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The even-odd hat problem

by

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Abstract. We answer a question of C. Hardin and A. Taylor concerning a hat-guessing game.

1. Introduction. The purpose of this paper is to answer a question stated by C. Hardin and A. Taylor in their paper [3]. Hardin and Taylor study *hat problems*, in which a set of players have hats of a variety of different colors placed on their heads. Each player can see the hats of some subset of the other players, but no player can see his own hat. Each player must try to guess the color of his own hat. There is no communication among the players once the hats are placed on their heads; in particular, no player hears the guesses made by any other players. However, the players may meet before the game begins to agree on strategies that they will use when formulating their guesses. The question Hardin and Taylor ask in each case is whether or not there are strategies the players can use that will guarantee that at least one player guesses correctly. For more background on hat problems we refer the reader to [2] and [3].

To formalize these problems, we let P be the set of players and C the set of available hat colors. A hat assignment is then represented by a function $f : P \to C$, and as usual we let ${}^P C$ denote the set of all such functions. We let V be the *visibility relation* among the players. In other words,

$$V = \{\langle p, q\rangle \in P^2 : p \text{ can see } q\text{'s hat}\}.$$

Since no player can see his own hat, V will be irreflexive. For any $p \in P$ we let

$$V(p) = \{q \in P : \langle p, q\rangle \in V\}.$$

Thus, $V(p)$ is the set of players whose hat p can see. A *strategy* for p is a function $S_p : {}^P C \to C$; we think of $S_p(f)$ as the color that p guesses when

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the hat assignment is f. Of course, when formulating this guess p should not be allowed to use information about the colors of hats that he cannot see. To enforce this condition, we require that if $f, g \in {}^P C$ and $f \restriction V(p) = g \restriction V(p)$ then $S_p(f) = S_p(g)$. We say that p *guesses correctly* using this strategy if $S_p(f) = f(p)$, which means that his guess matches the actual color of his hat.

A *predictor* is a sequence $S = \langle S_p : p \in P \rangle$ of strategies, one for each player. It is a *minimal predictor* if it guarantees that at least one player will guess correctly, no matter how the hat colors are assigned. In other words, a predictor S is a minimal predictor if for every function $f \in {}^P C$, there is some $p \in P$ such that $S_p(f) = f(p)$. (The motivation for this term is that if we measure the success of a predictor by the number of correct guesses it guarantees, then a minimal predictor is one that achieves at least the minimum positive degree of success.) With this terminology, we can now state more precisely the question we will discuss: Given sets P and C and a visibility relation V, does there exist a minimal predictor? (In Hardin and Taylor's notation, the statement that such a predictor exists can be written $\langle P, V \rangle \rightharpoonup \langle 1 \rangle_C$; however, we will not use that notation in this paper.)

Hardin and Taylor study a number of hat problems of this form. One class of problems that they focus on is the case in which $P = 2 \times \omega$ and $V = \{\langle \langle i, p \rangle, \langle j, q \rangle \rangle : i \neq j \text{ and } p < q\}$. In this situation we can think of the players as belonging to two denumerable teams, $\{0\} \times \omega$ and $\{1\} \times \omega$, with each player on one team able to see the higher-numbered players on the opposite team. Following Taylor and Hardin, we refer to this as the "even-odd context" (because one could also formalize this situation by letting the two teams consist of the even and odd natural numbers). The existence of a minimal predictor will now depend on the number of colors.

Hardin and Taylor prove that in the even-odd context there is a minimal predictor if C has 2 elements, there is no minimal predictor if $|C| \geq \aleph_2$, and the existence of a minimal predictor is independent of ZFC if $|C|$ is either $\aleph_0$ or $\aleph_1$. However, they leave the case $2 < |C| < \aleph_0$ unresolved (see [3, Theorems 4.1 and 5.5 and Question 6.4]). In this paper we resolve this last case. Our main theorem is:

THEOREM 1. *In the even-odd context with any finite number of colors, there is a minimal predictor.*

2. Proof of the theorem. In this section we assume that the number of colors is a fixed positive integer c. Of course, the problem is trivial if $c = 1$, so we may as well assume $c \geq 2$, and we can take the set C of colors to be $c = \{0, 1, \ldots, c - 1\}$. One of the key steps in our proof will be the following result of S. Butler, M. Hajiaghayi, R. Kleinberg, and T. Leighton (see [1, Theorem 7]). For completeness, we provide a proof.

