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Qualgebras and knotted 3-valent graphs

by

Victoria Lebed (Osaka)

Abstract. This paper is devoted to new algebraic structures, called *qualgebras* and *squandles*. Topologically, they emerge as an algebraic counterpart of knotted 3-valent graphs, just like quandles can be seen as an "algebraization" of knots. Algebraically, they are modeled after groups with conjugation and multiplication/squaring operations. We discuss basic properties of these structures, and introduce and study the notions of qualgebra/squandle 2-cocycles and 2-coboundaries. Knotted 3-valent graph invariants are constructed by counting qualgebra/squandle colorings of graph diagrams, and are further enhanced using 2-cocycles. A classification of size 4 qualgebras/squandles and a description of their second cohomology groups are given.

1. Introduction. A *quandle* is a set Q endowed with two binary operations $\lhd$ and $\widetilde{\lhd}$ satisfying the following axioms:

(1) *self-distributivity*: $(a \lhd b) \lhd c = (a \lhd c) \lhd (b \lhd c)$ RIII

(2) *invertibility*: $(a \lhd b) \widetilde{\lhd} b = (a \widetilde{\lhd} b) \lhd b = a$ RII

(3) *idempotence*: $a \lhd a = a$ RI

Since the operation $\widetilde{\lhd}$ can be deduced from $\lhd$ using (2), we shall often omit it from the definition. Originating from the work of topologists D. Joyce and S. V. Matveev [18, 27], this structure can be seen as an *algebraic counterpart of knots*. Indeed, consider colorings of the arcs of knot diagrams by elements of Q, according to the rule in Figure 1Ⓐ. This coloring rule is compatible

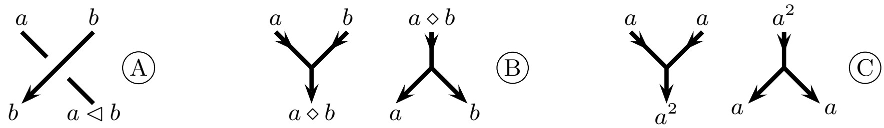

Fig. 1. Colorings by quandles, qualgebras and squandles

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with Reidemeister moves (Figure 3) if and only if Axioms (1)–(3) are satisfied: each axiom corresponds to the Reidemeister move indicated in the right column above. Thus the number of diagram colorings by a fixed quandle defines an invariant of underlying knots and links. This invariant can be strengthened by endowing each colored crossing with a *weight*, and considering the total weight of a colored diagram (Figure 4). The weights are calculated using a *quandle 2-cocycle* of Q according to a procedure suggested by Carter–Jelsovsky–Kamada–Langford–Saito [2]. From the algebraic viewpoint, the quandle structure can be regarded as an *axiomatization of the conjugation operation* in a group. Concretely, a group with the conjugation operation $a \lhd b = b^{-1}ab$ is always a quandle, and, as already noticed in [18], all the properties of conjugation that hold in every group are consequences of (1)–(3).

The aim of this paper is to find an *algebraic counterpart of knotted 3-valent graphs* (further simply called *graphs* for brevity) which would develop quandle ideas. To this end, we introduce the *qualgebra* structure. It is a quandle $(Q, \lhd)$ endowed with an additional binary operation $\diamond$ satisfying

(4)	*translation composability*:	$a \lhd (b \diamond c) = (a \lhd b) \lhd c$	RIV
(5)	*distributivity*:	$(a \diamond b) \lhd c = (a \lhd c) \diamond (b \lhd c)$	RVI
(6)	*semi-commutativity*:	$a \diamond b = b \diamond (a \lhd b)$	RV

Restricting oneself to *poleless* graphs (i.e., oriented graphs with only zip and unzip vertices, cf. Figure 7) and extending the quandle coloring rules 1Ⓐ to 3-valent vertices as shown in Figure 1Ⓑ, one gets rules compatible with Reidemeister moves for graphs (Figure 5) if and only if Axioms (4)–(6) are satisfied. Again, each axiom corresponds to one Reidemeister move (indicated on the right). Imitating what was done for quandle colorings of knots, one can thus define qualgebra coloring *counting invariants* for graphs. The latter can be upgraded to *weight invariants* using the qualgebra 2-cocycles introduced in this work. *Qualgebra 2-cocycles* consist of two maps—one for putting weights on crossings, and one for putting weights on 3-valent vertices (Figures 4 and 15); the weight of a colored diagram is obtained, as usual, by summing everything together.

A group with the operations $a \lhd b = b^{-1}ab$ and $a \diamond b = ab$ is a qualgebra. More precisely, the additional qualgebra axioms encode the *relations between conjugation and multiplication operations* in a group (Table 1). However, our qualgebra axioms do not imply any of the standard axioms defining a group. In particular, we shall give examples of 4-element qualgebras for which the operation $\diamond$ is non-cancellative, non-associative, and has no neutral element.

Besides defining qualgebras and constructing counting and weight invariants of graphs out of them, in this work we study some basic properties of qualgebras; give a complete classification of 4-element qualgebras (showing that a single quandle can be the base of numerous qualgebra structures with significantly different properties); and suggest the beginning of a qualgebra cohomology theory, calculating in particular the second cohomology group for 4-element qualgebras. Moreover, we compute certain qualgebra counting and weight invariants for some pairs of graphs, showing that these graphs can be distinguished using our methods.

In parallel with the qualgebra structure, we study the closely related *squandle* structure. It is defined as a quandle $(Q, \lhd)$ endowed with an additional unary operation $a \mapsto a^2$, obeying the following axioms (modeled after the properties of conjugation and squaring operations in a group):

$$a \lhd b^2 = (a \lhd b) \lhd b \qquad \text{RIV} \tag{7}$$

$$a^2 \lhd b = (a \lhd b)^2 \qquad \text{RVI} \tag{8}$$

A qualgebra with the squaring operation $a^2 = a \diamond a$ is an example of squandle. The coloring rule from Figure 1Ⓒ allows one to construct invariants of graphs by counting squandle colorings of their diagrams; weight invariants are obtained with the help of *squandle 2-cocycles*.

The terms "qualgebra" and "squandle" both come from the names of the two operations participating in the definition of these structures, zipped together as indicated in Figure 2.

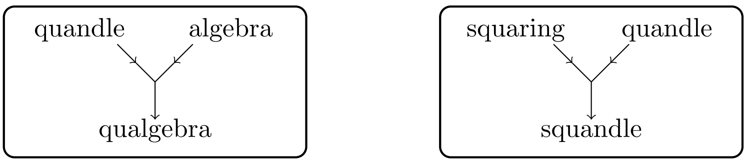

Fig. 2. The terms "qualgebra" and "squandle"

The paper is organized as follows. The language of colorings is developed in Section 2. It is illustrated with the famous example of quandle colorings of knot diagrams, from which some of our further constructions are inspired. We then turn to invariants of graphs which extend the quandle invariants of knots. In Section 3, after a brief survey of such extensions found in the literature, we propose a new one based on qualgebra colorings. Our invariants are defined for poleless graphs only, but they are proved to induce invariants of unoriented graphs. We further show that groups give an important source of qualgebra examples. Constructions from [15] and [5, 10, 9], close to but different from ours, are also discussed. The notion of squandle is introduced in Section 4, motivated by the concept of special colorings (with isosceles qualgebra colorings as the major example here). Squandle colorings are then used for distinguishing Kinoshita–Terasaka and standard Θ-curves. Section 5 contains a short study of basic properties of qualgebras and squandles, applied to a complete classification of qualgebras/squandles with four elements. One of the "exotic" structures obtained is next used for distinguishing two cuff graphs. Section 6 is devoted to the notions of qualgebra/squandle 2-cocycles and 2-coboundaries, as well as to the induced weight invariants of graphs. Qualgebra/squandle 2-cocycles and second cohomology groups are calculated for 4-element structures. The last section contains several suggestions for a further development of qualgebra ideas.

2. Coloring philosophy: the quandle example. One of the most natural and rich sources of invariants of certain topological objects (knots, braids, tangles, knotted graphs, knotted surfaces, etc.) is a study of their diagrams' colorings. If the coloring rules are carefully chosen, one can extract invariants of the underlying topological objects by studying such colorings—for instance, computing their total number, or some more sophisticated characteristics. In this section we develop a general framework for such coloring invariants and illustrate it with the celebrated quandle colorings for knots. A narrative style is preferred to a list of definitions here for better readability. The rest of the paper is devoted to applications of these coloring ideas to knotted 3-valent graphs.

Topological colorings, counting invariants, quandles. Let us now fix a class of 1-dimensional *diagrams* on a surface (e.g., familiar knot diagrams in $\mathbb{R}^2$). For this class of diagrams, choose several types of *special points*, with the local picture of a diagram around a special point being determined by the point type (crossing, points of local maximum, and graph vertices are typical examples). These local pictures are called *type patterns* (see Figure 1 for the examples of oriented crossing and 3-valent vertex patterns). We want to study diagrams up to special-point-preserving isotopy, and up to a set of local (i.e., realized inside a small ball) invertible moves, called *R-moves* (the example inspiring the name is that of Reidemeister moves for knots, see Figure 3). Diagrams related by isotopy and R-moves are called *R-equivalent*. This defines an equivalence relation on the set of diagrams, which corresponds in the cases of interest to isotopy equivalence for the underlying topological objects.

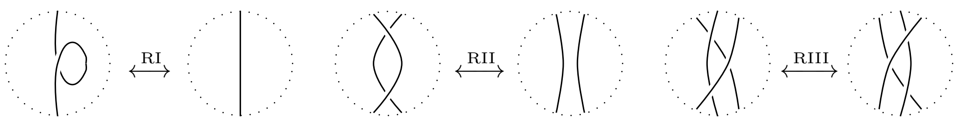

Fig. 3. Reidemeister moves for knot diagrams

An *arc* is a part of a diagram delimited by special points. Fix a set S (possibly with some algebraic structure), which we think of as the coloring set. An *S-coloring* of a diagram D is a map

$$\mathcal{C} : \mathscr{A}(D) \to S$$

from the set of its arcs to S, satisfying some prescribed *coloring rules* for arcs around special points. The set of such colorings of D is denoted by $\mathscr{C}_S(D)$. The notion of S-coloring extends from our class of diagrams to that of subdiagrams (for instance, those involved in an R-move) in the obvious way. In the pictures, an arc α is often decorated with its *color* $\mathcal{C}(\alpha)$.

Definition 2.1. S-coloring rules are *topological* if for any (sub)diagram D, any $\mathcal{C} \in \mathscr{C}_S(D)$, and any D' obtained from D by applying one R-move, there exists a unique coloring $\mathcal{C}' \in \mathscr{C}_S(D')$ coinciding with $\mathcal{C}$ outside the small ball where the R-move was effectuated.

Such coloring rules allow one to construct invariants under R-equivalence. The most basic ones are *counting invariants*:

Lemma 2.2. *Fix a class of diagrams, a set S, and topological S-coloring rules. For any R-equivalent diagrams D and D', there exists a* (*non-canonical*) *bijection between their S-coloring sets:*

$$\mathscr{C}_S(D) \overset{\text{bij}}{\longleftrightarrow} \mathscr{C}_S(D'). \tag{9}$$

In particular, the function $D \mapsto \#\mathscr{C}_S(D)$ (*where $\#X \in \mathbb{N} \cup \infty$ denotes the size of a set X*) *is well-defined on R-equivalence classes of diagrams.*

Thus, if R-equivalence of diagrams corresponds to the isotopy equivalence for underlying topological objects, the lemma produces invariants of these topological objects.

Proof. If D and D' differ by a single R-move, take the bijection from the definition of topological coloring rules. Composing these bijections, one gets the result for the case when D and D' differ by a sequence of R-moves. ∎

Before giving an example of topological coloring rules, we need a convention concerning orientations:

Convention 2.3. In a class of oriented diagrams, using unoriented strands in R-moves or coloring rules means imposing these moves or rules for all possible orientations.

Example 2.4. Consider the class of oriented knot diagrams in $\mathbb{R}^2$, crossings as the only type of special points, Reidemeister moves from Figure 3 as R-moves, a set Q endowed with a binary operation $\lhd$ as the coloring set, and Q-coloring rules from Figure 1Ⓐ. From the pioneering papers [18, 27], these rules are known to be topological if and only if the structure $(Q, \lhd)$ is a *quandle*, i.e., satisfies Axioms (1)–(3) (each of which corresponds to one Reidemeister move). A typical example consists of a group G with the conjugation operation $a \lhd b = b^{-1}ab$, called a *conjugation quandle.* Counting invariants for such colorings even by simplest finite quandles Q appear to be powerful and efficiently computable. Moreover, they are easily generalized to the diagrams of links and tangles, as well as to their virtual versions.

Weight invariants and quandle 2-cocycles. Let us return to the general setting of a class of diagrams endowed with topological S-coloring rules. Counting invariants are far from exploiting the full potential of bijection (9). The following concept allows one to extract more information out of (9). Here and below, A is an abelian group (e.g., $A = \mathbb{Z}$ or $A = \mathbb{Z}_p$).

Definition 2.5. A *weight function* ω is a collection of maps, one for each type of special points on our class of diagrams, associating an element of A to any S-colored pattern of the corresponding type. The ω*-weight* of an S-colored (sub)diagram $(D, \mathcal{C})$, denoted by $\mathcal{W}_\omega(D, \mathcal{C})$, is the sum of the values of ω on all its special points (we suppose the number of the latter to be finite). The weight function ω is called *Boltzmann* if for any R-move, the ω-weights of the two subdiagrams involved, S-colored correspondingly (in the sense of Definition 2.1), coincide.

Boltzmann weight functions allow one to upgrade counting invariants to what we call here *weight invariants*:

Lemma 2.6. *Fix a class of diagrams, a set S, topological S-coloring rules, and a Boltzmann weight function ω. Then the multi-sets of ω-weights of any R-equivalent diagrams D and D' coincide:*

$$\{\mathcal{W}_\omega(D, \mathcal{C}) \mid \mathcal{C} \in \mathscr{C}_S(D)\} = \{\mathcal{W}_\omega(D', \mathcal{C}') \mid \mathcal{C}' \in \mathscr{C}_S(D')\}. \tag{10}$$

In particular, restricted to the diagrams D for which the set $\mathscr{C}_S(D)$ is finite, the function

$$D \mapsto \sum_{\mathcal{C} \in \mathscr{C}_S(D)} t^{\mathcal{W}_\omega(D, \mathcal{C})} \in \mathbb{Z}[t^{\pm 1}]$$

is well-defined on R-equivalence classes of diagrams.

