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Realization and nonrealization of Poincaré duality quotients of $\mathbb{F}_2[x,y]$ as topological spaces

by

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Abstract. Let $\mathbf{d}_{2,0} = x^2y + xy^2$, $\mathbf{d}_{2,1} = x^2 + xy + y^2 \in \mathbb{F}_2[x,y]$ be the two Dickson polynomials. If a and b are positive integers, the ideal $(\mathbf{d}_{2,0}^a, \mathbf{d}_{2,1}^b) \subset \mathbb{F}_2[x,y]$ is invariant under the action of the mod 2 Steenrod algebra $\mathscr{A}^*$ if and only if when we write $b = 2^t \cdot k$ with k odd, then $a \le 2^t$. The quotient algebra $\mathbb{F}_2[x,y]/(\mathbf{d}_{2,0}^a, \mathbf{d}_{2,1}^b)$ is a Poincaré duality algebra and for such a and b admits an unstable action of $\mathscr{A}^*$. It has trivial Wu classes if and only if $a = 2^t$ for some $t \ge 0$ and $b = 2^t(2^s - 1)$ for some $s > 0$. We ask under what conditions on a and b, $\mathbb{F}_2[x,y]/(\mathbf{d}_{2,0}^a, \mathbf{d}_{2,1}^b)$ appears as the mod 2 cohomology of a manifold. In this note we show that for $a = 2^t = b$ there is a topological space whose cohomology is $\mathbb{F}_2[x,y]/(\mathbf{d}_{2,0}^{2^t}, \mathbf{d}_{2,1}^{2^t})$ if and only if $t = 0, 1, 2$, or 3, and in these cases the space may be taken to be a smooth manifold.

Let $\mathbf{d}_{2,0} = x^2y + xy^2, \mathbf{d}_{2,1} = x^2 + xy + y^2 \in \mathbb{F}_2[x,y]$ be the two Dickson polynomials. In [10] we determined which ideals of the form $(\mathbf{d}_{2,0}^a, \mathbf{d}_{2,1}^b) \subset \mathbb{F}_2[x,y]$ are invariant under the action of the mod 2 Steenrod algebra ([1]). The corresponding Poincaré duality algebras, viz. $\mathbb{F}_2[x,y]/(\mathbf{d}_{2,0}^a, \mathbf{d}_{2,1}^b)$, support the structure of an unstable algebra over the Steenrod algebra $\mathscr{A}^*$. As such they have *Wu classes* Wu_i defined by the requirement that $\mathrm{Sq}^i(u) = \mathrm{Wu}_i \cup u$ whenever $i + \deg(u)$ is the degree of the fundamental class of the Poincaré duality algebra (see e.g. [2]). Amongst the $\mathscr{A}^*$-unstable algebras $\mathbb{F}_2[x,y]/(\mathbf{d}_{2,0}^a, \mathbf{d}_{2,1}^b)$ those for which the Wu classes are trivial are of particular interest: this is for example the case if $a = 2^t = b$. We showed they can be used to study the so called *Hit Problem* (this problem has a large literature and we refer the reader to [16] and the reference list there). Namely, a monomial $x^\alpha y^\gamma \in \mathbb{F}_2[x,y]$ that represents a fundamental class for a quotient $\mathbb{F}_2[x,y]/(\mathbf{d}_{2,0}^a, \mathbf{d}_{2,1}^b)$ with trivial Wu classes is an $\mathscr{A}^*$-indecomposable. In the case of two variables mod 2 we thereby gave the known solution [11] to the

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([1]) We include in Section 3 an ad hoc derivation (after all we know the answer!) that avoids the theory developed in [10] to handle the general case.

problem of determining generators for $\mathbb{F}_2[x,y]$ as $\mathscr{A}^*$-module a fully new interpretation: see the comments following Theorem 1.1.

In this note we examine which of the $\mathscr{A}^*$-unstable algebras

$$\mathbb{F}_2[x,y]/(\mathbf{d}_{2,0}^{2^t},\mathbf{d}_{2,1}^{2^t})$$

can appear as the mod 2 cohomology of a topological space. The method that we use to construct those that do yields a closed smooth manifold.

We have tried to keep the notation as standard as possible: for any unexplained notations we refer to [15] and [14].

1. Recollections and statement of results. We denote by $\mathbf{d}_{2,0} = x^2y + xy^2, \mathbf{d}_{2,1} = x^2 + xy + y^2 \in \mathbb{F}_2[x,y]$ the two Dickson polynomials. They form a regular sequence, as do $\mathbf{d}_{2,0}^a, \mathbf{d}_{2,1}^b$ for any $a, b \in \mathbb{N}$. Hence the quotient algebra $\mathbb{F}_2[x,y]/(\mathbf{d}_{2,0}^a,\mathbf{d}_{2,1}^b)$ satisfies Poincaré duality (see e.g. [13] or [14, Theorem 6.5.1]). We introduce the notation $\mathfrak{d}(a,b) = (\mathbf{d}_{2,0}^a,\mathbf{d}_{2,1}^b)$ for the ideal in $\mathbb{F}_2[x,y]$ generated by $\mathbf{d}_{2,0}^a$ and $\mathbf{d}_{2,1}^b$, and write $H(a,b)$ for the quotient algebra $\mathbb{F}_2[x,y]/\mathfrak{d}(a,b)$. If the ideal $\mathfrak{d}(a,b)$ is invariant under the action of the mod 2 Steenrod algebra $\mathscr{A}^*$ then this algebra has Wu classes (see e.g. [2]). In [10] we proved the following result:

