A computation in Khovanov–Rozansky homology

by

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Abstract. We investigate the Khovanov–Rozansky invariant of a certain tangle and its compositions. Surprisingly the complexes we encounter reduce to ones that are very simple. Furthermore, we discuss a "local" algorithm for computing Khovanov–Rozansky homology and compare our results with those for the "foam" version of sl_3-homology.

1. Introduction. In a seminal work M. Khovanov and L. Rozansky [6] introduced a series of doubly-graded link homology theories with Euler characteristic the quantum sl_n-link polynomials. The construction relied on the theory of matrix factorizations, which was previously seen in the study of maximal Cohen–Macaulay modules on isolated hypersurface singularities. For $n = 2$ and $n = 3$, link homology theories with Euler characteristic the Jones polynomial and the quantum sl_3 polynomial were introduced earlier by M. Khovanov in [5] and [4] respectively. The constructions came in a very different guise, but it was easy to see that the matrix factorization version specialized to $n = 2$ agreed with what is now known as Khovanov homology. The sl_3 version is also known to be isomorphic to the matrix factorization version [9]. Variants of these theories were described in [1], [2], [8] as well as a number of other publications.

Using ideas from [3] we show that for certain classes of tangles, and hence for knots and links composed of these, the Khovanov–Rozansky complex reduces to one that is quite simple, namely one without any "thick" edges. In particular, we consider the tangle in Fig. 1 and show that its associated complex is homotopic to the one below, with some grading shifts and basic maps which we leave out for now. All of the necessary details of the construction will be provided in the next section.

The complexes for these knots and links are entirely "local," and to calculate the homology we only need to exploit the Frobenius structure of the
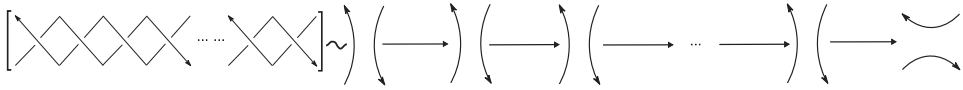
Fig. 1. Our main tangle and its reduced complex

underlying algebra assigned to the unknot. Hence, here the calculations and complexity are similar to that of sl_2-homology. We also discuss a general algorithm, basically the one described in [3], to compute these homology groups in a more time-efficient manner. We compare our results with similar computations in the version of sl_3-homology found in [4], which we refer to as the "foam" version (foams are certain types of cobordisms described in this paper), and giving an explicit isomorphism between the two versions.

The paper is structured as follows: in Section 2 we give a brief review of Khovanov–Rozansky homology, but assume that the reader is either familiar with the material or is willing to take a lot for granted; in Sections 3 through 5 we go through the main calculation; in Section 6 we discuss the algorithm and "foam" version of sl_3-homology.

2. A review of Khovanov–Rozansky homology

Matrix factorizations. Let $R = \mathbb{Q}[x_1, \ldots, x_n]$ be a graded polynomial ring in n variables with $\deg(x_i) = 2$, and let $\omega \in R$. A *matrix factorization* with *potential* ω is a collection of two free R-modules M^0 and M^1 and R-module maps $d^0 : M^0 \to M^1$ and $d^1 : M^1 \to M^0$ such that

$$d^0 \circ d^1 = \omega \, \mathrm{Id} \quad \text{and} \quad d^1 \circ d^0 = \omega \, \mathrm{Id}.$$

The d^i's are referred to as *differentials* and we often denote such a 2-complex by

$$M : \quad M^0 \xrightarrow{d^0} M^1 \xrightarrow{d^1} M^0.$$

Given two matrix factorizations M_1 and M_2 with potentials ω_1 and ω_2 respectively, their tensor product is given as the tensor product of complexes, and it is easy to see that $M_1 \otimes M_2$ is a matrix factorization with potential $\omega_1 + \omega_2$.

To keep track of minus signs, it is convenient to assign a label to the factorization and denote it by

$$M : \quad M(\emptyset) \xrightarrow{d^0} M(a) \xrightarrow{d^1} M(\emptyset),$$

so that the tensor product $M \otimes M$ of two factorizations can be written as

$$\begin{pmatrix} M(\emptyset) \\ M(ab) \end{pmatrix} \to \begin{pmatrix} M(a) \\ M(b) \end{pmatrix} \to \begin{pmatrix} M(\emptyset) \\ M(ab) \end{pmatrix}.$$

Here we are simply replacing M^0 by $M(\emptyset)$ and M^1 by a label such as $M(a)$; this will be useful below when we assign facorizations to plane graphs. See [6] for a more detailed treatment.

A homomorphism $f : M \to N$ of two factorizations is a pair of homomorphisms $f^0 : M^0 \to N^0$ and $f^1 : M^1 \to N^1$ such that the following diagram is commutative:

$$
\begin{array}{ccccc}
M^0 & \xrightarrow{d^0} & M^1 & \xrightarrow{d^1} & M^0 \\
\downarrow{\scriptstyle f^0} & & \downarrow{\scriptstyle f^1} & & \downarrow{\scriptstyle f^0} \\
N^0 & \xrightarrow{d^0} & N^1 & \xrightarrow{d^1} & N^0
\end{array}
$$

A homotopy h between maps $f, g : M \to N$ of factorizations is a pair of maps $h^i : M^i \to N^{i-1}$ such that $f - g = h \circ d_M + d_N \circ h$ where d_M and d_N are the differentials in M and N respectively. For a detailed treatment of matrix factorizations we refer the reader to [6].

Grading shifts. Let M be a matrix factorization as above, with M^0 and M^1 being $\mathbb{Z}$-graded modules over a $\mathbb{Z}$-graded ring, and let $k \in \mathbb{Z}$. Let $M\{k\}$ be the module M with degrees shifted up by k. By $M\langle k \rangle^i = M^{i+k}$ with $i+k$ taken mod 2 we denote the shift in homological grading coming from the factorization. Later we will see another homological grading of our complex, arising from the resolutions of a link diagram, and the shifted module there will be denoted by $M[k]$.

Planar graphs and matrix factorizations. Our graphs are embedded in a disk and have two types of edges, unoriented and oriented. Unoriented edges are called "thick" and drawn accordingly; each vertex adjoining a thick edge has either two oriented edges leaving it or two entering. In Fig. 3 left x_1, x_2 are outgoing and x_3, x_4 are incoming. Oriented edges are allowed to have marks and we also allow closed loops; points of the boundary are also referred to as marks. See for example Fig. 2 below. To such a graph Γ we assign a matrix factorization in the following manner:

To a thick edge t as in Fig. 3 (left) we assign a factorization C_t with potential $\omega_t = x_1^{n+1} + x_2^{n+1} - x_3^{n+1} - x_4^{n+1}$ over the ring $R_t = \mathbb{Q}[x_1, x_2, x_3, x_4]$. Since $x^{n+1} + y^{n+1}$ lies in the ideal generated by $x + y$ and xy we can write it as a polynomial $g(x + y, xy)$. Hence, ω_t can be written as

$$\omega_t = (x_1 + x_2 - x_3 - x_4)u_1 + (x_1 x_2 - x_3 x_4)u_2$$

where

$$u_1 = \frac{x_1^{n+1} + x_2^{n+1} - g(x_3 + x_4, x_1 x_2)}{x_1 + x_2 - x_3 - x_4},$$

$$u_2 = \frac{g(x_3 + x_4, x_1 x_2) - x_3^{n+1} - x_4^{n+1}}{x_1 x_2 - x_3 x_4}.$$
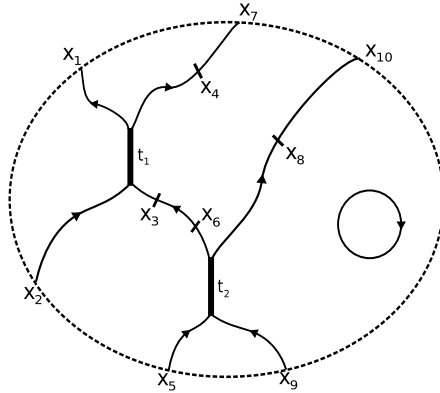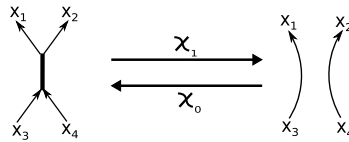
Fig. 2. A planar graph

Fig. 3. Maps χ_0 and χ_1

C_t is the tensor product of graded factorizations

$$R_t \xrightarrow{u_1} R_t\{1-n\} \xrightarrow{x_1+x_2-x_3-x_4} R_t$$

and

$$R_t \xrightarrow{u_2} R_t\{3-n\} \xrightarrow{x_1x_2-x_3x_4} R_t.$$

To an arc α bounded by marks oriented from j to i we assign the factorization L_j^i,

$$R_\alpha \xrightarrow{\pi_{ij}} R_\alpha \xrightarrow{x_i-x_j} R_\alpha,$$

where $R_\alpha = \mathbb{Q}[x_i, x_j]$ and

$$\pi_{ij} = \frac{x_i^{n+1} - x_j^{n+1}}{x_i - x_j}.$$

Finally, to an oriented loop with no marks we assign the complex $0 \to A \to 0 = A\langle 1 \rangle$ where $A = \mathbb{Q}[x]/(x^n)$. [Note: to a loop with marks we assign the tensor product of L_j^i's as above, but this turns out to be isomorphic to $A\langle 1 \rangle$ in the homotopy category.]

We define $C(\Gamma)$ to be the tensor product of C_t over all thick edges t, L_j^i over all edges α from i to j, and $A\langle 1 \rangle$ over all oriented markless loops. This tensor product is taken over appropriate rings such that $C[\Gamma]$ is a free module over $R = \mathbb{Q}[x_i]$ where the x_i's are marks. For example, to the graph in Fig. 2 we assign $C(\Gamma) = L_4^7 \otimes C_{t_1} \otimes L_6^3 \otimes C_{t_2} \otimes L_8^{10} \otimes A\langle 1 \rangle$ tensored over

$\mathbb{Q}[x_4]$, $\mathbb{Q}[x_3]$, $\mathbb{Q}[x_6]$, $\mathbb{Q}[x_8]$, $\mathbb{Q}$ respectively. $C(\Gamma)$ becomes a $\mathbb{Z} \oplus \mathbb{Z}_2$-graded complex with the $\mathbb{Z}_2$-grading coming from the factorization. It has potential $\omega = \sum_{i \in \partial \Gamma} \pm x_i^{n+1}$, where $\partial \Gamma$ is the set of all boundary marks and the $\pm$ is determined by whether the direction of the edge corresponding to x_i is towards or away from the boundary. [Note: if Γ is a closed graph the potential is zero.]

The maps χ_0 and χ_1. We now define maps between matrix factorizations associated to the thick edge and two disjoint arcs as in Fig. 3. Let Γ^0 correspond to the two disjoint arcs, and Γ^1 to the thick edge.

$C(\Gamma^0)$ is the tensor product of L_4^1 and L_3^2. If we assign labels a, b to L_4^1, L_3^2 respectively, the tensor product can be written as

$$\begin{pmatrix} R(\emptyset) \\ R(ab)\{2-2n\} \end{pmatrix} \xrightarrow{P_0} \begin{pmatrix} R(a)\{1-n\} \\ R(b)\{1-n\} \end{pmatrix} \xrightarrow{P_1} \begin{pmatrix} R(\emptyset) \\ R(ab)\{2-2n\} \end{pmatrix},$$

where

$$P_0 = \begin{pmatrix} \pi_{14} & x_2 - x_3 \\ \pi_{23} & x_4 - x_1 \end{pmatrix}, \qquad P_1 = \begin{pmatrix} x_1 - x_4 & x_2 - x_3 \\ \pi_{23} & -\pi_{14} \end{pmatrix},$$

$$\pi_{ij} = \sum_{k=0}^{n} x_i^k x_j^{n-k}.$$

Assigning labels a' and b' to the two factorizations in $C(\Gamma^1)$, we see that $C(\Gamma^1)$ is given by

$$\begin{pmatrix} R(\emptyset)\{-1\} \\ R(a'b')\{3-2n\} \end{pmatrix} \xrightarrow{Q_1} \begin{pmatrix} R(a')\{n\} \\ R(b')\{2-n\} \end{pmatrix} \xrightarrow{Q_2} \begin{pmatrix} R(\emptyset)\{-1\} \\ R(a'b')\{3-2n\} \end{pmatrix},$$

where

$$Q_1 = \begin{pmatrix} u_1 & x_1 x_2 - x_3 x_4 \\ u_2 & x_3 + x_4 - x_1 - x_2 \end{pmatrix},$$

$$Q_2 = \begin{pmatrix} x_1 + x_2 - x_3 - x_4 & x_1 x_2 - x_3 x_4 \\ u_2 & -u_1 \end{pmatrix}.$$

A map between $C(\Gamma^0)$ and $C(\Gamma^1)$ can be given by a pair of 2×2 matrices. Define $\chi_0 : C(\Gamma^0) \to C(\Gamma^1)$ by

$$U_0 = \begin{pmatrix} x_1 - x_3 & 0 \\ \frac{u_1 + x_1 u_2 - \pi_{23}}{x_1 - x_4} & 1 \end{pmatrix}, \qquad U_1 = \begin{pmatrix} x_1 & -x_3 \\ -1 & 1 \end{pmatrix},$$

and $\chi_1 : C(\Gamma^1) \to C(\Gamma^0)$ by

$$V_0 = \begin{pmatrix} 1 & 0 \\ \frac{u_1 + x_1 u_2 - \pi_{23}}{x_4 - x_1} & x_1 - x_3 \end{pmatrix}, \qquad V_1 = \begin{pmatrix} 1 & x_3 \\ 1 & x_1 \end{pmatrix}.$$

These maps have degree 1. Computing we see that $\chi_1\chi_0 = (x_1 - x_3)I$, where I is the identity matrix, i.e. $\chi_1\chi_0$ is multiplication by $x_1 - x_3$. Similarly $\chi_0\chi_1 = (x_4 - x_2)I$. [Note: these are specializations of the maps χ_0 and χ_1 given in [6], with $\lambda = 0$ and $\mu = 1$. As these maps are homotopic for any rational value of λ and μ we are free to do so.]

Define the trace $\varepsilon : \mathbb{Q}[x]/(x^n) \to \mathbb{Q}$ as $\varepsilon(x^i) = 0$ for $i \neq n - 1$ and $\varepsilon(x^{n-1}) = 1$. The unit $\iota : \mathbb{Q} \to \mathbb{Q}[x]/(x^n)$ is defined by $\iota(1) = 1$.