Proof. If D and D' differ by a single R-move, then Definition 2.1 describes a bijection between $\mathscr{C}_S(D)$ and $\mathscr{C}_S(D')$ such that the corresponding colorings $\mathcal{C}$ and $\mathcal{C}'$ differ only in small balls where the R-move is effectuated; Definition 2.5 then gives $\mathcal{W}_\omega(D, \mathcal{C}) = \mathcal{W}_\omega(D', \mathcal{C}')$, implying the desired multi-set equality. Iterating this argument, one gets the result for the case when D and D' differ by several R-moves. ∎

Example 2.7. Continuing Example 2.4, take a map $\chi : Q \times Q \to A$ and consider a weight function, still denoted by χ, that depends only on two of the colors around a crossing, as shown in Figure 4. In [2] this weight function was shown to be Boltzmann if and only if it satisfies the following axioms for all elements of Q (corresponding, respectively, to moves RIII and

RI; RII is automatic here):

$$\chi(a,b) + \chi(a \lhd b, c) = \chi(a \lhd c, b \lhd c) + \chi(a,c), \tag{11}$$

$$\chi(a,a) = 0. \tag{12}$$

Moreover, these conditions were interpreted as the definition of 2-*cocycles* from the celebrated *quandle cohomology* theory. In this theory, any map $\varphi : Q \to A$ defines a 2-*coboundary* via

$$\chi_\varphi(a,b) = \varphi(a \lhd b) - \varphi(a), \tag{13}$$

with $\mathcal{W}_{\chi_\varphi}$ vanishing on all Q-colored knot diagrams.

Fig. 4. Quandle 2-cocycle weight function for knot diagrams

Weight invariants of knots constructed out of quandle 2-cocycles are known as *quandle cocycle invariants*. They are even more efficient than quandle counting invariants, since the same small quandle can admit various 2-cocycles. Moreover, they are strictly stronger than quandle counting invariants since, unlike the latter, they can distinguish a knot from its mirror image. See [2, 3, 4, 13, 19, 29] and the references therein for more details.

3. Qualgebra coloring invariants of knotted 3-valent graphs. We now turn to our main object of study, the *knotted* 3-*valent graphs* (i.e., embeddings of abstract 3-valent graphs into $\mathbb{R}^3$) and their diagrams in $\mathbb{R}^2$; see Figures 13–14 for examples. We often use the word "graph" instead of "knotted 3-valent graph" for brevity. Two types of special points are relevant for graph diagrams: crossings and graph vertices. In 1989, L. H. Kauffman, S. Yamada, and D. N. Yetter independently [21, 31, 32] showed that the Reidemeister moves for knots (Figure 3), together with the three moves from Figure 5, describe graph isotopy in $\mathbb{R}^3$. We choose these six moves as R-moves; R-equivalence classes of graph diagrams thus correspond to isotopy classes of represented graphs.

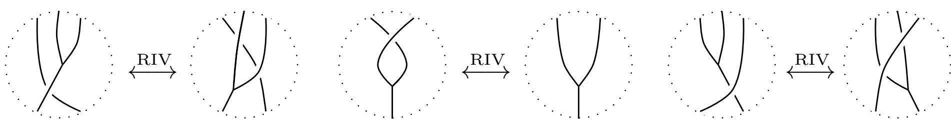

Fig. 5. Additional Reidemeister moves for knotted 3-valent graphs (note that the names of moves visually resemble the diagrams involved)

Since quandles worked so well for knots, we would like to use a quandle $(Q, \lhd)$ as the coloring set in the generalized setting of graphs as well. This section is thus devoted to the following question:

Question 3.1. Can one extend the Q-coloring rule from Figure 1Ⓐ to 3-valent vertices so that the resulting coloring rules for graphs are topological?

After a discussion of existing answers, we shall propose a new one. The coloring rule around crossings will always be that from Figure 1Ⓐ; hence we shall often omit it, and restrict our attention to vertices.

Colorings for graphs: existing approaches. Required coloring rules are easy to define geometrically for a conjugation quandle $(G, a \lhd b = b^{-1}ab)$. Choose a basepoint p "in front of" a diagram D of an oriented graph Γ. Consider the *Wirtinger presentation* of the graph group $\pi_1(\mathbb{R}^3 \setminus \Gamma; p)$ with one generator θ_α for each arc α of D, as shown in Figure 6Ⓐ. An (evident) relation is imposed on the generators around each special point. A representation of $\pi_1(\mathbb{R}^3 \setminus \Gamma; p)$ in G is a map $\mathcal{P}$ from $\{\theta_\alpha \mid \alpha \in \mathscr{A}(D)\}$ to G respecting these relations, which is equivalent to the map $\mathcal{C} : \alpha \mapsto \mathcal{P}(\theta_\alpha)$ being a coloring with respect to the coloring rules from Figures 1Ⓐ and 6Ⓑ (where a color or its inverse is chosen according to the arc being directed from or to the graph vertex). The coloring rules are topological, as can be seen via this graph group representation interpretation, or by direct verification. For any diagram D of Γ, one gets a bijection

$$\mathscr{C}_G(D) \overset{\text{bij}}{\longleftrightarrow} \operatorname{Hom}(\pi_1(\mathbb{R}^3 \setminus \Gamma), G).$$

These conjugation quandle colorings for graphs can be generalized in several ways. In 2010 M. Niebrzydowski [28] extended the rules from Figure 6Ⓑ to general quandles, as shown in Figure 6Ⓒ (we use the notation $\lhd^+ = \lhd$, $\lhd^- = \widetilde{\lhd}$; the choice in $\pm$ depends, as usual, on orientations). Another approach was recently proposed by A. Ishii [15]. He considered a quandle operation $\lhd$ on a disjoint union of groups $X = \bigsqcup_i G_i$, which is the conjugation when restricted to each G_i, and which satisfies some additional conditions. This structure is called a *multiple conjugation quandle* (*MCQ*). It includes as particular cases the usual conjugation quandles and *G-families of quandles*, defined in 2012 by Ishii–Iwakiri–Jang–Oshiro [16]. The coloring rule from Figure 6Ⓑ, where one requires a, b, and c to lie in the same group G_i, is topological for MCQ.

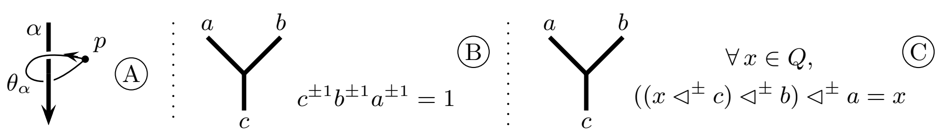

Fig. 6. Possible extensions of quandle colorings to graph diagrams

Poleless 3-valent graphs. The coloring rule we introduce in this work is another generalization of conjugation quandle colorings of graphs to a broader class of quandles. It is defined for graphs oriented in a special way:

Definition 3.2. An abstract or knotted oriented 3-valent graph is called *poleless* if it has only *zip* and *unzip* vertices (Figure 7).

In other words, one forbids source and sink vertices.

Fig. 7. Zip and unzip vertices for 3-valent graphs

For poleless graph diagrams, some of the R-moves can be discarded using the so-called *Turaev's trick* (see also [30] for a careful study of minimal generating sets of Reidemeister moves in the knot case):

Lemma 3.3. *Reidemeister moves* IV–VI *with orientations as in Figure* 8, *together with all oriented versions of moves* RI–RIII, *imply all remaining poleless versions of moves* RIV–RVI.

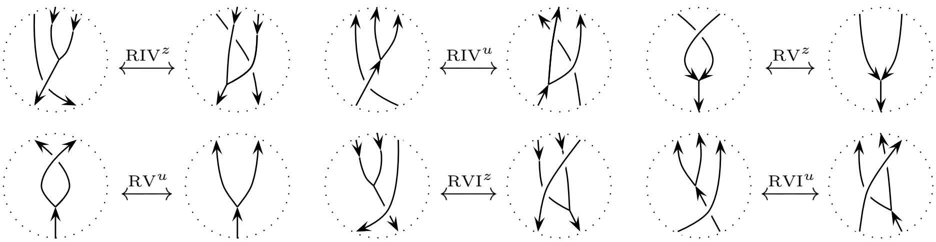

Fig. 8. Reidemeister moves for poleless graph diagrams (superscripts ^{z} and ^{u} refer to the zip or unzip vertex involved in the move)

Proof. Moves RIV^u and RV^u for certain alternative orientations are treated in Figure 9; other moves and orientations are dealt with in a similar way. ■

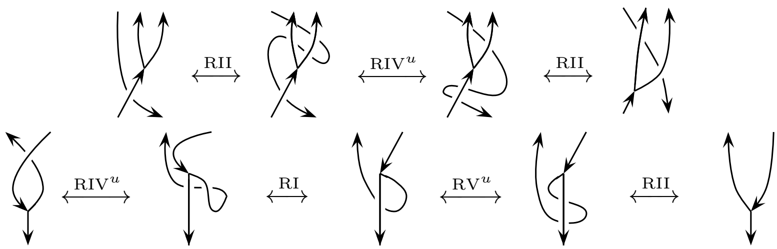

Fig. 9. Reidemeister moves RIV^u and RV^u for alternative orientations

Although our orientation restriction prevents one from working with arbitrary oriented graphs, unoriented graphs can be treated as follows:

Proposition 3.4. *Any 3-valent graph admits a poleless orientation.*

Proof. Take an abstract unoriented graph Γ with vertices of odd valency. A *path* is a sequence of pairwise distinct edges $e_1, \ldots, e_k$, the endpoints (s_i, t_i) of each e_i being ordered, such that t_i and s_{i+1} coincide for each $1 \leq i < k$. Choose a *maximal path* γ in Γ—i.e., γ is not a subpath of any longer path. Delete the edges of γ from Γ, and then delete all the resulting isolated vertices. One gets a graph $\Gamma \setminus \gamma$, whose vertices are still of odd valency. Indeed, the valency subtracted from internal vertices of γ is even (since we enter and leave them the same number of times); as for the first and last vertices, their full valencies are subtracted (otherwise γ would be extendible, and thus not maximal), so they are deleted. Now let Γ be 3-valent. Iterating the argument above, one presents Γ as a disjoint union of paths, with each vertex occurring at least once as a $t_i = s_{i+1}$. Orient each edge e_i in each path from s_i to t_i. One gets a poleless orientation of Γ. ■

Thus, in order to compare two unoriented graphs, it is sufficient to compare the sets of their poleless oriented versions.

A new coloring approach via qualgebras. Now, for poleless graph diagrams, consider the coloring rule from Figure 1Ⓑ, where $\diamond$ is a second binary operation on the quandle $(Q, \lhd)$. Trying to render these rules topological, one arrives at our central notion:

Definition 3.5. A set Q endowed with two binary operations $\lhd$ and $\diamond$ is called a *qualgebra* if it satisfies Axioms (1)–(6).

The term “qualgebra” comes from terms “quandle” and “algebra” zipped together, as shown in Figure 2. It underlines the presence of two interacting operations in this structure.

Remark 3.6. Our definition can be recast in a more structural way. Consider a set Q with binary operations $\lhd$ and $\diamond$, and define an operator

$$\sigma_\lhd : Q \times Q \to Q \times Q, \qquad (a, b) \mapsto (b, a \lhd b).$$

Then $(Q, \lhd, \diamond)$ is a qualgebra if and only if $(Q, \sigma_\lhd, \diamond)$ is a *braided algebra* which is *braided-commutative* but not necessarily associative, and such that the Yang–Baxter operator $\sigma_\lhd$ preserves the diagonal of Q. For precise definitions, see for instance [26].

Observe that Axiom (1) is a consequence of (4) and (6):

$$(a \lhd b) \lhd c \overset{(4)}{=} a \lhd (b \diamond c) \overset{(6)}{=} a \lhd (c \diamond (b \lhd c)) \overset{(4)}{=} (a \lhd c) \lhd (b \lhd c).$$

In what follows, we will include or omit this axiom according to our needs.

For further reference, let us also note the compatibility relations between the operations $\diamond$ and $\widetilde{\lhd}$.

LEMMA 3.7. *A qualgebra $(Q, \lhd, \diamond)$ enjoys the following properties:*

$$a \,\widetilde{\lhd}\, (b \diamond c) = (a \,\widetilde{\lhd}\, c) \,\widetilde{\lhd}\, b, \tag{14}$$

$$(a \diamond b) \,\widetilde{\lhd}\, c = (a \,\widetilde{\lhd}\, c) \diamond (b \,\widetilde{\lhd}\, c), \tag{15}$$

$$(a \,\widetilde{\lhd}\, b) \diamond b = b \diamond a. \tag{16}$$

Proof. Let us show (14), the proof for the remaining relations being similar. Applying (4) to the elements $a \,\widetilde{\lhd}\, (b \diamond c)$, b, and c, one gets

$$(a \,\widetilde{\lhd}\, (b \diamond c)) \lhd (b \diamond c) = ((a \,\widetilde{\lhd}\, (b \diamond c)) \lhd b) \lhd c.$$

The left-hand side equals a by (2). Now, apply the map $x \mapsto (x \,\widetilde{\lhd}\, c) \,\widetilde{\lhd}\, b$ to both sides:

$$(a \,\widetilde{\lhd}\, c) \,\widetilde{\lhd}\, b = ((((a \,\widetilde{\lhd}\, (b \diamond c)) \lhd b) \lhd c) \,\widetilde{\lhd}\, c) \,\widetilde{\lhd}\, b.$$

Using (2) for the right-hand side this time, one obtains (14). ∎

Now, returning to colorings of graphs, one gets

PROPOSITION 3.8. *Take a set Q endowed with two binary operations $\lhd$ and $\diamond$. The coloring rules from Figure 1Ⓐ&Ⓑ are topological if and only if $(Q, \lhd, \diamond)$ is a qualgebra.*

Proof. The equivalence between the compatibility of the coloring rule 1Ⓐ with Reidemeister moves I–III on the one hand, and Axioms (1)–(3) on the other hand, was discussed in Example 2.4. Let us turn to the remaining three moves, with orientations from Lemma 3.3. Analyzing move RIV^z (Figure 10), one notices that on each side the three colors on the top completely determine all the remaining colors, in particular the colors on the bottom. Then, the coloring bijection from Definition 2.1 occurs if and only if the induced bottom colors coincide on the two sides, which is equivalent to Axiom (4). An analogous argument shows that for move RIV^u, the coloring bijection is equivalent to Axiom (14), which, in the presence of (2), is the same as (4) (cf. the proof of Lemma 3.7).

