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Turning Borel sets into clopen sets effectively

by

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Abstract. We present the effective version of the theorem about turning Borel sets in Polish spaces into clopen sets while preserving the Borel structure of the underlying space. We show that under some conditions the emerging parameters can be chosen in a hyperarithmetical way and using this we obtain some uniformity results.

1. Introduction. One of the topics of effective descriptive set theory is the refinement of well-known theorems in recursive-theoretic terms. This "effective" version of a theorem is stronger than the original one, and as is often the case, it provides a uniformity result which does not seem to follow from the original statement. A typical example is *Suslin's Theorem*, which states that every bi-analytic subset of a Polish space is Borel, and its refinement, the *Suslin–Kleene Theorem*, which provides a recursive (and thus continuous) function $u : \mathcal{N} \times \mathcal{N} \to \mathcal{N}$, where $\mathcal{N} = \omega^\omega$, such that whenever α and β are codes of complementary analytic sets, say A and $\mathcal{X} \setminus A$, then $u(\alpha, \beta)$ is a Borel code of A (cf. [5, 7B.4] and [6]).

In this article we prove the effective version of the following well-known theorem of classical descriptive set theory: if $(\mathcal{X}, \mathcal{T})$ is a Polish space and A is a Borel subset of $\mathcal{X}$, then

(∗) there is a Polish topology $\mathcal{T}_\infty$ on $\mathcal{X}$ which extends $\mathcal{T}$, yields the same Borel sets as $\mathcal{T}$, and A is clopen in $(\mathcal{X}, \mathcal{T}_\infty)$ (Theorem 2.1).

Our proof will not follow the usual proof of the latter theorem. We will instead present a different proof, perhaps new. We will see that there are certain advantages of our approach which allow us to proceed with the effective version. Let us give a brief description of the usual proof. One starts with a Polish space $\mathcal{X}$, defines

$$\mathcal{S} = \{A \subseteq \mathcal{X} \mid A \text{ is Borel and satisfies } (*) \text{ above}\}$$

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and shows that $\mathcal{S}$ is a σ-algebra which contains the open sets and so it contains every Borel set. The effective version of this proof requires the notion of the effective σ-field (cf. [5, Section 7B]) and a rather messy encoding of Polish topologies as we proceed with the induction. Moreover this approach seems to provide little information on the best choice for the parameters which will emerge.

On the other hand our approach is based on the following result of Lusin–Suslin: every Borel subset of a Polish space is the injective continuous image of a closed subset of $\mathcal{N}$ ([1, 13.7]). So let us assume that $\mathcal{X}$ is a Polish space and that A is a Borel subset of $\mathcal{X}$. Then there are closed sets $F_1, F_2 \subseteq \mathcal{N}$ and continuous functions $\pi_1, \pi_2 : \mathcal{N} \to \mathcal{X}$ such that π_i is injective on F_i, $i = 1, 2$, $\pi_1[F_1] = A$ and $\pi_2[F_1] = \mathcal{X} \setminus A$. Now we define a distance function d_A on A in such a way that π_1 becomes an isometry, i.e. $d_A(x, y) = p_{\mathcal{N}}(\alpha, \beta)$, where $\alpha, \beta \in F_1$ with $\pi_1(\alpha) = x$ and $\pi_1(\beta) = y$. Similarly we define the distance function d_{A^c} on the complement of A and then we consider the direct sum $(A, d_A) \oplus (\mathcal{X} \setminus A, d_{A^c})$. The latter space has all the required properties ([1]).

The advantage of the latter proof is that it reduces the problem from Borel sets to closed sets where the verification of Cauchy-completeness is obvious. Moreover it is straightforward to effectivize, for the effective version of the Luslin–Suslin Theorem has been proved by Moschovakis (cf. [5, 4A.7]). Finally, as we will see, this approach provides very clear information about the emerging parameters.

In the rest of this section we recall the basic definitions and notation. We assume that the reader is familiar with recursion theory and effective descriptive set theory (cf. [5, Chapter 3]). In the next section we prove our main theorem (Theorem 2.1), and afterwards we examine the problem of choosing the emerging parameters in a Δ^1_1 way (cf. Theorems 3.9 and 3.11). We conclude this article with a related uniformity result about choosing the extended topology $\mathcal{T}_\infty$ in a Borel way (cf. Theorem 4.2). This is the only result of this article whose statement is purely classical in the sense that it involves no notions from effective theory. Its proof however is a corollary of almost all the preceding effective results.

Notation and definitions. By ω we mean the least infinite ordinal, which we identify with the set of natural numbers, and by ω_1 the least uncountable ordinal. We fix once and for all a recursive encoding $\langle\cdot\rangle$ of all

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([1]) One may express some reservations on whether our suggested proof is indeed independent of the original one, because one can derive the Lusin–Suslin Theorem from the theorem about turning Borel sets into clopen sets (cf. [1]). There are however straightforward proofs of the Lusin–Suslin Theorem: see for Example 1G.5 or—for a very different proof—4A.7 in [5]. Therefore the proof that we are now suggesting is indeed independent of the usual proof. Nevertheless—as it becomes clear from our comments—these two theorems are "equivalent" in the sense that we can deduce one from the other.

finite sequences of naturals by a natural number. The number 0 will be the code of the empty sequence. We denote by $\mathtt{Seq}$ the recursive set of all codes of finite sequences in ω. If $s \in \mathtt{Seq}$ is a code of $u = (u_0, \dots, u_{n-1})$ we define $\mathrm{lh}(s) = n$ and if $i < n$ we put $(s)_i = u_i$. If $s \in \mathtt{Seq}$ and $i \geq \mathrm{lh}(s)$ or if $s \notin \mathtt{Seq}$ and i is arbitrary we define $(s)_i$ to be 0. Finally we fix the following enumeration of the rational numbers:

$$q_s = (-1)^{(s)_0} \frac{(s)_1}{(s)_2 + 1}$$

for $s \in \omega$. By O we mean the usual Kleene's canonical set of ordinal notation.

We denote by $\mathcal{N}$ the space ω^ω of all infinite sequences of naturals with the product topology. The space $\mathcal{N}$ is the *Baire space*. The members of $\mathcal{N}$ will be denoted by lowercase Greek letters such as α, β etc. We fix the usual distance function $p_{\mathcal{N}}$ on $\mathcal{N}$ defined by

$$p_{\mathcal{N}}(\alpha, \beta) = ((\text{least } k \text{ with } \alpha(k) \neq \beta(k)) + 1)^{-1}$$

for $\alpha \neq \beta$. We may view every $\alpha \in \mathcal{N}$ as a code of an infinite sequence in $\mathcal{N}$. To be more specific, we define $(\alpha)_i(n) = \alpha(\langle i, n\rangle)$ for all $i \in \omega$ and so α gives rise to the sequence $((\alpha)_i)_{i\in\omega}$. Of course we may apply the inverse procedure: if $(\alpha_i)_{i\in\omega}$ is a sequence in $\mathcal{N}$ there is some $\alpha \in \mathcal{N}$ such that $(\alpha)_i = \alpha_i$ for all $i \in \omega$. For $\alpha, \beta \in \mathcal{N}$ we will denote by $\langle \alpha, \beta\rangle$ the unique $\gamma \in \mathcal{N}$ such that $(\gamma)_0 = \alpha$, $(\gamma)_1 = \beta$ and $\gamma(t) = 0$ if $t \neq \langle i, n\rangle$ for all $i = 0, 1$ and all $n \in \omega$.

We will often identify relations with the sets they define and write $P(x)$ instead of $x \in P$.

A topological space is a *Polish space* if it is separable and is metrizable by a complete distance function. We will call such a distance function a *suitable* distance function. We employ the standard hierarchy $(\underset{\sim}{\mathbf{\Sigma}}^0_\xi)_{\xi<\omega_1}$ of Borel sets in Polish spaces (cf. [1]).

Suppose now that $(\mathcal{X}, d)$ is a complete and separable metric space. A sequence $(x_n)_{n\in\omega}$ is a *recursive presentation of* $(\mathcal{X}, d)$ if

(I) the set $\{x_n \mid n \in \omega\}$ is dense in $\mathcal{X}$,

(II) the relations $P_<, P_\leq \subseteq \omega^4$ defined by

$$P_<(i, j, k, m) \iff d(x_i, x_j) < k(m + 1)^{-1},$$
$$P_\leq(i, j, k, m) \iff d(x_i, x_j) \leq k(m + 1)^{-1}$$

are recursive.

We say that $(\mathcal{X}, d)$ *admits a recursive presentation*, or that $(\mathcal{X}, d)$ *is recursively presented*, if there is a sequence $(x_n)_{n\in\omega}$ in $\mathcal{X}$ which satisfies the conditions (I) and (II).

For every complete space $(\mathcal{X}, d)$ with a recursive presentation $(x_n)_{n\in\omega}$ we consider the set

$$B(x_n, m, k) = \{x \in \mathcal{X} \mid d(x, x_n) < k(m+1)^{-1}\}.$$

The latter is either an empty set or the ball with center x_n and radius $m(m+1)^{-1}$. For $s \in \omega$ define

$$N(\mathcal{X}, s) = B(x_{(s)_0}, (s)_1, (s)_2).$$

The family $\{N(\mathcal{X}, s) \mid s \in \omega\}$ is the associated *neighborhood system of* $\mathcal{X}$ (with respect to $(x_n)_{n\in\omega}$ and d) and it is clear that it forms a basis for the topology of $\mathcal{X}$. We say that a Polish space $\mathcal{X}$ is recursively presented if there is a suitable distance function d such that the corresponding space $(\mathcal{X}, d)$ is recursively presented. When we refer to a recursively presented Polish space we will always assume that we are given a suitable distance function and a recursive presentation. Standard examples of recursively presented Polish spaces are the Baire space $\mathcal{N}$, the space of real numbers and ω.

The property of being recursively presented clearly carries over to finite products and sums of spaces, i.e., if $\mathcal{X}_1, \ldots, \mathcal{X}_n$ are recursively presented Polish spaces, then both $\mathcal{X}_1 \times \cdots \times \mathcal{X}_n$ and $\mathcal{X}_1 \oplus \cdots \oplus \mathcal{X}_n$ are recursively presented. We fix once and for all a scheme for passing from the recursive presentations of finitely many spaces to recursive presentations of their product and sum.

Suppose that $(\mathcal{X}, d)$ is complete and recursively presented. We say that $A \subseteq \mathcal{X}$ *is in* Σ^0_1 (or that A *is a* Σ^0_1 *set*, or that A *is semirecursive*) if there is a recursive function $f : \omega \to \omega$ such that $A = \bigcup_{s\in\omega} N(\mathcal{X}, f(s))$. In other words Σ^0_1 sets are the unions of a recursive collection from the family of our fixed open neighborhoods. This definition suggests that the property of being a Σ^0_1 set depends on the way we have encoded the basic neighborhoods $N(\mathcal{X}, s)$; however this is not the case (see [5, 3C.12]). (This property does depend of course on the distance function and the recursive presentation.) The set A *is in* Π^0_1 if $\mathcal{X} \setminus A$ is in Σ^0_1. Inductively we define the family of Σ^0_{n+1} subsets of $\mathcal{X}$ as the family of all sets which are the projection along ω of a Π^0_n subset of $\mathcal{X} \times \omega$, and Π^0_{n+1} sets as the complements of sets in Σ^0_{n+1}. The set A is in Σ^1_1 if it is the projection along $\mathcal{N}$ of a Π^0_1 subset of $\mathcal{X} \times \mathcal{N}$. The set A is in Π^1_1 if its complement is in Σ^1_1, and A is in Δ^1_1 if it is both in Σ^1_1 and Π^1_1. Of course all these definitions coincide with the usual ones of Kleene when $\mathcal{X} = \omega$ or $\mathcal{X} = \mathcal{N}$.

