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Provident sets and rudimentary set forcing

by

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Abstract. Using the theory of rudimentary recursion and provident sets expounded in [MB], we give a treatment of set forcing appropriate for working over models of a theory PROVI which may plausibly claim to be the weakest set theory supporting a smooth theory of set forcing, and of which the minimal model is Jensen's J_ω. Much of the development is rudimentary or at worst given by rudimentary recursions with parameter the notion of forcing under consideration. Our development eschews the power set axiom. We show that the forcing relation for $\dot{\Delta}_0$ wffs is propagated through our hierarchies by a rudimentary function, and we show that the construction of names for the values of rudimentary and rudimentarily recursive functions is similarly propagated. Our main result is that a set-generic extension of a provident set is provident.

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0. Introduction. There is a certain finitely axiomatisable theory which we call PROV, which is weaker than Kripke–Platek set theory KP, but stronger than Gandy–Jensen set theory GJ. All three theories are true in

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$\mathbf{HF} = V_\omega = J_1 = L_\omega$; if an axiom of infinity be added to each theory, giving the theories KPI, PROVI and GJI, the minimal transitive models are then respectively the Jensen fragments $J_{\omega_1^{CK}}$, J_ω and J_2.

The *provident* sets are **HF** and the transitive models of PROVI. We show that every provident set A supports the definition of the forcing relation $\Vdash^{\mathbb{P}}$ when $\mathbb{P} \in A$; our main result is that a set-generic extension of a provident set is provident.

For most of this paper we avoid use of the power set axiom; the paper [M4] discusses the problems and possibilities of set forcing over models of Mac Lane or of Zermelo set theory, two theories which include the power set axiom.

We draw on the notation (1) and results of [MB], and in particular we make heavy use of the rudimentary function $\mathbb{T}$ which was introduced in [M3]: its properties are that if u is transitive, then $\mathbb{T}(u)$ is transitive, with u both a member and a subset of it; every member of $\mathbb{T}(u)$ is a subset of u; further, the union over all n of $\mathbb{T}^n(u)$ is the rudimentary closure of $u \cup \{u\}$.

Provident sets. Let p be a set. Call a function $x \mapsto F(x)$ *p-rud-rec* (short for *p-rudimentarily recursive*) if there is a rud function H such that for every set x,

$$F(x) = H(p, F \restriction x).$$

Examples: the rank function, ϱ, and transitive closure, tcl, are $\varnothing$-rud-rec, meaning no parameters required; the evaluation $\mathrm{val}_{\mathcal{G}}(\cdot)$ of the names of a forcing language using a generic $\mathcal{G}$ is $\mathcal{G}$-rud-rec.

Rud recursion without parameters is treated in [MB, Section 4]; parameters are discussed in Section 5.

The axioms of PROV are such that its transitive models are those transitive sets A such that for each p, each p-rud-rec F and each $x \in A$, $F(x)$ is in A.

Let c be a transitive set; using $\mathbb{T}$ we define in §2 a hierarchy giving an initial segment of $L(c)$ by a recursion on the ordinals. The novelty of the definition is that the whole of c is not included at the start, but its members are fed in according to their rank: if we put $c_\nu = \{x \in c \mid \varrho(x) < \nu\}$, then the following c-rud recursion on the ordinals holds:

$$c_0 = 0, \quad c_{\nu+1} = c \cap \{x \mid x \subseteq c_\nu\}, \quad c_\lambda = \bigcup\{c_\nu \mid \nu < \lambda\} \text{ at limit } \lambda.$$

The *canonical progress towards c* is the hierarchy P^c_ν defined by setting

$$P^c_0 = \varnothing, \quad P^c_{\nu+1} = \mathbb{T}(P^c_\nu) \cup \{c_\nu\} \cup c_{\nu+1}, \quad P^c_\lambda = \bigcup\{P^c_\nu \mid \nu < \lambda\} \text{ at limit } \lambda.$$

(1) Examples: $(a, b)_2$, sometimes more informally written (a, b), is the Kuratowski ordered pair defined as $\{\{a\}, \{a, b\}\}$. $(a, b, c)_3$ is the Kuratowski ordered triple, defined as $(a, (b, c)_2)_2$. $\langle a, b, c\rangle$ denotes this function with domain $\{0, 1, 2\}$: $\{(a, 0)_2, (b, 1)_2, (c, 2)_2\}$.

0.0. Remark. As $c_\nu = c \cap P^c_\nu$, we might have defined P^c_ν by a single c-rudimentary recursion on ordinals:

$$P^c_0 = \varnothing, \quad P^c_{\nu+1} = \mathbb{T}(P^c_\nu) \cup \{c \cap P^c_\nu\} \cup (c \cap \{x \mid x \subseteq P^c_\nu\}), \quad P^c_\lambda = \bigcup_{\nu<\lambda} P^c_\nu.$$

The axiomatisation of PROV may then be summarised as

`extensionality`
+ `the empty set exists`
+ `all rudimentary functions are defined everywhere`
+ `every set has a rank`
+ `every set has a transitive closure`
+ `for every transitive` c `and ordinal` ν `the set` P^c_ν `exists`

Gentle functions. Let us review some material from [MB].

Definition. A *gentle* function is one of the form $G \circ F$ where G is rudimentary and F is rud-rec.

A *p-gentle* function is one of the form $G \circ F$ where G is rudimentary and F is p-rud-rec.

If A is a set, then a function is (A)-*gentle* if it is p-gentle for some $p \in A$.

To emphasise the absence of a parameter we may write $\varnothing$-*gentle* or, more accurately, $(\varnothing)$-*gentle*. In that paper, Proposition 4.5 gives Bowler's example of two rud-rec functions whose composite is not rud-rec, and Theorem 4.9 his result that the composite of two gentle functions is gentle.

A point to note is that the composition of two p-gentle functions is liable not to be p-gentle but will be q-gentle for some parameter q in the provident closure of $\{p\}$. So for provident A the composition of (A)-gentle functions is (A)-gentle.

Proposition 4.12 of [MB] proves that the characteristic function of a predicate B is gentle iff the separator $x \mapsto x \cap B$ is gentle. That yields the following principle:

0.1. Gentle Separation. *Let A be provident, a and p in A, and B a p-gentle predicate; then $a \cap B \in A$.*

Proof. There will be some attempt $f \in A$ at χ_B, that is, a fragment of χ_B, with a included in its domain, and then $x \cap \{y \mid f(y) = 1\}$ will be a set by Δ_0 separation. $\blacksquare_{0.1}$

Set forcing over provident sets. Let A be a transitive model of PROVI; let $\mathbb{P}$ be a separative partial ordering which is a member of A, and let $\mathbf{P}$ be the provident closure of $\{\mathbb{P}\}$. Many functions and relations involved in the development of forcing are if not actually $\mathbb{P}$-rud-rec, $(\mathbf{P})$-gentle.

The first goal of the paper is to prove that for each ℓ the forcing relation $p \Vdash^{\mathbb{P}} \varphi$, restricted to those sentences of the language of forcing that are $\dot{\Delta}_0$ and of length at most ℓ, is $(\mathbf{P})$-gentle.

The second goal is to analyse the construction of names for the values of functions applied to objects in a generic extension. We speak of this task as the construction of *nominators* for the functions concerned.

The first stage of that is to show that for each rudimentary function R, say of two variables, there is a $(\mathbf{P})$-gentle function $R^{\mathbb{P}}$ of two variables such that for $(A, \mathbb{P})$-generic $\mathcal{G}$ and all x, y in A,

$$\mathrm{val}_{\mathcal{G}}(R^{\mathbb{P}}(x, y)) = R(\mathrm{val}_{\mathcal{G}}(x), \mathrm{val}_{\mathcal{G}}(y)).$$

In many cases, we can do better, and indeed certain rudimentary functions have basic nominators.

We shall then find $\mathbb{P}$-rud-rec functions $\varrho^{\mathbb{P}}$ and $\mathrm{tcl}^{\mathbb{P}}$ that similarly build names for the rank and the transitive closure of a given object from its forcing name.

Finally we must build names for the stages of a progress $\nu \mapsto P^d_\nu$ for d a transitive set in the generic extension. Here we shall repeatedly use the principle that functions rudimentary recursive in the $(\mathbf{P})$-gentle ternary relation $p \Vdash^{\mathbb{P}} \underline{a} = \underline{b}$ are $(\mathbf{P})$-gentle, which follows from the results of Section 5 of [MB].

The main theorem will then follow easily for provident sets of the form P^e_θ, and will immediately extend to all provident sets containing $\mathbb{P}$, using the fact—a special case of [MB, Theorem 7.0]—that every provident set is the union of a directed family of sets each of the form P^c_θ.

In the rest of this paper *gentle* will normally be used to mean $(\mathbf{P})$-gentle: where it is necessary to emphasize the absence of parameters we shall write $\varnothing$-gentle.

1. Heuristic. We begin with some reminders of the general character of forcing: the present discussion is heuristic, to give the reader a feel for the way the forcing relation will operate. In particular, the methods used in this section for naming old and new objects are, dangerously, simpler than the methods of the formal development to be given in subsequent sections.

Suppose we face the following challenge:

given a transitive M, to find a transitive $N \supseteq M$ with $On \cap N = On \cap M$ but where N contains a subset of ω not in M.

If M and N are both provident, such an N will necessarily violate the axiom of constructibility, for $(L)^N = (L)^M$. Thus we are aiming to add a set $a \subseteq \omega$ to M.

1.0. We begin by asking questions about a. Suppose we have some information p about a: what statements about a will be forced to be true? For example if p is the statement that $5 \in a$, then `Not all members of` a `are even` is forced by p.

Our beginning intuition for forcing is the idea that we have pieces of partial information about the new object we are adding, and that we build up a picture of the new model from this partial information.

Our pieces of information are called *conditions*, and to start with we suppose that the collection of conditions is a set, $\boldsymbol{P}$. Experience shows that we should make the following assumptions about $\boldsymbol{P}$:

(1.0.0) $\boldsymbol{P}$ is partially ordered by a relation $\leq$; if $p \leq q$ we think that p contains **more** information than q.

(1.0.1) To get something interesting we allow the possibility of two conditions being *incompatible*: we say that p is *compatible* with q if there is some r stronger than both: $r \leq p$ & $r \leq q$; and we say that p is *incompatible* with q, in symbols $p \perp q$, if no such r exists.

(1.0.2) We assume that any condition can be strengthened in two incompatible ways:

$$\forall p \, \exists q_{\leq p} \, \exists r_{\leq p} \; q \perp r.$$

(1.0.3) We suppose that $\boldsymbol{P}$ has a greatest element $\boldsymbol{1}^{\mathbb{P}}$, where this condition is the one that gives us no information at all. Thus $\boldsymbol{1}^{\mathbb{P}}$ is compatible with every condition.

(1.0.4) Finally we suppose that $\mathbb{P} =_{\mathrm{df}} (\boldsymbol{P}, \boldsymbol{1}^{\mathbb{P}}, \leq)_3$ is *separative*, that is,

$$\forall p \, \forall q \; (p \not\leq q \;\Rightarrow\; \exists r_{\leq p} \; r \perp q).$$

Such a $\mathbb{P}$ is called a *notion of forcing*. Let us look at two examples.

Cohen's original forcing. We take

$$\boldsymbol{P} = \bigcup \{ {}^{n}2 \mid n \in \omega \} = {}^{<\omega}2,$$

the set of finite maps from ω to $2 = \{0, 1\}$.

We define the ordering by reverse inclusion:

$$p \leq^{\mathbb{P}} q \;\Leftrightarrow\; q \subseteq p.$$

This forcing is intended to add a "generic" $a : \omega \to 2$. We take a symbol $\dot{a}$ and use it as a name for the new a that we are trying to add.

The intended meaning of a condition $p : n \to \omega$ is that $\dot{a} \restriction n = p$. So if $n = 6$ and $p(3) = 1$, then p will force the statement that $\dot{a}(\hat{3}) = \hat{1}$.

We suppose that we are in one universe, which we call the ground model, describing a larger universe, which we call the generic extension; the new objects are only partially known to us, so we use dotted letters as names for them, as $\dot{a}$. The objects in the ground model are fully known to us, and we name them with hatted letters: thus $\hat{3}$ is our name for 3 in the language of forcing.

Continuing our discussion of Cohen's original forcing, we show now that the new real $\dot{a}$ is not the same as any old real $\hat{b}$: precisely, we prove the following:

1.1. Proposition. *Let $b : \omega \to 2$. Then $\forall p \, \exists q_{\leq p} \; q \Vdash^{\mathbb{P}} \hat{b} \neq \dot{a}$.*

Notice the topological flavour to this proposition: it is saying that the set of conditions forcing a certain statement is *dense*. Indeed we may topologize $\boldsymbol{P}$ so that is exactly what is happening.

Proof of 1.1. Given p, let $n = \mathrm{dom}(p)$; n is of course the least natural number not in the domain of p. Look at $b(n)$, and let

$$q = p \cup \{(1 - b(n), n)_2\}. \quad \blacksquare_{1.1}$$

So in any model in which everything that is forced on a dense set is true, $\dot{a}$ will be a new subset of ω.

This notion of density is central to the concept of forcing. One of the properties of the forcing relation, which we shall refer to as the *density property*, is that

$$p \Vdash^{\mathbb{P}} \varphi \iff \forall q_{\leq p} \exists r_{\leq q} \; r \Vdash^{\mathbb{P}} \phi.$$

As we progressively extend the definition of the forcing relation to ever wider classes of formulæ, we shall check at each stage that the density property and other characteristic properties of forcing are preserved.

Another example. Let η be an infinite ordinal. This time take

$$\boldsymbol{P} = \{p \mid \exists n_{\in\omega} \; p : n \xrightarrow{\text{1-1}} \eta\}.$$

As before, order by reverse inclusion:

$$p \leq^{\mathbb{P}} q \iff q \subseteq p.$$

This forcing adds a generic $\dot{f} : \hat{\omega} \xrightarrow{\text{1-1}} \hat{\eta}$: a condition p with domain n is a description of $\dot{f} \restriction \hat{n}$.

So

$$\mathbf{1}^{\mathbb{P}} \Vdash^{\mathbb{P}} \dot{f} \restriction \widehat{\ell(p)} = \hat{p}.$$

1.2. Exercise. $\mathbf{1}^{\mathbb{P}} \Vdash^{\mathbb{P}} \dot{f}$ `is` 1-1.

1.3. Proposition. $\mathbf{1}^{\mathbb{P}} \Vdash^{\mathbb{P}} \hat{\eta}$ `is countable`.

Proof. By a density argument. Given p, n not in $\mathrm{dom}(p)$ and an ordinal $\xi < \eta$, suppose that ξ is not in the image of p. We find $q \leq^{\mathbb{P}} p$ such that $n \in \mathrm{dom}(q)$ and $q(n) = \xi$.

Thus we have shown that $\forall \xi_{\leq\eta} \forall p \, \exists q_{\leq p} \; q \Vdash^{\mathbb{P}} \hat{\xi} \, \epsilon$ `the image of` $\dot{f}$.

2. Forcing in provident sets. Let M be a provident set, and $\mathbb{P} = (\boldsymbol{P}, \boldsymbol{1}^{\mathbb{P}}, \leq)_3$ a separative partial order in M, with a top point $\boldsymbol{1}^{\mathbb{P}}$. We suppose that $\omega \in M$.

We aim to define within M a relation $\Vdash$, more exactly $\Vdash^{\mathbb{P}}$, describing an extension $M[\mathcal{G}]$ of M, where $\mathcal{G}$ is an $(M, \mathbb{P})$-generic filter. Each object in M potentially names an element of $M[\mathcal{G}]$. $\Vdash$ is a relation between elements of $\boldsymbol{P}$ and sentences in a language of set theory that we shall gradually build up. In fact the full relation can only be defined schematically within M.

This language will start from two two-place relations $\mathtt{=}$ and ϵ and will broadly resemble the formal languages introduced in [M2]. We shall use our devices of dots and type-writer face as before; but the constants will play a different role, and hence we shall use a different mark. To each set x in the universe corresponds a name $\underline{x}$ for an object in the generic extension. Thus the statement $p \Vdash^{\mathbb{P}} \underline{x} \mathbin{\epsilon} \underline{y}$ expresses information about the evaluation of the objects x and y functioning as names of sets to be created in the forcing extension given by the notion of forcing $\mathbb{P}$.

2.0. Definition. $p \Vdash_0 \underline{a} \mathbin{\epsilon} \underline{b} \iff_{\mathrm{df}} (p, a) \in b$.

$\Vdash_0$ is our first approximation to the relation $\Vdash$.

2.1. Lemma. *If $p \Vdash_0 \underline{a} \mathbin{\epsilon} \underline{b}$ then $a \in \bigcup\bigcup b$.*

2.2. Definition. In future we shall write $\bigcup^2 x$ for $\bigcup\bigcup x$.

2.3. Lemma. *$\Vdash_0$ is Δ_0, indeed rudimentary.*

2.4. Remark. For relations, Δ_0 and rud are the same: cf. Devlin [De, VI.1.5].

2.5. Definition. $p \Vdash_1 \underline{a} \mathbin{\epsilon} \underline{b} \iff_{\mathrm{df}} \exists q_{\in \bigcup^2 b}\, [q \geq p \;\&\; (q, a) \in b]$.

2.6. Lemma. *For all $p \in \mathbb{P}$, a and b:*

(2.6.0) $p \Vdash_0 \underline{a} \mathbin{\epsilon} \underline{b} \Rightarrow p \Vdash_1 \underline{a} \mathbin{\epsilon} \underline{b}$;
(2.6.1) *if $p \Vdash_1 \underline{a} \mathbin{\epsilon} \underline{b}$ then $a \in \bigcup^2 b$;*
(2.6.2) *$\Vdash_1$ is rudimentary in $\mathbb{P}$.*

2.7. Lemma. *If $p \Vdash_1 \underline{a} \mathbin{\epsilon} \underline{b}$ and $r \leq p$ then $r \Vdash_1 \underline{a} \mathbin{\epsilon} \underline{b}$.*

This last statement shows that $\Vdash_1$ improves $\Vdash_0$ and starts to resemble a forcing relation.

We define the relation $p \Vdash \underline{b} \mathtt{=} \underline{c}$ by recursion:

2.8. Definition.

$$p \Vdash \underline{b} \mathtt{=} \underline{c} \iff_{\mathrm{df}}$$
$$\forall \beta_{\in \bigcup^2 b} \forall r_{\leq p} \left[r \Vdash_1 \underline{\beta} \mathbin{\epsilon} \underline{b} \Rightarrow \exists t_{\leq r} \exists \gamma_{\in \bigcup^2 c}\, (t \Vdash \underline{\beta} \mathtt{=} \underline{\gamma} \;\&\; t \Vdash_1 \underline{\gamma} \mathbin{\epsilon} \underline{c}) \right] \;\&$$
$$\forall \gamma_{\in \bigcup^2 c} \forall r_{\leq p} \left[r \Vdash_1 \underline{\gamma} \mathbin{\epsilon} \underline{c} \Rightarrow \exists t_{\leq r} \exists \beta_{\in \bigcup^2 b}\, (t \Vdash \underline{\gamma} \mathtt{=} \underline{\beta} \;\&\; t \Vdash_1 \underline{\beta} \mathbin{\epsilon} \underline{b}) \right].$$

The above definition is $\mathbb{P}$-rud recursive in a suitable sense, which we must now articulate, and therefore will succeed in provident sets of which $\mathbb{P}$ is a member, or, more generally, in $\mathbb{P}$-provident sets.

2.9. Definition. Let $\chi_{=}(p,b,c)$ be the characteristic function of the relation $p \Vdash^{\mathbb{P}} \underline{b} = \underline{c}$, so that it takes the value 1 if $p \Vdash^{\mathbb{P}} \underline{b} = \underline{c}$ and 0 otherwise. Our plan is to show that the graph of $\chi_{=}$ on transitive sets is definable by a $\mathbb{P}$-rudimentary recursion.

2.10. The Definability Lemma. *"f is a $\chi_{=}$ attempt" is $\Delta_0(\mathbb{P}, f)$.*

Proof. We must first say that everything in the domain of f is an ordered triple, of which the first component is a member of $\boldsymbol{P}$; and whenever $(p,b,c) \in \mathrm{Dom}(f)$ and β and γ are in b and c respectively, and $q \in \boldsymbol{P}$ then (q,c,b) and (q,β,γ) are in the domain too. But all that is $\Delta_0(\mathbb{P}, f)$.

Then we must say that f respects the recursive definition: but all that is also a Δ_0 statement about $\mathbb{P}$ and f. $\blacksquare_{2.10}$

2.11. The Propagation Lemma. *Let $F(u) = \chi_{=} \restriction (\mathbb{P} \times u \times u)$. There is a rudimentary function $H_{=}$ such that for any transitive P, if $P \subseteq P^+ \subseteq \mathcal{P}(P)$,*

$$F(P^+) = H_{=}(\mathbb{P}, F(P), P^+).$$

In the following argument, and elsewhere, $(\cdot)^3_i$ are basic "un-tripling" functions such that for a poset $\mathbb{P} = (\boldsymbol{P}, \boldsymbol{1}^{\mathbb{P}}, \leq)_3$, $(\mathbb{P})^3_0 = \boldsymbol{P}$, $(\mathbb{P})^3_1 = \boldsymbol{1}^{\mathbb{P}}$, and $(\mathbb{P})^3_2 = \leq$. On this occasion, but not in future, the restricted nature of a quantifier such as $\forall r_{\leq p}$ has been made manifest by re-writing it as $\forall r_{\in (\leq``\{p\})}$.

Proof of the Propagation Lemma. Let $\Psi(x,f,p,b,c)$ be the Δ_0 formula

$$\begin{aligned} &\forall \beta_{\in \bigcup^2 b} \forall r_{\in ((x)^3_2``\{p\})} \\ &\qquad \big[r \Vdash_1 \underline{\beta} \mathrel{\epsilon} \underline{b} \Rightarrow \exists t_{\in ((x)^3_2``\{r\})} \exists \gamma_{\in \bigcup^2 c} \, (f(t,\beta,\gamma) = 1 \;\&\; t \Vdash_1 \underline{\gamma} \mathrel{\epsilon} \underline{c})\big] \\ &\& \; \forall \gamma_{\in \bigcup^2 c} \forall r_{\in ((x)^3_2``\{p\})} \\ &\qquad \big[r \Vdash_1 \underline{\gamma} \mathrel{\epsilon} \underline{c} \Rightarrow \exists t_{\in ((x)^3_2``\{r\})} \exists \beta_{\in \bigcup^2 b} \, (f(t,\gamma,\beta) = 1 \;\&\; t \Vdash_1 \underline{\beta} \mathrel{\epsilon} \underline{b})\big]. \end{aligned}$$

Define $H_{=}(x,f,v)$ to be

$$\begin{aligned} &\big(\{0,1\} \times ((x)^3_0 \times (v \times v))\big) \\ &\cap \big(\{(1,p,b,c)_4 |_{p,b,c} \Psi(x,f,p,b,c)\} \cup \{(0,p,b,c)_4 |_{p,b,c} \neg\Psi(x,f,p,b,c)\}\big). \end{aligned}$$ $\blacksquare_{2.11}$

2.12. Definition.

$$\chi_{=} \upharpoonright Q =_{\mathrm{df}} \{\chi_{=} \restriction (\boldsymbol{P} \times u \times u) \mid u \text{ transitive } \;\&\; u \in Q\}.$$

2.13. Proposition. *Let e be a transitive set of which $\mathbb{P}$ is a member, and let η_0 be minimal such that $\mathbb{P} \in P^e_{\eta_0}$, so that $\eta_0 = \varrho(\mathbb{P}) + 1$. Then*

(2.13.0) $\chi_{=} \upharpoonright P^e_\omega \subseteq P^e_{\eta_0 + \omega}$;

(2.13.1) *there are integers $s_=$ and $g_=$ such that if λ is a limit ordinal, and $\chi_= \restriction P^e_\lambda \subseteq P^e_{\eta_0+\lambda}$, then for each $k \in \omega$, $\chi_= \restriction P^e_{\lambda+k} \in P^e_{\eta_0+\lambda+s_=+g_=k}$;*

(2.13.2) *for each limit ordinal λ, $\chi_= \restriction P^e_\lambda \subseteq P^e_{\eta_0+\lambda}$.*

Proof. Much as in [MB]. Iterating $H_=$ from $\chi_= \restriction \varnothing$ gives $\chi_= \restriction P^e_\omega \subseteq P^e_{\eta_0+\omega}$; and then continue the induction by showing for each limit λ that

[i] if $\chi_= \restriction P^e_\lambda \subseteq P^e_{\eta_0+\lambda}$ then $\chi_= \restriction P^e_\lambda$ is a member of $P^e_{\eta_0+\lambda+\omega}$;
[ii] if $\chi_= \restriction P^e_\lambda$ is a member of $P^e_{\eta_0+\lambda+\omega}$, then so is $\chi_= \restriction P^e_{\lambda+k}$ for each $k \in \omega$.