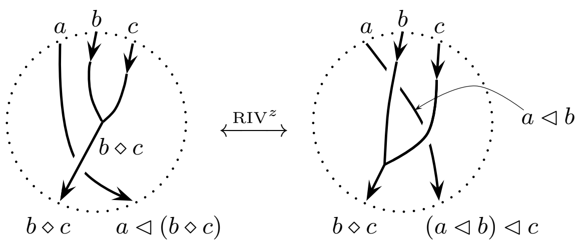

Fig. 10. Qualgebra axioms via coloring rules for graph diagrams

Similarly, one checks that for both the zip and unzip versions of RVI (or RV) the coloring bijection is equivalent to Axiom (5) (respectively, (6)). ∎

Remark 3.9. We could have used distinct operations $\diamond_z$ and $\diamond_u$ for coloring rules around zip and unzip vertices. However, our simplified choice already produces powerful invariants; moreover, it is natural if one thinks in terms of generalizations of (multiple) conjugation quandle colorings of graphs.

Lemma 2.2 now yields qualgebra coloring invariants for graphs:

Corollary 3.10. *Take a qualgebra* $(Q, \lhd, \diamond)$*, and consider the* Q*-coloring rules from Figure* 1Ⓐ&Ⓑ*. The quantity* $\#\mathscr{C}_Q(D)$ *does not depend on the choice of a diagram* D *representing a poleless* 3*-valent knotted graph* Γ*.*

Proof. Proposition 3.8 guarantees that the coloring rules in question are topological. Lemma 2.2 then tells us that the function $D \mapsto \#\mathscr{C}_Q(D)$ is well-defined on R-equivalence classes of diagrams, which, according to [21, 31, 32], correspond to isotopy classes of graphs. ∎

One thus gets a systematic way of producing invariants of poleless (or unoriented, cf. Proposition 3.4) graphs.

Group qualgebras. We now show that groups form an important source of qualgebras, playing also a significant motivational role.

Example 3.11. A conjugation quandle together with the group multiplication $a \diamond b = ab$ is a qualgebra, called a *group qualgebra*. The coloring rule from Figure 1Ⓑ repeats in this case the rule from Figure 6Ⓑ. Thus the graph invariants constructed using qualgebra colorings generalize those obtained from graph group representations.

While from the topological perspective quandle axioms (1)–(3) can be viewed as algebraic incarnations of Reidemeister moves for knots, from the algebraic viewpoint they are often interpreted as an axiomatization of the conjugation operation in a group. Concretely, if a relation involving only conjugation holds in every group, then it can be deduced from the quandle axioms; this follows from the inclusion of the free quandle on a set X into the free group on X, described for example in [18, 7]. In a similar way, as shown in the (proof of) Proposition 3.8, topologically additional qualgebra axioms (4)–(6) can be regarded as algebraic incarnations of specific R-moves for 3-valent graphs. Algebraically, they encode major relations between conjugation and multiplication operations in a group (cf. Table 1).

Table 1. Different viewpoints on quandles and qualgebras

abstract level	quandle axioms	specific qualgebra axioms
group level	conjugation	conjugation/multiplication interaction
topological level	moves RI–RIII	moves RIV–RVI

Remark 3.12. In contrast to what happens for quandles, not all the conjugation/multiplication relations in a group are captured by the qualgebra structure. For instance, the relation

$$(b \lhd a) \diamond (a \lhd b) = ((a \mathbin{\widetilde{\lhd}} b) \lhd a) \diamond b$$

holds in any group (both sides equal $a^{-1}bab^{-1}ab$). However, in [23] it is shown not to follow from qualgebra axioms (4)–(6). This is done by inspecting the free qualgebra on a set.

Example 3.13. New examples of qualgebras can be derived by considering subqualgebras of given qualgebras. In the case of group qualgebras, these are simply subsets closed under conjugation and multiplication operations, but not necessarily under taking inverse. For instance, the positive integers $\mathbb{N}$ form a subqualgebra of the group qualgebra of integers $\mathbb{Z}$.

Note that subqualgebras of group qualgebras do not necessarily contain the neutral element or inverses. However, they clearly remain associative:

Definition 3.14. A qualgebra $(Q, \lhd, \diamond)$ is called *associative* if the operation $\diamond$ is such, i.e., if for all elements of Q one has

$$(a \diamond b) \diamond c = a \diamond (b \diamond c). \tag{17}$$

Examples of non-associative qualgebras will be given in Section 5, showing that group qualgebras and their subqualgebras are far from covering all qualgebra examples.

Related constructions and "qualgebraizability". Our choice of qualgebra axioms was dictated by the desired applications to graph invariants. It gave an extremely rich structure. For instance, in Section 5 we will meet some exotic qualgebras with very "non-group-like" properties. Here we mention some related structures from the literature, appearing in different frameworks and exhibiting dissimilar properties.

First, observe that the associativity, absent from our topological picture, becomes relevant for *handlebody-knots* [14]. In particular it appears, together with some of our qualgebra axioms, in A. Ishii's definition of *multiple conjugation quandle*, which is tailored for producing handlebody-knot invariants. Algebraically, MCQs inherit many properties of groups, since they are formed by gluing several groups together.

Besides the topological and algebraic settings described above, Axioms (4)–(6) emerge in a completely different set-theoretic context. Namely, together with the associativity of $\diamond$ and the existence of a neutral element 1 for $\diamond$ satisfying $1 \lhd a = 1$ and $a \lhd 1 = a$ for all $a \in Q$, they define a (*right-*)*distributive monoid* (or, in other sources, RD algebra). Elementary embeddings, Laver tables, and extended braids admit rich RD monoid structures, motivating an extensive study of this concept (cf. [5, 10, 11, 6], or

[7, Chapter XI] for a comprehensive exposition). A weaker *augmented (right-)distributive system* structure of P. Dehornoy obeys only Axioms (1), (4), and (5); the major example here is that of parenthesized braids [8, 9]. Qualgebras are particular cases of augmented RD systems.

We finish with some remarks concerning the relations between quandle and qualgebra structures. Any quandle can be embedded (as a subquandle) into a qualgebra [23]. Further, some quandles can be upgraded to qualgebras using several different operations $\diamond$ (cf. Section 5 for examples). Here we give an example of a family of quandles which cannot be turned into qualgebras, and of a quandle admitting exactly one compatible operation $\diamond$.

Example 3.15. A *dihedral quandle* is the set $\mathbb{Z}/n\mathbb{Z}$ endowed with the operation $a \lhd b = 2b - a \pmod{n}$. Suppose that $\mathbb{Z}/n\mathbb{Z}$ is endowed with a second operation $\diamond$ satisfying (4). Then for all $a, b, c \in \mathbb{Z}/n\mathbb{Z}$, the element $(a \lhd b) \lhd c = 2c - 2b + a$ would coincide with $a \lhd (b \diamond c) = 2(b \diamond c) - a$, thus $2a = 2(b \diamond c) - 2c + 2b$ would not depend on a, which is impossible if $n \neq 2$.

Example 3.16. Consider the conjugation quandle of the symmetric group S_3. As usual, the operation $a \diamond b = ab$ turns it into a group quandle. Let us show that this is the only qualgebraization of this quandle. Indeed, Axiom (4) imposes the values of $(12) \lhd (a \diamond b)$ and $(123) \lhd (a \diamond b)$ for all $a, b \in S_3$; it remains to show that the values $(12) \lhd x$ and $(123) \lhd x$ uniquely identify an $x \in S_3$. This follows by direct computations:

$$(12) \lhd x = \begin{cases} (12) & \text{if } x \in \{\mathrm{Id}, (12)\}, \\ (23) & \text{if } x \in \{(132), (13)\}, \\ (13) & \text{if } x \in \{(123), (23)\}; \end{cases}$$

$$(123) \lhd x = \begin{cases} (123) & \text{if } x \in \{\mathrm{Id}, (123), (132)\}, \\ (213) & \text{if } x \in \{(12), (23), (13)\}. \end{cases}$$

4. Isosceles colorings and squandles. In concrete situations, one sometimes has to deal with pairs of graphs for which the Q-coloring counting invariants from Corollary 3.10 coincide for certain qualgebras Q, but which can be distinguished if only a particular kind of colorings is taken into account. After a short survey of the development of such *special coloring* ideas in the literature, we introduce a particular kind of qualgebra colorings, allowing one to distinguish, for instance, the two theta-curves from Figure 13.

Special colorings. Start with group coloring rules for arbitrary oriented graphs (Figures 1Ⓐ and 6Ⓑ). The most natural particular kind of corresponding colorings is the one where the colors of arcs adjacent to the same vertex coincide, up to taking inverse. This means using the coloring rule from Figure 11Ⓐ, where color a should be chosen for arcs oriented from the vertex, and color a^{-1} for the remaining ones. Such colorings can be traced back to C. Livingston's 1995 study of vertex constant graph groups [24]. These ideas were generalized in 2007 by T. Fleming and B. Mellor [12] to the case of *symmetric quandle*. The latter is a quandle Q endowed with a *good involution*, i.e., a map $\rho : Q \to Q$ satisfying, for all elements of Q,

$$\rho(\rho(a)) = a, \tag{18}$$

$$\rho(a) \lhd b = \rho(a \lhd b), \tag{19}$$

$$a \lhd \rho(b) = a \mathbin{\widetilde{\lhd}} b. \tag{20}$$

Symmetric quandles were defined by S. Kamada [20]. The basic example is our favourite conjugation quandle, with $\rho(a) = a^{-1}$. Now, for a symmetric quandle Q, Fleming–Mellor's coloring rule for graphs is presented in Figure 11(B); the notation $a^{+1} = a$, $a^{-1} = \rho(a)$ is used here, and the choice in ± 1 is controlled by the same rule as for group colorings. This rule generalizes that from Figure 11(A), and the corresponding colorings can be seen as special among the quandle colorings in the sense of 6(C). To see that one gets topological coloring rules, it suffices to verify that a special coloring remains so after an R-move and the corresponding coloring change (cf. the proof of Proposition 4.2). In 2010, M. Niebrzydowski [28] further generalized these ideas to an arbitrary quandle case.

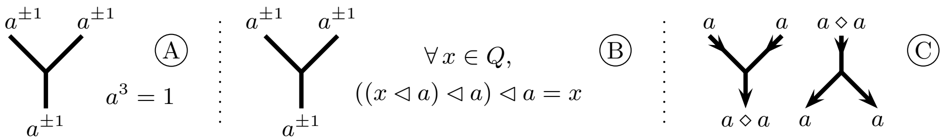

Fig. 11. Examples of special coloring

Isosceles colorings. We now return to qualgebra colorings for poleless graphs. The class of special colorings we propose to study here is the following:

Definition 4.1. Take a qualgebra $(Q, \lhd, \diamond)$ and a Q-colored poleless graph diagram $(D, \mathcal{C})$. A 3-valent vertex of D is called $\mathcal{C}$*-isosceles* if $\mathcal{C}$ assigns the same colors to its two adjacent co-oriented arcs. The coloring $\mathcal{C}$ itself is called *isosceles* if all vertices of D are $\mathcal{C}$-isosceles.

In other words, working with isosceles colorings means considering coloring rule 11(C).

Proposition 4.2. *Given a qualgebra* $(Q, \lhd, \diamond)$*, the coloring rules from Figures* 1(A) *and* 11(C) *are topological.*

Proof. Since isosceles colorings are particular instances of those from Proposition 3.8, which are controlled by topological rules, it suffices to check that an isosceles coloring remains so after an R-move and the corresponding coloring change. For moves RI–RIII and RV this is obvious, since they do not change the colors around isosceles trivalent vertices. Move RVI^u is treated in Figure 12: the top three colors determine all the remaining ones (note that

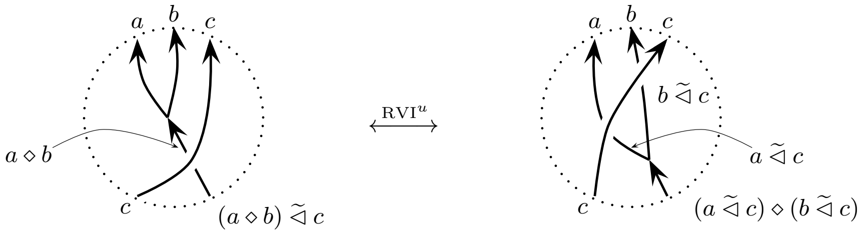

Fig. 12. Reidemeister move VI^u and induced colorings

the bottom colors coincide due to (15)), and for any of the two diagrams, being isosceles means satisfying $a = b$ (since the map $x \mapsto x \widetilde{\lhd} c$ is a bijection on Q). Moves RVI^z and RIV are treated similarly. ∎

Corollary 4.3. *Take a qualgebra $(Q, \lhd, \diamond)$. An invariant of poleless 3-valent knotted graphs can be constructed by assigning to such a graph the number of isosceles Q-colorings $\#\mathscr{C}_Q^{\mathrm{iso}}(D)$ of any of its diagrams D.*

Example 4.4. The Kinoshita–Terasaka Θ-curve Θ_{KT} and the standard Θ-curve Θ_{st} (Figure 13) often serve as a litmus test for new graph invariants. One of the reasons is the following: when any edge is removed from Θ_{KT}, the remaining two form the unknot, just as for Θ_{st}; however, the three edges of Θ_{KT} are knotted, in the sense that Θ_{KT} is not isotopic to Θ_{st}. These "partial unknottedness" phenomena are of the same nature as those exhibited by the Borromean rings.