The relations between $C(\Gamma)$'s mimic the graph skein relations (see for example [6]), and we list the ones needed below.

DIRECT SUM DECOMPOSITION 0:

where

$$D_0 = \sum_{i=0}^{n-1} x^i \iota \quad \text{and} \quad D_0^{-1} = \sum_{i=0}^{n-1} \varepsilon x^{n-1-i}.$$

By the pictures above, we really mean the complexes assigned to them, i.e. $\emptyset\langle 1 \rangle$ is the complex with $\mathbb{Q}$ sitting in homological grading 1 and the unknot is the complex $A\langle 1 \rangle$ as above. The map $x^i \iota$ is a composition of maps

$$A\langle 1 \rangle \xrightarrow{x^i} \langle 1 \rangle \xrightarrow{\iota} \emptyset\langle 1 \rangle,$$

where x^i is multiplication and ι is the unit map, i.e. $x^i \iota$ is the map

$$\mathbb{Q}[x]/(x^n) \xrightarrow{x^i} \mathbb{Q}[x]/(x^n) \xrightarrow{\iota} \mathbb{Q}.$$

Similar with εx^{n-1-i}. It is easy to check that the above maps are grading preserving and their composition is the identity.

DIRECT SUM DECOMPOSITION I:

where

$$D_1 = \sum_{i=0}^{n-2} \beta x_1^{n-i-2} \quad \text{and} \quad D_1^{-1} = \sum_{i=0}^{n-2} \sum_{j=0}^{i} x_1^j x_2^{i-j} \alpha$$

with $\alpha := \chi_0 \circ \iota'$ and $\beta := \varepsilon' \circ \chi_1$. Here $\iota' = \iota \otimes \text{Id}$ and $\varepsilon' = \varepsilon \otimes \text{Id}$; the Id corresponds to the arc with endpoints labeled by x_2, x_3, i.e. ι' is the map
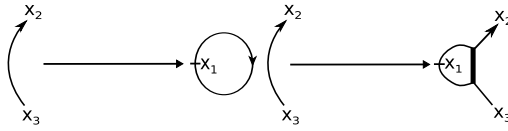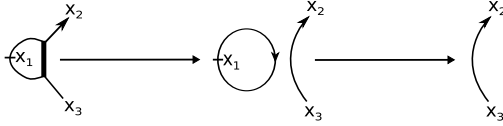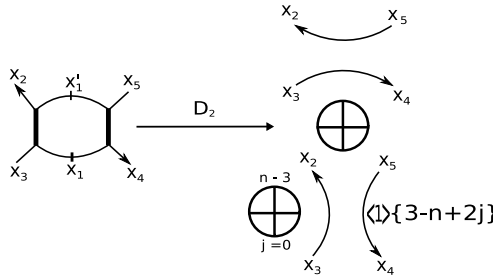
Fig. 4. The map α

Fig. 5. The map β

that includes the single arc diagram into one with the unkot and single arc disjoint (see Fig. 4). Similar with ε' in the right half of Fig. 5.

DIRECT SUM DECOMPOSITION II:

where

$$D_2 = S \oplus \sum_{j=0}^{n-3} \beta_j \quad \text{and} \quad \beta_j = \sum_{j=0}^{n-3} \beta \sum_{a+b+c=n-3-j} x_2^a x_4^b x_1^c.$$

Here β is given by the composition of two χ_1's, corresponding to the two thick edges on the left-hand side above, and the trace map ε (see Fig. 6). Finally, S is gotten by "merging" the thick edges together to form two disjoint horizontal arcs, as in the top right-hand corner above; an exact description of S will not really matter so we will not go into details and refer the interested reader to [6].
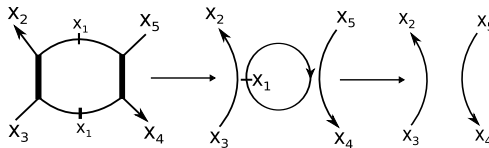

Fig. 6. The map β

Tangles and complexes. We resolve a crossing p in the two ways and assign to it a complex C^p depending on whether the crossing is positive or
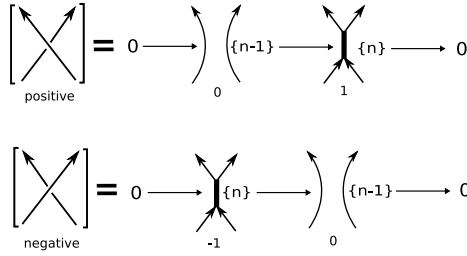
Fig. 7. Complexes associated to pos/neg crossings; the numbers below the diagrams are cohomological degrees.

negative. To a diagram D representing a tangle L we assign the complex $C(D)$ of matrix factorization which is the tensor product of C^p over all crossings p, of L^i_j over arcs $j \to i$, and of $A\langle 1\rangle$ over all crossingless markless circles in D. The tensor product is taken as before so that $C(D)$ is free and of finite rank as an R-module. This complex is $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_2$-graded.

THEOREM 1 (Khovanov–Rozansky, [6]). *The isomorphism class of $C(D)$ up to homotopy is an invariant of the tangle.*

If L is a link the cohomology groups are nontrivial only in degree equal to the number of components of L mod 2. Hence, the grading reduces to $\mathbb{Z} \oplus \mathbb{Z}$. The resulting cohomology groups are denoted by

$$H_n(D) = \bigoplus_{i,j \in \mathbb{Z}} H_n^{i,j}(D),$$

and the Euler characteristic of $H_n(D)$ is the quantum link polynomial $P_n(L)$, i.e.

$$P_n(L) = \sum_{i,j \in \mathbb{Z}} (-1)^i q^j \dim_{\mathbb{Q}} H_n^{i,j}(D).$$

The isomorphism classes of $H_n^{i,j}(D)$ depend only on the link L, and hence are invariants of the link.

Gaussian elimination for complexes

LEMMA 2. *If $\phi : B \to D$ is an isomorphism (in some additive category $\mathcal{C}$), then the four-term complex segment*

$$(1) \qquad \cdots \to [A] \xrightarrow{\binom{\alpha}{\beta}} \begin{bmatrix} B \\ C \end{bmatrix} \xrightarrow{\left(\begin{smallmatrix} \phi & \delta \\ \gamma & \epsilon \end{smallmatrix}\right)} \begin{bmatrix} D \\ E \end{bmatrix} \xrightarrow{(\mu\ \nu)} [F] \to \cdots$$

is isomorphic to the (direct sum) *complex segment*

$$(2) \qquad \cdots \to [A] \xrightarrow{\binom{0}{\beta}} \begin{bmatrix} B \\ C \end{bmatrix} \xrightarrow{\left(\begin{smallmatrix} \phi & 0 \\ 0 & \epsilon-\gamma\phi^{-1}\delta \end{smallmatrix}\right)} \begin{bmatrix} D \\ E \end{bmatrix} \xrightarrow{(0\ \nu)} [F] \to \cdots.$$

Both of these complexes are homotopy equivalent to the (*simpler*) *complex segment*

$$(3) \qquad \cdots \to [A] \xrightarrow{(\beta)} [C] \xrightarrow{(\epsilon - \gamma \phi^{-1} \delta)} [E] \xrightarrow{(\nu)} [F] \to \cdots .$$

