i \neq j, \\ \alpha_1,\dots,\alpha_k \in \mathbb{N}, k > 0\},$$

directed by the relation

$$\{(t_1,\alpha_1),\dots,(t_k,\alpha_k)\} \sqsubseteq \{(s_1,\beta_1),\dots,(s_m,\beta_m)\}$$

iff

$$\{t_1,\dots,t_k\} \subseteq \{s_1,\dots s_m\} \quad \text{and} \quad \forall i\ \forall j\ (t_i = s_j \Rightarrow \alpha_i \leq \beta_j).$$

If $v = \{(t_1,\alpha_1),\dots,(t_k,\alpha_k)\} \in \mathcal{A}$, write

$$T_v x = T_{t_1}^{\alpha_1} \cdots T_{t_k}^{\alpha_k} x,$$

and let

$$\mathcal{D} = \Big\{(v_n) \in \mathcal{A}^{\mathbb{N}} : \limsup_{n\to\infty} \sup_{x,y\in C} (\|T_{v_n}x - T_{v_n}y\| - \|x-y\|) \leq 0\Big\}.$$

Note that $\mathcal{D} \neq \emptyset$ since $(\{(t,n)\})_{n\in\mathbb{N}} \in \mathcal{D}$ for every $t \in I$. If $(v_n),(u_n) \in \mathcal{D}$, define $(v_n) \preceq (u_n)$ iff $v_n \sqsubseteq u_n$ for every $n \in \mathbb{N}$. It is not difficult to see that for every $(v_n),(u_n) \in \mathcal{D}$ there exists $(w_n) \in \mathcal{D}$ such that $(v_n) \preceq (w_n)$ and $(u_n) \preceq (w_n)$. Indeed, let

$$v_n = \{(t_1^{(n)},\alpha_1^{(n)}),\dots,(t_{k_n}^{(n)},\alpha_{k_n}^{(n)})\},$$
$$u_n = \{(s_1^{(n)},\beta_1^{(n)}),\dots,(s_{m_n}^{(n)},\beta_{m_n}^{(n)})\},$$

and put

$$w_n = \{(t_1^{(n)},\alpha_1^{(n)}),\dots,(t_{k_n}^{(n)},\alpha_{k_n}^{(n)}),(s_1^{(n)},\beta_1^{(n)}),\dots,(s_{m_n}^{(n)},\beta_{m_n}^{(n)})\}, \tag{3.1}$$

$n \in \mathbb{N}$ (to shorten notation, we use the convention that if $t_i = s_j$ for some i,j, then the pairs $(t_i,\alpha_i),(s_j,\beta_j)$ in w_n are identified with one pair $(t_i,\alpha_i+\beta_j)$). Notice that

$$\begin{aligned} s((T_{w_n})) &= \limsup_{n\to\infty} \sup_{x,y\in C} (\|T_{v_n}T_{u_n}x - T_{v_n}T_{u_n}y\| - \|x-y\|) \\ &\leq s((T_{v_n})) + s((T_{u_n})) \leq 0, \end{aligned}$$

where

$$s((T_{v_n})) = \limsup_{n\to\infty} \sup_{x,y\in C} (\|T_{v_n}x - T_{v_n}y\| - \|x-y\|).$$

Hence $(w_n) \in \mathcal{D}$ and clearly $(v_n) \preceq (w_n)$ and $(u_n) \preceq (w_n)$. Thus $(\mathcal{D},\preceq)$ is a directed set.

Let p be a free ultrafilter on $\mathbb{N}$. Then, for every $(x_n)_p,(y_n)_p \in \widetilde{C}$ and $(v_n) \in \mathcal{D}$,

$$\lim_{n\to p}(\|T_{v_n}x_n - T_{v_n}y_n\| - \|x_n-y_n\|) \leq \limsup_{n\to\infty} \sup_{x,y\in C} (\|T_{v_n}x - T_{v_n}y\| - \|x-y\|) \leq 0. \tag{3.2}$$

Therefore, we may define unambiguously a mapping $\widehat{T}_{(v_n)} : \widetilde{C} \to \widetilde{C}$ by setting

$$\widehat{T}_{(v_n)}(x_n)_p = (T_{v_n} x_n)_p. \tag{3.3}$$

It follows from (3.2) that $\widehat{T}_{(v_n)}$ is nonexpansive for every $(v_n) \in \mathcal{D}$.

