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Prediction problems and ultrafilters on ω

by

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Dedicated to the memory of James E. Baumgartner (1943–2011)

Abstract. We consider prediction problems in which each of a countably infinite set of agents tries to guess his own hat color based on the colors of the hats worn by the agents he can see, where who can see whom is specified by a graph V on ω. Our interest is in the case in which $\mathcal{U}$ is an ultrafilter on the set of agents, and we seek conditions on $\mathcal{U}$ and V ensuring the existence of a strategy such that the set of agents guessing correctly is of $\mathcal{U}$-measure one. A natural necessary condition is the absence of a set of agents in $\mathcal{U}$ for which no one in the set sees anyone else in the set. A natural sufficient condition is the existence of a set of $\mathcal{U}$-measure one so that everyone in the set sees a set of agents of $\mathcal{U}$-measure one. We ask two questions: (1) For which ultrafilters is the natural sufficient condition always necessary? (2) For which ultrafilters is the natural necessary condition always sufficient? We show that the answers are (1) p-point ultrafilters, and (2) Ramsey ultrafilters.

1. Introduction. Our set-theoretic notation is fairly standard. Each ordinal is the set of smaller ordinals, so $\omega = \{0, 1, 2, \dots\}$ and $2 = \{0, 1\}$. If X and Y are sets, then ${}^XY$ denotes the set of all functions from X to Y, $X - Y$ is the set-theoretic difference of X and Y, and $X^c = \omega - X$ when $X \subseteq \omega$. We let $\mathcal{P}(A)$ denote the power set of A, and $[\omega|^{<\omega}$ is the collection of finite subsets of ω. If f is a function and A is a subset of the domain of f, then $f|A$ is the restriction of f to the set A. We regard a graph V on ω as a subset of $\omega \times \omega$ in which $(x, y) \in V$ implies $x < y$. We let $V(x) = \{y \in \omega : (x, y) \in V\}$. If $s \in {}^X2$ for some $X \in [\omega|^{<\omega}$, then $[s] = \{f \in {}^\omega 2 : f|X = s\}$. These sets constitute a basis for a compact topology on ${}^\omega 2$ that makes it homeomorphic to the Cantor set.

We consider the standard hat problem in which ω is the set of agents and 2 is the set of hat colors. Each $g \in {}^\omega 2$ is a *coloring* which intuitively corresponds to the placing of colored hats on the agents, with "$g(x) = i$"

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being interpreted as "agent x has hat color i." Visibility is specified by a graph V on ω, where $(x, y) \in V$ is interpreted as "agent x can see (the hat worn by) agent y." If we fix such a *visibility graph*, then a *strategy for agent x* is a function $S_x : {}^\omega 2 \to 2$ such that $S_x(g) = S_x(h)$ whenever $g|V(x) = h|V(x)$. Intuitively, $S_x(g)$ is agent x's guess of his own hat color for the coloring g and his guess only depends on the hats worn by the agents he can see. A *predictor for V* is a sequence $P = \langle S_x : x \in \omega \rangle$ where, for each x, S_x is a strategy for agent x. Given a coloring g, agent x *guesses correctly for g with P* if $S_x(g) = g(x)$. For more background, see [HT08a], [HT08b], [HT09], and [HT10].

A *filter on ω* is a non-empty collection $\mathcal{F}$ of subsets of ω that is closed under finite intersections and the formation of supersets. It is *proper* if $\mathcal{F} \neq \mathcal{P}(\omega)$ and it is *non-principal* if it contains all cofinite subsets of ω. Whenever we say "filter" we mean "proper, non-principal filter on ω." An *ultrafilter $\mathcal{U}$ on ω* is a maximal filter; these are the filters for which exactly one of X and X^c is in $\mathcal{U}$ for each $X \subseteq \omega$. A set in $\mathcal{U}$ is said to be of *$\mathcal{U}$-measure one*; its complement is of *$\mathcal{U}$-measure zero*.

Two well-known classes of ultrafilters arise in what follows. An ultrafilter $\mathcal{U}$ is a *p-point* if for every $f \in {}^\omega\omega$ there exists a set $X \in \mathcal{U}$ such that $f|X$ is either constant or finite-to-one. If the conclusion can be strengthened from "finite-to-one" to "one-to-one," then $\mathcal{U}$ is a *Ramsey ultrafilter*. With the continuum hypothesis (or just Martin's Axiom), one can prove that Ramsey ultrafilters (and thus p-points) exist, but in ZFC one cannot even prove that p-points exist; see [W82]. Ramsey ultrafilters get their name from the fact that they can be characterized as precisely the ones for which every graph on ω will have either a complete subgraph in the ultrafilter or an independent subgraph in the ultrafilter.