The proof of [i] uses the Definability Lemma 2.10 and the presence of the parameter $\mathbb{P}$; and the proof of [ii] uses the Propagation Lemma 2.11. The integers $s_=$ and $g_=$ are the separational and generational delays calculated by [MB, Theorem 6.12 and Proposition 6.32]. $\blacksquare_{2.13}$

Propagation of $\chi_=$. The progress $(P^c_\nu)_\nu$ was defined in §6 of [MB] for c a transitive set.

We could continue to work with progresses of the above kind, but a problem would then arise at the end of the paper, in the proof that a set-generic extension of a provident set is provident. It is better to change tack now and work with other progresses, which might be called construction from e as a set and $\chi_=$ as a predicate, with the definition of $\chi_=$ evolving during the construction. An easy extension of [MB, Proposition 4.3] will assure us that the three functions we are about to define by simultaneous recursion are $\mathbb{P}$-rud-rec.

2.14. Definition. Let e be a transitive set of which $\mathbb{P}$ is a member, and let $\eta = \varrho(\mathbb{P})$. We define by a p-rudimentary recursion a sequence $((e_\nu, P^{e;=}_\nu, \chi^e_\nu)_3)_\nu$ of triples, thus obtaining a new progress $(P^{e;=}_\nu)_\nu$. For every ν, e_ν will be defined as before; for $\nu \leqslant \eta$ we set $P^{e;=}_\nu = P^e_\nu$; for $\nu < \eta$, we set $\chi^e_\nu = \varnothing$ but at η, we set $\chi^e_\eta = \chi_= \restriction P^e_\eta$, which will be a set by Proposition 2.13. Thereafter we set

$$\begin{aligned} e_{\nu+1} &= e \cap \{x \mid x \subseteq e_\nu\}, & e_\lambda &= \textstyle\bigcup_{\nu<\lambda} e_\nu, \\ P^{e;=}_{\nu+1} &= \mathbb{T}(P^{e;=}_\nu) \cup \{e_\nu\} \cup e_{\nu+1} \cup \{\chi^e_\nu \cap P^{e;=}_\nu\}, & P^{e;=}_\lambda &= \textstyle\bigcup_{\nu<\lambda} P^{e;=}_\nu, \\ \chi^e_{\nu+1} &= H_=(\mathbb{P}, \chi^e_\nu, P^{e;=}_{\nu+1}), & \chi^e_\lambda &= \textstyle\bigcup_{\nu<\lambda} \chi^e_\nu. \end{aligned}$$

2.15. Proposition. *Let e be transitive, with $\mathbb{P} \in e$, and let θ be indecomposable and strictly greater than $\varrho(\mathbb{P})$. Then $P^{e;=}_\theta = P^e_\theta$.*

Proof. First consider the *special case* that $\theta > \varrho(e)$. By [MB, Proposition 6.35], P^e_θ is provident and therefore supports all p-rud recursions with $p \in P^e_\theta$; the sequence of triples $((e_\nu, P^{e;=}_\nu, \chi^e_\nu)_3)_\nu$ is defined by such a recursion, with parameter the triple $(e, \mathbb{P}, \chi^e_{\varrho(\mathbb{P})})_3$. So the left side is included in the right. On the other hand, $(P^{e;=}_\nu)_{\nu\leqslant\theta}$ is a θ-progress, continuous at θ;

$e \in P^{e;=}_{\varrho(e)+1}$; and so by [MB, Proposition 6.35], the right side is included in the left.

Now for the *general case*: the special case tells us that for each ζ with $\varrho(\mathbb{P})<\zeta<\theta$, $P^{e_\zeta;=}_\theta = P^{e_\zeta}_\theta$. Taking the union over all such ζ gives $P^{e_\theta;=}_\theta = P^{e_\theta}_\theta$; the equalities $P^{e;=}_\theta = P^{e_\theta;=}_\theta$ and $P^{e_\theta}_\theta = P^{e}_\theta$, proved (as) in [MB, Proposition 6.35], complete the proof. $\blacksquare_{2.15}$

This reconstruction of P^e_θ shortens the delay for most χ^e_ν:

2.16. Proposition. *For any ordinal $\nu \geqslant \eta$, and any limit ordinal $\lambda > \eta$:*

(2.16.0) $\chi^e_\nu = \chi_= \restriction P^{e;=}_\nu$;
(2.16.1) $\chi^e_\nu \subseteq P^{e;=}_{\nu+6}$;
(2.16.2) $\chi^e_\lambda \subseteq P^{e;=}_\lambda$;
(2.16.3) $\chi_= \restriction P^{e;=}_\nu \in P^{e;=}_{\nu+12}$.

Proof. (2.16.0) is true by definition for $\nu = \eta$; thereafter, the function $H_=$ preserves its truth at successor stages, and at limit stages, we simply take unions on both sides.

(2.16.1): The '6' reflects the delay in the creation of Kuratowski ordered pairs.

(2.16.2): At limit stages, that delay no longer exists.

(2.16.3): Fix $\nu \geqslant \eta$. Let $\chi^+ = \chi^e_{\nu+6} \cap P^{e;=}_{\nu+6}$; then $\chi^+ \in P^{e;=}_{\nu+7}$ by definition of the progress. Since $\chi^e_\nu \subseteq P^{e;=}_{\nu+6}$, $\chi^e_\nu = \chi^+ \cap (2 \times (\boldsymbol{P} \times (P^{e;=}_\nu \times P^{e;=}_\nu)))$.

By [MB, Lemmata 6.15 and 6.16], $x, y \in P^{e;=}_\zeta \;\Rightarrow\; x \cap y = x \smallsetminus (x \smallsetminus y) \in P^{e;=}_{\zeta+2}$ & $x \times y \in P^{e;=}_{\zeta+3}$.

$P^{e;=}_\nu \in P^{e;=}_{\nu+1}$, so $P^{e;=}_\nu \times P^{e;=}_\nu \in P^{e;=}_{\nu+4}$, $\boldsymbol{P} \times (P^{e;=}_\nu \times P^{e;=}_\nu) \in P^{e;=}_{\nu+7}$ and $\{0,1\} \times \big(\boldsymbol{P} \times (P^{e;=}_\nu \times P^{e;=}_\nu)\big) \in P^{e;=}_{\nu+10}$.

We conclude that $\chi^e_\nu \in P^{e;=}_{\nu+12}$. $\blacksquare_{2.16}$

Propagation of χ_ϵ. We may now define $p \Vdash \underline{a} \,\epsilon\, \underline{b}$:

2.17. Definition.

$$p \Vdash \underline{a} \,\epsilon\, \underline{b} \;\Leftrightarrow_{\mathrm{df}}\; \forall s_{\leq p} \exists t_{\leq s} \exists \beta_{\in \bigcup^2 b} \big[t \Vdash \underline{\beta} = \underline{a} \;\&\; t \Vdash_1 \underline{\beta} \,\epsilon\, \underline{b} \big].$$

2.18. Remark. This is not a definition by recursion: indeed, it is visibly rudimentary in $p \Vdash \underline{b} = \underline{c}$.

2.19. Definition. Let $\chi_\epsilon(p, a, b)$ be the characteristic function of the relation $p \Vdash^{\mathbb{P}} \underline{a} \,\epsilon\, \underline{b}$.

2.20. Proposition. *There is a natural number s_ϵ such that for each ordinal $\nu \geqslant \eta$, $\chi_\epsilon \restriction P^{e;=}_\nu \in P^{e;=}_{\nu+s_\epsilon}$, where e and η are as in Definition* 2.14.

Proof. There are rudimentary functions R and S such that

$$\chi_\epsilon \restriction P^{e;=}_\nu = 2 \times \mathrm{Dom}(\chi^e_\nu) \cap \big(\{(1,p,a,b)_4 \mid R(p,a,b,\chi^e_\nu) = 1\} \\ \cup \{(0,p,a,b)_4 \mid R(p,a,b,\chi^e_\nu) = 0\}\big) = S(\chi^e_\nu).$$

We may take $s_\epsilon = 12 + c_S$. $\blacksquare_{2.20}$

Familiar properties of forcing. We check as our definition of forcing develops that it has the expected density properties, and we establish familiar properties of equality and the substitution properties of = for ϵ:

2.21. Proposition. *If* $p \Vdash \underset{\sim}{b} = \underset{\sim}{c}$ *and* $q \le p$ *then* $q \Vdash \underset{\sim}{b} = \underset{\sim}{c}$.
$p \Vdash \underset{\sim}{a} = \underset{\sim}{b} \;\Leftrightarrow\; \forall q_{\le p} \exists r_{\le q}\; r \Vdash \underset{\sim}{a} = \underset{\sim}{b}$.

2.22. Proposition. *For all* $p \in \mathbb{P}$, a, b *and* c:

(2.22.0) $p \Vdash \underset{\sim}{b} = \underset{\sim}{b}$;
(2.22.1) *if* $p \Vdash \underset{\sim}{b} = \underset{\sim}{a}$ *then* $p \Vdash \underset{\sim}{a} = \underset{\sim}{b}$;
(2.22.2) *if* $p \Vdash \underset{\sim}{a} = \underset{\sim}{b}$ *and* $p \Vdash \underset{\sim}{b} = \underset{\sim}{c}$ *then* $p \Vdash \underset{\sim}{a} = \underset{\sim}{c}$.

Proof. (0) Let b be a counter-example of minimal rank. The definition of $p \Vdash \underset{\sim}{b} = \underset{\sim}{b}$ involves various r, $\beta \in \bigcup^2 b$, for which $r \Vdash \underset{\sim}{\beta} = \underset{\sim}{\beta}$ by the minimality condition on b.

(1) From the symmetry of the definition.

(2) If $q \Vdash_1 \underset{\sim}{\alpha} \,\epsilon\, \underset{\sim}{a}$ then $\exists r_{\le q}\, \exists \beta\ (r \Vdash_1 \underset{\sim}{\beta} \,\epsilon\, \underset{\sim}{b}\ \&\ r \Vdash \underset{\sim}{\alpha} = \underset{\sim}{\beta})$, so $\exists s_{\le r}\, \exists \gamma\ (s \Vdash_1 \underset{\sim}{\gamma} \,\epsilon\, \underset{\sim}{c}\ \&\ s \Vdash \underset{\sim}{\beta} = \underset{\sim}{\gamma})$; the t we seek is s; $s \Vdash \underset{\sim}{\alpha} = \underset{\sim}{\beta} \wedge \underset{\sim}{\beta} = \underset{\sim}{\gamma}$; so, assuming we have minimised the rank of a possible failure b, $s \Vdash \underset{\sim}{\alpha} = \underset{\sim}{\gamma}$ and $s \Vdash_1 \underset{\sim}{\gamma} \,\epsilon\, \underset{\sim}{c}$, as required. $\blacksquare_{2.22}$

2.23. Lemma. $q \le p\ \&\ p \Vdash \underset{\sim}{a} \,\epsilon\, \underset{\sim}{b} \;\Rightarrow\; q \Vdash \underset{\sim}{a} \,\epsilon\, \underset{\sim}{b}$.

2.24. Lemma. $p \Vdash \underset{\sim}{a} \,\epsilon\, \underset{\sim}{b} \;\Leftrightarrow\; \forall q_{\le p} \exists r_{\le q}\; r \Vdash \underset{\sim}{a} \,\epsilon\, \underset{\sim}{b}$.

2.25. Proposition. *If* $p \Vdash_1 \underset{\sim}{a} \,\epsilon\, \underset{\sim}{b}$ *then* $p \Vdash \underset{\sim}{a} \,\epsilon\, \underset{\sim}{b}$.

Proof. Let $r \le p$; take $s = r$ and $\beta = a$; then $s \Vdash \underset{\sim}{\beta} = \underset{\sim}{a}$ and $s \Vdash_1 \underset{\sim}{\beta} \,\epsilon\, \underset{\sim}{b}$. $\blacksquare_{2.25}$

2.26. Proposition. *If* $p \Vdash \underset{\sim}{a} \,\epsilon\, \underset{\sim}{b}$ *and* $p \Vdash \underset{\sim}{a} = \underset{\sim}{c}$ *then* $p \Vdash \underset{\sim}{c} \,\epsilon\, \underset{\sim}{b}$.

Proof. Let $s \le p$. We seek $t \le s$ and $\beta \in \bigcup^2 b$ such that $t \Vdash \underset{\sim}{\beta} = \underset{\sim}{c}$ and $t \Vdash_1 \underset{\sim}{\beta} \,\epsilon\, \underset{\sim}{b}$. We know that there are $t \le s$ and $\beta \in \bigcup^2 b$ such that $t \Vdash \underset{\sim}{\beta} = \underset{\sim}{a}$ and $t \Vdash_1 \underset{\sim}{\beta} \,\epsilon\, \underset{\sim}{b}$; since $p \Vdash \underset{\sim}{a} = \underset{\sim}{c}$ and $t \le p$, $t \Vdash \underset{\sim}{\beta} = \underset{\sim}{c}$. $\blacksquare_{2.26}$

2.27. Proposition. *If* $p \Vdash \underset{\sim}{a} \,\epsilon\, \underset{\sim}{b}$ *and* $p \Vdash \underset{\sim}{b} = \underset{\sim}{d}$, *then* $p \Vdash \underset{\sim}{a} \,\epsilon\, \underset{\sim}{d}$.

Proof. Let $s \le p$. We seek $t \le s$ and $\delta \in \bigcup^2 d$ such that $t \Vdash \underset{\sim}{\delta} = \underset{\sim}{a}$ and $t \Vdash_1 \underset{\sim}{\delta} \,\epsilon\, \underset{\sim}{d}$. Since $p \Vdash \underset{\sim}{a} \,\epsilon\, \underset{\sim}{b}$, there are $r_{\le s}$ and $\beta \in \bigcup^2 b$ such that $r \Vdash \underset{\sim}{\beta} = \underset{\sim}{a}\ \&\ r \Vdash_1 \underset{\sim}{\beta} \,\epsilon\, \underset{\sim}{b}$. Since $p \Vdash \underset{\sim}{b} = \underset{\sim}{d}$ and $r \le p$, there are $t \le r$ and $\delta \in \bigcup^2 d$ such that $t \Vdash \underset{\sim}{\beta} = \underset{\sim}{\delta}$ and $t \Vdash_1 \underset{\sim}{\delta} \,\epsilon\, \underset{\sim}{d}$; as $t \Vdash \underset{\sim}{\beta} = \underset{\sim}{\delta}$ and $t \Vdash \underset{\sim}{\beta} = \underset{\sim}{a}$, $t \Vdash \underset{\sim}{\delta} = \underset{\sim}{a}$. $\blacksquare_{2.27}$

Forcing the negation of a statement. We have defined $p \Vdash \dot\Phi$ for $\dot\Phi$ of the form $\underline{x} \mathrel{\epsilon} \underline{y}$ or $\underline{x} = \underline{y}$, and now wish to define $p \Vdash \neg\dot\Phi$ in these cases. The definition is characteristic of forcing; and we will maintain it as we extend our definition of forcing to ever larger classes of formulæ.

2.28. Definition. $p \Vdash \neg\dot\Phi \iff \forall q_{\leq p}\, q \nVdash \dot\Phi$.

We shall use this definition in our next proposition: notice that it has the immediate consequence that

$$\forall p\, \exists q_{\leq p} \left[q \Vdash \dot\Phi \vee q \Vdash \neg\dot\Phi\right].$$

Further it renders *Modus Ponens* effective:

2.29. Proposition. *If $p \Vdash \Phi \to \Psi$ and $p \Vdash \Phi$ then $p \Vdash \Psi$.*

Proof. if $p \Vdash \neg\Phi \vee \Psi$ then $\forall q_{\leq p} \exists r_{\leq q}\, (r \Vdash \neg\Phi \vee r \Vdash \Psi)$; but the first alternative is impossible for $r \leq p$ if $p \Vdash \Phi$, and so $r \Vdash \Psi$; by density $p \Vdash \Psi$. $\blacksquare_{2.29}$

2.30. Remark. We have given this proof now; it only applies, of course, to those formulæ for which a forcing definition has been given.

Installing the ground model. Before we turn to the definition of the generic extension of a model, we identify objects that will serve as names for the elements of the ground model, and thus ensure that our generic structure is indeed an extension of our ground model. It is convenient to assume that $\boldsymbol{1}^{\mathbb{P}}$ is actually the ordinal 1.

2.31. Definition. Set $\hat{y} =_{\mathrm{df}} \{(\boldsymbol{1}^{\mathbb{P}}, \hat{x}) \mid x \in y\}$.

This is a rudimentary recursion in the parameter $\boldsymbol{1}^{\mathbb{P}}$, being of the form

$$F(a) = G(\boldsymbol{1}^{\mathbb{P}}, F \restriction a)$$

where G is the rudimentary function $(i, f) \mapsto \{i\} \times \mathrm{Im}(f)$; thus with our convention that $\boldsymbol{1}^{\mathbb{P}} = 1$, the recursion may be regarded as pure.

2.32. Lemma. *If $q \Vdash_1 \underline{\xi} \mathrel{\epsilon} \underline{\hat{x}}$ then $\exists a_{\in x}\, \xi = \hat{a}$.*

Proof. If $q \leq p$ and $(p, \xi) \in \hat{x}$ then $p = \boldsymbol{1}^{\mathbb{P}}$ and $\xi = \hat{a}$ for some $a \in x$. $\blacksquare_{2.32}$

2.33. Proposition. *For all x and y, the following hold:*

$$\begin{aligned}
x \in y &\Rightarrow \boldsymbol{1}^{\mathbb{P}} \Vdash \underline{\hat{x}} \mathrel{\epsilon} \underline{\hat{y}},\\
x = y &\Rightarrow \boldsymbol{1}^{\mathbb{P}} \Vdash \underline{\hat{x}} = \underline{\hat{y}},\\
\exists p\, p \Vdash \underline{\hat{x}} = \underline{\hat{y}} &\Rightarrow x = y,\\
x \neq y &\Rightarrow \boldsymbol{1}^{\mathbb{P}} \Vdash \neg(\underline{\hat{x}} = \underline{\hat{y}}),\\
\exists p\, p \Vdash \underline{\hat{x}} \mathrel{\epsilon} \underline{\hat{y}} &\Rightarrow x \in y,\\
x \notin y &\Rightarrow \boldsymbol{1}^{\mathbb{P}} \Vdash \neg(\underline{\hat{x}} \mathrel{\epsilon} \underline{\hat{y}}).
\end{aligned}$$

Proof. If $x \in y$, then $(\boldsymbol{1}^{\mathbb{P}}, \hat{x}) \in \hat{y}$, so $\boldsymbol{1}^{\mathbb{P}} \Vdash_0 \underline{\hat{x}} \mathrel{\epsilon} \underline{\hat{y}}$ and so $\boldsymbol{1}^{\mathbb{P}} \Vdash \underline{\hat{x}} \mathrel{\epsilon} \underline{\hat{y}}$. If $x = y$, $\hat{x} = \hat{y}$ and so $\boldsymbol{1}^{\mathbb{P}} \Vdash \underline{\hat{x}} = \underline{\hat{y}}$, by 2.22.

We prove the third line inductively: suppose $p \Vdash \underline{\hat{x}} = \underline{\hat{y}}$ for $a \in x$ and $r \leq p$, $r \Vdash_1 \underline{\hat{a}} \mathrel{\epsilon} \underline{\hat{x}}$, there will be a b and $s \leq r$ with $s \Vdash \underline{\hat{a}} = \underline{b}$ and $s \Vdash_1 \underline{b} \mathrel{\epsilon} \underline{\hat{y}}$; but then $b = \hat{c}$ for some $c \in y$; so $s \Vdash \underline{\hat{a}} = \underline{\hat{c}}$, so by induction $a = c$; and thus $x \subseteq y$; similarly $y \subseteq x$ and so $x = y$.

The next line is the contrapositive, by definition of forcing for negation.

If $p \Vdash \underline{\hat{x}} \mathrel{\epsilon} \underline{\hat{y}}$ then for some $r \leq p$ and some b, $r \Vdash \underline{\hat{x}} = \underline{b}$ and $r \Vdash_1 \underline{b} \mathrel{\epsilon} \underline{\hat{y}}$ so for some $c \in y$, b is $\hat{c}$; so $r \Vdash \underline{\hat{x}} = \underline{\hat{c}}$; by line 3, $x = c$, and thus $x \in y$.

Thus the fifth line is proved; and the sixth is its contrapositive. $\blacksquare_{2.33}$

3. Extension of the definition of forcing to all $\dot{\Delta}_0$ sentences. So far we have set up the beginnings of a definition of forcing, for atomic sentences and their negations. We wish to extend the definition of $\Vdash$ to all $\dot{\Delta}_0$ sentences of the forcing language, on these lines:

3.0. Proposed Definition.

$$\begin{aligned} p \Vdash \varphi \wedge \vartheta \ &\Leftrightarrow\ p \Vdash \varphi \ \&\ p \Vdash \vartheta, \\ p \Vdash \neg\varphi \ &\Leftrightarrow\ \forall q_{\leq p}\, q \nVdash \varphi, \\ p \Vdash \textstyle\bigwedge \mathfrak{x}_{\epsilon \underline{y}}\, \varphi(\mathfrak{x}) \ &\Leftrightarrow\ \forall q_{\leq p} \forall (s, \beta)_{\in y} \big(q \leq s \ \Rightarrow\ \exists r_{\leq q}\, r \Vdash \varphi[\underline{\beta}] \big). \end{aligned}$$

The Forcing Theorem for $\dot{\Delta}_0$ sentences will follow if we can arrange that such classes as $\boldsymbol{P} \cap \{p \mid_p p \Vdash^{\mathbb{P}} \varphi\}$ and $\boldsymbol{P} \cap \{p \mid_p p \Vdash^{\mathbb{P}} \neg\varphi\}$ are sets, where φ is a $\dot{\Delta}_0$ sentence of the language of forcing.

The annotated language $\mathcal{L}$. We must describe our language of forcing in greater detail. The first step is to define in some recursive manner a language $\mathcal{L} \subseteq \mathbf{HF}$, which is a first-order language with no constants other than those that we shall call *token constants*, with the two binary predicate symbols = and ϵ, connectives $\neg$ and $\wedge$, and the restricted quantifier $\bigwedge \mathfrak{x}_{\epsilon \mathfrak{y}}$, where in the rules of formation $\mathfrak{y}$ is required to be a distinct variable from $\mathfrak{x}$. There are no unrestricted quantifiers. Other propositional connectives and the existential restricted quantifier $\bigvee \mathfrak{x}_{\epsilon \mathfrak{y}}$ may be introduced by definition.

We shall need the customary notions of free and bound occurrence of a variable in a formula, and we imagine that each formula of $\mathcal{L}$ is accompanied by an annotation saying which occurrences of variables are bound by which occurrences of quantifiers. As we build up formulæ, we have to update the annotations, and we imagine all that going on inside $\mathbf{HF}$.

We then define $\mathcal{L}^u$ as the language resulting from $\mathcal{L}$ when all token constants are replaced by constants $\underline{a}$ for $a \in u$, and $\mathcal{E}^u$ to be the set of sentences of $\mathcal{L}^u$, meaning those wffs with no free variables. If u is rud closed and non-empty, and the map $a \mapsto \underline{a}$ is basic, then $\mathcal{L} \subseteq \mathcal{L}^u \subseteq u$.

We shall denote by $\mathrm{Subst}(\varphi, \mathfrak{x}/\underline{a})$ the result of substituting the constant $\underline{a}$ for the free occurrences of the formal variable $\mathfrak{x}$ in the formula φ; we give the familiar recursive definition of this process below, but in fact we shall think of it as accomplished by passing from $\varphi \in \mathcal{L}^u$ to the corresponding formula φ_0 in $\mathcal{L}$, with the constants of φ replaced (in a 1-1 manner) by token constants, then looking up inside **HF** the free occurrences of $\mathfrak{x}$ in φ_0 and replacing them by occurrences of an as yet unused token constant, and then reverting to a formula of $\mathcal{L}^u$ by replacing each occurrence of the new token constant by an occurrence of $\underline{a}$, and reversing the other replacements of constants of φ. In this way Subst will become a function that is rudimentary in the subset of **HF** which is recursive in the ordinary sense and which codes all the necessary information about $\mathcal{L}$. As that subset is recursive, we shall simply say that Subst is rudimentary.

3.1. Definition. We define the tree-rank τ of a formula, the substitution of a constant for a free occurrence of a variable in a formula, and, when the formula is a sentence, the set $\mathrm{Rub}(\varphi)$ of sentences to which reference will be made when deciding whether $p \Vdash \varphi$. In the following, $\mathfrak{x}$ and $\mathfrak{y}$ are distinct formal variables.