A function $f : \mathcal{X} \to \mathcal{Y}$ between recursively presented Polish spaces is *recursive* (or Σ^0_1*-recursive*) if the relation $R^f \subseteq \mathcal{X} \times \omega$ defined by

$$R^f(x, s) \iff f(x) \in N(\mathcal{Y}, s)$$

is in Σ^0_1. Similarly the function f is Δ^1_1*-recursive* if the set R^f is in Δ^1_1.

An important notion is the one of relativization with respect to some parameter. Suppose that $\mathcal{X}$ and $\mathcal{Y}$ are recursively presented Polish spaces and that $y \in \mathcal{Y}$. A subset A of $\mathcal{X}$ is in $\Sigma^0_1(y)$ if there is some $P \subseteq \mathcal{Y} \times \mathcal{X}$ such that A is the y-section of P, i.e., $A = P_y := \{x \in \mathcal{X} \mid P(x, y)\}$. Similarly one defines the classes of sets $\Sigma^0_n(y)$, $\Pi^0_n(y)$, $\Sigma^1_1(y)$ and $\Pi^1_1(y)$. The class $\Delta^1_1(y)$ is $\Sigma^1_1(y) \cap \Pi^1_1(y)$. By replacing Δ^1_1 with $\Delta^1_1(y)$ one defines the class of $\Delta^1_1(y)$-recursive functions and similarly one defines the class of $\Sigma^0_1(y)$-recursive or simply y-recursive functions.

For any pointclass Γ and for any recursively presented Polish space $\mathcal{X}$, a point $x \in \mathcal{X}$ *is in* Γ or *it is a* Γ *point* exactly when the relation $U \subseteq \omega$ defined by $U(s) \Leftrightarrow x \in N(\mathcal{X}, s)$ is in Γ. Suppose that $x \in \Gamma$; we say that

- x *is recursive* (resp. *recursive in* y) if $\Gamma = \Sigma^0_1$ (resp. $\Sigma^0_1(y)$),
- x *is hyperarithmetical* (resp. *hyperarithmetical in* y) if $\Gamma = \Delta^1_1$ (resp. $\Delta^1_1(y)$),
- x *is arithmetical* (resp. *arithmetical in* y) if $\Gamma = \Sigma^0_n$ (resp. $\Sigma^0_n(y)$) for some $n \geq 1$.

If x is hyperarithmetical in y and y is hyperarithmetical in x we say that x and y have *the same hyperdegree*. We will also consider the cases where the above x and y are in fact subsets of ω by identifying a set $P \subseteq \omega$ with its characteristic function χ_P, which is a member of the recursively presented Polish space 2^ω. We say for example that P *is hyperarithmetical in* Q if $\chi_P \in \Delta^1_1(\chi_Q)$. The latter is equivalent to saying that the function $\chi_P : \omega \to 2$ is $\Delta^1_1(\chi_Q)$-recursive.

The relativization applies to recursive presentations as well. Consider a point $\varepsilon \in \mathcal{N}$. A sequence $(x_n)_{n\in\omega}$ in a complete metric space $(\mathcal{X}, d)$ is an *ε-recursive presentation of* $(\mathcal{X}, d)$ if the above conditions (I) and (II) are satisfied with the modification that the relations $P_<$ and $P_\leq$ are now ε-recursive. One can then repeat all previous definitions by replacing everywhere the term "recursive" with "ε-recursive". For example a subset A of an ε-recursively presented Polish space $\mathcal{X}$ is *ε-semirecursive* if there is an ε-recursive function $f : \omega \to \omega$ such that $A = \bigcup_{s\in\omega} N(\mathcal{X}, f(s))$. We will denote the class of ε-semirecursive sets by $\Sigma^0_1(\varepsilon)$. There is a potential double meaning for $\Sigma^0_1(\varepsilon)$, because every recursive presentation is also an ε-recursive presentation for any $\varepsilon \in \mathcal{N}$. One can check however (with the help of [5, 3C.5]) that the ε-recursive unions of basic neighborhoods $N(\mathcal{X}, s)$—where $\mathcal{X}$ is recursively presented—are exactly the ε-sections of Σ^0_1 subsets of $\mathcal{N} \times \mathcal{X}$. So no conflict arises.

It is not true that every Polish space is recursively presented but it is easy to see that every Polish space is recursively presented in some parameter ε. All theorems about recursively presented Polish spaces are transferred to ε-recursively presented Polish spaces. The latter claim is the *Relativiza-*

tion Principle. This principle is fundamental for the applications of effective descriptive set theory to "classical mathematics" such as Theorem 4.2. For more information refer to [5, 3I].

A function $f : (\mathcal{X}, d_{\mathcal{X}}) \to (\mathcal{Y}, d_{\mathcal{Y}})$ is an *isometry* if $d_{\mathcal{Y}}(f(x), f(y)) = d_{\mathcal{X}}(x, y)$ for all $x, y \in \mathcal{X}$ and f is surjective. If the spaces $(\mathcal{X}, d_{\mathcal{Y}})$ and $(\mathcal{Y}, d_{\mathcal{Y}})$ are complete and recursively presented, and the function f is recursive, it is not hard to verify that a set A is in Γ exactly when $f[A]$ is in Γ, where Γ is any of the classes $\Sigma^0_n(\alpha)$, $\Pi^0_n(\alpha)$, $\Sigma^1_1(\alpha)$, $\Pi^1_1(\alpha)$ and $\Delta^1_1(\alpha)$.

We are going to use some fundamental theorems of effective descriptive set theory including (but not restricted to) the Theorem on Restricted Quantification (4D.3), the Effective Perfect Set Theorem (4F.1), the Strong Δ-Selection Principle (4D.6) and the theorem about the existence of Δ^1_1 points inside Π^1_1 non-meager sets (4F.20). (All these references are from [5].) We should mention explicitly the following result.

THEOREM 1.1 (cf. [5, 4A.7]). *Every Δ^1_1 subset of a recursively presented Polish space is the recursive injective image of a Π^0_1 subset of $\mathcal{N}$.*

In order to estimate the emerging parameters we need some fundamental results of higher recursion theory, which are due to Kleene and Spector. We give a brief presentation of these results. The reader may refer to [5] and to [8] for more information about their proofs, as well as to the original articles. (Below we give more detailed references.)

A set $P \subseteq \omega$ is Π^1_1-*complete* if P is in Π^1_1 and for all sets $Q \subseteq \omega$ in Π^1_1 there exists a recursive function $f : \omega \to \omega$ such that for all $n \in \omega$ we have

$$Q(n) \iff P(f(n)).$$

We shall give a typical example of a Π^1_1-complete set (although this is not straightforward to verify), but let us assume for the moment the Π^1_1-complete sets do exist and let us examine some basic facts about them. It is clear from the definition that every subset of ω in $\Sigma^1_1 \cup \Pi^1_1$ is recursive in every Π^1_1-complete set. It is also easy to see that a Π^1_1-complete set is not a Σ^1_1 set, for otherwise every Π^1_1 subset of ω would be in Σ^1_1 as well.

Another important fact about Π^1_1-complete sets is that they have the same hyperdegree. This follows from the following result of Spector.

THEOREM 1.2 (Spector, cf. [9]). *If A and B are Π^1_1 subsets of ω and B is not in Δ^1_1, then $A \in \Delta^1_1(B)$. (Equivalently any two sets in $\Pi^1_1 \setminus \Delta^1_1$ have the same hyperdegree.)*

We will present a proof of the latter theorem using some tools from [5]. Suppose that $\mathcal{X}$ is a recursively presented Polish space and that P is a non-empty subset of $\mathcal{X}$. A *norm* on P is any function $\varphi : P \to$ Ordinals. A Π^1_1-*norm* on P is a norm φ (on P) for which there exist two relations $\leq_\Sigma$

and $\leq_\Pi$ in Σ^1_1 and Π^1_1 respectively such that for all $x \in \mathcal{X}$ and all $y \in P$ we have

$$[P(x) \ \&\ \varphi(x) \leq \varphi(y)] \ \Leftrightarrow\ x \leq_\Sigma y \ \Leftrightarrow\ x \leq_\Pi y.$$

The following is an important fact about Π^1_1 sets.

Theorem 1.3 (cf. [5, 4B.2]). *Every Π^1_1 set admits a Π^1_1-norm.*

Now we are ready to prove Spector's Theorem.

Proof of Theorem 1.2. Consider a Π^1_1-complete set $P \subseteq \omega$. It suffices to show that every $C \subseteq \omega$ in $\Pi^1_1 \setminus \Delta^1_1$ has the same hyperdegree as P.

Suppose that $C \subseteq \omega$ is in $\Pi^1_1 \setminus \Delta^1_1$ and that $f : \omega \to \omega$ is a recursive function such that $n \in C \Leftrightarrow f(n) \in P$ for all $n \in \omega$. Clearly C is recursive in P. Consider a Π^1_1-norm φ on P. The claim is that the set $\{\varphi(f(n)) \mid n \in C\}$ is unbounded in $\{\varphi(m) \mid m \in P\}$. Indeed if this were not the case, there would be some $m_0 \in P$ such that $\varphi(f(n)) \leq \varphi(m_0)$ for all $n \in C$. Using the key property of a Π^1_1-norm we have

$$n \in C \ \Leftrightarrow\ n \in C \ \&\ \varphi(f(n)) \leq \varphi(m_0) \ \Leftrightarrow\ f(n) \leq_\Sigma m_0.$$

The latter implies that C is in Σ^1_1, which contradicts the fact that C is not in Δ^1_1.

Therefore for all $m \in P$ there exists $n \in C$ such that $\varphi(m) < \varphi(f(n))$. From this and from the key property of a Π^1_1-norm it follows that

$$\begin{aligned} m \in P \ &\Leftrightarrow\ (\exists n)[n \in C \ \&\ \varphi(m) \leq \varphi(f(n))] \\ &\Leftrightarrow\ (\exists n)[n \in C \ \&\ m \leq_\Sigma f(n)] \ \Leftrightarrow\ (\exists n)[n \in C \ \&\ m \leq_\Pi f(n)]. \end{aligned}$$

The latter two equivalences show that P is in $\Sigma^1_1(C) \cap \Pi^1_1(C) = \Delta^1_1(C)$. ∎

We now present a typical example of a Π^1_1-complete set. We consider the standard enumeration of the partial recursive functions from ω to ω. By $\{e\}$ we mean the eth term of this enumeration.

Definition 1.4 (Kleene, cf. [2] and [3]). Define the condition $\mathcal{A}(X, Y)$, where $X \subseteq \omega$ and $Y \subseteq \omega \times \omega$, to hold exactly when

(III) $1 \in X$,
(IV) $(\forall y)[y \in X \to 2^y \in X \ \&\ (y, 2^y) \in Y]$,
(V) $(\forall e)\{$if for all $n \in \omega$, $\{e\}(n)$ is defined and $\{e\}(n) \in X$ then $3 \cdot 5^e \in X \ \&\ (\forall n)[(\{e\}(n), 3 \cdot 5^e) \in Y]\}$,
(VI) $(\forall x, y, z)[(x, y) \in Y \ \&\ (y, z) \in Y \to (x, z) \in Y]$.

Kleene's O and the relation $<_O$ are defined as follows:

$$x \in O \ \Leftrightarrow\ (\forall X, Y)[\mathcal{A}(X, Y) \to x \in X],$$
$$(x, y) \in {<_O} \ \Leftrightarrow\ (\forall X, Y)[\mathcal{A}(X, Y) \to (x, y) \in Y]$$

for all $x, y \in \omega$. It is not hard to verify that $\mathcal{A}(O, <_O)$ holds, so that $(O, <_O)$ is the least pair (X, Y) satisfying conditions (III)–(VI).

Kleene's O serves as an ordinal notation system (cf. [7]). However we will not pursue this direction here, since what we are interested in is the following.

THEOREM 1.5 (Kleene, cf. [3] or [8]). *Kleene's O is a Π^1_1-complete set.*

So from Theorems 1.2 and 1.5 and the comments about Π^1_1-complete sets we have:

(VII) every subset of ω in Π^1_1 is recursive in O,
(VIII) every subset of ω in $\Pi^1_1 \setminus \Delta^1_1$ has the same hyperdegree as O.

