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Selections and weak orderability

by

Michael Hrušák and **Iván Martínez-Ruiz** (Morelia)

Dedicated to Jan Pelant

Abstract. We answer a question of van Mill and Wattel by showing that there is a separable locally compact space which admits a continuous weak selection but is not weakly orderable. Furthermore, we show that a separable space which admits a continuous weak selection can be covered by two weakly orderable spaces. Finally, we give a partial answer to a question of Gutev and Nogura by showing that a separable space which admits a continuous weak selection admits a continuous selection for all finite sets.

1. Introduction. The study of continuous selections was initiated by E. Michael in his seminal 1951 paper [14]. He considered the *hyperspace* 2^X of all non-empty closed subsets of a topological space X equipped with the *Vietoris topology*, i.e. the topology on 2^X generated by sets of the form

$$\langle U; V_0, \dots, V_n\rangle = \{F \in 2^X : F \subseteq U \text{ and } F \cap V_i \neq \emptyset \text{ for any } i \leq n\},$$

where $U, V_0, \dots, V_n$ are open subsets of X. A function φ defined on 2^X (or some subspace of 2^X) is a *selection* if $\varphi(F) \in F$ for every member of its domain. A selection is *continuous* if it is continuous with respect to the Vietoris topology. In particular, a *weak selection* is a selection defined on $[X]^2$, the set of all two-element subsets of X.

The general question studied in Michael's and subsequent articles is: *When does a space admit a continuous* (*weak*) *selection?* In his paper, Michael has shown that a sufficient condition for a space X to admit a continuous weak selection is that it admits a weaker topology generated by a linear order, i.e. that the space is *weakly orderable*. The natural question, whether this characterizes spaces which admit continuous weak selections, implicit in Michael's paper, was stated explicitly in a paper by J. van Mill

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and E. Wattel [15]. Michael himself showed that the answer is positive for connected spaces and that for compact connected spaces the existence of a continuous weak selection on X is equivalent to *orderability* of X. J. van Mill and E. Wattel showed the same for all compact spaces (not necessarily connected). Building on work of G. Artico, U. Marconi, J. Pelant, L. Rotter and M. Tkachenko [1], S. García Ferreira and M. Sanchis [7] showed that a pseudocompact space X admits a continuous weak selection if and only if the Čech–Stone compactification βX is orderable, and consequently, if and only if X is *suborderable* (or a GO-space). Improving on Michael's result stated above, T. Nogura and G. Shakhmatov proved in [16] that a locally connected space X admits a continuous weak selection if and only if it is orderable. Recently, V. Gutev and T. Nogura [11], in a very nice survey article on the *selection problem*, restated van Mill–Wattel's question and asked, in particular, whether a locally compact space admitting a continuous weak selection is weakly orderable.

Here we will answer both questions in the negative by constructing a separable locally compact Tikhonov space which admits a continuous weak selection but is not weakly orderable. In fact, our space is a *Ψ-space*, a natural space associated to a carefully constructed almost disjoint family on a countable set.

We further study the existence of continuous weak selections on separable spaces. We show that, although the answer to van Mill–Wattel's question is negative even for separable spaces, every separable space admitting a continuous weak selection can be covered by two weakly orderable subspaces. As a corollary, we provide a partial answer to another question of Gutev and Nogura [10] by showing that a separable space admitting a continuous weak selection admits in fact a continuous selection for all finite sets.

All spaces considered here are at least Hausdorff. In general, all our spaces X are Tikhonov, though the hyperspaces 2^X typically are not. Given a space X, we will work with the following special subsets of 2^X, where $n \geq 1$:

$$\mathcal{F}_n(X) = \{F \in 2^X : |F| \leq n\}, \qquad [X]^n = \{F \in 2^X : |F| = n\},$$
$$\mathrm{Fin}(X) = \bigcup\{\mathcal{F}_n(X) : n \in \omega\}, \quad \mathcal{K}(X) = \{F \in 2^X : F \text{ is compact}\}.$$

We will denote by $\mathrm{Sel}(\mathcal{A})$ the collection of continuous selections for $\mathcal{A}$. In particular, $\mathrm{Sel}(\mathcal{F}_2(X))$ consists of all continuous weak selections on the space X. It is easy to see that X admits a continuous weak selection if and only if there is a continuous function $\varphi : X^2 \to X$ such that $\varphi(x,y) = \varphi(y,x) \in \{x,y\}$ for every $x, y \in X$. We will refer to such a φ also as a weak selection.

Given an ordered set $(X, \leq)$ and $x \in X$, we denote by $(\leftarrow, x)_{\leq}$ the *initial segment* and by $(x, \rightarrow)_{\leq}$ the *final segment* determined by x, respectively;

i.e. $(\leftarrow, x)_\leq = \{y \in X : y < x\}$ and $(x, \rightarrow)_\leq = \{y \in X : x < y\}$. Similarly, $(\leftarrow, x]_\leq = X \setminus (x, \rightarrow)_\leq$ and $[x, \rightarrow)_\leq = X \setminus (\leftarrow, x)_\leq$.

Our set-theoretic notation is mostly standard and follows [13]. In particular, ω stands for the set of all natural numbers (finite ordinals) and $[\omega]^\omega$ the set of all infinite subsets of ω. $A \subseteq^* B$ denotes that A is *almost contained in* B, i.e. $A \setminus B$ is finite; $A =^* B$ means that $A \subseteq^* B$ and $B \subseteq^* A$. Recall also that if $\mathcal{C} \subseteq [\omega]^\omega$, then $A \in [\omega]^\omega$ is a *pseudointersection of* $\mathcal{C}$ if $A \subseteq^* C$ for every $C \in \mathcal{C}$.

Concerning weak selections we introduce the following notation. Let X and Y be sets and $\psi : [X]^2 \to X$ and $\varphi : [Y]^2 \to Y$ weak selections. We will say that ψ and φ are *isomorphic*, $\psi \approx \varphi$, if there is a bijection $\varrho : X \to Y$ such that $\psi(\{a, b\}) = \varphi(\{\varrho(a), \varrho(b)\})$ for every $a, b \in X$. We will also say that ψ is *embedded* in φ if $\psi \approx \varphi \restriction [A]^2$ for some $A \subseteq X$. Let φ be a weak selection on a set X and let $x, y \in X$. We will denote by $x \to_\varphi y$ the condition $\varphi(x, y) = y$. If $A, B \subseteq X$, we will say that B *dominates* A *with respect to* φ, denoted by $A \rightrightarrows_\varphi B$, if $a \to_\varphi b$ for all $a \in A$ and $b \in B$. We will also say that A and B are *aligned* with respect to φ, and write $A \parallel_\varphi B$, if $A \rightrightarrows_\varphi B$ or $B \rightrightarrows_\varphi A$. Given $A, B \in [\omega]^\omega$ and ψ a weak selection on ω, we will say that B *almost dominates* A *with respect to* ψ (or simply that B *almost dominates* A if ψ is clear from context), and write $A \rightrightarrows^*_\psi B$, if there is a $k \in \omega$ such that $A \setminus k \rightrightarrows_\psi B \setminus k$. We will also say that A and B are *almost aligned* with respect to ψ, denoted by $A \parallel^*_\psi B$, if $A \rightrightarrows^*_\psi B$ or $B \rightrightarrows^*_\psi A$. If $n \in \omega$ then we will say that A *is almost dominated by* $\{n\}$, written $A \rightrightarrows^*_\psi \{n\}$, whenever $A \setminus k \rightrightarrows_\psi \{n\}$ for some $k \in \omega$. In a similar way, we define $\{n\} \rightrightarrows^*_\psi A$ and $\{n\} \parallel^*_\psi A$. When the selection is clear from context, we suppress the use of the subscript. Given a weak selection φ, a triple $\{a, b, c\}$ is called a 3-*cycle* if either $a \to b \to c \to a$ or $c \to b \to a \to c$.

2. A solution to van Mill and Wattel's question. In this section we will answer van Mill and Wattel's question by constructing a separable locally compact Tikhonov space which admits a continuous weak selection, yet is not weakly orderable. In fact, our space is going to be a Mrówka–Isbell space associated to a certain almost disjoint family on a countable set.

2.1. *Extensions of selections to Mrówka–Isbell spaces.* Recall that a family $\mathcal{A} \subseteq [\omega]^\omega$ is *almost disjoint* (AD) if any two distinct elements of $\mathcal{A}$ have finite intersection. A family $\mathcal{A}$ is *MAD* if it is *AD* and maximal with respect to this property.

The *Mrówka–Isbell space* $\Psi(\mathcal{A})$ associated to an *AD* family $\mathcal{A}$ is defined as follows: The underlying set is $\omega \cup \mathcal{A}$, all the elements of ω are isolated and the basic neighborhoods of $A \in \mathcal{A}$ are of the form $\{A\} \cup (A \setminus F)$ for some finite set $F \subseteq \omega$.

It follows immediately from the definition that $\Psi(\mathcal{A})$ is a first countable and locally compact space. It is well known and easy to see that $\Psi(\mathcal{A})$ is pseudocompact if and only if the family $\mathcal{A}$ is *MAD* [4]. Continuous selections on Mrówka–Isbell spaces were considered in [12], where it was shown that $\Psi(\mathcal{A})$ does not admit a continuous weak selection if $\mathcal{A}$ is *MAD*, and that $\Psi(\mathcal{A})$ does not admit a selection for $2^{\Psi(\mathcal{A})}$ for any uncountable $\mathcal{A}$.

The next easy lemma characterizes when a weak selection on ω extends to a continuous weak selection on $\Psi(\mathcal{A})$.