- ψ atomic:

$$\begin{aligned}&\tau(\psi) = 0, \quad \mathrm{Rub}(\psi) = \varnothing,\\ &\mathrm{Subst}(\mathfrak{x} = \mathfrak{x}, \mathfrak{x}/\underline{\alpha}) = \underline{\alpha} = \underline{\alpha}, \quad \mathrm{Subst}(\mathfrak{x} \epsilon \mathfrak{x}, \mathfrak{x}/\underline{\alpha}) = \underline{\alpha} \epsilon \underline{\alpha},\\ &\mathrm{Subst}(\mathfrak{x} = \mathfrak{y}, \mathfrak{x}/\underline{\alpha}) = \underline{\alpha} = \mathfrak{y}, \quad \mathrm{Subst}(\mathfrak{y} = \mathfrak{x}, \mathfrak{x}/\underline{\alpha}) = \mathfrak{y} = \underline{\alpha},\\ &\mathrm{Subst}(\mathfrak{x} \epsilon \mathfrak{y}, \mathfrak{x}/\underline{\alpha}) = \underline{\alpha} \epsilon \mathfrak{y}, \quad \mathrm{Subst}(\mathfrak{y} \epsilon \mathfrak{x}, \mathfrak{x}/\underline{\alpha}) = \mathfrak{y} \epsilon \underline{\alpha}.\end{aligned}$$

- $\psi = \vartheta \wedge \varphi$:

$$\begin{aligned}&\tau(\psi) = \max(\tau(\vartheta), \tau(\varphi)) + 1, \quad \mathrm{Rub}(\psi) = \{\vartheta, \varphi\},\\ &\mathrm{Subst}(\psi, \mathfrak{x}/\alpha) = \mathrm{Subst}(\vartheta, \mathfrak{x}/\alpha) \wedge \mathrm{Subst}(\varphi, \mathfrak{x}/\alpha).\end{aligned}$$

- $\psi = \neg\vartheta$:

$$\begin{aligned}&\tau(\psi) = \tau(\vartheta) + 1, \quad \mathrm{Rub}(\psi) = \{\vartheta\},\\ &\mathrm{Subst}(\psi, \mathfrak{x}/\alpha) = \neg\mathrm{Subst}(\vartheta, \mathfrak{x}/\alpha).\end{aligned}$$

- $\psi = \bigwedge \mathfrak{y}_{\epsilon\mathfrak{x}}\ \vartheta$:

$$\begin{aligned}&\tau(\psi) = \tau(\vartheta) + 1,\\ &\mathrm{Subst}(\psi, \mathfrak{x}/\underline{\alpha}) = \bigwedge \mathfrak{y}_{\epsilon\underline{\alpha}}\ \mathrm{Subst}(\vartheta, \mathfrak{x}/\underline{\alpha}).\end{aligned}$$

- $\psi = \bigwedge \mathfrak{y}_{\epsilon\underline{a}}\ \vartheta$:

$$\begin{aligned}&\tau(\psi) = \tau(\vartheta) + 1, \quad \mathrm{Rub}(\psi) = \{\mathrm{Subst}(\vartheta, \mathfrak{x}/\underline{\alpha}) \mid \exists p_{\in \boldsymbol{P}}\ (p, \alpha) \in a\},\\ &\mathrm{Subst}(\psi, \mathfrak{x}/\underline{\alpha}) = \bigwedge \mathfrak{y}_{\epsilon\underline{a}}\ \mathrm{Subst}(\vartheta, \mathfrak{x}/\underline{\alpha}).\end{aligned}$$

3.2. Remark. As we have defined it above, viewing formulæ as trees, substitution is $\varnothing$-rud-rec. But we have seen that we may treat substitution as rudimentary, by viewing formulæ as annotated sequences.

3.3. Remark. To go from $\bigwedge \mathfrak{x}_{\in \underline{a}}\, \varphi(\mathfrak{x})$ to $\{\varphi[\underline{\alpha}] \mid \exists s_{\in \boldsymbol{P}}\, (s, \alpha) \in a\}$ is to form the image of the substitution function, and is thus rudimentary. The annotations will tell us where are the free occurrences of $\mathfrak{x}$ in φ.

3.4. Definition. Let $\chi^{\ell}_{\Vdash} \restriction u$, for u a transitive set and $k \in \omega$, be the characteristic function of the forcing relation restricted to those $\dot{\Delta}_0$ sentences φ of the forcing language with $\tau(\varphi) \leqslant \ell$ and all a with $\underline{a}$ occurring in φ being in u.

Let $\chi_{\Vdash} \restriction u$ be $\bigcup_{\ell<\omega} \chi^{\ell}_{\Vdash}$.

3.5. Proposition. *For u transitive and e and η as in Definition* 2.14*:*

(3.5.0) *$\chi^{0}_{\Vdash} \restriction u$ is rudimentary in $\chi_{=} \restriction u$;*

(3.5.1) *for each ℓ, $\chi^{\ell+1}_{\Vdash} \restriction u$ is rudimentary in $\chi^{\ell}_{\Vdash} \restriction u$, and thus rudimentary in $\chi_{=} \restriction u$;*

(3.5.2) *for each ℓ there is a natural number s_ℓ such that for each ordinal $\nu \geqslant \eta$, $\chi^{\ell}_{\Vdash} \restriction P^{e;=}_{\nu} \in P^{e;=}_{\nu+s_\ell}$;*

(3.5.3) *for each limit ordinal $\lambda > \eta$, $\chi_{\Vdash} \restriction P^{e;=}_{\lambda}$ is total and is a set, being definable over $P^{e;=}_{\lambda}$.*

Proof. For (3.5.0), note that $\chi^{0}_{\Vdash} \restriction u$ is rudimentary in $\chi_{=} \restriction u$ and $\chi_{\epsilon} \restriction u$, which is rudimentary in $\chi_{=} \restriction u$.

(3.5.1): The passage from $\chi^{\ell}_{\Vdash} \restriction u$ to $\chi^{\ell+1}_{\Vdash} \restriction u$ is rudimentary in $\mathbb{P}$ and Subst, being given uniformly by these clauses:

$$\chi^{\ell+1}_{\Vdash}(p, \varphi \wedge \vartheta, \underline{\vec{b}}, \underline{\vec{c}}) = \inf\{\chi^{\ell}_{\Vdash}(p, \varphi, \underline{\vec{b}}), \chi^{\ell}_{\Vdash}(p, \vartheta, \underline{\vec{c}})\},$$

$$\chi^{\ell+1}_{\Vdash}(p, \neg\varphi, \underline{\vec{a}}) = 1 - \sup\{\chi^{\ell}_{\Vdash}(q, \varphi, \underline{\vec{a}}) \mid_q q \leq p\},$$

$$\chi^{\ell+1}_{\Vdash}(p, \textstyle\bigwedge \mathfrak{x}_{\epsilon \mathfrak{y}}\, \varphi, \mathfrak{y}/\underline{a}, \underline{\vec{b}}) = \begin{cases} 1 & \text{if } \forall q_{\leq p} \forall (s,\alpha)_{\in a} \\ & (q \leq s \ \Rightarrow\ \exists r_{\leq q}\, \chi^{\ell}_{\Vdash}(r, \varphi, \mathfrak{x}/\underline{\alpha}, \mathfrak{y}/\underline{a}, \underline{\vec{b}}) = 1), \\ 0 & \text{otherwise.} \end{cases}$$

The notation $\mathfrak{y}/\underline{a}$ indicates that the free occurrences of the variable $\mathfrak{y}$ have been replaced by occurrences of the constant $\underline{a}$. Only certain substitutions have been indicated explicitly: we think of wffs as accompanied by annotations, as described above; so, for example, in the longest line of the above equations, where visually φ should be $\mathrm{Subst}(\varphi, \mathfrak{x}/\underline{\alpha})$, the details of the substitutions would be in the annotations.

(3.5.2): Argue as in the proofs of Proposition 2.16 and 2.20, using (3.5.1).

(3.5.3) now follows. $\blacksquare_{3.5}$

Once we know that there is at least one rud-closed transitive v containing $\mathbb{P}$ with $\chi_{\Vdash} \restriction v$ total and a set, we may advance the definition of $\chi_{\Vdash}$ more generally, as we now show in Proposition 3.11.

Rudimentary generation of the sentences of the forcing language. Suppose that u is transitive, that $u \subseteq u^+ \subseteq \mathcal{P}(u)$, and that S is a subset of $\mathcal{E}^u$, so that S is a set of sentences, all of whose constants are in u. We define a ternary function $H_{\mathcal{E}}$ which will yield a larger set of sentences, all of whose constants are in u^+, thus:

$$\begin{aligned} &H_{\mathcal{E}}(\mathbb{P}, S, u^+) =_{\mathrm{df}} \\ &\quad S \cup \{\underline{a} \doteq \underline{b} \mid a, b \in u^+\} \\ &\qquad \cup \{\underline{a} \mathrel{\epsilon} \underline{b} \mid a, b \in u^+\} \\ &\qquad \cup \{\varphi \wedge \vartheta \mid \varphi, \vartheta \in S\} \\ &\qquad \cup \{\neg\varphi \mid \varphi \in S\} \\ &\qquad \cup \{\textstyle\bigwedge \mathfrak{x}_{\epsilon \underline{a}}\, \varphi \mid a \in u^+ \;\&\; \forall (p, \alpha)_{\in a}\, (p \in \boldsymbol{P} \;\Rightarrow\; \mathrm{Subst}(\varphi, \mathfrak{x}/\underline{\alpha}) \in S)\}. \end{aligned}$$

3.6. Lemma. *$H_{\mathcal{E}}$ is rudimentary in the parameter* **HF**; *more exactly, it is rudimentary in the subset of* **HF** *that codes the annotation, described above, of formulæ of the language $\mathcal{L}$.*

3.7. Lemma. $S \subseteq \mathcal{E}^u \;\Rightarrow\; H_{\mathcal{E}}(\mathbb{P}, S, u^+) \subseteq \mathcal{E}^{u^+}$.

3.8. Lemma. *For each $\varphi \in H_{\mathcal{E}}(\mathbb{P}, S, u^+)$,* $\mathrm{Rub}(\varphi) \subseteq S$.

3.9. Lemma. *There is a rudimentary function $H_{\Vdash}$ such that for every $\mathbb{P}$, u and S as above,*

$$\chi_{\Vdash} \restriction H_{\mathcal{E}}(\mathbb{P}, S, u^+) = H_{\Vdash}(\mathbb{P}, \chi_{\Vdash} \restriction S, u^+).$$

Proof sketch. $H_{\Vdash}$ would be built from the rudimentary function $H_{\doteq}$ supplied by Lemma 2.11 and other rudimentary functions suggested by the clauses of the Proposed Definition 3.0. $\blacksquare_{3.9}$

3.10. Lemma. *Suppose that $(u_n)_{n \leqslant \omega}$ is a strict continuous progress, and that u_0 is a rud-closed transitive set. Suppose that $\mathcal{E}_0 = \mathcal{E}^{u_0}$, that for each n, $\mathcal{E}_{n+1} = H_{\mathcal{E}}(\mathbb{P}, \mathcal{E}_n, u_{n+1})$, and put $\mathcal{E}_\omega = \bigcup_{n<\omega} \mathcal{E}_n$. Then $\mathcal{E}_\omega = \mathcal{E}^{u_\omega}$.*

Proof. Note that if $\vartheta \in \mathrm{Rub}(\varphi)$ then $\tau(\vartheta) < \tau(\varphi)$. Now prove by induction on $k \in \omega$ that for each $n \in \omega$, if $\tau(\varphi) = k$ and all a with $\underline{a}$ occurring in φ are in u_{n+1} then $\varphi \in \mathcal{E}_{n+k+1}$. $\blacksquare_{3.10}$

3.11. Proposition. *Suppose that $(u_n)_{n \leqslant \omega}$ is a strict continuous progress, with u_0 a rud-closed transitive set containing $\mathbb{P}$ and $\chi_{\Vdash} \restriction \mathcal{E}^{u_0}$ total and definable over u_0. Then the same is true of u_ω.*

Proof. By Lemma 3.10, any $\varphi \in \mathcal{E}^{u_\omega} \smallsetminus \mathcal{E}_0$ is in some $\mathcal{E}_{n+1}$, and $\mathrm{Rub}(\varphi) \subseteq \mathcal{E}_n$. By Lemma 3.9, we can build $\chi_{\Vdash} \restriction \mathcal{E}_{n+1}$ inside u_ω. So the definition of $p \Vdash \varphi$ over u_ω will take the expected form "there is a set $\mathcal{E}$ closed under Rub with $\phi \in \mathcal{E}$ and a function χ defined on $\mathcal{E}$ that satisfies the recursive definition of $\chi_{\Vdash}$; moreover, $\chi(p, \varphi, \vec{a}) = 1$." $\blacksquare_{3.11}$

A further speed-up. In [MB, Section 9], a definition of truth for $\dot{\Delta}_0$ formulæ is given that begins by de-nesting restricted quantifiers. That method could be used here: one would first define $p \Vdash^{\mathbb{P}} \varphi$ for quantifier-free sentences φ and then extend that definition to de-nested $\dot{\Delta}_0$ sentences, adapting appropriately the steps of [MB, §9]. Finally one would argue that every $\dot{\Delta}_0$ sentence is logically equivalent to a de-nested one.

This device then makes it evident that for each $\varphi \in \mathcal{L}$ and constants for $\vec{a}$, $p \Vdash^{\mathbb{P}} \varphi[\underline{\vec{a}}]$ is rudimentary in an appropriate fragment of $\chi_{=}$ and thus a gentle predicate of p, ϕ, and $\vec{a}$.

Forcing $\dot{\Delta}_0$ statements

3.12. Proposition. *For every $\dot{\Delta}_0$ wff φ,*

$$q \le p \Vdash \varphi \;\Rightarrow\; q \Vdash \varphi,$$
$$p \Vdash \varphi \;\Leftrightarrow\; \forall q_{\le p} \exists r_{\le q}\, q \Vdash \varphi,$$
$$p \Vdash \textstyle\bigvee \mathfrak{x}_{\epsilon \underline{y}}\, \varphi \;\Leftrightarrow\; \forall q_{\le p} \exists r_{\le q} \exists (t, \beta)_{\in y}\, (r \le t \;\&\; r \Vdash \varphi[\underline{\beta}]),$$
$$p \Vdash \varphi[\underline{\alpha}] \wedge \underline{\alpha} = \underline{\beta} \;\Rightarrow\; p \Vdash \varphi[\underline{\beta}].$$

We shall often use the following general principle in our development.

3.13. Proposition. *Suppose we have a name z such that*

$$\forall p \forall \alpha\, (p \Vdash_0 \underline{\alpha} \mathrel{\epsilon} \underline{z} \;\Rightarrow\; p \Vdash \varphi[\underline{\alpha}])$$

for some formal wff φ. Then $\forall p \forall \alpha\, (p \Vdash \underline{\alpha} \mathrel{\epsilon} \underline{z} \;\Rightarrow\; p \Vdash \varphi[\underline{\alpha}])$.

Proof. We gradually weaken the hypothesis. Suppose that $p \Vdash_1 \underline{\alpha} \mathrel{\epsilon} \underline{z}$. Then for some $q \ge p$, $(q, \alpha) \in z$, so $q \Vdash_0 \underline{\alpha} \mathrel{\epsilon} \underline{z}$; so q and therefore also p forces $\varphi[\underline{\alpha}]$.

Now suppose that $p \Vdash \underline{\alpha} \in \underline{z}$. This tells us that

$$\forall q_{\le p} \exists r_{\le q} \exists \beta\; r \Vdash_1 \underline{\beta} \mathrel{\epsilon} \underline{z} \;\&\; r \Vdash \underline{\alpha} = \underline{\beta}.$$

So for such r, $r \Vdash \varphi[\underline{\beta}]$; and so $r \Vdash \varphi[\underline{\alpha}]$. The class of such r being dense below p, $p \Vdash \varphi[\underline{\alpha}]$. $\blacksquare_{3.13}$

Axioms of Extensionality and Foundation. We may now prove that the Axiom of Extensionality is forced: that reduces to proving the following, which presents no difficulty.

3.14. Proposition. *If $p \Vdash \bigwedge \mathfrak{x}_{\epsilon \underline{a}}\; \mathfrak{x} \mathrel{\epsilon} \underline{b}$ and $p \Vdash \bigwedge \mathfrak{x}_{\epsilon \underline{b}}\; \mathfrak{x} \mathrel{\epsilon} \underline{a}$, then $p \Vdash \underline{a} = \underline{b}$.*

3.15. Proposition. $\Vdash$ Foundation.

Proof. Given x, consider $A = \{a \mid \exists p_{\in \boldsymbol{P}}\ (p,a) \in x\}$. Then A is a Δ_0 subclass of $\bigcup^2 x$ and therefore a set; assuming it is non-empty, let c be an element of A of minimal rank. Then $c \in A$ but $\bigcup^2 c \cap A$ is empty; so if $(p,c) \in x$, $p \Vdash \underline{c} \mathrel{\epsilon} \underline{x}\ \&\ \underline{c} \mathbin{\dot\cap} \underline{x} \mathrel{\dot=} \dot\varnothing$. Thus

$$\Vdash \underline{x} \mathrel{\dot{\neq}} \dot\varnothing \to \bigvee \mathfrak{y}_{\epsilon \underline{x}}\ \mathfrak{y} \mathbin{\dot\cap} \underline{x} \mathrel{\dot=} \dot\varnothing. \quad \blacksquare_{3.15}$$

Preservation of $\dot\Delta_0$ statements about the ground model. Now that we have defined forcing for $\dot\Delta_0$ statements and have seen how elements of the ground model are named in the language of forcing, we may verify that $\dot\Delta_0$ statements about them, if true, are forced.

3.16. Lemma. $\forall p_{\in \boldsymbol{P}} \forall y \bigl(p \Vdash \bigwedge \mathfrak{x}_{\epsilon \underline{\hat y}}\ \varphi \Leftrightarrow \forall x_{\in y}\ p \Vdash \varphi(\underline{\hat x})\bigr)$.

Proof.

$$\begin{aligned} p \Vdash \textstyle\bigwedge \mathfrak{x}_{\epsilon \underline{\hat y}}\ \varphi\ &\Leftrightarrow\ \forall q_{\le p} \forall (s,\beta)_{\in \hat y}\ \bigl(q \le s\ \Rightarrow\ \exists r_{\le q}\, r \Vdash \varphi[\underline{\beta}]\bigr)\\ &\Leftrightarrow\ \forall q_{\le p} \forall x_{\in y}\ \bigl(\exists r_{\le q}\, r \Vdash \varphi(\underline{\hat x})\bigr)\\ &\Leftrightarrow\ \forall x_{\in y} \forall q_{\le p}\ \bigl(\exists r_{\le q}\, r \Vdash \varphi(\underline{\hat x})\bigr)\\ &\Leftrightarrow\ \forall x_{\in y}\ p \Vdash \varphi(\underline{\hat x}). \quad \blacksquare_{3.16} \end{aligned}$$

3.17. Proposition. *Let $\Phi(x_1, \ldots, x_n)$ be a $\dot\Delta_0$ statement true of $a_1, \ldots, a_n$. Then $\boldsymbol{1}^{\mathbb{P}} \Vdash \dot\Phi[\underline{\widehat{a_1}}, \ldots, \underline{\widehat{a_n}}]$.*

Proof. The cases that Φ are either atomic or the negation of atomic are covered by Proposition 2.33. We then proceed by induction on the length of Φ; propositional connectives are easily handled, as a 0-1 law applies in this context; and the last lemma covers restricted quantifiers. $\blacksquare_{3.17}$

The above is a schema expressed in the metalanguage: the version when we quantify over $\dot\Delta_0$ wffs in the language of discourse would read

3.18. Proposition. *Let $\varphi(x_1, \ldots, x_n)$ be a $\dot\Delta_0$ statement such that $\overset{0}{\models} \varphi[a_1, \ldots, a_n]$. Then $\boldsymbol{1}^{\mathbb{P}} \Vdash \varphi[\underline{\widehat{a_1}}, \ldots, \underline{\widehat{a_n}}]$.*

4. Generic extensions of a transitive set. Let M be a transitive set and $\boldsymbol{P}$ a notion of forcing in M. The aim in life of an $(M, \boldsymbol{P})$-generic filter $\mathcal{G}$ is to create a transitive set $M[\mathcal{G}]$ out of the Lindenbaum algebra of the language of forcing, with the property that what is true in the model is what is forced by some $p \in \mathcal{G}$. That principle is known as the Forcing Theorem.

4.0. Definition. Δ is *dense open* in $\boldsymbol{P}$ if

$$\forall p_{\in \boldsymbol{P}} \exists q_{\in \Delta}\ q \le p \quad \text{and} \quad \forall p_{\in \Delta} \forall q_{\le p}\ q \in \Delta.$$

4.1. Definition. $\mathcal{G} \subseteq \boldsymbol{P}$ is $(M, \mathbb{P})$-*generic* if $\forall p_{\in \mathcal{G}} \forall q_{\in \mathcal{G}} \exists r_{\in \mathcal{G}}\ r \le p\ \&\ r \le q$, $\forall p_{\in \mathcal{G}} \forall q\, p \le q\ \Rightarrow\ q \in \mathcal{G}$, and $\mathcal{G} \cap \Delta \neq \emptyset$ for each $\Delta \in M$ that is dense open in $\boldsymbol{P}$.

Formally, those definitions work for any transitive set M.

For forcing over models of ZF that definition of *generic* would suffice to prove the Forcing Theorem; but for models of certain weaker theories such as KP, it is inadequate: we shall find in a later section that if M is admissible, we must require $\mathcal{G}$ to meet all dense open subsets of $\boldsymbol{P}$ that are unions of a Σ_1 and a Π_1 class over M if we are to show that $M[\mathcal{G}]$ will be admissible. Ironically, for models of the still weaker theory PROVI, all is well again: we shall show in this section that if a filter $\mathcal{G}$ is generic as defined above, then the Forcing Theorem will hold for $\dot{\Delta}_0$ formulæ. We assume henceforth that M models PROVI.

We know from its being a filter that $\mathcal{G}$ will be *consistent* in the sense that for no sentence φ of the language $\mathcal{L}^{\mathbb{P}}$ can there be a $p \in \mathcal{G}$ with $p \Vdash^{\mathbb{P}} \varphi$ and a $q \in \mathcal{G}$ with $q \Vdash^{\mathbb{P}} \neg\varphi$. Let $\Delta(\varphi) = \{p \in \boldsymbol{P} \mid p \Vdash^{\mathbb{P}} \varphi \vee p \Vdash^{\mathbb{P}} \neg\varphi\}$; this is a dense open subclass of $\boldsymbol{P}$. If φ is Δ_0, $\Delta(\varphi)$ will be a set of M by gentle separation. If $\mathcal{G}$ meets $\Delta(\varphi)$, then some $p \in \mathcal{G}$ decides φ in the sense that it forces either φ or $\neg\varphi$. We shall refer to this property as the *completeness* of $\mathcal{G}$.

4.2. Definition. Suppose now that $\mathcal{G}$ is $(M, \mathbb{P})$-generic. Define (externally to M) $\mathrm{val}_{\mathcal{G}} : M \to V$ by

$$\mathrm{val}_{\mathcal{G}}(b) = \{\mathrm{val}_{\mathcal{G}}(a) \mid \exists p_{\in \mathcal{G}}\ (p, a) \in b\}.$$

4.3. Remark. This is a rudimentary recursion with parameter $\mathcal{G}$: $\phi(b) = H(\mathcal{G}, \phi \restriction b)$ where $H(g, x) =_{\mathrm{df}} x``(\mathrm{Dom}(x))``g)$. In [M4] it is shown that certain transitive models of Zermelo set theory fail to support such recursions: thus it is necessary to assume that PROVI is true in the "background" set theory.

4.4. Remark. An immediate consequence of the definition is that if $p \in \mathcal{G}$ and either $p \Vdash_0 \underline{\alpha} \,\epsilon\, \underline{a}$ or $p \Vdash_1 \underline{\alpha} \,\epsilon\, \underline{a}$, then $\mathrm{val}_{\mathcal{G}}(\alpha) \in \mathrm{val}_{\mathcal{G}}(a)$.

4.5. Proposition. *For all a and b the following hold:*

$$\mathrm{val}_{\mathcal{G}}(a) = \mathrm{val}_{\mathcal{G}}(b) \iff \exists p_{\in G}\ p \Vdash \underline{a} = \underline{b}, \tag{4.6}$$

$$\mathrm{val}_{\mathcal{G}}(a) \in \mathrm{val}_{\mathcal{G}}(b) \iff \exists p_{\in G}\ p \Vdash \underline{a} \,\epsilon\, \underline{b}. \tag{4.7}$$

We divide the proof into four lemmata.

4.8. Lemma. *If $p \in \mathcal{G}$ and $p \Vdash \underline{a} = \underline{b}$ then* $\mathrm{val}_{\mathcal{G}}(a) = \mathrm{val}_{\mathcal{G}}(b)$.

Proof by induction on rank. Let $x \in \mathrm{val}_{\mathcal{G}}(a)$. Let $(q_0, \alpha) \in a$ with $q_0 \in \mathcal{G}$ and $\mathrm{val}_{\mathcal{G}}(\alpha) = x$. Let $q \in \mathcal{G}$ be below both p and q_0. Consider the class

$$\boldsymbol{P} \cap \{r \mid r \le q \;\&\; \exists \beta_{\in \bigcup^2 b}\ r \Vdash_1 \underline{\beta} \,\epsilon\, \underline{b} \;\&\; r \Vdash \underline{\alpha} = \underline{\beta}\}.$$

That is dense below q, and is a set by $\dot{\Delta}_0$ separation, once one has replaced the predicate $r \Vdash \underline{\alpha} = \underline{\beta}$ by an evaluation by an appropriate fragment of $\chi_{=}$.