We can now prove a counterpart of Lemma 2.2.

LEMMA 3.1. *Let $\{T_t : t \in I\}$ be a commuting family of asymptotically nonexpansive mappings in the intermediate sense acting on a nonempty bounded closed and convex subset C of a Banach space X. Suppose that there exists $\widetilde{y} \in \widetilde{C}$ such that*

$$\widehat{T}_{(v_n)}\widetilde{y} = \widetilde{y} \tag{3.4}$$

for all $(v_n) \in \mathcal{D}$. Let $\|\widetilde{y} - \dot{x}_0\| < \delta$ for some $x_0 \in C$ and $\delta > 0$. Then for every $\varepsilon > 0$ there exist $x \in C$ and $n \in \mathbb{N}$ such that $\|x - x_0\| < \delta$ and $\|T_u x - x\| < \varepsilon$ for every $u \in \mathcal{D}'(n)$, where

$$\mathcal{D}'(n) = \Big\{ v = \{(t_1, \alpha_1), \ldots, (t_k, \alpha_k)\} \in \mathcal{A} : \sup_{x,y \in C} (\|T_v x - T_v y\| - \|x - y\|) \le \frac{1}{n+1} \Big\}.$$

Proof. Since $\|\widetilde{y} - \dot{x}_0\| < \delta$, there exists a sequence (y_n) in C such that $\|y_n - x_0\| < \delta$ for all $n \in \mathbb{N}$ and $\widetilde{y} = (y_n)_p$. Assume, contrary to our claim, that there exists $\varepsilon_0 > 0$ such that for every $x \in C$ and $n \in \mathbb{N}$ there exists $u \in \mathcal{D}'(n)$ such that $\|x - x_0\| \ge \delta$ or $\|T_u x - x\| \ge \varepsilon_0$. We shall define a sequence $(u_n) \in \mathcal{D}$ by induction. For $n = 0$ and $y_0 \in C$, there exists $u_0 \in \mathcal{D}'(0)$ such that $\|T_{u_0} y_0 - y_0\| \ge \varepsilon_0$. Suppose that we have chosen $u_0 \in \mathcal{D}'(0), u_1 \in \mathcal{D}'(1), \ldots, u_n \in \mathcal{D}'(n)$ such that $\|T_{u_i} y_i - y_i\| \ge \varepsilon_0$ for $i = 0, 1, \ldots, n$. By assumption, since $\|y_{n+1} - x_0\| < \delta$, there exists $u_{n+1} \in \mathcal{D}'(n+1)$ such that $\|T_{u_{n+1}} y_{n+1} - y_{n+1}\| \ge \varepsilon_0$. Thus we obtain a sequence $(u_n) \in \mathcal{A}^{\mathbb{N}}$ such that $\|T_{u_n} y_n - y_n\| \ge \varepsilon_0$ and

$$\sup_{x,y \in C} (\|T_{u_n} x - T_{u_n} y\| - \|x - y\|) \le \frac{1}{n+1}$$

for all $n \in \mathbb{N}$. Hence

$$\limsup_{n \to \infty} \sup_{x,y \in C} (\|T_{u_n} x - T_{u_n} y\| - \|x - y\|) = 0,$$

i.e., $(u_n) \in \mathcal{D}$. But this contradicts (3.4), since $\|\widehat{T}_{(u_n)}\widetilde{y} - \widetilde{y}\| \ge \varepsilon_0$. ∎

We will also make use of the following simple observation.

LEMMA 3.2. *For every $(u_n) \in \mathcal{D}$ and $\tilde{x}, \tilde{y} \in \widetilde{C}$,*

$$\limsup_{(v_n) \in \mathcal{D}} \|\widehat{T}_{(v_n)}\tilde{x} - \widehat{T}_{(u_n)}\tilde{y}\| = \limsup_{(v_n) \in \mathcal{D}} \|\widehat{T}_{(v_n)}\widehat{T}_{(u_n)}\tilde{x} - \widehat{T}_{(u_n)}\tilde{y}\|.$$