Here the capital letters are arbitrary columns of objects in $\mathcal{C}$ and all Greek letters are arbitrary matrices representing morphisms with the appropriate dimensions, domains and ranges (*all the matrices are block matrices*); *moreover*, $\phi : B \to D$ *is an isomorphism, i.e. it is invertible.*

Proof. The matrices in complexes (1) and (2) differ by a change of bases, and hence the complexes are isomorphic. (2) and (3) differ by the removal of a contractible summand; hence, they are homotopy equivalent. ∎

3. The basic calculation. We first consider the complex associated to the tangle T in Fig. 8 with the appropriate maps χ_0 and χ_1 left out.
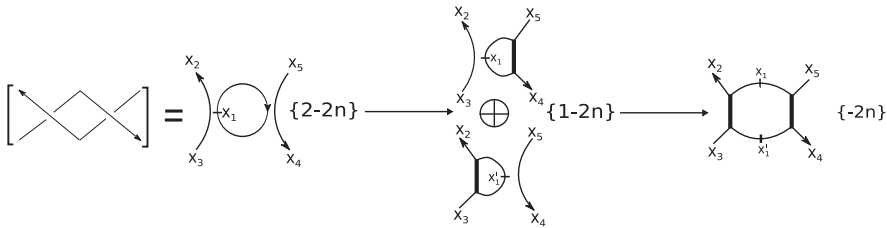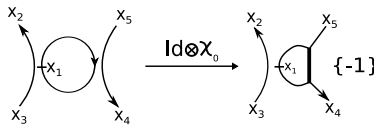

Fig. 8. The tangle T and its complex

We first look at the following part of the complex and, for simplicity, leave out the overall grading shifts until later:

We apply Direct Sum Decompositions 0 and I and end up with the picture of Fig. 9 where the maps F_1 and F_2 are isomorphisms.
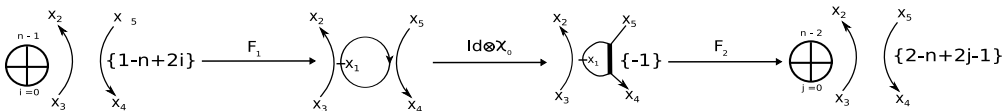

Fig. 9. First part of the complex for T with decompositions

Explicitly,

$$F_1 = \sum_{i=0}^{n-1} \mathrm{Id} \otimes x_1^i \iota \otimes \mathrm{Id} \quad \text{and} \quad F_2 = \sum_{j=0}^{n-2} \mathrm{Id} \otimes \beta_j.$$

Composing the maps we get

$$F_2 \circ (\mathrm{Id} \otimes \chi_0) \circ F_1 = \Big(\sum_{j=0}^{n-2} \mathrm{Id} \otimes \beta_j \Big) \circ (\mathrm{Id} \otimes \chi_0) \circ \Big(\sum_{i=0}^{n-1} \mathrm{Id} \otimes x_1^i \iota \otimes \mathrm{Id} \Big)$$

$$= \Big(\sum_{j=0}^{n-2} \mathrm{Id} \otimes \beta_j \Big) \circ \Big(\sum_{i=0}^{n-1} \mathrm{Id} \otimes (\chi_0 \circ (x_1^i \iota \otimes \mathrm{Id})) \Big)$$

$$= \sum_{j=0}^{n-2} \sum_{i=0}^{n-1} \mathrm{Id} \otimes (\beta_j \circ \chi_0 \circ (x_1^i \iota \otimes \mathrm{Id}))$$

$$= \sum_{j=0}^{n-2} \sum_{i=0}^{n-1} \mathrm{Id} \otimes (\varepsilon'(x_1 - x_4) x_1^{n+i-j-2})$$

$$= \sum_{j=0}^{n-2} \sum_{i=0}^{n-1} \mathrm{Id} \otimes (\varepsilon'(x_1^{n+i-j-1} - x_4 x_1^{n+i-j-2}))$$

$$= \sum_{j=0}^{n-2} \sum_{i=0}^{n-1} \mathrm{Id} \otimes \underbrace{(\varepsilon(x_1^{n+i-j-1}) - x_4 \varepsilon(x_1^{n+i-j-2}))}_{\Theta} .$$

To go from line 3 to 4 and 4 to 5, recall that $\beta_j = \varepsilon' \circ \chi_1 x_1^{n-j-2}$ and $\chi_1 \circ \chi_0 = x_1 - x_4 = x_1 - x_5$. [Note: for lack of better notation, we use "$\sum$" to indicate both a map from a direct sum and an actual sum, as seen above indexed i and j respectively.]

Now $\Theta = \mathrm{Id}$ if $i = j$, $-x_4$ if $i = j+1$, and 0 otherwise; $F_2 \circ (\mathrm{Id} \otimes \chi_0) \circ F_1$ is given by the following $(n-1) \times (n-1)$ matrix:

$$\begin{bmatrix} \mathrm{Id} & -x_4 & 0 & \ldots & \ldots & 0 \\ 0 & \mathrm{Id} & -x_4 & 0 & \ldots & 0 \\ \vdots & & \ddots & \ddots & & \vdots \\ \vdots & & & \ddots & \ddots & \vdots \\ 0 & & \ldots & 0 & \mathrm{Id} & -x_4 \end{bmatrix}.$$

Using Gaussian Elimination for complexes it is easy to see that, up to homotopy, only the top degree term survives. By degree, we mean one with respect to the above grading shifts.
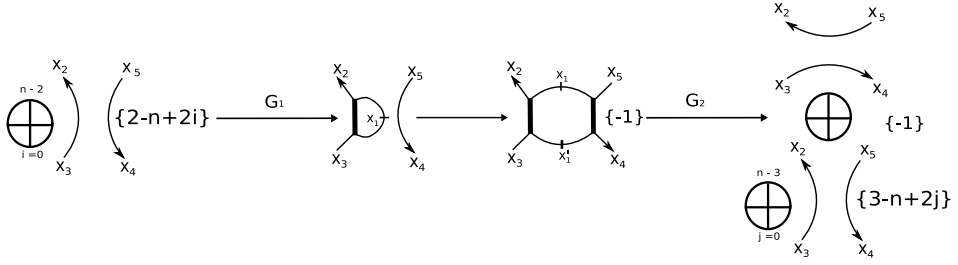
Now we look at the following subcomplex:
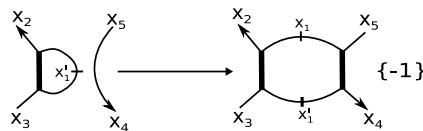
Fig. 10. The second part of the complex for T with decompositions