It is therefore met by $\mathcal{G}$; so let $r \in \mathcal{G}$ be below q and $\beta \in \bigcup^2 b$ with $r \Vdash \underline{\alpha} = \underline{\beta}$ and $r \Vdash_1 \underline{\beta} \mathrel{\epsilon} \underline{b}$.

From the second property of r, $\mathrm{val}_{\mathcal{G}}(\beta) \in \mathrm{val}_{\mathcal{G}}(b)$, and from the first, applying the induction hypothesis, $\mathrm{val}_{\mathcal{G}}(\alpha) = \mathrm{val}_{\mathcal{G}}(\beta)$; thus $x \in \mathrm{val}_{\mathcal{G}}(b)$; as x was arbitrary, $\mathrm{val}_{\mathcal{G}}(a) \subseteq \mathrm{val}_{\mathcal{G}}(b)$.

A similar argument shows that $\mathrm{val}_{\mathcal{G}}(b) \subseteq \mathrm{val}_{\mathcal{G}}(a)$. $\blacksquare_{4.8}$

4.9. Lemma. *If $p \in \mathcal{G}$ and $p \Vdash \underline{a} \mathrel{\epsilon} \underline{b}$ then* $\mathrm{val}_{\mathcal{G}}(a) \in \mathrm{val}_{\mathcal{G}}(b)$.

Proof. $\{r \in \boldsymbol{P} \mid \exists\beta_{\in\bigcup^2 b}\, r \Vdash \underline{\beta} = \underline{a} \;\&\; r \Vdash_1 \underline{\beta} \mathrel{\epsilon} \underline{b}\}$ is dense below p and is a set, and so there is an $r \in \mathcal{G}$ and a $\beta \in \bigcup^2 b$ such that $r \Vdash \underline{\beta} = \underline{a}$, which by the previous lemma implies that $\mathrm{val}_{\mathcal{G}}(\beta) = \mathrm{val}_{\mathcal{G}}(a)$, and, by Remark 4.4, such that $\mathrm{val}_{\mathcal{G}}(\beta) \in \mathrm{val}_{\mathcal{G}}(b)$; so $\mathrm{val}_{\mathcal{G}}(a) \in \mathrm{val}_{\mathcal{G}}(b)$. $\blacksquare_{4.9}$

4.10. Lemma. *If* $\mathrm{val}_{\mathcal{G}}(a) = \mathrm{val}_{\mathcal{G}}(b)$, *then for some $p \in \mathcal{G}$, $p \Vdash \underline{a} = \underline{b}$.*

Proof by induction. We show first that $\exists p_{\in\mathcal{G}}\, p \Vdash \bigwedge \mathfrak{x}_{\epsilon\underline{a}}\, \mathfrak{x} \mathrel{\epsilon} \underline{b}$. If not, then by density, $\exists p_{\in\mathcal{G}}\, p \Vdash \bigvee \mathfrak{x}_{\epsilon\underline{a}}\, \neg(\mathfrak{x} \mathrel{\epsilon} \underline{b})$; indeed there will then exist $p \in \mathcal{G}$ and $\alpha \in \bigcup^2 a$ such that $p \Vdash_1 \underline{\alpha} \mathrel{\epsilon} \underline{a}$ and $p \Vdash \neg(\underline{\alpha} \mathrel{\epsilon} \underline{b})$. Given such p and α, $\mathrm{val}_{\mathcal{G}}(\alpha) \in \mathrm{val}_{\mathcal{G}}(a)$; so there is a $\beta \in \bigcup^2 b$ and a $q \in \mathcal{G}$ with $(q, \beta) \in b$, and $\mathrm{val}_{\mathcal{G}}(\alpha) = \mathrm{val}_{\mathcal{G}}(\beta)$.

By the induction hypothesis, there will be an $r \in \mathcal{G}$, which we may suppose to be below both q and p, such that $r \Vdash \underline{\alpha} = \underline{\beta}$ and $r \Vdash_1 \underline{\beta} \mathrel{\epsilon} \underline{b}$; so $r \Vdash \underline{\alpha} \mathrel{\epsilon} \underline{b}$, contrary to our hypothesis on p.

A similar argument will show that $\exists p_{\in\mathcal{G}}\, p \Vdash \bigwedge \mathfrak{x}_{\epsilon\underline{b}}\, \mathfrak{x} \mathrel{\epsilon} \underline{a}$; and we may now invoke the fact that Extensionality is forced, to conclude that there is a $p \in \mathcal{G}$ such that $p \Vdash \underline{a} = \underline{b}$. $\blacksquare_{4.10}$

4.11. Lemma. *If* $\mathrm{val}_{\mathcal{G}}(a) \in \mathrm{val}_{\mathcal{G}}(b)$, *then for some $p \in \mathcal{G}$, $p \Vdash \underline{a} \mathrel{\epsilon} \underline{b}$.*

Proof. The hypothesis implies that there are $q \in \mathcal{G}$ and c such that $(q, c) \in b$ and $\mathrm{val}_{\mathcal{G}}(a) = \mathrm{val}_{\mathcal{G}}(c)$. By the previous lemma, there is a p_0 in $\mathcal{G}$ such that $p \Vdash \underline{a} = \underline{c}$; then if $p \in \mathcal{G}$ is below both p_0 and q, then $p \Vdash_1 \underline{c} \mathrel{\epsilon} \underline{b}$ and so $p \Vdash \underline{a} \mathrel{\epsilon} \underline{b}$. $\blacksquare_{4.11}$

The proof of Proposition 4.5 is now complete. $\blacksquare_{4.5}$

4.12. Definition. $M^{\mathbb{P}}[\mathcal{G}] =_{\mathrm{df}} \{\mathrm{val}_{\mathcal{G}}(a) \mid a \in M\}$.

We check that $M \cup \{\mathcal{G}\} \subseteq M^{\mathbb{P}}[\mathcal{G}]$.

4.13. Proposition. *For all $x \in M$, $\hat{x} \in M$ and* $\mathrm{val}_{\mathcal{G}}(\hat{x}) = x$; *hence $M \subseteq M^{\mathbb{P}}[\mathcal{G}]$.*

Proof. By the providence of M and an easy application of Proposition 2.33. $\blacksquare_{4.13}$

We have a canonical name for $\mathcal{G}$:

4.14. Definition. $\dot{\mathcal{G}} =_{\mathrm{df}} \{(p, \hat{p}) \mid p \in \boldsymbol{P}\}$.

Note that $\dot{\mathcal{G}} \in M$ as M is provident and $a \mapsto \hat{a}$ is rud-rec.

4.15. Proposition. $\mathrm{val}_{\mathcal{G}}(\dot{\mathcal{G}}) = \mathcal{G}$.

Proof. Both sides equal $\{\mathrm{val}_{\mathcal{G}}(\hat{p}) \mid p \in \mathcal{G}\}$. $\blacksquare_{4.15}$

4.16. Corollary. $\mathcal{G} \in M^{\mathbb{P}}[\mathcal{G}]$ *and hence*

$$M \cup \{\mathcal{G}\} \subseteq M^{\mathbb{P}}[\mathcal{G}] \subseteq \mathrm{Prov}(M \cup \{\mathcal{G}\}).$$

4.17. The Forcing Theorem. *Given A, $\mathbb{P}$ and $\mathcal{G}$, for each $\dot{\Delta}_0$ formula φ and $a_1, \dots a_n$ in A:*

$$A^{\mathbb{P}}[\mathcal{G}] \models \varphi[\mathrm{val}_{\mathcal{G}}(a_1), \dots, \mathrm{val}_{\mathcal{G}}(a_n)] \;\Leftrightarrow\; \exists p_{\in \mathcal{G}}\, (p \Vdash^{\mathbb{P}} \varphi[\underline{a_1}, \dots, \underline{a_n}])^A.$$

Proof. First, the case of atomic φ is covered by Proposition 4.5.

For Boolean conjunctions: $\exists s_{\in \mathcal{G}}\, (s \le p \;\&\; s \le q)$ iff p and q are both in $\mathcal{G}$.

For negations: there is a dense class to be met by $\mathcal{G}$, and we must show that the class in question is a set. Note that $\boldsymbol{P} \cap \{t \mid \neg\exists r \le t\ (r \Vdash \varphi[\underline{\beta}])\}$ is a member of the provident set that is the ground model: take an attempt at $\chi_{=}$ that covers all; then this set is obtainable as a separator that is Δ_0 in that attempt.

Now consider the problem of a restricted quantifier (2). Suppose $p \Vdash \bigwedge \mathfrak{x}_{\epsilon \underline{b}}\, \varphi(\mathfrak{x})$, and let $\mathbf{b} = \mathrm{val}_{\mathcal{G}}(b)$. Suppose $A[\mathcal{G}] \models \tilde{\mathbf{a}} \,\epsilon\, \tilde{\mathbf{b}}$ where $\mathbf{a} = \mathrm{val}_{\mathcal{G}}(a)$. Then there is a q in $\mathcal{G}$ and an η such that $q \Vdash_1 \underline{\eta} \,\epsilon\, \underline{b}$ and $q \Vdash \underline{a} = \underline{\eta}$. Then densely below q, there are r such that $r \Vdash \varphi[\underline{\eta}]$. So some such r is in $\mathcal{G}$; so $A[\mathcal{G}] \models \varphi[\mathrm{val}_{\mathcal{G}}(\eta)]$; but $\mathrm{val}_{\mathcal{G}}(\eta) = \mathrm{val}_{\mathcal{G}}(a) = \mathbf{a}$. Thus $A[\mathcal{G}] \models \bigwedge \mathfrak{x}_{\epsilon \tilde{\mathbf{b}}}\, \varphi(\mathfrak{x})$.

Conversely, suppose that $A[\mathcal{G}] \models \bigwedge \mathfrak{x}_{\epsilon \tilde{\mathbf{b}}}\, \varphi(\mathfrak{x})$, and suppose that $b \in A$ with $\mathbf{b} = \mathrm{val}_{\mathcal{G}}(b)$. Let

$$X = \mathbb{P} \cap \{t \mid \exists \beta_{\in \bigcup^2 b}\ t \Vdash_1 \underline{\beta} \,\epsilon\, \underline{b} \;\&\; \neg\exists r_{\le t}\ (r \Vdash \phi[\underline{\beta}])\}.$$

X is a set and is downwards closed, i.e. open in the usual topology on $\mathbb{P}$. Let $\Delta = X \cup \{p \mid \mathcal{O}_p \cap X = \varnothing\}$, where $\mathcal{O}_p = \{q \mid q \le p\}$.

Δ is a dense open set, and so meets $\mathcal{G}$. Let $p \in \mathcal{G} \cap \Delta$. If $p \in X$, then for some $\beta \in \bigcup^2 b$, $p \Vdash_1 \underline{\beta} \,\epsilon\, \underline{b}$, but for no $r \le p$ does $r \Vdash \varphi[\underline{\beta}]$; so $p \Vdash \neg\varphi[\underline{\beta}]$; so $A[\mathcal{G}] \models \neg\varphi[\mathrm{val}_{\mathcal{G}}(\beta)]$; but $\mathrm{val}_{\mathcal{G}}(\beta) \in \mathrm{val}_{\mathcal{G}}(b)$.

Thus $p \notin X$; and so there is no q below p with $q \in X$. So

$$\forall q_{\le p}\, \forall \beta_{\in \bigcup^2 b}\, \big(q \Vdash_1 \underline{\beta} \,\epsilon\, \underline{b} \;\Rightarrow\; \exists r_{\le q}\ r \Vdash \varphi[\underline{\beta}]\big),$$

which says precisely that $p \Vdash \bigwedge \mathfrak{x}_{\epsilon \underline{b}}\, \varphi(\mathfrak{x})$. $\blacksquare_{4.17}$

5. Construction of nominators for rudimentary functions. Our aim in this section is to prove a theorem about the construction of names in a provident set for the values of a rudimentary function in a set-generic

(2) For explanation of the notation, please see the footnote on page 141.

extension of that set. We state it now, and shall later give more precise formulations of particular cases.

5.0. THEOREM. *Let R be a rudimentary function of some number of arguments. Then there is a function $R^{\mathbb{P}}$, of the same number of arguments, with the property that if A is a provident set and $\mathbb{P} \in A$ a notion of forcing, then A is closed under $R^{\mathbb{P}}$ and, further, if $\mathcal{G}$ is an $(A, \mathbb{P})$-generic, then (to take the case of a function of two variables) for all x and y in A,* $\mathrm{val}_{\mathcal{G}}(R^{\mathbb{P}}(x, y)) = R(\mathrm{val}_{\mathcal{G}}(x), \mathrm{val}_{\mathcal{G}}(y))$.

DEFINITION. We shall call the function $R^{\mathbb{P}}$ the *nominator* of the function R. Usually its definition is uniform in $\mathbb{P}$ and A. We shall use the phrase "Cohen term" to speak of the value of a nominator given some arguments.

5.1. COROLLARY. *Let A be provident, $\mathbb{P} \in A$ and $\mathcal{G}$ $(A, \mathbb{P})$-generic. Then $A[\mathcal{G}]$ is rud-closed and so a model of* GJ_0.

Proof of the Corollary. Suppose (to take a function of two variables) that $R(x, y)$ is a rudimentary function and that x and y are in $A[\mathcal{G}]$. Then there are a and b in A so that $x = \mathrm{val}_{\mathcal{G}}(a)$ and $y = \mathrm{val}_{\mathcal{G}}(b)$. Applying the nominator of R, the corresponding Cohen term $R^{\mathbb{P}}(a, b)$ exists in A: let $z = \mathrm{val}_{\mathcal{G}}(R^{\mathbb{P}}(a, b))$. Then $z \in A[\mathcal{G}]$, and by the theorem $R(x, y) = z$. Since $R(x, y) = z$ is a $\dot{\Delta}_0$ statement, and therefore absolute for transitive sets containing x, y and z, we know that it is true in $A[\mathcal{G}]$ that $R(x, y) = z$. $\blacksquare_{5.1}$

Some general lemmata about forcing

5.2. LEMMA. *If $a \subseteq b$,* $\mathrm{val}_{\mathcal{G}}(a) \subseteq \mathrm{val}_{\mathcal{G}}(b)$.

Proof.

$$\begin{aligned} \mathrm{val}_{\mathcal{G}}(a) &= \{\mathrm{val}_{\mathcal{G}}(\alpha) \mid \exists p_{\in \mathcal{G}}\ (p, \alpha) \in a\} \\ &\subseteq \{\mathrm{val}_{\mathcal{G}}(\beta) \mid \exists p_{\in \mathcal{G}}\ (p, \beta) \in b\} = \mathrm{val}_{\mathcal{G}}(b). \ \blacksquare_{5.2} \end{aligned}$$

5.3. LEMMA. *Let u be transitive. Then* $\mathrm{val}_{\mathcal{G}}(u)$ *is transitive.*

Proof. If $x \in \mathrm{val}_{\mathcal{G}}(u)$, then $\exists \alpha\, \exists p_{\in \mathcal{G}}\ (p, \alpha) \in u\ \&\ \mathrm{val}_{\mathcal{G}}(\alpha) = x$. But $\alpha \in \bigcup^2 u \subseteq u$ so $\alpha \subseteq u$; so $\mathrm{val}_{\mathcal{G}}(\alpha) \subseteq \mathrm{val}_{\mathcal{G}}(u)$. $\blacksquare_{5.3}$

5.4. We note alternative ways of naming an object. For given y, put

$$\begin{aligned} \mathcal{A}_0(y) &= \{(p, x) \mid p \Vdash_0 \underline{x} \mathrel{\epsilon} \underline{y}\}, \\ \mathcal{A}_1(y) &= \{(p, x) \mid p \Vdash_1 \underline{x} \mathrel{\epsilon} \underline{y}\}, \\ \mathcal{A}(y) &= \{(p, x) \mid p \Vdash \underline{x} \mathrel{\epsilon} \underline{y}\}. \end{aligned}$$

5.5. REMARK. $\mathcal{A}_0(y) \subseteq y$; $\mathcal{A}_0(y) \subseteq \mathcal{A}_1(y) \subseteq \boldsymbol{P} \times \bigcup^2 y$, so $\mathcal{A}_0(y)$ and $\mathcal{A}_1(y)$ are sets, whereas $\mathcal{A}(y)$ is a proper class whose intersection with a set will be a set provided one has rud-rec separation.

5.6. Remark. $\mathcal{A}_0(\mathcal{A}_1(y)) = \mathcal{A}_1(y) = \mathcal{A}_1(\mathcal{A}_0(y)) = \mathcal{A}_1(\mathcal{A}_1(y))$ and $\mathcal{A}_0(\mathcal{A}_0(y)) = \mathcal{A}_0(y)$.

5.7. Lemma. *If* $q \Vdash_1 \underline{w} \mathrel{\epsilon} \underline{\mathcal{A}_1(y)}$, *then* $q \Vdash_1 \underline{w} \mathrel{\epsilon} \underline{y}$; *if* $q \Vdash_1 \underline{w} \mathrel{\epsilon} \underline{y}$, *then* $q \Vdash_0 \underline{w} \mathrel{\epsilon} \underline{\mathcal{A}_1(y)}$.

5.8. Lemma. *Let* $\mathbb{P} \in \mathbf{M}$ *and let* $\mathcal{G}$ *be* $(\mathbf{M}, \mathbb{P})$*-generic. Then if* $y \in \mathbf{M}$, $\mathrm{val}_{\mathcal{G}}(\mathcal{A}_0(y)) = \mathrm{val}_{\mathcal{G}}(\mathcal{A}_1(y)) = \mathrm{val}_{\mathcal{G}}(y)$.

Proof. That $\mathrm{val}_{\mathcal{G}}(y) = \mathrm{val}_{\mathcal{G}}(A_0(y)) \subseteq \mathrm{val}_{\mathcal{G}}(\mathcal{A}_1(y))$ is immediate from the definition of $\mathrm{val}_{\mathcal{G}}(\cdot)$ and Lemma 5.2. It remains to show that $\mathrm{val}_{\mathcal{G}}(\mathcal{A}_1(y)) \subseteq \mathrm{val}_{\mathcal{G}}(y)$.

Suppose that $p \in \mathcal{G}$ and $z \in M$ are such that $p \Vdash \underline{z} \mathrel{\epsilon} \underline{\mathcal{A}_1(y)}$ and $p \Vdash \underline{z} \not\mathrel{\epsilon} \underline{y}$. Then there are $w \in M$ and $q \in \mathcal{G}$ with $q \leq p$, $q \Vdash_1 \underline{w} \mathrel{\epsilon} \underline{\mathcal{A}_1(y)}$, (so by the previous lemma, $q \Vdash \underline{w} \mathrel{\epsilon} \underline{y}$), but also $q \Vdash \underline{w} = \underline{z}$ and therefore $q \Vdash \underline{w} \not\mathrel{\epsilon} \underline{y}$, a contradiction. $\blacksquare_{5.8}$

The proof. To prove Theorem 5.0 we begin by working through the list, given in [M3, p. 166] and discussed in greater detail at the beginning of [MB, §2], of the nine functions $R_0, \ldots, R_8$, the auxiliary function A_{14}, and some others, and show for each function how, given names (in the ground model) for its arguments in the generic extension, we may build names for its values. Certain of the nominators are rudimentary, even basic, functions of their arguments, others will be $\mathbb{P}$-gentle. We shall do the rudimentary ones first. We assume that $\mathbb{P} = (\boldsymbol{P}, \boldsymbol{1}^{\mathbb{P}}, \leq^{\mathbb{P}})_3$.

Although we shall not always obtain a nominator for a rudimentary function as a rudimentary function of the names of its arguments, as we shall see with $x \setminus y$, we may check as we go that we always find a nominator that is rudimentary in the relation $\chi^{\ell}_{\Vdash}$ for some ℓ.

5.9. Remark. Composition of rudimentary nominators will of course be rudimentary.

A basic nominator for $R_0 : x, y \mapsto \{x, y\}$

5.10. Definition. $\{a, b\}^{\mathbb{P}} =_{\mathrm{df}} \{(\boldsymbol{1}^{\mathbb{P}}, a)_2, (\boldsymbol{1}^{\mathbb{P}}, b)_2\}$.

5.11. Lemma. $\{a, b\}^{\mathbb{P}}$ *is a basic function of the variables shown.*

5.12. Proposition. $\mathrm{val}_{\mathcal{G}}(\{a, b\}^{\mathbb{P}}) = \{\mathrm{val}_{\mathcal{G}}(a), \mathrm{val}_{\mathcal{G}}(b)\}$.

5.13. Corollary. $\boldsymbol{1}^{\mathbb{P}} \Vdash^{\mathbb{P}} \underline{\{a, b\}^{\mathbb{P}}} = \dot{\{}\underline{a}, \underline{b}\dot{\}}$.

Basic nominators for ordered pairs and triples. Those can be obtained by composition.

5.14. Definition. $\{x\}^{\mathbb{P}} =_{\mathrm{df}} \{(\boldsymbol{1}^{\mathbb{P}}, x)_2\}$.

5.15. Definition. $(x, y)_2^{\mathbb{P}} =_{\mathrm{df}} \{\{x\}^{\mathbb{P}}, \{x, y\}^{\mathbb{P}}\}^{\mathbb{P}}$.

5.16. Definition. $A_2^{\mathbb{P}}(x, y, z) =_{\mathrm{df}} \{x, (y, z)_2^{\mathbb{P}}\}^{\mathbb{P}}$.

5.17. Definition. $(x, y, z)^{\mathbb{P}}_3 =_{\mathrm{df}} (x, (y, z)^{\mathbb{P}}_2)^{\mathbb{P}}_2$.

5.18. Lemma. *The four functions just introduced are basic functions of the variables shown.*

A basic nominator for $x \cup y$

5.19. Definition. $x \cup^{\mathbb{P}} y =_{\mathrm{df}} x \cup y$.

A basic nominator for $R_2 : x \mapsto \bigcup x$

5.20. Definition.

$$\bigcup\nolimits^{\mathbb{P}} x =_{\mathrm{df}} (\boldsymbol{P} \times \bigcup\nolimits^5 x) \cap \{(p, \alpha) \mid \exists (q, \beta)_{\in x}\, (p \leq q \;\&\; p \Vdash_1 \underline{\alpha} \epsilon \underline{\beta})\}.$$

5.21. Remark. $\bigcup^{\mathbb{P}}$ is a basic function, being the application of a Δ_0 separator.

5.22. Lemma. *If $p \Vdash_0 \underline{\gamma} \epsilon \underline{\bigcup^{\mathbb{P}} x}$, then $(p, \gamma)_2 \in \boldsymbol{P} \times \bigcup^5 x$.*

5.23. Lemma. *If $p \Vdash_1 \underline{\gamma} \epsilon \underline{\bigcup^{\mathbb{P}} x}$, then $(p, \gamma)_2 \in \boldsymbol{P} \times \bigcup^5 x$.*

5.24. Remark. Hence if $p \Vdash^{\mathbb{P}} \underline{\gamma} \in \underline{\bigcup^{\mathbb{P}} x}$, then there are many β and t (dense below p) for which $t \Vdash \underline{\beta} = \underline{\gamma}$ and $t \Vdash_1 \underline{\beta} \epsilon \underline{\bigcup^{\mathbb{P}} x}$.

5.25. Proposition. $\mathrm{val}_{\mathcal{G}}(\bigcup^{\mathbb{P}} x) = \bigcup \mathrm{val}_{\mathcal{G}}(x)$.

5.26. Corollary. $\boldsymbol{1}^{\mathbb{P}} \Vdash^{\mathbb{P}} \underline{\bigcup^{\mathbb{P}} x} = \dot{\bigcup} \underline{x}$.

A basic nominator for $R_4 : x, y \mapsto x \times y$

5.27. Definition.

$$x \times^{\mathbb{P}} y =_{\mathrm{df}} \{(p, (\alpha, \beta)^{\mathbb{P}}_2) \mid p \Vdash_1 \underline{\alpha} \epsilon \underline{x} \;\&\; p \Vdash_1 \underline{\beta} \epsilon \underline{y}\}.$$

5.28. Remark. $\cdot \times^{\mathbb{P}} \cdot$ is evidently rudimentary, but it is actually basic in $\mathbb{P}$, being the result of applying a Δ_0 separator to the set $\boldsymbol{P} \times [\{\boldsymbol{1}^{\mathbb{P}}\} \times [\{\boldsymbol{1}^{\mathbb{P}}\} \times (x \cup y)]^{\leqslant 2}]^{\leqslant 2}$.

5.29. Proposition. $\mathrm{val}_{\mathcal{G}}(x \times^{\mathbb{P}} y) = \mathrm{val}_{\mathcal{G}}(x) \times \mathrm{val}_{\mathcal{G}}(y)$.

5.30. Corollary. $\boldsymbol{1}^{\mathbb{P}} \Vdash^{\mathbb{P}} \underline{x \times^{\mathbb{P}} y} = \underline{x} \dot{\times} \underline{y}$.

A basic nominator for $[x]^1$

5.31. Definition. $F^{\mathbb{P}}_1(x) =_{\mathrm{df}} \{(p, \{\alpha\}^{\mathbb{P}})_2 \mid (p, \alpha)_2 \in x\}$.

Again, this rudimentary term can be shown to be basic, using the fact that $F^{\mathbb{P}}_1(x) \subseteq \boldsymbol{P} \times (\{\boldsymbol{1}^{\mathbb{P}}\} \times \bigcup^2 x)$.