LEMMA 2. *There are disjoint finite sets A and B such that in the hat problem with $P = A \cup B$, $V = (A \times B) \cup (B \times A)$, and c colors, there is a minimal predictor.*

Proof. Let $B = c - 1$ and $A = \{X \subseteq {}^B c : |X| = c\}$. Of course, B and A are finite, with cardinalities $c - 1$ and $\binom{c^{c-1}}{c}$, respectively. Let P and V be defined as in the lemma. According to the definition of V, all of the players in A can see the hats of all of the players in B (but no others), and vice versa. Any hat assignment $f : P \to c$ can be thought of as a union $f = f_A \cup f_B$, where $f_A = f{\restriction}A$ and $f_B = f{\restriction}B$.

We first explain the strategies that the players in A will use. Fix a linear ordering $\prec$ on ${}^B c$. Consider a player $X \in A$, and write $X = \{h_0, h_1, \dots, h_{c-1}\}$, where $h_0 \prec h_1 \prec \dots \prec h_{c-1}$. For any hat assignment $f : P \to c$, we define $S_X(f)$ as follows:

$$S_X(f) = \begin{cases} i & \text{if } f_B = h_i, \\ 0 & \text{if } f_B \notin X. \end{cases}$$

In other words, a player $X \in A$ can be thought of as a list of c possible assignments of hats to the players in B. His strategy is to see if the hat assignment he can see, namely f_B, is one of the assignments in this list. If so, he guesses the corresponding color, and if not he guesses 0.

Next we define the strategies for the players in B. Suppose $f : P \to c$ is a hat assignment. For any function $h \in {}^B c$, let $f^h = f_A \cup h$. The functions f^h, for all $h \in {}^B c$, are the hat assignments that the players in B think are possible, based on the colors of the hats they can see.

Let W be the set of all $h \in {}^B c$ such that under the hat assignment f^h, no player in A would guess correctly. That is,

$$W = \{h \in {}^B c : \text{for all } X \in A,\ S_X(f^h) \neq f^h(X)\}.$$

Note that if $f_B \notin W$ then for some $X \in A$, $S_X(f^{f_B}) = f^{f_B}(X)$. But $f^{f_B} = f_A \cup f_B = f$, so this means $S_X(f) = f(X)$, and therefore X guesses correctly. Thus, we need only worry about the case $f_B \in W$. It will suffice to define the strategies for the players in B so as to guarantee that if $f_B \in W$, then one of the players in B guesses correctly.

Let $m = |W|$, and write $W = \{h_0, h_1, \dots, h_{m-1}\}$, where $h_0 \prec h_1 \prec \dots \prec h_{m-1}$. We claim that $m < c$. To see why, suppose $m \geq c$, and let $X = \{h_0, h_1, \dots, h_{c-1}\} \subseteq W$. Let $j = f(X) = f_A(X)$, the color of X's hat under the assignment f. Then according to the definition of S_X,

$$S_X(f^{h_j}) = j = f_A(X) = f^{h_j}(X),$$

contradicting the fact that $h_j \in W$. Thus $m < c$, so $m \leq c - 1$.

For any player $i \in B$, we now define the strategy S_i as follows:

$$S_i(f) = \begin{cases} h_i(i) & \text{if } i < m, \\ 0 & \text{otherwise.} \end{cases}$$

Notice that W depends only on f_A and not on f_B, so the players in B can compute $W = \{h_0, h_1, \ldots, h_{m-1}\}$, and therefore this is an admissible strategy. If $f_B \in W$ then there is some $i < m$ such that $f_B = h_i$. Since $m \leq c - 1$, we have $i \in B$, and

$$S_i(f) = h_i(i) = f_B(i) = f(i),$$

so i guesses correctly, as required. ∎

We are now ready to prove our main theorem.

Proof of Theorem 1. Partition the set $\{1\} \times \omega$ into infinitely many sets $B_0, B_1, \ldots,$ each of cardinality $c - 1$. Choose disjoint sets $A_0, A_1, \ldots$ contained in $\{0\} \times \omega$, each of cardinality $\binom{c^{c-1}}{c}$, so that for every i, $B_i \times A_i \subseteq V$. Thus, every player in B_i can see the hats of all players in A_i.

Our plan is to have each pair (A_i, B_i) play according to the strategies in the lemma. But there is a problem with this plan: the players in A_i cannot see the hats of the players in B_i. Our solution to this problem will be to have the players in A_i guess the colors of the hats in B_i. If they guess correctly for any i, then, by the lemma, some player in $A_i \cup B_i$ will guess his hat color correctly, as required.