THEOREM 1.1. *For $a, b \in \mathbb{N}$ the ideal $\mathfrak{d}(a,b)$ is invariant under the action of the mod 2 Steenrod algebra $\mathscr{A}^*$ if and only if when we write $b = 2^t \cdot k$ with k odd, then $a \le 2^t$. The quotient algebra $H(a,b)$ has trivial Wu classes if and only if $a = 2^t$ and $b = 2^t(2^s - 1)$ for some $s > 0$.*

This result follows quite naturally from a number of general principles established in [10, Parts II and III]. A reason why the quotient algebras $H(a,b)$ with trivial Wu classes are particularly interesting is that monomials representing their fundamental class are $\mathscr{A}^*$-indecomposable in $\mathbb{F}_2[x,y]$ (see e.g. [11] and [16]). In fact we show in [10, Part V] that choosing one representative for the fundamental class of each such algebra and adjoining the distinct products of the form $x^{2^s-1}y^{2^t-1}$ for $s, t \in \mathbb{N}_0$ one obtains a basis for the $\mathscr{A}^*$-indecomposables of $\mathbb{F}_2[x,y]$. For the sake of completeness, and to make this short note independent of [10], we present an ad hoc proof of Theorem 1.1 in Section 3.

In this note we consider the problem of realizing the quotient algebras $H(2^t,2^t)$ for $t \in \mathbb{N}_0$, which support an $\mathscr{A}^*$-action, as the $\mathbb{F}_2$-cohomology of a topological space. We prove:

THEOREM 1.2. *The unstable $\mathscr{A}^*$-algebra $H(2^t,2^t) \cong \mathbb{F}_2[x,y]/(\mathbf{d}_{2,0}^{2^t},\mathbf{d}_{2,1}^{2^t})$ occurs as a cohomology algebra if and only if $t = 0, 1, 2, 3$.*

The proof of this result occupies the next section.

2. Constructions and nonexistence results. The first result provides us with a means of constructing examples of spaces realizing $H(2^t, 2^t)$ as an algebra over the Steenrod algebra for small t. We denote by $\mathbb{RP}(n)$ the n-dimensional real projective space. We write $x \in H^1(\mathbb{RP}(n); \mathbb{F}_2)$ for the nonzero element and $\xi\downarrow\mathbb{RP}(n)$ for the canonical line bundle; so the first Stiefel–Whitney class $w_1(\xi)$ is x. If $\eta\downarrow X$ is a real vector bundle over the space X then $\mathbb{RP}(\eta\downarrow X)$ denotes the total space of the associated real projective space bundle $\mathbb{RP}(\eta)\downarrow X$.

Proposition 2.1. *If there exists a vector bundle* $\tau_t\downarrow\mathbb{RP}(3\cdot 2^t - 1)$ *of dimension* 2^{t+1} *having total Stiefel–Whitney class* $w(\tau_t) = 1 + x^{2^t} + x^{2^{t+1}} \in H^{**}(\mathbb{RP}(3\cdot 2^t - 1); \mathbb{F}_2)$ *then*

$$\begin{aligned} &H^*(\mathbb{RP}(\tau_t\downarrow\mathbb{RP}(3\cdot 2^t - 1)); \mathbb{F}_2) \\ &\qquad = \mathbb{F}_2[x,y]/((x^2y + xy^2)^{2^t}, (x^2 + xy + y^2)^{2^t}) = H(2^t, 2^t). \end{aligned}$$

For $t = 0, 1, 2, 3$ *such a bundle exists, and hence* $H(2^t, 2^t)$ *occurs as a cohomology algebra for* $t = 0, 1, 2, 3$.

Proof. Suppose $\tau_t\downarrow\mathbb{RP}(3\cdot 2^t - 1)$ is a vector bundle as in the statement. We employ the projective bundle theorem, [6, Chapter 16, Theorem 2.5], to compute the mod 2 cohomology of the total space $\mathbb{RP}(\tau_t\downarrow\mathbb{RP}(3\cdot 2^t - 1))$ of the corresponding real projective bundle and find

$$\begin{aligned} H^*(\mathbb{RP}(\tau_t\downarrow\mathbb{RP}(3\cdot 2^t - 1)); \mathbb{F}_2) &\cong \frac{H^*(\mathbb{RP}(3\cdot 2^t - 1); \mathbb{F}_2)[y]}{(\sum_{i=0}^{2^{t+1}} y^i w_{2^{t+1}-i}(\tau_t))} \\ &\cong \frac{\mathbb{F}_2[x,y]}{(x^{3\cdot 2^t}, y^{2^{t+1}} + y^{2^t}x^{2^t} + x^{2^{t+1}})}. \end{aligned}$$

Note that

$$\begin{aligned} x^{2^t}(y^{2^{t+1}} + y^{2^t}x^{2^t} + x^{2^{t+1}}) &= (x^{2^t}y^{2^{t+1}} + y^{2^t}x^{2^{t+1}}) + x^{2^{t+1}+2^t} \\ &= (xy^2 + x^2y)^{2^t} + x^{3\cdot 2^t} \end{aligned}$$

so we can rewrite the cohomology as

$$\mathbb{F}_2[x,y]/((x^2y + xy^2)^{2^t}, (x^2 + xy + y^2)^{2^t}),$$

which is $H(2^t, 2^t)$.