5.32. Proposition. $\Vdash^{\mathbb{P}} \dot{[}\underline{x}\dot{]}^1 = \underline{F^{\mathbb{P}}_1(x)}$.

Proof. If $q \Vdash_1 \underline{z} \epsilon \underline{F^{\mathbb{P}}_1(x)}$, then there is a $p \geq q$ with $(p, z)_2 \in F^{\mathbb{P}}_1(x)$, so that there is an α with $(p, \alpha)_2 \in x$, $\boldsymbol{1}^{\mathbb{P}} \Vdash^{\mathbb{P}} \underline{z} = \dot{\{}\underline{\alpha}\dot{\}}$ and $q \Vdash_1 \underline{\alpha} \epsilon \underline{x}$.

Conversely, if $q \Vdash^{\mathbb{P}} \bigvee\mathfrak{y}_{\epsilon\underline{x}}\ \underline{z} \doteq \dot{\{}\mathfrak{y}\dot{\}}$, then there are y and $r \le q$ with $r \Vdash_1 \underline{y} \epsilon \underline{x}$ and $r \Vdash^{\mathbb{P}} \underline{z} \doteq \dot{\{}\underline{y}\dot{\}}$. So $(p, y)_2 \in x$ for some $p \ge r$, so that $(p, \{y\}^{\mathbb{P}})_2 \in F_1^{\mathbb{P}}(x)$, $r \Vdash_1 \underline{\{y\}^{\mathbb{P}}} \epsilon \underline{F_1^{\mathbb{P}}(x)}$ and so $r \Vdash^{\mathbb{P}} \underline{z} \epsilon \underline{F_1^{\mathbb{P}}(x)}$. ■5.32

A basic nominator for $[x]^{\leqslant 2}$

5.33. REMARK. $[x]^{\leqslant 2}$ is easier to get than $[x]^2$, because the latter will require us to be certain that two names denote different things; we could obtain such a term by using the identity $[x]^{\leqslant 2} = \cup(x \times x)$; the following is slightly simpler.

5.34. DEFINITION. $F_{\leqslant 2}^{\mathbb{P}}(x) =_{\mathrm{df}} \{(r, \{\alpha, \beta\}^{\mathbb{P}}) \mid r \Vdash_1 \underline{\alpha} \epsilon \underline{x}\ \&\ r \Vdash_1 \underline{\beta} \epsilon \underline{x}\}$.

5.35. REMARK. That function is basic since its value is a Δ_0 subset of $\boldsymbol{P} \times \bigcup((\{\boldsymbol{1}^{\mathbb{P}}\} \times \bigcup^2 x) \times (\{\boldsymbol{1}^{\mathbb{P}}\} \times \bigcup^2 x))$.

5.36. PROPOSITION. $\Vdash \dot{[}\underline{x}\dot{]}^{\leqslant 2} \doteq \underline{F_{\leqslant 2}^{\mathbb{P}}(x)}$.

Proof. If $t \Vdash^{\mathbb{P}} \underline{a} \epsilon \underline{F_{\leqslant 2}^{\mathbb{P}}(x)}$, there are $s \le t$ and b such that $s \Vdash \underline{a} \doteq \underline{b}$ and $s \Vdash_1 \underline{b} \epsilon \underline{F_{\leqslant 2}^{\mathbb{P}}(x)}$. So there are $r \ge s$, α and β with $b = \{\alpha, \beta\}^{\mathbb{P}}$, and conditions p and q with

$$(r \le p\ \&\ r \le q\ \&\ (p, \alpha) \in x\ \&\ (q, \beta) \in x),$$

so that $s \Vdash^{\mathbb{P}} \underline{a} \doteq \underline{b} \doteq \dot{\{}\underline{\alpha}, \underline{\beta}\dot{\}} \epsilon \dot{[}\underline{x}\dot{]}^{\leqslant 2}$.

If $s \Vdash^{\mathbb{P}} \underline{a} \epsilon \dot{[}\underline{x}\dot{]}^{\leqslant 2}$ then there are $t \le s$, α, β such that $t \Vdash_1 \underline{\alpha} \epsilon \underline{x}$ & $t \Vdash_1 \underline{\beta} \epsilon \underline{x}$ & $t \Vdash^{\mathbb{P}} \underline{a} \doteq \dot{\{}\underline{\alpha}, \underline{\beta}\dot{\}}$, so that there are p and q with $t \le p$, $t \le q$, $(p, \alpha) \in x$ and $(q, \beta) \in x$; so $(t, \{\alpha, \beta\}^{\mathbb{P}})_2 \in F_{\leqslant 2}^{\mathbb{P}}(x)$, so $t \Vdash^{\mathbb{P}} \underline{a} \epsilon \underline{F_{\leqslant 2}^{\mathbb{P}}(x)}$. ■5.36

A basic nominator for $u^\star$**.** We recall the definition:

$$u^\star =_{\mathrm{df}} u \cup [u]^{\leqslant 2} \cup (u \times u),$$

and that for u transitive, $u^\star$ is transitive.

Then a basic nominator for it can be found by composition using the preceding ones.

Gentle nominators for the other rudimentary generators. We show that for the remaining rud generators we get nominators of the form $G(\mathbb{P}, x, y) \cap A$ where G is a rudimentary function and A is a separator that is rudimentary in an appropriate segment of $\chi_{=}$; such nominators will be gentle by the principle of gentle separation.

REMARK. That that should be so is suggested by our theory of companions, at least for DB functions. Each of them has a 2-companion W that is generated by $\bigcup$ and $\times$ and is therefore such that $W^{\mathbb{P}}$ is basic; so if $R(x, y) \subseteq W(\{x, y\})$, then we may expect $R^{\mathbb{P}}(x, y)$ to be of the form

$W^{\mathbb{P}}(\{x,y\}^{\mathbb{P}} \cap \{(p,\alpha) \mid p \Vdash \underline{\alpha} \epsilon \dot{R}(\underline{x},\underline{y})\}$; and as $z \in R(x,y)$ is Δ_0, $\underline{\alpha} \epsilon \dot{R}(\underline{x},\underline{y})$ will be $\dot{\varDelta}_0$.

A gentle nominator for $R_1 : x, y \mapsto x \smallsetminus y$. Set

$$x \smallsetminus^{\mathbb{P}} y =_{\mathrm{df}} \mathcal{A}_1(x) \cap \{(p,\alpha) \mid p \Vdash^{\mathbb{P}} \underline{\alpha} \not\epsilon \underline{y}\}.$$

Then $x \smallsetminus^{\mathbb{P}} y$, being a subclass of $\boldsymbol{P} \times \bigcup^2 x$, will be a set if $\boldsymbol{P}$ is, being the application of a separator that is Δ_0 in some appropriate attempt at χ_ϵ.

Let $z = x \smallsetminus^{\mathbb{P}} y$. For each p and α with $p \Vdash_0 \underline{\alpha} \epsilon \underline{z}$, $p \Vdash \alpha \not\epsilon \underline{y} \wedge \underline{\alpha} \epsilon \underline{x}$ so by our general principle, the same is true for each p and α with $p \Vdash \alpha \epsilon z$. Hence $\Vdash \forall t_{\epsilon \underline{z}} [t \epsilon x \wedge t \not\epsilon y]$.

Conversely, suppose that $q \Vdash \underline{\beta} \epsilon \underline{x} \wedge \underline{\beta} \not\epsilon \underline{y}$. We seek $\bar{s} \leq r$ and $(\bar{t}, \alpha) \in z$ with $\bar{s} \leq \bar{t}$ and $\bar{s} \Vdash \underline{\alpha} = \underline{\beta}$. We know that

$$\exists s_{\leq r} \exists (t,\alpha)_{\in x} \; s \leq t \;\&\; s \Vdash \underline{\alpha} = \underline{\beta}$$

so that for such an s, $s \Vdash_1 \underline{\alpha} \epsilon \underline{x}$ and $s \Vdash \underline{\alpha} \not\epsilon \underline{y}$, so $(s,\alpha) \in z$. Hence we may take $\bar{s} = \bar{t} = s$.

A gentle nominator for $R_3 : x \mapsto \mathrm{Dom}(x)$

5.37. Definition.

$$\mathrm{Dom}^{\mathbb{P}}(x) =_{\mathrm{df}} (\mathbb{P} \times \bigcup^{10} x) \cap \big\{(p,\alpha)_2 \bigm| \forall q_{\leq p} \exists r_{\leq q} \exists \beta_{\in \bigcup^{10} x} \; r \Vdash^{\mathbb{P}} \underline{(\beta,\alpha)_2^{\mathbb{P}}} \epsilon \underline{x}\big\}.$$

5.38. Proposition. $\mathit{1}^{\mathbb{P}} \Vdash^{\mathbb{P}} \underline{R_3^{\mathbb{P}}(x)} = \dot{R}_3(\underline{x})$.

A gentle nominator for $R_5 : x \mapsto x \cap \{(a,b)_2 \mid_{a,b} a \in b\}$

5.39. Definition.

$$R_5^{\mathbb{P}}(x) =_{\mathrm{df}} \{(p,\gamma) \mid \exists \alpha_{\in \bigcup^{10} x} \exists \beta_{\in \bigcup^{10} x} \; \gamma = (\alpha,\beta)_2^{\mathbb{P}} \;\&\; p \Vdash^{\mathbb{P}} \underline{\alpha} \epsilon \underline{\beta}\}.$$

That is a set since $(\alpha,\beta)_2^{\mathbb{P}}$ is basic, so we can easily find a companion (i.e. a bounding set), and then apply the separator induced by the relation $p \Vdash \underline{\alpha} \epsilon \underline{\beta}$.

5.40. Proposition. $\mathit{1}^{\mathbb{P}} \Vdash^{\mathbb{P}} \underline{R_5^{\mathbb{P}}(x)} = \dot{R}_5(\underline{x})$.

A gentle nominator for $R_6 : x \mapsto \{(b,a,c)_3 \mid (a,b,c)_3 \in x\}$

5.41. Definition.

$$R_6^{\mathbb{P}}(x) =_{\mathrm{df}} \big\{(p,\delta) \bigm| \exists (q,\tau)_{\in x} \exists \alpha_{\in \bigcup^{\mathfrak{l}} x} \exists \beta_{\in \bigcup^{\mathfrak{m}} x} \exists \gamma_{\in \bigcup^{\mathfrak{n}} x} \\ \big[q \geq p \;\&\; \delta = (\beta,\alpha,\gamma)_3^{\mathbb{P}} \;\&\; p \Vdash^{\mathbb{P}} \underline{\tau} = \underline{(\alpha,\beta,\gamma)_3^{\mathbb{P}}}\big]\big\}.$$

We have defined $(\cdot,\cdot,\cdot)_3^{\mathbb{P}}$ above; it is basic; so we can use it to predict the whereabouts of δ; $\mathfrak{l}, \mathfrak{m}, \mathfrak{n}$ must then be given appropriate values.

5.42. Proposition. $\mathit{1}^{\mathbb{P}} \Vdash^{\mathbb{P}} \underline{R_6^{\mathbb{P}}(x)} = \dot{R}_6(\underline{x})$.

A gentle nominator for $R_7 : x \mapsto \{(b,c,a)_3 \mid (a,b,c)_3 \in x\}$

5.43. Definition.

$$R_7^{\mathbb{P}}(x) =_{\mathrm{df}} \big\{(p,\delta) \bigm| \exists (q,\tau)_{\in x}\, \exists \alpha_{\in \bigcup^{\mathfrak{l}} x}\, \exists \beta_{\in \bigcup^{\mathfrak{m}} x}\, \exists \gamma_{\in \bigcup^{\mathfrak{n}} x} \\ \big[\, q \geq p \;\&\; \delta = (\beta,\gamma,\alpha)_3^{\mathbb{P}} \;\&\; p \Vdash^{\mathbb{P}} \underline{\tau} \mathrel{\mathbf{=}} \underline{(\alpha,\beta,\gamma)_3^{\mathbb{P}}}\,\big]\big\}.$$

5.44. Proposition. $\boldsymbol{1}^{\mathbb{P}} \Vdash^{\mathbb{P}} \underline{R_7^{\mathbb{P}}(x)} \mathrel{\mathbf{=}} \dot{R}_7(\underline{x})$.

A gentle nominator for $A_{14} : x, w \mapsto x``\{w\}$

5.45. Definition.

$$A_{14}^{\mathbb{P}}(x,w) =_{\mathrm{df}} \big(\boldsymbol{P} \times \bigcup^{10} x\big) \cap \big\{(p,\alpha) \bigm| p \Vdash^{\mathbb{P}} \underline{(\alpha,w)_2^{\mathbb{P}}} \mathrel{\epsilon} \underline{x}\big\}.$$

5.46. Proposition. $\boldsymbol{1}^{\mathbb{P}} \Vdash^{\mathbb{P}} \underline{A_{14}^{\mathbb{P}}(x,w)} \mathrel{\mathbf{=}} \dot{A}_{14}(\underline{x},\underline{w})$.

A gentle nominator for $R_8 : x, y \mapsto \{x``\{w\} \mid_w w \in y\}$

5.47. Definition.

$$R_8^{\mathbb{P}}(x,y) =_{\mathrm{df}} \big\{(p,\gamma) \bigm| \exists (q,\beta)_{\in y}\, p \leq q \;\&\; \gamma = A_{14}^{\mathbb{P}}(x,\beta)\big\}.$$

5.48. Lemma. *$R_8^{\mathbb{P}}(x,y)$ is a set.*

Proof. Define

$$F(x,y) =_{\mathrm{df}} ((\boldsymbol{P} \times \bigcup^{10} x) \times \bigcup^{2} y) \\ \cap \{((p,\alpha)_2,\beta)_2 \bigm| p \in \boldsymbol{P} \;\&\; p \Vdash^{\mathbb{P}} \underline{\{\alpha,\beta\}_2^{\mathbb{P}}} \mathrel{\epsilon} \underline{x}\}.$$

Note that if u is any transitive set containing $\mathbb{P}$, x and y, $F \restriction (u \times u)$ is rudimentary in $\chi_{\mathbf{=}} \restriction (\boldsymbol{P} \times u \times u)$.

Set $G(x,y,\beta) = A_{14}(F(x,y),\beta)$. Then G is rudimentary in F and $\beta \in \bigcup^2 y \;\Rightarrow\; A_{14}^{\mathbb{P}}(x,\beta) = G(x,y,\beta)$.

Now set $H(x,y) = \{G(x,y,\beta) \mid \beta \in \bigcup^2(y)\}$. Then H is rudimentary in G, and $R_8^{\mathbb{P}}(x,y)$ is the result of applying a Δ_0 separator to $\boldsymbol{P} \times H(x,y)$. Thus there is a rudimentary function E such that for all such u, $R_8^{\mathbb{P}} \restriction (u \times u) = E\big(\chi_{\mathbf{=}} \restriction (\boldsymbol{P} \times u \times u)\big)$. $\blacksquare_{5.48}$

5.49. Proposition. $\boldsymbol{1}^{\mathbb{P}} \Vdash^{\mathbb{P}} \underline{R_8^{\mathbb{P}}(x,y)} \mathrel{\mathbf{=}} \dot{R}_8(\underline{x},\underline{y})$.

5.50. Remark. Suppose that $Q(\vec{x}) = R(S(\vec{x}), T(\vec{x}))$, where Q, R, S, T are rudimentary, and that we have already found functions $R^{\mathbb{P}}$, $S^{\mathbb{P}}$, $T^{\mathbb{P}}$ as in the statement of the theorem. We may obtain $Q^{\mathbb{P}}$ by composition: define $Q^{\mathbb{P}}(\vec{x}) = R^{\mathbb{P}}(S^{\mathbb{P}}(\vec{x}), T^{\mathbb{P}}(\vec{x}))$.

The proof of Theorem 5.0 as stated is now complete. $\blacksquare_{5.0}$

We must now prove that each of these functions is of uniform finite delay.

Propagation of nominators for rudimentary functions

5.51. Proposition. *Let R be a rudimentary function of some number of arguments, and let $R^{\mathbb{P}}$ be the corresponding nominator. There is a natural number s_R such that whenever e is a transitive set with $\mathbb{P} \in e$, and ν is an ordinal not less than $\varrho(\mathbb{P})$,*

$$R^{\mathbb{P}} \restriction P_\nu^{e;=} \in P_{\nu+s_R}^{e;=}.$$

Proof. We have seen that the nominators for R_0, R_2 and R_4 proved to be themselves rudimentary, and hence s_R can be taken in these cases to be 1 plus the rudimentary constant for $u \mapsto R \restriction u$. The nominators corresponding to the other functions in the standard generating set are all rudimentary in appropriate fragments of $\chi_=$, and so the proof in those cases follows from the corresponding result for $\chi_=$. We give the argument for R_8.

Let $\nu \geqslant \eta$. We know that $\chi_= \restriction P_\nu^{e;=} \in P_{\nu+12}^{e;=}$, and that for some rudimentary function E, $R_8^{\mathbb{P}} \restriction P_\nu^{e;=} = E(\chi_= \restriction P_\nu^{e;=})$, so we may take $s_{R_8} = 12 + c_E$.

Once the theorem has been established for the nine generators, it remains only to observe that the property in question is preserved under composition. If, for example, $Q(\vec{x}) = R(S(\vec{x}), T(\vec{x}))$, then s_Q can be taken to be $c_R + \max\{c_S, c_T\} + c_\circ$, where $c_\circ$ is the constant of the rudimentary function that composes fragments of $R^{\mathbb{P}}$, $S^{\mathbb{P}}$ and $T^{\mathbb{P}}$ to a fragment of $Q^{\mathbb{P}}$. $\blacksquare_{5.51}$

5.52. Remark. At this point we know that all the axioms of GJ_0 are forced by the trivial condition.

No new ordinals! There is a long-established principle that a generic extension will contain no ordinals not in the ground model. In [M4] an admittedly pathological example of forcing over an improvident but transitive model of Zermelo set theory is presented where this principle breaks down. So our task here is to present a proof, working in the theory PROVI, of the following:

5.53. Proposition. *If $p \Vdash \underline{x} \mathrel{\epsilon} \dot{On}$ then $\exists q_{\leq p} \exists \zeta_{\leq \varrho(x)}\, q \Vdash \hat{\underline{\zeta}} = \underline{x}$.*

Plainly the statement of the proposition requires every ordinal to have a hat; but hatting is $\mathbf{1}^{\mathbb{P}}$-rud-rec, so available in PROVI. Proposition 3.16 then yields

5.54. Proposition. *For each ordinal η, $\mathbf{1}^{\mathbb{P}} \Vdash^{\mathbb{P}} \hat{\eta}$ `is an ordinal`.*

The second requirement is that there should be enough set theory to prove that the principle of trichotomy for two ordinals is forced. So let ζ and η be ordinals.

5.55. Lemma. *Either $\zeta \cap \eta = \eta$ or $\zeta \cap \eta \in \eta$.*

Proof. $\eta \smallsetminus \zeta$ if non-empty has, by Foundation, a least element, ξ say; then show that $\xi = \eta \cap \zeta$. $\blacksquare_{5.55}$

5.56. Trichotomy for ordinals. *$\zeta \in \eta$, $\zeta = \eta$ or $\eta \in \zeta$.*

Proof. Consider the four statements

1a	$\zeta \cap \eta = \eta,$
1b	$\zeta \cap \eta \in \eta,$
2a	$\zeta \cap \eta = \zeta,$
2b	$\zeta \cap \eta \in \zeta.$

We know that [(1a or 1b) and (2a or 2b)] holds. Of the four possibilities, (1b and 2b) is impossible, as it would imply $\zeta \cap \eta \in \zeta \cap \eta$, contradicting Foundation; the three disjuncts of the proposition correspond to (1b and 2a), (1a and 2a), (1a and 2b). $\blacksquare_{5.56}$

The final requirement is that rank should be definable in the ground model; but ϱ is $\varnothing$-rud-rec.

5.57. Lemma. $\Vdash^{\mathbb{P}} \dot{On}$ $\texttt{is transitive}$.

5.58. Lemma. *There are no $p \in \boldsymbol{P}$ and x such that $p \Vdash \underline{x} \mathrel{\epsilon} \dot{On} \wedge \underline{\widehat{\varrho(x)}} \mathrel{\epsilon} \underline{x}$.*

Proof. Suppose such an x exists; let it be a member of the transitive set u. By rewriting its definition in terms of the attempts $\chi_{\Vdash} \restriction u$, $\varrho \restriction u$ and $\hat{\cdot} \restriction \varrho(u)$, we see that the class

$$u \cap \{x \mid \exists p_{\in \boldsymbol{P}}\, p \Vdash \underline{x} \mathrel{\epsilon} \dot{On} \;\&\; p \Vdash \underline{\widehat{\varrho(x)}} \mathrel{\epsilon} \underline{x}\}$$

is a set by Δ_0 separation, and non-empty by the initial supposition. Call that set A.

Let x be a member of A with $\varrho(x)$ minimal. Then $x \in A$ and $\bigcup^2 x \cap A = \varnothing$. Let $\eta = \varrho(x)$, and let $p \Vdash \underline{x} \mathrel{\epsilon} \dot{On} \wedge \underline{\hat{\eta}} \mathrel{\epsilon} \underline{x}$. So $\exists q_{\leq p} \exists r_{\geq q}\, (r, y) \in x$ & $q \Vdash \underline{\hat{\eta}} = \underline{y}$.

Let $\zeta = \varrho(y)$. Since $y \in \bigcup^2 x$, $\zeta \in \eta$, so by Proposition 3.17, $\boldsymbol{1}^{\mathbb{P}} \Vdash^{\mathbb{P}} \underline{\hat{\zeta}} \mathrel{\epsilon} \underline{\hat{\eta}}$; $q \Vdash \underline{\hat{\eta}} = \underline{y}$, so $q \Vdash \underline{\widehat{\varrho(y)}} \mathrel{\epsilon} \underline{y}$; so $y \in A$, in contradiction to the choice of x. $\blacksquare_{5.58}$

We complete the proof of Proposition 5.53 by noting that the law of trichotomy for ordinals is forced:

5.59. Lemma. $\Vdash^{\mathbb{P}}$ $\texttt{Trichotomy for ordinals}$.

Proof. We have just seen that Trichotomy for ordinals is provable in GJ_0, and we know that all axioms of GJ_0 are forced. $\blacksquare_{5.59}$

Now Lemma 5.58 implies that if $p \Vdash^{\mathbb{P}} \underline{x} \mathrel{\epsilon} \dot{On}$, and $\eta = \varrho(x)$ then $p \Vdash^{\mathbb{P}} \underline{\hat{\eta}} \not\mathrel{\epsilon} \underline{x}$. By Trichotomy, $p \Vdash^{\mathbb{P}} \underline{x} \mathrel{\epsilon} \underline{\hat{\eta}} \vee \underline{x} = \underline{\hat{\eta}}$; which implies that there are $q \leq p$ and $\zeta \leqslant \eta$ with $q \Vdash^{\mathbb{P}} \underline{x} = \underline{\hat{\zeta}}$ as required. $\blacksquare_{5.53}$

5.60. Remark. In Section 6 of [M2], a forcing contruction is done over a non-standard model $\mathfrak{N}$, and it was there blithely stated without proof that the generic extension would bring no new "ordinals". Fortunately the model $\mathfrak{N}$ was power-admissible, and therefore certainly a model of PROVI, which is a sub-theory of KP, so that the present remarks justify that blithe statement; that is reassuring in view of the models presented in [M4].

We record two related arguments.

5.61. LEMMA. *$\varrho(\mathrm{val}_{\mathcal{G}}(x)) \leqslant \varrho(x)$.*

Proof.

$$\begin{aligned}\varrho(\mathrm{val}_{\mathcal{G}}(x)) &= \sup\{\varrho(\mathrm{val}_{\mathcal{G}}(y)) + 1 \mid (1, y) \in x\}\\ &\leqslant \sup\{\varrho(y) + 1 \mid (1, y) \in x\}\\ &\leqslant \varrho(x). \quad \blacksquare_{5.61}\end{aligned}$$

5.62. LEMMA. *If $p \Vdash \underline{\hat{\zeta}} \mathrel{\epsilon} \underline{x}$ then $\zeta < \varrho(x)$.*

Proof by eps-recursion on x. If $p \Vdash \underline{\hat{\zeta}} \mathrel{\epsilon} \underline{x}$ then there are q and r with $q \leq p$, $q \leq r$, $(r, y) \in x$ and $q \Vdash \underline{\hat{\zeta}} = \underline{y}$. Hence for all $\eta < \zeta$, $q \Vdash \underline{\hat{\eta}} \mathrel{\epsilon} \underline{y}$ and so by induction, $\eta < \varrho(y)$, so $\zeta \leq \varrho(y) < \varrho(x)$. $\blacksquare_{5.62}$

6. Construction of rudimentarily recursive nominators for rank and transitive closure. Rank and transitive closure are pure rud-rec; we show here that $\mathbb{P}$-rud-rec nominators exist for them.