Suppose now that $\mathcal{F}$ is a filter on ω and that we have a visibility graph V on ω such that $V(x) \in \mathcal{F}$ for every $x \in \omega$. Then we can construct a predictor for the corresponding hat problem as follows. Consider the equivalence relation $\equiv$ on ${}^\omega 2$ wherein $g \equiv h$ if $\{x \in \omega : g(x) = h(x)\} \in \mathcal{F}$. The axiom of choice guarantees the existence of a function $\Phi : {}^\omega 2 \to {}^\omega 2$ such that $\Phi(g) \equiv g$, and $\Phi(g) = \Phi(h)$ iff $g \equiv h$. We now set $S_x(g) = \Phi(g)(x)$. Notice that this makes sense, because agent x can determine g's equivalence class. Intuitively, the agents are agreeing on a representative from each equivalence class and guessing as if the true coloring is this representative. It is now easy to see that the set of agents guessing correctly is in $\mathcal{F}$; we call a predictor with this property an *$\mathcal{F}$-predictor* (and when $\mathcal{F}$ is the collection of cofinite subsets of an infinite set, we call it a *finite-error predictor*). This argument is really a weakened version of arguments put forth by Galvin [G65] (see also [GP76]) in the mid-1960s and by Gabay-O'Connor [HT08a] in the mid-2000s; it works equally well if there are κ colors for an arbitrary cardinal κ.

If $\mathcal{U}$ is an ultrafilter on ω, then $\mathcal{U} \times \mathcal{U}$ is the ultrafilter on $\omega \times \omega$ wherein $V \in \mathcal{U} \times \mathcal{U}$ if $\{x : \{y : (x, y) \in V\} \in \mathcal{U}\} \in \mathcal{U}$. Notice that if we regard V as a visibility graph, then $V \in \mathcal{U} \times \mathcal{U}$ precisely when we have a $\mathcal{U}$-measure one set of agents who each see a $\mathcal{U}$-measure one set of agents. The argument in the previous paragraph shows that $V \in \mathcal{U} \times \mathcal{U}$ is a sufficient condition for the existence of a $\mathcal{U}$-predictor; we call this the *natural sufficient condition for the existence of a $\mathcal{U}$-predictor.*

There is also a *natural necessary condition for the existence of a $\mathcal{U}$-predictor*: The visibility graph contains no independent set of $\mathcal{U}$-measure one. The point is that we can always make every agent in an independent set guess incorrectly by first placing hats of color zero on everyone's head, and then changing the hat color for those agents in the independent set who guessed correctly.

With these preliminaries, we can define the classes of ultrafilters in which we are interested.

DEFINITION 1.1. An *SIN-ultrafilter* $\mathcal{U}$ is one for which the natural sufficient condition for the existence of a $\mathcal{U}$-predictor ($V \in \mathcal{U} \times \mathcal{U}$) is also necessary.

An *NIS-ultrafilter* $\mathcal{U}$ is one for which the natural necessary condition for the existence of a $\mathcal{U}$-predictor (no independent set in $\mathcal{U}$) is also sufficient.

"SIN" stands for "sufficient is necessary" and "NIS" stands for "necessary is sufficient." We read each of these prefixes letter-by-letter ("S-I-N" instead of "sin") and thus use the article "an" as opposed to "a." In Section 2 we show that the SIN-ultrafilters are precisely the p-points, and that the NIS-ultrafilters are precisely the Ramsey ultrafilters. Some additional remarks are in Section 3.

I thank Chris Hardin for many discussions concerning these and related problems. My own interest in ultrafilters goes back to some joint work [BT78] in the late 1970s with my thesis advisor, Jim Baumgartner.

2. The main results. We begin by characterizing the SIN-ultrafilters.

THEOREM 2.1. *An ultrafilter is an SIN-ultrafilter iff it is a p-point ultrafilter.*

Proof. Suppose first that $\mathcal{U}$ is not a p-point and choose $f \in {}^{\omega}\omega$ such that f is neither constant on a set in $\mathcal{U}$ nor finite-to-one on a set in $\mathcal{U}$. Let V be the graph on ω with $(x, y) \in V$ if $x < y$ and $f(x) = f(y)$. No agent sees a set in $\mathcal{U}$, and thus $V \notin \mathcal{U} \times \mathcal{U}$. However, for each $k \in \omega$, the agents in $f^{-1}(\{k\})$ have a finite-error predictor among themselves because each agent in $f^{-1}(\{k\})$ sees all but finitely many other agents in $f^{-1}(\{k\})$. The combined use of these predictors will ensure that the set of errors is a set upon which the function f is finite-to-one, and thus not in $\mathcal{U}$. Hence, we have a $\mathcal{U}$-predictor even though each agent sees only a set of agents of $\mathcal{U}$-measure zero.