Including all the isomorphisms we have the complex in Fig. 10, with $G_1 = \sum_{i=0}^{n-2} \alpha_i \otimes \mathrm{Id}$ and $G_2 = S \oplus \sum_{j=0}^{n-3} \beta_j$ (S is the saddle map).
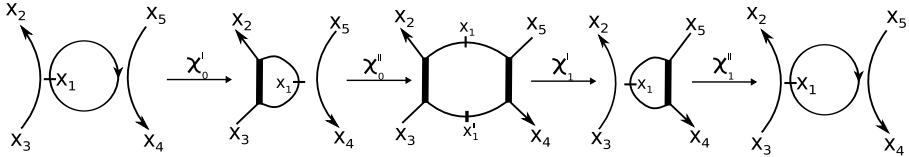
Composing these maps we get

$$G_2 \circ \chi_0'' \circ G_1$$

$$= \Big(S \oplus \sum_{j=0}^{n-3} \beta_j\Big) \circ \chi_0'' \circ \Big(\sum_{i=0}^{n-2} \alpha_i \otimes \mathrm{Id}\Big)$$

$$= \Big(S \oplus \sum_{j=0}^{n-3} \beta \sum_{a+b+c=n-3-j} x_2^a x_4^b x_1^c\Big) \circ \chi_0'' \circ \Big(\sum_{i=0}^{n-2} \sum_{k=0}^{i} x_1^k x_2^{i-k} \alpha \otimes \mathrm{Id}\Big)$$

$$= \Big(S \oplus \sum_{j=0}^{n-3} \varepsilon' \circ \chi_1'' \circ \chi_1' \sum_{a+b+c=n-3-j} x_2^a x_4^b x_1^c\Big) \circ \chi_0'' \circ \Big(\sum_{i=0}^{n-2} \sum_{k=0}^{i} x_1^k x_2^{i-k} \chi_0' \circ \iota' \otimes \mathrm{Id}\Big)$$

$$= \overline{S} \oplus \sum_{j=0}^{n-3} \sum_{i=0}^{n-2} \varepsilon' \circ \chi_1'' \circ \chi_1' \chi_0'' \circ \chi_0' \circ \Big(\sum_{a+b+c=n-3-j} x_2^a x_4^b x_1^c\Big)\Big(\sum_{k=0}^{i} x_1^k x_2^{i-k}\Big)\iota'$$

$$= \overline{S} \oplus \underbrace{\sum_{j=0}^{n-3} \sum_{i=0}^{n-2} \varepsilon'(x_1^2 - x_1 x_2 - x_1 x_4 + x_2 x_4)\Big(\sum_{a+b+c=n-3-j} x_2^a x_4^b x_1^c\Big)\Big(\sum_{k=0}^{i} x_1^k x_2^{i-k}\Big)\iota'}_{\Omega}$$

where

$$(4) \qquad \overline{S} = S \circ \chi_0'' \circ \Big(\sum_{i=0}^{n-2} \sum_{k=0}^{i} x_1^k x_2^{i-k} \chi_0' \circ \iota' \otimes \mathrm{Id}\Big).$$

To go from line 4 to 5 we recall what these χ's are:

We have $\chi_1'' \circ \chi_0'' \circ \chi_1' \circ \chi_0' = (x_4 - x_1)(x_2 - x_1) = x_1^2 - x_1 x_2 - x_1 x_4 + x_2 x_4$, so now we just have to figure what happens with Ω.

CLAIM. *If $i < j$ then $\Omega = 0$ and if $i = j$ then $\Omega = \mathrm{Id}$.*

Proof. This is just a simple check. The only thing to note is that $\Omega \neq 0$ iff one of the following occurs:

1) $c + k = n - 1$,
2) $c + k + 1 = n - 1$,
3) $c + k + 2 = n - 1$.

So $i < j \Rightarrow k < j$, so say $c + k = n - 1$. Then $a + b + c = a + b + n - 1 - k = n - 3 - j$, so $a + b = -2 + k - j < 0$, a contradiction, since a, b, c are nonnegative integers. The other two cases are similar.

From the above we see that we need k at least equal to j. So if $i = j = k$ and $c + k + 2 = n - 1$ then $a + b + c = a + b + n - 3 - k = n - 3 - j$, so $a + b = 0$ and $\Omega = \mathrm{Id}$. The other two cases force $a + b < 0$. ∎

So the matrix for Ω looks like

$$\begin{bmatrix} \mathrm{Id} & * & * & *\,*\,* & *\,*\,* & * & * \\ 0 & \mathrm{Id} & * & *\,*\,* & *\,*\,* & * & * \\ \vdots & 0 & \ddots & \ddots & & \vdots & \vdots \\ \vdots & & & \ddots & \ddots & \vdots & \vdots \\ \vdots & & & & \ddots & * & * \\ 0 & & \cdots & \cdots & 0 & \mathrm{Id} & * \end{bmatrix}.$$

Using Gaussian Elimination we see that only the entry corresponding to $i = n - 2$ survives and the original complex is homotopic to
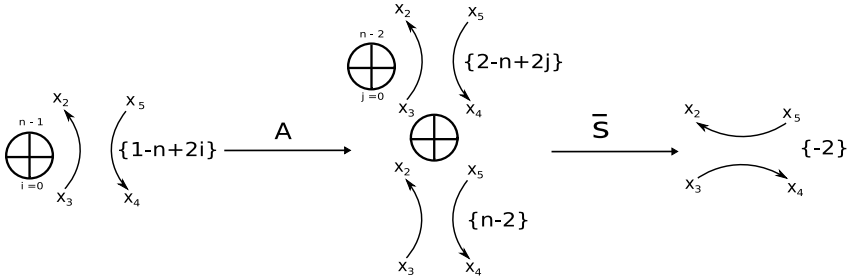
where

$$A = \begin{bmatrix} \mathrm{Id} & -x_4 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & \mathrm{Id} & -x_4 & 0 & \cdots & 0 & 0 \\ \vdots & & \ddots & \ddots & & \vdots & \vdots \\ \vdots & & & \ddots & \ddots & \vdots & \vdots \\ \vdots & & & & \mathrm{Id} & -x_4 & 0 \\ 0 & & \cdots & \cdots & 0 & \mathrm{Id} & -x_4 \\ 0 & & \cdots & \cdots & 0 & -\mathrm{Id} & x_2 \end{bmatrix}.$$

This is just our original matrix Θ but with one more row for the extra term, for which the entries are computed identically as we have already done. We reduce the complex in Fig. 8, insert the overall grading shifts and arrive at our desired conclusion, shown in Fig. 11.
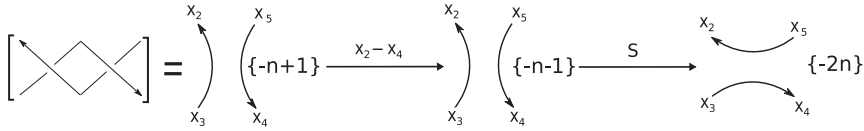

Fig. 11. The reduced complex for tangle T

Note: to convince ourselves that the map S above is indeed the "saddle" map as prescribed, we need only know that the hom-space of degree zero maps between the two rightmost diagrams above is 1-dimensional, in the homotopy category, and then argue that the map is nonzero. This can be done by say closing off the two ends of the tangle above so that we have a nonstandard diagram of the unknot and looking at the cohomology of the associated complex. We leave the details to the reader and refer to [6] for hom-space calculations.

