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The multifractal box dimensions of typical measures

by

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Abstract. We compute the typical (in the sense of Baire's category theorem) multifractal box dimensions of measures on a compact subset of $\mathbb{R}^d$. Our results are new even in the context of box dimensions of measures.

1. Introduction

1.1. Formulation of the problem. The origin of this paper goes back to the work [MR02] of J. Myjak and R. Rudnicki, where they investigate the box dimensions of typical measures. To state their result, we need to introduce some terminology. Let K be a compact subset of $\mathbb{R}^d$, and let $\mathcal{P}(K)$ be the set of Borel probability measures on K; we endow $\mathcal{P}(K)$ with the weak topology. By a property true for a typical measure of $\mathcal{P}(K)$, we mean a property which is satisfied by a dense G_δ-set of elements of $\mathcal{P}(K)$.

For a subset $E\subset\mathbb{R}^d$, we denote the lower box dimension of E and the upper box dimension of E by $\underline{\dim}_{\mathrm{B}}(E)$ and $\overline{\dim}_{\mathrm{B}}(E)$, respectively. Also, for a probability measure μ, we define the *small* and *big lower* (resp. *upper*) *multifractal box dimensions* of μ by

$$\underline{\dim}_{*,\mathrm{B}}(\mu)=\inf_{\mu(E)>0}\underline{\dim}_{\mathrm{B}}(E),\qquad \underline{\dim}^*_{\mathrm{B}}(\mu)=\lim_{\varepsilon>0}\inf_{\mu(E)>1-\varepsilon}\underline{\dim}_{\mathrm{B}}(E),$$
$$\overline{\dim}_{*,\mathrm{B}}(\mu)=\inf_{\mu(E)>0}\overline{\dim}_{\mathrm{B}}(E),\qquad \overline{\dim}^*_{\mathrm{B}}(\mu)=\lim_{\varepsilon>0}\inf_{\mu(E)>1-\varepsilon}\overline{\dim}_{\mathrm{B}}(E).$$

Finally, we define the *local upper box dimension of* K by

$$\overline{\dim}_{\mathrm{B,loc}}(K)=\inf_{x\in K}\inf_{r>0}\overline{\dim}_{\mathrm{B}}(K\cap B(x,r)).$$

Theorem A (Myjak and Rudnicki). *Let K be a compact subset of $\mathbb{R}^d$. Then a typical measure $\mu\in\mathcal{P}(K)$ satisfies*

$$\underline{\dim}_{*,\mathrm{B}}(\mu)=\underline{\dim}^*_{\mathrm{B}}(\mu)=0,$$
$$\overline{\dim}_{\mathrm{B,loc}}(K)\le\overline{\dim}_{*,\mathrm{B}}(\mu)\le\overline{\dim}^*_{\mathrm{B}}(\mu)\le\overline{\dim}_{\mathrm{B}}(K).$$

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The result concerning the upper multifractal box dimension does not solve completely the problem for compact sets even as simple as $K = \{0\} \cup [1,2]$. In this case we just find that, typically,

$$0 \le \overline{\dim}_{*,\mathrm{B}}(\mu) \le \overline{\dim}^*_{\mathrm{B}}(\mu) \le 1.$$

In particular, we do not know whether the interval $[0,1]$ is the shortest possible, or whether $\overline{\dim}_{*,\mathrm{B}}(\mu)$ and $\overline{\dim}^*_{\mathrm{B}}(\mu)$ coincide for a typical measure.

Our original aim was to answer this question. To do that, we need to introduce the *maximal local upper box dimension* of a set E, defined by

$$\overline{\dim}_{\mathrm{B,loc,max}}(E) = \sup_{y\in E,\,\rho>0} \overline{\dim}_{\mathrm{B,loc}}(E\cap B(y,\rho)).$$

Our first main result now reads:

THEOREM 1.1. *Let K be a compact subset of $\mathbb{R}^d$. Then a typical measure $\mu \in \mathcal{P}(K)$ satisfies*

$$\overline{\dim}_{*,\mathrm{B}}(\mu) = \overline{\dim}_{\mathrm{B,loc}}(K), \qquad \overline{\dim}^*_{\mathrm{B}}(\mu) = \overline{\dim}_{\mathrm{B,loc,max}}(K).$$

Applying this theorem with $K = \{0\} \cup [1,2]$, we find that a typical measure $\mu \in \mathcal{P}(K)$ satisfies

$$\overline{\dim}_{*,\mathrm{B}}(\mu) = 0 \quad \text{and} \quad \overline{\dim}^*_{\mathrm{B}}(\mu) = 1.$$

1.2. Multifractal box dimensions. In [Ols11], L. Olsen has put the work of Myjak and Rudnicki in a more general context, that of multifractal box dimensions of measures, which is interesting by itself. Fix a Borel probability measure π on $\mathbb{R}^d$ with support K. For a bounded subset E of K, the *multifractal box dimensions* of E with respect to π are defined as follows. For $r > 0$ and a real number q, write

$$\mathbf{N}^q_\pi(E,r) = \inf_{(B(x_i,r))\text{ is a cover of }E} \sum_i \pi(B(x_i,r))^q.$$

The *lower* and *upper covering multifractal box dimensions* of E of order q with respect to π are defined by

$$\underline{\dim}^q_{\pi,\mathrm{B}}(E) = \liminf_{r\to 0} \frac{\log \mathbf{N}^q_\pi(E,r)}{-\log r}, \qquad \overline{\dim}^q_{\pi,\mathrm{B}}(E) = \limsup_{r\to 0} \frac{\log \mathbf{N}^q_\pi(E,r)}{-\log r}.$$

Let now $\mu \in \mathcal{P}(K)$. We define the *small* and *big lower multifractal box dimensions* of μ of order q with respect to the measure π (resp. the *small* and *big upper multifractal box dimensions* of μ of order q with respect to the measure π) by

$$\underline{\dim}^q_{*,\pi,\mathrm{B}}(\mu) = \inf_{\mu(E)>0} \underline{\dim}^q_{\pi,\mathrm{B}}(E), \qquad \underline{\dim}^{*,q}_{\pi,\mathrm{B}}(\mu) = \lim_{\varepsilon>0} \inf_{\mu(E)>1-\varepsilon} \underline{\dim}^q_{\pi,\mathrm{B}}(E),$$
$$\overline{\dim}^q_{*,\pi,\mathrm{B}}(\mu) = \inf_{\mu(E)>0} \overline{\dim}^q_{\pi,\mathrm{B}}(E), \qquad \overline{\dim}^{*,q}_{\pi,\mathrm{B}}(\mu) = \lim_{\varepsilon>0} \inf_{\mu(E)>1-\varepsilon} \overline{\dim}^q_{\pi,\mathrm{B}}(E).$$

Multifractal box dimensions of measures play a central role in multifractal analysis. For instance, the multifractal box dimensions of measures in $\mathbb{R}^d$ having some degree of self-similarity have been intensively studied (see [Fal97] and the references therein). In [Ols11], L. Olsen gives estimates of the typical multifractal box dimensions of measures, in the spirit of Myjak and Rudnicki. To state his result, we need a few definitions. Firstly, the *upper moment scaling* of π is the function $\tau_\pi : \mathbb{R} \to \mathbb{R}$ defined by

$$\tau_\pi(q) = \overline{\dim}^q_{\pi,\mathrm{B}}(K).$$

The *local upper multifractal box dimension of K of order q* is defined by

$$\overline{\dim}^q_{\pi,\mathrm{B},\mathrm{loc}}(K) = \inf_{x\in K} \inf_{r>0} \overline{\dim}^q_{\pi,\mathrm{B}}(K \cap B(x,r)).$$

This last quantity will also be called the *local upper moment scaling* of π and will be denoted by $\tau_{\pi,\mathrm{loc}}(q)$. Finally, let

$$\overline{D}_\pi(-\infty) = \limsup_{r\to 0} \frac{\log \inf_{x\in K} \pi(B(x,r))}{\log r},$$

$$\underline{D}_\pi(+\infty) = \liminf_{r\to 0} \frac{\log \sup_{x\in K} \pi(B(x,r))}{\log r}.$$

Recall also that a measure π on $\mathbb{R}^d$ is called a *doubling measure* provided there exists $C > 0$ such that

$$\sup_{x\in \mathrm{supp}(\pi)} \sup_{r>0} \frac{\pi(B(x,2r))}{\pi(B(x,r))} \le C.$$

We can now give Olsen's result.