Now suppose that $\mathcal{U}$ is a p-point and V is a graph on ω for which $V \notin \mathcal{U} \times \mathcal{U}$. Thus, the set $X = \{x \in \omega : V(x) \notin \mathcal{U}\} \in \mathcal{U}$. Let $\langle S_x : x \in \omega \rangle$ be any predictor. We will produce a coloring g for which a set of agents of $\mathcal{U}$-measure one guesses incorrectly.

Let $Y = \{y \in \omega : \exists x \in X \text{ such that } (x, y) \in V\}$. Thus, Y consists of those agents who are seen by a (smaller) agent in X.

CASE 1: $Y \notin \mathcal{U}$. Because $Y \notin \mathcal{U}$, we know that $Y^c \in \mathcal{U}$, and thus $X \cap Y^c = X - Y \in \mathcal{U}$. But $X - Y$ is an independent set and, as pointed out earlier, it is easy to produce a coloring g for which everyone in an independent set guesses incorrectly. Thus we are done in Case 1.

CASE 2: $Y \in \mathcal{U}$. Choose $f \in {}^{\omega}\omega$ such that for each $y \in Y$ we have that $f(y)$ is the least $x \in X$ such that $(x, y) \in V$. Because $\mathcal{U}$ is a p-point, there exists a set $Z \in \mathcal{U}$ such that $f|Z$ is either constant or finite-to-one. If $f(Z) = \{x\}$, then $f(Y \cap Z) = \{x\}$, and by choosing any $y \in Y \cap Z$ (which is in $\mathcal{U}$ and thus non-empty), we see that we must have $x \in X$. But then $\{y \in Y \cap Z : x < y\} \subseteq V(x)$, and thus $V(x) \in \mathcal{U}$, contrary to the definition of X. It thus follows that $f|Z$ is finite-to-one. We will now produce a coloring that makes all the agents in Z guess incorrectly.

For each $x \in Z$ let $W_x \subseteq {}^{\omega}2$ be defined as follows: $h \in W_x$ iff $h(y) = 0$ for all $y \in Z^c$ and $S_x(h) \neq h(x)$. Thus W_x consists of those colorings that place hats of color zero on all the agents not in Z and place hats on the agents in Z so that agent x guesses incorrectly.

CLAIM 1. *W_x is a closed subset of ${}^{\omega}2$.*

If $h \notin W_x$, then either $h(x) = 1$ for some $x \in Z^c$ (in which case $h \in [h|\{x\}] \subseteq {}^{\omega}2 - W_x$) or $h(x) = 0$ for every $x \in Z^c$ and $S_x(h) = h(x)$ (in which case $h \in [h|((V(x) \cup \{x\}) \cap Z)] \subseteq {}^{\omega}2 - W_x$).

CLAIM 2. *The sets W_x have the finite-intersection property.*

Suppose $x_1 < \cdots < x_k$. Starting with color zero hats on everyone, we can begin at agent x_k and change his hat color (if necessary) so that he guesses incorrectly. We now move to agent x_{k-1} and do the same; this does not affect agent x_k's guess. Continuing down to agent x_1 yields a coloring in $W_{x_1} \cap \cdots \cap W_{x_k}$.

By compactness, there exists a coloring g that is in W_x for each $x \in Z$. This coloring makes everyone in Z guess incorrectly, and thus completes the proof. ∎

We now characterize the NIS-ultrafilters.

THEOREM 2.2. *An ultrafilter is an NIS-ultrafilter iff it is a Ramsey ultrafilter.*

Proof. Suppose first that $\mathcal{U}$ is a Ramsey ultrafilter and that V is a graph on ω having no independent subgraph with vertex set in $\mathcal{U}$. Because $\mathcal{U}$ is Ramsey it follows that there is a complete subgraph with vertex set $X \in \mathcal{U}$. But now we know that the agents in X have a finite-error strategy among themselves and this ensures correct guesses by a set of agents in $\mathcal{U}$ as desired.

Suppose now that $\mathcal{U}$ is not a Ramsey ultrafilter, and choose f such that f is neither constant on any set in $\mathcal{U}$ nor one-to-one on any set in $\mathcal{U}$. We want to produce a graph V satisfying the natural necessary condition (no independent set in $\mathcal{U}$) but for which there is no $\mathcal{U}$-predictor. We consider two cases:

CASE 1: There exists a set $Y \in \mathcal{U}$ such that $f|Y$ is finite-to-one. Let V be the graph on ω in which $(x, y) \in V$ iff $x < y$ and $f(x) = f(y)$. If Z is an independent set in V, then $f|Z$ is one-to-one, and so $Z \notin \mathcal{U}$. Now as in the proof of the previous theorem, we deduce that each agent in Y sees only finitely many other agents in Y, and so for every predictor, we have a hat coloring for which every agent in Y guesses incorrectly. This shows that in this case, the necessary condition is not sufficient.