6.0. LEMMA. *Let A be provident and closed under F and F“: for example if F is gentle. Then*

$$\mathrm{val}_{\mathcal{G}}(\{(p, F(y)) \mid_{p,y} (p, y) \in x\}) = \{\mathrm{val}_{\mathcal{G}}(F(y)) \mid_y \exists p_{\in G}\ (p, y) \in x\}.$$

Proof. Let $Z = \{(p, F(y)) \mid_{p,y} (p, y) \in x\}$. Then Z is in A by the hypotheses concerning the closure of A under F, F“ and related functions. Then $\mathrm{val}_{\mathcal{G}}(Z) = \{\mathrm{val}_{\mathcal{G}}(z) \mid_z \exists p_{\in \mathcal{G}}\, (p, z) \in Z\}$.

So if $w \in \mathrm{val}_{\mathcal{G}}(Z)$, then $\exists z\, \exists p_{\in \mathcal{G}}\, [w = \mathrm{val}_{\mathcal{G}}(z)\ \&\ (p, z) \in Z]$. So $\exists y\ [(p, y) \in x\ \&\ z = F(y)]$. So $w = \mathrm{val}_{\mathcal{G}}(F(y))$ where for some $p \in \mathcal{G}$, $(p, y) \in x$. So the LHS is contained in the RHS.

Conversely, if $(p, y) \in x$ and $p \in \mathcal{G}$, then $(p, F(y)) \in Z$ and $p \in \mathcal{G}$; so $\mathrm{val}_{\mathcal{G}}(F(y)) \in \mathrm{val}_{\mathcal{G}}(Z)$. $\blacksquare_{6.0}$

Let $S(\cdot)$ be the basic function $z \mapsto z \cup \{z\}$.

6.1. LEMMA. *There is a rud function $S^{\mathbb{P}}(\cdot)$ such that $\mathrm{val}_{\mathcal{G}}(S^{\mathbb{P}}(x)) = S(\mathrm{val}_{\mathcal{G}}(x))$.*

Proof. By composition. $\blacksquare_{6.1}$

6.2. DEFINITION. $\varrho^{\mathbb{P}}(x) =_{\mathrm{df}} \bigcup^{\mathbb{P}}\{(p, S^{\mathbb{P}}(\varrho^{\mathbb{P}}(y)) \mid (p, y) \in x\ \&\ p \in \boldsymbol{P}\}$.

6.3. REMARK. $\varrho^{\mathbb{P}}$ is rud-rec in the parameter $\mathbb{P}$.

6.4. LEMMA. *Let A be provident, and $\mathbb{P} \in A$. For all $x \in A$,*

$$\mathrm{val}_{\mathcal{G}}(\varrho^{\mathbb{P}}(x)) = \varrho(\mathrm{val}_{\mathcal{G}}(x)).$$

REMARK. That all makes sense: if x is in A, the name $\varrho^{\mathbb{P}}(x)$ is in A. Note that $\varrho(\mathrm{val}_{\mathcal{G}}(x))$ is evaluated in the universe. At present we do not know that the evaluation can be carried out in $A^{\mathbb{P}}[G]$.

Proof.

$$\begin{aligned}
&\varrho(\mathrm{val}_{\mathcal{G}}(x)) \\
&\quad = \bigcup\{\varrho(y) + 1 \mid_y y \in \mathrm{val}_{\mathcal{G}}(x)\} && \text{definition of } \varrho \\
&\quad = \bigcup\{\varrho(\mathrm{val}_{\mathcal{G}}(w)) + 1 \mid_w \exists p_{\in G}\ (p, w) \in x\} && \text{definition of } \mathrm{val}_{\mathcal{G}}(x) \\
&\quad = \bigcup\{\mathrm{val}_{\mathcal{G}}(\varrho^{\mathbb{P}}(w)) + 1 \mid_w \exists p_{\in G}\ (p, w) \in x\} && \text{induction hypothesis} \\
&\quad = \bigcup\{\mathrm{val}_{\mathcal{G}}(S^{\mathbb{P}}(\varrho^{\mathbb{P}}(w))) \mid_w \exists p_{\in G}\ (p, w) \in x\} && \text{property of } S^{\mathbb{P}} \\
&\quad = \bigcup \mathrm{val}_{\mathcal{G}}(\{(p, S^{\mathbb{P}}(\varrho^{\mathbb{P}}(w))) \mid_{p,w} (p, w) \in x\}) && \text{by Lemma 6.0} \\
&\quad = \mathrm{val}_{\mathcal{G}}({\textstyle\bigcup}^{\mathbb{P}}(\{(p, S^{\mathbb{P}}(\varrho^{\mathbb{P}}(w))) \mid_{p,w} (p, w) \in x\})) && \text{property of } {\textstyle\bigcup}^{\mathbb{P}} \\
&\quad = \mathrm{val}_{\mathcal{G}}(\varrho^{\mathbb{P}}(x)) \quad \text{by the definition of } \varrho^{\mathbb{P}}. \ \blacksquare_{6.4}
\end{aligned}$$

6.5. DEFINITION. $\mathrm{tcl}^{\mathbb{P}}(x) =_{\mathrm{df}} x \cup^{\mathbb{P}} \bigcup^{\mathbb{P}}(\{(p, \mathrm{tcl}^{\mathbb{P}}(z)) \mid (p, z) \in x\})$.

6.6. REMARK. $\mathrm{tcl}^{\mathbb{P}}$ is rud-rec in the parameter $\mathbb{P}$.

6.7. LEMMA. *Let A be provident, and $\mathbb{P} \in A$. For all $x \in A$,*

$$\mathrm{val}_{\mathcal{G}}(\mathrm{tcl}^{\mathbb{P}}(x)) = \mathrm{tcl}(\mathrm{val}_{\mathcal{G}}(x)).$$

Proof. By similar reasoning:

$$\begin{aligned}
&\mathrm{tcl}(\mathrm{val}_{\mathcal{G}}(x)) \\
&\ = \mathrm{val}_{\mathcal{G}}(x) \cup \bigcup\{\mathrm{tcl}(y) \mid_y y \in \mathrm{val}_{\mathcal{G}}(x)\} && \text{definition of tcl} \\
&\ = \mathrm{val}_{\mathcal{G}}(x) \cup \bigcup\{\mathrm{tcl}(y) \mid_y \exists p_{\in G} \exists z\, (p, z) \in x \ \&\ y = \mathrm{val}_{\mathcal{G}}(z)\} \\
& && \text{definition of } \mathrm{val}(x) \\
&\ = \mathrm{val}_{\mathcal{G}}(x) \cup \bigcup\{\mathrm{tcl}(\mathrm{val}_{\mathcal{G}}(z)) \mid_z \exists p_{\in G}\ (p, z) \in x\} \\
&\ = \mathrm{val}_{\mathcal{G}}(x) \cup \bigcup\{\mathrm{val}_{\mathcal{G}}(\mathrm{tcl}^{\mathbb{P}}(z)) \mid_z \exists p_{\in G}\ (p, z) \in x\} && \text{induction hypothesis} \\
&\ = \mathrm{val}_{\mathcal{G}}(x) \cup \bigcup \mathrm{val}_{\mathcal{G}}(\{(p, \mathrm{tcl}^{\mathbb{P}}(z)) \mid_{p,z} (p, z) \in x\}) && \text{Lemma 6.0} \\
&\ = \mathrm{val}_{\mathcal{G}}(x \cup^{\mathbb{P}} {\textstyle\bigcup}^{\mathbb{P}}(\{(p, \mathrm{tcl}^{\mathbb{P}}(z)) \mid_{p,z} (p, z) \in x\})) && \text{properties of } \cup^{\mathbb{P}} \text{ and } {\textstyle\bigcup}^{\mathbb{P}} \\
&\ = \mathrm{val}_{\mathcal{G}}(\mathrm{tcl}^{\mathbb{P}}(x)) \quad \text{by the definition of } \mathrm{tcl}^{\mathbb{P}}. \ \blacksquare_{6.7}
\end{aligned}$$

6.8. PROPOSITION. *Let A be provident, and $\mathbb{P} \in A$. Let G be $(A, \mathbb{P}, \dot{\Delta}_0)$-generic. Then $A^{\mathbb{P}}[G]$ is closed under rank and transitive closure.*

6.9. REMARK. A similar result will hold whenever F is rud-rec, given by G, where $G^{\mathbb{P}}$ is rudimentary; $G^{\mathbb{P}}$ may be permitted to have as a parameter a name $\underline{a}$ for a parameter $\mathrm{val}_{\mathcal{G}}(a)$ in the extension.

We pause for breath. The next stage will be to show that the generic extension is closed under the formation of certain canonical progresses; but we digress to discuss the case of primitive recursively closed sets, which is now easy.

7. Construction of primitive recursive nominators for primitive recursive functions. Jensen and Karp give, following Gandy, this definition: there are some initial functions, which are all rudimentary; two versions of substitution: $F(\vec{x},\vec{y}) = G(\vec{x}, H(\vec{x}), \vec{y})$ and $F(\vec{x},\vec{y}) = G(H(\vec{x}),\vec{y})$; and this recursion schema:

$$F(z,\vec{x}) = G(\bigcup\{F(u,\vec{x}) \mid_u u \in z\}, z, \vec{x}).$$

7.0. Lemma. *Let A be transitive and primitive recursively closed, and let F be primitive recursive. Then*

$$\mathrm{val}_{\mathcal{G}}(\{(p, F(y)) \mid_{p,y} (p,y) \in x\}) = \{\mathrm{val}_{\mathcal{G}}(F(y)) \mid_y \exists p_{\in G}\ (p,y) \in x\}.$$

Proof. As before. $\blacksquare_{7.0}$

For notational simplicity there is only one x in the following, but it could easily be replaced by a finite sequence.

7.1. Proposition. *Let A be transitive and primitive recursively closed. Let $\mathbb{P} \in A$, and let $\mathcal{G}$ be $(A,\mathbb{P})$-generic. Suppose that $G(f,z,x)$ is primitive recursive, and that it has a nominator, $G^{\mathbb{P}}$, primitive recursive in the parameter $\mathbb{P}$, so that for all f,z,x in A,*

$$\mathrm{val}_{\mathcal{G}}(G^{\mathbb{P}}(f,z,x)) = G(\mathrm{val}_{\mathcal{G}}(f), \mathrm{val}_{\mathcal{G}}(z), \mathrm{val}_{\mathcal{G}}(x)).$$

Suppose that $F(z,x) = G(\bigcup\{F(u,x) \mid u \in z\}, z, x)$. Define $F^{\mathbb{P}}$ by

$$F^{\mathbb{P}}(z,x) = G^{\mathbb{P}}(\textstyle\bigcup^{\mathbb{P}}(\{(p, F^{\mathbb{P}}(u,x)) \mid_{p,u} (p,u) \in z\}), z, x).$$

Then $F^{\mathbb{P}}$ is primitive recursive in the parameter $\mathbb{P}$, and for all z,x in A,

$$\mathrm{val}_{\mathcal{G}}(F^{\mathbb{P}}(z,x)) = F(\mathrm{val}_{\mathcal{G}}(z), \mathrm{val}_{\mathcal{G}}(x));$$

so that $F^{\mathbb{P}}$ is a primitive recursive nominator for F.

Proof. For fixed x by recursion on z:

$$\begin{aligned}
&F(\mathrm{val}_{\mathcal{G}}(z), \mathrm{val}_{\mathcal{G}}(x))\\
&\quad = G\big(\bigcup\{F(w, \mathrm{val}_{\mathcal{G}}(x)) \mid_w w \in \mathrm{val}_{\mathcal{G}}(z)\}, \mathrm{val}_{\mathcal{G}}(z), \mathrm{val}_{\mathcal{G}}(x)\big)\\
&\quad = G\big(\bigcup\{F(\mathrm{val}_{\mathcal{G}}(u), \mathrm{val}_{\mathcal{G}}(x)) \mid_u \exists p_{\in \mathcal{G}}\ (p,u) \in z\}, \mathrm{val}_{\mathcal{G}}(z), \mathrm{val}_{\mathcal{G}}(x)\big)\\
&\quad = G\big(\bigcup\{\mathrm{val}_{\mathcal{G}}(F^{\mathbb{P}}(u,x)) \mid_u \exists p_{\in \mathcal{G}}\ (p,u) \in z\}, \mathrm{val}_{\mathcal{G}}(z), \mathrm{val}_{\mathcal{G}}(x)\big)\\
&\quad = G\big(\bigcup \mathrm{val}_{\mathcal{G}}(\{(p, F^{\mathbb{P}}(u,x)) \mid_{p,u} (p,u) \in z\}), \mathrm{val}_{\mathcal{G}}(z), \mathrm{val}_{\mathcal{G}}(x)\big)\\
&\quad = G\big(\mathrm{val}_{\mathcal{G}}(\textstyle\bigcup^{\mathbb{P}}(\{(p, F^{\mathbb{P}}(u,x)) \mid_{p,u} (p,u) \in z\})), \mathrm{val}_{\mathcal{G}}(z), \mathrm{val}_{\mathcal{G}}(x)\big)\\
&\quad = G^{\mathbb{P}}\big(\textstyle\bigcup^{\mathbb{P}}(\{(p, F^{\mathbb{P}}(u,x)) \mid_{p,u} (p,u) \in z\}), z, x\big)\\
&\quad = \mathrm{val}_{\mathcal{G}}(F^{\mathbb{P}}(z,x)).\ \blacksquare_{7.1}
\end{aligned}$$

The above confirms an observation made some years ago by Jensen:

7.2. Corollary. *A set-generic extension of a primitive recursively closed set is primitive recursively closed.*

8. Construction of nominators for the stages of a progress. Let e be a transitive set in the ground model of which $\mathbb{P}$ is a member, and let θ be indecomposable, exceeding $\eta =_{\mathrm{df}} \varrho(e)$. Then P^e_θ is provident. Let $\dot{d}$ be the Cohen term $\hat{e} \cup \{\dot{\mathcal{G}}\}^{\mathbb{P}}$, so that $\mathrm{val}_{\mathcal{G}}(\dot{d})$ will be the transitive set $d = e \cup \{\mathcal{G}\}$.

8.0. Remark. $\dot{d}$ will be a member of $P^e_{\varrho(\mathbb{P})+k}$ for some (small) k, given the definition of $\dot{\mathcal{G}}$, our convention that $\boldsymbol{1}^{\mathbb{P}} = 1$ and the fact that $\hat{\cdot}$ is $\mathbb{1}$-rud-rec.

Our task is to build for each $\nu < \theta$ a name $N(\nu)$ for the stage P^d_ν of the progress towards d.

A simplified progress. Now $\varrho(\mathcal{G}) \leqslant \varrho(\boldsymbol{P}) < \eta$, so that for $\nu \geqslant \eta$, $d_\nu = e_\nu \cup \{\mathcal{G}\}$. It might be that $\varrho(\mathcal{G}) < \varrho(\boldsymbol{P})$; to avoid building names which make allowance for that uncertainty, we shall build names for the terms of a slightly different progress $(Q^d_\nu)_\nu$.

8.1. Definition. For $\nu < \eta$,

$$Q^d_\nu = P^e_\nu, \qquad Q^d_\eta = P^e_\eta \cup \{\mathcal{G}\};$$

for $\nu \geqslant \eta$,

$$Q^d_{\nu+1} = \mathbb{T}(Q^d_\nu) \cup \{d_\nu\} \cup d_{\nu+1}, \qquad Q^d_\lambda = \bigcup_{\nu<\lambda} Q^d_\nu \quad \text{for } \lambda = \bigcup \lambda > \eta.$$

8.2. Proposition. *If θ is indecomposable, then Q^d_θ is provident and equals P^d_θ.*

Proof. By [MB, Theorem 6.34]. $\blacksquare_{8.2}$

Names using dynamic predicates. With that in mind, we now define names $N(\nu)$ such that $\mathrm{val}_{\mathcal{G}}(N(\nu)) = Q^d_\nu$.

8.3. Definition. $\dot{d}_\nu =_{\mathrm{df}} \widehat{e_\nu} \cup \{\dot{\mathcal{G}}\}^{\mathbb{P}}$ for $\nu \geqslant \eta$. Moreover, for $\nu < \eta$,

$$N(\nu) = \widehat{P^e_\nu}, \qquad N(\eta) = \widehat{P^e_\eta} \cup \{\dot{\mathcal{G}}\}^{\mathbb{P}};$$

for $\nu \geqslant \eta$,

$$N(\nu+1) = \mathbb{T}^{\mathbb{P}}(N(\nu)) \cup \{\dot{d}_\nu\} \cup \dot{d}_{\nu+1},$$
$$N(\lambda) = \bigcup\nolimits^{\mathbb{P}} \{(\boldsymbol{1}^{\mathbb{P}}, N(\nu)) \mid \nu < \lambda\} \quad \text{for } \lambda = \bigcup \lambda > \eta.$$

8.4. Lemma. *For $\nu \geqslant \eta$, $N(\nu) \in P^{e;=}_{\nu+\omega}$.*

Proof by cases. For $\nu = \eta$, by inspection; for successor ordinals, by knowledge of the birthday of $\mathbb{T}^{\mathbb{P}}$; for limit λ by knowledge of the delay of $\bigcup^{\mathbb{P}}$. $\blacksquare_{8.4}$

8.5. Proposition. *Each $N(\nu)$ for $\nu < \theta$ is in P^e_θ.*

Proof. All those names are in $P_\theta^{e;=}$, which was shown in Proposition 2.15 to equal P_θ^e. $\blacksquare_{8.5}$

8.6. PROPOSITION. *Let $\mathcal{G}$ be $(P_\theta^e, \mathbb{P})$ generic and let $\nu < \theta$. Then*

$$\mathrm{val}_{\mathcal{G}}(N(\nu)) = Q_\nu^d.$$

Proof. By induction on ν. $\blacksquare_{8.6}$

Pre-nominators for rud-rec functions

8.7. We may use the above idea to construct what one might call a pre-nominator of a rud-rec function.

Suppose that in $M[\mathcal{G}]$, F is given by G and the parameter $g = \mathrm{val}_{\mathcal{G}}(\gamma)$. Let $H = H_G$ be the rudimentary function ([MB, 5.9]) such that for transitive P and P^+ with $P^+ \subseteq \mathcal{P}(P)$, $F \restriction P^+ = H(g, F \restriction P, P^+)$.

Define for $\nu < \eta$, $E(\nu) = \widehat{F \restriction P_\nu^e}$; thereafter define

$$E(\nu + 1) = H^{\mathbb{P}}(\delta, E(\nu), N(\nu)),$$

and for limit λ, $E(\lambda) = \bigcup^{\mathbb{P}}\{(\mathit{1}^{\mathbb{P}}, E(\nu)) \mid_\nu \nu < \lambda\}$. Then an easy induction shows that for every ν, $\mathrm{val}_{\mathcal{G}}(E(\nu)) = F \restriction Q_{\nu+1}^d$.

8.8. REMARK. To show that the sequences $(N(\nu))_\nu$ and $(E(\nu))_\nu$ are $(\mathbf{P})$-gentle, one could repeat their definition and those of $(P_n^{e;=}u)_\nu$ and $\mathbb{T}^{\mathbb{P}}$ by a single simultaneous rudimentary recursion, a method seen in Definition 2.14 above and the proof of [MB, Proposition 4.19]. With a little extra work one would then get a $(\mathbf{P})$-gentle nominator for F.

9. Generic extensions of provident sets and of Jensen fragments. We are now in a position to prove the following theorem:

9.0. THEOREM. *Let θ be an indecomposable ordinal strictly greater than the rank of a transitive set e which contains the notion of forcing, $\mathbb{P}$. Let $\mathcal{G}$ be $(P_\theta^e, \mathbb{P})$-generic. Then $(P_\theta^e)^{\mathbb{P}}[\mathcal{G}] = P_\theta^{e\cup\{\mathcal{G}\}}$ and hence is provident.*

Proof. $(P_\theta^e)^{\mathbb{P}}[\mathcal{G}]$ contains $P_\theta^{e\cup\{\mathcal{G}\}}$, as we have for each $\nu < \theta$ built a name in P_θ^e that evaluates under $\mathcal{G}$ to $Q_\nu^{e\cup\{\mathcal{G}\}}$, and we know by Proposition 8.2 that $Q_\theta^{e\cup\{\mathcal{G}\}}$ equals $P_\theta^{e\cup\{\mathcal{G}\}}$.

For the converse direction, we know that $P_\theta^{e\cup\{\mathcal{G}\}}$ is provident, and has $\mathcal{G}$ as a member and hence can support the $\mathcal{G}$-rudimentary recursion defining $\mathrm{val}_{\mathcal{G}}(\cdot)$. Further $P_\theta^{e\cup\{\mathcal{G}\}}$ includes $(P_\nu^e)_\nu$, which is defined by an e-rudimentary recursion, and so includes $(P_\theta^e)^{\mathbb{P}}[\mathcal{G}]$. $\blacksquare_{9.0}$

REMARK. Thus, in this special case, a generic extension of a model of PROVI is a model of PROVI. We shall use this result to establish it more generally.

Remark. Theorem 9.0 remains true if the hypothesis on θ is weakened to requiring that $\theta > \varrho(\mathbb{P})$.

Proof that a generic extension of a provident set is provident

9.1. Theorem. *Let A be provident, $\mathbb{P} \in A$ and $\mathcal{G}$ $(A, \mathbb{P})$-generic. Then $A^{\mathbb{P}}[\mathcal{G}]$ is provident.*

Proof. Let $\theta =_{\mathrm{df}} On \cap A$ and let $T = \{c \mid c \in A \ \&\ c$ is transitive & $\mathbb{P} \in c\}$. Then

$$A = \bigcup \{P^c_\theta \mid c \in T\},$$

since the union on the right contains each element of A and is contained in A. It follows that

$$A^{\mathbb{P}}[\mathcal{G}] = \bigcup_{c \in T} (P^c_\theta)^{\mathbb{P}}[\mathcal{G}].$$

By Theorem 9.0, as each P^c_θ is provident and contains $\mathbb{P}$,

$$A^{\mathbb{P}}[\mathcal{G}] = \bigcup_{c \in T} P^{c \cup \{\mathcal{G}\}}_\theta$$

and each $P^{c \cup \{\mathcal{G}\}}_\theta$ is provident. Now [MB, Theorem 6.7 and Proposition 6.35] yields

Lemma. *If θ is indecomposable and D is a collection of transitive sets each of rank less than θ and such that the pair of any two is a member of a third, then $\bigcup_{d \in D} P^d_\theta$ is provident.*

Take $D = \{c \cup \{\mathcal{G}\} \mid c \in T\}$ to complete the proof. $\blacksquare_{9.1}$

Remark. Theorem 9.0 and Corollary 4.16 give the elegant characterization noted by Bowler, that if A is provident, $\mathbb{P} \in A$ and $\mathcal{G}$ is $(A, \mathbb{P})$-generic then $A^{\mathbb{P}}[\mathcal{G}] = \mathrm{Prov}(A \cup \{\mathcal{G}\})$.

Generic extensions of $\mathbb{P}$-provident sets. Our methods will prove the following more general result:

9.2. Theorem. *Let λ be a limit ordinal $\geqslant \omega^2$. Write η for the largest indecomposable not greater than λ. Let T be a set of transitive sets, and put $A = \bigcup_{c \in T} P^c_\lambda$ and $B = \bigcup_{c \in T} P^c_\eta$. Suppose that both A and B are closed under pairing. Then for any $\mathbb{P} \in B$, forcing with $\mathbb{P}$ is definable over A, and if $\mathcal{G}$ is $(A, \mathbb{P})$-generic, then $A[\mathcal{G}]$ is $\mathbb{P}$-provident; indeed $A[\mathcal{G}]$ will be q-provident for every q in $B[\mathcal{G}]$.*

Proof. In these circumstances, B will be provident, so forcing with $\mathbb{P}$ over B is by now well-established; each of the relations and functions $\chi_{=}$, $\chi_{\Vdash}$, $R^{\mathbb{P}}$ and $F^{\mathbb{P}}$ will progress through each P^c_λ for $c \in T$, and the closure of A under pairing shows that these functions and relations will be total on A. $\blacksquare_{9.2}$

Genericity at every limit level

9.3. Proposition. *Let e be a transitive set with $\mathbb{P} \in e$. Let θ be indecomposable, greater than $\varrho(e)$. Let λ be a limit ordinal not less than θ. Let $\kappa \geqslant \lambda + \omega$, and let $\mathcal{G}$ be $(P^e_\kappa, \mathbb{P})$-generic. Put $d = e \cup \{\mathcal{G}\}$; then d is also transitive of rank $< \theta$. Suppose that $P^e_\lambda[\mathcal{G}] = P^d_\lambda$. Then $P^e_{\lambda+\omega}[\mathcal{G}] = P^d_{\lambda+\omega}$.*

Proof. At this level, where we are above the rank of both e and d, $P^e_{\nu+1} = \mathbb{T}(P^e_\nu)$ and $P^d_{\nu+1} = \mathbb{T}(P^d_\nu)$.