Theorem B (Olsen). *Let π be a Borel probability measure on $\mathbb{R}^d$ with compact support K.*

(1) *A typical measure $\mu \in \mathcal{P}(K)$ satisfies*

$$-q\underline{D}_\pi(+\infty) \le \underline{\dim}^q_{*,\pi,\mathrm{B}}(\mu) \le \underline{\dim}^{*,q}_{\pi,\mathrm{B}}(\mu) \le -q\overline{D}_\pi(-\infty) \quad \textit{for all } q \le 0,$$
$$-q\overline{D}_\pi(-\infty) \le \underline{\dim}^q_{*,\pi,\mathrm{B}}(\mu) \le \underline{\dim}^{*,q}_{\pi,\mathrm{B}}(\mu) \le -q\underline{D}_\pi(+\infty) \quad \textit{for all } q \ge 0.$$

(2) *If π is a doubling measure, then a typical measure $\mu \in \mathcal{P}(K)$ satisfies*

$$\tau_{\pi,\mathrm{loc}}(q) \le \overline{\dim}^q_{*,\pi,\mathrm{B}}(\mu) \le \overline{\dim}^{*,q}_{\pi,\mathrm{B}}(\mu) \le \tau_\pi(q) \quad \textit{for all } q \le 0.$$

If moreover K does not contain isolated points, then this result remains true for all $q \in \mathbb{R}$.

For $q = 0$, this implies in particular Myjak and Rudnicki's theorem.

1.3. Statement of our main results. Of course, the questions asked after Theorem A also make sense in this more general context. To answer them, we have to introduce the *maximal local upper moment scaling* of π, which is defined by

$$\tau_{\pi,\mathrm{loc},\max}(q) = \sup_{y\in K,\rho>0} \overline{\dim}^{q}_{\pi,\mathrm{B},\mathrm{loc}}(K\cap B(y,\rho)).$$

THEOREM 1.2. *Let π be a doubling Borel probability measure on $\mathbb{R}^d$ with compact support K. Then a typical measure $\mu\in\mathcal{P}(K)$ satisfies, for any $q\in\mathbb{R}$,*

$$\overline{\dim}^{q}_{*,\pi,\mathrm{B}}(\mu)=\tau_{\pi,\mathrm{loc}}(q),\qquad \overline{\dim}^{*,q}_{\pi,\mathrm{B}}(\mu)=\tau_{\pi,\mathrm{loc},\max}(q).$$

Putting $q=0$, we retrieve Theorem 1.1.

We can also observe that Olsen's theorem does not settle completely the typical values of the lower multifractal box dimensions. For instance, when computed for a self-similar compact set K satisfying the open set condition (see below) and an associated self-similar measure π, the values of $\underline{D}_\pi(+\infty)$ and $\overline{D}_\pi(-\infty)$ are in general different. Moreover, it has been pointed out in [Bay12] that, given a fixed compact set $K\subset\mathbb{R}^d$, a typical probability measure $\pi\in\mathcal{P}(K)$ satisfies $\overline{D}_\pi(-\infty)=+\infty$ and $\underline{D}_\pi(+\infty)=0$!

We have been able to compute the typical value of the big lower multifractal box dimension of a measure. As before, we need to introduce some definitions, which are uniform versions of $\overline{D}_\pi(-\infty)$ and $\underline{D}_\pi(+\infty)$. Let π be a Borel probability measure with support K. Define

$$\overline{D}_{\pi,\mathrm{unif}}(-\infty)=\inf_N \inf_{\substack{y_1,\dots,y_N\in K\\ \rho>0}} \limsup_{r\to0} \inf_{i=1,\dots,N} \frac{\log(\inf_{x\in B(y_i,\rho)}\pi(B(x,r)))}{\log r},$$

$$\underline{D}_{\pi,\mathrm{unif}}(+\infty)=\sup_N \sup_{\substack{y_1,\dots,y_N\in K\\ \rho>0}} \liminf_{r\to0} \sup_{i=1,\dots,N} \frac{\log(\sup_{x\in B(y_i,\rho)}\pi(B(x,r)))}{\log r}.$$

THEOREM 1.3. *Let π be a Borel probability measure with compact support K. Then a typical measure $\mu\in\mathcal{P}(K)$ satisfies*

$$\underline{\dim}^{*,q}_{\pi,\mathrm{B}}(\mu)=\begin{cases}-q\overline{D}_{\pi,\mathrm{unif}}(-\infty) & \text{provided } q\geq 0,\\ -q\underline{D}_{\pi,\mathrm{unif}}(+\infty) & \text{provided } q\leq 0.\end{cases}$$

Unfortunately, we do not have a similar result for the small lower multifractal box dimensions. We have just been able to improve Olsen's inequality. This improvement is sufficient to conclude for self-similar compact sets. We need to introduce the following quantities. Let π be a Borel probability measure with compact support K. Define

$$\begin{aligned}&\overline{D}_{\pi,\mathrm{unif},\max}(-\infty)\\ &\quad=\sup_{\substack{z\in K\\ \kappa>0}} \inf_{\substack{y_1,\dots,y_N\in B(z,\kappa)\\ \rho>0}} \limsup_{r\to0} \inf_{i=1,\dots,N} \frac{\log(\inf_{x\in B(y_i,\rho)}\pi(B(x,r)))}{\log r},\end{aligned}$$

$$\overline{D}_{\pi,\max}(-\infty) = \sup_{\substack{y\in K\\ \rho>0}} \limsup_{r\to 0} \frac{\log \inf_{x\in B(y,\rho)} \pi(B(x,r))}{\log r},$$

$$\underline{D}_{\pi,\mathrm{unif},\min}(+\infty)$$
$$= \inf_{\substack{z\in K\\ \kappa>0}} \sup_{\substack{y_1,\dots,y_N\in B(z,\kappa)\\ \rho>0}} \liminf_{r\to 0} \sup_{i=1,\dots,N} \frac{\log(\sup_{x\in B(y_i,\rho)} \pi(B(x,r)))}{\log r},$$

$$\underline{D}_{\pi,\min}(+\infty) = \inf_{\substack{y\in K\\ \rho>0}} \liminf_{r\to 0} \frac{\log \sup_{x\in B(y,\rho)} \pi(B(x,r))}{\log r}.$$

THEOREM 1.4. *Let π be a Borel probability measure with compact support K. Then a typical measure $\mu\in\mathcal{P}(K)$ satisfies*

$$-q\overline{D}_{\pi,\max}(-\infty) \le \underline{\dim}^q_{*,\pi,\mathrm{B}}(\mu) \le -q\overline{D}_{\pi,\mathrm{unif},\max}(-\infty) \quad \textit{provided } q\ge 0,$$
$$-q\underline{D}_{\pi,\min}(+\infty) \le \underline{\dim}^q_{*,\pi,\mathrm{B}}(\mu) \le -q\underline{D}_{\pi,\mathrm{unif},\min}(-\infty) \quad \textit{provided } q\le 0.$$

Although the above quantities are not very engaging, they can be easily computed for regular measures π. This is for instance the case for self-similar measures on self-similar compact sets. To show this fix an integer $M\ge 2$. For any $m=1,\dots,M$, let $S_m:\mathbb{R}^d\to\mathbb{R}^d$ be a contracting similarity with Lipschitz constant $r_m\in(0,1)$. Let $(p_1,\dots,p_M)$ be a probability vector. We define K and π as the self-similar compact set and the self-similar measure associated with the list $(S_1,\dots,S_M,p_1,\dots,p_M)$, i.e. K is the unique nonempty compact subset of $\mathbb{R}^d$ such that

$$K=\bigcup_m S_m(K),$$

and π is the unique Borel probability measure on $\mathbb{R}^d$ such that

$$\pi=\sum_m p_m \pi\circ S_m^{-1}$$

(see for instance [Fal97]). It is well known that $\operatorname{supp}\pi = K$. We say that the list $(S_1,\dots,S_M)$ satisfies the *Open Set Condition* if there exists an open and nonempty bounded subset U of $\mathbb{R}^d$ with $S_mU\subset U$ for all m, and $S_mU\cap S_lU=\emptyset$ for all l,m with $l\ne m$.