4. Basic tensor product calculation. We now consider our tangle T composed with itself, i.e. the tangle gotten by taking two copies of T
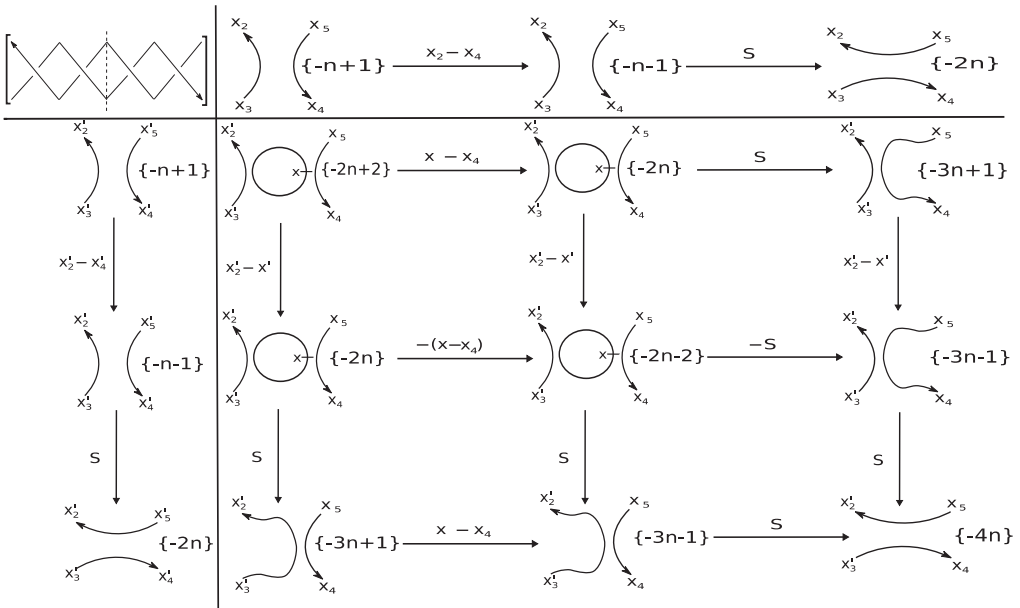

Fig. 12. Complex for the tensor product

and gluing the rightmost ends of one to the leftmost of the other. On the
complex level this corresponds to taking the tensor product of the complex
for T with itself while keeping track of the associated markings.

Note that when we take the tensor product we need to keep track of
markings. For example, in the leftmost entry of the tensored complex, $x_2 = x_5' = x_4' = x_3$, which we denote simply by x, etc.

As before, we decompose entries in the complex into direct sums of sim-
pler objects, compute the differentials and reduce using Gaussian Elimina-
tion. In a number of instances we will restrict ourselves to the $n = 3$ case, as
the general case works in exactly the same way with the computation more
cumbersome.

We break the computation up based on homological grading.
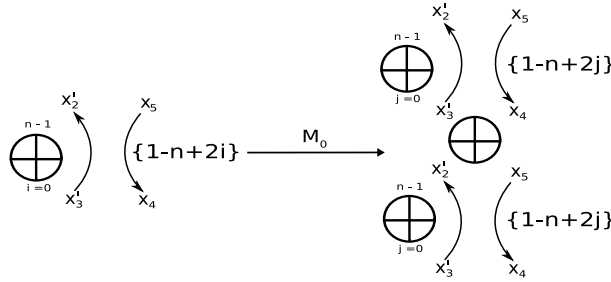
DEGREE 0:

Fig. 13. Degree 0 to 1

where

$$
M_0 = \begin{bmatrix} \displaystyle\sum_{i,j=0}^{n-1} \mathrm{Id} \otimes \varepsilon(x^{n+i-j} - x^{n-1+i-j}x_4)\iota \otimes \mathrm{Id} \\ \displaystyle\sum_{i,j=0}^{n-1} \mathrm{Id} \otimes \varepsilon(x_2' x^{n-1+i-j} - x^{n+i-j})\iota \otimes \mathrm{Id} \end{bmatrix}.
$$

For $n = 3$ we have the following:

$$
\begin{bmatrix} -x_4 & 0 & 0 \\ \mathrm{Id} & -x_4 & 0 \\ 0 & \mathrm{Id} & -x_4 \\ x_2' & 0 & 0 \\ -\mathrm{Id} & x_2' & 0 \\ 0 & -\mathrm{Id} & x_2' \end{bmatrix}
\overset{\text{reduce}}{\rightsquigarrow}
\begin{bmatrix} \mathrm{Id} & -x_4 \\ -x_4^2 & 0 \\ x_2' x_4 & 0 \\ x_2' - x_4 & 0 \\ -\mathrm{Id} & x_2' \end{bmatrix}
\overset{\text{reduce}}{\rightsquigarrow}
\begin{bmatrix} 0 \\ x_2' x_4^2 \\ x_2' x_4 - x_4^2 \\ x_2' - x_4 \end{bmatrix} = \overline{M}_0
$$

[Note: we first permute the rows in the first half of the matrix so that the
Id maps appear on the diagonal.]

The general case is exactly the same, i.e. in the leftmost matrix above, the upper and lower 3×3 matrices become expanded to similar $n \times n$ matrices. Hence, the complex reduces to:
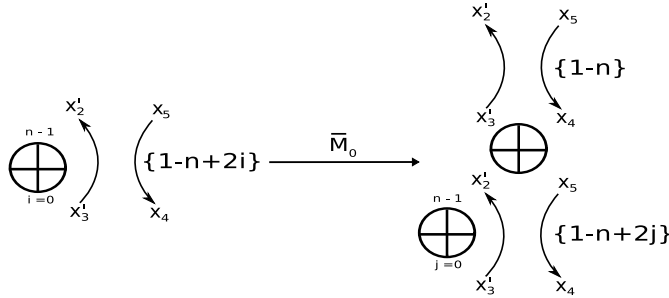

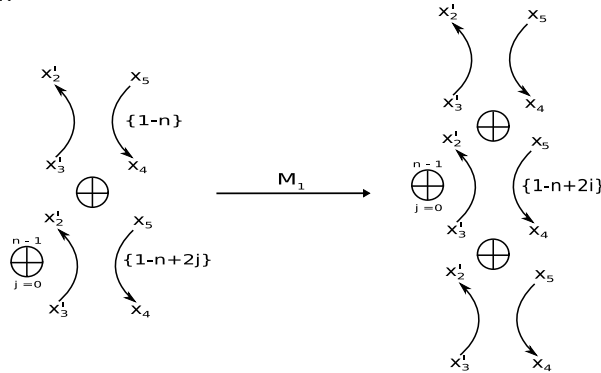Fig. 14. Degree 0 to 1

DEGREE 1:

Fig. 15. Degree 1 to 2

with

$$
M_1 := \begin{bmatrix} \mathrm{Id} \otimes S \circ \iota \otimes \mathrm{Id} & \{0\}_{1 \times n} \\ M_1^a & M_1^b \\ \{0\}_{n \times 1} & M_1^c \end{bmatrix}
$$

where

$$
M_1^a = \sum_{j=0}^{n-1} \mathrm{Id} \otimes \varepsilon(x_2' x^{n-1-j} - x^{n-j}) \iota \otimes \mathrm{Id},
$$

$$
M_1^b = \sum_{i,j=0}^{n-1} \mathrm{Id} \otimes \varepsilon(x_4 x^{n-1-j+i} - x^{n-j+i}) \iota \otimes \mathrm{Id},
$$

$$
M_1^c = \sum_{i=0}^{n-1} \mathrm{Id} \otimes x^i S \circ \iota \otimes \mathrm{Id}.
$$