Proof. Fix $(u_n) \in \mathcal{D}$, $\tilde{x}, \tilde{y} \in \widetilde{C}$, and notice that for every $(v_n) \in \mathcal{D}$,

$$\sup_{(\bar{w}_n)\succeq(w_n)} \|\widehat{T}_{(\bar{w}_n)}\tilde{x} - \widehat{T}_{(u_n)}\tilde{y}\| = \sup_{(\bar{v}_n)\succeq(v_n)} \|\widehat{T}_{(\bar{v}_n)}\widehat{T}_{(u_n)}\tilde{x} - \widehat{T}_{(u_n)}\tilde{y}\|,$$

where (w_n) is defined by (3.1). Hence

$$\limsup_{(v_n)\in\mathcal{D}} \|\widehat{T}_{(v_n)}\tilde{x} - \widehat{T}_{(u_n)}\tilde{y}\| \le \limsup_{(v_n)\in\mathcal{D}} \|\widehat{T}_{(v_n)}\widehat{T}_{(u_n)}\tilde{x} - \widehat{T}_{(u_n)}\tilde{y}\|.$$

The reverse inequality is obvious since $\widehat{T}_{(v_n)}\widehat{T}_{(u_n)}\tilde{x} = \widehat{T}_{(w_n)}\tilde{x}$ and $(v_n) \preceq (w_n)$. ■

We are now in a position to prove the following generalization of Theorem 2.3.

THEOREM 3.3. *Suppose C is a nonempty bounded closed and convex subset of a Banach space X with SFPP and $\mathcal{T} = \{T_t : t \in I\}$ is a commuting family of asymptotically nonexpansive mappings in the intermediate sense acting on C. Then there exists $x \in C$ such that $T_t x = x$ for every $t \in I$ (a common fixed point for $\mathcal{T}$).*

Proof. We partly follow the reasoning in the proof of Theorem 2.3. Assume that X has the super fixed point property for nonexpansive mappings. Let $\mathcal{T} = \{T_t : t \in I\}$ be a commuting family of asymptotically nonexpansive mappings in the intermediate sense acting on a nonempty bounded closed and convex set $C \subset X$. We can assume that $\operatorname{diam} C = 1$. Take a free ultrafilter p on $\mathbb{N}$, $(v_n) \in \mathcal{D}$ and define $\widehat{T}_{(v_n)}$ by (3.3). Notice that for every $(v_n), (u_n) \in \mathcal{D}$ and any $(x_n)_p \in \widetilde{C}$,

$$(\widehat{T}_{(v_n)} \circ \widehat{T}_{(u_n)})(x_n)_p = (\widehat{T}_{(u_n)} \circ \widehat{T}_{(v_n)})(x_n)_p.$$

It follows from Bruck's theorem that there exists $\widetilde{y}_0 \in \widetilde{C}$ such that $\widehat{T}_{(v_n)}\widetilde{y}_0 = \widetilde{y}_0$ for all $(v_n) \in \mathcal{D}$. Fix $\varepsilon < 1$ and $x_0 \in C$. We shall define by induction a sequence (n_j) of natural numbers and a sequence (x_j) of elements in C such that

(3.5)
$$\|x_j - x_{j-1}\| < 3\varepsilon^{j-1} \text{ and } \|T_u x_j - x_j\| < \varepsilon^j \quad \text{for every } u \in \mathcal{D}'(n_j), j \ge 1.$$

By Lemma 3.1, there exist $x_1 \in C$ and $n_1 \in \mathbb{N}$ such that $\|T_u x_1 - x_1\| < \varepsilon$ for every $u \in \mathcal{D}'(n_1)$ and $\|x_1 - x_0\| \le \operatorname{diam} C < 3$.

Suppose that we have chosen natural numbers $n_1, \ldots, n_j$ and $x_1, \ldots, x_j \in C$ $(j \ge 1)$ such that

(3.6)
$$\|x_i - x_{i-1}\| < 3\varepsilon^{i-1} \text{ and } \|T_u x_i - x_i\| < \varepsilon^i \text{ for every } u \in \mathcal{D}'(n_i), 1 \le i \le j.$$

Let

$$D_j = \Big\{ \widetilde{y} = (y_n)_p \in \widetilde{C} : \limsup_{(v_n)\in\mathcal{D}} \|\widehat{T}_{(v_n)}\dot{x}_j - \widetilde{y}\| \le \varepsilon^j \Big\},$$

where $\dot{x}_j = (x_j, x_j, \ldots)_p$ and

$$\limsup_{(v_n)\in\mathcal{D}} \|\widehat{T}_{(v_n)}\dot{x}_j - \widetilde{y}\| = \inf_{(v_n)\in\mathcal{D}} \sup_{(u_n)\succeq(v_n)} \lim_{n\to p} \|T_{u_n}x_j - y_n\|.$$