Theorems 1.3 and 1.4 imply the following more appealing corollary:

COROLLARY 1.5. *Let K and π be as above, and assume that the Open Set Condition is satisfied. Let*

$$s_{\min}=\min_m \frac{\log p_m}{\log r_m} \quad \textit{and} \quad s_{\max}=\max_m \frac{\log p_m}{\log r_m}.$$

Then a typical measure $\mu \in \mathcal{P}(K)$ *satisfies*

$$\underline{\dim}^q_{*,\pi,\mathrm{B}}(\mu) = \underline{\dim}^{*,q}_{\pi,\mathrm{B}}(\mu) = \begin{cases} -s_{\max} q & \text{for any } q \geq 0, \\ -s_{\min} q & \text{for any } q \leq 0. \end{cases}$$

This improves Theorem 2.1 of [Ols11], which just says that a typical $\mu \in \mathcal{P}(K)$ satisfies

$$\begin{aligned} -s_{\max} q \leq \underline{\dim}^q_{*,\pi,\mathrm{B}}(\mu) \leq \underline{\dim}^{*,q}_{\pi,\mathrm{B}}(\mu) \leq -s_{\min} q \quad & \text{for all } q \geq 0, \\ -s_{\min} q \leq \underline{\dim}^q_{*,\pi,\mathrm{B}}(\mu) \leq \underline{\dim}^{*,q}_{\pi,\mathrm{B}}(\mu) \leq -s_{\max} q \quad & \text{for all } q \leq 0. \end{aligned}$$

1.4. Organization of the paper. In Section 2, we summarize all the results which will be needed throughout the paper. Section 3 is devoted to the proof of Theorem 1.2. The proofs of Theorems 1.3 and 1.4 share some similarities. They will be given in Section 4, together with application to self-similar measures.

2. Preliminaries. Throughout this paper, $\mathcal{P}(K)$ will be endowed with the weak topology. It is well known (see for instance [Par67]) that this topology is completely metrizable by the Fortet–Mourier distance defined as follows. Let $\mathrm{Lip}(K)$ denote the family of Lipschitz functions $f : K \to \mathbb{R}$, with $|f| \leq 1$ and $\mathrm{Lip}(f) \leq 1$, where $\mathrm{Lip}(f)$ denotes the Lipschitz constant of f. We endow $\mathcal{P}(K)$ with the metric L defined by

$$L(\mu,\nu) = \sup_{f\in \mathrm{Lip}(K)} \left| \int f\,d\mu - \int f\,d\nu \right|$$

for any $\mu,\nu \in \mathcal{P}(K)$. In particular, for $\mu \in \mathcal{P}(K)$ and $\delta > 0$, $B_L(\mu,\delta) = \{\nu \in \mathcal{P}(K);\ L(\mu,\nu) < \delta\}$ will stand for the ball with center at μ and radius equal to δ.

We shall repeatedly use the following lemma.

Lemma 2.1. *For any* $\alpha \in (0,1)$ *and any* $\beta > 0$, *there exists* $\eta > 0$ *such that, for any Borel subset* E *of* K *and any* $\mu,\nu \in \mathcal{P}(K)$,

$$L(\mu,\nu) < \eta \ \Rightarrow\ \mu(E) \leq \nu(E(\alpha)) + \beta,$$

where $E(\alpha) = \{x \in K;\ \mathrm{dist}(x,E) < \alpha\}$.

Proof. We set

$$f(t) = \begin{cases} \alpha & \text{provided } t \in \overline{E}, \\ \alpha - \mathrm{dist}(x,E) & \text{provided } 0 < \mathrm{dist}(x,E) \leq \alpha, \\ 0 & \text{otherwise.} \end{cases}$$

Then f is Lipschitz, with $|f| \leq 1$ and $\mathrm{Lip}(f) \leq 1$. Thus, if $L(\mu,\nu) < \eta$,

$$\mu(E) \leq \frac{1}{\alpha}\int f\,d\mu \leq \frac{1}{\alpha}\left[\int f\,d\nu + \eta\right] \leq \nu(E(\alpha)) + \frac{\eta}{\alpha}.$$

Hence, it suffices to take $\eta = \alpha\beta$. ∎

An application of Lemma 2.1 is the following result on open subsets of $\mathcal{P}(K)$:

LEMMA 2.2. *Let $x \in K$, $a \in \mathbb{R}$ and $r > 0$. Then the set $\{\mu \in \mathcal{P}(K);\ \mu(B(x,r)) > a\}$ is open.*

Proof. If $a \notin [0,1)$, then the set is either empty or equal to $\mathcal{P}(K)$. Otherwise, let $\mu \in \mathcal{P}(K)$ be such that $\mu(B(x,r)) > a$. One may find $\varepsilon > 0$ such that $\mu(B(x,(1-\varepsilon)r)) > a$. Thus the result follows from Lemma 2.1 applied with $E = B(x,(1-\varepsilon)r)$, $\alpha = \varepsilon r$ and $\beta = (\mu(B(x,(1-\varepsilon)r)) - a)/2$. ∎

Finally, we will use the fact that some subsets of $\mathcal{P}(K)$ are dense in $\mathcal{P}(K)$ (see e.g. [Ols05, Lemma 2.2.4]):

LEMMA 2.3. *Let $(x_n)_{n\geq 1}$ be a dense sequence in K and let $(\eta_n)_{n\geq 1}$ be a sequence of positive real numbers going to zero. For each $n \geq 1$ and each $i \in \{1,\dots,n\}$, let $\mu_{n,i} \in \mathcal{P}(K)$ be such that $\operatorname{supp}(\mu_{n,i}) \subset K \cap B(x_i,\eta_n)$. Then, for any $m \geq 1$, the set $\bigcup_{n\geq m}\{\sum_{i=1}^n p_i\mu_{n,i};\ p_i > 0,\ \sum_i p_i = 1\}$ is dense in $\mathcal{P}(K)$.*

3. The typical upper multifractal box dimensions.

This section is devoted to the proof of Theorem 1.2.

3.1. Doubling measures, packings and coverings.

When π is a doubling measure, it will be convenient to express the multifractal box dimensions of a set using packings instead of coverings. For $E \subset \mathbb{R}^d$, recall that a family of balls $(B(x_i,r))$ is called a *centred packing* of E if $x_i \in E$ for all i and $|x_i - x_j| > 2$ for all $i \neq j$. We then define

$$\mathbf{P}_\pi^q(E,r) = \sup_{(B(x_i,r))\text{ is a packing of }E} \sum_i \pi(B(x_i,r))^q.$$

When π is a doubling measure, $\underline{\dim}_{\pi,\mathrm{B}}^q(E)$ and $\overline{\dim}_{\pi,\mathrm{B}}^q(E)$ can be defined using packings (see [Ols11]):

LEMMA 3.1. *Let π be a doubling Borel probability measure on $\mathbb{R}^d$ with support K. Then*

$$\underline{\dim}_{\pi,\mathrm{B}}^q(E) = \liminf_{r\to 0} \frac{\log \mathbf{P}_\pi^q(E)}{-\log r}, \qquad \overline{\dim}_{\pi,\mathrm{B}}^q(E) = \limsup_{r\to 0} \frac{\log \mathbf{P}_\pi^q(E)}{-\log r}$$

for all $E \subset K$ and all $q \in \mathbb{R}$.

One of the advantages of using packings instead of coverings is that it helps us to obtain regularity of the map $q \mapsto \overline{\dim}_{\pi,\mathrm{B}}^q(E)$, as shown in the following lemma.

LEMMA 3.2. *Let π be a doubling Borel probability measure on $\mathbb{R}^d$ with support K, and let $E \subset K$.*

(1) *The map* $q \mapsto \overline{\dim}^q_{\pi,\mathrm{B}}(E)$ *is nonincreasing, convex and therefore continuous.*
(2) *The maps* $q \mapsto \tau_{\pi,\mathrm{loc}}(q)$ *and* $q \mapsto \tau_{\pi,\mathrm{loc},\max}(q)$ *are nonincreasing.*

Proof. Part (1) is Lemma 4.2 of [Ols11] and part (2) is trivial. ∎

As a first application, we show that, in order to find a residual subset $\mathcal{R}$ of $\mathcal{P}(K)$ such that any $\mu \in \mathcal{R}$ satisfies the conclusions of Theorem 1.2 for any $q \in \mathbb{R}$, it suffices to find a residual subset which works for a fixed $q \in \mathbb{R}$.