Lemma 2.1. *Let φ be a weak selection on ω and let $\mathcal{A}$ be an almost disjoint family. Then φ extends* (*uniquely*) *to a continuous weak selection on $\Psi(\mathcal{A})$ if and only if*

(1) $A \parallel_{\varphi}^{*} B$ *for all* $A \neq B \in \mathcal{A}$,
(2) $\{n\} \parallel_{\varphi}^{*} A$ *for all* $n \in \omega$ *and* $A \in \mathcal{A}$.

Our plan for constructing the space is to first find a suitable weak selection on ω and then to carefully construct an *AD* family to which the selection extends. It should be noted here that such a selection has to be rather complicated. Consider, for instance, a weak selection on ω defined by $\varphi(\{m,n\}) = \min\{m,n\}$. Then φ cannot be extended to any *AD* family which has more than one element, as no two infinite subsets of ω are almost aligned with respect to φ.

2.2. *Universal weak selection.* We will describe a sufficiently complex weak selection on ω here. It is, in fact, the most complicated selection on ω and is an "oriented" version of Rado's random graph (see [2], [5] and [17]). It can be easily defined as a Fraïse limit; here we define it directly from a countable independent family. Recall that a family $\mathcal{I} \subseteq [\omega]^{\omega}$ is *independent* if $\bigcap \mathcal{F} \setminus \bigcup \mathcal{F}'$ is infinite for any finite disjoint subsets $\mathcal{F}, \mathcal{F}'$ of $\mathcal{I}$.

Proposition 2.2. *There is a weak selection $\varphi : [\omega]^2 \to \omega$ with the following extension property*:

($\mathcal{D}$) *For any disjoint $F, G \in [\omega]^{<\omega}$, there is an $n \in \omega \setminus (F \cup G)$ such that* $F \rightrightarrows_{\varphi} \{n\} \rightrightarrows_{\varphi} G$.

Proof. Let $\mathcal{J} = \{J_n : n \in \omega\} \subseteq [\omega]^{\omega}$ be an independent family. Recursively define a family $\mathcal{I} = \{I_n : n \in \omega\}$ in the following way:

- $I_0 = J_0$;
- $I_{n+1} = (J_{n+1} \setminus \{k \leq n : n+1 \in I_k\}) \cup \{k \leq n : n+1 \notin I_k\}$.

For every $n \in \omega$, the set $I_n \in \mathcal{I}$ is obtained by finite changes of J_n, guaranteeing that $\mathcal{I}$ is also an independent family such that $n \in I_m$ if and only if $m \notin I_n$, for all $n, m \in \omega$. Let $\varphi : [\omega]^2 \to \omega$ be defined by $\varphi(\{n,m\}) = n$ if and only if $n \in I_m$.

To conclude the proof, it is enough to verify that φ satisfies $(\mathcal{D})$; but this follows from the fact that $\mathcal{I}$ is independent: if $F, G \in [\omega]^{<\omega}$ are disjoint then $F \rightrightarrows_\varphi \{k\} \rightrightarrows_\varphi G$ for any $k \in (\bigcap_{n\in F} I_n) \cap (\bigcap_{m\in G}(\omega \setminus I_m))$. ∎

In what follows, φ will denote the weak selection described in the previous proposition and will be called the *universal weak selection*. The next proposition gathers basic facts about the universal weak selection and is a direct translation of basic properties of the random graph [2]. We include the proof for the sake of completeness. Let $\mathcal{R} = \{A \subseteq \omega : \varphi\restriction[A]^2 \approx \varphi\}$.

PROPOSITION 2.3. *Let φ be the universal weak selection. Then:*

(a) *φ is, up to isomorphism, the unique weak selection with property $\mathcal{D}$.*
(b) *Every weak selection ψ on ω can be embedded in φ.*
(c) *Given any partition $\{P_0, P_1\}$ of ω, there is an $i \in 2$ such that $P_i \in \mathcal{R}$.*
(d) *If $F, G \in [\omega]^{<\omega}$ are disjoint, then*

$$\{k \in \omega \setminus (F \cup G) : F \rightrightarrows_\varphi \{k\} \rightrightarrows_\varphi G\} \in \mathcal{R}.$$

Proof. (a) and (b) follow by an application of the back-and-forth argument. To verify (c), suppose the contrary and let $\{P_0, P_1\}$ be a partition of ω such that neither P_0 nor P_1 is in $\mathcal{R}$. As $\varphi\restriction[P_i]^2$ does not satisfy $(\mathcal{D})$, we can find disjoint $F_i, G_i \in [P_i]^{<\omega}$ such that for each $n \in P_i$ either n does not dominate F_i or n is not dominated by G_i. Since $(\mathcal{D})$ is satisfied by φ, there is an $m \in \omega$ so that $F_0 \cup F_1 \rightrightarrows \{m\} \rightrightarrows G_0 \cup G_1$. However, $m \in P_0$ or $m \in P_1$, a contradiction in either case. Finally, to verify (d), suppose that for a couple F, G of finite disjoint subsets of ω, the set $A = \{k \in \omega \setminus (F \cup G) : F \rightrightarrows \{k\} \rightrightarrows G\}$ is not in $\mathcal{R}$. It follows by (c) that $\omega \setminus A \in \mathcal{R}$ and so one can find $n \in \omega \setminus A$ that dominates F and is dominated by G; but this n must also be in A, which is a contradiction. ∎

We are now interested in studying the universal weak selection in relation to linear orders on ω. Let $\le$ be a linear order on a set X and let $Y \subseteq X$ be infinite. We will say that the set Y is *monotone* if either there is a downward closed set $S \subseteq X$ such that $Y \subseteq S$ and $Y \cap (\leftarrow, s)_\le$ is finite for every $s \in S$, or there is an upward closed set $T \subseteq X$ such that $Y \subseteq T$ and $Y \cap (t, \rightarrow)_\le$ is finite for every $t \in T$.

PROPOSITION 2.4. *Let φ be the universal selection and let $\preccurlyeq$ be a linear order on ω. If $X \subseteq \omega$ belongs to $\mathcal{R}$, then there are $X_0, X_1 \in [X]^\omega$ such that*

(1) $X_0 \cap X_1 = \emptyset$,
(2) $X_0 \rightrightarrows X_1$,
(3) $X_0 \cup X_1$ *is monotone.*

Proof. If $X \cap (\leftarrow, 0)_\preccurlyeq \in \mathcal{R}$, then define $M_0 = X \cap (\leftarrow, 0)_\preccurlyeq$; otherwise let $M_0 = X \cap [0, \rightarrow)_\preccurlyeq$. As $X \in \mathcal{R}$, in either case $M_0 \in \mathcal{R}$ by 2.3(e). Choose distinct $a_0, b_0, c_0 \in M_0$ so that $\{a_0, b_0, c_0\}$ is a 3-cycle in M_0. Choose now

$x_0, y_0 \in \{a_0, b_0, c_0\}$ such that $x_0 \prec y_0$ and $x_0 \to y_0$ and define the set $D_1 = \{n \in M_0 : x_0 \to n \to y_0\} \setminus \{x_0, y_0\}$, which, by Proposition 2.3(d), is in $\mathcal{R}$. As before, let $M_1 = D_1 \cap (\leftarrow, 1)_{\preccurlyeq}$ if $D_1 \cap (\leftarrow, 1)_{\preccurlyeq} \in \mathcal{R}$, and $M_1 = D_1 \cap [1, \to)_{\preccurlyeq}$ otherwise. Choose $a_1, b_1, c_1 \in M_1$ so that $\{a_1, b_1, c_1\}$ is a 3-cycle in M_1 and pick $x_1, y_1 \in \{a_1, b_1, c_1\}$ such that $x_1 \to y_1$ and $y_1 \prec x_1$. Notice that $\{x_0, x_1\} \rightrightarrows \{y_0, y_1\}$.

Following this procedure, we can form recursively $\{M_n : n \in \omega\} \subseteq \mathcal{R}$ and disjoint subsets $W_0 = \{x_n : n \in \omega\}, W_1 = \{y_n : n \in \omega\} \in [X]^\omega$ such that for every $n \in \omega$, $M_{n+1} \subseteq M_n$, $\{x_0, x_1, \ldots, x_n\} \rightrightarrows \{y_0, y_1, \ldots, y_n\}$, $x_n \prec y_n$ whenever n is even, and $y_n \prec x_n$ if n is odd. Moreover, the set $S = \{n \in \omega : M_n \subseteq (n, \to)_{\preccurlyeq}\}$, if infinite, is $\preccurlyeq$-downward closed, while $T = \{n \in \omega : M_n \subseteq (\leftarrow, n)_{\preccurlyeq}\}$ is $\preccurlyeq$-upward closed, if it is infinite. Notice also that $(W_0 \cup W_1) \cap (\leftarrow, k)_{\preccurlyeq}$ is finite for every $k \in S$, as also is $(W_0 \cup W_1) \cap (k, \to)_{\preccurlyeq}$ for every $k \in T$.

To conclude the proof, notice that either $W_0 \cap S$ and $W_1 \cap S$ are both infinite, or both $W_0 \cap T$ and $W_1 \cap T$ are. To see this, suppose, e.g., that $W_0 \cap S$ is finite. As $S \cup T = \omega$, there is some $k \in \omega$ such that for all $n \geq k$, $x_n \in T$. Whenever $m \geq k$ is even, then $x_m \prec y_m$ and as T is $\preccurlyeq$-upward closed, also $y_m \in T$. If both $W_0 \cap S, W_1 \cap S$ are infinite, define $X_0 = W_0 \cap S$ and $X_1 = W_1 \cap S$, if not, let $X_0 = W_0 \cap T$ and $X_1 = W_1 \cap T$. The recursion guarantees that whenever $k \geq n$, then $x_k, y_k \in M_n$, and consequently the set $X_0 \cup X_1$ is monotone. ∎

2.3. *The construction.* Here we will show how to construct an almost disjoint family $\mathcal{B}$ such that the universal selection extends to $\Psi(\mathcal{B})$, yet $\Psi(\mathcal{B})$ is not weakly orderable. The next lemma shows that the universal selection can be extended to a large almost disjoint family.