Notice that for every $(v_n) \in \mathcal{D}$ and $\eta > 0$, there exists $k \in \mathbb{N}$ such that $\sup_{x,y\in C}(\|T_{v_n}x - T_{v_n}y\| - \|x-y\|) < \eta$ for every $n > k$. Hence $v_n \in \mathcal{D}'(n_j)$ for sufficiently large n and applying the induction assumption (3.6) gives $\lim_{n\to p}\|T_{v_n}x_j - x_j\| \le \varepsilon^j$ for every $(v_n) \in \mathcal{D}$. It follows that $\dot{x}_j \in D_j$ and D_j is a nonempty closed and convex subset of $\widetilde{C}$. By Lemma 3.2, for fixed $(u_n) \in \mathcal{D}$ and $\widetilde{y} \in D_j$,

$$\begin{aligned}\limsup_{(v_n)\in\mathcal{D}} \|\widehat{T}_{(v_n)}\dot{x}_j - \widehat{T}_{(u_n)}\widetilde{y}\| &= \limsup_{(v_n)\in\mathcal{D}} \|\widehat{T}_{(v_n)}\widehat{T}_{(u_n)}\dot{x}_j - \widehat{T}_{(u_n)}\widetilde{y}\| \\ &\le \limsup_{(v_n)\in\mathcal{D}} \|\widehat{T}_{(v_n)}\dot{x}_j - \widetilde{y}\| \le \varepsilon^j,\end{aligned}$$

and hence $\widehat{T}_{(u_n)}(D_j) \subset D_j$ for every $(u_n) \in \mathcal{D}$. By Bruck's theorem, there exists $\widetilde{y}_j \in D_j$ such that $\widehat{T}_{(v_n)}\widetilde{y}_j = \widetilde{y}_j$ for all $(v_n) \in \mathcal{D}$. It is easy to see that $\|\widetilde{y}_j - \dot{x}_j\| < 3\varepsilon^j$ and, by Lemma 3.1, there exist $x_{j+1} \in C$ and $n_{j+1} \in \mathbb{N}$ such that $\|x_{j+1} - x_j\| < 3\varepsilon^j$ and $\|T_u x_{j+1} - x_{j+1}\| < \varepsilon^{j+1}$ for every $u \in \mathcal{D}'(n_{j+1})$. Thus we obtain by induction a sequence (n_j) of natural numbers and a sequence (x_j) of elements in C such that (3.5) is satisfied. It follows that (x_j) is a Cauchy sequence converging to some $x \in C$.

Fix $T_t \in \mathcal{T}$ and notice that for every n_j, there exists k_j such that $\{(t,n)\} \in \mathcal{D}'(n_j)$ for $n > k_j$, since T_t is asymptotically nonexpansive in the intermediate sense. Applying (3.5) gives

$$\limsup_{n\to\infty} \|T_t^n x_j - x_j\| \le \varepsilon^j, \ j \ge 1.$$

Furthermore,

$$\begin{aligned}\|T_t^n x - x\| &\le (\|T_t^n x - T_t^n x_j\| - \|x - x_j\|) + \|T_t^n x_j - x_j\| + 2\|x_j - x\| \\ &\le \sup_{x,y\in C}(\|T_t^n x - T_t^n y\| - \|x-y\|) + \|T_t^n x_j - x_j\| + 2\|x_j - x\|\end{aligned}$$

for every $j, n \ge 1$, and consequently $\limsup_{n\to\infty}\|T_t^n x - x\| = 0$. Since T_t is continuous, $T_t x = x$. ∎

Remark. It was proved in [5, Th. 4] that if C is a nonempty weakly compact convex subset of a Banach space X and every asymptotically nonexpansive mapping of C has the (ω)-fixed point property (which is a little stronger than the fixed point property), then the set of common fixed points of any commuting family of asymptotically nonexpansive mappings acting on C is a nonexpansive retract of C. It is not known whether a similar conclusion can be drawn under the assumptions of Theorem 3.3.

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