Conditions (VII) and (VIII) are all that we are going to need about Kleene's O. Therefore one may replace O in all of its upcoming occurrences with any Π^1_1-complete set (2).

2. The effective version. In this section we present our main theorem, which is the effective version of the theorem about turning a Borel subset of a Polish space into a clopen set.

THEOREM 2.1. *Suppose that $(\mathcal{X}, \mathcal{T})$ is a recursively presented Polish space, d is a suitable distance function for $(\mathcal{X}, \mathcal{T})$, and A is a Δ^1_1 subset of $\mathcal{X}$. Then there exist an $\varepsilon_A \in \mathcal{N}$, which is recursive in O, and a Polish topology $\mathcal{T}_\infty$ with suitable distance function d_∞, which extends $\mathcal{T}$ and has the following properties:*

(1) *The Polish space $(\mathcal{X}, \mathcal{T}_\infty)$ is ε_A-recursively presented.*
(2) *The set A is a $\Delta^0_1(\varepsilon_A)$ subset of $(\mathcal{X}, d_\infty)$.*
(3) *If $B \subseteq \mathcal{X}$ is a $\Delta^1_1(\alpha)$ subset of $(\mathcal{X}, d)$, where $\alpha \in \mathcal{N}$, then B is a $\Delta^1_1(\varepsilon_A, \alpha)$ subset of $(\mathcal{X}, d_\infty)$.*
(4) *If $B \subseteq \mathcal{X}$ is a $\Delta^1_1(\varepsilon_A, \alpha)$ subset of $(\mathcal{X}, d_\infty)$, where $\alpha \in \mathcal{N}$, then B is a $\Delta^1_1(\varepsilon_A, \alpha)$ subset of $(\mathcal{X}, d)$.*

Proof. From Theorem 1.1 there are Π^0_1 sets $F_1, F_2 \subseteq \mathcal{N}$ and recursive functions $\pi_1, \pi_2 : \mathcal{N} \to \mathcal{X}$ such that π_i is one-to-one on F_i, $i = 1, 2$, and $A = \pi_1[F_1]$, $A^c = \pi_2[F_2]$; here A^c stands for the complement of A in $\mathcal{X}$. We may assume that $A \neq \emptyset, \mathcal{X}$ for otherwise the result is immediate. So $F_1, F_2 \neq \emptyset$.

Now we define a distance function on A such that the function π_1 becomes an isometry. Let $p_{\mathcal{N}}$ be the usual distance function on the Baire space $\mathcal{N}$, and $x, y \in A$ and α, β be the unique members of F_1 such that

(2) Another (perhaps more natural) example of a Π^1_1-complete set is *Spector's W* (cf. [9]), which is defined as follows:

$$W = \{e \in \omega \mid \{e\} \text{ is total and the set } \{(n,m) \mid \{e\}(\langle n,m\rangle) = 0\} \text{ is a well-ordering}\}.$$

$x = \pi_1(\alpha)$ and $y = \pi_1(\beta)$. Put

$$d_A(x, y) = d_A(\pi_1(\alpha), \pi_1(\beta)) = p_{\mathcal{N}}(\alpha, \beta).$$

We do the same for A^c, i.e. for $x, y \in A^c$ we define

$$d_{A^c}(x, y) = d_{A^c}(\pi_2(\alpha), \pi_2(\beta)) = p_{\mathcal{N}}(\alpha, \beta)$$

where α, β are the unique members of F_2 such that $x = \pi_2(\alpha)$ and $y = \pi_2(\beta)$.

Let p_{F_i} be the restriction of the distance function $p_{\mathcal{N}}$ on $F_i \times F_i$, $i = 1, 2$. Since the sets F_i are closed the metric spaces (F_i, p_{F_i}) are separable and complete, $i = 1, 2$. Also since the functions $\pi_i \restriction F_i$ are isometries we see that the spaces (A, d_A), (A^c, d_{A^c}) are separable and complete as well.

Now we are going to define $\varepsilon_1, \varepsilon_2 \in \mathcal{N}$ in which the metric spaces (A, d_A), (A^c, d_{A^c}) admit a recursive presentation, respectively. Recall the set $\mathtt{Seq} \subseteq \omega$ of the codes of all finite sequences of ω. For $s \in \mathtt{Seq}$ we define

$$N_s = \{\alpha \in \mathcal{N} \mid \alpha(i) = (s)_i \ \forall i < \mathrm{lh}(s)\}.$$

We adopt the notation $s\,\hat{}\,k$ for the natural number $\langle (s)_0, \ldots, (s)_{\mathrm{lh}(s)-1}, k\rangle$, where $s \in \mathtt{Seq}$ and $k \in \omega$. Define

$$\varepsilon_1(s) = 1 \iff \mathtt{Seq}(s) \ \& \ F_1 \cap N_s \neq \emptyset$$

and $\varepsilon_1(s) = 0$ otherwise. Notice that for all s with $\varepsilon_1(s) = 1$ there always exists some $k \in \omega$ such that $\varepsilon_1(s\,\hat{}\,k) = 1$.

We now define a sequence $(\alpha_s)_{s\in\omega}$ which is dense in F_1. First for $\varepsilon_1(s) = 1$ define $\alpha_s(i) = (s)_i$ for all $i < \mathrm{lh}(s)$, and for $i \geq \mathrm{lh}(s)$,

$$\alpha_s(i) = (\mu k)[\varepsilon_1(\langle \alpha_s(0), \ldots, \alpha_s(i-1), k\rangle) = 1].$$

It is clear that for all s with $\varepsilon_1(s) = 1$ we have $\alpha_s \in N_s$ and

$$N_{\langle \alpha_s(0), \ldots, \alpha_s(i-1)\rangle} \cap F_1 \neq \emptyset$$

for all $i \in \omega$. Since F_1 is closed it follows that $\alpha_s \in N_s \cap F_1$ for all $s \in \mathtt{Seq}$ such that $N_s \cap F_1 \neq \emptyset$. Therefore the sequence $(\alpha_s)_{\{s:\ \varepsilon_1(s)=1\}}$ is dense in F_1. Now pick the least natural s, call it s_0, for which $\varepsilon_1(s) = 1$, and define $\alpha_s = \alpha_{s_0}$ for all s for which $\varepsilon_1(s) = 0$.

We will prove that the sequence $(\alpha_s)_{s\in\omega}$ is an ε_1-recursive presentation of (F_1, p_{F_1}). First notice that the relation

$$P(s, t, i) \iff \alpha_s(i) = \alpha_t(i)$$

is recursive in ε_1. To see this, notice that the definitions above can be reformulated so that s becomes a variable, i.e. there is a $\Sigma^0_1(\varepsilon_1)$-recursive function $f : \omega \times \omega \to \omega$ such that $f(s, i) = \alpha_s(i)$ for all s, i.

Now we claim that the relations $Q \subseteq \omega \times \omega$ and $R \subseteq \omega \times \omega$ defined by

$$\begin{aligned} Q(s, t, i) &\iff P(s, t, i) \ \& \ (\forall j < i)\neg P(s, t, j) \\ &\iff i \text{ is the least } k \text{ for which } \alpha_s(k) \neq \alpha_t(k) \end{aligned}$$

and

$$R(s,t) \;\Leftrightarrow\; \alpha_s = \alpha_t$$

are recursive in ε_1. This is clear for Q, since P is recursive in ε_1. We prove the claim for R. If $\varepsilon_1(s) = \varepsilon_1(t) = 1$ then the equality between α_s and α_t can be verified by only finite values; in particular

$$\begin{aligned}\alpha_s = \alpha_t \;\Leftrightarrow\; & \big[s \sqsubseteq t \;\&\; (\forall i)[\mathrm{lh}(s) \leq i < \mathrm{lh}(t) \rightarrow \alpha_s(i) = \alpha_t(i)]\big] \vee \\ & \big[t \sqsubseteq s \;\&\; (\forall i)[\mathrm{lh}(t) \leq i < \mathrm{lh}(s) \rightarrow \alpha_s(i) = \alpha_t(i)]\big].\end{aligned}$$

Similarly if $\varepsilon_1(s) = 1$ and $\varepsilon_1(t) = 0$ then

$$\alpha_s = \alpha_t \Leftrightarrow \alpha_s = \alpha_{s_0}.$$

If $\varepsilon_1(s) = \varepsilon_1(t) = 0$ then clearly $\alpha_s = \alpha_{s_0} = \alpha_t$. From these remarks it follows that the relation R is recursive in ε_1. We now compute

$$\begin{aligned}P_<(s,t,m,k) \;\Leftrightarrow\; & p_{F1}(\alpha_s,\alpha_t) < \frac{m}{k+1} \\ \Leftrightarrow\; & [R(s,t) \;\&\; m > 0] \\ & \vee (\exists i)\big[\alpha_s(i) \neq \alpha_t(i) \;\&\; (\forall j < i)[\alpha_s(j) = \alpha_t(j)] \\ & \;\&\; k+1 < (i+1)m\big] \\ \Leftrightarrow\; & [R(s,t) \;\&\; m > 0] \vee (\exists i)[Q(s,t,i) \;\&\; k+1 < (i+1)m] \\ \Leftrightarrow\; & [R(s,t) \;\&\; m > 0] \vee (\forall i)[Q(s,t,i) \rightarrow k+1 < (i+1)m].\end{aligned}$$

Thus the relation $P_<$ is recursive in ε_1. It is clear that $P_\leq$ is recursive in ε_1 as well. Therefore the metric space (F_1, p_{F_1}) is recursively presented in ε_1. Since π_1 is an isometry between (F_1, p_{F_1}) and (A, d_A) it follows that the sequence $(\pi_1(\alpha_s))_{s\in\omega}$ is an ε_1-recursive presentation of (A, d_A). Similarly we define ε_2 for the metric space (A^c, d_{A^c}) with the ε_2-recursive presentation $(\pi_1(\alpha_s))_{s\in\omega}$. We take

$$\varepsilon_A = \langle \varepsilon_1, \varepsilon_2 \rangle$$

and we check that ε_A is recursive in O. Define

$$P_1(s) \Leftrightarrow F_1 \cap N(\mathcal{X}, s) \neq \emptyset \quad \text{and} \quad P_2(s) \Leftrightarrow F_2 \cap N(\mathcal{X}, s) \neq \emptyset.$$

Notice that

$$\begin{aligned}\varepsilon_A(\langle i, s\rangle) = 1 \;\Leftrightarrow\; & [i = 0 \;\&\; \varepsilon_1(s) = 1] \vee [i = 1 \;\&\; \varepsilon_2(s) = 1] \\ \Leftrightarrow\; & [i = 0 \;\&\; P_1(s)\;] \vee [i = 1 \;\&\; P_2(s) = 1].\end{aligned}$$

From this it follows that ε_A is recursive in a Σ^1_1 subset of ω. From the property (VII) mentioned in the Introduction we deduce that ε_A is recursive in Kleene's O.

We now define the topology $\mathcal{T}_\infty$ on $\mathcal{X}$ as the direct sum of (A, d_A) and (A^c, d_{A^c}), i.e. $V \in \mathcal{T}_\infty$ if and only if $V \cap A$ is d_A-open and $V \cap A^c$ is d_{A^c}-open. As we mentioned in the Introduction, $(\mathcal{X}, \mathcal{T}_\infty)$ is Polish and recursively presented in ε_A. A suitable dense sequence is

$$x_{\langle i,s\rangle} = \begin{cases} \pi_1(\alpha_s) & \text{if } i = 0, \\ \pi_2(\beta_s) & \text{if } i = 1, \end{cases}$$

and $x_t = \alpha_{s_0}$ in any other case.