It remains to prove that for $0 \le t \le 3$ such a bundle $\tau_t\downarrow\mathbb{RP}(3\cdot 2^t - 1)$ exists. To this end consider the bundle $\underbrace{\xi \oplus \ldots \oplus \xi}_{3\cdot 2^t}\downarrow\mathbb{RP}(3\cdot 2^t - 1)$. Its total Stiefel–Whitney class is

$$w(\underbrace{\xi \oplus \ldots \oplus \xi}_{3\cdot 2^t}) = w(\xi)^{3\cdot 2^t} = (1+x)^{3\cdot 2^t} = (1+x)^{2^{t+1}}(1+x)^{2^t} = 1 + x^{2^t} + x^{2^{t+1}}$$

because $x^{3\cdot 2^t} = 0 \in H^*(\mathbb{RP}(3\cdot 2^t - 1))$. The dimension of this bundle is $3\cdot 2^t$.

If this bundle had 2^t linearly independent cross sections, then we could write

$$\underbrace{\xi \oplus \ldots \oplus \xi}_{\longleftarrow 3\cdot 2^t \longrightarrow} \cong \tau_t \oplus \mathbb{R}^{2^t} \downarrow \mathbb{RP}(3 \cdot 2^t - 1)$$

and τ_t would be the sought for bundle.

Denote by $s(k,n)$ the maximum number of linearly independent cross sections of the bundle $k\xi\downarrow\mathbb{RP}(n)$. The geometric dimension of $k\xi\downarrow\mathbb{RP}(n)$ is therefore $k - s(k,n)$, i.e., we may write $k\xi \cong \widetilde{\xi} \oplus \mathbb{R}^{s(k,n)}\downarrow\mathbb{RP}(n)$ for a suitable vector bundle $\widetilde{\xi}$ of dimension $k - s(k,n)$. According to K. Y. Lam, [7, Theorem 1.12] or [8, Theorem 2.1], $s(n+1,n) = \varrho(n+1)$, where $\varrho(m)$ is the Radon–Hurwitz number: if $m = 2^{\alpha+4\beta}(2\gamma+1)$ where $0 \le \alpha \le 3$ then $\varrho(m) = 2^\alpha + 8\beta$. We use this to compute the geometric dimension of the bundle $\underbrace{\xi \oplus \ldots \oplus \xi}_{\longleftarrow 3\cdot 2^t \longrightarrow}\downarrow\mathbb{RP}(3 \cdot 2^t - 1)$.

For small values of t we obtain the following table.

t	$\mathbb{RP}(3 \cdot 2^t - 1)$	$3 \cdot 2^t$	$\varrho(3 \cdot 2^t)$	g.d.$(3 \cdot 2^t\xi\downarrow\mathbb{RP}(3 \cdot 2^t - 1))$	2^{t+1}
0	$\mathbb{RP}(2)$	3	1	2	2
1	$\mathbb{RP}(5)$	6	2	4	4
2	$\mathbb{RP}(11)$	12	4	8	8
3	$\mathbb{RP}(23)$	24	8	16	16
4	$\mathbb{RP}(47)$	48	9	39	32

In the table g.d.$(-)$ denotes the geometric dimension of the bundle in parentheses. The difference of the entries in the columns headed $3 \cdot 2^t$ and $\varrho(3 \cdot 2^t)$ is the geometric dimension listed in the fifth column. The last column shows the geometric dimension needed to construct the desired vector bundle. From this table we deduce the existence of the bundle $\tau_t\downarrow\mathbb{RP}(3 \cdot 2^t - 1)$ for $0 \le t \le 3$. ∎

Note that the table occurring in the preceding proof suggests that the $\mathscr{A}^*$-unstable algebra $H(16,16)$ does not occur as the cohomology of a topological space. This is in fact the case: namely, the remaining examples of unstable $\mathscr{A}^*$-algebras of the form $H(2^t, 2^t)$ do not occur as the mod 2 cohomology of a space. To demonstrate this, suppose that $X(t)$ is a topological space that has mod 2 cohomology isomorphic to $H(2^t, 2^t)$ as an algebra over the Steenrod algebra. Let $f_t : X(t) \to \mathbb{RP}(\infty) \times \mathbb{RP}(\infty)$ be a map that realizes the natural epimorphism

$$\mathbb{F}_2[x,y] \to H(2^t, 2^t) = \mathbb{F}_2[x,y]/(\mathbf{d}_{2,0}^{2^t}, \mathbf{d}_{2,1}^{2^t})$$

in mod 2 cohomology. Let $F(t)$ be the homotopy fibre of f_t. The space $\mathbb{RP}(\infty) \times \mathbb{RP}(\infty)$ has fundamental group $\mathbb{Z}/2 \times \mathbb{Z}/2$. The composition factors of a finite-dimensional vector space over a field of characteristic p acted on by a finite p-group are trivial. It follows from [3] that we may employ the

Eilenberg–Moore spectral sequence to compute the mod 2 cohomology of $F(t)$ (see e.g. [12], whose notations we employ).