Note: $x^i S \circ \iota$ here is equal to multiplication by $x_2'^i$. Expanding we get

$$
\begin{bmatrix}
\text{Id} & 0 & \ldots & \ldots & \ldots & 0 \\
x_2' & x_4 & 0 & \ldots & \ldots & 0 \\
-\text{Id} & -\text{Id} & x_4 & 0 & \ldots & \vdots \\
0 & \ldots & \ddots & \ddots & \ldots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \ddots & 0 \\
0 & 0 & \ldots & 0 & -\text{Id} & x_4 \\
0 & \text{Id} & x_2' & \ldots & \ldots & x_2'^{n-1}
\end{bmatrix}
\overset{\text{reduce}}{\rightsquigarrow}
\begin{bmatrix}
x_4 & 0 & \ldots & \ldots & 0 \\
-\text{Id} & x_4 & 0 & \ldots & \vdots \\
0 & \ddots & \ddots & \ldots & \vdots \\
\vdots & \vdots & \ddots & \ddots & 0 \\
0 & \ldots & 0 & -\text{Id} & x_4 \\
\text{Id} & x_2' & \ldots & \ldots & x_2'^{n-1}
\end{bmatrix}
$$

$$
\overset{\text{row moves}}{\rightsquigarrow}
\begin{bmatrix}
-\text{Id} & x_4 & 0 & \ldots & \vdots \\
0 & -\text{Id} & x_4 & \ldots & \vdots \\
\vdots & \vdots & \ddots & \ddots & 0 \\
0 & \ldots & 0 & -\text{Id} & x_4 \\
x_4 & 0 & \ldots & \ldots & 0 \\
\text{Id} & x_2' & \ldots & \ldots & x_2'^{n-1}
\end{bmatrix}
\overset{\text{reduce}}{\rightsquigarrow}
\begin{bmatrix}
-\text{Id} & x_4 & \ldots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
0 & \ldots & -\text{Id} & x_4 \\
x_4^2 & \ldots & \ldots & 0 \\
x_2' + x_4 & \ldots & \ldots & x_2'^{n-1}
\end{bmatrix}
$$

$$
\overset{\text{reduce}}{\rightsquigarrow}
\begin{bmatrix}
0 \\
\sum_{i=0}^{n-1} x_2'^i x_4^{n-1-i}
\end{bmatrix}
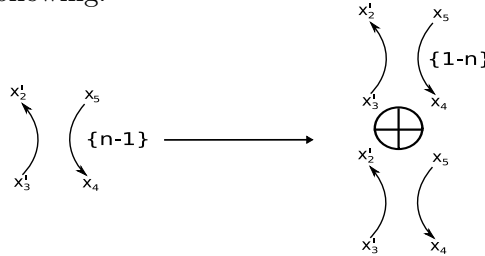$$

and we have the following:

Fig. 16. Degree 1 to 2
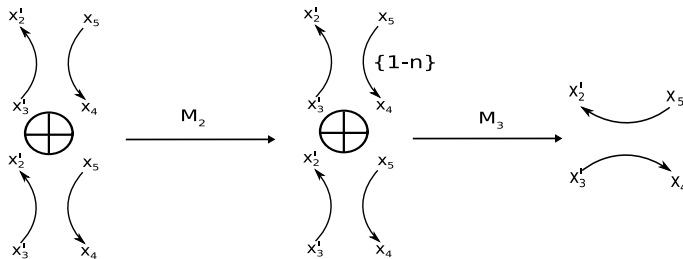
DEGREE 2 AND 3: The complex now is pretty simple:

Fig. 17. Degree 2 and 3

$$M_2 = \begin{bmatrix} -(\mathrm{Id} \otimes S \circ \iota) \otimes \mathrm{Id} & \mathrm{Id} \otimes (S \circ \iota \otimes \mathrm{Id}) \\ 0 & x'_2 - x_4 \end{bmatrix}, \quad M_3 = [S \quad S].$$

All we have to do is note that $\mathrm{Id} \otimes S \circ \iota \otimes \mathrm{Id} = \mathrm{Id}$ reduce, insert the grading shifts and arrive at the desired conclusion, shown in Fig. 18, with $A = \sum_{i=0}^{n-1} x_2'^i x_4^{n-1-i}$.

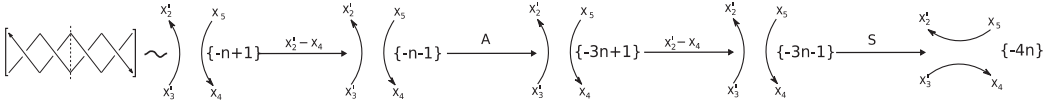
Fig. 18. The tensor complex

5. The general case. We suppose by induction that the k-fold tensor product of our basic complex has the form as above in Fig. 18 with alternating maps $x'_2 - x_4$ and A, the last map being the saddle cobordism S, and investigate what happens when we add one more iteration. As before, this corresponds to tensoring with another copy of the reduced complex for tangle T, i.e. the one in Fig. 11, but as we will see below, "most" of this new complex is null-homotopic and it suffices to consider only the part depicted in Fig. 19. Note that here the bottom row is a subcomplex which is isomorphic to that of the top tangle and we claim that, up to homotopy, this plus two more terms in leftmost homological degree is exactly what survives. The remaining calculation is left to clear up this statement and we begin by taking a look at the part of the complex depicted in Fig. 19,
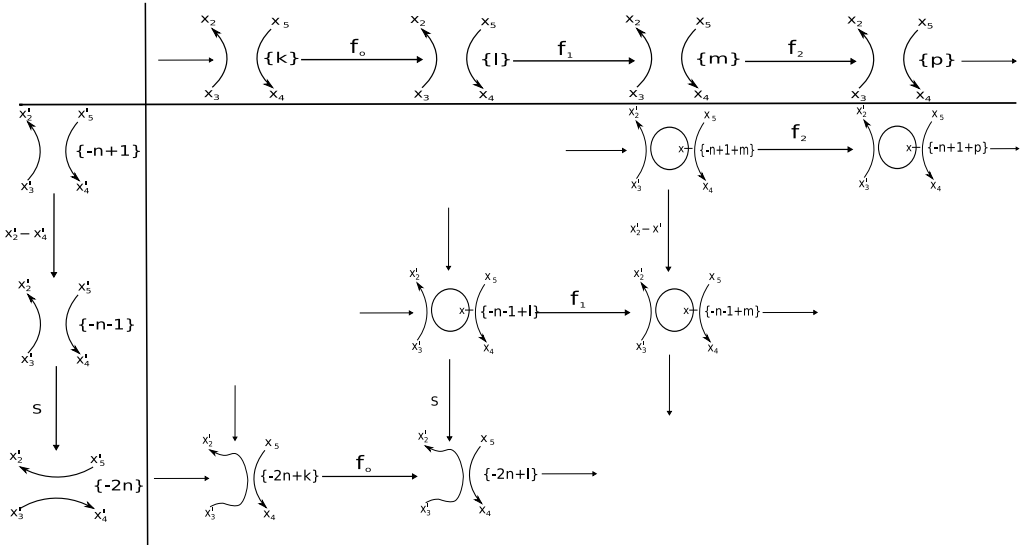

Fig. 19. Tensoring the complex with another copy of the basic tangle T
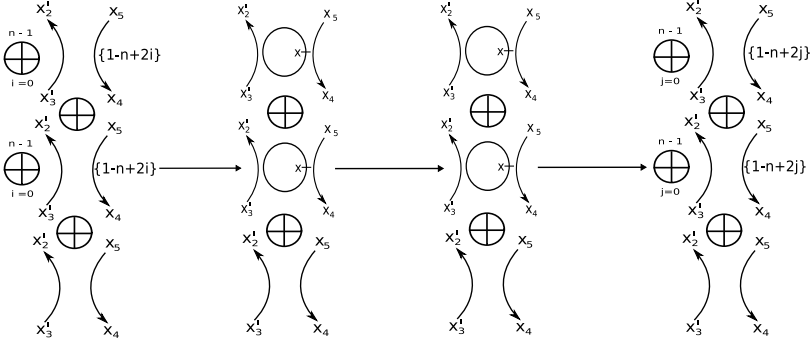
shown in Fig. 20.