Clearly A is $\Delta^0_1(\varepsilon_A)$ in $(\mathcal{X}, d_\infty)$, where d_∞ is the distance function which arises from the direct sum of (A, d_A) and (A^c, d_{A^c}). We prove that the topology $\mathcal{T}_\infty$ extends $\mathcal{T}$. Let V be in $\mathcal{T}$. We will show that $V \cap A$ is d_A-open. The proof for $V \cap A^c$ is similar. Let $y = \pi_1(\alpha) \in V \cap A$, with $\alpha \in F_1$. Since $\pi_1 : \mathcal{N} \to (\mathcal{X}, d)$ is continuous there is some $k \in \omega$ such that for all $\beta \in \mathcal{N}$ with $p_{\mathcal{N}}(\alpha, \beta) < 1/(k+1)$ we have $\pi_1(\beta) \in V$. We claim that the d_A-ball of center y and radius $1/(k+1)$ is contained in V. Let $z = \pi_1(\beta)$, with $\beta \in F_1$, be such that $d_A(y, z) < 1/(k+1)$. Then

$$p_{\mathcal{N}}(\alpha, \beta) = d_A(\pi_1(\alpha), \pi_1(\beta)) = d_A(y, z) < \frac{1}{k+1},$$

hence $z = \pi_1(\beta) \in V$.

Now we deal with the rest of the assertions. It is not hard to see that every recursive function $\pi : \mathcal{N} \to (\mathcal{X}, d)$ is $\Delta^1_1(\varepsilon_A)$-recursive as a function from $\mathcal{N}$ to $(\mathcal{X}, d_\infty)$ (by considering the standard recursive presentation of $\mathcal{N}$ as an ε_A-recursive presentation). It follows from [5, 4A.7 and 4D.7] that every Δ^1_1 subset of $(\mathcal{X}, d)$ is a $\Delta^1_1(\varepsilon_A)$ subset of $(\mathcal{X}, d_\infty)$. This settles assertion (3).

To prove assertion (4), suppose B is a $\Delta^1_1(\varepsilon_A)$ subset of $(\mathcal{X}, d_\infty)$. The set $\pi_1^{-1}[B \cap A] \subseteq F_1 \subseteq \mathcal{N}$ is in $\Delta^1_1(\varepsilon_A)$ since π_1 is recursive as a function from $\mathcal{N}$ to $(\mathcal{X}, d)$ and therefore $\Delta^1_1(\varepsilon_A)$-recursive as a function from $\mathcal{N}$ to $(\mathcal{X}, d_\infty)$. Since π_1 is one-to-one on $\pi_1^{-1}[B \cap A]$, again from [5, 4D.7] we deduce that $B \cap A = \pi_1[\pi_1^{-1}[B \cap A]]$ is a $\Delta^1_1(\varepsilon_A)$ subset of $(\mathcal{X}, d)$. Similarly we prove that $B \cap A^c$ is a $\Delta^1_1(\varepsilon_A)$ subset of $(\mathcal{X}, d)$ as well. Therefore $B = (B \cap A) \cup (B \cap A^c)$ is a $\Delta^1_1(\varepsilon_A)$ subset of $(\mathcal{X}, d)$. ∎

3. Computing the parameter. As we have seen, one can choose the parameter ε of Theorem 2.1 to be recursive in O. Since we are dealing mostly with Δ^1_1 sets, a natural question to ask is whether we can choose a hyperarithmetical such ε. We will see that the latter is not necessarily true even if we start with a Σ^0_1 set which we want to turn into Δ^0_1. Nevertheless we will show that in some cases, choosing a hyperarithmetical such ε is possible (cf. Theorem 3.9). Moreover we will see that, under some specific assumptions about the set we start with and about our underlying space, we can characterize the case where the choice of a hyperarithmetical ε is possible (cf. Theorem 3.11).

DEFINITION 3.1. Suppose that $(\mathcal{X}, \mathcal{T})$ is a recursively presented Polish space, d is a suitable distance function, and A is a Δ^1_1 subset of $\mathcal{X}$.

(1) We say that $\varepsilon \in \mathcal{N}$ is a *good parameter for* A if all conclusions of Theorem 2.1 are satisfied if we take ε_A as ε. The latter means that there is a Polish topology $\mathcal{T}_\infty$ on $\mathcal{X}$ which extends $\mathcal{T}$; the space $(\mathcal{X}, \mathcal{T}_\infty)$ is ε-recursively presented; the set A is $\Delta^0_1(\varepsilon)$ in $(\mathcal{X}, d_\infty)$, where d_∞ is a suitable distance function; every B which is a $\Delta^1_1(\alpha)$ subset of $(\mathcal{X}, d)$ is a $\Delta^1_1(\varepsilon, \alpha)$ subset of $(\mathcal{X}, d_\infty)$; and every B which is a $\Delta^1_1(\varepsilon, \alpha)$ subset of $(\mathcal{X}, d_\infty)$ is a $\Delta^1_1(\alpha, \varepsilon)$ subset of $(\mathcal{X}, d)$.

(2) We say that *the class* Δ^1_1 *is dense in* A if whenever some $V \in \mathcal{T}$ intersects A then V intersects A in a Δ^1_1 point of $(\mathcal{X}, d)$, i.e.

$$(\forall V \in \mathcal{T})\big[V \cap A \neq \emptyset \Rightarrow (\exists x \in \Delta^1_1[(\mathcal{X}, d)])[x \in V \cap A]\big].$$

The next proposition gives a necessary condition for a Δ^1_1 set to admit a good parameter in Δ^1_1.

PROPOSITION 3.2. *If a Δ^1_1 subset A of a recursively presented Polish space $(\mathcal{X}, \mathcal{T})$ admits a good parameter in Δ^1_1 then the class Δ^1_1 is dense both in A and in the complement $\mathcal{X} \setminus A$ with respect to $\mathcal{T}$.*

Proof. Let ε be a good parameter for A in Δ^1_1, $\mathcal{T}_\infty$ be the corresponding extension of the topology $\mathcal{T}$, and d and d_∞ be the corresponding suitable distance functions. Suppose that $V \in \mathcal{T}$ intersects A. Since $\mathcal{T}_\infty$ extends $\mathcal{T}$ we find that $V \in \mathcal{T}_\infty$. From the choice of ε and $\mathcal{T}_\infty$ we also deduce that $A \in \mathcal{T}_\infty$. Thus $V \cap A$ is a non-empty $\mathcal{T}_\infty$-open set. Since $(\mathcal{X}, \mathcal{T}_\infty)$ is recursively presented in ε, the set $V \cap A$ contains a point, say x_0, which is recursive in ε with respect to $(\mathcal{X}, d_\infty)$. It follows that $\{x_0\}$ is a $\Delta^1_1(\varepsilon)$ subset of $(\mathcal{X}, d_\infty)$, and therefore $\{x_0\}$ is a $\Delta^1_1(\varepsilon)$ subset of $(\mathcal{X}, d)$. Thus x_0 is a $\Delta^1_1(\varepsilon)$ point of $(\mathcal{X}, d)$. Since $\varepsilon \in \Delta^1_1$ we conclude that x_0 is a Δ^1_1 point of $(\mathcal{X}, d)$.

The proof for $\mathcal{X} \setminus A$ is similar. ■

REMARK 3.3. It follows from the previous proposition that a Δ^1_1 set A may not admit a good parameter in Δ^1_1. To see this take $\mathcal{X} = \mathcal{N}$ and A any non-empty Π^0_1 set with no Δ^1_1 members.

Our next goal is to find sufficient conditions under which a given Δ^1_1 set admits a good parameter in Δ^1_1. We will work inside a special category of spaces.

DEFINITION 3.4. A recursively presented Polish space $(\mathcal{X}, \mathcal{T})$ is *recursively zero-dimensional* if there is a distance function d on $\mathcal{X}$ which generates the topology $\mathcal{T}$, witnesses that $(\mathcal{X}, \mathcal{T})$ is recursively presented with recursive presentation the sequence $(r_i)_{i \in \omega}$, and the relation $I \subseteq \mathcal{X} \times \omega \times \omega$ defined by

$$I(x, i, s) \;\Leftrightarrow\; d(x, r_i) < q_s$$

is recursive. By replacing "recursive" with "ε-recursive" one defines the notion of an *ε-recursively zero-dimensional space.*

It is well-known that every zero-dimensional Polish space is topologically isomorphic to a closed subset of the Baire space (cf. [1, 7.8]). The next lemma is the effective analogue of this statement.

LEMMA 3.5. *For every recursively zero-dimensional Polish space $\mathcal{X}$ there is a recursive injection $f : \mathcal{X} \to \mathcal{N}$ such that the set $\mathcal{Y} := f[\mathcal{X}]$ is in $\Pi^0_1(\varepsilon)$ for some $\varepsilon \in \Delta^0_2$. Moreover the inverse function $f^{-1} : \mathcal{Y} \to \mathcal{X}$ is computed by a semirecursive subset of $\mathcal{N} \times \omega \times \omega$ on $\mathcal{Y}$, i.e. there is a semirecursive $R \subseteq \mathcal{N} \times \omega \times \omega$ such that for all $\alpha \in \mathcal{Y}$ we have*

$$d(f^{-1}(\alpha), r_i) < q_s \;\Leftrightarrow\; R(\alpha, i, s).$$

In particular the inverse function f^{-1} is continuous.

Proof. Fix a suitable distance function d on $\mathcal{X}$. By replacing d with $d/(1+d)$ we may assume that $d \leq 1$. First we claim that there is a recursive relation $A \subseteq \mathcal{X} \times \mathtt{Seq}$ such that:

(1) $A_0 = \mathcal{X}$,
(2) $A_s = \bigcup_{n\in\omega} A_{s\hat{\ }n}$,
(3) $A_{s\hat{\ }n} \cap A_{s\hat{\ }m} = \emptyset$ for all $n \neq m$, and
(4) each A_s has diameter less than $1/2^{\mathrm{lh}(s)}$,

where $A_s = \{x \in \mathcal{X} \mid A(x, s)\}$. Notice that each A_s is a clopen set and that we do not exclude the case $A_s = \emptyset$.

We now prove the claim. For all $i, n \in \omega$ we denote by $N(i, n)$ the open d-ball with center r_i and radius $1/(n + 1)$, so that the relation $I \subseteq \mathcal{X} \times \omega \times \omega$ defined by

$$I(x, i, n) \;\Leftrightarrow\; x \in N(i, n)$$

is recursive. Using Kleene's Recursion Theorem we obtain a recursive function $b : \mathcal{X} \times \mathtt{Seq} \to 2$ such that $b(x, 0) = 1$ for all $x \in \mathcal{X}$ and

$$b(x, s\hat{\ }k) = 1 \;\Leftrightarrow\; b(x, s) = 1 \;\&\; x \in N(k, 2^{\mathrm{lh}(s)+2}) \;\&\; b(r_k, s) = 1$$

for all $x \in \mathcal{X}$, $s \in \mathtt{Seq}$ and $k \in \omega$, so that if we define

$$B_s = \{x \in \mathcal{X} \mid b(x, s) = 1\}$$

then $B_0 = \mathcal{X}$ and

$$B_{s\hat{\ }k} = \begin{cases} B_s \cap N(k, 2^{\mathrm{lh}(s)+2}) & \text{if } r_k \in B_s, \\ \emptyset & \text{otherwise} \end{cases}$$

for all $s \in \mathtt{Seq}$ and $k \in \omega$. It is clear that the diameter of B_s is less than $2^{-\mathrm{lh}(s)}$, and B_s is a clopen set. Also, $B_s = \bigcup_{n\in\omega} B_{s\hat{\ }n}$. To see this let $x \in B_s$; since B_s is open we can choose $r_k \in B_s$ with $d(x, r_k) < (2^{\mathrm{lh}(s)+2} + 1)^{-1}$.