Suppose $t > 0$. The Eilenberg–Moore spectral sequence for the fibration $F(t) \hookrightarrow X(t) \xrightarrow{f_t} \mathbb{RP}(\infty) \times \mathbb{RP}(\infty)$ has as E_2-term

$$E_2^{*,*} \cong \mathrm{Tor}^{*,*}_{\mathbb{F}_2[x,y]}(\mathbb{F}_2[x,y]/(\mathbf{d}^{2^t}_{2,0}, \mathbf{d}^{2^t}_{2,1}), \mathbb{F}_2).$$

Since $\mathbf{d}^{2^t}_{2,0}, \mathbf{d}^{2^t}_{2,1}$ is a regular sequence in $\mathbb{F}_2[x,y]$ this may be computed with the Koszul complex (see e.g. [14, Section 6.1] or [12, Part I]) and the result is

$$E_2^{*,*} \cong E(s^{-1}(\mathbf{d}^{2^t}_{2,0}), s^{-1}(\mathbf{d}^{2^t}_{2,1})),$$

i.e., a bigraded exterior algebra on two generators $s^{-1}(\mathbf{d}^{2^t}_{2,0})$ of bidegree $(-1, 3\cdot 2^t)$ and $s^{-1}(\mathbf{d}^{2^t}_{2,1})$ of bidegree $(-1, 2^{t+1})$. From this it easily follows that $E_2 = E_\infty$. The action of the Steenrod algebra on $E_2 = E_\infty$ satisfies $\mathrm{Sq}^{2^t}(s^{-1}(\mathbf{d}^{2^t}_{2,1})) = s^{-1}(\mathbf{d}^{2^t}_{2,0})$. For degree reasons the extension problem as algebra over the Steenrod algebra from E_∞ to $H^*(F(t), \mathbb{F}_2)$ is trivial. Therefore

$$H^*(F(t); \mathbb{F}_2) \cong \mathrm{Tot}(E(s^{-1}(\mathbf{d}^{2^t}_{2,0}), s^{-1}(\mathbf{d}^{2^t}_{2,1})))$$

where $\mathrm{Tot}(E(s^{-1}(\mathbf{d}^{2^t}_{2,0}), s^{-1}(\mathbf{d}^{2^t}_{2,1})))$ is a graded exterior algebra on generators $s^{-1}(\mathbf{d}^{2^t}_{2,0})$ of degree $3\cdot 2^t - 1$ and $s^{-1}(\mathbf{d}^{2^t}_{2,1})$ of degree $2^{t+1} - 1$. Since

$$\mathrm{Sq}^{2^t} : H^{2^{t+1}-1}(F(t), \mathbb{F}_2) \to H^{3\cdot 2^t - 1}(F(t), \mathbb{F}_2)$$

is nonzero and $H^i(F(t), \mathbb{F}_2) = 0$ for $2^{t+1} - 1 < i < 3\cdot 2^t - 1$ the solution to the Hopf invariant one problem [1] shows that this is impossible if $t > 3$.

Therefore, combining this discussion with Proposition 2.1 we have proven Theorem 1.2.

One further interesting aspect of this circle of examples and nonexamples is to examine the spaces $F(t)$ for $t = 0, 1, 2,$ and 3 which occurred after the proof of Proposition 2.1, and ask if we can identify them. Indeed, with the help of an e-mail exchange with Fred Cohen and John Hubbuck, we can to some extent. If we denote the exterior generators of $H^*(F(i); \mathbb{F}_2)$ by $u_{2^{t+1}-1}$ and $u_{3\cdot 2^t - 1}$, for $i = 1, 2, 3$, and the single generator for $H^*(F(0); \mathbb{F}_2)$ by u_1, so that $\mathrm{Sq}^{2^t} u_{2^{t+1}-1} = u_{3\cdot 2^t - 1}$, respectively $\mathrm{Sq}^1(u_1) = u_1^2$, then

$$\begin{aligned}
H^*(F(0); \mathbb{F}_2) &\cong F_2[u_1]/(u_1^3) \cong H^*(\mathbb{SO}(3); \mathbb{F}_2),\\
H^*(F(1); \mathbb{F}_2) &\cong E(u_3, u_5) \cong H^*(\mathbb{SU}(3); \mathbb{F}_2),\\
H^*(F(2); \mathbb{F}_2) &\cong E(u_7, u_{11}) \cong H^*(\mathbb{Sp}(3)/\mathbb{Sp}(1); \mathbb{F}_2),\\
H^*(F(3); \mathbb{F}_2) &\cong E(u_{15}, u_{23}) \cong H^*(F_4/G_2; \mathbb{F}_2).
\end{aligned}$$

The first two examples are realized by Lie groups which are homotopy unique as H-spaces (see e.g. [4]). Of the remaining two examples, it is known that no space with the stated cohomology can be an H-space (see [5, Theorem 5.4]). We are unaware of any reference that these spaces are homotopy unique at the prime 2.