PROPOSITION 3.3. *Let* π *be a doubling Borel probability measure on* $\mathbb{R}^d$ *with support* K. *Then there exists a countable set* $\mathbf{Q} \subset \mathbb{R}$ *such that*

$$\bigcap_{q\in\mathbb{R}}\{\mu \in \mathcal{P}(K);\ \tau_{\pi,\mathrm{loc}}(q) \le \overline{\dim}^q_{*,\pi,\mathrm{B}}(\mu)\} = \bigcap_{q\in\mathbf{Q}}\{\mu \in \mathcal{P}(K);\ \tau_{\pi,\mathrm{loc}}(q) \le \overline{\dim}^q_{*,\pi,\mathrm{B}}(\mu)\},$$

$$\bigcap_{q\in\mathbb{R}}\{\mu \in \mathcal{P}(K);\ \tau_{\pi,\mathrm{loc}}(q) \ge \overline{\dim}^q_{*,\pi,\mathrm{B}}(\mu)\} = \bigcap_{q\in\mathbf{Q}}\{\mu \in \mathcal{P}(K);\ \tau_{\pi,\mathrm{loc}}(q) \ge \overline{\dim}^q_{*,\pi,\mathrm{B}}(\mu)\},$$

$$\bigcap_{q\in\mathbb{R}}\{\mu \in \mathcal{P}(K);\ \tau_{\pi,\mathrm{loc},\max}(q) \le \overline{\dim}^{*,q}_{\pi,\mathrm{B}}(\mu)\} = \bigcap_{q\in\mathbf{Q}}\{\mu \in \mathcal{P}(K);\ \tau_{\pi,\mathrm{loc},\max}(q) \le \overline{\dim}^{*,q}_{\pi,\mathrm{B}}(\mu)\}$$

$$\bigcap_{q\in\mathbb{R}}\{\mu \in \mathcal{P}(K);\ \tau_{\pi,\mathrm{loc},\max}(q) \ge \overline{\dim}^{*,q}_{\pi,\mathrm{B}}(\mu)\} = \bigcap_{q\in\mathbf{Q}}\{\mu \in \mathcal{P}(K);\ \tau_{\pi,\mathrm{loc},\max}(q) \ge \overline{\dim}^{*,q}_{\pi,\mathrm{B}}(\mu)\}.$$

Proof. Let $\mathbf{Q}_1$ (resp. $\mathbf{Q}_2$) be the set of points of discontinuity of $\tau_{\pi,\mathrm{loc}}$ (resp. of $\tau_{\pi,\mathrm{loc},\max}$). $\mathbf{Q}_1$ and $\mathbf{Q}_2$ are at most countable. Set $\mathbf{Q} = \mathbf{Q}_1 \cup \mathbf{Q}_2 \cup \mathbb{Q}$.

The first equality is already contained in [Ols11, Prop. 4.3]. Regarding the second one, let $\mu \in \mathcal{P}(K)$ be such that $\tau_{\pi,\mathrm{loc}}(q) \ge \overline{\dim}^q_{*,\pi,\mathrm{B}}(\mu)$ for any $q \in \mathbf{Q}$, and let us fix $q \in \mathbb{R}\backslash\mathbf{Q}$. Let (q_n) be a sequence in $\mathbb{Q}$ increasing to q. For each n, we may find E_n with $\mu(E_n) > 0$ and $\overline{\dim}^{q_n}_{\pi,\mathrm{B}}(E_n) \le \tau_{\pi,\mathrm{loc}}(q_n) + 1/n$. For n large enough, we get, by continuity of $\tau_{\pi,\mathrm{loc}}$ at q,

$$\overline{\dim}^q_{\pi,\mathrm{B}}(E_n) \le \overline{\dim}^{q_n}_{\pi,\mathrm{B}}(E_n) \le \tau_{\pi,\mathrm{loc}}(q) + \delta$$

for any fixed $\delta > 0$, so that $\overline{\dim}^q_{*,\pi,\mathrm{B}}(\mu) \le \tau_{\pi,\mathrm{loc}}(q)$.

The proof of the third equality goes along the same lines and is left to the reader. Regarding the last one, let $\mu \in \mathcal{P}(K)$ be such that $\tau_{\pi,\mathrm{loc},\max}(q) \ge$

$\overline{\dim}^{*,q}_{\pi,\mathrm{B}}(\mu)$ for any $q \in \mathbf{Q}$, and let us fix $q \in \mathbb{R}$. Let $\varepsilon > 0$, $\delta > 0$ and let $(q_n) \subset \mathbf{Q}$ be a sequence decreasing to q. Let also $(\varepsilon_n) \subset (0, +\infty)$ be such that $\sum_n \varepsilon_n < \varepsilon$. For each n, we may find $E_n \subset K$ such that $\mu(E_n) > 1 - \varepsilon_n$ and $\overline{\dim}^{q_n}_{\pi,\mathrm{B}}(E_n) \geq \tau_{\pi,\mathrm{loc,max}}(q_n) + \delta$. Set $E = \bigcap_n E_n$ so that $\mu(E) > 1 - \varepsilon$ and observe that, by continuity of $\overline{\dim}^{q}_{\pi,\mathrm{B}}(E)$,

$$\overline{\dim}^{q}_{\pi,\mathrm{B}}(E) \leq \liminf_n \tau_{\pi,\mathrm{loc,max}}(q_n) + \delta \leq \tau_{\pi,\mathrm{loc,max}}(q) + \delta. \ \blacksquare$$

We conclude this section by pointing out that, working with doubling measures, we can also add a dilation factor when studying the multifractal dimensions.

Lemma 3.4. *Let π be a doubling Borel probability measure on $\mathbb{R}^d$ with compact support K. Let $c > 0$, $E \subset K$ and $q \in \mathbb{R}$. Then*

$$\overline{\dim}^{q}_{\pi,\mathrm{B}}(E) = \limsup_{r \to 0} \frac{\log \sup_{(B(x_i,r))\ \text{is a packing of}\ E} \sum_i \pi(B(x_i, cr))^q}{-\log r}.$$

3.2. The lower bounds. In this subsection, we fix $q \in \mathbb{R}$. We shall prove, at the same time, that quasi-all measures $\mu \in \mathcal{P}(K)$ satisfy

$$(\mathbf{A})\ \overline{\dim}^{q}_{*,\pi,\mathrm{B}}(\mu) \geq \tau_{\pi,\mathrm{loc}}(q), \qquad (\mathbf{B})\ \overline{\dim}^{*,q}_{\pi,\mathrm{B}}(\mu) \geq \tau_{\pi,\mathrm{loc,max}}(q)$$

(we shall prove $(\mathbf{A})$ since we want to dispense with the assumption "K has no isolated points").

If we want to prove $(\mathbf{A})$, we consider $t < \tau_{\pi,\mathrm{loc}}(q)$ and we set $F = G = K$. If we want to prove $(\mathbf{B})$, then we consider $t < \tau_{\pi,\mathrm{loc,max}}(q)$, a pair $(y, \kappa) \in K \cap (0, +\infty)$ such that $\overline{\dim}^{q}_{\pi,\mathrm{B,loc}}(B(y, \kappa) \cap K) > t$, and we set $F = K \cap B(y, \kappa)$, $G = K \cap B(y, \kappa/2)$.

Let now $x \in K$ and $s > 0$.