Then $x \in B_s \cap N(k, 2^{\mathrm{lh}(s)+2}) = B_{s^\frown k}$. We apply the Recursion Theorem one more time to get a recursive set $A \subseteq \mathcal{X} \times \mathtt{Seq}$ such that $A_0 = \mathcal{X}$ and

$$A_{s^\frown k} = \Big(B_{s^\frown k} \setminus \bigcup_{i<k} B_{s^\frown i}\Big) \cap A_s$$

for all $s \in \mathtt{Seq}$ and $k \in \omega$, where A_s is the sth section of A. It is clear that conditions (1)–(4) are satisfied for this family $(A_s)_{s\in\mathtt{Seq}}$.

Having proved our claim we define $f : \mathcal{X} \to \mathcal{N}$ as follows:

$$\begin{aligned} f(x)(n) = {} & \text{the unique } i \text{ for which there is (a unique) } s \in \mathtt{Seq} \text{ such that} \\ & x \in A_{s^\frown i} \text{ and } \mathrm{lh}(s) = n, \end{aligned}$$

for all $x \in \mathcal{X}$ and $n \in \omega$. It is easy to check that $f(x)\restriction n$ is the unique $s \in \mathtt{Seq}$ of length n such that $x \in A_s$, and that $f(x)(n)$ is the unique i such that $x \in A_{f(x)\restriction n\,^\frown (i)}$. Clearly the function f is recursive and injective.

We now define $\varepsilon(s) = 1$ exactly when there exists i such that $r_i \in A_s$, and 0 otherwise. It is clear that $\varepsilon \in \Delta^0_2$. Since each A_s is clopen we see that $\varepsilon(s) = 1$ exactly when $A_s \neq \emptyset$. Moreover one can verify that

$$\alpha \in f[\mathcal{X}] \iff (\forall n)[A_{\alpha\restriction n} \neq \emptyset] \iff (\forall n)[\varepsilon(\alpha\restriction n) = 1].$$

So the set $\mathcal{Y} := f[\mathcal{X}]$ is a $\Pi^0_1(\varepsilon)$ subset of $\mathcal{N}$.

Finally we prove the assertion about f^{-1}. Suppose that $\alpha \in \mathcal{Y}$ and $x = f^{-1}(\alpha)$. We claim that

$$d(x, r_i) < q_s \iff (\exists n, j)[r_j \in A_{\alpha\restriction n} \ \&\ d(r_j, r_i) < q_s - 1/2^n].$$

For the left-to-right direction we choose natural numbers n, j such that $2/2^n < q_s - d(x, r_i)$ and $r_j \in A_{\alpha\restriction n}$. (The set $A_{\alpha\restriction n}$ is non-empty since $\alpha \in \mathcal{Y}$.) As both r_j and $x = f^{-1}(\alpha)$ are members of $A_{\alpha\restriction n}$ we have $d(r_j, x) < 1/2^n$. Then

$$d(r_j, r_i) \leq d(r_j, x) + d(x, r_i) < 1/2^n + d(x, r_i) < q_s - 1/2^n.$$

For the right-to-left direction we use $x = f^{-1}(\alpha) \in A_{\alpha\restriction n}$ for all n to compute

$$d(x, r_i) \leq d(x, r_j) + d(r_j, r_i) < 1/2^n + q_s - 1/2^n = q_s.$$

Thus the equivalence is proved. Take now

$$R(\alpha, i, s) \iff (\exists n, j)[r_j \in A_{\alpha\restriction n} \ \&\ d(r_j, r_i) < q_s - 1/2^n],$$

and we are done. ∎

It would be interesting to see if the parameter ε in the previous proof can be chosen to be recursive.

Remark 3.6. Suppose that $\mathcal{X}$ is a recursively zero-dimensional Polish space and that $\mathcal{Y}$ and $f : \mathcal{X} \to \mathcal{Y}$ are as in the previous lemma.

(1) Consider a recursive presentation for $\mathcal{X}$, say $(x_i)_{i\in\omega}$, and define

$$R(i, j) \iff x_i \neq x_j.$$

The metric space $(\mathcal{Y}, p_{\mathcal{N}})$ admits a recursive presentation. To see this define $y_i = f(x_i)$ for all $i \in \omega$ and notice that the sequence $(y_i)_{i\in\omega}$ is dense in $\mathcal{Y}$. Moreover for all i, j with $x_i \neq x_j$ there is exactly one triple (u, n, m), where u is a finite sequence of naturals and $n \neq m \in \omega$, such that $x_i, x_j \in A_u$, $x_i \in A_{u\hat{\ }(n)}$ and $x_j \in A_{u\hat{\ }(m)}$. So the least natural k for which $f(x_i)(k) \neq f(x_j)(k)$ is exactly the length of the finite sequence u. We define $g(i,j) =$ *the length of* u if $R(i,j)$, and $g(i,j) = 0$ otherwise. The function g is recursive and from the definition of $p_{\mathcal{N}}$ it is clear that $p_{\mathcal{N}}(y_i, y_j) = p_{\mathcal{N}}(f(x_i), f(x_j)) = (g(i,j)+1)^{-1}$ for $x_i \neq x_j$.

(2) Consider the parameter $\varepsilon \in \Delta^0_2$ of the proof of the previous lemma, so that $\mathcal{Y}$ is in $\Pi^0_1(\varepsilon)$. For every $A \subseteq \mathcal{X}$, if A is in Π^0_1 the set $f[A]$ is a $\Pi^0_1(\varepsilon)$ subset of $\mathcal{N}$, and if A is in Σ^0_{n+1} the set $f[A]$ is a Σ^0_{n+1} subset of $\mathcal{N}$ for all $n \geq 1$. To see this consider for all $k \in \omega$ the space $\mathcal{X} \times \omega^k$ (for $k = 0$ we just mean the space $\mathcal{X}$) and the function $\tilde{f} : \mathcal{X} \times \omega^k \to \mathcal{N} \times \omega^k$ defined by $\tilde{f}(x, \vec{z}) = (f(x), \vec{z})$. Since $\mathcal{Y}$ is a $\Pi^0_1(\varepsilon)$ subset of $\mathcal{N}$ one can show that for every $k \in \omega$ and for every $A \subseteq \mathcal{X} \times \omega^k$ in Π^0_1 the set $\tilde{f}[A]$ is a $\Pi^0_1(\varepsilon)$ subset of $\mathcal{N} \times \omega^k$. From this and the fact that $\varepsilon \in \Delta^0_2$ one can show, by induction on $n \geq 1$, that for all $k \in \omega$ and all $A \subseteq \mathcal{X} \times \omega^k$ in Σ^0_{n+1} the set $\tilde{f}[A]$ is a Σ^0_{n+1} subset of $\mathcal{N} \times \omega^k$.

The proof of Theorem 1.1 (cf. [5]) shows that every Δ^1_1 subset of $\mathcal{N}$ is in fact the injective image under the projection of a Π^0_1 subset of $\mathcal{N} \times \mathcal{N}$. Thus in the case of Δ^1_1 subsets of $\mathcal{N}$ the function π of Theorem 1.1 can be chosen to be an open mapping, i.e. to carry open sets to open sets. But since π is injective on F and not necessarily injective on $\mathcal{N}$, we cannot conclude that the inverse function $\pi^{-1} : A \to F$ is continuous. For reasons that will become clear later on, the continuity of the inverse function will be necessary for our purposes. This property though puts a limitation on the Δ^1_1 set we start with.

REMARK 3.7. Suppose that $A \subseteq \mathcal{N}$ is in Δ^1_1 and that there is a Π^0_1 subset of $\mathcal{N}$, say F, and a recursive function $\pi : F \to \mathcal{N}$ such that $\pi[F] = A$, π is injective on F and the inverse function $\pi^{-1} : A \to F$ is continuous. Then A is a G_δ subset of $\mathcal{N}$.

To see why the latter holds define the distance function d on A by $d(\alpha, \beta) = p_{\mathcal{N}}(\pi^{-1}(\alpha), \pi^{-1}(\beta))$ for all $\alpha, \beta \in A$. Since both functions π and π^{-1} are continuous it follows that the topology on A induced by d is exactly $\{V \cap A \mid V \text{ open in } \mathcal{N}\}$, i.e. the relative topology of $\mathcal{N}$ on A. Moreover since F is closed it follows that (A, d) is Cauchy-complete. Therefore the set A with the relative topology induced by $\mathcal{N}$ is a Polish space. It follows from Theorem 3.11 in [1] that A is a G_δ set.

What is perhaps more important is that the previous remark has a converse.

Lemma 3.8. *For every $A \subseteq \mathcal{N}$ in Π^0_2 there is an $F \subseteq \mathcal{N} \times \mathcal{N}$ in Π^0_1 such that*

$$A(\alpha) \iff (\exists \beta)F(\alpha,\beta) \iff (\exists \text{ unique } \beta)F(\alpha,\beta)$$

for all $\alpha \in \mathcal{N}$, and the function $\mathrm{pr}^{-1} : A \to F$, $\mathrm{pr}^{-1}(\alpha) = (\alpha,\beta)$, where β is such that $F(\alpha,\beta)$, is continuous.

As usual there is a similar version of Lemma 3.8 with respect to some parameter α. For reasons of exposition we refrain from stating this parameterized version.

Proof of Lemma 3.8. Since A is in Π^0_2, using [5, 3C.4] there is a recursive $R \subseteq \mathcal{N} \times \omega \times \omega$ such that

$$A(\alpha) \iff (\forall n)(\exists m)R(\alpha,n,m)$$

for all $\alpha \in \mathcal{N}$. Define

$$F(\alpha,\beta) \iff (\forall n)[R(\alpha,n,\beta(n)) \ \& \ (\forall k < \beta(n))\neg R(\alpha,n,k)]$$

for all $\alpha,\beta \in \mathcal{N}$. It is clear that F is in Π^0_1 and that for all $\alpha \in \mathcal{N}$ there is at most one β such that $F(\alpha,\beta)$. Moreover

$$A(\alpha) \iff (\exists \beta)F(\alpha,\beta)$$

for all $\alpha,\beta \in \mathcal{N}$. Now we prove that the inverse function $\mathrm{pr}^{-1} : A \to F$, $\mathrm{pr}^{-1}(\alpha) = (\alpha,\beta)$, is continuous. Clearly it suffices to prove that the function $g : A \to \mathcal{N}$, $g(\alpha) = \beta$, where β is such that $F(\alpha,\beta)$, is continuous. So take any α and β with $F(\alpha,\beta)$ and $N \in \omega$. For all $n \in \omega$ we consider the set

$$V_n = \{\alpha' \mid R(\alpha',n,\beta(n)) \ \& \ (\forall k < \beta(n))\neg R(\alpha',n,k)\}.$$

Since $F(\alpha,\beta)$ and R is recursive the set $V = \bigcap_{n<N} V_n$ is an open neighborhood of α. It is then clear that for all $\alpha' \in V$ and all β' with $F(\alpha',\beta')$ we have $\beta' \restriction N = \beta \restriction N$. ∎

We are now ready for the next theorem.

Theorem 3.9. *Suppose that $(\mathcal{X},\mathcal{T})$ is a recursively zero-dimensional Polish space with no isolated points and that A is a countable Δ^1_1 subset of $\mathcal{X}$. Then there exists an $\varepsilon \in \Delta^1_1$ and a Polish topology $\mathcal{T}_\infty$ with suitable distance function d_∞, which extends $\mathcal{T}$ and has the following properties:*

1. *The Polish space $(\mathcal{X},\mathcal{T}_\infty)$ is ε-recursively presented.*
2. *The set A is a $\Delta^0_1(\varepsilon)$ subset of $(\mathcal{X},d_\infty)$.*
3. *Every $B \subseteq \mathcal{X}$ is a $\Delta^1_1(\alpha)$ subset of $(\mathcal{X},d)$, where d is a suitable distance function for $(\mathcal{X},\mathcal{T})$ and $\alpha \in \mathcal{N}$, exactly when B is a $\Delta^1_1(\varepsilon,\alpha)$ subset of $(\mathcal{X},d_\infty)$.*

Proof. For notational purposes we interchange A with $\mathcal{X} \setminus A$, for it is the case of a co-countable Δ^1_1 set which is interesting. So let us assume that

$\mathcal{X} \setminus A$ is countable. Since A has a countable complement and it is Δ^1_1 there is a Δ^1_1 sequence $(x_n)_{n\in\omega}$ in $\mathcal{X}$ such that $\mathcal{X} \setminus A = \{x_n \mid n \in \omega\}$. Using this it is easy to find some $\varepsilon_D \in \Delta^1_1$ such that $\mathcal{X} \setminus A$ is in $\Sigma^0_2(\varepsilon_D)$ and so A is in $\Pi^0_2(\varepsilon_D)$. To see this we remark that $x \notin A \Leftrightarrow (\exists n)(\forall s)[x \in N(\mathcal{X}, s) \to x_n \in N(\mathcal{X}, s)]$. Define then $\varepsilon_D(\langle n, s\rangle) = 1 \Leftrightarrow x_n \in N(\mathcal{X}, s)$, and 0 otherwise.