3. A proof of Theorem 1.1. If $\mathbf{Sq} = 1 + \mathrm{Sq}^1 + \ldots + \mathrm{Sq}^k + \ldots$ is the total Steenrod operation then (see e.g. [14, Chapter 9, Section 4])

$$(\circledast)\qquad \begin{aligned} \mathbf{Sq}(\mathbf{d}_{2,0}) &= \mathbf{d}_{2,0}(1 + \mathbf{d}_{2,1} + \mathbf{d}_{2,0}), \\ \mathbf{Sq}(\mathbf{d}_{2,1}) &= \mathbf{d}_{2,0} + \mathbf{d}_{2,1}(1 + \mathbf{d}_{2,1}). \end{aligned}$$

From these formulae it is routine to verify the following:

LEMMA 3.1. *The ideal* $\mathfrak{d}(a,b) = (\mathbf{d}_{2,0}^a, \mathbf{d}_{2,1}^b) \subset \mathbb{F}_2[x,y]$ *is invariant under the action of the mod 2 Steenrod algebra* $\mathscr{A}^*$ *if and only if upon writing* $b = 2^t \cdot k$ *with* k *odd one has* $a \le 2^t$. ∎

Although the proof of Theorem 1.1 in this section is ad hoc, a number of interesting results occur along the way. We begin with the observation that $\{1, x, y, y^2, xy, xy^2\}$ projects to an $\mathbb{F}_2$-basis for the quotient algebra $H(1,1) = \mathbb{F}_2[x,y]/(\mathbf{d}_{2,0}, \mathbf{d}_{2,1})$. Since the Dickson polynomials $\mathbf{d}_{2,0}, \mathbf{d}_{2,1} \in \mathbb{F}_2[x,y]$ form a regular sequence the algebra $\mathbb{F}_2[x,y]$ is a free $\mathbb{F}_2[\mathbf{d}_{2,0}^a, \mathbf{d}_{2,1}^b]$-module for any $a, b \in \mathbb{N}$. Using the nested chain of subalgebras $\mathbb{F}_2[\mathbf{d}_{2,0}^a, \mathbf{d}_{2,1}^b] \subseteq \mathbb{F}_2[\mathbf{d}_{2,0}, \mathbf{d}_{2,1}] \subset \mathbb{F}_2[x,y]$ we see that

$$(\star)\qquad \{\mathbf{d}_{2,0}^\lambda \cdot \mathbf{d}_{2,1}^\mu \cdot h \mid 0 \le \lambda < a,\ 0 \le \mu < b, \text{ and } h \in \{1, x, y, y^2, xy, xy^2\}\}$$

is a basis for $\mathbb{F}_2[x,y]$ as an $\mathbb{F}_2[\mathbf{d}_{2,0}^a, \mathbf{d}_{2,1}^b]$-module, and hence projects to an $\mathbb{F}_2$-basis for $\mathbb{F}_2[x,y]/(\mathbf{d}_{2,0}^a, \mathbf{d}_{2,1}^b)$. The element $\mathbf{d}_{2,0}^{a-1} \cdot \mathbf{d}_{2,1}^{b-1} \cdot xy^2$ represents a fundamental class of this quotient algebra.

LEMMA 3.2. *The graded vector subspace of* $\mathbb{F}_2[x,y]$ *defined by*

$$M = \mathrm{Span}_{\mathbb{F}_2}\{\mathbf{d}_{2,0}^\lambda \cdot \mathbf{d}_{2,1}^\mu \cdot h \mid \lambda, \mu \in \mathbb{N}_0 \text{ and } h \in \{1, x, y, y^2, xy\}\} \subset \mathbb{F}_2[x,y]$$

is closed under the action of the Steenrod algebra on $\mathbb{F}_2[x,y]$. (*N.B.* $h = xy^2$ *is excluded from the list.*)

Proof. Direct computation gives

$$(\maltese)\qquad \begin{aligned} \mathbf{Sq}(x) &= x + x^2 = x + \mathbf{d}_{2,1} \cdot 1 + xy + y^2, \\ \mathbf{Sq}(y) &= y + y^2, \\ \mathbf{Sq}(xy) &= (x + x^2)(y + y^2) = xy + \mathbf{d}_{2,0} \cdot 1 + \mathbf{d}_{2,0} \cdot (x + y) + \mathbf{d}_{2,1} \cdot xy, \\ \mathbf{Sq}(y^2) &= y^2(1 + y^2) = y^2 + \mathbf{d}_{2,0} \cdot y + \mathbf{d}_{2,1} \cdot y^2. \end{aligned}$$

Since $\mathbf{Sq}$ is multiplicative, for $\lambda, \mu \in \mathbb{N}_0$ and $h \in \mathbb{F}_2[x,y]$ one has

$$\mathbf{Sq}(\mathbf{d}_{2,0}^\lambda \cdot \mathbf{d}_{2,1}^\mu \cdot h) = \mathbf{Sq}(\mathbf{d}_{2,0})^\lambda \cdot \mathbf{Sq}(\mathbf{d}_{2,1})^\mu \cdot \mathbf{Sq}(h),$$

and the result follows from ($\circledast$) and ($\maltese$). ∎

LEMMA 3.3. *Suppose that the ideal* $\mathfrak{d}(a,b) = (\mathbf{d}_{2,0}^a, \mathbf{d}_{2,1}^b) \subset \mathbb{F}_2[x,y]$ *is closed under the action of the Steenrod algebra. Then the graded vector subspace of* $\mathbb{F}_2[x,y]/\mathfrak{d}(a,b)$ *defined by*

$$N = \operatorname{Span}_{\mathbb{F}_2}\{\mathbf{d}_{2,0}^{\lambda} \cdot \mathbf{d}_{2,1}^{\mu} \cdot h \mid 0 \le \lambda < a,\ 0 \le \mu < b,\ \textit{and}\ h \in \{1, x, y, y^2, xy\}\}$$

is closed under the action of the Steenrod algebra on $\mathbb{F}_2[x,y]/\mathfrak{d}(a,b)$. (*N.B. Again* $h = xy^2$ *is excluded from the list.*) *Hence the Wu class* Wu_k *of this algebra is nontrivial if and only if there are integers* $\lambda, \mu \in \mathbb{N}_0$ *with* $0 \le \lambda < a$, $0 \le \mu < b$ *such that* $\mathrm{Sq}^k(\mathbf{d}_{2,0}^{\lambda} \cdot \mathbf{d}_{2,1}^{\mu} \cdot xy^2) = \mathbf{d}_{2,0}^{a-1} \cdot \mathbf{d}_{2,1}^{b-1} \cdot xy^2$.