- If $x \notin F$, then we set $\mu_{x,s} = \delta_x$ and $r_{x,s} = s$.
- If $x \in F$, then $\overline{\dim}^{q}_{\pi,\mathrm{B}}(B(x, s) \cap F) > t$, so that we may choose $r_{x,s}$ in $(0, s)$ satisfying

$$t < \frac{\log \mathbf{P}^q_\pi(B(x, s) \cap F, r_{x,s})}{-\log r_{x,s}}.$$

Thus, there exists a finite set $\Lambda_{x,s} \subset B(x, s) \cap F$ which consists of points at distance at least $2r_{x,s}$ and satisfying

$$\sum_{z \in \Lambda_{x,s}} \pi(B(z, r_{x,s}))^q \geq r_{x,s}^{-t}.$$

We then set

$$\mu_{x,s} = \frac{1}{\sum_{z \in \Lambda_{x,s}} \pi(B(z, r_{x,s}))^q} \sum_{z \in \Lambda_{x,s}} \pi(B(z, r_{x,s}))^q \delta_z.$$

Observe that, in both cases, $\mathrm{supp}(\mu_{x,s}) \subset B(x, s)$.

Let us denote by $\mathcal{F}$ the set of nonempty finite subsets of K. For $A\in\mathcal{F}$, we write

$$\mathcal{Q}(A)=\Big\{(p_x)_{x\in A};\ p_x\in(0,1),\ \sum_{x\in A}p_x=1\Big\}.$$

Next, for $A\in\mathcal{F}$ and $\mathbf{p}=(p_x)_{x\in A}\in\mathcal{Q}(A)$, we denote

$$\mu_{A,\mathbf{p},s}=\sum_{x\in A}p_x\mu_{x,s},\qquad r_{A,s}=\inf_{x\in A}r_{x,s}\in(0,s).$$

An application of Lemma 2.3 shows that, for any sequence (η_n) decreasing to zero and for any $m\geq 1$,

$$\bigcup_{n\geq m}\bigcup_{A\in\mathcal{F}}\bigcup_{\mathbf{p}\in\mathcal{Q}(A)}\{\mu_{A,\mathbf{p},\eta_n}\}$$

is dense in $\mathcal{P}(K)$. Finally, for any $A\in\mathcal{F}$, any $\mathbf{p}\in\mathcal{Q}(A)$, any $s>0$ and any $\varepsilon>0$, we consider a real number $\eta_{A,\mathbf{p},s,\varepsilon}>0$ such that any $\mu\in\mathcal{P}(K)$ with $L(\mu,\mu_{A,\mathbf{p},s})<\eta_{A,\mathbf{p},s,\varepsilon}$ also satisfies, for any $E\subset K$,

$$\mu_{A,\mathbf{p},s}(E(r_{A,s}/2))\geq\mu(E)-\varepsilon.$$

We now set

$$\mathcal{R}=\bigcap_{m\geq1}\bigcup_{n\geq m}\bigcup_{A\in\mathcal{F}}\bigcup_{\mathbf{p}\in\mathcal{Q}(A)}B_L(\mu_{A,\mathbf{p},1/n},\eta_{A,\mathbf{p},1/n,1/n})\cap\{\mu\in\mathcal{P}(K);\ \mu(G)>0\}.$$

This is a dense G_δ-subset of $\mathcal{P}(K)$ and we pick $\mu\in\mathcal{R}$. We shall prove that either

$$(\mathbf{A})\ \overline{\dim}^q_{*,\pi,\mathrm{B}}(\mu)\geq t\quad\text{or}\quad(\mathbf{B})\ \overline{\dim}^{*,q}_{\pi,\mathrm{B}}(\mu)\geq t.$$

In case $(\mathbf{A})$, let $E\subset K$ with $\mu(E)>0$ and let $E'=E$. In case $(\mathbf{B})$, we begin by fixing $\varepsilon>0$ such that any subset E of K satisfying $\mu(E)\geq1-\varepsilon$ also satisfies $\mu(E\cap G)>0$. Then we let $E\subset K$ with $\mu(E)\geq1-\varepsilon$ and define $E'=E\cap G$. In both cases, we are going to show that $\overline{\dim}^q_{\pi,\mathrm{B}}(E')\geq t$.

Since $\mu\in\mathcal{R}$ we may find sequences $(A_n)\subset\mathcal{F}$, $(\mathbf{p}_n)$ with $\mathbf{p}_n\in\mathcal{Q}(A_n)$, and (s_n) going to zero such that

$$\mu\in B_L(\mu_{A_n,\mathbf{p}_n,s_n},\eta_{A_n,\mathbf{p}_n,s_n,s_n}).$$

For convenience, we set $r_n=r_{A_n,s_n}$, $\eta_n=\eta_{A_n,\mathbf{p}_n,s_n,s_n}$ and $E'_n=E'(r_n/2)$. Our assumption on η_n ensures that

$$\mu_{A_n,\mathbf{p}_n,s_n}(E'_n)\geq\mu(E')-s_n\geq\frac12\mu(E')$$

provided n is large enough. By construction of $\mu_{A_n,\mathbf{p}_n,s_n}$, we may find $x_n\in A_n$ such that $\mu_{x_n,s_n}(E'_n)\geq\frac12\mu(E')$. Moreover, x_n also belongs to F. This is clear in case $(\mathbf{A})$, and in case $(\mathbf{B})$, it follows from

$$\mu_{x_n,s_n}(E'_n)\leq\delta_{x_n}(G(\kappa/2))=\delta_{x_n}(F)=0,$$

provided $x_n \notin F$ and n is large enough so that $E'_n \subset G(\kappa/2)$. Hence, by definition of μ_{x_n,s_n} when $x_n \in F$, we obtain

$$\begin{aligned}\sum_{z\in\Lambda_{x_n,\,s_n}\cap E'_n} \pi(B(z,r_{x_n,s_n}))^q &\geq \frac{1}{2}\mu(E')\Big(\sum_{z\in\Lambda_{x_n,s_n}} \pi(B(z,r_{x_n,s_n}))^q\Big)\\ &\geq \frac{1}{2}\mu(E')r_{x_n,s_n}^{-t}.\end{aligned}$$

Now for any $z \in \Lambda_{x_n,s_n} \cap E'_n$, there exists $x_z \in E$ with $\|x_z - z\| \leq \frac{1}{2}r_n \leq r_{x_n,s_n}$. It is then not hard to show that $(B(x_z, r_{x_n,s_n}/2))_{z\in\Lambda_{x_n,s_n}\cap E'_n}$ is a centred packing of E. Indeed, for $u \neq v$ in Λ_{x_n,s_n},

$$\|x_u-x_v\| \geq \|u-v\|-\|u-x_u\|-\|v-x_v\| \geq 2r_{x_n,s_n}-\frac{r_{x_n,s_n}}{2}-\frac{r_{x_n,s_n}}{2} = r_{x_n,s_n}.$$

We also observe that, for any $z \in \Lambda_{x_n,s_n} \cap E'_n$,

$$B(x_z, r_{x,s_n}/2) \subset B(z, r_{x_n,s_n}) \subset B(x_z, 2r_{x_n,s_n}).$$

Summarizing, we have found a packing $(B(u,r))_{u\in\Lambda}$ of E' with r as small as we want, and a constant $c_0 \in \mathbb{R}$ ($c_0 = 2$ if $q \geq 0$, and $c_0 = 1/2$ if $q \leq 0$) so that

$$\sum_{u\in\Lambda} \pi(B(u,c_0 r))^q \geq \frac{1}{2}\mu(E')r^{-t}.$$

This yields $\overline{\dim}^q_{\pi,\mathrm{B}}(E') \geq t$ and concludes this part of the proof.