We first show how one can reduce the problem to subspaces of $\mathcal{N}$. We consider the embedding $f : \mathcal{X} \to \mathcal{N}$ of Lemma 3.5 and an $\varepsilon_0 \in \Delta^0_2$ such that the set $\mathcal{Y} := f[\mathcal{X}]$ is in $\Pi^0_1(\varepsilon_0)$. We take $\varepsilon^* = \langle \varepsilon_0, \varepsilon_D\rangle \in \Delta^1_1$ and using (the parameterized version of) Remark 3.6 we deduce that $f[A]$ is a $\Pi^0_2(\varepsilon^*)$ subset of $\mathcal{N}$.

We consider the set $\mathcal{Y}$ with the usual distance function $p_{\mathcal{N}}$ of the Baire space. From Remark 3.6 the space $(\mathcal{Y}, p_{\mathcal{N}})$ is recursively presented in some arithmetical parameter, which we may assume that we have included in the previous ε^*. Moreover the space $(\mathcal{Y}, p_{\mathcal{N}})$ has no isolated points for otherwise $(\mathcal{X}, d)$ would have had isolated points. The set $\mathcal{Y} \setminus f[A] = f[\mathcal{X} \setminus A]$ is a countable subset of $\mathcal{Y}$ and since $\mathcal{Y}$ has no isolated points it follows that $f[A]$ is co-meager in $\mathcal{Y}$.

Thus we have an $\varepsilon^* \in \Delta^1_1$ such that the space $(\mathcal{Y}, p_{\mathcal{N}})$ is recursively presented in ε^*, the set $f[A]$ is a $\Pi^0_2(\varepsilon^*)$ subset of $\mathcal{N}$, and $f[A]$ is co-countable and co-meager in $\mathcal{Y}$. We will show that there is a Polish topology $\mathcal{S}_\infty$ on $\mathcal{Y}$ with suitable distance function p_∞ and an $\varepsilon \in \Delta^1_1$ such that all conclusions are satisfied for this ε and for $\mathcal{Y}$, p_∞, $\mathcal{S}_\infty$ and $f[A]$ in place of $\mathcal{X}$, d_∞, $\mathcal{T}_\infty$ and A respectively. Then we can return to $\mathcal{X}$ using the function f. More specifically we define $d_\infty(x, y) = p_\infty(f(x), f(y))$ for all $x, y \in \mathcal{X}$, and we define an ε-recursive presentation for $(\mathcal{X}, d_\infty)$ via the ε-recursive presentation of $(\mathcal{Y}, p_\infty)$. Therefore we turn f into an ε-recursive isometry. All conclusions are then satisfied.

We now go back to the proof of Theorem 2.1 and we show that the ε_1 and ε_2 defined there are in fact in Δ^1_1. Since $f[A]$ is a $\Pi^0_2(\varepsilon^*)$ subset of $\mathcal{N}$, from the parameterized version of Lemma 3.8 there is an $F_1 \subseteq \mathcal{N} \times \mathcal{N}$ in $\Pi^0_1(\varepsilon^*)$ and a function $\pi_1 : \mathcal{N} \times \mathcal{N} \to \mathcal{N}$ such that:

- π_1 is ε^*-recursive,
- π_1 is injective on F_1,
- $\pi_1[F_1] = f[A]$, and
- the function $\pi_1^{-1} : f[A] \to F_1$ is continuous.

In order to match the notation of the proof of Theorem 2.1 we view F_1 as a subset of $\mathcal{N}$.

We now show that the ε_1 of the proof of Theorem 2.1 is in Δ^1_1. Define the relation $R_1 \subseteq \texttt{Seq}$ by

$$R_1(s) \;\Leftrightarrow\; N_s \cap F_1 \neq \emptyset,$$

where $N_s = \{\alpha \in \mathcal{N} \mid \alpha(i) = (s)_i \ \forall i < \mathrm{lh}(s)\}$ for all $s \in \mathtt{Seq}$. We need to show that R_1 is Δ^1_1. Since $\varepsilon^* \in \Delta^1_1$ and F_1 is in $\Pi^0_1(\varepsilon^*)$ it is clear that R_1 is in Σ^1_1. We now claim that

$$R_1(s) \iff (\exists \alpha \in \Delta^1_1)[\alpha \in N_s \cap F_1].$$

By the Theorem on Restricted Quantification [5, 4D.3] it follows from the latter equivalence that R_1 is in Π^1_1. The right-to-left implication is clear, so assume that $R_1(s)$. Pick some $\alpha_0 \in N_s \cap F_1$. In particular $\alpha_0 = \pi_1^{-1}(y_0)$ for some $y_0 \in f[A]$. Since the function $\pi_1^{-1} : f[A] \to F_1$ is continuous there is some basic open (and thus recursive) neighborhood $V \subseteq \mathcal{N}$ such that $y_0 \in V$ and $\pi_1^{-1}(y) \in N_s \cap F_1$ for all $y \in V \cap f[A]$. Since $f[A]$ is co-meager in $\mathcal{Y}$ and $V \cap \mathcal{Y}$ is non-empty and open in $\mathcal{Y}$ we see that $V \cap f[A]$ is non-meager in $\mathcal{Y}$. Moreover $V \cap f[A]$ is a $\Pi^0_2(\varepsilon^*)$—and thus $\Pi^1_1(\varepsilon^*)$—subset of $\mathcal{Y}$. From the parameterized version of [5, 4F.20] there is some $y \in V \cap f[A]$ which is a $\Delta^1_1(\varepsilon^*)$ point of $\mathcal{Y}$. Since $\varepsilon^* \in \Delta^1_1$ it is easy to check that y is a Δ^1_1 point of $\mathcal{N}$ and so $\alpha = \pi_1^{-1}(y) \in N_s \cap F_1$ is also in Δ^1_1. Thus the equivalence is proved.

Now we deal with the complement of $f[A]$ in $\mathcal{Y}$ and subsequently with ε_2. From Theorem 1.1 there is a set $F_2 \subseteq \mathcal{N}$ in Π^0_1 and a recursive function $\pi_2 : \mathcal{N} \to \mathcal{X}$ such that $\pi_2[F_2] = \mathcal{Y} \setminus f[A]$ and π_2 is injective on F_2. Since $\mathcal{Y} \setminus f[A]$ is countable, so is F_2. It follows from the Effective Perfect Set Theorem [5, 4F.1] that F_2 consists of Δ^1_1 points. Hence

$$N_s \cap F_2 \neq \emptyset \iff (\exists \alpha \in \Delta^1_1)[\alpha \in N_s \cap F_2].$$

From this it follows that $\varepsilon_2 \in \Delta^1_1$. Now we define $\varepsilon = \langle \varepsilon^*, \varepsilon_1, \varepsilon_2 \rangle \in \Delta^1_1$ and we continue as in the proof of Theorem 2.1. ∎

Notice that in the previous proof there is no need to choose the function π_2 so that its inverse function is continuous. The reason is our countability hypothesis which ensures that the corresponding closed set F_2 consists of Δ^1_1 points. Suppose for the moment that we drop the countability hypothesis in favor of A and its complement being G_δ and Δ^1_1 sets. Assume moreover that Δ^1_1 is dense in both A and $\mathcal{X} \setminus A$. Then from (the parameterized version of) Lemma 3.8 we can always choose the above functions π_1 and π_2 so that their inverses are continuous functions and then we can repeat the previous proof. The only problem with this idea is that we have to make sure that our application of Lemma 3.8 stays within the level of hyperarithmetical parameters. Under the countability hypothesis that is clear, for a countable Δ^1_1 set is easily in $\Sigma^0_2(\varepsilon^*)$ for some hyperarithmetical ε^*. But now it is not so clear that the G_δ sets A and $\mathcal{X} \setminus A$ we start with (which are also in Δ^1_1) are in fact in $\Pi^0_2(\varepsilon^*)$ for some hyperarithmetical ε^*. The latter assertion is true thanks to a deep result of Louveau (cf. [4]).

THEOREM 3.10 (Louveau). *Suppose that $\mathcal{X}$ is a recursively presented Polish space and that $A \subseteq \mathcal{X}$ is Δ_1^1 and $\underset{\sim}{\mathbf{\Pi}}_\xi^0$ for some $\xi < \omega_1$. Then there exists an $\varepsilon \in \Delta_1^1$ such that A is in $\Pi_\xi^0(\varepsilon)$.*

We can now proceed to the next theorem.

THEOREM 3.11. *Suppose that $(\mathcal{X}, \mathcal{T})$ is a recursively zero-dimensional Polish space and that A is a Δ_1^1 subset of $\mathcal{X}$ which is also in $\underset{\sim}{\mathbf{\Delta}}_2^0$. Then:*

(1) *The set A admits a good parameter in Δ_1^1 if and only Δ_1^1 is dense in both A and $\mathcal{X} \setminus A$.*
(2) *If A does not admit a good parameter in Δ_1^1, then O is hyperarithmetic in every good parameter for A.*

Proof. From Louveau's Theorem there is an $\varepsilon_D \in \Delta_1^1$ such that both A and $\mathcal{X} \setminus A$ are in $\Pi_2^0(\varepsilon_D)$. We consider the embedding $f : \mathcal{X} \to \mathcal{N}$ of Lemma 3.5 and an $\varepsilon_0 \in \Delta_2^0$ such that the set $\mathcal{Y} := f[\mathcal{X}]$ is in $\Pi_1^0(\varepsilon_0)$. Consider also an arithmetical ε_0' such that the space $(\mathcal{Y}, p_{\mathcal{N}})$ admits an ε_0'-recursive presentation and define $\varepsilon^* = \langle \varepsilon_D, \varepsilon_0, \varepsilon_0' \rangle$. It is clear that $\varepsilon^* \in \Delta_1^1$, the space $(\mathcal{Y}, p_{\mathcal{N}})$ is ε^*-recursively presented and the sets $f[A]$ and $\mathcal{Y} \setminus f[A]$ are in $\Pi_2^0(\varepsilon^*)$.

As pointed out in the proof of Theorem 3.9 every good parameter for $f[A]$ (as a subset of $\mathcal{Y}$) is also a good parameter for A, for we can define the new distance function on $\mathcal{X}$ in such a way that the function f becomes an isometry and recursive with respect to this good parameter. Using the same method one can see that the converse is also true modulo the parameter ε^*, i.e. if ε is a good parameter for A then $\langle \varepsilon^*, \varepsilon \rangle$ is a good parameter for $f[A]$. Thus A admits a good parameter in Δ_1^1 exactly when $f[A]$ does.

We apply the parameterized version of Lemma 3.8 to get sets $F_i \subseteq \mathcal{N}$ in $\Pi_1^0(\varepsilon^*)$ and ε^*-recursive functions $\pi_i : \mathcal{N} \to \mathcal{X}$, $i = 1, 2$, such that:

- π_i is injective on F_i for $i = 1, 2$,
- $\pi_1[F_1] = f[A]$, $\pi_2[F_2] = \mathcal{Y} \setminus f[A]$, and
- the inverses $\pi_1^{-1} : f[A] \to F_1$ and $\pi_2^{-1} : \mathcal{Y} \setminus f[A] \to F_2$ are continuous.