Proof. Since N is the image of M of Lemma 3.2 under the quotient map $\mathbb{F}_2[x,y] \to \mathbb{F}_2[x,y]/\mathfrak{d}(a,b)$, which is an $\mathscr{A}^*$-module homomorphism, the first assertion follows from Lemma 3.2. The second assertion follows from the first assertion, the fact that the set $(\star)$ projects to an $\mathbb{F}_2$-basis for $\mathbb{F}_2[x,y]/(\mathbf{d}_{2,0}^a, \mathbf{d}_{2,1}^b)$, and that the element $\mathbf{d}_{2,0}^{a-1} \cdot \mathbf{d}_{2,1}^{b-1} \cdot xy^2$ represents a fundamental class of this quotient algebra. ∎

Proof of Theorem 1.1. Consider the Poincaré duality algebra $H(a,b)$, where $b = 2^t \cdot c$ for some odd number c, and $0 < a \le 2^t$. First we show that if $a \ne 2^t$ or if c is not of the form $2^s - 1$ for some $s > 0$, then $H(a,b)$ has nontrivial Wu classes.

To this end recall that a representative of the fundamental class is given by $\mathbf{d}_{2,0}^{a-1}\mathbf{d}_{2,1}^{b-1} \cdot xy^2$. We will describe elements in $H(a,b)$ that are mapped to this fundamental class by a Steenrod operation.

First note that

$$\begin{aligned}
\mathrm{Sq}^1(xy^2) &= x^2y^2 = \mathbf{d}_{2,0} \cdot x + \mathbf{d}_{2,0} \cdot y + \mathbf{d}_{2,1} \cdot xy,\\
\mathrm{Sq}^2(xy^2) &= xy^4 = \mathbf{d}_{2,0} \cdot xy + \mathbf{d}_{2,1} \cdot xy^2,\\
\mathrm{Sq}^3(xy^2) &= x^2y^4\\
&= \mathbf{d}_{2,0}^2 \cdot 1 + \mathbf{d}_{2,0}\mathbf{d}_{2,1} \cdot x + \mathbf{d}_{2,0}\mathbf{d}_{2,1} \cdot y + \mathbf{d}_{2,1}^2 \cdot xy + \mathbf{d}_{2,0} \cdot xy^2,
\end{aligned}$$

so that we have $\mathbf{Sq}(xy^2) \equiv (1 + \mathbf{d}_{2,1} + \mathbf{d}_{2,0}) \cdot xy^2 + E$ where the error term E lies in N (here N is as in Lemma 3.3).

Consider the expression $\mathbf{Sq}(\mathbf{d}_{2,0}^{a-1}\mathbf{d}_{2,1}^{b-1-a} \cdot xy^2)$. Since N is trivial in the degree of the fundamental class, $\mathrm{Sq}^{2a}(\mathbf{d}_{2,0}^{a-1}\mathbf{d}_{2,1}^{b-1-a} \cdot xy^2)$ is the homogeneous part in degree $3(a-1) + 2(b-1) + 3$ of

$$(\maltese) \qquad \mathbf{d}_{2,0}^{a-1}(1 + \mathbf{d}_{2,1} + \mathbf{d}_{2,0})^a(\mathbf{d}_{2,1}(1 + \mathbf{d}_{2,1}) + \mathbf{d}_{2,0})^{b-1-a} \cdot xy^2.$$

Because $\mathbf{d}_{2,0}^a \cdot xy^2 = 0$ in $H(a,b)$, expression $(\maltese)$ is the same as

$$\mathbf{d}_{2,0}^{a-1}(1 + \mathbf{d}_{2,1})^a(\mathbf{d}_{2,1}(1 + \mathbf{d}_{2,1}))^{b-1-a} \cdot xy^2 = \mathbf{d}_{2,0}^{a-1}\mathbf{d}_{2,1}^{b-1-a}(1 + \mathbf{d}_{2,1})^{b-1} \cdot xy^2.$$

The homogeneous part in degree $3(a-1) + 2(b-1) + 3$ of this expression is $\binom{b-1}{a}\mathbf{d}_{2,0}^{a-1}\mathbf{d}_{2,1}^{b-1} \cdot xy^2$. If $a < 2^t$ then $\binom{b-1}{a}$ is 1, and so in this case $\mathrm{Sq}^{2a}(\mathbf{d}_{2,0}^{a-1}\mathbf{d}_{2,1}^{b-1-a} \cdot xy^2) = \mathbf{d}_{2,0}^{a-1}\mathbf{d}_{2,1}^{b-1} \cdot xy^2$.

So suppose that $a = 2^t$ but that c is not of the form $2^s - 1$. Let $j_c := \min\{i \geq 0 \mid \alpha_i(c) = 0\}$, where $\alpha_i(c)$ is the coefficient of 2^i in the dyadic expansion of c. Since c is odd, $j_c > 0$.