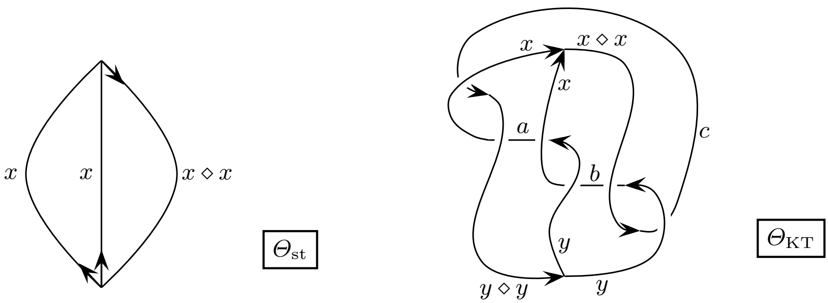

Fig. 13. Isosceles colorings for diagrams of standard and Kinoshita–Terasaka Θ-curves

Now, for these two Θ-curves, consider the isosceles Q-colorings of their diagrams D_{KT} and D_{st}, depicted in Figure 13. The diagram D_{st} (as well as all the other poleless versions of the underlying unoriented diagram) has $\#Q$ isosceles Q-colorings: the co-oriented arcs can be colored with any color x, and the remaining arc gets the color $x \diamond x$. As for D_{KT}, the coloring rule 11Ⓒ around 3-valent vertices is taken into consideration in Figure 13, and the rule 1Ⓐ around crossings gives the relations

$$(*)\begin{cases} a = x \lhd (y \diamond y) = y \lhd x, \\ b = x \mathbin{\widetilde{\lhd}} y = y \mathbin{\widetilde{\lhd}} (x \diamond x), \\ c = (y \diamond y) \lhd x = (x \diamond x) \mathbin{\widetilde{\lhd}} y. \end{cases}$$

Thus, $\#\mathscr{C}^{\mathrm{iso}}_{Q}(D_{\mathrm{KT}})$ is the number of solutions of $(*)$ in x and y. For any $q \in Q$, one has a solution $x = y = q$ (cf. Lemma 5.10). In order to find other isosceles colorings of D_{KT}, let us try the simplest case of a group qualgebra Q and of its order 3 elements x and y. System $(*)$ is now equivalent to a single equation $xyx = yxy$. In the symmetric group S_4, distinct order 3 elements $x = (123)$ and $y = (432)$ give a solution. One obtains

$$\#\mathscr{C}^{\mathrm{iso}}_{S_4}(D_{\mathrm{KT}}) > \#S_4 = \#\mathscr{C}^{\mathrm{iso}}_{S_4}(D_{\mathrm{st}}).$$

Since, as mentioned above, $\#\mathscr{C}^{\mathrm{iso}}_{S_4}(D_{\mathrm{st}})$ is the same for all poleless versions of D_{st}, one concludes that Θ_{KT} and Θ_{st} are distinct as unoriented graphs.

A variation of qualgebra ideas. Restricting our attention to isosceles colorings only, we do not exploit the whole structure of qualgebra. Indeed, the only values of $a \diamond b$ we need are those for $a = b$. In other words, we use only the "squaring" part $\varsigma : a \mapsto a \diamond a$ of the operation $\diamond$. Pursuing this remark, we consider unary operations ς for which the coloring rule 1Ⓒ is topological:

Definition 4.5. A set Q endowed with a binary operation $\lhd$ and a unary operation ς (which we often denote by $a \mapsto a^2$) is called a *squandle* if it satisfies Axioms (1)–(3) and (7)–(8).

The term "squandle" (similarly to the term "qualgebra") comes from terms "square" and "quandle" zipped together, as in Figure 2.

Let us also note the compatibility relations between ς and $\widetilde{\lhd}$:

Lemma 4.6. *A squandle $(Q, \lhd, \varsigma)$ enjoys the following properties:*

$$a \mathbin{\widetilde{\lhd}} b^2 = (a \mathbin{\widetilde{\lhd}} b) \mathbin{\widetilde{\lhd}} b, \tag{21}$$

$$a^2 \mathbin{\widetilde{\lhd}} b = (a \mathbin{\widetilde{\lhd}} b)^2. \tag{22}$$

Example 4.7. A qualgebra $(Q, \lhd, \diamond)$ gives rise to a squandle $(Q, \lhd, \varsigma : a \mapsto a \diamond a)$. Moreover, the subsquandles of the latter (which are not necessarily subqualgebras) can be of interest. In particular, conjugation and the squaring operation $a \mapsto a^2$ in a group turn it into a squandle, called a *group squandle.* Axioms (7)–(8) can now be seen as an abstraction of the relations between conjugation and squaring in a group.

Now, considering squandle colorings, one gets the following results, with the statements and proofs analogous to the qualgebra case:

PROPOSITION 4.8. *Take a set Q endowed with a binary operation $\lhd$ and a unary operation ς. The coloring rules from Figure 1Ⓐ&Ⓒ are topological if and only if $(Q, \lhd, \varsigma)$ is a squandle.*

COROLLARY 4.9. *Take a squandle $(Q, \lhd, \varsigma)$ and consider the Q-coloring rules 1Ⓐ & Ⓒ. The (possibly infinite) quantity $\#\mathscr{C}_Q(D)$ does not depend on the choice of a diagram D representing a poleless 3-valent knotted graph Γ.*

EXAMPLE 4.10. Let us resume Example 4.4. In the symmetric group S_4, consider the subset S_4^3 of cycles of length 3. It contains eight elements, and it is closed under conjugation and squaring. Hence S_4^3 is a size 8 subsquandle of the group squandle of S_4 (but not a subqualgebra, since it does not contain $\mathrm{Id} = (123)^3$). Calculations from Example 4.4 show that $\#\mathscr{C}_{S_4^3}(D_{\mathrm{st}}) = \#S_4^3 = 8$, and that $\#\mathscr{C}_{S_4^3}(D_{\mathrm{KT}})$ is the number of solutions of $xyx = yxy$ in S_4^3. Now, for any x, the pair (x, x) is a solution, while (x, x^{-1}) is not. Further, we have seen that the cycles (123) and (432) form a solution, and one checks that (123) and (423) do not. A conjugation argument allows one to conclude that for a fixed x_0, precisely half of the pairs (x_0, y) are solutions, which totals to $\#\mathscr{C}_{S_4^3}(D_{\mathrm{KT}}) = 8 \cdot 4 = 32$. Thus, although this example gives nothing new about the graphs Θ_{KT} and Θ_{st} (the group qualgebra of S_4 was sufficient to distinguish them), it does show that with squandle colorings, actual computation of counting invariants can be much easier.

5. Qualgebras and squandles with four elements. In this section we completely describe qualgebras and squandles with four elements, discovering abundant examples even in such a small size.

General properties. Some general facts about qualgebras and squandles are first due.

NOTATION 5.1. Given a quandle $(Q, \lhd)$ (in particular, a qualgebra or a squandle) and an $a \in Q$, denote by $\mathcal{S}_a$ the *right translation* map $x \mapsto x \lhd a$. These maps are written on the right of their arguments: $(x)\mathcal{S}_a = x \lhd a$.

Most axioms of quandle-like structures can be expressed in terms of these right translations, allowing one to work with symmetric groups instead of abstract structures. This approach was extensively used for quandles in [25]. Here we apply similar ideas to qualgebras and squandles.

DEFINITION 5.2. A map between two qualgebras/squandles $f : Q \to R$ is called a *qualgebra/squandle morphism* if it respects the qualgebra/squandle structure, in the sense of

$$f(a \lhd_Q b) = f(a) \lhd_R f(b), \quad f(a \diamond_Q b) = f(a) \diamond_R f(b) \quad (\text{or } f(a^2) = f(a)^2).$$

LEMMA 5.3. *Given a qualgebra* $(Q, \lhd, \diamond)$ *or a squandle* $(Q, \lhd, \varsigma)$*, the map*

$$\mathcal{S} : Q \to \operatorname{Aut}(Q), \qquad a \mapsto \mathcal{S}_a, \tag{23}$$

is a well-defined qualgebra/squandle morphism from Q *to* $\operatorname{Aut}(Q)$*, the latter being the group qualgebra/squandle of the group of qualgebra/squandle automorphisms of* Q*.*

Proof. We prove the assertion for qualgebras; the one for squandles is analogous. One should first show that any $\mathcal{S}_a$ is a qualgebra automorphism. Indeed, it is invertible due to Axiom (2), its inverse $\mathcal{S}_a^{-1}$ being the map $x \mapsto x \,\widetilde{\lhd}\, a$, and it respects $\lhd$ and $\diamond$ due to (1) and (5) respectively.

It remains to prove that $\mathcal{S}$ is a qualgebra morphism. The relation $\mathcal{S}_{a \diamond b} = \mathcal{S}(a)\mathcal{S}(b)$ directly follows from (4). Next, for any $x \in Q$ one calculates

$$\begin{aligned}(x)\mathcal{S}_{a \lhd b} &= x \lhd (a \lhd b) = ((x \,\widetilde{\lhd}\, b) \lhd b) \lhd (a \lhd b) = ((x \,\widetilde{\lhd}\, b) \lhd a) \lhd b \\ &= (((x)\mathcal{S}_b^{-1})\mathcal{S}_a)\mathcal{S}_b = (x)(\mathcal{S}_a \lhd \mathcal{S}_b)\end{aligned}$$

(we have used quandle axioms (1)–(3) and the definition of $\lhd$ as conjugation in the group qualgebra $\operatorname{Aut}(Q)$). Hence $\mathcal{S}_{a \lhd b} = \mathcal{S}(a) \lhd \mathcal{S}(b)$. ∎

LEMMA 5.4. *For a finite qualgebra* Q*, the image* $\mathcal{S}(Q)$ *of the map* (23) *is a subgroup of* $\operatorname{Aut}(Q)$*.*

Proof. Since $\mathcal{S}$ is a qualgebra morphism (Lemma 5.3), its image $\mathcal{S}(Q)$ is a subqualgebra of the group qualgebra $\operatorname{Aut}(Q)$, which is finite since Q is finite. Let us now show that, in general, a non-empty finite subqualgebra R of a group qualgebra G is in fact a subgroup. Indeed, R is closed under product since it is a subqualgebra; it contains the unit 1 of the group G since $1 = a^p$, where a is any element of R and p is its order in G; and it contains all the inverses, since, with the previous notation, $a^{-1} = a^{p-1}$. ∎

Below we will see that this lemma can be false for squandles.

In a study of a qualgebra or squandle, the understanding of its local structure can be useful.

NOTATION 5.5. Take a qualgebra or a squandle Q and an $a \in Q$.

- The subqualgebra/subsquandle of Q generated by a is denoted by Q_a.
- The set of fixed points x of $\mathcal{S}_a$ (i.e., $(x)\mathcal{S}_a = x$) is denoted by $\operatorname{Fix}(a)$.
- The set of elements x of Q fixing a (in the sense that $(a)\mathcal{S}_x = a$) is denoted by $\operatorname{Stab}(a)$.

LEMMA 5.6. *Take a qualgebra/squandle* Q*, and an* $a \in Q$*. Both sets* $\operatorname{Fix}(a)$ *and* $\operatorname{Stab}(a)$ *are subqualgebras/subsquandles of* Q *containing* Q_a*.*

Proof. $\mathrm{Fix}(a)$ is a subqualgebra/subsquandle of Q since $\mathcal{S}_a$ is a qualgebra/squandle automorphism of Q. As for $\mathrm{Stab}(a)$, note that the set $\widetilde{\mathrm{Stab}}(a)$ of the maps in $\mathrm{Aut}(Q)$ fixing a is a subgroup of $\mathrm{Aut}(Q)$, hence also a subqualgebra/subsquandle, so $\mathrm{Stab}(a)$, which is its preimage $\mathcal{S}^{-1}(\widetilde{\mathrm{Stab}}(a))$ under the qualgebra/squandle morphism $\mathcal{S}$, is a subqualgebra/subsquandle of Q (cf. Lemma 5.3). Further, both $\mathrm{Fix}(a)$ and $\mathrm{Stab}(a)$ contain a by the idempotence axiom (3). Since they are subqualgebras/subsquandles of Q, they have to include the whole Q_a. ∎

LEMMA 5.7. *Endow a set Q with the trivial quandle operation $a \lhd_0 b = a$. Then any unary operation ς completes it to a squandle. Further, a binary operation $\diamond$ completes it to a qualgebra if and only if $\diamond$ is commutative.*

Proof. With the trivial quandle operation, all qualgebra and squandle axioms automatically hold true except for (6), which is equivalent to the commutativity of $\diamond$. ∎

DEFINITION 5.8. The qualgebras/squandles from the lemma above are called *trivial*.

Colorings by trivial qualgebras/squandles do not distinguish over- and under-crossings, hence the corresponding counting invariants can capture only the underlying abstract graph and not the way it is knotted in $\mathbb{R}^3$. However, weight invariants can be sensible to the knotting information even for trivial structures.

PROPOSITION 5.9. *A non-trivial qualgebra or squandle has ≥ 4 elements.*

Proof. Let a be an element of a qualgebra/squandle Q defining a non-trivial right translation map $\mathcal{S}_a$. Then $\mathcal{S}_{a^2} = \mathcal{S}_a^2 \neq \mathcal{S}_a$, so $\mathrm{Fix}(a)$ contains elements $a \neq a^2$ (cf. Lemma 5.6). Further, since $S_a \in \mathrm{Aut}(Q)$ is not the identity, ≥ 2 elements of Q lie outside $\mathrm{Fix}(a)$. Altogether, one gets ≥ 4 elements. ∎

We finish by showing that every qualgebra/squandle is “locally trivial”:

LEMMA 5.10. *Take a qualgebra $(Q, \lhd, \diamond)$ or a squandle $(Q, \lhd, \varsigma)$, and an $a \in Q$. The subqualgebra/subsquandle Q_a of Q generated by a is trivial. In the qualgebra case, the restriction of operation $\diamond$ to Q_a is commutative.*

Proof. Lemma 5.6 shows that every $x \in Q_a$ fixes a. Thus, the set $\mathrm{Fix}(x)$ contains a; but, being a subqualgebra/subsquandle of Q (again by Lemma 5.6), it contains the whole Q_a. The triviality of $\lhd$ restricted to Q_a follows. The commutativity of $\diamond$ on Q_a now follows from Lemma 5.7. ∎

Classification of qualgebras of size 4. Since trivial qualgebras/squandles were completely described in Lemma 5.7, only non-trivial structures are studied in the remainder of this section.