Lemma 2.5. *There is an AD family $\mathcal{A} \subseteq [\omega]^\omega$ such that*:

(1) $|\mathcal{A}| = \mathfrak{c}$,
(2) $\mathcal{A} \subseteq \mathcal{R}$,
(3) $A \parallel^* B$ *for every* $A \neq B \in \mathcal{A}$.

Proof. Consider the complete binary tree $2^{<\omega}$ and for every $f \in 2^\omega$, consider the branch determined by f, $A_f = \{f \restriction n : n \in \omega\}$. For $f, g \in 2^{<\omega}$, we write $f \perp g$ if there is an $n \in \omega$ so that $f(n) \neq g(n)$, and $f \not\perp g$ whenever either $f \subseteq g$ or $g \subseteq f$. Define the weak selection ψ on $2^{<\omega}$ by $\psi(\{f, g\}) = g$ if and only if either $f \not\perp g$ and $\varphi(\{|f|, |g|\}) = |g|$, or $f \perp g$ and $f(f \vartriangle g) = 0$, where $f \vartriangle g = \min\{k \in \omega : f(k) \neq g(k)\}$.

By the universality of φ, we can suppose that ψ is embedded in φ. It is easy to see that $A_f \in \mathcal{R}$ for every $f \in 2^\omega$. Moreover, $(A_f \setminus f \vartriangle g) \rightrightarrows (A_g \setminus f \vartriangle g)$ if $f(f \vartriangle g) = 0$, and $(A_g \setminus f \vartriangle g) \rightrightarrows (A_f \setminus f \vartriangle g)$ otherwise, which implies that $A_f \parallel^* A_g$. Therefore $\mathcal{A} = \{A_f : f \in 2^\omega\}$ is the required family. ∎

Let $\mathcal{A} = \{A_\alpha : \alpha < \mathfrak{c}\}$ be the almost disjoint family constructed in the lemma. Next we will show how to refine $\mathcal{A}$ to "kill" all potential linear orders on ω. To that end, enumerate the collection of all linear orders on ω as $\{\leq_\alpha : \alpha < \mathfrak{c}\}$.

LEMMA 2.6. *For every $\alpha < \mathfrak{c}$, there are $X_0^\alpha, X_1^\alpha \in [A_\alpha]^\omega$ such that*

(1) $X_0^\alpha \cap X_1^\alpha =^* \emptyset$,
(2) $X_0^\alpha \parallel^* X_1^\alpha$,
(3) *for every $n \in \omega$ and $i \in 2$, $X_i^\alpha \parallel^* \{n\}$,*
(4) $X_0^\alpha \cup X_1^\alpha$ *is $\leq_\alpha$-monotone.*

Proof. Fix $\alpha < \mathfrak{c}$. By Lemma 2.5, $A_\alpha \in \mathcal{R}$ and by Proposition 2.4 we can find $X_0, X_1 \in [A_\alpha]^\omega$ such that $X_0 \rightrightarrows X_1$ and $X_0 \cup X_1$ is $\leq_\alpha$-monotone. Since for every $x \in X_0$, either $x \to 0$ or $0 \to x$, there is an infinite $C_0 \subseteq X_0$ such that $C_0 \parallel \{0\}$. Proceeding recursively, construct a family $\mathcal{C} = \{C_n : n \in \omega\}$ of infinite subsets of X_0 such that for every $n \in \omega$, $C_{n+1} \subseteq C_n$ and $C_n \parallel \{n\}$. Let X_0^α be a pseudointersection of $\mathcal{C}$, i.e. $X_0^\alpha \in [X_0]^\omega$ is such that $C_n \setminus X_0^\alpha$ is finite for every $n \in \omega$. Analogously, construct a family $\mathcal{E} = \{E_n : n \in \omega\}$ of infinite subsets of X_1 such that $E_{n+1} \subseteq E_n$ and $E_n \parallel \{n\}$ for every $n \in \omega$. Therefore, if X_1^α is a pseudointersection of $\mathcal{E}$, then X_0^α, X_1^α satisfy (1)–(3) by construction, and (4) follows from the fact that both sets are infinite subsets of X_0 and X_1, which satisfy 2.4(3). ∎

We are now ready to prove the main result of the paper.

THEOREM 2.7. *There is a separable, first countable, locally compact space which admits a continuous weak selection but is not weakly orderable.*

Proof. Let $\mathcal{B} = \{X_0^\alpha, X_1^\alpha : \alpha < \mathfrak{c}\}$, where X_i^α is as in Lemma 2.6 for $i \in 2$, and consider $X = \Psi(\mathcal{B})$, the Mrówka–Isbell space associated to $\mathcal{B}$.

By Lemmas 2.5 and 2.6, φ satisfies the conditions of Lemma 2.1, hence there is a (unique) continuous weak selection $\overline{\varphi}$ on $\Psi(\mathcal{B})$ extending the universal weak selection φ.

To conclude the proof, it is enough to verify that X is not weakly orderable. Aiming at a contradiction, suppose that there exists a linear order $\sqsubseteq$ on X whose induced topology is coarser than the topology on X. Let $\alpha < \mathfrak{c}$ be such that $\sqsubseteq\restriction[\omega]^2 = \leq_\alpha$ and suppose, without loss of generality, that for the points $X_0^\alpha, X_1^\alpha \in \Psi(\mathcal{B})$ the inequality $X_0^\alpha \sqsubseteq X_1^\alpha$ holds. By Lemma 2.6, the infinite set $X_0^\alpha \cup X_1^\alpha$ is $\leq_\alpha$-monotone. Assume that $S \subseteq \omega$ is downward closed, contains $X_0^\alpha \cup X_1^\alpha$ and for every $s \in S$, $(\leftarrow, s)_{\leq_\alpha} \cap (X_0^\alpha \cup X_1^\alpha)$ is finite. If there is an $s \in S$ with $X_0^\alpha \sqsubseteq s$, then $(\leftarrow, s)_\sqsubseteq$ is an $\sqsubseteq$-open interval containing the point X_0^α, which meets the set X_0^α in finitely many points. However, this contradicts the assumption that the $\sqsubset$-order topology on X is coarser that the original one. On the other hand, if $S \subseteq (\leftarrow, X_0^\alpha)_\sqsubseteq$, then the interval $(X_0^\alpha, \to)_\sqsubseteq$ contains the point X_1^α and is disjoint from the set X_1^α, which leads to the same contradiction.

The case when $X_0^\alpha \cup X_1^\alpha$ is contained in an upward directed set T is treated analogously.

We have proved that the topology determined by the order $\sqsubseteq$ cannot be coarser than that of X, and therefore X is not weakly orderable. ∎

Remark 2.8. *There is a space $X \subseteq \beta\omega$ which admits a continuous weak selection but is not weakly orderable.*

Fix $\mathcal{A}$ as in Theorem 2.7 and pick $p_A \in A^*$ for every $A \in \mathcal{A}$. The space $X = \omega \cup \{p_A : A \in \mathcal{A}\}$ is as required.

3. Weak selections on separable spaces. If X is a weakly orderable space, then it not only admits a weak selection, but also a selection for $\mathcal{K}(X)$: the function $\min\restriction\mathcal{K}(X)$ is a selection for $\mathcal{K}(X)$. Motivated by this, Gutev and Nogura asked in [10] the following question:

Does there exist a space X that admits a continuous weak selection, but $\mathrm{Sel}(\mathcal{F}_n(X)) = \emptyset$ *for some $n > 2$?*

This question is still open, even for $n = 3$. In this section we will prove that for certain spaces, including separable spaces, the existence of a continuous weak selection implies that $\mathrm{Sel}(\mathrm{Fin}(X)) \neq \emptyset$, providing a partial negative answer to the question. In particular, the example presented in Theorem 2.7 admits even a continuous selection for all compact sets. We can conclude that there are spaces that are not weakly orderable even when $\mathrm{Sel}(\mathcal{K}(X)) \neq \emptyset$.

3.1. *2-to-1 maps onto ordered spaces.* Next we show that for separable spaces the existence of weak selections implies the existence of a 2-to-1 continuous map onto an ordered space. In particular, even though there are separable spaces which admit a weak selection and are not weakly orderable, they can always be covered by two weakly orderable subspaces.

Costantini [3] considered a similar analysis of weak selections on separable spaces having a dense set of isolated points. Gutev [9] proved that every second countable space which admits a continuous weak selection is weakly orderable.

Proposition 3.1. *Let ψ be a continuous weak selection defined on X and let $x, y, z \in X$ be such that $\{x, y, z\}$ is a 3-cycle with respect to ψ. Then there is a (canonical) partition $\mathcal{P}$ of X so that $|\mathcal{P}| = 5$ and each $P \in \mathcal{P}$ is clopen and satisfies $|\{x, y, z\} \cap P| \leq 1$.*

Proof. Suppose that $x \to y \to z \to x$ and consider the following sets:

$$\begin{aligned}
P_0 &= \{w \in X \setminus \{y, z\} : z \to w \to y\},\\
P_1 &= \{w \in X \setminus \{x, z\} : x \to w \to z\},\\
P_2 &= \{w \in X \setminus \{x, y\} : y \to w \to x\},\\
P_3 &= \{w \in X : \{x, y, z\} \rightrightarrows \{w\}\},\\
P_4 &= \{w \in X : \{w\} \rightrightarrows \{x, y, z\}\}.
\end{aligned}$$

It is not difficult to prove that $\mathcal{P} = \{P_i : i < 5\}$ is a partition of X and, by continuity of ψ, P_i is open (and so clopen) for every $i < 5$. Also, $x \in P_0$, $y \in P_1$ and $z \in P_2$. ∎

An immediate consequence of the previous proposition is that if X is a connected space admitting a continuous weak selection ψ, then it does not admit 3-cycles with respect to ψ and so X is weakly orderable.