Consider the expression $\mathrm{Sq}^{2^{j_c+t+1}}(\mathbf{d}_{2,0}^{2^t-1}\mathbf{d}_{2,1}^{2^t(c-2^{j_c})-1} \cdot xy^2)$. With the same arguments as before, we see that it is just the homogeneous part in degree $3(2^t-1)+2(2^tc-1)+3$ of

$$\mathbf{d}_{2,0}^{2^t-1}\mathbf{d}_{2,1}^{2^t(c-2^{j_c})-1}(1+\mathbf{d}_{2,1})^{2^t(c-2^{j_c}+1)-1} \cdot xy^2,$$

which is

$$\binom{2^t(c-2^{j_c}+1)-1}{2^t \cdot 2^{j_c}}\mathbf{d}_{2,0}^{2^t-1}\mathbf{d}_{2,1}^{2^tc-1} \cdot xy^2.$$

The binomial coefficient is nontrivial so that indeed we have

$$\mathrm{Sq}^{2^{j_c+t+1}}(\mathbf{d}_{2,0}^{2^t-1}\mathbf{d}_{2,1}^{2^t(c-2^{j_c})-1} \cdot xy^2) = \mathbf{d}_{2,0}^{2^t-1}\mathbf{d}_{2,1}^{2^tc-1} \cdot xy^2.$$

It remains to verify that $H(a,b)$ does have trivial Wu classes if $a = 2^t$ and $b = 2^t(2^s-1)$. Denote the monomial $\mathbf{d}_{2,0}^{2^t-1}\mathbf{d}_{2,1}^{2^t(2^s-1)-1} \cdot xy^2$ representing the fundamental class of the algebra $H(2^t, 2^t(2^s-1))$ by $[H]$. It suffices to show that the coefficient of $[H]$ in $\mathbf{Sq}(\mathbf{d}_{2,0}^{\lambda}\mathbf{d}_{2,1}^{\mu} \cdot xy^2)$ is zero unless $\lambda = 2^t - 1$ and $\mu = 2^t(2^s-1)-1$.

We divide the work into three cases: let $0 \leq j < t$ and assume that $\alpha_i(\lambda+1) = 0$ and $\alpha_i(\mu) = 1$ for all $0 \leq i < j$. We will show that

(i) If $\alpha_j(\lambda+1) = 0$ and $\alpha_j(\mu) = 1$ then the coefficient of $[H]$ in $\mathbf{Sq}(\mathbf{d}_{2,0}^{\lambda}\mathbf{d}_{2,1}^{\mu} \cdot xy^2)$ is the same as its coefficient in

$$\mathbf{d}_{2,1}^{2^{j+1}-1}\mathbf{Sq}(\mathbf{d}_{2,0}^{\lambda}\mathbf{d}_{2,1}^{\mu-(2^{j+1}-1)} \cdot xy^2).$$

It is zero otherwise.

(ii) If $\alpha_i(\mu) = 1$ for all $0 \leq i \leq t$, then the coefficient of $[H]$ in $\mathbf{Sq}(\mathbf{d}_{2,0}^{2^t-1}\mathbf{d}_{2,1}^{\mu} \cdot xy^2)$ is zero.

(iii) Let $t < i < s+t$ and assume that $\alpha_t(\mu) = 0$ and $\alpha_m(\mu) = 1$ for all $0 \leq m < t$ and all $t < m < i$. If $\alpha_i(\mu) = 1$ then the coefficient of $[H]$ in $\mathbf{Sq}(\mathbf{d}_{2,0}^{2^t-1}\mathbf{d}_{2,1}^{\mu} \cdot xy^2)$ is the same as its coefficient in

$$\mathbf{d}_{2,1}^{2^{i+1}-1-2^t}\mathbf{Sq}(\mathbf{d}_{2,0}^{2^t-1}\mathbf{d}_{2,1}^{\mu-(2^{i+1}-1-2^t)} \cdot xy^2).$$

It is zero otherwise.

The proofs of all three assertions are very similar so we supply details only for the first, which is done by induction on j. If $j > 0$ then by induction it suffices to consider the coefficient η_j of $h_j := \mathbf{d}_{2,0}^{2^t-1}\mathbf{d}_{2,1}^{2^t(2^s-1)-2^j} \cdot xy^2$ in

$$(*) \quad \mathbf{Sq}(\mathbf{d}_{2,0}^{\lambda}\mathbf{d}_{2,1}^{\mu-(2^j-1)} \cdot xy^2) = \mathbf{d}_{2,0}^{\lambda}(1+\mathbf{d}_{2,1}+\mathbf{d}_{2,0})^{\lambda+1}(\mathbf{d}_{2,1}(1+\mathbf{d}_{2,1})+\mathbf{d}_{2,0})^{\mu-(2^j-1)} \cdot xy^2.$$

We show that η_j is equal to the coefficient of h_j in the expression $\mathbf{d}_{2,1}^{2^{j+1}-1}(\mathbf{Sq}(\mathbf{d}_{2,0}^{\lambda}\mathbf{d}_{2,1}^{\mu-(2^{j+1}-1)}\cdot xy^2))$ if $\alpha_j(\lambda+1)=0$ and $\alpha_j(\mu)=1$, and is zero otherwise. There are four subcases corresponding to the four possible values of $\alpha_j(\lambda+1)$ and $\alpha_j(\mu)$.