We start with a full list of non-trivial qualgebra structures on a 4-element set $P = \{p, q, r, s\}$ (up to isomorphism). Our description uses the involution

(24) $$(p)\tau = q, \quad (q)\tau = p, \quad (r)\tau = r, \quad (s)\tau = s.$$

PROPOSITION 5.11. *The operations*

$$\begin{array}{lll} x \lhd r = (x)\tau, & x \lhd y = x \text{ if } y \neq r; & \\ r \diamond r = s, & r \diamond x = x \diamond r = r \text{ if } x \neq r, & \\ s \diamond s = s, & q \diamond s = s \diamond q \in \{p, q, s\}, & p \diamond s = s \diamond p = (q \diamond s)\tau, \\ p \diamond q = q \diamond p = s, & q \diamond q \in \{p, q, s\}, & p \diamond p = (q \diamond q)\tau \end{array}$$

define a qualgebra structure on the set P for any choices of $q \diamond s$ and $q \diamond q$ in $\{p, q, s\}$. The nine structures thus obtained are pairwise non-isomorphic, and describe, up to isomorphism, all non-trivial qualgebras with four elements.

To better feel the qualgebra structures from the proposition, think of the element r as the r̲otation $p \leftrightarrow q$, and of s as the s̲quare (of r).

Proof. Fix a qualgebra structure on P. Observe first that for any $x \in P$, one has $\#\mathrm{Fix}(x) \geq 2$. Indeed, otherwise the subqualgebra P_x generated by x, which is contained in $\mathrm{Fix}(x)$ due to Lemma 5.6, would consist of x itself only, and so, according to Lemma 5.3, $\mathcal{S}(\{P_x\}) = \{\mathcal{S}_x\}$ would be a 1-element subqualgebra of $\mathrm{Aut}(P) \subseteq S_4$, which is possible only if $\mathcal{S}_x = \mathrm{Id}$, giving $\#\mathrm{Fix}(x) = 4$.

Now, the condition $\#\mathrm{Fix}(x) \geq 2$ implies that $\mathcal{S}_x$ moves at most two elements of P, so it is a transposition or the identity. But then $\mathcal{S}(P)$ is a subgroup of S_4 (Lemma 5.4) containing nothing except transpositions and the identity, hence either $\mathcal{S}(P) = \{\mathrm{Id}\}$ (and thus the qualgebra is trivial), or, without loss of generality,

$$\mathcal{S}(P) = \{\mathrm{Id}, \tau\},$$

with, say, $\mathcal{S}_r = \tau$. We next show that $\mathcal{S}^{-1}(\tau)$ consists of r only. Indeed, $\mathcal{S}(P_r)$ is a subqualgebra of $\mathrm{Aut}(P)$ (Lemma 5.3) contained in $\mathcal{S}(\mathrm{Fix}(r))$ (Lemma 5.6), so $\mathcal{S}(\mathrm{Fix}(r)) = \{\mathcal{S}(r), \mathcal{S}(s)\} = \{\tau, \mathcal{S}_s\}$ should include $\tau^2 = \mathrm{Id}$, hence $\mathcal{S}_s = \mathrm{Id}$, implying $s \notin \mathcal{S}^{-1}(\tau)$. As for p and q, they are not fixed by τ, so they cannot lie in $\mathcal{S}^{-1}(\tau)$.

We can thus restrict our analysis to the case $\mathcal{S}_r = \tau$ and $\mathcal{S}_y = \mathrm{Id}$ for $y \neq r$. This choice of $\lhd$ guarantees (2) and (3). Axiom (1) can be checked directly, but we prefer recalling that it is a consequence of (4)–(6).

Let us now analyze the specific qualgebra axioms (4)–(6). First, (4) translates as $\mathcal{S}_{b \diamond c} = \mathcal{S}_b \mathcal{S}_c$, which here means that $r \diamond x = x \diamond r = r$ for all $x \neq r$, while all other products take value in $\{p, q, s\}$. Next, (5) is equivalent to all maps from $\mathcal{S}(P)$ respecting the operation $\diamond$, which here translates as $(a \diamond b)\tau = (a)\tau \diamond (b)\tau$. This means that $r \diamond r$ and $s \diamond s$ are both τ-stable, so, lying in $\{p, q, s\}$, they can equal only s; this gives nothing new when one of a, b is r and the other is not; and it divides the remaining ordered couples into pairs, with the product for one couple from the pair determined by that for the other (e.g., $p \diamond s = (q \diamond s)\tau$). Finally, (6) is automatic when one of the elements a and b is r and the other is p or q, and for the other couples it means the commutativity of $\diamond$. In particular, this commutativity gives $p \diamond q = q \diamond p$, which, combined with $(p \diamond q)\tau = (p)\tau \diamond (q)\tau = q \diamond p$, implies that $p \diamond q$ is τ-stable, so, lying in $\{p, q, s\}$, it can equal only s. Putting all these conditions together, one gets the description of $\diamond$ given in the statement.

It remains to check that the nine qualgebra structures obtained are pairwise non-isomorphic. Let $f : P \to P$ be a bijection intertwining the structures $(\lhd, \diamond_1)$ and $(\lhd, \diamond_2)$ from our list. Since r is the only element of P with $\mathcal{S}_a \neq \mathrm{Id}$, one has $(r)f = r$, and also $(s)f = (r \diamond_1 r)f = r \diamond_2 r = s$. Two options emerge: either $(q)f = q$ and $(p)f = p$, in which case $\diamond_1$ and $\diamond_2$ automatically coincide; or $(q)f = p$ and $(p)f = q$, that is, $f = \tau$, in which case one has

$$x \diamond_2 y = ((x)f^{-1} \diamond_1 (y)f^{-1})f = ((x)\tau^{-1} \diamond_1 (y)\tau^{-1})\tau = x \diamond_1 y,$$

since the right translation $\tau = \mathcal{S}_r$ respects $\diamond_1$. One concludes that there are no isomorphisms between different qualgebra structures from our list. ■

Properties and examples. In spite of very close definitions, the nine structures above exhibit quite different algebraic properties. Some of them are studied below.

PROPOSITION 5.12. *The operations* $\diamond$ *from Proposition* 5.11 *are*

- *all commutative;*
- *never cancellative;*
- *unital if and only if* $q \diamond s = s \diamond q = q$ *and* $p \diamond s = s \diamond p = p$*;*
- *associative if and only if* $q \diamond s = s \diamond q = p \diamond s = s \diamond p = s$ *and either* $q \diamond q = p \diamond p = s$*, or* $q \diamond q = q$ *and* $p \diamond p = p$*;*
- *never unital associative.*

Proof. The commutativity can be read off from the explicit definition of $\diamond$. The non-cancellativity follows from the "absorbing" property of r with respect to $\diamond$.

Further, the relations $q \diamond p = s$ and $r \diamond s = r$ imply that s is the only possible neutral element. Examining the definition of $\diamond$, one sees that it is indeed so if and only if the value of $q \diamond s = s \diamond q$ is chosen to be q (implying $p \diamond s = s \diamond p = (q \diamond s)\tau = (q)\tau = p$).

Associativity is trickier to deal with. First, if $\diamond$ is associative, then

$$s \diamond q = (r \diamond r) \diamond q = r \diamond (r \diamond q) = r \diamond r = s.$$

Since $(s)\tau = s$, this implies $q \diamond s = p \diamond s = s \diamond p = s$. Next, $q \diamond q$ cannot be p, since this would give

$$\begin{aligned} q &= (p)\tau = (q \diamond q)\tau = (q)\tau \diamond (q)\tau = p \diamond p = p \diamond (q \diamond q) \\ &= (p \diamond q) \diamond q = s \diamond q = s. \end{aligned}$$

Thus, either $q \diamond q = p \diamond p = s$, or $q \diamond q = q$ and $p \diamond p = p$. It remains to show that these two operations $\diamond$ are indeed associative. Consider the direct product $\mathbb{Z}_4^{\times 3}$ endowed with term-by-term multiplication $\cdot$, and define an injection $P \hookrightarrow \mathbb{Z}_4^{\times 3}$ by

$$p \mapsto (a, 0, 1), \quad q \mapsto (0, a, 1), \quad r \mapsto (0, 0, 3), \quad s \mapsto (0, 0, 1),$$

for some $a \neq 0$. One easily checks that this injection intertwines $\diamond$ and $\cdot$, where one takes $a = 2$ for the choice $q \diamond q = p \diamond p = s$, and $a = 1$ for $q \diamond q = q$, $p \diamond p = p$. Thus the associativity of $\cdot$ implies that of $\diamond$.

To conclude, notice that if a unital associative $\diamond$ existed, then it would satisfy the incompatible conditions $q \diamond s = q$ and $q \diamond s = s$. ∎

Thus, three non-trivial qualgebra structures with four elements are unital, and two are associative. Further, none of these qualgebras can be a subqualgebra of a group qualgebra because of non-cancellativity.

Example 5.13. Let us now use a 4-element qualgebra to distinguish the standard cuff graph C_{st} from the Hopf cuff graph C_{H}. Consider their diagrams D_{st} and D_{H} depicted in Figure 14, and choose the qualgebra P from Proposition 5.11 with $q \diamond q = s$ and $q \diamond s = q$. The multiplication $\diamond$ of this qualgebra can be briefly described by saying that it is commutative with a neutral element s, that the element r absorbs everything but itself (in the sense that $r \diamond x = r$), and that $x \diamond y = s$ for $x = y$ and for $x = (y)\tau$.

With the orientation in Figure 14, the coloring rules for D_{st} around 3-valent vertices read $b \diamond a = a$ and $b \diamond c = c$. Every orientation of D_{st} is poleless, and an orientation change results only in an argument inversion in one or all of the conditions above; since $\diamond$ is commutative, this preserves the conditions. Summarizing, for any orientation of D_{st} one gets a bijection

$$\mathscr{C}_P(D_{\mathrm{st}}) \stackrel{\mathrm{bij}}{\longleftrightarrow} \{(a, b, c) \in P \mid b \diamond a = a,\ b \diamond c = c\}.$$

Now, the equation $b \diamond a = a$ (and similarly $b \diamond c = c$) has six solutions in P: either b is the unit s, and a is arbitrary; or b is p or q, and $a = r$. Searching for pairs of solutions with the same b, one gets

$$\mathscr{C}_Q(D_{\mathrm{st}}) \stackrel{\mathrm{bij}}{\longleftrightarrow} \{(a, s, c) \mid a, c \in Q\} \sqcup \{(r, b, r) \mid b \in \{p, q\}\},$$

and so $\#\mathscr{C}_P(D_{\mathrm{st}}) = 4 \cdot 4 + 2 = 18$.

Consider now the Hopf cuff graph diagram D_{H}, oriented as in Figure 14. The coloring rule around crossings allows one to express a' and c' as $c' = c \lhd a$, $a' = a \mathbin{\widetilde{\lhd}} c'$. In our qualgebra P, all translations $\mathcal{S}_x$ (cf. Notation 5.1) are either the identity or τ, so they are pairwise commuting involutions, implying $a' = a \mathbin{\widetilde{\lhd}} c' = (a)\mathcal{S}_{c\lhd a}^{-1} = (a)\mathcal{S}_{c\lhd a} = (a)(\mathcal{S}_c \lhd \mathcal{S}_a) = (a)\mathcal{S}_c = a \lhd c$. Around 3-valent vertices, coloring rules read $b \diamond a = a'$ and $b \diamond c = c'$, yielding

$$\mathscr{C}_P(D_{\mathrm{H}}) \overset{\text{bij}}{\longleftrightarrow} \{(a,b,c) \in P \mid b \diamond a = a \lhd c,\ b \diamond c = c \lhd a\}.$$

The latter system admits no solutions with $b = r$. For $b = s$, the equations become $a = a \lhd c$ and $c = c \lhd a$, for which the solutions are all pairs (a, c) except $a = r$, $c \in \{p, q\}$ or vice versa. In the remaining case $b \in \{p, q\}$, the only possibility is $a = c = r$. Summarizing, one gets

$$\mathscr{C}_P(D_{\mathrm{H}}) \overset{\text{bij}}{\longleftrightarrow} \{(a,s,c) \mid a, c \in \{p,q,s\}\} \sqcup \{(r,s,r),(r,s,s),(s,s,r)\} \sqcup \{(r,b,r) \mid b \in \{p,q\}\},$$

so $\#\mathscr{C}_P(D_{\mathrm{H}}) = 3\cdot 3 + 3 + 2 = 14 \neq \#\mathscr{C}_P(D_{\mathrm{st}})$. With the orientation remarks made for D_{st}, Corollary 3.10 distinguishes the two unoriented cuff graphs.

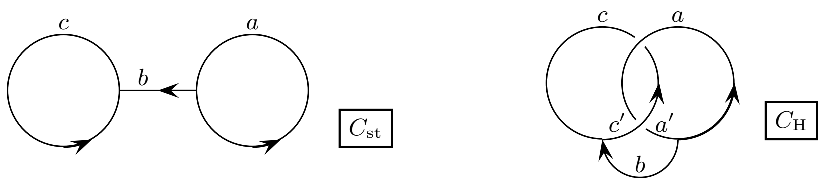

Fig. 14. Qualgebra colorings for the diagrams of the standard and Hopf cuff graphs

Classification of squandles of size 4. We now turn to non-trivial 4-element squandles.

Proposition 5.14. *Any non-trivial squandle with four elements is isomorphic either to*

- *the subsquandle* $S_3^2 = \{\mathrm{Id}, (12), (23), (13)\}$ *of the group squandle of the symmetric group* S_3*; or to*
- *the set* $P = \{p, q, r, s\}$ *with the following operations (using the involution* τ *from* (24)*):*

$$x \lhd r = (x)\tau, \quad x \lhd y = x \text{ if } y \neq r;$$
$$r^2 = s^2 = s, \quad q^2 \in \{p,q,s\}, \quad p^2 = (q^2)\tau.$$

The four structures thus obtained are pairwise non-isomorphic.