(i) $P^d_\lambda = \mathrm{val}_{\mathcal{G}}(P^e_\lambda)$: for as λ is a limit ordinal, P^e_λ is rud-closed. Hence
(ii) $\mathbb{T}(P^d_\lambda) = \mathrm{val}_{\mathcal{G}}(\mathbb{T}^{\mathbb{P}}(P^e_\lambda)) \in P^e_{\lambda+\omega}[\mathcal{G}]$ as, $\mathbb{T}^{\mathbb{P}}$ being (P^e_θ)-gentle, $\mathbb{T}^{\mathbb{P}}(P^e_\lambda) \in P^e_{\lambda+\omega}$. Iterating $\mathbb{T}$, we see that $P^d_{\lambda+\omega} \subseteq P^e_{\lambda+\omega}[\mathcal{G}]$.
(iii) In the other direction, both e and $\mathcal{G}$ are in P^d_θ; as $\theta \leqslant \lambda$, $P^d_{\lambda+\omega}$ will be both e- and $\mathcal{G}$-provident; so $P^e_{\lambda+\omega} \subseteq P^d_{\lambda+\omega}$, and therefore as $\mathrm{val}_{\mathcal{G}}(\cdot)$ is $\mathcal{G}$-rud-rec, $P^e_{\lambda+\omega}[\mathcal{G}] \subseteq P^d_{\lambda+\omega}$. $\blacksquare_{9.3}$

Consider now the special case that $e = T_\eta$, a fragment of the constructible hierarchy, and let ξ be the least indecomposable not less than η. The above argument yields

9.4. Lemma. *Let $\mathbb{P} \in T_\eta$. Suppose that ν is a limit ordinal $\geqslant \xi$, that $\mathcal{G}$ is $(T_{\nu+\omega}, \mathbb{P})$-generic, and that $T^{\mathbb{P}}_\nu[\mathcal{G}] = P^d_\nu$, where $d = T_\eta \cup \{\mathcal{G}\}$. Then $T^{\mathbb{P}}_{\nu+\omega}[\mathcal{G}] = P^d_{\nu+\omega}$.*

Then, reverting to Jensen's notation, that gives us the following

9.5. Theorem. *Let $\mathbb{P} \in J_\xi$, where ξ is indecomposable. Let $\mathcal{G}$ be $(L, \mathbb{P})$-generic. Then for each ordinal $\zeta \geq \xi$, $J_\zeta(\mathcal{G}) = J^{\mathbb{P}}_\zeta[\mathcal{G}]$; in particular each set in $J_\zeta(\mathcal{G})$ is $\mathrm{val}_{\mathcal{G}}(a)$ for some $a \in J_\zeta$.*

Here is an application, which fleshes out an argument outlined in a letter from Sy Friedman.

9.6. Proposition. *Let $\theta < \eta \leqslant \zeta < \xi$ be ordinals, with η indecomposable. Suppose that $\mathbb{P} \in J_\eta$ and that $\mathcal{G}$ is $(J_\xi, \mathbb{P})$-generic. Let $x \subseteq \theta$, with $x \in J_\xi$ and $x \in J_\zeta(\mathcal{G})$. Then $x \in J_\zeta$.*

Proof. We have $\hat{x} \in J_\xi$. Since $J^{\mathbb{P}}_\zeta[\mathcal{G}] = J_\zeta(\mathcal{G})$, $x = \mathrm{val}_{\mathcal{G}}(y)$ for some $y \in J_\zeta$. Therefore some condition p in $\mathcal{G}$ forces $\underline{y} = \underline{\hat{x}}$; so $x = \{\nu < \theta \mid p \Vdash \underline{\hat{\nu}} \in \underline{y}\}$. The map $\nu \mapsto \hat{\nu}$, restricted to the ν less than θ, is in $J_{\theta+1}$, and the relation $p \Vdash \underline{\hat{\nu}} \in \underline{z}$ is rudimentary in $\chi_=$; an appropriate segment of that characteristic function is in J_ζ, by propagation starting from η; and therefore $x \in J_\zeta$. $\blacksquare_{9.6}$

A similar argument leads to an extension of a result in [MB]:

9.7. Proposition. *If a set c is such that there is a progress $(Q^c_\nu)_\nu$ given by a c-rudimentary recursion with $Q^c_\lambda = L_\lambda$ for every limit $\lambda \geqslant \omega$, then c is not set-generic over L.*

Proof. Suppose to the contrary that $\mathcal{G}$ is $(L, \mathbb{P})$-generic where $\mathbb{P} \in L$ and $c \in L^{\mathbb{P}}[\mathcal{G}]$. Let η be indecomposable with $\mathbb{P} \in L_\eta$, $c \in L_\eta^{\mathbb{P}}[\mathcal{G}]$, and $\eta \geqslant \omega^\omega$, so that $\eta = \omega\eta$, and therefore $L_\eta = J_\eta = T_\eta$. Let $\lambda = \eta + \omega$.

9.8. REMARK. (i) $T_\lambda^{\mathbb{P}}[\mathcal{G}]$ is c-provident and so includes $Q_\lambda^c = L_\lambda$;

(ii) so if $A \in L_\lambda$, $A = \mathrm{val}_{\mathcal{G}}(a)$ for some $a \in T_\lambda$, and there will be a condition p in the generic filter $\mathcal{G}$ with $p \Vdash^{\mathbb{P}} \underline{\hat{A}} = \underline{a}$;

(iii) for such p and any $z \in L$, $p \Vdash^{\mathbb{P}} \underline{\hat{z}} \epsilon \underline{a} \iff z \in A$.

From our analysis in §2 of the forcing relation, we know that the characteristic function of the predicate $p \Vdash^{\mathbb{P}} \underline{\hat{z}} \epsilon \underline{a}$ is $\mathbb{P}$-gentle; and therefore from the discussion in [MB, §6] of the rate of convergence of a gentle function in a progress—see in particular the proof of [MB, Theorem 6.38]—we know the following:

9.9. PROPOSITION. *There is a natural number s such that if $(P_m)_{m<\omega}$ is a strict progress, with $\mathbb{P} \in P_0$, P_0 $\mathbb{P}$-provident, $p \in \mathbb{P}$, then for any n and any $a \in P_n$,*

$$P_n \cap \{z \mid_z p \Vdash^{\mathbb{P}} \underline{\hat{z}} \epsilon \underline{a}\} \in P_{(n+1)s}.$$

9.10. LEMMA. *There is a primitive recursive function $f : \omega \times \omega \to \omega$ such that for each $k, n < \omega$, and $A \in L$, if $A \subseteq L_{\eta+k}$ and $A = \mathrm{val}_{\mathcal{G}}(a)$ for some $a \in T_{\eta+n}$ then $A \in T_{\eta+f(k,n)}$.*

Proof. Case $k = 0$. We suppose that $A \subseteq L_\eta = T_\eta$ and for some $a \in T_{\eta+n}$, $A = \mathrm{val}_{\mathcal{G}}(a)$.

Then there is a condition $p \in \mathcal{G}$ with $p \Vdash^{\mathbb{P}} \hat{A} = a$. So, using Remark (iii) and applying the proposition to the strict progress $(T_{\eta+m})_{m<\omega}$, we shall have

$$A = T_{\eta+n} \cap \{z \mid_z p \Vdash^{\mathbb{P}} \underline{\hat{z}} \epsilon \underline{a}\} \in T_{\eta+(n+1)s}.$$

Therefore we may set $f(0, n) = (n + 1)s$.

Case $k = \ell+1$. We suppose that $A \subseteq L_{\eta+\ell+1}$ and that for some $a \in T_{\eta+n}$, $A = \mathrm{val}_{\mathcal{G}}(a)$.

Then there is a condition $p \in \mathcal{G}$ with $p \Vdash^{\mathbb{P}} \hat{A} = a$.

Let $B \in A$. Then $B \subseteq L_{\eta+\ell}$ and for some $b \in T_{\eta+n}$, $B = \mathrm{val}_{\mathcal{G}}(b)$.

[Actually b will be in $\bigcup\bigcup a$ and therefore in $T_{\eta+n-3}$, if $n \geqslant 3$. So we are giving a little ground: unimportant in the present case, but perhaps a fact to be stored for the future.]

So by the case $k = \ell$, $B \in T_{\eta+f(\ell,n)}$. Thus

$$A = T_{\eta+f(\ell,n)} \cap \{z \mid_z p \Vdash^{\mathbb{P}} \underline{\hat{z}} \epsilon \underline{a}\} \in T_{\eta+(f(\ell,n)+1)s}.$$

So we may set $f(\ell + 1, n) = (f(\ell, n) + 1)s$. $\blacksquare_{9.10}$

The lemma is proved, and Remark 9.8(ii), implies, under the current hypotheses on c, that $L_\lambda \subseteq T_\lambda$; but that is false, as $L_{\eta+1} \in L_\lambda$ but

$L_{\eta+1} \notin T_\lambda = J_{\eta+1}$, by the proof of [MB, Proposition 6.68]. Thus c cannot be set-generic over L. $\blacksquare_{9.7}$

10. Persistence of certain systems. We suppose throughout that we are forcing over a provident set (or model, meaning possibly ill-founded) M by some $\mathbb{P} \in M$. Our first task is for each of the schemes of foundation, collection and separation, to show that if they hold in M then in $M[\mathcal{G}]$. We call that the phenomenon of *persistence*. Establishing the persistence of a given axiom divides into two steps: first, showing that the axiom is forced; and second, showing that what is forced is true in the generic extension. The first step generally uses the truth of the axiom concerned, possibly in an apparently stronger form, in the ground model. The second step succeeds if the generic meets all appropriately definable dense subclasses of $\mathbb{P}$, which it will do given the truth in the ground model of sufficiently strong instances of the separation scheme.

Extension of the definition of forcing to all formulæ. We may extend the definition of forcing, schematically, to all formulæ, thus:

10.0. Definition. $p \Vdash \bigwedge \mathfrak{x}\, \dot{\Phi} \iff \forall x\, p \Vdash \dot{\Phi}(\underline{x})$.

10.1. Proposition. $p \Vdash \bigvee \mathfrak{x}\, \Phi \iff \forall q_{\le p} \exists r_{\le q} \exists x\; r \Vdash \Phi(\underline{x})$.
$p \Vdash \bigvee \mathfrak{x}_{\epsilon \underline{y}}\, \Phi \iff \forall q_{\le p} \exists r_{\le q} \exists (t, \beta)_{\in y}\, (r \le t\ \&\ r \Vdash \dot{\Phi}(\underline{\beta}))$.

10.2. Proposition. $p \Vdash \Phi \iff \forall q_{\le p} \exists r_{\le q}\; r \Vdash \Phi$.
$\Vdash \underline{x} = \underline{\alpha} \wedge \Phi(\underline{\alpha}) \to \Phi(\underline{x})$.

Proof. Already proved for atomic wffs, an easy induction thereafter. $\blacksquare_{10.2}$

10.3. Exercise. Show that if $p \Vdash \bigwedge \mathfrak{x}\, (\Phi \to \Psi(\mathfrak{x}))$ and Φ is a sentence of $\mathcal{L}^{\mathbb{P}}$ (so that, intuitively, $\mathfrak{x}$ has no free occurrence in Φ), then $p \Vdash \Phi \to \bigwedge \mathfrak{x}\, \Psi(\mathfrak{x})$.

This exercise, coupled with our remarks about *Modus Ponens* above, ensures that we may apply mathematical reasoning to statements in our forcing language.

Complexity of some classes of formulæ. For a class Γ of wffs, such as $\Sigma_{\mathfrak{k}}$ or $\Pi_{\mathfrak{k}}$, let $\Sigma_1\Gamma$ be the class of wffs of the form $\exists x\, \Psi$ where $\Psi \in \Gamma$; let $\Delta_0\Gamma$ be the class of those wffs where a formula from Γ is preceded by a finite string of restricted quantifiers; and let $\Vdash \Gamma$ be the class of wffs of the form $p \Vdash^{\mathbb{P}} \dot{\Phi}$ where Φ is in Γ.

10.4. Some computations for $\varphi\ \dot{\Delta}_0$:

(10.4.0) $p \Vdash^{\mathbb{P}} \varphi$ is a $\mathbb{P}$-gentle predicate of p and ϕ, and therefore $\Delta_1^{\mathsf{PROVI}}$;
(10.4.1) $p \Vdash^{\mathbb{P}} \bigwedge \mathfrak{x}\, \varphi(\mathfrak{x})$ is $\forall x\, p \Vdash^{\mathbb{P}} \varphi[\underline{x}]$, and thus Π_1^{PROVI};
(10.4.2) $p \Vdash^{\mathbb{P}} \bigvee \mathfrak{x}\, \varphi(\mathfrak{x})$ is $\forall q_{\le p} \exists r_{\le q} \exists x\; r \Vdash^{\mathbb{P}} \varphi[\underline{x}]$, and thus $\Delta_0\Sigma_1^{\mathsf{PROVI}}$;

(10.4.3) $p \Vdash^{\mathbb{P}} \bigwedge \mathfrak{x}_{\epsilon a} \bigvee \mathfrak{y}\ \varphi$ is $\forall x_{\in \bigcup^2 a} \forall q_{\leq p} \exists r_{\leq q} \exists t\ \big(q \Vdash_1 \underline{x} \epsilon \underline{a} \Rightarrow r \Vdash^{\mathbb{P}} \varphi[\underline{t}, \underline{x}]\big)$, and thus $\Delta_0 \Sigma_1^{\mathsf{PROVI}}$ and Σ_1^{KPI};

(10.4.4) $p \Vdash^{\mathbb{P}} \bigwedge \mathfrak{y} \bigvee \mathfrak{x}\ \varphi(\mathfrak{x}, \mathfrak{y})$ is $\forall y \forall q_{\leq p} \exists r_{\leq q} \exists x\ r \Vdash^{\mathbb{P}} \varphi[\underline{x}, \underline{y}]$, and thus Π_2^{PROVI}.

More generally for positive $\mathfrak{k}$, all $\Vdash \Pi_{\mathfrak{k}}$ predicates are $\Pi_{\mathfrak{k}}^{\mathsf{PROVI}}$, as Δ_0 quantifiers such as $\forall q_{\leq p} \exists r_{\leq q}$ will usually be interpolated between successive unrestricted quantifiers $\forall\exists$ of the $\Pi_{\mathfrak{k}}$ formula, and thus absorbed, while $\Vdash \Sigma_{\mathfrak{k}}$ predicates will be $\Delta_0 \Sigma_{\mathfrak{k}}^{\mathsf{PROVI}}$.

When appropriate forms of Collection hold in M, those initial restricted quantifiers can also be absorbed; for example Δ_0-collection implies that every $\Delta_0\Sigma_1$ formula is equivalent to a Σ_1 formula, so that $\Vdash^{\mathbb{P}} \Sigma_1$ predicates are Σ_1^{KPI}. Thus (in KP) if $\mathbb{P}$ is a set, $\Vdash^{\mathbb{P}}$ restricted to Σ_1 wffs will be Σ_1; and in ZF, forcing for Σ_n wffs is Σ_n.

Remark. Initial restricted quantifiers can also be absorbed in contexts such as $V = L$ when the lemma of Sy Friedman discussed in §5 of [M2] holds.

Completeness and consistency. By the *consistency* of forcing we shall mean the principle that no condition forces both a formula and its negation; by the *completeness* of forcing we shall mean the principle that every condition can be refined to one that decides a given formula.

The consistency and completenes of forcing are maintained by our system of definitions as we progressively enlarge the definition of forcing to wider classes of formulæ: they follow from the way forcing handles negation.

Problem. Let us take $\mathcal{G}$-completeness to mean that each statement is decided by some $p \in \mathcal{G}$. The class of conditions that decide is certainly dense; but is it a set?

$\boldsymbol{P} \cap \{p \mid p \Vdash \Phi \vee p \Vdash \neg\Phi\}$ will be a set if we have enough separation. If Φ is $\dot{\Delta}_0$, it will be a set by gentle separation. For other Φ, it will be a set if we have both $(\Vdash \Phi)$-separation and $(\Delta_0 \neg \Vdash \Phi)$-separation.

10.5. Remark. There is a point to be made here, similar to the problem of defining truth for all formulæ. We have defined the forcing relation $\chi_{\Vdash}$ for all $\dot{\Delta}_0$ sentences φ of our formal language, essentially by a single recursion, so that every set is a member of a transitive set u such that $\chi_{\Vdash} \restriction \mathcal{E}^u$ is total and a set. A definition of truth for a single $\dot{\Delta}_0$ sentence is achieved by recursion over the transitive closure of the sets named in that sentence; in the case of forcing we must start our recursion from the transitive closure of the union of $\{\mathbb{P}\}$ and some set containing each set a of which the name $\underline{a}$ occurs in the sentence.

But we are not able to make a single definition for all formulæ with arbitrarily many unrestricted quantifiers, but must introduce a sequence of definitions schematically. This would become very apparent in the Boolean-valued presentation of forcing, where truth values are assigned in a complete Boolean algebra, and we must invoke the axiom of replacement for each quantifier to see that the supremum over a class is actually the supremum over a set.

Persistence of Foundation

10.6. Proposition. $(\Delta_0 \Vdash^{\mathbb{P}} \Gamma)$-*foundation implies that* Γ-*foundation is forced.*

Proof. Let Φ be in Γ. Let $A =_{\mathrm{df}} \{a \mid \exists p_{\in \boldsymbol{P}}\, p \Vdash^{\mathbb{P}} \Phi[\underline{a}]\}$. Then A is $(\Delta_0 \Vdash^{\mathbb{P}} \Phi)$.

Form $C =_{\mathrm{df}} \{c \mid c \in A \vee \exists d_{\in c}\, d \in A \vee \exists d_{\in c} \exists e_{\in d}\, e \in A\}$. Then C is also $(\Delta_0 \Vdash^{\mathbb{P}} \Gamma)$, so, assuming $A \neq \varnothing$, let c be C-minimal. Then for some p, $p \Vdash^{\mathbb{P}} \Phi[\underline{c}]$, but p will also force that c is Φ-minimal. $\blacksquare_{10.6}$

Persistence of Collection. It is convenient to take Collection in the form $\forall x_{\in a} \exists y\, \Phi \ \Rightarrow\ \exists b \forall x_{\in a} \exists y_{\in b}\, \Phi$.

Suppose that $p \Vdash \bigwedge \mathfrak{x}_{\epsilon \underline{a}} \bigvee \mathfrak{y}^{\cdot}\, \Phi$. Let $B(p,s) =_{\mathrm{df}} \boldsymbol{P} \cap \{q \mid_q q \leq p \,\&\, q \leq s)$. Note that B is $\mathbb{P}$-rudimentary. Then

$$\forall (s,\beta)_{\in a} \forall q_{\in B(p,s)} \exists r_{\leq q} \exists y\, r \Vdash^{\mathbb{P}} \dot{\Phi}(\underline{y}, \underline{\beta}).$$

After massaging the displayed formula and applying $(\Vdash \Phi)$-collection we shall reach a b such that

$$\forall (s,\beta)_{\in a} \forall q_{\in B(p,s)} \exists r_{\leq q} \exists y_{\in b}\, r \Vdash^{\mathbb{P}} \dot{\Phi}(\underline{y}, \underline{\beta}).$$

Let $w = \{(\boldsymbol{1}^{\mathbb{P}}, y)_2 \mid_y y \in b\}$. Then $p \Vdash \bigwedge \mathfrak{x}_{\epsilon \underline{a}} \bigvee \mathfrak{y}_{\epsilon \underline{w}}\, \dot{\Phi}$, as required. We have proved:

10.7. Proposition. $(\Vdash \Gamma)$-*collection implies that* Γ-*collection is forced.*

Remark. Collection is more natural than replacement in the context of forcing.

Persistence of Separation. Given a and $\dot{\Phi}$, let $b = (\boldsymbol{P} \times \bigcup^2 a) \cap \big\{(p,\alpha)_2 \ \big|_{p,\alpha}\ p \Vdash \dot{\Phi}[\underline{\alpha}]\big\}$. By $(\Delta_0 \Vdash \dot{\Phi})$-separation, b is a set. I assert that $\Vdash b = \{\mathfrak{x} \epsilon \underline{a} \mid_{\mathfrak{x}} \dot{\Phi}(\mathfrak{x})\}$. Note first that (trivially) if $q \Vdash_0 \underline{\alpha} \epsilon \underline{b}$ then $q \Vdash \dot{\Phi}[\underline{\alpha}]$. Proposition 3.13 tells us that if $q \Vdash \underline{\alpha} \epsilon \underline{b}$ then $q \Vdash \dot{\Phi}[\underline{\alpha}]$, so that $\Vdash \underline{\alpha} \epsilon \underline{b} \to \dot{\Phi}[\underline{\alpha}]$.

In the other direction, suppose that $p \Vdash \underline{\alpha} \epsilon \underline{a} \wedge \dot{\Phi}[\underline{\alpha}]$; then $\forall q_{\leq p} \exists r_{\leq q} \exists \gamma\, (r \Vdash_1 \underline{\gamma} \epsilon \underline{a} \,\&\, r \Vdash \underline{\gamma} = \underline{\alpha})$; so $\gamma \in \bigcup^2 a \,\&\, r \Vdash \dot{\Phi}[\underline{\gamma}]$; so $(r,\gamma)_2 \in b$ so $r \Vdash_0 \underline{\gamma} \in \underline{b}$; so as $r \Vdash \underline{\gamma} = \underline{\alpha}$, $r \Vdash \underline{\alpha} \in \underline{b}$. By density, $p \Vdash \underline{\alpha} \epsilon \underline{b}$.

Hence $\Vdash (\underline{\alpha} \epsilon \underline{a} \wedge \dot{\Phi}[\underline{\alpha}]) \leftrightarrow \underline{\alpha} \epsilon \underline{b}$ and $\Vdash b = \{\mathfrak{x} \epsilon \underline{a} \mid_{\mathfrak{x}} \dot{\Phi}(\mathfrak{x})\}$. We have proved:

10.8. Proposition. $(\Delta_0 \Vdash \Gamma)$*-separation implies that* Γ*-separation is forced.*

Towards the Forcing Theorem

10.9. Definition. For a formula Φ, let $\mathsf{FT}(\Phi)$ be the principle that "what is forced is true": namely that if Φ is forced by some $p \in \mathcal{G}$, then Φ is true in $M[\mathcal{G}]$; let $\mathsf{TF}(\Phi)$ be the principle that "what is true is forced", namely that if Φ is true in $M[\mathcal{G}]$, then Φ is forced by some $p \in \mathcal{G}$; and let $\mathsf{TFT}(\Phi)$, the Forcing Theorem for Φ, be the conjunction of $\mathsf{FT}(\Phi)$ and $\mathsf{TF}(\Phi)$.

Let $\mathsf{FT}(\Gamma)$ mean that $\mathsf{FT}(\Phi)$ holds for each Φ in Γ, and similarly for $\mathsf{TF}(\Gamma)$ and $\mathsf{TFT}(\Gamma)$.

10.10. Remark. Proposition 4.5 proved TFT for atomic wffs, and Theorem 4.17 proved $\mathsf{TFT}(\dot{\Delta}_0)$.

Propagating FT and TF: the rôle of Separation

10.11. Theorem. *For any* $\mathfrak{k}$*:*

(10.11.0) *if* $\mathsf{TF}(\Pi_{\mathfrak{k}})$ *then* $\mathsf{TF}(\Sigma_{\mathfrak{k}+1})$*;*
(10.11.1) *if* $\mathsf{FT}(\Pi_{\mathfrak{k}})$ *and* $\mathcal{G}$ *meets every* $(\Sigma_1 \Vdash \Pi_k)^M$*-subclass of conditions, then* $\mathsf{FT}(\Sigma_{\mathfrak{k}+1})$.

Proof. Let $\Psi(\cdot,\cdot)$ be $\Pi_{\mathfrak{k}}$. If $M[\mathcal{G}] \models \bigvee\mathfrak{y}\ \Psi(\mathfrak{y}; \mathrm{val}_{\mathcal{G}}(x))$ then $\exists y_{\in M}\, M[\mathcal{G}] \models \Psi[\mathrm{val}_{\mathcal{G}}(y), \mathrm{val}_{\mathcal{G}}(x)]$; so by $\mathsf{TF}(\Pi_{\mathfrak{k}})$ some p in $\mathcal{G}$ forces $\Psi[\underline{y}, \underline{x}]$; and then this p trivially forces $\bigvee\mathfrak{y}\,\Psi(\mathfrak{y}; \underline{x}]$. Thus (10.11.0) is proved.

If $p_0 \in \mathcal{G}$ forces $\bigvee\mathfrak{y}\ \Psi(\mathfrak{y}; \underline{x}]$ then the class $E =_{\mathrm{df}} \{p \mid \exists y\ p \Vdash \Psi(\underline{y}; \underline{x}]\}$, is dense below p_0; E is definable in M by a $(\Sigma_1 \Vdash \Pi_{\mathfrak{k}})$ formula, and therefore meets $\mathcal{G}$. Let $p \in E \cap \mathcal{G}$. Then for some $y \in M$, $p \Vdash \Psi[\underline{y}, \underline{x}]$, so by $\mathsf{FT}(\Pi_{\mathfrak{k}})$, $M[\mathcal{G}] \models \Psi[\mathrm{val}_{\mathcal{G}}(y), \mathrm{val}_{\mathcal{G}}(x)]$, whence $M[\mathcal{G}] \models \bigvee\mathfrak{y}\ \Psi(\mathfrak{y}; \mathrm{val}_{\mathcal{G}}(x)]$, proving (10.11.1). $\blacksquare_{10.11}$

10.12. Proposition. *For any* Φ*:*

(10.12.0) *if* $\mathcal{G}$ *decides* Φ, $\mathsf{FT}(\Phi)$ *implies* $\mathsf{TF}(\neg\Phi)$*;*
(10.12.1) $\mathsf{TF}(\Phi)$ *implies* $\mathsf{FT}(\neg\Phi)$

Proof. Suppose $\mathsf{FT}(\Phi)$, and that $\neg\Phi$ is true in $M[\mathcal{G}]$; then Φ is not true there, so not $\mathcal{G}$-forced, so something in $\mathcal{G}$ forces $\neg\Phi$, by $\mathcal{G}$-completeness.