Recall that a relation $R \subseteq X \times X$ is *total* if for every $a, b \in X$, $(a, b) \in R$ or $(b, a) \in R$.

Proposition 3.2. *Let X be a separable space that admits a continuous weak selection ψ. Then there is a closed, reflexive, total and transitive relation $R \subseteq X \times X$ such that $|\{z \in X : (x, z) \in R$ and $(z, x) \in R\}| \leq 2$.*

Proof. Let $D = \{d_n : n \in \omega\}$ be a countable dense subset of X and let $\mathcal{T} = \{T_n : n \in \omega\}$ be an enumeration of all triples $T \in [D]^3$ that are 3-cycles with respect to ψ. For every $n \in \omega$, let $\mathcal{E}_n$ be the canonical partition determined by the 3-cycle T_n, defined in the proof of Proposition 3.1.

Define recursively closed relations $R_n \subseteq X \times X$ for every $n \in \omega$ as follows:

Let $R_0 = X \times X$ and suppose that R_n is a closed, reflexive and total relation with the following property:

There is a unique finite family $\mathcal{C}_n = \{C_0, \ldots, C_{k_n}\}$ of closed subsets of X such that for $x, y \in X$ and $i < j \leq k_n$:

(1) $X = \bigcup\{C_l : l \leq k_n\}$,
(2) $\mathcal{C}_n$ is a refinement of the partition $\mathcal{E}_{n-1}$, where $\mathcal{E}_{-1} = \{X\}$,
(3) if $x, y \in C_i$ then $(x, y) \in R_n \cap R_n^{-1}$,
(4) if $x \in C_i$ and $y \in C_j \setminus C_i$, then $(y, x) \notin R_n$,
(5) if $C_i \cap C_j \neq \emptyset$ then $C_i \cap C_j = \{d_l\}$ for some $l < n$, and $C_i \cap C_j \neq \emptyset$ only if $j - i = 1$,
(6) if $d \in C_j \cap \{d_l : l < n\}$ and $z \to d$ for some $z \in C_j \setminus \{d\}$, then $C_j \rightrightarrows \{d\}$,
(7) if $d \in C_j \cap \{d_l : l < n\}$ and $d \to z$ for some $z \in C_j \setminus \{d\}$, then $\{d\} \rightrightarrows C_j$.

The conditions (3) and (4) guarantee uniqueness of the family $\mathcal{C}_n$. Also, the family $\mathcal{C}_n$ determines the relation R_n in a natural way: $(x, y) \notin R_n$ if and only if there are $i < j \leq k_n$ such that $y \in C_i$ and $x \in C_j \setminus C_i$.

Let $i \leq k_n$ be such that $d_n \in C_i$. If $\{d_n\}$ is an isolated point, define $C_{i,0} = C_i \cap \{x \in X \setminus \{d_n\} : d_n \to x\}$, $C_{i,1} = \{d_n\}$ and $C_{i,2} = C_i \cap \{x \in X \setminus \{d_n\} : x \to d_n\}$. In this case, let

$$S_n = R_n \setminus \{(x,y) : x \in C_{i,l},\ y \in C_{i,s} \text{ and } 0 \leq s < l \leq 2\}.$$

If d_n is not isolated, let $C_{i,0} = C_i \cap \{x \in X : d_n \to x\}$ and $C_{i,1} = C_i \cap \{x \in X : x \to d_n\}$. Define then

$$S_n = R_n \setminus \{(x,y) : x \in C_{i,1} \setminus \{d_n\},\ y \in C_{i,0} \setminus \{d_n\}\}.$$

The relation S_n is reflexive and total. Moreover, S_n is also closed. To prove this, let $(x,y) \notin S_n$. Since R_n is closed and $S_n \subseteq R_n$, we can suppose that $(x,y) \in R_n$. Therefore, $x, y \in C_i$, $x \to d_n$ and $d_n \to y$. Let U_x and U_y be disjoint neighborhoods of x and y respectively such that $U_x \subseteq (C_i \cup C_{i+1}) \setminus (C_{i-1} \cup C_{i+2})$ and $U_y \subseteq (C_{i-1} \cup C_i) \setminus (C_{i-2} \cup C_{i+1})$ (take $U_x = \{d_n\}$ if $x = d_n$ and d_n is isolated, and analogously for U_y). This is possible because of condition (5). Finally, let $U'_x = U_x \cap \{z \in X \setminus \{d_n\} : z \to d_n\}$ and $U'_y = U_y \cap \{z \in X \setminus \{d_n\} : d_n \to z\}$ (again, let $U'_x = \{d_n\}$ if d_n is isolated). Then $(U'_x \times U'_y) \cap S_n = \emptyset$ and so the relation S_n is closed.

Let $\mathcal{C}'_n = \{C'_0, \ldots, C'_{k_n+2}\}$, where $C'_j = C_j$ if $0 \leq j < i$, $C'_i = C_{i,0}$, $C'_{i+1} = C_{i,1}$, $C_{i+2} = C'_{i,2}$, and $C'_j = C_{j-2}$ for $i + 2 < j \leq k_n + 2$, where $C'_{i+2} = \emptyset$ if d_n is not isolated.

By construction, S_n together with the collection $\mathcal{C}'_n$ has properties (1)–(7), except possibly (2). We only need to refine this relation so as to obtain a refinement of $\mathcal{E}_n$.

Fix $j \leq k'_n$, where $k'_n = k_n + 2$ if d_n is isolated and $k'_n = k_n + 1$ otherwise. We will find a partition $\mathcal{D}_j$ of C'_j which refines $\mathcal{E}_n$ and consists of closed sets. For this, we consider all possible cases:

Case 1: There is an $E \in \mathcal{E}_n$ such that $C'_j \subseteq E$. In this case, let $\mathcal{D}_j = \{D_{j,0}\}$, where $D_{j,0} = C'_j$.

Case 2: $\{E \in \mathcal{E}_n : E \cap C'_j \neq \emptyset\} = \{E_0, \ldots, E_t\}$ with $0 < t \leq 5$, $C'_j \cap \{d_l : l \leq n\} = \{d\}$, $\{d\} \rightrightarrows C'_j$ and $d \in E_t$. Let $\mathcal{D}_j = \{D_{j,l} : 0 \leq l \leq t\}$, where $D_{j,l} = E_l \cap C'_j$ for every l.

Case 3: $\{E \in \mathcal{E}_n : E \cap C'_j \neq \emptyset\} = \{E_0, \ldots, E_t\}$ with $0 < t \leq 5$, $C'_j \cap \{d_l : l \leq n\} = \{d\}$, $C'_j \rightrightarrows \{d\}$ and $d \in E_0$. Let $\mathcal{D}_j = \{D_{j,l} : 0 \leq l \leq t\}$, where $D_{j,l} = E_l \cap C'_j$ for every l.

Case 4: $\{E \in \mathcal{E}_n : E \cap C'_j \neq \emptyset\} = \{E_0, \ldots, E_t\}$ with $0 < t \leq 5$, $C'_j \cap \{d_l : l \leq n\} = \{d, d'\}$, $d' \to d$ and $d, d' \in E_0$. Let $x \in C'_j \setminus E_0$. By (6) and (7), $d' \to x$ and $x \to d$. In this case, let $\mathcal{D}_j = \{D_{j,0}, \ldots, D_{j,t+1}\}$, where $D_{j,0} = C'_j \cap E_0 \cap \{y \in X : x \to y\}$, $D_{j,l} = C'_j \cap E_l$ for every $1 \leq l \leq t$ and $D_{j,t+1} = C'_j \cap E_0 \cap \{y \in X : y \to x\}$.

Case 5: $\{E \in \mathcal{E}_n : E \cap C'_j \neq \emptyset\} = \{E_0, \ldots, E_t\}$ with $0 < t \leq 5$, $C'_j \cap \{d_l : l \leq n\} = \{d, d'\}$, $d' \to d$, $d \in E_0$ and $d' \in E_t$. In this case, let $\mathcal{D}_j = \{D_{j,0}, \ldots, D_{j,t}\}$, where $D_{j,l} = C'_j \cap E_l$ for every l.

Let $\mathcal{C}_{n+1} = \bigcup\{\mathcal{D}_j : j \leq k'_n\}$. Notice that we can enumerate $\mathcal{D}$ as $\{D_j : j \leq k_{n+1}\}$ so that for every $i \leq j \leq k_{n+1}$, if $x \in D_i$ and $y \in D_j$ then $(x, y) \in S_n$. Finally, define the relation

$$R_{n+1} = S_n \setminus \{(x, y) : y \in D_i,\ x \in D_j \text{ and } i < j\}.$$

Then $R_{n+1} \subseteq S_n$ and $\mathcal{C}_{n+1}$ refines $\mathcal{E}_n$. Further, R_{n+1} is reflexive, total, and it can be proved, in an analogous way to the case of S_n, that R_{n+1} is closed. Moreover, R_{n+1} together with the collection $\mathcal{C}_{n+1}$ satisfies (1)–(7).