CASE 2: f is not finite-to-one on any set in $\mathcal{U}$. Let V be the graph on ω in which $(x, y) \in V$ iff $x < y$ and $f(x) > f(y)$. Suppose $Z \in \mathcal{U}$; we will show that Z is not an independent set in V. We know that $f|Z$ is not finite-to-one, so we can choose p such that infinitely many points of Z map to p. But $\{x \in Z : f(x) \le p\} \notin \mathcal{U}$ and so we can choose $x \in Z$ such that $f(x) > p$. But now if we choose $y \in Z$ such that $y > x$ and $f(y) = p$, then $x, y \in Z$, $x < y$, and $f(x) > f(y)$ so we have $(x, y) \in V$. This shows that V has no independent set in $\mathcal{U}$.

But now, given any predictor, we can start with color zero hats on everyone and then successively change the hats on the agents in $f^{-1}(\{0\})$ so that these agents all guess incorrectly, and then change the hats on the agents in $f^{-1}(\{1\})$ so that these agents all guess incorrectly—without affecting the guesses of the agents in $f^{-1}(\{0\})$—and so on until we have a coloring for which every agent guesses incorrectly. ∎

3. Additional remarks. We conclude with six brief comments.

P-point ultrafilters are also characterized by the following property: For each countable collection $\mathcal{X}$ of sets in $\mathcal{U}$ there is a set $Y \in \mathcal{U}$ such that $Y - X$ is finite for each $X \in \mathcal{X}$. This property, in fact, explains the name, since it directly shows these ultrafilters to be p-points (in the topological sense) in the space $\beta\omega - \omega$. One can use this property in the proof of Theorem 2.1 to obtain the set $Z \in \mathcal{U}$ such that $Z - V(x)^c$ is finite for each $x \in X$.

An ultrafilter $\mathcal{U}$ is a *q-point* if for each $f \in {}^{\omega}\omega$ that is finite-to-one, there exists a set $X \in \mathcal{U}$ such that $f|X$ is one-to-one. Thus, $\mathcal{U}$ is a Ramsey ultrafilter iff it is both a p-point and a q-point. The two cases occurring in the proof of Theorem 2.2 correspond (respectively) to $\mathcal{U}$ not being a q-point and $\mathcal{U}$ not being a p-point.

Intuitively, the existence of a "successful" predictor depends on the graph V providing a "large amount of visibility." Theorem 2.1 is saying that an ultrafilter $\mathcal{U}$ is a p-point precisely when the collection of graphs providing sufficient visibility for a $\mathcal{U}$-predictor is itself an ultrafilter (on $\omega \times \omega$—and it is indeed $\mathcal{U} \times \mathcal{U}$ in this case).

One can generalize the questions we asked to the context of filters on ω. An *SIN-filter $\mathcal{F}$ on ω* is one for which the obvious sufficient condition for the existence of an $\mathcal{F}$-predictor ($V \in \mathcal{F} \times \mathcal{F}$) is also necessary. An *NIS-filter $\mathcal{F}$ on ω* is one for which the obvious necessary condition for the existence of an $\mathcal{F}$-predictor (all independent sets are of $\mathcal{F}$-measure zero) is also sufficient.

Christopher Hardin [H10] has provided an elegant argument showing that the filter of cofinite subsets of ω is an NIS-filter (that is, for every graph on ω with no infinite independent set, there is a finite-error predictor). We do not have a characterization of NIS-filters. On the other hand, it is not hard to see (as we now demonstrate) that if $\mathcal{F}$ is an SIN-filter then $\mathcal{F}$ is an ultrafilter. To see this, suppose that $\mathcal{F}$ is not an ultrafilter and let $X \subseteq \omega$ be such that neither X nor X^c is in $\mathcal{F}$. Consider the graph V in which $(x, y) \in V$ if $x < y$ and both x and y are in X or both x and y are in X^c. Then, for every x, we have $V(x) \notin \mathcal{F}$ but the agents in X have a finite-error predictor among themselves as do the agents in X^c. Thus, there is a finite-error predictor and hence an $\mathcal{F}$-predictor, showing that the sufficient condition is not necessary.

Finally, it is also the case that although we worked exclusively with two-color hat problems, the number of colors, provided it is at least two, played no role in any of our results. That is, the positive results were based on the construction in Section 1 which, as pointed out there, works equally well with κ colors.

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