Fig. 20. Decomposing the entries of the general tensor product

Of course, we have once again decomposed the complex and left out the overall grading shifts until later.

The above composition of maps is

$$\begin{bmatrix} M^a & \{0\}_{n\times n} & \{0\}_{n\times 1} \\ M^b & M^c & \{0\}_{n\times 1} \\ \{0\}_{1\times n} & M^d & f_0 \end{bmatrix}$$

where

$$M^a = \sum_{i,j=0}^{n-1} \mathrm{Id} \otimes \varepsilon f_2 x^{n-1-j+i} \iota \otimes \mathrm{Id}, \quad M^c = -\sum_{i,j=0}^{n-1} \mathrm{Id} \otimes \varepsilon f_1 x^{n-1-j+i} \iota \otimes \mathrm{Id},$$

$$M^b = \sum_{i,j=0}^{n-1} \mathrm{Id} \otimes \varepsilon x^{n-1-j}(x_2' - x)x^i \iota \otimes \mathrm{Id}, \quad M^d = \sum_{j=0}^{n-1} \mathrm{Id} \otimes x^{n-1-j} S \circ \iota \otimes \mathrm{Id}.$$

Expanding, with $f_0 = f_2 = x - x_4$ and $f_1 = \sum_{m=0}^{n-1} x^m x_4^{n-1-m}$, we get the following submatrices:

$$M^a = \begin{bmatrix} -x_4 & 0 & \dots & \dots & 0 \\ \mathrm{Id} & -x_4 & 0 & \dots & \vdots \\ 0 & \mathrm{Id} & -x_4 & \dots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \mathrm{Id} & -x_4 \end{bmatrix}, \quad M^b = \begin{bmatrix} x_2' & 0 & \dots & \dots & 0 \\ -\mathrm{Id} & x_2' & 0 & \dots & \vdots \\ 0 & -\mathrm{Id} & x_2' & \dots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & -\mathrm{Id} & x_2' \end{bmatrix},$$

$$M^c = -\begin{bmatrix} x^{n-1}x_4^{n-1} & 0 & \dots & \dots & 0 \\ * & x^{n-1}x_4^{n-1} & 0 & \dots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ * & \dots & * & x^{n-1}x_4^{n-1} & 0 \\ \mathrm{Id} & * & \dots & \dots & x^{n-1}x_4^{n-1} \end{bmatrix}.$$

Now this might look like a mess to reduce, but the thing to notice is that, in the corresponding summand in our decomposition, the first matrix above kills off all but the topmost degree terms (with respect to the decomposition-induced grading shifts), whereas the Id map found in the bottom-left corner of the second kills off precisely the topmost degree term. As the maps alternate when we increase cohomological grading and none of the reductions affect the bottom row (this is easy to see due to the 0's found in the first row), up to homotopy the bottom row remains altered only by a grading shift.

As far as the beginning and the end of the complex are concerned, we have already done those computations when we looked at the 2-fold tensor product. Hence, we arrive at our desired conclusion, shown in Fig. 21 with $A = \sum_{i=0}^{n-1} x_2'^i x_4^{n-1-i}$.
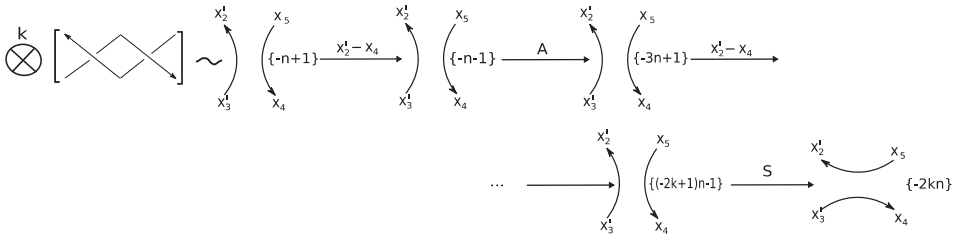

Fig. 21. The complex of the k-fold tensor product

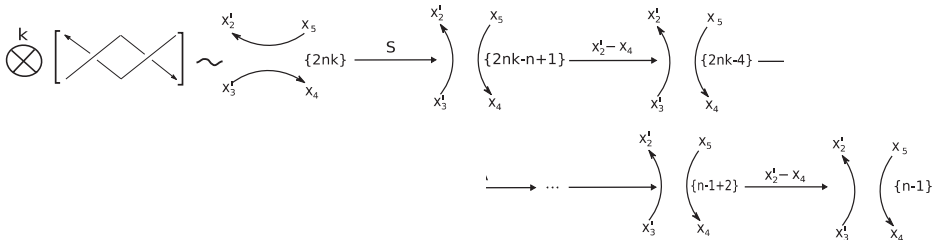Similarly we see that the tangle gotten by flipping all the crossings is as in Fig. 22.

Fig. 22. The complex of the k-fold tensor product

6. Remarks. Following [2] we can propose a similar "local" algorithm for computing Khovanov–Rozansky homology. Start with a knot or link diagram and reduce it locally using the Direct Sum Decompositions found. Then put all the pieces back together and end up with a complex where the objects are just circles, which we can further reduce to a complex of empty sets with grading shifts, i.e. direct sums of $\mathbb{Q}$ where the maps are matrices with rational entries. Since a multiplication map $\mathbb{Q} \to \mathbb{Q}$ is either zero

or an isomorphism we can use Gaussian Elimination, as above, to further reduce this complex to one where all the differentials are zero. The computational advantage of such an algorithm is described in more detail in [2]. Unfortunately no such program exists to our knowledge.

Furthermore, for the examples of tangles we consider here the computational complexity is similar to that of sl_2-homology. As there are no more "thick edges" in any resolution, only Direct Sum Decomposition 0 is necessary to reduce the complex to $\mathbb{Q}$-vector spaces and matrices between them. Potentially a modification of the existing programs could allow one to compute a large collection of examples composed from these tangles.

We have done a similar computation for the "foam" version of sl_3-homology introduced in [4]. Here the nodes in the cube of resolutions are generated by maps from the empty graph to the one at the corresponding node, with some relations, and the maps are given by cobordisms between these trivalent graphs. The decompositions mimic the ones we find here, when specializing to $n = 3$, as do the relations on the maps. Reducing the complex as before we find that it is identical to the one found above when specialized to the $n = 3$ case. Hence, any link that can be decomposed into the above tangles has exactly the same homology groups for the "foam" and matrix factorization version. This provides a rather vast number of examples where the isomorphism between the two theories is completely explicit. Furthermore, there did not seem to be any particular advantage in using the "foam" construction for these calculations and the author expects using "foams" to be only more cumbersome in general. However, the virtual crossings construction in the appendix of [7] is a bit easier to handle and further investigation of this could lead to other similar reductions in complexity. Analogous results can also be obtained for the $(2, 2n + 1)$ tangles with reversed orientation or for $(2, k)$ tangles with the "standard" orientation, but in these cases "thick" edges are unavoidable unless using the construction from [7].

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