Let $R = \bigcap\{R_n : n \in \omega\}$. Then R is closed, total and reflexive, since each R_n is.

Before showing that R is transitive, let us record two properties of R.

Fact 1. *If $x, y \in R$ and there is a $d \in D$ so that $x \to d \to y$ and d belongs to a 3-cycle with respect to ψ, then $(x, y) \notin R \cap R^{-1}$.*

Let $n \in \omega$ be such that $d \in T_n$. Since $\{x, y\} \not\parallel \{d\}$, the points x and y do not belong to the same element of the partition $\mathcal{E}_n$ and so either $(x, y) \notin R_{n+1}$ or $(y, x) \notin R_{n+1}$.

As a consequence, if $x, y \in X$ and there is a $z \in X$ so that $\{x, y, z\}$ forms a 3-cycle, then $(x, y) \notin R \cap R^{-1}$.

Fact 2. *For any $x \in X$, the set $P_x = \{z \in X : (x, z) \in R \cap R^{-1}\}$ contains at most two points.*

Suppose that $x, y, z \in P_x$ and $x \to y \to z$ (the other cases are treated in the same way). By density of D, there is a $d \in D$ so that $x \to d \to z$. Let $k = \min\{l \in \omega : \{x, z\} \not\parallel \{d_l\}\}$ and let $\mathcal{C}_k$ be the collection determined by the relation R_k and satisfying (1)–(7). Since $y \in P_x$, there is a $C \in \mathcal{C}_k$ such that $x, y \in C$. Then $d_k \notin C$, because otherwise $(x, y) \notin R_{k+1}$ and so $(x, y) \notin R$. Hence, because of the way R_k is constructed, there is an $l < k$ such that either $\{x, y, d_k\} \not\parallel \{d_l\}$; or x, y and d_k do not belong to the same element of the partition $\mathcal{E}_l$ determined by the 3-cycle T_l; or there are $E \in \mathcal{E}_l$ and $w \in X \setminus E$ with $x, y, d_k \in E$, $\{x, y\} \parallel \{w\}$ and $\{x, y, d_k\} \not\parallel \{w\}$. In any case, we can find a 3-cycle $T \in [D]^3$ with $d_k \in T$. Thus by Fact 1, either $(x, y) \notin R$ or $(y, x) \notin R$, which is a contradiction. This proves Fact 2.

Finally, to prove that R is transitive, let $x, y, z \in X$ be such that $(x, y) \in R$ and $(y, z) \in R$. Aiming at a contradiction, suppose that $(z, x) \notin R$. Let $n \in \omega$ be such that $(z, x) \notin R_n$, and let $\mathcal{C}_n = \{C_i : i \leq k_n\}$ be the family determined by R_n. Three cases are possible:

Case 1: $(x, y) \in R \cap R^{-1}$. Since $(z, x) \notin R_n$, there are $i < j \leq k_n$ such that $z \in C_i$ and $x \in C_j \setminus C_i$. But (x, y), (y, x) and (x, z) are in R_n, so $j = i+1$

and $y \in C_i \cap C_{i+1}$. Notice that although z and y are indistinguishable until step n (i.e. $(y,z) \in R_n \cap R_n^{-1}$), they must eventually be separated. Since $y = d_l$ for some $l < n$, we have $y \to z$ and, by construction, $(z,y) \in R$, which is false.

Case 2: $(y,z) \in R \cap R^{-1}$. Analogous to Case 1.

Case 3: $(y,x) \notin R$ and $(z,y) \notin R$. As before, we can find an $n \in \omega$ such that if $\mathcal{C}_n$ is the collection of closed subsets determined by R_n, then there are $i < j < k$ with $z \in C_i$, $y \in C_j \setminus C_i$ and $x \in C_k \setminus C_j$. Since $k > i+1$, we have $C_i \cap C_k = \emptyset$ and so $x \in C_k \setminus C_i$. This implies that $(x,z) \notin R_n$, which is again impossible. ∎

The following result states that if a separable space X admits a continuous weak selection, then it is *almost weakly orderable*, in the sense that it can be covered by two weakly orderable sets.

Corollary 3.3. *Let X be a separable space that admits a continuous weak selection ψ. Then there are an orderable space L and a continuous function $f : X \to L$ satisfying*:

(i) $|f^{-1}[\{y\}]| \leq 2$ *for every* $y \in Y$ (*i.e.*, f *is* $\leq$2-*to*-1),
(ii) *if* $\{x_0, x_1, x_2\}$ *is a* 3-*cycle with respect to* ψ, *then* $f{\restriction}\{x_0, x_1, x_2\}$ *is injective.*

Proof. Let R be the closed relation constructed in Proposition 3.2, let D be the countable dense set used in this construction, and let $\mathcal{T} = \{T \in [D]^3 : T \text{ is a 3-cycle}\}$.

Define a relation $\sim_R$ on X as follows:

$$x \sim_R y \quad \text{if and only if} \quad P_x = P_y,$$

where $P_z = \{w \in X : (w,z) \in R \cap R^{-1}\}$ for $z \in X$.

By Proposition 3.2, if $x \in X$ then P_x contains at most two points, and $|P_x| = 1$ when x is in D and is isolated. Therefore, $\sim_R$ is an equivalence relation on X and each equivalence class $[x]_{\sim_R}$ contains at most two points.

Define an order $<$ on the set $L = X/{\sim_R}$ in the natural way:

$$[x]_{\sim_R} < [y]_{\sim_R} \quad \text{if and only if} \quad (x,y) \in R \text{ and } (y,x) \notin R.$$

Then $(L,<)$ is a linear order.

Define the function $f : X \to L$ by $f(x) = [x]_{\sim_R}$. It is clear that f is a $\leq$2-to-1 function. Moreover, if $\{x,y,z\}$ is a 3-cycle with respect to the weak selection ψ, then by continuity of ψ and density of D, we can find $T \in \mathcal{T}$ such that, if $\mathcal{E}$ is the partition determined by T, then none of the points x, y and z belongs to the same element $E \in \mathcal{E}$. Hence $f{\restriction}\{x,y,z\}$ is injective. Finally, continuity of f follows because R is closed in $X \times X$. ∎

3.2. *Selections for finite sets.* Here we prove that a separable space which admits a continuous weak selection admits, in fact, a continuous selection for all finite sets. The first result in this direction belongs to J. Steprāns [18], who showed that a separable space X with a dense set of isolated points which admits a continuous weak selection also admits a continuous selection on $\mathcal{F}_3(X)$.

We will use the following result, obtained by Gutev in [8].

PROPOSITION 3.4 (Gutev). *Let X be a space such that* $\mathrm{Sel}(\mathcal{F}_n(X)) \neq \emptyset$ *and there exists a continuous selection $\varrho : [X]^{n+1} \to X$ for some $n \in \omega$. Then* $\mathrm{Sel}(\mathcal{F}_{n+1}(X)) \neq \emptyset$.

To prove Proposition 3.4, Gutev used the notion of decisive partitions of finite sets with respect to a weak selection ψ, where a partition $\mathcal{P}$ of a finite set F is *decisive* if $A \parallel_\psi B$ for every $A, B \in \mathcal{P}$. Whenever $|F| \geq 2$, $\mathrm{di}(F, \psi)$ is defined as the minimal cardinality of a decisive partition $\mathcal{P}$ of F with at least two elements, and $\mathrm{di}(F, \psi) = 1$ if $|F| = 1$. He proved that if $|\mathrm{di}(F, \psi)| \geq 3$, then F has a unique decisive partition $\mathcal{M}$ with $|\mathcal{M}| = \mathrm{di}(F, \psi)$. Moreover, any other decisive partition of F refines $\mathcal{M}$. He also showed that the function $\mathrm{di}_\psi : \mathrm{Fin}(X) \setminus \mathcal{F}_1(X) \to \omega$, defined by $\mathrm{di}_\psi(F) = \mathrm{di}(F, \psi)$, is continuous.

PROPOSITION 3.5. *Let X be a space that admits a continuous weak selection ψ. If there are an orderable space Y and a continuous function $f : X \to Y$ such that*:

(i) $|f^{-1}[\{y\}]| \leq 2$ *for every* $y \in Y$ (*i.e.*, f *is* $\leq$2-*to*-1),
(ii) *if* $\{x_0, x_1, x_2\}$ *is a* 3-*cycle with respect to* ψ, *then* $f{\restriction}\{x_0, x_1, x_2\}$ *is injective*,

then there is a sequence $\{\psi_n : n \geq 2\}$ *of compatible continuous selections such that* $\psi_n \in \mathrm{Sel}(\mathcal{F}_n(X))$ *for every* $n \in \omega$.

Proof. We define $\psi_2 = \psi$ and argue by induction on $n > 2$. For $n = 3$, we define a function $\psi_3 : \mathcal{F}_3(X) \to X$ by cases. Let $F \subseteq X$.

CASE 1: If $|F| \leq 2$, then define $\psi_3(F) = \psi(F)$.

CASE 2: If $|F| = 3$ and there is an $x \in F$ such that $\psi(\{x, y\}) = x$ for every $y \in F$, then define $\psi_3(F) = x$.

CASE 3: If $F = \{x_0, x_1, x_2\}$ is a 3-cycle with respect to ψ, then let $\psi_3(F) = x$, where $f(x) = \min\{f(x_0), f(x_1), f(x_2)\}$.

By (ii), ψ_3 is well defined. Clearly ψ_3 is continuous at $\{x\}$ for every $x \in X$, so we only have to verify continuity at $F \subseteq X$ if $|F| \geq 2$.