3.3. The upper bounds. We now turn to the proof of the upper bounds in Theorem 1.2, which are simpler. As before, we fix $q \in \mathbb{R}$. We first show that a generic $\mu \in \mathcal{P}(K)$ satisfies

$$\overline{\dim}^q_{*,\pi,\mathrm{B}}(\mu) \leq \tau_{\pi,\mathrm{loc}}(q).$$

Indeed, let $t > \tau_{\pi,\mathrm{loc}}(q)$. There exists $x_t \in K$ and $r_t > 0$ such that

$$\overline{\dim}^q_{\pi,\mathrm{B}}(B(x_t,r_t)) \leq t.$$

We set $\mathcal{U}_t = \{\mu \in \mathcal{P}(K);\ \mu(B(x_t,r_t)) > 0\}$. Then $\mathcal{U}_t$ is dense and open. Moreover, any $\mu \in \mathcal{U}_t$ satisfies $\overline{\dim}^q_{*,\pi,\mathrm{B}}(\mu) \leq t$. The residual set we are looking for is thus given by

$$\mathcal{R} = \bigcap_{t\in\mathbb{Q},\, t>\tau_{\pi,\mathrm{loc}}(q)} \mathcal{U}_t.$$

We now show that a generic $\mu \in \mathcal{P}(K)$ satisfies

$$\overline{\dim}^{*,q}_{\pi,\mathrm{B}}(\mu) \leq \tau_{\pi,\mathrm{loc,max}}(q).$$

As before, let $t > \tau_{\pi,\mathrm{loc,max}}(q)$. We just need to prove that a generic $\mu \in \mathcal{P}(K)$ satisfies $\overline{\dim}^{*,q}_{\pi,\mathrm{B}}(\mu) \leq t$. Let (y_n) be a dense sequence of distinct points in K, and let (κ_n) be a sequence decreasing to zero. For each n, we may find

$x_n \in B(y_n,\kappa_n)$ and $r_n>0$ such that $\overline{\dim}^q_{\pi,\mathrm{B}}(B(x_n,r_n))\leq t$. We may assume that the sequence (r_n) is going to zero.

We set

$$\Lambda_n=\Big\{\sum_{i=1}^n p_i\delta_{x_i};\ \sum_{i=1}^n p_i=1,\ p_i>0\Big\},$$

so that, by Lemma 2.3, the set $\bigcup_{n\geq m}\Lambda_n$ is dense for any integer $m\geq 1$. Moreover, Lemma 2.1 tells us that, for any $m\geq 1$, one may find $\eta_m>0$ such that, for any $\mu\in\Lambda_n$, and any $\nu\in\mathcal{P}(K)$ with $L(\mu,\nu)<\eta_m$,

$$\nu\Big(\bigcup_{i=1}^n B(x_i,r_n)\Big)\geq \mu\Big(\bigcup_{i=1}^m B(x_i,r_n/2)\Big)-\frac{1}{m}\geq 1-\frac{1}{m}.$$

We then set

$$\mathcal{R}=\bigcap_{m\geq 1}\bigcup_{n\geq m}\bigcup_{\mu\in\Lambda_n}B_L(\mu,\eta_m).$$

We observe that $\mathcal{R}$ is a dense G_δ-set. Pick $\nu\in\mathcal{R}$ and $\varepsilon>0$. Let also $m\geq 1$ with $1/m\leq\varepsilon$. We may find $n\geq m$ and $\mu\in E_n$ such that $L(\mu,\nu)<\eta_m$. Thus, defining $E=\bigcup_{i=1}^n B(x_i,r_n)$, we get

$$\nu(E)\geq 1-\frac{1}{m}\geq 1-\varepsilon,\qquad \overline{\dim}^q_{\pi,\mathrm{B}}(E)\leq t.$$

Therefore, $\overline{\dim}^{*,q}_{\pi,\mathrm{B}}(\nu)\leq t$.

4. The typical lower multifractal box dimensions. This section is devoted first to proving Theorem 1.3. We begin with a lemma which helps us to avoid the assumption "π is a doubling measure" throughout the proofs.

Lemma 4.1. *Let π be a Borel probability measure with compact support K. Then*

$$\begin{aligned}&\overline{D}_{\pi,\mathrm{unif}}(-\infty)\\&\quad=\inf_N\inf_{\substack{y_1,\dots,y_N\in K\\ \rho>0}}\limsup_{r\to 0}\inf_{i=1,\dots,N}\frac{\log(\inf_{B(x,r)\cap B(y_i,\rho)\neq\emptyset}\pi(B(x,r)))}{\log r},\\&\overline{D}_{\pi,\max}(-\infty)=\sup_{\substack{y\in K\\ \rho>0}}\limsup_{r\to 0}\frac{\log\inf_{B(x,r)\cap B(y,\rho)\neq\emptyset}\pi(B(x,r))}{\log r}.\end{aligned}$$

Proof. Let $t>\overline{D}_{\pi,\mathrm{unif}}(-\infty)$. One may find $y_1,\dots,y_N\in K$, $\rho>0$, $\alpha>0$ such that, for any $r\in(0,\alpha)$, there exists $i\in\{1,\dots,N\}$ such that any $x\in B(y_i,\rho)$ satisfies

$$\frac{\log(\pi(B(x,r)))}{\log r}\leq t. \tag{4.1}$$

We set $\rho_0 = \rho/2$ and $\alpha_0 = \min(\rho_0, \alpha)$. Let $r \in (0, \alpha_0)$, let $i \in \{1, \ldots, N\}$ be as above and let $x \in K$ with $B(x, r) \cap B(y_i, \rho_0) \neq \emptyset$. Then $x \in B(y_i, \rho)$ so that (4.1) holds true. Thus, since $t > \overline{D}_{\pi,\mathrm{unif}}(-\infty)$ is arbitrary,

$$\inf_N \inf_{\substack{y_1,\ldots,y_N \in K \\ \rho>0}} \limsup_{r\to 0} \inf_{i=1,\ldots,N} \frac{\log(\inf_{B(x,r)\cap B(y_i,\rho)\neq\emptyset} \pi(B(x,r)))}{\log r} \\ \leq \overline{D}_{\pi,\mathrm{unif}}(-\infty).$$

The opposite inequality is trivial, and the proof of the second assertion follows exactly the same lines. ■

4.1. Proof of Theorem 1.3, part 1. In this subsection, we shall prove that a generic measure $\mu \in \mathcal{P}(K)$ satisfies

$$\underline{\dim}^{*,q}_{\pi,\mathrm{B}}(\mu) \geq \begin{cases} -q\overline{D}_{\pi,\mathrm{unif}}(-\infty) & \text{provided } q \geq 0, \\ -q\underline{D}_{\pi,\mathrm{unif}}(+\infty) & \text{provided } q \leq 0. \end{cases}$$

Firstly, let $t > \overline{D}_{\pi,\mathrm{unif}}(-\infty)$ and let us prove that a generic $\mu \in \mathcal{P}(K)$ satisfies $\underline{\dim}^{*,q}_{\pi,\mathrm{B}}(\mu) \geq -qt$ for any $q \geq 0$. Let $N \geq 1$, $y_1, \ldots, y_N \in K$ and $\rho > 0$ be such that

$$\limsup_{r\to 0} \inf_{i=1,\ldots,N} \frac{\log(\inf_{B(x,r)\cap B(y_i,\rho)\neq\emptyset} \pi(B(x,r)))}{\log r} < t.$$

We set $\mathcal{U} = \bigcap_{i=1}^N \{\mu \in \mathcal{P}(K);\ \mu(B(y_i, \rho)) > 0\}$. Then $\mathcal{U}$ is a dense and open subset of $\mathcal{P}(K)$; let us pick $\mu \in \mathcal{U}$. There exists $\varepsilon > 0$ such that $\mu(E) > 1-\varepsilon$ implies $\mu(E \cap B(y_i, \rho)) > 0$ for any $i = 1, \ldots, N$. Let now $E \subset K$ with $\mu(E) > 1-\varepsilon$ and let r be sufficiently small. There exists $i \in \{1, \ldots, N\}$ such that

$$\frac{\log(\inf_{B(x,r)\cap B(y_i,\rho)\neq\emptyset} \pi(B(x,r)))}{\log r} < t.$$

Now,

$$\begin{aligned} \log \mathbf{N}^q_\pi(E, r) &\geq \log \mathbf{N}^q_\pi(E \cap B(y_i, \rho), r) \\ &\geq \log\Big(\inf_{B(x,r)\cap B(y_i,\rho)\neq\emptyset} \pi(B(x,r))^q\Big) \geq qt \log r. \end{aligned}$$

Hence, $\underline{\dim}^q_{\pi,\mathrm{B}}(E) \geq -qt$, which yields $\underline{\dim}^{*,q}_{\pi,\mathrm{B}}(\mu) \geq -qt$.