Proof. Fix a squandle structure on P. Repeating verbatim the beginning of the proof of Proposition 5.11, one shows that, for any $x \in P$, $\mathcal{S}_x$ is a transposition or the identity. Forgetting trivial squandles, which correspond to $\mathcal{S}(P) = \{\mathrm{Id}\}$, we consider three remaining cases:

1. There are two intersecting transpositions—say, (p, q) and (q, r)—in $\mathcal{S}(P)$. Then $\mathcal{S}(P)$ also contains $(p, q) \lhd (q, r) = (p, r)$ and $(p, q)^2 =$ Id. Since P itself has only four elements, $\mathcal{S}$ is an injection, so P is isomorphic to the subsquandle $\{\mathrm{Id}, (12), (23), (13)\}$ of S_4 (it is indeed a subsquandle, being closed under conjugation and squaring). Omitting the element 4, one interprets the latter as the subsquandle S_3^2 of S_3.
2. There are two non-intersecting transpositions—say, (p, q) and (r, s)—in $\mathcal{S}(P)$. A fixed point argument gives $(p, q) \in \{\mathcal{S}_r, \mathcal{S}_s\}$ and $(r, s) \in \{\mathcal{S}_p, \mathcal{S}_q\}$—say, $\mathcal{S}_r = (p, q)$ and $\mathcal{S}_p = (r, s)$. Consider now the possible values of r^2. According to Lemma 5.6, one has $r^2 \in \mathrm{Fix}(r) = \{r, s\}$. Since $\mathcal{S}_{r^2} = (\mathcal{S}_r)^2 = \mathrm{Id} \neq \mathcal{S}_r$, the only possibility left is $r^2 = s$, with $\mathcal{S}_s = \mathrm{Id}$. But then (1) leads to a contradiction: $(q \lhd r) \lhd p = p \lhd p = p$ and $(q \lhd p) \lhd (r \lhd p) = q \lhd s = q$.
3. The only remaining situation is $\mathcal{S}(P) = \{\mathrm{Id}, \tau\}$ with, say, $\mathcal{S}_r = \tau$. Repeating once again an argument from the proof of Proposition 5.11, one concludes that the operation $\lhd$ is defined by $\mathcal{S}_r = \tau$ and $\mathcal{S}_x = \mathrm{Id}$ for $x \neq r$, and satisfies (1)–(3). Thus only specific squandle axioms (7)–(8) remain to be checked. First, (7) translates as $\mathcal{S}_{b^2} = \mathcal{S}_b^2$, which here means that $x^2 \in \{p, q, s\}$ for all $x \in P$. Next, (8) is equivalent to all maps from $\mathcal{S}(P)$ respecting the squaring operation, which here translates as $(a^2)\tau = ((a)\tau)^2$. This means that $p^2 = (q^2)\tau$, and that r^2 and s^2 are both τ-stable, hence equal to s (since they lie in $\{p, q, s\}$). One thus gets the desired description of ς.

The four structures obtained are shown to be mutually non-isomorphic in the same way as was done for qualgebras in Proposition 5.11. ∎

Observe that the first structure from the proposition is a subsquandle of a finite group squandle which is not a subgroup; thus Lemma 5.4 does not hold for squandles. The three remaining structures are induced from the qualgebras from Proposition 5.11 according to Example 4.7.

6. Qualgebra 2-cocycles and weight invariants of graphs. We now return to the general setting of a qualgebra $(Q, \lhd, \diamond)$ and Q-colorings of poleless knotted 3-valent graph diagrams, according to coloring rules from Figure 1Ⓐ&Ⓑ. The aim of this section is to extract weight invariants out of such colorings.

Qualgebra 2-cocycles as Boltzmann weight functions. Recall the type of weight functions used for quandle colorings of knot diagrams (Example 2.7): a fixed map $\chi : Q \times Q \to A$ (where A is an abelian group) is applied to the colors of two arcs adjacent to a crossing (Figure 4). Trying to treat 3-valent vertices of a graph diagram colored by a qualgebra Q in a similar way, take a map $\lambda : Q \times Q \to A$, and let (χ, λ) be a weight function defined in Figures 4 and 15. Note that the colors we evaluate χ or λ on determine all the colors around a crossing or a vertex. Remark also that different maps λ and $\curlyvee$ could be chosen for unzip and zip vertices; our choice simplifies calculations, while conserving abundant examples. To make notation easier to follow, we denote the components of a weight function by Greek letters with a shape referring to that of the corresponding special points.

Fig. 15. Weight function for qualgebra-colored graph diagrams

Proposition 6.1. *Take a qualgebra* $(Q, \lhd, \diamond)$ *and maps* $\chi, \lambda : Q \times Q \to A$. *The weight function* (χ, λ) *described above* (*and depicted in Figures* 4 *and* 15) *is Boltzmann if and only if it satisfies, for all elements of* Q, *Axioms* (11)–(12) *together with three additional ones:*

(25) $\chi(a, b \diamond c) = \chi(a, b) + \chi(a \lhd b, c),$

(26) $\chi(a \diamond b, c) + \lambda(a \lhd c, b \lhd c) = \chi(a, c) + \chi(b, c) + \lambda(a, b),$

(27) $\chi(a, b) + \lambda(a, b) = \lambda(b, a \lhd b).$

Proof. One should check when each of the six R-moves, combined with the induced coloring transformation (Definition 2.1), leaves the (χ, λ)-weights unchanged. For moves RI–RIII, this is known to be equivalent to Axioms (11)–(12) for χ (cf. Example 2.7). Figure 16 deals with the zip versions of moves RIV, RVI, and RV, which are shown to preserve weights if and only if (25) (respectively, (26) or (27)) is satisfied. The unzip versions are similar due to our choice of weight function around zip and unzip vertices, and to relations (14)–(16) allowing one to treat $\widetilde{\lhd}$ like $\lhd$. ∎

Axioms (25)–(26) also appear in the homology theory of G-families of quandles, developed by J. S. Carter and A. Ishii in their 2012 preprint [1].

Definition 6.2. For a qualgebra Q, a pair of maps (χ, λ) satisfying Axioms (11)–(12) and (25)–(27) is called an (A-valued) *qualgebra* 2-*cocycle* of Q; the term will be commented on below. The set of all qualgebra 2-cocycles of Q is denoted by $Z^2(Q, A)$.

Lemma 2.6 now yields weight invariants for graphs:

Corollary 6.3. *Take a qualgebra* $(Q, \lhd, \diamond)$ *and an* A*-valued qualgebra* 2-*cocycle* (χ, λ). *Consider* Q*-coloring rules from Figure* 1Ⓐ&Ⓑ *and the weight function* (χ, λ) *from Figures* 4 *and* 15. *The multi-set* $\{ \mathcal{W}_{(\chi,\lambda)}(D, \mathcal{C}) \mid \mathcal{C} \in \mathscr{C}_Q(D) \}$ *does not depend on the choice of a diagram* D *representing a poleless* 3*-valent knotted graph* Γ.

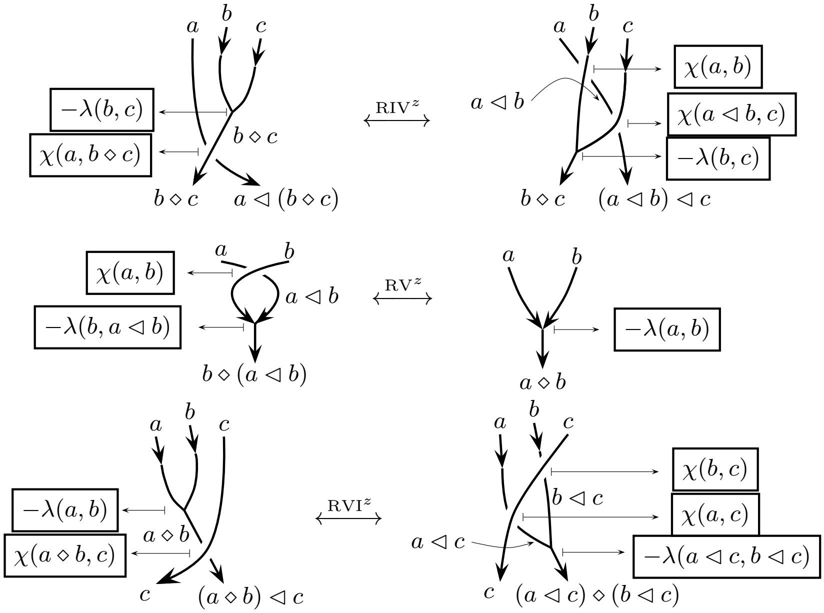

Fig. 16. Obtaining axioms for qualgebra 2-cocycles

Proof. Proposition 3.8 guarantees that our coloring rules are topological, and Proposition 6.1 tells us that our weight function is Boltzmann. Lemma 2.6 then asserts that the multi-set in question is well-defined on R-equivalence classes of diagrams, which correspond to isotopy classes of graphs. ∎

One thus gets a systematic way of producing invariants of poleless (or unoriented, cf. Proposition 3.4) graphs, which sharpen the counting invariants from Corollary 3.10.

More on qualgebra 2-cocycles: properties and examples. We start with an easy observation on the structure of $Z^2(Q,A)$:

LEMMA 6.4. *The space $Z^2(Q,A)$ of qualgebra 2-cocycles of a qualgebra Q is an abelian group under pointwise addition of the two components; in other words, the sum $(\chi,\lambda) = (\chi',\lambda') + (\chi'',\lambda'')$ is defined by*

$$\chi(a,b) = \chi'(a,b) + \chi''(a,b), \qquad \lambda(a,b) = \lambda'(a,b) + \lambda''(a,b).$$

Moreover, for a Q-colored graph diagram $(D,\mathcal{C})$, the following map is linear:

$$Z^2(Q,A) \to A, \qquad (\chi,\lambda) \mapsto \mathcal{W}_{(\chi,\lambda)}(D,\mathcal{C}).$$

Proof. An easy standard verification using, for the first assertion, the linearity of all qualgebra 2-cocycle axioms, and for the second, the linearity of our qualgebra coloring rules. ∎

Recall that in the definition of a qualgebra, the self-distributivity axiom is redundant. For qualgebra 2-cocycles, some axioms can be omitted as well:

LEMMA 6.5. *Take a qualgebra* Q *and two maps* $\chi, \lambda : Q \times Q \to A$. *Relation* (11) *for these maps follows from* (26) *and* (27), *and relation* (12) *is a consequence of* (27).

Proof. Putting $b = a$ in (27) and using the idempotence of a, one gets (12).

To deduce (11) from (26) and (27), one can either use a direct computation, or argue diagrammatically. We opt for the latter. Consider a sequence of moves RVI and RV from Figure 17. Endow the first and the last diagrams from the figure with the unique colorings extending the partial ones indicated in the figure, and the intermediate diagrams with the induced colorings (cf. Proposition 3.8). Relations (26) and (27) imply, according to the proof of Proposition 6.1, that the (χ, λ)-weights of all five diagrams coincide. But the (χ, λ)-weights of the first and the last diagrams, decreased by $\lambda(a, b)$, are precisely the (χ, λ)-weights of the two sides of an RIII move with colors a, b, c on the top. Recalling that move RIII preserves the χ-weights if and only if (11) holds (cf. Example 2.7), we finish the proof. ■

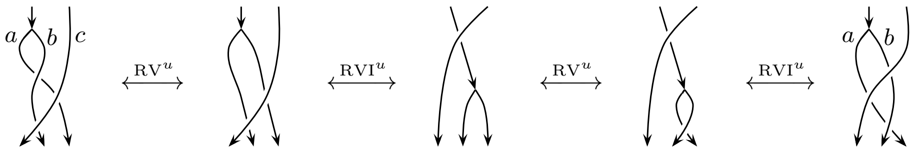

Fig. 17. Move RIII as a sequence of moves RVI and RV

It is thus sufficient to keep only Axioms (25)–(27) in the definition of qualgebra 2-cocycles, simplifying their investigation.

EXAMPLE 6.6. Let us explore qualgebra 2-cocycles with zero χ-part. In this situation, Axioms (25)–(27) become

$$\lambda(a \lhd c, b \lhd c) = \lambda(a, b), \tag{28}$$

$$\lambda(a, b) = \lambda(b, a \lhd b). \tag{29}$$

Relation (28) implies that $\lambda(b, a \lhd b) = \lambda(b \lhd b, a \lhd b) = \lambda(b, a)$, thus the maps $(0, \lambda)$ form a 2-cocycle if and only if λ is a symmetric invariant (in the sense of (28)) form on Q. The simplest example of such a form is the constant map $\lambda_\alpha(a, b) = \alpha$, $\alpha \in A$; in this case, $\mathcal{W}_{(0,\lambda_\alpha)}(D, \mathcal{C})$ does not depend on the coloring $\mathcal{C}$ and counts the difference between the numbers of unzip and zip vertices. Another example is the Kronecker delta $\delta_\alpha(a, b) = \alpha$ if $a = b$, and 0 otherwise; $\mathcal{W}_{(0,\delta_\alpha)}(D, \mathcal{C})$ counts the difference between the numbers of $\mathcal{C}$-isosceles unzip and zip vertices (cf. Definition 4.1).