Suppose that Case 1 occurs and that $F = \{x_0, x_1\} \subseteq X$ is such that $\psi_3(\{x_0, x_1\}) = \psi(\{x_0, x_1\}) = x_0$. By continuity of ψ, there are disjoint neighborhoods U and V of x_0 and x_1 respectively such that $\psi(\{u, v\}) \subseteq U$

for all $u \in U$ and $v \in V$. Then $\psi_3[\langle U, V\rangle] \subseteq U$, which guarantees continuity of ψ_3 at F.

If $F = \{x_0, x_1, x_2\}$ and $\psi(\{x_0, x_j\}) = x_0$ for $j \in 3$ then, again by continuity of ψ, we can find disjoint neighborhoods U_0, U_1 and U_2 of x_0, x_1 and x_2 respectively such that $\psi[\langle U_0, U_j\rangle] \subseteq U_0$ for $j < 3$. If $F' \in \mathcal{U} = \mathcal{F}_3(X) \cap \langle U_0, U_1, U_2\rangle$ then clearly F' is as in case (ii) and $\psi_3[\mathcal{U}] \subseteq U_0$, which guarantees continuity at F.

Finally, let $F = \{x_0, x_1, x_2\} \subseteq X$ be a 3-cycle with respect to ψ. Let U_0, U_1 and U_2 be pairwise disjoint neighborhoods of x_0, x_1 and x_2 respectively.

By (ii), $f(x_i) \neq f(x_j)$ whenever $i \neq j$, and we may suppose that $f(x_0) < f(x_1) < f(x_2)$. Consider disjoint intervals I_0, I_1 and I_2 in the orderable space Y such that $f(x_i) \in I_i$ for $i < 3$. Then $\mathcal{U} = \langle f^{-1}[I_0] \cap U_0, f^{-1}[I_1] \cap U_1, f^{-1}[I_2] \cap U_2\rangle$ is a neighborhood of F, every $F' \in \mathcal{U} \cap \mathcal{F}_3(X)$ is a 3-cycle and $\psi_3[\mathcal{U}] \subseteq f^{-1}[I_0]$, which implies that ψ_3 is continuous at F.

Suppose now that we have defined continuous selections $\psi_k : \mathcal{F}_k(X) \to X$ for $k \leq n$ such that $\psi_{s+1} \restriction \mathcal{F}_s(X) = \psi_s$ for every $s < n$. Again, we will define a selection $\psi_{n+1} : \mathcal{F}_n(X) \to X$ by cases:

Case 1: Suppose that $F \in \mathcal{F}_{n+1}(X)$ and $\mathrm{di}(F, \psi) = 2$. According to [6], there is a unique decisive partition $\mathcal{P} = \{P_0, P_1\}$ such that $P_0 \rightrightarrows P_1$ and $\mathcal{P}$ is minimal in the sense that if $\mathcal{M} = \{M_0, M_1\}$ is another decisive partition with $M_0 \rightrightarrows M_1$ then $P_1 \subseteq M_1$. Define then $\psi_{n+1}(F) = \psi_n(P_1)$.

Case 2: Suppose that $F \in \mathcal{F}_{n+1}(X)$ and $2 < \mathrm{di}(F, \psi) \leq n$. Let

$$\mathcal{P} = \{P_0, \ldots, P_{k-1}\}$$

be the only decisive partition of F of cardinality k, where $k = \mathrm{di}(F, \psi)$. Define then $\psi_{n+1}(F) = \psi_n(\{\psi_n(P_i) : i < k\})$.

Case 3: Suppose that $F \in [X]^{n+1}$ and $\mathrm{di}(F, \psi) = n + 1$. If $x, y \in F$ then there must be a $z \in F$ such that $\{x, y, z\}$ is a 3-cycle with respect to ψ, because otherwise the partition $\mathcal{P} = \{\{x, y\}\} \cup \{\{z\} : z \in F \setminus \{x, y\}\}$ would be a decisive partition of F with cardinality n, which is not possible. This implies that the function f restricted to the set F is injective. In this case we define $\psi_{n+1}(F) = x$, where $f(x) = \min\{f(y) : y \in F\}$.

Cases 1 and 2 are defined exactly in the same way as in the proof of Proposition 3.4 and the proof of continuity of ψ_{n+1} in any of these cases is given in [8]. Therefore, to conclude the proof of continuity of ψ_{n+1}, it is enough to consider Case 3. The proof of continuity in this case is very similar to that of Case 3 for ψ_3. Thus, ψ_{n+1} is a continuous selection for $\mathcal{F}_{n+1}(X)$.

To conclude the inductive step, we must show that ψ_{n+1} is an extension of ψ_n. Let $F \subseteq X$ be such that $|F| \leq n$. If $\mathrm{di}(F, \psi) = n$ then $\psi_{n+1}(F) = \psi_n(\{\psi_n(\{x\}) : x \in F\}) = \psi_n(F)$. Otherwise, if $\mathrm{di}(F, \psi) < n$, let $\mathcal{P}$ be the decisive partition with $|\mathcal{P}| = \mathrm{di}(F,\psi)$. Since $|\mathcal{P}| < n$ and $|P| < n$ for every $P \in \mathcal{P}$, the inductive hypothesis yields

$$\psi_{n+1}(F) = \psi_n(\{\psi_n(P) : P \in \mathcal{P}\}) = \psi_{n-1}(\{\psi_{n-1}(P) : P \in \mathcal{P}\}) = \psi_n(F). \blacksquare$$

THEOREM 3.6. *Let X be a space that admits a continuous weak selection ψ, Y an orderable space and $f : X \to Y$ a continuous function as in Proposition* 3.5. *Then* $\mathrm{Sel}(\mathrm{Fin}(X)) \neq \emptyset$.

Proof. Let $\{\psi_n : n \geq 2\}$ be a sequence of compatible continuous selections as in Proposition 3.5 and define $\Phi = \bigcup_{n\geq 2} \psi_n$. Clearly Φ is a selection. Let us prove that it is continuous.

CLAIM. *Let $F \in \mathrm{Fin}(X)$ and let $\mathcal{M}$ be a decisive partition of F. Then $\Phi(F) = \Phi(\{\Phi(M) : M \in \mathcal{M}\})$.*

We argue by induction on $|F|$. Clearly the result is true if $|F| = 2$. Assume that it holds for every $E \subseteq X$ with $|E| \leq n$ and let $F \in [X]^{n+1}$. Let $\mathcal{M}$ be a decisive partition of F and let $G = \{\Phi(M) : M \in \mathcal{M}\}$. The result is evidently true if $\mathrm{di}(F,\psi) = n + 1$ (i.e. $\mathcal{M} = \{\{x\} : x \in F\}$), so we can suppose that $\mathrm{di}(F,\psi) \leq n$ and $|\mathcal{M}| \leq n$. We will consider separately the cases when $\mathrm{di}(F,\psi) = 2$ and when $\mathrm{di}(F,\psi) > 2$.

Suppose first that $\mathrm{di}(F,\psi) = 2$ and let $\mathcal{P} = \{P_0, P_1\}$ be the decisive partition of F, with $P_0 \rightrightarrows P_1$ and minimal as in the proof of Proposition 3.5. Then $\Phi(F) = \Phi(P_1)$.

If $\mathcal{M} = \{M_0, M_1\}$ and $M_0 \rightrightarrows M_1$ then $\Phi(G) = \Phi(M_1)$. By minimality of $\mathcal{P}$, we have $P_1 \subseteq M_1$. Notice that $\{P_1, M_1 \setminus P_1\}$ is a decisive partition of M_1 such that $M_1 \setminus P_1 \rightrightarrows P_1$ and $|M_1| \leq n$. Then, by the inductive hypothesis, $\Phi(M_1) = \Phi(\{\Phi(P_1), \Phi(M_1 \setminus P_1)\}) = \psi(\{\Phi(P_1), \Phi(M_1 \setminus P_1)\}) = \Phi(P_1)$ and then the result also holds.

Suppose now that $\mathcal{M} = \{M_0, \ldots, M_{k-1}\}$ and $2 < k < n$. Define $\mathcal{M}_0 = \{M \in \mathcal{M} : M \cap P_0 \neq \emptyset\}$ and $\mathcal{M}_1 = \{M \in \mathcal{M} : M \cap P_1 \neq \emptyset\}$. Notice that $|\mathcal{M}_0 \cap \mathcal{M}_1| \leq 1$, because otherwise $\mathcal{M}$ would not be a decisive partition. If $\mathcal{M}_0$ and $\mathcal{M}_1$ are disjoint then $\mathcal{M}_i$ is a decisive partition of P_i for $i \in 2$. If $N_i = \{\Phi(M) : M \in \mathcal{M}_i\}$ for $i \in 2$, then also $\mathcal{N} = \{N_0, N_1\}$ is a decisive partition of G such that $N_0 \rightrightarrows N_1$. Therefore, by inductive hypothesis, $\Phi(G) = \Phi(N_1)$. But, since $\mathcal{M}_1$ is a decisive partition of P_1 and $|M_1| \leq n$, we have $\Phi(N_1) = \Phi(P_1)$. On the other hand, if $|\mathcal{M}_0 \cap \mathcal{M}_1| = 1$, let $M^* \in \mathcal{M}_0 \cap \mathcal{M}_1$. In this case, $\{M^* \cap P_1\} \cup (\mathcal{M}_0 \cup \mathcal{M}_1) \setminus \{M^*\}$ is a decisive partition of $F' = F \setminus (M^* \cap P_0)$. Again, by inductive hypothesis, $\Phi(M^*) = \Phi(M^* \cap P_1)$, which implies that $\Phi(G) = \Phi(F')$. But $\{P_1, F' \setminus P_1\}$ is also a decisive partition of F' with $F' \setminus P_1 \rightrightarrows P_1$, which implies that $\Phi(F') = \Phi(P_1)$.