If $\alpha_j(\lambda+1)=0$ and $\alpha_j(\mu)=0$, then the right hand side of (⁎) can be expanded into a sum of terms of the form $\mathbf{d}_{2,0}^{\omega}\mathbf{d}_{2,1}^{\sigma}\cdot xy^2$ with $2^{j+1}\,|\,\sigma$. Consequently, the coefficient of h_j in this expression is zero.

If $\alpha_j(\lambda+1)=1$ and $\alpha_j(\mu)=0$, then we have

$$\mathbf{Sq}(\mathbf{d}_{2,0}^{\lambda}\mathbf{d}_{2,1}^{\mu-(2^j-1)}\cdot xy^2)=(1+\mathbf{d}_{2,1})^{2^j}\cdot E+\mathbf{d}_{2,0}^{2^j}\cdot E$$

with $E=\mathbf{d}_{2,0}^{\lambda}(1+\mathbf{d}_{2,1}+\mathbf{d}_{2,0})^{\lambda+1-2^j}(\mathbf{d}_{2,1}(1+\mathbf{d}_{2,1})+\mathbf{d}_{2,0})^{\mu-(2^j-1)}\cdot xy^2$. The expressions $(1+\mathbf{d}_{2,1})^{2^j}\cdot E$ and $\mathbf{d}_{2,0}^{2^j}\cdot E$ can be expanded into sums of terms of the form $\mathbf{d}_{2,0}^{\omega}\mathbf{d}_{2,1}^{\sigma}\cdot xy^2$ which satisfy $\alpha_j(\omega)=0$ and $2^{j+1}\,|\,\sigma$ respectively. So again the coefficient of h_j is zero.

If $\alpha_j(\lambda+1)=1$ and $\alpha_j(\mu)=1$, then we write

$$\mathbf{Sq}(\mathbf{d}_{2,0}^{\lambda}\mathbf{d}_{2,1}^{\mu-(2^j-1)}\cdot xy^2)=(\mathbf{d}_{2,1}+\mathbf{d}_{2,1}^{3}+\mathbf{d}_{2,0}^{2})^{2^j}\cdot E'+(1+\mathbf{d}_{2,1}^{2^{j+1}})\mathbf{d}_{2,0}^{2^j}\cdot E'$$

with $E'=\mathbf{d}_{2,0}^{\lambda}(1+\mathbf{d}_{2,1}+\mathbf{d}_{2,0})^{\lambda+1-2^j}(\mathbf{d}_{2,1}(1+\mathbf{d}_{2,1})+\mathbf{d}_{2,0})^{\mu-(2^{j+1}-1)}\cdot xy^2$. Again, the expressions $(\mathbf{d}_{2,1}+\mathbf{d}_{2,1}^{3}+\mathbf{d}_{2,0}^{2})^{2^j}\cdot E'$ and $(1+\mathbf{d}_{2,1}^{2^{j+1}})\mathbf{d}_{2,0}^{2^j}\cdot E'$ can be expanded into sums of terms of the form $\mathbf{d}_{2,0}^{\omega}\mathbf{d}_{2,1}^{\sigma}\cdot xy^2$ which satisfy $\alpha_j(\omega)=0$ and $2^{j+1}\,|\,\sigma$ respectively. Hence also in this case the coefficient of h_j is zero.

If $\alpha_j(\lambda+1)=0$ and $\alpha_j(\mu)=1$, then we write

$$\mathbf{Sq}(\mathbf{d}_{2,0}^{\lambda}\mathbf{d}_{2,1}^{\mu-(2^j-1)}\cdot xy^2)=(\mathbf{d}_{2,1}^{2^j}+\mathbf{d}_{2,1}^{2^{j+1}}+\mathbf{d}_{2,0}^{2^j})\cdot E''$$

with $E''=\mathbf{d}_{2,0}^{\lambda}(1+\mathbf{d}_{2,1}+\mathbf{d}_{2,0})^{\lambda+1}(\mathbf{d}_{2,1}(1+\mathbf{d}_{2,1})+\mathbf{d}_{2,0})^{\mu-(2^{j+1}-1)}\cdot xy^2$. The expression $\mathbf{d}_{2,1}^{2^j}\cdot E''$ is the same as $\mathbf{d}_{2,1}^{2^j}\mathbf{Sq}(\mathbf{d}_{2,0}^{\lambda}\mathbf{d}_{2,1}^{\mu-(2^{j+1}-1)}\cdot xy^2)$, and $(\mathbf{d}_{2,1}^{2^{j+1}}+\mathbf{d}_{2,0}^{2^j})\cdot E''$ can be expanded into a sum of terms of the form $\mathbf{d}_{2,0}^{\omega}\mathbf{d}_{2,1}^{\sigma}\cdot xy^2$ which satisfy $2^{j+1}\,|\,\sigma$. Hence the coefficient of h_j in $\mathbf{Sq}(\mathbf{d}_{2,0}^{\lambda}\mathbf{d}_{2,1}^{\mu-(2^j-1)}\cdot xy^2)$ is the same as its coefficient in $\mathbf{d}_{2,1}^{2^j}\mathbf{Sq}(\mathbf{d}_{2,0}^{\lambda}\mathbf{d}_{2,1}^{\mu-(2^{j+1}-1)}\cdot xy^2)$.

To start the induction, recycle the above argument with $j=0$. ■

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