We are now ready to prove (1). The left-to-right implication is Proposition 3.2, so we prove the converse. Consider the previous functions π_i and the sets F_i, $i = 1, 2$. We repeat the steps of the proof of Theorem 2.1. Since Δ_1^1 is dense in A and $\mathcal{X} \setminus A$ and since f is recursive and injective, it is easy to check that Δ_1^1 is dense both in $f[A]$ and in $\mathcal{Y} \setminus f[A]$. Using the density argument as in the previous proof we can see that the parameters ε_1 and ε_2 (with the notation of the proof of Theorem 2.1) are in fact in Δ_1^1. Thus the fraction $\varepsilon = \langle \varepsilon^*, \varepsilon_1, \varepsilon_2 \rangle$ is a good parameter for $f[A]$ which is in Δ_1^1 and so from the previous comments A admits a good parameter in Δ_1^1.

Now let us prove (2). Since A (and hence $f[A]$) does not admit a good parameter in Δ^1_1, at least one of the relations $R_1, R_2 \subseteq \mathtt{Seq}$ defined by

$$R_i(s) \iff N_s \cap F_i \neq \emptyset,$$

where $i = 1, 2$ and $N_s = \{\alpha \in \mathcal{N} \mid \alpha(j) = (s)_j \ \forall j < \mathrm{lh}(s)\}$, say R_1, is not in Δ^1_1. (For otherwise both ε_1 and ε_2 would be in Δ^1_1 and so as above $f[A]$ would admit a good parameter in Δ^1_1.) It is clear that R_1 is a Σ^1_1 set. From the property (VIII) of the Introduction it follows that R_1 has the same hyperdegree as the one of Kleene's O and hence so does its characteristic function ε_1. Now let ε be any good parameter for A. We will prove that R_1 is in $\Delta^1_1(\varepsilon)$.

Notice that $\langle \varepsilon^*, \varepsilon \rangle$ is a good parameter for $f[A]$. We claim that

$$R_1(s) \iff (\exists \alpha \in \Delta^1_1(\varepsilon))[\alpha \in N_s \cap F_1].$$

To see this, assume that $N_s \cap F_1 \neq \emptyset$; therefore $\pi_1[N_s \cap F_1] \neq \emptyset$. Since the function $\pi_1^{-1} : f[A] \to F_1$ is continuous the set $\pi_1[N_s \cap F_1]$ is open in $f[A]$. Hence there is an open $V \subseteq \mathcal{N}$ such that $\pi_1[N_s \cap F_1] = V \cap f[A] \neq \emptyset$. The set $V \cap f[A]$ is open in the extended topology, which is $\langle \varepsilon^*, \varepsilon \rangle$-recursively presented, and therefore it contains a point, say y_0, which is recursive in $\langle \varepsilon^*, \varepsilon \rangle$. It follows that the singleton $\{y_0\}$ is a $\Delta^1_1(\langle \varepsilon^*, \varepsilon \rangle)$ subset of $(\mathcal{Y}, p_\infty)$, where p_∞ is a suitable distance function for the extended topology. Therefore $\{y_0\}$ is a $\Delta^1_1(\langle \varepsilon^*, \varepsilon \rangle)$ subset of $(\mathcal{Y}, p_{\mathcal{N}})$. Since $y_0 \in V \cap f[A] = \pi_1[N_s \cap F_1]$ there is some $\alpha_0 \in N_s \cap F_1$ such that $y_0 = \pi_1(\alpha_0)$. Using the fact that π_1 is injective on F_1 we deduce for $\beta \in \mathcal{N}$ that

$$\beta = \alpha_0 \iff \beta \in F_1 \ \& \ \pi_1(\beta) \in \{y_0\}.$$

Since the function π_1 is recursive it follows from the above equivalence that the point α_0 is in $\Delta^1_1(\langle \varepsilon^*, \varepsilon \rangle) = \Delta^1_1(\varepsilon)$ and so the claim has been proved.

From the Theorem on Restricted Quantification [5, 4D.3] the set R_1 is in $\Pi^1_1(\varepsilon)$. Since R_1 is in Σ^1_1 we conclude that R_1 is in $\Delta^1_1(\varepsilon)$. ∎

COROLLARY 3.12. *Suppose that $(\mathcal{X}, \mathcal{T})$ is a recursively zero-dimensional Polish space and that A is a Δ^1_1 subset of $\mathcal{X}$ which is also in $\underset{\sim}{\mathbf{\Delta}}^0_2$. Then A admits a good parameter which either is in Δ^1_1 or has the same hyperdegree as O.*

Proof. This is immediate from Theorems 2.1 and 3.11. ∎

It would be interesting to see if the previous theorem can be extended to sets which are above the level of $\underset{\sim}{\mathbf{\Delta}}^0_2$ sets.

Before we close this section it is perhaps worth mentioning that the idea of using the Π^0_1 sets F_i and the recursive functions π_i is not always the best way to compute the complexity of the parameter, for in some cases it is easier to use directly the properties of the set we start with and of the underlying space.

Proposition 3.13. *Suppose that $(\mathcal{X}, \|\cdot\|)$ is a separable vector space (real or complex). Consider the induced distance function $d(x,y) = \|x-y\|$, $x, y \in \mathcal{X}$, and assume that the metric space $(\mathcal{X}, d)$ is recursively presented. Then every basic open ball of $\mathcal{X}$ admits a good parameter in Δ^1_1.*

Proof. It is enough to prove the conclusion for $A = \{x \in \mathcal{X} \mid d(x,0) < 1\}$. Define

$$d_A(x,y) = d(x,y) + \left| \frac{1}{d(x,A^c)} - \frac{1}{d(y,A^c)} \right|$$

for all $x, y \in A$. Then d_A is a distance function on A which generates the same topology as the restriction of the topology of the norm on A (cf. the proof of Theorem 3.11 in [1]). Moreover the metric space (A, d_A) is complete. It is easy to find a Δ^1_1 sequence $(x_n)_{n\in\omega}$ in A such that the set $\{x_n \mid n \in \omega\}$ is dense in A. Then $(x_n)_{n\in\omega}$ is a presentation of (A, d_A) which is recursive in some hyperarithmetical parameter.

Now we deal with the complement of A. Let d_{A^c} be the restriction of the distance function d to $\mathcal{X} \setminus A$. Since the latter set is closed it follows that $(\mathcal{X} \setminus A, d_{A^c})$ is complete and separable. Moreover it admits a presentation which is recursive in some hyperarithmetical parameter, for there is a Δ^1_1 sequence $(y_n)_{n\in\omega}$ with $\|y_n\| > 1$ for all $n \in \omega$, which is dense in $\mathcal{X} \setminus A$. Finally consider the direct sum $(A, d_A) \oplus (\mathcal{X} \setminus A, d_{A^c})$. ∎

4. Uniformity results. We conclude this article with the uniformity result mentioned in the Introduction.

Definition 4.1. We denote by $\mathrm{Df}(\omega)$ the set of all distance functions on ω and by $[(\omega, d)]$ the completion of the metric space (ω, d), where $d \in \mathrm{Df}(\omega)$. It is clear that every complete and separable metric space which is an infinite set is isometric to a space of the form $[(\omega, d)]$ for some $d \in \mathrm{Df}(\omega)$, so the set $\mathrm{Df}(\omega)$ can characterize up to isometry all complete and separable infinite metric spaces. Every $d \in \mathrm{Df}(\omega)$ can be encoded by a member of $\mathcal{N}$, for d is just a double sequence of real numbers. In our case it is enough to work with distance functions on ω which take only rational values; these are a bit shorter to encode.

Suppose that $d \in \mathrm{Df}(\omega)$ takes only rational values. We define $\beta_d \in \mathcal{N}$ as follows: $\beta_d(\langle i,j,m,n\rangle) = 1 \Leftrightarrow d(i,j) = m/(n+1)$, and $\beta_d(t)$ is 0 in any other case. We say that $\beta \in \mathcal{N}$ *encodes the complete and separable space* $(\mathcal{X}, d^{\mathcal{X}})$ if there is a distance function d with rational values such that $\beta = \beta_d$ and $(\mathcal{X}, d^{\mathcal{X}})$ is isometric to $[(\omega, d)]$.

We will also use the following encoding of y-recursive functions from $\mathcal{N}$ to $\mathcal{N}$, where y varies through a recursively presented Polish space $\mathcal{Y}$, as given in [5, 7A]. For all recursively presented Polish spaces $\mathcal{X}$ consider the $G^{\mathcal{X}} \subseteq \omega \times \mathcal{X}$ as defined in [5, 3H.1], which is in Σ^0_1 and is universal for

$\Sigma^0_1 \upharpoonright \mathcal{X}$. Fix now a recursively presented Polish space $\mathcal{Y}$. For every function $\pi : \mathcal{N} \to \mathcal{N}$ which is y-recursive for some $y \in \mathcal{Y}$ there is some $e \in \omega$ such that for all $s \in \omega$ we have $\pi(\alpha) \in N(\mathcal{N}, s) \Leftrightarrow G^{\mathcal{Y}\times\mathcal{N}\times\omega}(e, y, \alpha, s)$. We consider such a number e to be a *code of the function* π. We define the set $\mathrm{Cod}^{\mathcal{Y}} \subseteq \omega \times \mathcal{Y}$ by

$$\mathrm{Cod}^{\mathcal{Y}}(e, y) \Leftrightarrow (\forall\alpha)(\exists \text{ unique } \beta)(\forall s)[\beta \in N(\mathcal{N}, s) \leftrightarrow G^{\mathcal{Y}\times\mathcal{N}\times\omega}(e, y, \alpha, s)].$$

It is easy to verify that $\mathrm{Cod}^{\mathcal{Y}}$ is a Π^1_1 set and so the set of codes of (total) y-recursive functions is in $\Pi^1_1(y)$.

THEOREM 4.2. *Suppose that $\mathcal{Z}$ is a Polish space, $\mathcal{X}$ is a closed subset of $\mathcal{N}$ and that P is a Borel subset of $\mathcal{Z} \times \mathcal{X}$ such that every section P_z is neither a finite nor a co-finite set* (3). *Assume moreover that*

($*$) *for all $z \in \mathcal{Z}$ the z-section P_z is either countable or co-countable.*

Then there is a Borel-measurable function $f : \mathcal{Z} \to \mathcal{N}$ and a family $(d^{\mathcal{X}}_z)_{z\in\mathcal{Z}}$ of distance functions on $\mathcal{X}$ such that for all $z \in \mathcal{Z}$ the following are true:

(a) *the metric space $(\mathcal{X}, d^{\mathcal{X}}_z)$ is complete and separable,*
(b) *$f(z)$ encodes the space $(\mathcal{X}, d^{\mathcal{X}}_z)$,*
(c) *the topology induced by $d^{\mathcal{X}}_z$ extends the original one,*
(d) *the section P_z is $d^{\mathcal{X}}_z$-clopen, and*
(e) *a subset of $\mathcal{X}$ is Borel in the original topology exactly when it is Borel in the topology induced by $d^{\mathcal{X}}_z$.*

Proof. Fix for the moment some $z \in \mathcal{Z}$ and let $F_i \subseteq \mathcal{N}$ be closed and $\pi_i : \mathcal{N} \to \mathcal{X}$ be continuous such that π_i is injective on F_i for $i = 1, 2$, $\pi_1[F_1] = P_z$ and $\pi_2[F_2] = \mathcal{X} \setminus P_z$. As usual we define the distance functions d^1_z and d^2_z on P_z and $\mathcal{X} \setminus P_z$ respectively so that the π_i's become isometries. The space $(\mathcal{X}, p_z) := (P_z, d^1_z) \oplus (\mathcal{X} \setminus P_z, d^2_z)$ satisfies the conclusions (c)–(e). Consider now sequences $(\alpha^i_n)_{n\in\omega}$ of distinct terms which are dense in F_i, $i = 1, 2$. It follows that the sequence $(\pi_1(\alpha^1_0), \pi_2(\alpha^2_0), \pi_1(\alpha^1_1), \pi_2(\alpha^2_1), \ldots)$ is dense in $(\mathcal{X}, p_z)$. We define $d \equiv d(z) \in \mathrm{Df}(\omega)$ by $d(2i, 2j) = p_{\mathcal{N}}(\alpha^1_i, \alpha^1_j)$, $d(2i+1, 2j+1) = p_{\mathcal{N}}(\alpha^2_i, \alpha^2_j)$ and $d(2i, 2j+1) = d(2i+1, 2j) = 2$. It is clear that $[(\omega, d(z))]$ is isometric to $(\mathcal{X}, p_z)$. Notice that $d(z)$ takes only rational values. We consider the $\beta_{d(z)}$ which encodes $d(z)$ as in Definition 4.1. We will show that there is a Borel way of choosing this $\beta_{d(z)}$.