Finally, suppose that $2 < \mathrm{di}(F,\psi) \leq n$ and let $\mathcal{P}$ be the decisive partition of F of cardinality $\mathrm{di}(F,\psi)$. For every $P \in \mathcal{P}$, we have $|P| \leq n$ and $\mathcal{M}_P = \{M \in \mathcal{M} : M \subseteq P\}$ is a decisive partition of P. Therefore,

$\Phi(P) = \Phi(\{\Phi(M) : M \in \mathcal{M}_P\})$. Note also that $\mathcal{P}' = \{G \cap P : P \in \mathcal{P}\}$ is a decisive partition of G. By the inductive hypothesis, $\Phi(G) = \Phi(\{\Phi(G \cap P) : P \in \mathcal{P}\})$. Now, for every $P \in \mathcal{P}$, we have $\Phi(G \cap P) = \Phi(\{\Phi(M) : M \subseteq P\}) = \Phi(\{\Phi(M) : M \in \mathcal{M}_P\}) = \Phi(P)$. We conclude that $\Phi(F) = \Phi(G)$.

To prove continuity of Φ, let $F = \{x_i : i < n\} \subseteq X$ and let U be a neighborhood of $\Phi(F)$. By continuity of ψ_n, there is a pairwise disjoint decisive family $\{U_i : i < n\}$ of open subsets of X such that $x_i \in U_i$ for every $i < n$ and $\psi_n(G) \subseteq U$ for every $G \in \mathcal{U} \cap \mathcal{F}_n(X)$, where $\mathcal{U} = \langle U_0, \dots, U_{n-1} \rangle$. Moreover, if $G \in \mathrm{Fin}(X) \cap \mathcal{U}$ then $\{U_i \cap G_i : i < n\}$ is a decisive partition of G, which implies, by the previous fact, that

$$\Phi(G) = \Phi(\{\Phi(U_i \cap G_i) : i < n\}) = \psi_n(\{\Phi(U_i \cap G_i) : i < n\}) \subseteq U. \ \blacksquare$$

The following result is an immediate consequence of Corollary 3.3 and Theorem 3.6, and provides a partial answer to Gutev and Nogura's question.

Corollary 3.7. *Let X be a separable space that admits a continuous weak selection. Then* $\mathrm{Sel}(\mathrm{Fin}(X)) \neq \emptyset$.

After proving that every continuous weak selection on a separable space can be extended to a selection for all finite sets, it is natural to ask if there is also a continuous selection for all compact sets. Notice that an example that would answer this question in the negative cannot be weakly orderable. The following result proves that, in particular, the space in Theorem 2.7 is not such an example.

Proposition 3.8. *There is a separable space X which admits a continuous selection on $\mathcal{K}(X)$ but is not weakly orderable.*

Proof. Let $\mathcal{B}$ be the almost disjoint family introduced in Theorem 2.7 and let $X = \Psi(\mathcal{B})$. Consider also the weak selection φ on X defined there and let Φ be the continuous selection on $\mathrm{Fin}(X)$ determined by Corollary 3.7. We can suppose, by Ramsey's Theorem, that for every $B \in \mathcal{B}$ either $\varphi{\restriction}[B]^2 = \min$ or $\varphi{\restriction}[B]^2 = \max$.

We define a selection ϱ on $\mathcal{K}(X)$ pointwise. Let $K \in \mathcal{K}(X)$. Then there are integers $q \le s$, a finite set $\{B_0, \dots, B_s\} \subseteq \mathcal{B}$, a family $\{A_i \in [B_i]^{\le\omega} : i \le q\}$ and a finite subset $F \subseteq \omega \setminus \bigcup\{B_i : i \le s\}$ such that $K = F \cup \bigcup\{A_i : i \le q\} \cup \{B_0, \dots, B_s\}$. Let

$$k = \min\{n \in \omega : \{B_i \setminus n : i \le s\} \cup \{\{x\} : x \in F \cup G_n\} \text{ is decisive}\},$$

where $G_n = \bigcup\{A_j \cap n : j \le q\}$ for every $n \in \omega$.

Enumerate the set $F \cup G_k$ as $\{m_0, \dots, m_t\}$ and, for every $i \le s$, choose $x_i \in K$ so that $x_i = \min(A_i \setminus k)$ if $(A_i \setminus k) \cap K \neq \emptyset$ and $\varphi{\restriction}[B_i]^2 = \min$, and $x_i = B_i$ otherwise. Define

$$\varrho(K) = \Phi(\{x_i : i \le s\} \cup \{m_j : j \le t\}).$$

To prove continuity of ϱ, let U be a neighborhood of $\varrho(K)$. By continuity of Φ, we can find an $r \in \omega$ such that $r \geq k$ and if for every $i \leq s$, $U_i = \{x_i\}$ when $x_i \in \omega$ and $U_i = \{B_i\} \cup B_i \setminus r$ whenever $x_i = B_i$, then the neighborhood $\mathcal{U} = \langle U_1, \ldots, U_s, \{m_0\}, \ldots, \{m_t\}\rangle \cap \mathrm{Fin}(X)$ satisfies $\Phi[\mathcal{U}] \subseteq U$.

Enumerate $\bigcup\{(A_j \cap r) \setminus k : j \leq q\}$ as $\{m_{t+1}, m_{t+2}, \ldots, m_v\}$ and let

$$\mathcal{V} = \langle \{B_0\} \cup B_0 \setminus r, \ldots, \{B_s\} \cup B_s \setminus r, U_0, \ldots, U_s, \{m_0\}, \ldots, \{m_v\}\rangle.$$

Notice that $\mathcal{V}$ is a neighborhood of K. To conclude the proof, let us prove that $\varrho[\mathcal{V}] \subseteq U$. Let K' be a compact subset of X contained in $\mathcal{V}$. Then there are integers $u, w \in j+1$ with $u \leq w$, $z_0, z_1, \ldots, z_w \in j+1$, a family $\{A'_{z_j} \subseteq B_{z_j} : j \leq u\}$ and a finite subset $F' \subseteq \omega \setminus \bigcup\{B_{z_j} : j \leq w\}$ such that $K' = F' \cup \bigcup\{A'_{z_j} : j \leq u\} \cup \{B_{z_j} : j \leq w\}$. As before, let

$$l = \min\{n \in \omega : \{B_{z_j} \setminus n : j \leq w\} \cup \{\{x\} : x \in F' \cup G'_n\} \text{ is decisive}\},$$

where $G'_n : \bigcup\{A_j \cap n : j \leq u\}$ for every $n \in \omega$.

It is clear that $l \leq k$. For every $j \leq w$, let $y_j = \min(A_{z_j} \cap l)$ if $\varphi\restriction[B_{z_j}]^2 = \min$ and $(B_{z_j} \setminus l) \cap K' \neq \emptyset$, and let $y_j = B_{z_j}$ in any other case. Notice that if $\varrho(K) = B_{z_j}$ for some $j \leq w$ and $\varrho\restriction[B_{z_j}] = \min$, then $(K' \cap k) \cap (A_{z_j} \setminus l) = \emptyset$, because otherwise if $M_j = (K' \cap k) \cap (B_{n_j} \setminus l)$ is nonempty then $\{M_j \cup \{\varrho(K)\}\} \cup (\{\{x_i\} : i \leq s\} \setminus \varrho(K)) \cup \{\{m\} : m \in \{m_0, \ldots, m_v\} \setminus M_j\}$ would be a decisive partition of $\{x_i : i \leq s\} \cup \{m_i : i \leq v\}$ and $\Phi(M_j \cup \{\varrho(K)\}) = \min M_j$, which is not possible. Therefore, in this case $y_j = \varrho(K)$.

Then $\Phi(\{y_j : j \leq w\} \cup (F' \cup G'_l)) = \Phi(\{y_j : j \leq w\} \cup (F' \cup G'_l) \cup \{x_i : i \leq s\} \cup \{m_j : j \leq v\})$. But $\{y_j : j \leq w\} \cup (F' \cup G'_l) \cup \{x_i : i \leq s\} \cup \{m_j : j \leq v\} \subseteq \mathcal{U}$, which implies that $\varrho(K') \subseteq U$. We conclude that ϱ is continuous and so $\mathrm{Sel}(\mathcal{K}(X)) \neq \emptyset$. ∎

Remark 3.9. The space X described in Remark 2.8 also has this property, as $\mathcal{K}(X) = \mathrm{Fin}(X)$.

3.3. *Another example.* Here we prove that the existence of a continuous selection for triples does not guarantee, even for separable spaces, the existence of a continuous weak selection.

Proposition 3.10. *There is a separable space that admits a continuous selection for* $[X]^3$ *but* $\mathrm{Sel}(\mathcal{F}_2(X)) = \emptyset$.

Proof. Identify ω with $2^{<\omega}$. For every $f \in 2^\omega$ let $A_f = \{f\restriction n : n \in \omega\}$ be the branch determined by f and let $\mathcal{A} = \{A_f : f \in 2^\omega\}$. Enumerate the AD family $\mathcal{A}$ as $\{A_\alpha : \alpha < \mathfrak{c}\}$. Enumerate also the set of all weak selections on $2^{<\omega}$ as $\{f_\alpha : \alpha < \mathfrak{c}\}$.

For every $\alpha < \mathfrak{c}$ define $g_\alpha : [A_\alpha]^2 \to 2$ as follows:

$$g_\alpha(\{f\restriction m, f\restriction n\}) = \begin{cases} 0 & \text{if } f_\alpha(f\restriction m, f\restriction n) = f\restriction \min\{m,n\}, \\ 1 & \text{if } f_\alpha(f\restriction m, f\restriction n) = f\restriction \max\{m,n\}, \end{cases}$$

where $f \in 2^\omega$ and $A_\alpha = A_f$.