Suppose $\mathsf{TF}(\Phi)$, and that $\neg\Phi$ is forced; then (by $\mathcal{G}$-consistency, which is automatic) nothing in $\mathcal{G}$ forces Φ, so Φ is not true, so $\neg\Phi$ is true. $\blacksquare_{10.12}$

10.13. Corollary. *For any* $\mathfrak{k}$*:*

(10.13.0) $\mathsf{TF}(\Sigma_{\mathfrak{k}})$ *implies* $\mathsf{FT}(\Pi_{\mathfrak{k}})$*;*

(10.13.1) *if* $\mathcal{G}$ *decides all* $\Sigma_{\mathfrak{k}}$ *statements,* $\mathsf{FT}(\Sigma_{\mathfrak{k}})$ *implies* $\mathsf{TF}(\Pi_{\mathfrak{k}})$*.*

We get the following pattern of implications, where $\Rightarrow$ indicates that there is no extra assumption on $\mathcal{G}$, and $\rightarrow$ indicates that there is, $\mathcal{G}\mathrm{m}\Sigma_{\mathfrak{k}+1}$ meaning that $\mathcal{G}$ is assumed to meet every dense $(\Sigma_1 \Vdash^{\mathbb{P}} \Pi_{\mathfrak{k}})^M$ subclass of conditions and $\mathcal{G}\mathrm{d}\Sigma_{\mathfrak{k}}$ meaning that $\mathcal{G}$ is assumed to decide every $\Sigma_{\mathfrak{k}}$ statement.

$$\mathsf{TF}(\Pi_0) \;\Rightarrow\; \mathsf{TF}(\Sigma_1) \;\Rightarrow\; \mathsf{FT}(\Pi_1) \underset{\mathcal{G}\mathrm{m}\Sigma_2}{\rightarrow} \mathsf{FT}(\Sigma_2) \underset{\mathcal{G}\mathrm{d}\Sigma_2}{\rightarrow} \mathsf{TF}(\Pi_2) \;\Rightarrow\; \mathsf{TF}(\Sigma_3) \ldots$$

$$\mathsf{FT}(\Pi_0) \underset{\mathcal{G}\mathrm{m}\Sigma_1}{\rightarrow} \mathsf{FT}(\Sigma_1) \underset{\mathcal{G}\mathrm{d}\Sigma_1}{\rightarrow} \mathsf{TF}(\Pi_1) \;\Rightarrow\; \mathsf{TF}(\Sigma_2) \;\Rightarrow\; \mathsf{FT}(\Pi_2) \underset{\mathcal{G}\mathrm{m}\Sigma_3}{\rightarrow} \mathsf{FT}(\Sigma_3) \ldots$$

$$\mathsf{TFT}(\dot{\Delta}_0) \underset{\mathcal{G}\mathrm{m}\Sigma_1}{\rightarrow} \mathsf{TFT}(\Sigma_1) \underset{\mathcal{G}\mathrm{d}\Sigma_1}{\rightarrow} \mathsf{TFT}(\Pi_1) \underset{\mathcal{G}\mathrm{m}\Sigma_2}{\rightarrow} \mathsf{TFT}(\Sigma_2) \underset{\mathcal{G}\mathrm{d}\Sigma_2}{\rightarrow} \mathsf{TFT}(\Pi_2) \underset{\mathcal{G}\mathrm{m}\Sigma_3}{\rightarrow} \mathsf{TFT}(\Sigma_3) \ldots$$

10.14. Remark. Suppose that $\mathcal{G}$ meets every dense member of M, a model of Provi. Then "$\mathcal{G}$ meets every dense $(\Sigma_1 \Vdash \Pi_{\mathfrak{k}})$ subclass of $\boldsymbol{P}$" will hold if $\Sigma_{\mathfrak{k}+1}$-separation holds in M; and $\mathcal{G}$ will decide all $\Sigma_{\mathfrak{k}}$ statements if $\Delta_0\Sigma_{\mathfrak{k}}$-separation holds in M, or (when $\mathfrak{k} > 0$), if $\Sigma_{\mathfrak{k}}$-separation and $\Pi_{\mathfrak{k}-1}$-collection hold in M.

10.15. Corollary. *Any set-generic extension of a provident set which models full separation will also be provident and model full separation; moreover the forcing theorem will hold for all formulæ.*

Proof. 10.8 tells us that all instances of the separation scheme are forced. Then 10.14 with 10.11 will tell us that $\mathsf{TFT}(\Pi_{\mathfrak{k}})$ implies $\mathsf{TFT}(\Sigma_{\mathfrak{k}+1})$, and 10.14 with 10.13 will tell us that $\mathsf{TFT}(\Sigma_{\mathfrak{k}})$ implies $\mathsf{TFT}(\Pi_{\mathfrak{k}})$. $\mathsf{TFT}(\Pi_0)$ is $\mathsf{TFT}(\dot{\Delta}_0)$ which was Theorem 4.17, so we have TFT for all formulæ of the forcing language; so that all instances of the separation scheme are true in the extension. $\blacksquare_{10.15}$

Persistence of the power set axiom and of provident Zermelo set theory. The persistence of the power set axiom, among others, is proved in [M4, §4]. Combining that result with those above, we get

10.16. Theorem. *Any set-generic extension of a provident set which models* Z *will also be provident and model* Z*; and the forcing theorem will hold for all wffs.*

Summary. Collection for Φ in the extension follows from $(\Vdash \Phi)$-collection in M with $\mathsf{TF}(\Delta_0\Sigma_1\Phi)$ and $\mathsf{FT}(\Delta_0\Phi)$.

Foundation for Φ in the extension follows from $(\Delta_0 \Vdash \Phi)$-foundation in M with $\mathsf{TFT}(\Phi)$.

Separation for Φ in the extension follows from $(\Delta_0 \Vdash \dot{\Phi})$-separation in M with $\mathsf{TFT}(\Phi)$.

The persistence of KPI

10.17. THEOREM. *If M is admissible, $\mathbb{P} \in M$ and G is an $(M, \mathbb{P})$-generic filter meeting each dense open subclass of M that is the union of a $\Sigma_1(M)$ and a $\Pi_1(M)$ class, then $M^{\mathbb{P}}[G]$ is admissible.*

Proof. The system KPI, as presented in [M2], may be obtained by adding to the axioms of PROVI the schemes of Π_1-foundation and Δ_0-collection. As we have proved the persistence of PROVI, it only remains to discuss those two schemes.

Let us first remark that the hypotheses imply that $\mathsf{FT}(\Sigma_1)$ is available, since every $(\Sigma_1 \Vdash^{\mathbb{P}} \dot{\Delta}_0)^M$ class is Σ_1^M, since M is admissible.

Suppose you have Δ_0-collection in the ground model. For $\Phi\ \dot{\Delta}_0$, $\Vdash^{\mathbb{P}} \Phi$ is a gentle predicate, hence $\Sigma_1^{\mathsf{PROVI}}$, and Σ_1-collection follows from Δ_0-collection in PROVI. So from Proposition 10.7, we know that $\dot{\Delta}_0$-collection is forced. It remains to show that it is true in the extension.

Suppose therefore that Ψ is Δ_0, that $\mathbf{a} = \mathrm{val}_{\mathcal{G}}(a) \in M[G]$ (3) and that

$$M[G] \models \bigwedge\mathfrak{x}_{\epsilon\tilde{\mathbf{a}}} \bigvee\mathfrak{y}\ \Psi(\mathfrak{x}, \mathfrak{y}).$$

Let

$$\Delta =_{\mathrm{df}} \{p \mid p \Vdash^{\mathbb{P}} \bigwedge\mathfrak{x}_{\epsilon\underline{a}} \bigvee\mathfrak{y}\ \Psi \vee \exists(q,\alpha)_{\in a}\ p \le q\ \&\ p \Vdash^{\mathbb{P}} \bigwedge\mathfrak{y}\ \neg\Psi(\underline{\alpha}, \mathfrak{y}).$$

Then Δ is dense open, and is the union of a Σ_1^{KP} class and a Π_1^{KP} class. By hypothesis, $G \cap \Delta \neq 0$.

Hence there must be a $p \in G$ such that $p \Vdash^{\mathbb{P}} \bigwedge\mathfrak{x}_{\epsilon\underline{a}} \bigvee\mathfrak{y}\ \Psi$, the other half of the dense set being excluded by our assumption on $M[G]$. But we know from Proposition 10.7 that then $p \Vdash^{\mathbb{P}} \bigvee\mathfrak{v} \bigwedge\mathfrak{x}_{\epsilon\underline{a}} \bigvee\mathfrak{y}_{\epsilon\mathfrak{v}}\ \Psi(\mathfrak{x}, \mathfrak{y})$; as $p \in G$, and the statement being forced is Σ_1, $\mathsf{FT}(\Sigma_1)$ tells us that $M^{\mathbb{P}}[G] \models \bigvee\mathfrak{v} \bigwedge\mathfrak{x}_{\epsilon\tilde{\mathbf{a}}} \bigvee\mathfrak{y}_{\epsilon\mathfrak{v}}\ \Psi(\mathfrak{x}, \mathfrak{y})$, as required.

Let Φ be Π_1. Suppose you have Π_1-foundation in the ground model. By (10.4.1) using Δ_0-collection, that gives $(\Delta_0 \Vdash \Pi_1)$-foundation there, using which we have seen how to construct a name c for a member of the extension that is forced to be Φ-minimal, so that $\Phi[\underline{c}]$ and $\forall\mathfrak{x}_{\epsilon\underline{c}}\ \neg\Phi(\mathfrak{x})$ are both forced by some member of $\mathcal{G}$. To complete the proof we use $\mathsf{FT}(\Pi_1)$, which is free, and $\mathsf{FT}(\dot{\Delta}_0\Sigma_1)$, which is $\mathsf{FT}(\Sigma_1)$ given Δ_0-collection, and which will hold as $\mathcal{G}$ meets every dense Σ_1 subclass of $\boldsymbol{P}$. $\blacksquare_{10.17}$

10.18. REMARK. We know that Δ_0-separation will hold in the extension by the closure under basic separators. An alternative argument is to say that

(3) A comment on the notation: a is in M and is a name in the forcing for $\mathbf{a}$; $\underline{a}$ is used in the forcing language to remind us that a is not being spoken of as itself but as a name for an as yet uncreated object; on the other hand, once the model $M[G]$ exists we may discuss what sentences are true in it, in terms of the usual truth predicate $\models$ and the associated language; $\tilde{\mathbf{a}}$ is a name for $\mathbf{a}$ in that language.

it would follow from Δ_1-separation in the ground model, which fortunately is a theorem of KP, together with $\mathsf{TFT}(\Delta_0)$, which is free.

The persistence of $\mathsf{KPI} + \Sigma_1$-separation

10.19. Remark. To get Σ_1-separation, we need $(\Delta_0 \Vdash \Sigma_1)$-separation in the ground model—fortunately $(\Delta_0 \Vdash \Sigma_1)$ formulæ are Σ_1^{KPI}—and $\mathsf{TFT}(\Sigma_1)$, which is available as $\mathsf{TF}(\Sigma_1)$ is free, and $\mathsf{FT}(\Sigma_1)$ will hold by Σ_1-separation.

A teasing question

10.20. Remark. We have seen that we can reduce the amount of separation required to hold in M in proving the forcing theorem if, instead, we require the generic $\mathcal{G}$ to meet certain dense definable classes. On the other hand, that device apparently cannot be used to show that separation holds in the extension where it did not hold in the ground model; which raises the following question:

10.21. Problem. Can a set-generic extension satisfy more separation than held in the ground model?

Here is one case, suggested by Kai Hauser in conversation, where a negative answer holds:

10.22. Proposition. *Suppose that M is admissible and of the form P^c_θ for some transitive set c. Let $\mathbb{P} \in M$, and let $\mathcal{G}$ be $(M, \mathbb{P})$-generic. If Σ_1 separation holds in $M[\mathcal{G}]$, then it holds in M.*

Proof. Inside $M[G]$, M is a $\Sigma_1(c)$ class, and so if a and b are in M, the class $a \cap \{x \mid_x \exists y_{\in M}\ \Psi(x, y, b)\}$ is a set, d say, of $M[\mathcal{G}]$. For some $\nu < \theta$, $d \in P^c_\nu[\mathcal{G}]$. $\forall x_{\in d} \exists \xi\ (\exists y_{\in P^c_\xi}\ \Psi(x, y, b))$, so by Σ_1-collection in $M[\mathcal{G}]$, there is a $\zeta < \theta$ such that for $z = P^c_\zeta \in M$, $d = a \cap \{x \mid_x \exists y_{\in z}\ \Psi(x, y, b)\} \in M$ by Δ_0-separation. $\blacksquare_{10.22}$

11. Definition of generic filter and extension in the ill-founded case. In Section 6 of [M2], forcing over an ill-founded model is used to obtain an independence result for H. Friedman's theory of power admissibility; and pages 192–193 of [M2] contain some general remarks on such forcing. We go over some of this ground again as [M2] was written before the development of the theory of rudimentary recursion and in any case is chiefly concerned with set theories with the power set axiom.

11.0. Suppose we have a countable ill-founded model $\mathfrak{M} = \langle M, R\rangle$ of PROVI, and $\mathbb{P} = (\boldsymbol{P}, \leq)$ in M which $\mathfrak{M}$ believes to be a separative poset. Working in $\mathfrak{M}$ we can, as above, define a forcing relation $\Vdash^{\mathbb{P}}$.

The definition of an $(\mathfrak{M}, \mathbb{P})$-generic $\mathcal{F}$ will be much as before, but let us simplify our discussion by requiring $\mathcal{F}$ to meet every $\mathfrak{M}$-definable subclass of $\boldsymbol{P}$.

In the case when M is transitive, we are able to define $\mathrm{val}_{\mathcal{G}}(\cdot)$ by a rudimentary recursion; but in our present context, the model $\mathfrak{M}$ is ill-founded, and so *prima facie* we cannot carry out that recursive definition. Instead we treat as a definition what in the transitive case was Proposition 4.5:

11.1. Definition. Define for all a and b in $\mathfrak{M}$ the following equivalence relation:

$$a \equiv_{\mathcal{F}} b \iff \exists p_{\in \mathcal{F}}\ p \Vdash \underline{a} = \underline{b}.$$

Let $Q = Q_{\mathcal{F}}$ be the set of equivalence classes. Write $[a]_{\mathcal{F}}$ for the $\equiv_{\mathcal{F}}$-equivalence class of $a \in M$.

Define a relation $\in_{\mathcal{F}}$ on Q by

$$[a]_{\mathcal{F}} \in_{\mathcal{F}} [b]_{\mathcal{F}} \iff \exists p_{\in \mathcal{F}}\ p \Vdash \underline{a} \epsilon \underline{b}.$$

11.2. Remark. That that relation is independent of the chosen representives a, b of their equivalence classes follows from Propositions 2.26 and 2.27 established within $\mathfrak{M}$.

11.3. Then $\mathfrak{Q} =_{\mathrm{df}} (Q, \in_{\mathcal{F}})$ is a perfectly reasonable countable set with a two-place relation on it, and we can ask which of the sentences of the forcing language are true in that model when we interpret = by equality and ϵ by $\in_{\mathcal{F}}$.

The Forcing Theorem in the general case. We wish to prove, to take the case of formulæ with two free variables, that

$$(Q, \in_{\mathcal{F}}) \models \Phi[(a)_{\mathcal{F}}, (b)_{\mathcal{F}}] \iff \exists p_{\in \mathcal{F}} (M, R) \models p \Vdash \Phi[\underline{a}, \underline{b}].$$

11.4. Remark. That notation hints at a conflict of language level. We have $\dot{\Delta}_0$ wffs which are sets, and over the set of which we can quantify; we are using these wffs, when their formal free variables are interpreted by constants, in two contexts; in our current universe, for which we have a truth definition $\overset{0}{\models}$ and in the generic extension via the definition of forcing $p \Vdash \varphi$.

To resolve that conflict for $\dot{\Delta}_0$ wffs we should formulate the theorem thus:

11.5. The Forcing Theorem. *For $\mathfrak{M}$, $\mathcal{F}$ and $\mathfrak{Q}$ as above, and for every $\dot{\Delta}_0$ formula $\varphi(\mathfrak{x}, \mathfrak{y})$ and a, b in M,*

$$(Q, \in_{\mathcal{F}}) \models \overset{0}{\models} \varphi[(a)_{\mathcal{F}}, (b)_{\mathcal{F}}] \iff \exists p_{\in \mathcal{F}}\ (M, R) \models p \Vdash \varphi[\underline{a}, \underline{b}].$$

When unrestricted quantifiers are then "added by hand", the Forcing Theorem will extend schematically with no further problem.

The proof will follow that for the transitive case, which relied entirely on the fact that $\mathcal{F}$ meets all the necessary dense classes, and made no use of the well-foundedness of the model under consideration.

Once that has been done, we may strengthen the ties between $\mathfrak{Q}$ and $\mathfrak{N}$, by showing that we may treat $\mathfrak{Q}$ as an extension of $\mathfrak{N}$ by considering the map $x \mapsto [\hat{x}]$; we may also show that $\mathcal{F}$ is in $\mathfrak{Q}$, being $[\dot{\mathcal{F}}]$. Here $\hat{x}$ is the canonical forcing name for the member x of the ground model, defined recursively inside $\mathfrak{N}$, (using which we may define a predicate $\hat{V}$ of the forcing language for membership of the ground model) and $\dot{\mathcal{F}}$ is the canonical forcing name for the generic being added.

11.6. The proof given in §5 that the generic extension has no new ordinals will go through in this case. So loosely we may say that the extension $\mathfrak{Q}$ is no more ill-founded than is the starting model $\mathfrak{M}$. Further, $\mathfrak{Q}$ considers itself to be a generic extension of $\mathfrak{M}$ via $\boldsymbol{P}$ and $\mathcal{F}$, the corresponding statement about $\hat{\boldsymbol{P}}$ and $\dot{\mathcal{F}}$ being forced. Hence inside $\mathfrak{Q}$ the recursive definition of $\mathrm{val}_{\mathcal{F}} : M \to Q$ by

$$\mathrm{val}_{\mathcal{F}}(b) = \{\mathrm{val}_{\mathcal{F}}(a) \mid \exists p_{\in \mathcal{F}}\ (p, a)_2 \in b\}$$

succeeds, using the predicate $\hat{V}$ identifying the members of M.

11.7. Proposition. $[a]_{\mathcal{F}} = \mathrm{val}_{\mathcal{F}}(a)$.

Proof. Use recursion inside the ill-founded model $\mathfrak{Q}$ of PROVI. $\blacksquare_{11.7}$

The author is much indebted to the anonymous referee for his meticulous reading of the manuscript and his thoughtful comments, which will now be addressed, on the past and possible future of work of this kind.

11.8. Historical note. Since Cohen's creation of forcing as a construction of extensions of models of full ZF, many people have examined the possibility of forcing over models of weaker systems of set theory, to say nothing of those who have transplanted Cohen's ideas to other areas of enquiry outside set theory. Forcing over admissible sets was studied briefly by Barwise in his 1967 Stanford thesis, at greater length by Jensen in an originally unpublished treatise [J3] on admissibility that contained a proof of his celebrated "sequence-of-admissibles" theorem, in Steel's 1978 paper [St1], in Sacks' study [Sa], and in numerous writings of Sy Friedman such as his papers [F1] and [F2], which latter expounds *inter alia* that result of Jensen. In this connection, the referee draws my attention to the paper of Carlson [Ca] and the expositions of Ershov (the paper [E1], with a correction following a critical review by Blass; and the book [E2] which discusses forcing over models of KPU) and of Zarach [Z2].

Remark. Carlson [Ca] proves using constructibility and forcing arguments that for many set theories T and any Δ_0 formula Φ, if $\mathsf{T} + \mathsf{AC}$ proves `for all` x `there is a unique` y `such that` $\Phi(x, y)$ then so does T. The results of the present paper, [MB] and [M2] show that Carlson's theorem will also hold for Provi, for $\mathsf{Z} + \mathsf{KP}$, and for some other theories. Whether it will hold for Zermelo set theory Z, for $\mathsf{Z} + \mathsf{Provi}$ or for $\mathsf{KP}^{\mathcal{P}}$, the theory of "power-admissibility" discussed in Section 6 of [M2], seem delicate questions.

Remark. The paper of Hauser [Ha] and the as yet unpublished notes of Steel [St2] contain explorations of forcing over transitive sets which, whilst not required to be admissible, are nevertheless assumed to possess certain fine-structural properties.

11.9. A paper of Feferman gave an application of forcing in the context of second order arithmetic; this theme was developed in an expository article of Scott, and in lectures by Jensen at the 1967 UCLA meeting. The referee suggests that a bridge between the work of Feferman and the ideas of this paper might result if it were to be shown, as is indeed the case, that all axioms of Provi are theorems of the set-theoretic variant $\mathsf{ATR}_0^{\mathrm{set}}$ described in [Si2, §VII.3], of the well-known system ATR_0. With his permission we report that François Dorais is investigating this question and wrote as follows on January 11th, 2014:

> If HC denotes the statement that every set is countable then $\mathsf{Provi} + \mathsf{HC}$ is bi-interpretable with ACA_0^+ and $\mathsf{Provi} + \mathsf{HC} + \mathsf{Mostowski\ Collapse}$ is bi-interpretable with ATR_0. The bi-interpretations are actually very strong. Here are the precise results, seen from the model-theoretic perspective:
>
> (1) If V, W are two models of $\mathsf{Provi} + \mathsf{HC}$ with isomorphic $(\omega, P(\omega), RWO)$, where RWO is the class of wellorderings of ω that have ordinal-valued rank functions, then V and W are globally isomorphic.
>
> (2) If $(N, P(N))$ is a model of ACA_0 and RWO is a subclass of the wellorderings of N (as understood from within the model) which contains the usual ordering of N, is closed under isomorphism, initial segments, addition, and such that arithmetic transfinite recursion is possible along any element of RWO, then there is a model V of $\mathsf{Provi} + \mathsf{HC}$ whose $(\omega, P(\omega), RWO)$ from (1) is isomorphic to this $(N, P(N), RWO)$.
>
> (3) If V is a model of $\mathsf{Provi} + \mathsf{HC}$ and RWO is the class of wellorderings of ω that have ordinal-valued rank functions, then RWO contains the usual ordering of ω, is closed under isomorphism, initial segments, addition, and, seen as a model of second-order arithmetic, $(\omega, P(\omega))$ satisfies arithmetic transfinite recursion along any element of RWO. In other words, $(\omega, P(\omega))$ with RWO satisfies the hypotheses of (2).
>
> (4) A model $(N, P(N))$ of ACA_0 admits a class RWO as described in (2) if and only if $(N, P(N))$ satisfies ACA_0^+, in which case we can take RWO to be the smallest class of wellorderings of N that contains the usual ordering of N and is closed under isomorphism, initial segments and addition.

> The results for ATR_0 follow in the same way by taking *RWO* to be the class of all wellorderings of N or ω since the Mostowski Collapse ensures that every wellordering has an ordinal-valued rank function. It follows that PROVI and ACA_0^+ have the same proof-theoretic ordinal: the first fixed point of the epsilon function. Similarly, PROVI + Mostowski Collapse and ATR_0 have the Feferman–Schütte ordinal Γ_0. All of these results only assume set-foundation in PROVI ; I'm still unsure what happens with Π_1-foundation.

Something of the interplay between analysis and set theory is to be seen in a paper of Zarach and an unpublished manuscript of Gandy.

11.10. The referee also draws attention to the use of class forcing over admissible sets, which, it is hoped, might form the subject of a further paper. An early paper on class forcing in the context of Morse–Kelley theory is [Ch]. Of the papers of Zarach, [Z1] cites a preprint form of [Ch]. It discusses forcing with classes in the context of ZF-. [Z2] cites [Z1] and [Ch]: it discusses set forcing over admissible sets and certain cases of class forcing. [Z4] cites [Z1], [Z2] and [Z3]; it does both set and class forcing over models of ZF-. [Z3] cites none of the above, but it might be re-read in the light of the theory of rudimentary recursion.

A possible line of attack is this: suppose that M is a transitive model of some class theory, so that M has members of all ranks $\leqslant \lambda$, where λ is a limit ordinal. For example, in the case of Morse–Kelley, M might be $V_{\kappa+1}$ where κ is a strongly inaccessible cardinal. Pass to the provident closure, N of M, as defined in [M4] and [MB]. N will be of height $\lambda\omega$. Now the class forcing one had in mind for M will be a member of N, and therefore we can treat the problem as one of set forcing over the provident set N. The attraction of this approach is that names for members of N can be explicitly laid out, since the ordinals in N are all of the form $\lambda n + \zeta$ where $\zeta < \lambda$.

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