Qualgebra 2-cocycles for trivial qualgebras. We next explicitly describe the structure of the abelian group $Z^2(Q, A)$ of qualgebra 2-cocycles for a trivial qualgebra Q (Definition 5.8):

PROPOSITION 6.7. *Take a trivial qualgebra* $(Q, \lhd_0, \diamond)$. *Endow* Q *with any linear order. Let* $\mathrm{ABF}(Q, A)$ *be the abelian group of* A*-valued anti-symmetric bilinear forms* χ *on* Q *(i.e.,* $\chi(a, b) + \chi(b, a) = 0$ *and* $\chi(a, b \diamond c) = \chi(a, b) + \chi(a, c)$*), and let* $\mathrm{SF}(Q, A)$ *be the abelian group of* A*-valued symmetric forms* λ *on* Q $(\lambda(a, b) = \lambda(b, a))$. *Then* $Z^2(Q, A)$ *is a direct sum of* $\mathscr{L} = \{\varLambda_\lambda = (0, \lambda) \mid \lambda \in \mathrm{SF}(Q, A)\}$ *and of* $\mathscr{X} = \{X_\chi = (\chi, \lambda_\chi) \mid \chi \in \mathrm{ABF}(Q, A)\}$, *where*

$$\lambda_\chi(a, b) = \begin{cases} 0 & \text{if } a \le b, \\ \chi(b, a) & \text{otherwise.} \end{cases}$$

Proof. According to Lemma 6.5, we are looking for maps $\chi, \lambda : Q \times Q \to A$ satisfying Axioms (25)–(27). Using the triviality of the quandle operation $\lhd_0$, and renaming the variables in (26), we can rewrite the axioms as

$$\chi(a, b \diamond c) = \chi(a, b) + \chi(a, c), \tag{30}$$

$$\chi(b \diamond c, a) = \chi(b, a) + \chi(c, a), \tag{31}$$

$$\chi(a, b) = \lambda(b, a) - \lambda(a, b). \tag{32}$$

The last one implies that χ is anti-symmetric, which makes (31) a consequence of (30), and also shows that $\chi \in \mathrm{ABF}(Q, A)$. It thus suffices to consider Axioms (30) and (32) only. The maps X_χ and $\varLambda_\lambda$ clearly satisfy them. Moreover, $\mathscr{L}$ is a subgroup of $Z^2(Q, A)$ by construction, and so is $\mathscr{X}$, since $X_\chi + X_{\chi'} = X_{\chi+\chi'}$. The intersection of $\mathscr{X}$ and $\mathscr{L}$ is trivial: $X_\chi = \varLambda_\lambda$ implies $\chi = 0$, hence $X_\chi = 0$. To see that the two generate the whole $Z^2(Q, A)$, note that, as shown above, one has $\chi \in \mathrm{ABF}(Q, A)$ for any $(\chi, \lambda) \in Z^2(Q, A)$; then $(\chi, \lambda) - X_\chi$ is of the form $(0, \lambda')$ and still lies in $Z^2(Q, A)$, so, due to (32), it satisfies $\lambda'(a, b) = \lambda'(b, a)$, hence $(0, \lambda') \in \mathscr{L}$. ∎

One thus gets an abelian group isomorphism $Z^2(Q, A) \cong \mathrm{SF}(Q, A) \oplus \mathrm{ABF}(Q, A)$ for any trivial qualgebra Q.

EXAMPLE 6.8. Returning to Example 6.6, one sees that the part $\mathscr{L}$ of $Z^2(Q, A)$ always contains the cocycles $\varLambda_{\lambda_\alpha}$ and $\varLambda_{\delta_\alpha}$.

Note that for finite Q, the part $\mathscr{L}$ of $Z^2(Q, \mathbb{Z})$ has a basis $\{\varLambda_{x,y} = (0, \lambda_{x,y}) \mid x \le y\}$, where $\lambda_{x,y}$ takes value 1 on (possibly coinciding) pairs (x, y) and (y, x), and 0 elsewhere. Moreover, the part $\mathscr{X}$ of $Z^2(Q, \mathbb{Z})$ becomes trivial, since $\mathrm{ABF}(Q)$ reduces to the zero map: indeed, for $\chi \in \mathrm{ABF}(Q, \mathbb{Z})$ the bilinearity implies that $\chi(a, b \diamond b) = 2\chi(a, b)$, thus if χ takes a non-zero value $\chi(a, b)$, then it also takes arbitrarily large (or small) values $2^k \chi(a, b)$, $k \in \mathbb{N}$, which contradicts the finiteness of Q. However, the part $\mathscr{X}$ of $Z^2(Q, A)$ can be non-trivial even for finite Q if, for instance, $A = \mathbb{Z}_p$.

Qualgebra 2-cocycles for size 4 qualgebras. We now study $Z^2(P,A)$ for non-trivial 4-element qualgebras P.

Proposition 6.9. *Let $(P,\lhd,\diamond)$ be any of the nine 4-element qualgebras from Proposition 5.11. Then $Z^2(P,A)\cong A^8$ as abelian groups.*

Proof. Lemma 6.5 tells us to look for maps $\chi,\lambda : P\times P\to A$ satisfying Axioms (25)–(27).

Start with Axiom (25). For $c=r$ and $b\neq r$, one has $b\diamond c=r$, $a\lhd b=a$, so (25) $\Leftrightarrow$ $\chi(a,b)=0$. One gets the first relation describing 2-cocycles:

$$\forall x,\ \forall y\neq r,\qquad \chi(x,y)=0. \tag{33}$$

The case $b=r$, $c\neq r$ leads to the same relation. For $b,c\neq r$, one has $b\diamond c\neq r$, so (33) implies (25). In the remaining case $b=c=r$, one gets $\chi(a,r\diamond r)=\chi(a,r)+\chi(a\lhd r,r)$. The right side simplifies to $\chi(a,r)+\chi((a)\tau,r)$, the left one reduces to $\chi(a,r\diamond r)=\chi(a,s)=0$ due to (33). One obtains

$$\chi(p,r)+\chi(q,r)=0,\qquad 2\chi(r,r)=2\chi(s,r)=0. \tag{34}$$

We now turn to Axiom (27). If $b\neq r$, then $a\lhd b=a$, and, using (33), our axiom becomes $\lambda(a,b)=\lambda(b,a)$. This also holds true for $b=r$, $a\neq r$ by a symmetry argument, and trivially for $a=b=r$. Summarizing, one gets

$$\forall x,y,\qquad \lambda(x,y)=\lambda(y,x). \tag{35}$$

For $b=r$, (27) becomes $\chi(a,r)=\lambda(r,(a)\tau)-\lambda(a,r)$, or, separating different values of a and using the symmetry (35) of λ,

$$\chi(r,r)=\chi(s,r)=0, \tag{36}$$

$$\lambda(p,r)-\lambda(q,r)=\chi(q,r), \tag{37}$$

and $\lambda(q,r)-\lambda(p,r)=\chi(p,r)$, which is a consequence of (37) and (34) and is thus discarded. The relation $2\chi(r,r)=2\chi(s,r)=0$ above follows from (36) and is discarded as well.

It remains to analyze Axiom (26). For $c\neq r$ or for $c=r$ with $a,b\in\{r,s\}$, one has $a\lhd c=a$, $b\lhd c=b$, so everything vanishes due to (33). Take $c=r$. If $\{a,b\}=\{p,q\}$, then $a\diamond b=s$ (hence $\chi(a\diamond b,c)=\chi(s,r)=0$ due to (36)), $\chi(a,r)+\chi(b,r)=0$ because of (34), and $\lambda(a\lhd c,b\lhd c)=\lambda((a)\tau,(b)\tau)=\lambda(b,a)=\lambda(a,b)$; all of these imply our axiom. If $a=b=q$, then one gets

$$\lambda(p,p)-\lambda(q,q)=2\chi(q,r)-\chi(q\diamond q,r). \tag{38}$$

The case $a=b=p$ leads to the same relation by (34). For $a=r$, $b\in\{p,q\}$, one has $a\diamond b=r$, and our axiom becomes $\lambda(r,(b)\tau)=\chi(b,r)+\lambda(r,b)$, which is equivalent to (37) (due to (34)–(35)). The case $b=r$, $a\in\{p,q\}$ is analogous. If $a=s$ and $b=q$, then our axiom becomes $\chi(s\diamond q,r)+\lambda(s,p)=$

$\chi(q,r) + \lambda(s,q)$, or else

(39) $$\lambda(p,s) - \lambda(q,s) = \chi(q,r) - \chi(q \diamond s, r).$$

The cases $a = s$, $b = p$ or $b = s$, $a \in \{p,q\}$ lead to the same relation.

Putting everything together, one concludes that (χ, λ) is a 2-cocycle for P if and only if the maps $\chi, \lambda : P \times P \to A$ satisfy (33)–(39). Note that $\chi(q \diamond q, r)$ equals $\chi(q,r)$, $-\chi(q,r)$ or 0, according to $q \diamond q$ being chosen as q, p or s, and similarly for $\chi(q \diamond s, r)$. Thus, one sees that the eight values $\chi(q,r)$, $\lambda(q,r)$, $\lambda(q,s)$, $\lambda(q,q)$, $\lambda(q,p)$, $\lambda(r,r)$, $\lambda(s,r)$, and $\lambda(s,s)$ can be chosen arbitrarily, and the other values of χ and λ are deduced from these in a unique way. This gives $Z^2(P,A) \cong A^8$. ∎

Qualgebra 2-coboundaries. Recall the definition $\chi_\varphi(a,b) = \varphi(a \lhd b) - \varphi(a)$ of a 2-coboundary for quandles, with an arbitrary map $\varphi : Q \to A$ (Example 2.7). It can be interpreted as the difference between the total weight $\varphi(b) + \varphi(a \lhd b)$ at the bottom of the diagram describing the quandle coloring rule around a crossing, and the total weight $\varphi(a) + \varphi(b)$ at the top of this diagram (Figure 1Ⓐ). Trying to treat the coloring rule around a 3-valent vertex (Figure 1Ⓑ) in a similar way, one gets a good candidate for the notion of qualgebra 2-coboundary:

Definition 6.10. For a qualgebra Q and a map $\varphi : Q \to A$, the pair of maps $(\chi_\varphi, \lambda_\varphi)$ defined by

$$\chi_\varphi(a,b) = \varphi(a \lhd b) - \varphi(a), \qquad \lambda_\varphi(a,b) = \varphi(a) + \varphi(b) - \varphi(a \diamond b)$$

is called an (A-valued) *qualgebra* 2-*coboundary* of Q. The set of all qualgebra 2-coboundaries of Q is denoted by $B^2(Q,A)$.

Proposition 6.11. *Given a qualgebra* $(Q, \lhd, \diamond)$*, the set of its qualgebra* 2*-coboundaries* $B^2(Q,A)$ *is an abelian subgroup of* $Z^2(Q,A)$*. Moreover, for any* Q*-colored graph diagram* $(D, \mathcal{C})$ *and any* 2*-coboundary* (χ, λ)*, the weight* $\mathcal{W}_{(\chi,\lambda)}(D, \mathcal{C})$ *is zero.*

Before giving a proof, we write explicitly the weights of crossings and vertices constructed out of the maps χ_φ and λ_φ according to the rules from Figures 4 and 15; see Figure 18.

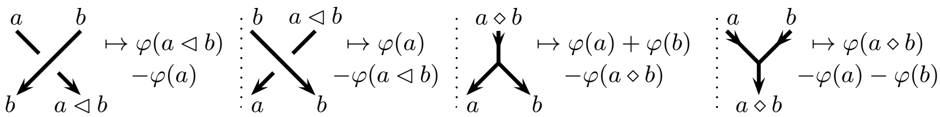

Fig. 18. Weight function for the maps χ_φ and λ_φ

Proof. We first show that a qualgebra 2-coboundary $(\chi_\varphi, \lambda_\varphi)$ of Q is also a qualgebra 2-cocycle. One can either check Axioms (25)–(27) directly, or develop the "total weight increment" argument which leads to the definition of qualgebra 2-coboundaries. Indeed, the $(\chi_\varphi, \lambda_\varphi)$-weight (Figure 18) of the Q-colored diagrams that appear in R-moves is the difference between the total φ-weight at the bottom and at the top of these diagrams (or vice versa). Since the bottom/top colors are the same for both diagrams involved in an R-move, these diagrams have the same $(\chi_\varphi, \lambda_\varphi)$-weights. Hence, according to (the proof of) Proposition 6.1, $(\chi_\varphi, \lambda_\varphi)$ is a qualgebra 2-cocycle.

We have thus showed that $B^2(Q, A) \subseteq Z^2(Q, A)$. To see that it is an abelian subgroup, observe that $(\chi_\varphi, \lambda_\varphi) + (\chi_{\varphi'}, \lambda_{\varphi'}) = (\chi_{\varphi+\varphi'}, \lambda_{\varphi+\varphi'})$.

Take now a Q-colored graph diagram $(D, \mathcal{C})$ and a 2-coboundary $(\chi_\varphi, \lambda_\varphi)$. As shown above, the latter is also a 2-cocycle, and hence defines a Boltzmann weight function. We shall now prove that the total χ_φ-weight of the crossings of $(D, \mathcal{C})$ kills the total λ_φ-weight of its 3-valent vertices, implying that $\mathcal{W}_{(\chi_\varphi, \lambda_\varphi)}(D, \mathcal{C}) = 0$.

Consider an edge e of D, and analyse how the color behaves when one moves along e. The color changes from a to $a \lhd b$ or $a \mathrel{\widetilde{\lhd}} b$ when e goes under a b-colored arc (depending on the orientation of the latter) and stays constant otherwise. Observe that $\varphi(a \lhd^{\pm 1} b) - \varphi(a)$ is precisely the χ_φ-weight of the crossing where the color changes. Hence the total weight of all the crossings of D is the sum $\sum_e [\varphi(\mathcal{C}(t(e))) - \varphi(\mathcal{C}(s(e)))]$ taken over all the edges e of D, where $s(e)$ and $t(e)$ are, respectively, the first and the last arcs of e. Since each edge starts and finishes at a 3-valent vertex, this sum can be reorganized as the sum $\sum_v \sum_{\alpha \in \mathscr{A}(v)} \pm\varphi(\alpha)$ taken over all the vertices v of D, where $\mathscr{A}(v)$ is the set of arcs adjacent to v, and $\varphi(\alpha)$ is taken with the sign $-$ if α is directed from v, and $+$ otherwise. On the other hand, the total weight of all the 3-valent vertices is the sum of the same form, but with opposite sign conventions (Figure 18). ∎

Example 6.12. Let us describe a qualgebra 2-coboundary $(\chi_\varphi, \lambda_\varphi)$ for a trivial qualgebra $(Q, \lhd_0, \diamond)$ (Definition 5.8). Its χ-component is zero: $\chi_\varphi(a, b) = \varphi(a \lhd_0 b) - \varphi(a) = \varphi(a) - \varphi(a) = 0$. Its λ-component is the symmetric form $\lambda_\varphi(a, b) = \varphi(a) + \varphi(b) - \varphi(a \diamond b)$ (recall that $\diamond$ is commutative for trivial qualgebras). Thus our 2-coboundaries have the form $\Lambda_{\lambda_\varphi}$, where φ runs through all maps from Q to A, and they all lie in the $\mathscr{L}$-part of $Z^2(Q, A)$ (Proposition 6.7).