By Ramsey's Theorem, there is a g_α-homogeneous set $B_\alpha \in [A_\alpha]^\omega$ so that $g''_\alpha[B_\alpha]^2 = \{i\}$ for some $i \in 2$. Let $\{B^0_\alpha, B^1_\alpha\}$ be a partition of B_α such that $|B^i_\alpha| = \omega$ for $i \in 2$ and consider the AD family $\mathcal{B} = \{B^0_\alpha, B^1_\alpha : \alpha < \mathfrak{c}\}$. Let $X = \Psi(\mathcal{B})$, the Mrówka–Isbell space associated to $\mathcal{B}$.

We define a relation $\leq$ on X in the following way:

$$x \leq y \quad \text{if and only if} \quad \begin{cases} x = y, \text{ or} \\ x, y \in 2^{<\omega} \text{ and } x \subseteq y, \text{ or} \\ x = f{\restriction}n \in 2^{<\omega} \text{ and } y = B^i_f \text{ for some } i \in 2. \end{cases}$$

It is clear that $\leq$ is reflexive, antisymmetric and transitive.

If $x \not\leq y$ and $y \not\leq x$, we will write $x \perp y$. Now, to any $x, y \in X$ with $x \perp y$ we can associate an element $\Delta_{x,y}$ of $\omega \cup \{\omega\}$ as follows:

$$\Delta_{x,y} = \begin{cases} \min\{n : x(n) \neq y(n)\} & \text{if } x, y \in 2^{<\omega}, \\ \min\{n : x(n) \neq f(n)\} & \text{if } x \in 2^{<\omega} \text{ and } y = B^i_f \text{ for some } i \in 2, \\ \min\{n : f(n) \neq g(n)\} & \text{if } x = B^i_f,\ y = B^j_g \text{ with } i, j \in 2 \text{ and } f \neq g, \\ \omega & \text{if } \{x, y\} = \{B^0_f, B^1_f\} \text{ for some } f \in S. \end{cases}$$

Notice that if $x \perp y$ and $y \leq z$ then $x \perp z$ and $\Delta_{x,y} = \Delta_{x,z}$.

We define $\varrho : [X]^3 \to X$ by $\varrho(\{x, y, z\}) = x$ if either $x \leq y$ and $x \leq z$, or $x \perp y$, $x \perp z$ and $\Delta_{x,y} = \Delta_{x,z}$.

Let us first prove that ϱ is well defined. Let $F = \{x, y, z\} \in [X]^3$. Notice that F has at most one element comparable with all its elements. In this case, the function is well defined by construction. So we can suppose that $x \perp y$ and $x \perp z$. If $y \leq z$ then $\Delta_{x,y} = \Delta_{x,z}$, and since y and z are comparable, we have $\varrho(\{x, y, z\}) = x$. In the same way, if $x \perp z$ and $z \leq y$ then $\varrho(\{x, y, z\}) = x$. Therefore, we can suppose that $x \perp y$, $x \perp z$ and $y \perp z$. If $\Delta_{x,y} = \Delta_{x,z}$ then $\Delta_{y,z} > \Delta_{x,y}$ and so $\varrho(\{x, y, z\}) = x$. Otherwise, if $\Delta_{x,y} < \Delta_{x,z}$ then $\Delta_{y,z} = \Delta_{x,y}$ and so $\varrho(\{x, y, z\}) = y$. Finally, if $\Delta_{x,y} > \Delta_{x,z}$ then $\Delta_{y,z} = \Delta_{x,z}$ and $\varrho(\{x, y, z\}) = z$.

To prove that ϱ is continuous, let $\{x, y, z\} \in [X]^3$ and suppose that $\varrho(\{x, y, z\}) = x$.

Case 1: *$x \leq y$ and $x \leq z$.* Since $x \in 2^{<\omega}$, there are $f \in S$ and $n \in \omega$ such that $x = f{\restriction}n$. If $y = f{\restriction}m$ for some $m > n$, then let $U_y = \{f{\restriction}m\}$. Otherwise, if $y = B^i_f$ for some $i \in 2$, let $U_y = \{y\} \cup (B^i_f \setminus \{f{\restriction}k : k \leq n\})$. In a similar way, we can consider a neighborhood U_z for z. It is not difficult to verify that $\mathcal{U} = \langle \{x\}, U_y, U_z \rangle$ is a neighborhood of $\{x, y, z\}$ with $\varrho[\mathcal{U}] = \{x\}$.

Case 2: *$x \perp y$, $x \perp z$ and $\Delta_{x,y} = \Delta_{x,z}$.* Suppose first that $x \in 2^{<\omega}$ and let $U_x = \{x\}$. Let $U_y = \{y\}$ if $y \in 2^{<\omega}$, and $U_y = \{y\} \cup (B^j_g \setminus \{g{\restriction}k : k \leq \Delta_{x,y}\})$ if $y = B^j_g$ for $g \in S$ and $j \in 2$. Define U_z in the same form. Finally, consider the neighborhood $\mathcal{U} = \langle U_x, U_y, U_z \rangle$ of $\{x, y, z\}$. Notice that for every $y_0 \in U_y$

and $z_0 \in U_z$, we have $x \perp y_0$, $x \perp z_0$ and $\Delta_{x,y_0} = \Delta_{x,z_0} = \Delta_{x,y}$. Therefore, $\varrho[\mathcal{U}] = \{x\}$.

On the other hand, suppose that $x = B^i_f$ for some $f \in S$ and $i \in 2$, and let U be a neighborhood of x. We can find $n \in \omega$ such that $\{x\} \cup (B^i_f \setminus \{f{\restriction}k : k \leq n\}) \subseteq U$. Let $m = \max\{n, \Delta_{x,y}\}$ and $U_x = \{x\} \cup (B^i_f \setminus \{f{\restriction}k : k \leq m\})$. If $y \in 2^{<\omega}$, consider the neighborhood $U_y = \{y\}$. If otherwise $y = B^j_g$ for some $g \in S$ and $j \in 2$, let $U_y = \{y\} \cup (B^j_g \setminus \{g{\restriction}k : k \leq m\})$. In a similar way, we can find a neighborhood U_z for z. As before, if $\mathcal{U} = \langle U_x, U_y, U_z\rangle$, it is not hard to verify that $\varrho[\mathcal{U}] \subseteq U_x \subseteq U$ and we conclude that ϱ is continuous at $\{x, y, z\}$.

Finally, to prove that X does not admit a continuous weak selection, let h be any weak selection on X. Then $h{\restriction}2^{<\omega} = f_\alpha$ for some $\alpha < \mathfrak{c}$. Let $f \in 2^\omega$ with $A_\alpha = A_f$ and assume, without loss of generality, that $h(\{B^0_\alpha, B^1_\alpha\}) = B^0_\alpha$. Let $\mathcal{U}$ be a basic neighborhood of (B^0_α, B^1_α). We can find a $k \in \omega$ such that $(B^0_\alpha \setminus \{f{\restriction}l : l < k\}) \cap (B^1_\alpha \setminus \{f{\restriction}l : l < k\}) = \emptyset$ and $\langle\{B^0_\alpha\} \cup (B^0_\alpha \setminus \{f{\restriction}l : l < k\}), \{B^1_\alpha\} \cup (B^1_\alpha \setminus \{f{\restriction}l : l < k\})\rangle \subseteq \mathcal{U}$. If $f_\alpha(\{f{\restriction}m, f{\restriction}n\}) = f{\restriction}\min\{m, n\}$ for any $f{\restriction}n, f{\restriction}m \in B_\alpha$, choose $n, m \in \omega$ with $n > m$, $f{\restriction}n \in B^0_\alpha \setminus \{f{\restriction}l : l < k\}$ and $f{\restriction}m \in B^1_\alpha \setminus \{f{\restriction}l : l < k\}$. Then $(f{\restriction}n, f{\restriction}m) \in \mathcal{U}$ and $h(\{f{\restriction}n, f{\restriction}m\}) = f{\restriction}m \notin B^0_\alpha$. In the other case, if $g''_\alpha[B_\alpha]^2 = \{1\}$, choose $n, m \in \omega$ with $n < m$, $f{\restriction}n \in B^0_\alpha \setminus \{f{\restriction}l : l < k\}$ and $f{\restriction}m \in B^1_\alpha \setminus \{f{\restriction}l : l < k\}$. Then $(f{\restriction}n, f{\restriction}m) \in \mathcal{U}$ and $h(\{f{\restriction}n, f{\restriction}m\}) = m \notin B^1_\alpha$. We conclude that h is not continuous at (B^0_α, B^1_α). ∎

4. Questions. We conclude with some open questions.

QUESTION 4.1 (Gutev–Nogura). *Is there a space X which admits a continuous weak selection but not a selection for $[X]^{\leq n}$ for some $n > 2$?*

Corollary 3.7 shows that if there is such a space, it cannot be separable.

QUESTION 4.2. *Is every space which admits a continuous weak selection a continuous $\leq$2-to-1-preimage of an ordered space?*

QUESTION 4.3. *Does every* (*separable*) *space which admits a continuous weak selection admit a continuous selection for all compact sets?*

QUESTION 4.4. *Does there exist a second countable space X that admits a continuous weak selection for $[X]^n$ for some $n \in \omega$, but does not admit a continuous weak selection?*

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Instituto de Matemáticas
UNAM
Apartado Postal 61-3
Xangari, 58089
Morelia, Michoacán, México
E-mail: michael@matmor.unam.mx
ivan@matmor.unam.mx

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