Let us assume for simplicity that $\mathcal{Z}$ is recursively presented, $\mathcal{X}$ is recursively zero-dimensional and P is in Δ^1_1, so that every section P_z is in $\Delta^1_1(z)$. We will apply the Strong Δ-Selection Principle [5, 4D.6], where the underlying pointclass is $\Gamma = \Pi^1_1$. From our hypothesis ($*$) and the construction in the proof of Theorem 3.9 it follows that for all $z \in \mathcal{Z}$ there are $\varepsilon \in \Delta^1_1(z)$, sets $F_i \subseteq \mathcal{N}$ in $\Pi^0_1(\varepsilon, z)$ and (ε, z)-recursive functions $\pi_i : \mathcal{N} \to \mathcal{X}$, $i = 1, 2$, such

(3) The term "finite" also applies to the empty set.

that $\pi_1[F_1] = P_z$, $\pi_2[F_2] = \mathcal{X} \setminus P_z$ and π_i is injective on F_i, $i = 1, 2$. Moreover there are $\Delta^1_1(z)$ sequences $(\alpha^i_n)_{n\in\omega}$, which are dense in F_i for $i = 1, 2$ and—as we may assume—consist of distinct terms.

We consider the space Tr of trees on the natural numbers (cf. [1, 4.32]). This is easily a recursively presented Polish space. Moreover every $F \subseteq \mathcal{N}$ in $\Pi^0_1(z)$ is the body $[T]$ of some z-recursive tree T; this is immediate from [5, 4A.1]. We define the relation $R_1 \subseteq \mathcal{Z} \times \omega \times \mathrm{Tr} \times \mathcal{N}^2$ as follows:

$$\begin{aligned} R_1(z, e, T, \varepsilon, \alpha) \;\Leftrightarrow\; & T \text{ is an } (\varepsilon, z)\text{-recursive tree} \\ & \;\&\; e \text{ encodes a total } (\varepsilon, z)\text{-recursive function} \\ & \quad \pi : \mathcal{N} \to \mathcal{N} \text{ which is injective on } [T] \\ & \;\&\; \pi[[T]] = P_z \;\&\; ((\alpha)_i)_{i\in\omega} \text{ is dense in } [T]. \end{aligned}$$

Similarly we define the relation R_2 by replacing P_z with $\mathcal{X} \setminus P_z$. From our previous comments and from the fact that $P_z \neq \emptyset, \mathcal{X}$ it follows that for all $z \in \mathcal{Z}$ there are $(e_i, T_i, \varepsilon_i, \alpha_i) \in \Delta^1_1(z)$ such that $R_i(z, e_i, T_i, \varepsilon_i, \alpha_i)$ for $i = 1, 2$.

Suppose for the moment that we have proved that the R_i's are Π^1_1 sets. Then from the Strong Δ-Selection Principle there are Δ^1_1-recursive functions $g = (g_1, g_2, g_3, g_4), h = (h_1, h_2, h_3, h_4) : \mathcal{Z} \to \omega \times \mathrm{Tr} \times \mathcal{N}^2$ such that for all $z \in \mathcal{Z}$ we have $R_1(z, g(z))$ and $R_2(z, h(z))$. Let $(\alpha^1_i)_{i\in\omega}$ and $(\alpha^2_i)_{i\in\omega}$ be the sequences which arise from $g_4(z)$ and $h_4(z)$ respectively. As above we define $d \equiv d(z) \in \mathrm{Df}(\omega)$ as follows: $d(2i, 2j) = p_{\mathcal{N}}(\alpha^1_i, \alpha^1_j)$, $d(2i + 1, 2j + 1) = p_{\mathcal{N}}(\alpha^2_i, \alpha^2_j)$ and $d(2i, 2j + 1) = d(2i + 1, 2j) = 2$. Finally we define $f(z) = \beta_{d(z)}$ for all $z \in \mathcal{Z}$. It is clear that f is Δ^1_1-recursive and thus it is Borel-measurable. According to the previous comments this function f satisfies the conclusions (a)–(e).

So it remains to verify that the sets R_1 and R_2 are indeed in Π^1_1. We consider the set $\mathrm{Cod}^{\mathcal{N}\times\mathcal{Z}} \equiv \mathrm{Cod}$ as in the comments following Definition 4.1. We also define the relation $\mathrm{Gr} \subseteq \omega \times \mathcal{N} \times \times \mathcal{N}^2$ by

$$\mathrm{Gr}(e, \varepsilon, z, \alpha, \beta) \;\Leftrightarrow\; (\forall s)[\beta \in N(\mathcal{N}, s) \leftrightarrow G^{\mathcal{N}^2\times\mathcal{Z}\times\omega}(e, \varepsilon, z, \alpha, s)].$$

In other words, when $\mathrm{Cod}(e, \varepsilon, z)$ holds, $\mathrm{Gr}(e, \varepsilon, z, \alpha, \beta)$ means that β is the image of α under the (ε, z)-recursive function encoded by e. Clearly Gr is in Δ^1_1. It is now easy to see that e encodes an (ε, z)-recursive function $\pi : \mathcal{N} \to \mathcal{N}$ which is injective on $[T]$, and $\pi[[T]] = P_z$ exactly when

$$\begin{aligned} & \mathrm{Cod}(e, \varepsilon, z) \;\&\; (\forall\alpha, \beta)[\alpha \in [T] \;\&\; \mathrm{Gr}(e, \varepsilon, z, \alpha, \beta) \to \beta \in P_z], \\ & \forall(\alpha_1, \alpha_2, \beta_1, \beta_2)(\forall i = 1, 2)[\mathrm{Gr}(e, \varepsilon, z, \alpha_i, \beta_i) \;\&\; \alpha_1 \neq \alpha_2 \to \beta_1 \neq \beta_2], \\ & (\forall\beta)(\exists\alpha \in \Delta^1_1(\varepsilon, z, \beta))[\beta \in P_z \to \alpha \in [T] \;\&\; \mathrm{Gr}(e, \varepsilon, z, \alpha, \beta)]. \end{aligned}$$

(Here we are using the fact that for all $\beta \in P_z$ the unique $\alpha \in [T]$ for which $\pi(\alpha) = \beta$ is in $\Delta^1_1(\varepsilon, z, \beta)$.) Using a similar method with universal sets one can prove that the relation $Q_1 \subseteq \mathcal{N} \times \mathcal{Z} \times \mathrm{Tr}$ defined by

$$Q_1(\varepsilon, z, T) \iff T \text{ is } (\varepsilon, z)\text{-recursive}$$

is in Δ^1_1. Moreover it is easy to check using the definition that the relation $Q_2 \subseteq \mathcal{N} \times \mathrm{Tr}$ defined by $Q_2(\alpha, T) \Leftrightarrow ((\alpha)_s)_{s\in\omega}$ *is dense in* $[T]$ is in Π^1_1. It follows that R_1 is in Π^1_1. Similarly one shows that R_2 is in Π^1_1. ∎

Remark 4.3. Our assumption about P_z being either countable or co-countable was in order to be able to apply the proof of Theorem 3.9. The analogous assumption can be added to the statement of Theorem 4.2 in order to be able to apply the proof of Theorem 3.11 as well. To be more specific the conclusion of Theorem 4.2 is still true if the hypothesis $(*)$ in its statement is replaced by the following. For all $z \in \mathcal{Z}$ *one* of the following conditions applies:

- P_z is a $\utilde{\mathbf{\Delta}}^0_2$ set and $\Delta^1_1(z)$ is dense both in P_z and in $\mathcal{X} \setminus P_z$,
- P_z is countable,
- P_z is co-countable.

We point out that the condition "$\Delta^1_1(z)$ is dense both in P_z and in $\mathcal{X}\setminus P_z$" is met if there is a σ-finite Borel measure μ on $\mathcal{X}$ such that when P_z meets an open set V in an uncountable set, then $\mu(P_z \cap V) > 0$, and similarly for $\mathcal{X} \setminus P_z$ (provided that $P \in \Delta^1_1$). To see the latter, one needs to use the following result of Tanaka and Sacks, which is the counterpart of the Thomason–Hinman Theorem for measure (cf. [5, 4F.20] and the discussion following it): if μ is a σ-finite Borel measure on a recursively presented Polish space and $P \subseteq \mathcal{X}$ is a Π^1_1 set of positive μ-measure, then there exists some $x \in \Delta^1_1$ such that $x \in P$.

One may ask about a uniformity result which follows just from the proof of Theorem 2.1, i.e. without any additional assumptions about the z-sections P_z. We can still get a uniformizing function f but we would not be able to infer that it is Borel-measurable.

As usual the symbol $\utilde{\mathbf{\Pi}}^1_1$ stands for the class of co-analytic sets and $\utilde{\mathbf{\Sigma}}^1_2$ for the class of sets which are continuous images of $\utilde{\mathbf{\Pi}}^1_1$ sets. A set A is in $\utilde{\mathbf{\Delta}}^1_2$ if both A and its complement are in $\utilde{\mathbf{\Sigma}}^1_2$.

Theorem 4.4. *Suppose that $\mathcal{Z}$ is a Polish space, $\mathcal{X}$ is a closed subset of $\mathcal{N}$, and P is a Borel subset of $\mathcal{Z} \times \mathcal{X}$ such that every section P_z is neither finite nor co-finite. Then there is a $\utilde{\mathbf{\Delta}}^1_2$-measurable function $f : \mathcal{Z} \to \mathcal{N}$ such that for all $z \in \mathcal{Z}$ the conclusions* (a)–(e) *of Theorem* 4.2 *are satisfied.*

Proof. Consider the sets R_1 and R_2 of the proof of Theorem 4.2. As is shown there, these sets are $\utilde{\mathbf{\Pi}}^1_1$ sets and for all $z \in \mathcal{Z}$, since $P_z \neq \emptyset, \mathcal{X}$, it follows from the proof of Theorem 2.1 that there are $(e_i, T_i, \varepsilon_i, \alpha_i)$ (not necessarily in $\Delta^1_1(z)$) such that $R_i(z, e_i, T_i, \varepsilon_i, \alpha_i)$ for $i = 1, 2$. Now we apply the Uniformization Property of $\underset{\sim}{\mathbf{\Pi}}^1_1$ in order to get functions

$$g = (g_1, g_2, g_3, g_4), h = (h_1, h_2, h_3, h_4) : \mathcal{Z} \to \omega \times \mathrm{Tr} \times \mathcal{N}^2$$

whose graphs are $\underset{\sim}{\mathbf{\Pi}}^1_1$ sets such that for all $z \in \mathcal{Z}$ we have $R_1(z, g(z))$ and $R_2(z, h(z))$. It follows easily that the functions g and h are $\underset{\sim}{\mathbf{\Delta}}^1_2$-measurable and from the closure properties of $\underset{\sim}{\mathbf{\Delta}}^1_2$ we deduce that the arising function f is $\underset{\sim}{\mathbf{\Delta}}^1_2$-measurable as well. ∎

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