The proof for $q < 0$ is similar, but now we have to take $t < \underline{D}_{\pi,\mathrm{unif}}(+\infty)$. As before, there exist $y_1, \ldots, y_N \in K$, $\rho > 0$ and $\alpha > 0$ such that, for any $r \in (0, \alpha)$, there exists $i \in \{1, \ldots, N\}$ with

$$\frac{\log(\sup_{B(x,r)\cap B(y_i,\rho)\neq\emptyset} \pi(B(x,r)))}{\log r} > t.$$

We then carry on the same proof mutatis mutandis, except that now

$$\log \mathbf{N}^q_\pi(E, r) \geq q \log\Big(\sup_{B(x,r)\cap B(y_i,\rho)\neq\emptyset} \pi(B(x,r))\Big).$$

4.2. Proof of Theorem 1.3, part 2. In this subsection, we shall prove that a generic measure $\mu\in\mathcal{P}(K)$ satisfies

$$\underline{\dim}^{*,q}_{\pi,\mathrm{B}}(\mu)\le\begin{cases}-q\overline{D}_{\pi,\mathrm{unif}}(-\infty) & \text{provided } q\ge 0,\\ -q\underline{D}_{\pi,\mathrm{unif}}(+\infty) & \text{provided } q\le 0.\end{cases}$$

We just consider the case $q\ge 0$ and let $t<\overline{D}_{\pi,\mathrm{unif}}(-\infty)$. Let also (y_n) be a dense sequence in K, let (ρ_n) be a sequence decreasing to zero, and let (ε_n) be a sequence of positive real numbers with $\sum_n\varepsilon_n<1$. By assumption, for any $n\ge 1$, we may find $r_n\in(0,n^{-n})$ and points $x_1^n,\dots,x_n^n$ with $x_i^n\in B(y_i,\rho_n)$ such that, for any $i=1,\dots,n$,

$$\log(\pi(B(x_i^n,r_n)))\le t\log r_n.$$

We set

$$\varLambda_n=\Big\{\sum_{i=1}^n p_i\delta_{x_i^n};\ \sum_i p_i=1,\ p_i>0\Big\}$$

so that $\bigcup_{n\ge m}\varLambda_n$ is dense in $\mathcal{P}(K)$ for any $m\ge 1$. We also set $E_n=\{x_1^n,\dots,x_n^n\}$ so that $\mu(E_n)=1$ for any $\mu\in\varLambda_n$. Lemma 2.1 gives us a real number $\eta_n>0$ such that

$$\forall\mu\in\varLambda_n,\ L(\mu,\nu)<\eta_n\ \Rightarrow\ \nu(E_n(r_n))>1-\varepsilon_n.$$

We let $F_n=E_n(r_n)$ and we consider the dense G_δ-set

$$\mathcal{R}=\bigcap_{m\ge 1}\bigcup_{n\ge m}\bigcup_{\mu\in\varLambda_n}B_L(\mu,\eta_n).$$

Pick $\nu\in\mathcal{R}$. There exists a sequence (n_k) going to $+\infty$ and a sequence (μ_{n_k}) with $L(\nu,\mu_{n_k})<\eta_{n_k}$ for any k. Hence, $\nu(F_{n_k})>1-\varepsilon_{n_k}$. We define $G_l=\bigcap_{k\ge l}F_{n_k}$ so that $\nu(G_l)\to 1$ as $l\to+\infty$. On the other hand, for any $k\ge l$,

$$G_l\subset F_{n_k}\subset\bigcup_{i=1}^{n_k}B(x_i^{n_k},r_{n_k}).$$

Using this covering of G_l, we get

$$\log\mathbf{N}^q_\pi(G_l,r_{n_k})\le\sum_{i=1}^{n_k}\pi(B(x_i^{n_k},r_{n_k}))^q\le n_k r_{n_k}^{qt}.$$

Taking the logarithm and then the lim inf yields

$$\underline{\dim}^q_{\pi,\mathrm{B}}(G_l)\le -tq.$$

Since $\nu(G_l)$ can be arbitrarily close to 1, this implies $\underline{\dim}^{*,q}_{\pi,\mathrm{B}}(\nu)\le -qt$.

4.3. Proof of Theorem 1.4, part 1. We turn to the study of the small lower multifractal dimensions of a generic measure. More specifically, in this subsection, we prove that a generic $\mu\in\mathcal{P}(K)$ satisfies $\underline{\dim}^q_{*,\pi,\mathrm{B}}(\mu)\ge$

$-q\overline{D}_{\pi,\max}(-\infty)$ for any $q \geq 0$. Hence, let $t > \overline{D}_{\pi,\max}(-\infty)$. Let $(y_n)_n$ be a dense sequence in K and let $(\rho_n)_n$ be a sequence of positive real numbers decreasing to zero. Let us fix $n \geq 1$. One may find $\alpha_n > 0$ such that, for any $r \in (0, \alpha_n)$, $k \in \{1, \ldots, n\}$, and $x \in K$ such that $B(x,r) \cap B(y_k, \rho_n) \neq \emptyset$,

$$\log \pi(B(x,r)) \geq t \log r.$$

We then set

$$\Lambda_n = \Big\{ \sum_{i=1}^{n} p_i \delta_{y_i};\, p_i > 0,\, \sum_i p_i = 1 \Big\}, \qquad F_n = \{y_1, \ldots, y_n\}.$$

Any $\mu \in \Lambda_n$ satisfies $\mu(F_n) = 1$. Hence, we may find $\eta_n > 0$ such that $\nu(F_n(\rho_n)) > 1 - 1/n$ provided $L(\mu,\nu) < \eta_n$. We finally consider

$$\mathcal{R} = \bigcap_{m\geq 1} \bigcup_{n \geq m} \bigcup_{\mu \in \Lambda_n} B_L(\mu, \eta_n).$$

Pick ν in the dense G_δ-set $\mathcal{R}$ and let $E \subset K$ with $\nu(E) > 0$. We may find n as large as we want such that $\nu(E \cap F_n(\rho_n)) > 0$. Now, for any $r \in (0, \alpha_n)$,

$$\begin{aligned} \log \mathbf{N}_\pi^q(E,r) &\geq \log \mathbf{N}_\pi^q(E \cap F_n(\rho_n), r) \\ &\geq \log\Big(\inf_{B(x,r)\cap F_n(\rho_n) \neq \emptyset} \pi(B(x,r))^q \Big) \geq qt \log r. \end{aligned}$$

Hence, $\underline{\dim}^q_{*,\pi,\mathrm{B}}(\nu) \geq -qt$.

4.4. Proof of Theorem 1.4, part 2. We conclude the proof of Theorem 1.4 by showing that a generic $\mu \in \mathcal{P}(K)$ satisfies $\underline{\dim}^q_{*,\pi,\mathrm{B}}(\mu) \leq -q\overline{D}_{\pi,\mathrm{unif},\max}(-\infty)$ for any $q \geq 0$. We begin by fixing $t < \overline{D}_{\pi,\mathrm{unif},\max}(-\infty)$. There exist $z \in K$ and $\kappa > 0$ such that

$$t < \inf_{\substack{y_1,\ldots,y_N \in B(z,\kappa) \\ \rho > 0}} \limsup_{r \to 0} \inf_{i=1,\ldots,N} \frac{\log(\inf_{x \in B(y_i,\rho)} \pi(B(x,r)))}{\log r}.$$

The proof now follows part 2 of the proof of Theorem 1.3, except that we "localize" it in $K \cap B(z,\kappa)$. Specifically, we now consider a dense sequence (y_n) in $K \cap B(z,\kappa)$. We construct the sequences (ρ_n), (ε_n), (r_n) and (x_n^i) as above, but starting from this sequence (y_n) and from the property

$$\forall n \geq 1,\ \limsup_{r\to 0} \inf_{i=1,\ldots,n} \frac{\log(\inf_{x\in B(y_i,\rho_n)} \pi(B(x,r)))}{\log r} \geq t.$$

We also ask that for any $n \geq 1$ and any $i \in \{1, \dots, n\}$, $B(x_i^n, r_n)$ is contained in $B(z, \kappa)$. Next, for any $n \geq 1$, we now set

$$\begin{aligned} \Lambda_n &= \Big\{\lambda \sum_{i=1}^{n} p_i \delta_{x_i^n} + (1-\lambda)\theta;\ \lambda, p_i \in (0,1),\ \sum_i p_i = 1,\ \theta \in \mathcal{P}(K), \\ &\qquad\qquad\qquad\qquad \operatorname{supp}(\theta) \cap B(z, \kappa + 2r_n) = \emptyset\Big\}, \\ E_n &= \{x_1^n, \dots, x_n^n\}, \qquad F_n = E_n(r_n). \end{aligned}$$