For each i, by enumerating the second coordinates of the elements of B_i in increasing order we can put B_i in one-to-one correspondence with $c - 1$, and we can therefore think of the colors of the hats of the players in B_i as being given by a function $h_i \in {}^{c-1}c$. Fix a linear ordering $\prec$ of ${}^{c-1}c$, and let h be the $\prec$-least element of ${}^{c-1}c$ such that for infinitely many i, $h_i = h$. Although the players in A_i do not know h_i, they do know h_j for all but finitely many j, so they can determine h. The players in A_i guess that the hat assignment for the players in B_i is h. For infinitely many i we have $h_i = h$, and therefore the players in A_i correctly guess the colors of the hats of the players in B_i. Thus some player (indeed, infinitely many players) will guess his hat color correctly. ∎

3. The axiom of choice. The proof in Section 2 does not use the axiom of choice. But if we are willing to assume AC, then an alternative proof is possible. The proof relies on a fact that may be of independent interest, so we present it in this section.

One of the complications of the even-odd context, as defined by Hardin and Taylor, is that players on each team see only the *higher-numbered* players on the other team. A simpler scenario would be one in which all players on each team can see *all* players on the other team. To distinguish these two scenarios, in this section we will refer to Hardin and Taylor's even-odd context, as defined in Section 1, as the *upward even-odd context*, and we define the *full even-odd context* to be the context in which $P = 2 \times \omega$ and $V = \{\langle\langle i,p\rangle,\langle j,q\rangle\rangle : i \neq j\}$. We now show that, as far as the existence of minimal predictors is concerned, there is no difference between these contexts:

THEOREM 3. *For any set of colors C, there is a minimal predictor in the upward even-odd context if and only if there is a minimal predictor in the full even-odd context.*

Proof. One direction is clear: if there is a minimal predictor in the upward even-odd context, then there is one in the full even-odd context as well, because players in the full even-odd context can simply ignore the extra information that is available to them and use the predictor for the upward even-odd context.

Now suppose there is a minimal predictor for the full even-odd context. We first observe that there must be a predictor for the full even-odd context that guarantees *infinitely many* correct guesses. To see why, partition ω into infinitely many infinite sets $\{A_i : i \in \omega\}$ and let $P_i = 2 \times A_i$. We now let the players in P_i use a minimal predictor among themselves, ignoring the players in P_j for $j \neq i$. For each i, at least one player in P_i will guess correctly, so altogether infinitely many players guess correctly.

Thus, we may assume that we have a predictor S for the full even-odd context that guarantees infinitely many correct guesses. To define a minimal predictor for the upward even-odd context, consider a hat assignment $f : 2 \times \omega \to C$. Define $f_0, f_1 : \omega \to C$ by $f_i(p) = f(i,p)$. Thus, f_0 and f_1 are the hat assignments for the two teams in the even-odd context. Define an equivalence relation $\equiv$ on ${}^{\omega}C$ by letting $g \equiv h$ if and only if $\{p \in \omega : g(p) \neq h(p)\}$ is finite. By the axiom of choice, choose a representative from each equivalence class, and let $\widehat{g}$ denote the representative of the equivalence class of g. In particular, $\widehat{f_0}$ and $\widehat{f_1}$ are functions that disagree with f_0 and f_1 in only finitely many places, and we can combine them into a function $\widehat{f} : 2 \times \omega \to C$ that disagrees with f only finitely often by letting $\widehat{f}(i,p) = \widehat{f_i}(p)$. Notice that in the upward even-odd context, a player $\langle i,p\rangle$ cannot see all of the hat assignment f_{1-i} for the opposite team, but he can see all but finitely many values of it, which is enough to determine $\widehat{f_{1-i}}$. Thus, we can define a predictor $\widehat{S}$ for the upward even-odd context as follows:

$$\widehat{S}_{\langle i,p\rangle}(f) = S_{\langle i,p\rangle}(\widehat{f}\,).$$

By assumption, there are infinitely many players $\langle i,p\rangle$ such that $S_{\langle i,p\rangle}(\widehat{f}\,) = \widehat{f}(i,p)$. Since $\widehat{f}$ and f disagree only finitely many times, this means that there are infinitely many players $\langle i,p\rangle$ such that $\widehat{S}_{\langle i,p\rangle}(f) = f(i,p)$. Thus, $\widehat{S}$ is a minimal predictor for the upward even-odd context. ■

The relevance of this result to Theorem 1 is that Lemma 2 immediately implies the existence of a minimal predictor in the full even-odd context with finitely many colors: simply choose $\binom{c^{c-1}}{c}$ players from one team and $c-1$ players from the other team, let them play the strategies from Lemma 2, and ignore all other players. Applying Theorem 3, we get another proof of Theorem 1.

References

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