Towards a qualgebra homology theory. Proposition 6.11 legitimates the following

Definition 6.13. For a qualgebra Q, the quotient abelian group

$$H^2(Q, A) = Z^2(Q, A)/B^2(Q, A)$$

is called the second (A-valued) *qualgebra cohomology group* of Q.

Proposition 6.11 and Lemma 6.4 imply that the Q-colored graph diagram weight $\mathcal{W}_{[(\chi,\lambda)]}(D,\mathcal{C})$ is well-defined for equivalence classes $[(\chi,\lambda)] \in H^2(Q,A)$. (Note that this need not be true for subdiagrams.)

We now calculate the second qualgebra cohomology groups for non-trivial 4-element qualgebras. We remark that the result is the same for all the nine structures. Note also the possibility of torsion.

PROPOSITION 6.14. *Let* $(P,\lhd,\diamond)$ *be any of the* 4*-element qualgebras from Proposition* 5.11*. Then* $B^2(P,A) \cong A^4$ *and* $H^2(P,A) \cong A/2A \oplus A^4$*.*

Proof. We give a proof only for the case when A is a ring. The proof in the general case follows the same lines, but is less readable.

According to the proof of Proposition 6.9, a basis of $Z^2(P,A) \cong A^8$ can be constructed as follows. For the ith generator, let the ith of the values $\chi(q,r)$, $\lambda(q,r)$, $\lambda(q,s)$, $\lambda(q,q)$, $\lambda(q,p)$, $\lambda(r,r)$, $\lambda(s,r)$, and $\lambda(s,s)$ be 1, and the other vanish; calculate the remaining values of χ and λ via relations (33)–(39). Denote this basis by $\mathscr{B} = (\varepsilon^{\chi}_{q,r}, \varepsilon_{q,r}, \varepsilon_{q,s}, \varepsilon_{q,q}, \varepsilon_{q,p}, \varepsilon_{r,r}, \varepsilon_{s,r}, \varepsilon_{s,s})$.

Consider the sub-A-module Z' of $Z^2(P,A)$ with a basis

$$\mathscr{B}' = (\varepsilon^{\chi}_{q,r}, \varepsilon_{q,r}, \varepsilon_{q,s}, \varepsilon_{q,q}, \varepsilon_{q,p}, 2\varepsilon_{r,r}, \varepsilon_{s,r} - \varepsilon_{r,r}, \varepsilon_{s,s}).$$

The "Dirac maps" $\varphi_a : P \to A$, $a \in P$, defined by $\varphi_a(a) = 1$ and $\varphi_a(x) = 0$ for $x \neq a$, form a basis of the A-module of maps $\varphi : P \to A$. Hence the pairs of maps $\varepsilon_a = (\chi_{\varphi_a}, \lambda_{\varphi_a})$, $a \in P$, generate $B^2(P,A)$. We will now show

$$\mathscr{B}'' = (\varepsilon_p, \varepsilon_q, \varepsilon_r, \varepsilon_s, \varepsilon_{q,s}, \varepsilon_{q,q}, \varepsilon_{q,p}, \varepsilon_{s,s})$$

to be an alternative basis of Z'. This will give a 4-element basis $(\varepsilon_p, \varepsilon_q, \varepsilon_r, \varepsilon_s)$ of $B^2(P,A)$ and a 4-element basis $([\varepsilon_{q,s}], [\varepsilon_{q,q}], [\varepsilon_{q,p}], [\varepsilon_{s,s}])$ of $Z'/B^2(P,A)$ (here and afterwards the square brackets stand for equivalence classes of pairs of maps). Moreover, by construction $Z^2(P,A)/Z' \cong A/2A$, and $[\varepsilon_{r,r}]$ is its generator. Putting together all the pieces, one gets

$$H^2(P,A) = Z^2(P,A)/B^2(P,A) \cong Z^2(P,A)/Z' \oplus Z'/B^2(P,A) \cong A/2A \oplus A^4.$$

In order to show that $\mathscr{B}''$ is indeed a basis, in Table 2 we calculate for the 2-coboundaries ε_a the eight values which completely determine a 2-cocycle. In the table, exactly one α_i and one β_j equal 1, while the other are zero; this depends on the values of $q \diamond s$ and $q \diamond q$ in our P.

Table 2. Essential components of the 2-coboundaries ε_a

	$\chi(q,r)$	$\lambda(q,r)$	$\lambda(r,r)$	$\lambda(s,r)$	$\lambda(q,s)$	$\lambda(q,q)$	$\lambda(q,p)$	$\lambda(s,s)$
ε_p	1	0	0	0	$-\alpha_1$	$-\beta_1$	1	0
ε_q	-1	1	0	0	$1-\alpha_2$	$2-\beta_2$	1	0
ε_r	0	0	2	0	0	0	0	0
ε_s	0	0	-1	1	$1-\alpha_3$	$-\beta_3$	-1	1

Adding some linear combinations of the 2-cocycles $\varepsilon_{q,s}$, $\varepsilon_{q,q}$, $\varepsilon_{q,p}$, and $\varepsilon_{s,s}$, one can transform the ε_a's into 2-cocycles $\widetilde{\varepsilon}_a$ for which the value table can be obtained from Table 2 by replacing everything to the right of the middle vertical bar by zeroes. Since the eight values in the table completely determine a 2-cocycle, the elements of $\mathscr{B}'$ can be expressed in terms of those of $\mathscr{B}''$ as follows:

$$\varepsilon^{\chi}_{q,r} = \widetilde{\varepsilon}_p, \qquad \varepsilon_{s,r} - \varepsilon_{r,r} = \widetilde{\varepsilon}_s,$$
$$\varepsilon_{q,r} = \widetilde{\varepsilon}_q + \widetilde{\varepsilon}_p, \quad 2\varepsilon_{r,r} = \widetilde{\varepsilon}_r.$$

Now, $\mathscr{B}''$ is a basis of Z' since $\mathscr{B}'$ is such. ■

REMARK 6.15. The weight invariants corresponding to the 2-cocycles $\varepsilon_{q,s}, \varepsilon_{q,q}, \varepsilon_{q,p}, \varepsilon_{s,s}$, and $\varepsilon_{r,r}$, whose classes modulo $B^2(P,A)$ generate $H^2(P,A)$, have an easy combinatorial description. Namely, $\mathcal{W}_{\varepsilon_{q,s}}(D,\mathcal{C})$ counts the difference between the numbers of unzip and zip vertices whose adjacent co-oriented arcs are colored with either s and p, or s and q; $\mathcal{W}_{\varepsilon_{q,p}}(D,\mathcal{C})$ counts a similar difference for arcs colored with p and q; finally, $\mathcal{W}_{\varepsilon_{q,q}}(D,\mathcal{C})$ (respectively, $\mathcal{W}_{\varepsilon_{s,s}}(D,\mathcal{C})$ or $\mathcal{W}_{\varepsilon_{r,r}}(D,\mathcal{C})$) counts a similar difference for both arcs having the same color p or q (respectively, s or r). Observe that $\mathcal{W}_{\varepsilon_{r,r}}(D,\mathcal{C})$ always vanishes: the r-colored edges of D and adjacent vertices form a 2-valent subgraph, and such a subgraph has the same number of sources and sinks (corresponding, respectively, to unzip and zip vertices in D). Now, Proposition 6.11 and Lemma 6.4 imply that the four non-vanishing invariants above contain all the information one can deduce from non-trivial 4-element qualgebra colorings of graphs using the Boltzmann weight method.

One would expect second qualgebra cohomology groups described above to fit into a complete *qualgebra cohomology theory*, extending the celebrated quandle cohomology theory. However, the author knows how to construct such a theory for *non-commutative qualgebras* only (that is, one keeps Axioms (1)–(5), but not the semi-commutativity (6)). Topologically, this structure corresponds to *rigid-vertex* poleless 3-valent graphs, for which move RV should be removed from the list of Reidemeister moves (cf. also [21]). 2-cocycles for this structure are defined by Axioms (11)–(12) and (25)–(26) (omitting (27)), and they give all Boltzmann weight functions for rigid-vertex graph diagrams. Our cohomology construction is based on the *braided system* concept from [22]; details will appear in a separate publication.

Squandle 2-cocycles. Weight invariants can also be constructed out of squandle colorings, by a procedure that very closely repeats what we have done for qualgebra colorings. We shall now briefly present relevant definitions and results; all the details and proofs can be easily adapted from the qualgebra case.

Definition 6.16. For a squandle Q, an (A-valued) *squandle* 2*-cocycle* of Q is a pair of maps $\chi : Q\times Q \to A$, $\lambda : Q \to A$ satisfying Axioms (11)–(12) together with two additional ones:

$$\chi(a, b^2) = \chi(a, b) + \chi(a \lhd b, b), \qquad \chi(a^2, b) + \lambda(a \lhd b) = 2\chi(a, b) + \lambda(a).$$

The abelian group of all squandle 2-cocycles of Q is denoted by $Z^2(Q, A)$.

Note that Axioms (11)–(12) can no longer be omitted from the definition.

Proposition 6.17. *Take a squandle Q and maps $\chi : Q \times Q \to A$, $\lambda : Q \to A$. The weight function constructed out of (χ, λ) according to Figures* 4 *and* 19 *is Boltzmann if and only if $(\chi, \lambda) \in Z^2(Q, A)$.*

Fig. 19. Weight function for squandle-colored graph diagrams

Corollary 6.18. *Take a squandle Q and a squandle* 2*-cocycle (χ, λ). Consider the Q-coloring rules from Figure* 1Ⓐ&Ⓒ *and the weight function from Figures* 4 *and* 19*, still denoted by (χ, λ). Then the multi-set $\{\, \mathcal{W}_{(\chi,\lambda)}(D, \mathcal{C}) \mid \mathcal{C} \in \mathscr{C}_Q(D)\,\}$ does not depend on the choice of a diagram D representing a poleless* 3*-valent knotted graph Γ.*

Definition 6.19. For a squandle Q and a map $\varphi : Q \to A$, the pair of maps $(\chi_\varphi, \lambda_\varphi)$ defined by

$$\chi_\varphi(a, b) = \varphi(a \lhd b) - \varphi(a), \qquad \lambda_\varphi(a) = 2\varphi(a) - \varphi(a^2)$$

is called an (A-valued) *squandle* 2*-coboundary* of Q. The abelian group of all squandle 2-coboundaries of Q is denoted by $B^2(Q, A)$.

Proposition 6.20. *For a squandle Q, the set of its squandle* 2*-coboundaries $B^2(Q, A)$ is a subgroup of $Z^2(Q, A)$. Moreover, for any Q-colored graph diagram $(D, \mathcal{C})$ and any* 2*-coboundary (χ, λ), the weight $\mathcal{W}_{(\chi,\lambda)}(D, \mathcal{C})$ is zero.*

Definition 6.21. The second (A-valued) *squandle cohomology group* of a squandle Q is the quotient abelian group $H^2(Q, A) = Z^2(Q, A)/B^2(Q, A)$.

Example 6.22. For trivial squandles, 2-coboundaries have the form $(0, \lambda_\varphi)$ with $\lambda_\varphi(a) = 2\varphi(a) - \varphi(a^2)$, and 2-cocycles have the form (χ, λ), where λ is arbitrary and χ satisfies

$$\chi(a, b^2) = \chi(a^2, b) = 2\chi(a, b), \qquad \chi(a, a) = 0.$$

All $\mathbb{Z}$-valued 2-cocycles of finite trivial squandles have a zero χ-part.

Example 6.23. Recall the 4-element squandles from Proposition 5.14. Arguments analogous to those used to prove Propositions 6.9 and 6.14 show that for all these squandles, $Z^2(Q,A) \cong B^2(Q,A) \cong A^4$. For cohomology, one has $H^2(Q,A) \cong A/2A$, except for the squandle of the second type with $q^2 = s$, in which case $H^2(Q,A) \cong A/2A \oplus A/2A$.

7. Going further. This is the first in a series of publications devoted to qualgebras and squandles. A lot of work remains to be done on the algebraic as well as on the topological sides.

First, we are currently working on an algebraic study of qualgebras and squandles [23]: their general properties, free structures, the "qualgebraization" of familiar quandles (cf. Example 3.15), conceptual examples, a classification of all structures in small size. Sizes 5 and 6 are still doable by hand, and contain a large variety of examples. It would be interesting to calculate the induced invariants for reasonably "small" graphs. General qualgebra and squandle cohomology theories would be of interest (cf. Section 6).

There is also a variation of qualgebra/squandle structure called *symmetric qualgebra/squandle*. It includes an involution ρ compatible with both the quandle operation $\lhd$ (in the sense of Axioms (18)–(20)) and the qualgebra/squandle operation. Symmetric quandles were invented by S. Kamada [20] in order to extend quandle coloring invariants of oriented knots to unoriented ones; they were later used by Y. Jang and K. Oshiro [17] to extend quandle coloring invariants of oriented graphs (with coloring rules from Figure 6Ⓒ) to unoriented ones. Similarly, our symmetric qualgebras/squandles are tailored for coloring unoriented knotted 3-valent graph diagrams, and lead to invariants of such graphs. Together with the usual group examples, one finds numerous examples even in small size.

Lastly, a variation of coloring ideas includes assigning colors to diagram regions, and not only arcs, with a relevant notion of topological coloring rules. Such colorings are called *shadow colorings* in the quandle case, and corresponding counting and weight invariants prove to be extremely powerful for knots. The same can be done for graphs by introducing the notions of *qualgebra/squandle modules* (used for coloring regions), *qualgebra/squandle* 2*-cocycles with coefficients* (used for fabricating Boltzmann weight functions), and constructing counting and weight invariants out of these. Regarding a qualgebra/squandle as a module over itself, one naturally gets a definition of *qualgebra/squandle* 3*-cocycles* (without coefficients), suggesting one more step towards a qualgebra/squandle cohomology theory.

A detailed study of these constructions and their topological applications will appear in a separate publication.

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Victoria Lebed
OCAMI, Osaka City University
3-3-138 Sugimoto-cho, Sumiyoshi-ku
Osaka, 558-8585, Japan
E-mail: lebed.victoria@gmail.com

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