It is not hard to show that, for any $m \geq 1$, the set $\bigcup_{n \geq m} \Lambda_n$ remains dense in $\mathcal{P}(K)$. Moreover, for any $\mu \in \Lambda_n$, we may find $\eta_{n,\mu} > 0$ such that

$$L(\nu, \mu) < \eta_{n,\mu} \ \Rightarrow\ \begin{cases} \nu(F_n) \geq \lambda(1-\varepsilon_n), \\ \nu(B(z,\kappa)) \leq \lambda(1-\varepsilon_n)^{-1}. \end{cases}$$

Let $\mathcal{R}$ be the dense G_δ-subset of $\mathcal{P}(K)$ defined by

$$\mathcal{R} = \bigcap_{m \geq 1} \bigcup_{n \geq m} \bigcup_{\mu \in \Lambda_n} B_L(\mu, \delta_{n,\mu}) \cap \{\nu \in \mathcal{P}(K);\ \nu(B(z,\kappa)) > 0\}.$$

Let $\nu \in \mathcal{R}$ and let (n_k) be a sequence growing to $+\infty$ such that

$$\nu(F_{n_k}) \geq (1 - \varepsilon_{n_k})^2 \nu(B(z,\kappa))$$

for any $k \geq 1$. We finally define $G = \bigcap_n F_{n_k}$. Since any F_n is contained in $B(z,\kappa)$, the previous inequality ensures that $\nu(G) > 0$ provided (ε_n) goes sufficiently fast to 0. On the other hand, for any $k \geq 1$,

$$G \subset F_{n_k} \subset \bigcup_{i=1}^{n_k} B(x_i^{n_k}, r_{n_k}).$$

This yields (see part 2 of the proof of Theorem 1.3)

$$\mathbf{N}_\pi^q(G, r_{n_k}) \leq n_k r_{n_k}^{qt}$$

so that $\underline{\dim}^q_{*,\pi,\mathrm{B}}(\nu) \leq -qt$.

4.5. Application to self-similar sets. We now show how to apply Theorems 1.3 and 1.4 to self-similar compact sets. Let $M \geq 2$, and let $S_1, \dots, S_M : \mathbb{R}^d \to \mathbb{R}^d$ be contracting similarities with respective ratios $r_1, \dots, r_M \in (0,1)$. Let $(p_1, \dots, p_M)$ be a probability vector. Let K be a nonempty compact subset of $\mathbb{R}^d$ and let π be the probability measure in $\mathcal{P}(K)$ satisfying

$$K = \bigcup_{m=1}^{M} S_i(K), \qquad \pi = \sum_{m=1}^{M} p_i \pi \circ S_m^{-1}.$$

We just need to prove the following proposition.

Proposition 4.2. *Let K and π be as above and assume that the Open Set Condition is satisfied. Define*

$$s_{\min} = \min_m \frac{\log p_m}{\log r_m} \quad \textit{and} \quad s_{\max} = \max_m \frac{\log p_m}{\log r_m}.$$

Then

$$\overline{D}_{\pi,\mathrm{unif}}(-\infty) = \overline{D}_{\pi,\mathrm{unif},\max}(-\infty) = \overline{D}_{\pi,\max}(-\infty) = s_{\max},$$
$$\underline{D}_{\pi,\mathrm{unif}}(+\infty) = \underline{D}_{\pi,\mathrm{unif},\min}(+\infty) = \underline{D}_{\pi,\min}(-\infty) = s_{\min}.$$

Proof. We just give the proof of the first inequality. It is straightforward to check that

$$\overline{D}_{\pi,\max}(-\infty) \geq \overline{D}_{\pi,\mathrm{unif},\max}(-\infty) \geq \overline{D}_{\pi,\mathrm{unif}}(-\infty).$$

Thus we just need to prove that

$$\overline{D}_{\pi,\mathrm{unif}}(-\infty) \geq s_{\max} \quad \text{and} \quad \overline{D}_{\pi,\max}(-\infty) \leq s_{\max}.$$

Without loss of generality, we may assume that the diameter of K is less than 1. We shall use standard notations which can be found e.g. in [Fal97]. For a word $\mathbf{m} = (m_1, \dots, m_n)$ in $\{1, \dots, M\}^n$ of length n, let

$$S_{\mathbf{m}} = S_{m_1} \circ \cdots \circ S_{m_n}, \qquad p_{\mathbf{m}} = p_{m_1} \times \cdots \times p_{m_n}, \qquad r_{\mathbf{m}} = r_{m_1} \times \cdots \times r_{m_n}.$$

If the word $\mathbf{m}$ is infinite, then $S_{\mathbf{m}}(K) = \bigcap_{i=1}^{+\infty} S_{m_i}(K)$ reduces to a single point $x_{\mathbf{m}} \in K$ and each point of K is uniquely defined by such a word. Let now $y \in K$, $\rho > 0$ and let l be such that $\frac{\log p_l}{\log r_l} = s_{\max}$. There exists a word $\mathbf{m} = (m_1, \dots, m_n)$ such that $S_{\mathbf{m}}(K) \subset B(y, \rho)$. We then define

$$\overline{\mathbf{m}} = (m_1, \dots, m_n, l, \dots), \qquad \overline{\mathbf{m}_k} = (m_1, \dots, m_n, l, \dots, l)$$

where l appears k times at the end of $\overline{\mathbf{m}_k}$. We define x_y as $S_{\overline{\mathbf{m}}}(K)$. Now, for any $k \geq 1$, there exists $z \in K$ such that $x = S_{\overline{\mathbf{m}_k}} z$, so that $B(x_y, r_{\overline{\mathbf{m}_k}}) = S_{\overline{\mathbf{m}_k}}(B(z, 1))$. Now the definition of π and the open set condition ensure that

$$\pi(S_{\overline{\mathbf{m}_k}}(B(z, 1))) = p_{\overline{\mathbf{m}_k}} \pi(B(z, 1)) = p_{\overline{\mathbf{m}_k}}$$

since the diameter of K is less than 1. Thus, for any $k \geq 1$,

$$\pi(B(x_y, r_l^{k+n})) \leq \pi(B(x_y, r_{\overline{\mathbf{m}_k}})) \leq p_{\overline{\mathbf{m}_k}} = p_{m_1} \dots p_{m_n} p_l^k.$$

Finally, let $N \geq 1$, let $y_1, \dots, y_N \in K$ and let $\rho > 0$. To each y_i, we can associate a word $\mathbf{m}^i$ of length n^i and a point x_i as above. Let $n = \max(n^i)$. Then for any $i = 1, \dots, N$,

$$\frac{\log \pi(B(x_i, r_l^{k+n}))}{(k+n) \log r_l} \geq \frac{C}{k+n} + \frac{k}{k+n} s_{\max}$$

where C does not depend on k. Letting $k \to +\infty$ gives $\overline{D}_{\pi,\mathrm{unif}}(-\infty) \geq s_{\max}$.

On the other hand, it is well known that $\overline{D}_{\pi}(-\infty) \leq s_{\max}$ (see for instance [Pat97]). By the homogeneity of self-similar sets and self-similar measures, this implies $\overline{D}_{\pi,\max}(-\infty) \leq s_{\max}$. ∎

References

[Bay12] F. Bayart, *How behave the typical L^q-dimensions of measures?*, arXiv:1203.2813, 2012.

[Fal97] K. Falconer, *Techniques in Fractal Geometry*, Wiley, 1997.

[MR02] J. Myjak and R. Rudnicki, *On the box dimension of typical measures*, Monatsh. Math. 136 (2002), 143–150.

[Ols05] L. Olsen, *Typical L^q-dimensions of measures*, Monatsh. Math. 146 (2005), 143–157.

[Ols11] L. Olsen, *Typical multifractal box dimensions of measures*, Fund. Math. 211 (2011), 245–266.

[Par67] K. R. Parthasarathy, *Probability Measures on Metric Spaces*, Probab. Math. Statist., Academic Press, 1967.

[Pat97] N. Patzschke, *Self-conformal multifractal measures*, Adv. Appl. Math. 19 (1997), 486–513.

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