FUNDAMENTA
MATHEMATICAE
177 (2003)

# Generating varieties for the triple loop space of classical Lie groups

by

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**Abstract.** For $G = SU(n), Sp(n)$ or $\mathrm{Spin}(n)$, let $C_G(SU(2))$ be the centralizer of a certain $SU(2)$ in $G$. We have a natural map $J : G/C_G(SU(2)) \to \Omega_0^3 G$. For a generator $\alpha$ of $H_*(G/C_G(SU(2)); \mathbb{Z}/2)$, we describe $J_*(\alpha)$. In particular, it is proved that $J_* : H_*(G/C_G(SU(2)); \mathbb{Z}/2) \to H_*(\Omega_0^3 G; \mathbb{Z}/2)$ is injective.

**1. Introduction.** The purpose of this paper is to study an analogue of Bott's theorem on generating varieties when $S^1$ is replaced by $SU(2)$ and $\Omega G$ is replaced by $\Omega_0^3 G$, where $G$ is a classical Lie group. We will explain the motivation for the study later. We recall Bott's theorem. Let $G$ be a compact simply connected Lie group. For a circle $S^1$ on $G$, let $C_G(S^1)$ be the centralizer of $S^1$ in $G$. We associate with it a map $f : G/C_G(S^1) \to \Omega G$ defined by $f(gC_G(S^1))(x) = gxg^{-1}x^{-1}$, where $x \in S^1$. According to Bott [2], if $S^1$ is a suitable circle (more precisely, $S^1$ determines an element of $H_1(T; \mathbb{Z})$, where $T$ is a maximal torus of $G$ containing $S^1$, and if the element is dual to a long root), then ($S^1$ becomes a generating circle and) $f$ has the property that the image of $f_* : H_*(G/C_G(S^1); \mathbb{Z}) \to H_*(\Omega G; \mathbb{Z})$ generates the Pontryagin ring $H_*(\Omega G; \mathbb{Z})$. We call $G/C_G(S^1)$ a *generating variety*.

Let $SU(2)$ be a subgroup of $G$ and $C_G(SU(2))$ its centralizer in $G$. Hereafter we abbreviate $C_G(SU(2))$ to $C$ or $C_G$ and consider homology with $\mathbb{Z}/2$-coefficients. We have a map

$$J : G/C \to \Omega_0^3 G$$

defined by

$$J(gC)(x) = gxg^{-1}x^{-1},$$

where $x \in SU(2)$. For $G = SU(2)$, $J$ is essentially the well known $J$-homomorphism $J : SO(3) \to \Omega_1^3 S^3$. At present, the Pontryagin ring $H_*(\Omega_0^3 G; \mathbb{Z}/2)$

2000 *Mathematics Subject Classification*: Primary 58D27; Secondary 53C07, 55R40.
*Key words and phrases*: generating variety, instantons, triple loop space.

is known for all $G$. (See [3] for $G = SU(n)$, [4] for $G = \mathrm{Spin}(n)$, [5] for $G = Sp(n)$ and [6] for exceptional Lie groups.) The ring is polynomial with infinitely many generators and the generators are constructed as follows. From a finite number of elements $b_{\mu_1}, \dots, b_{\mu_r}$ of $H_*(\Omega_0^3 G; \mathbb{Z}/2)$, all the generators are constructed by applying the homology operations $Q_1$, $Q_2$ (and possibly $Q_3$). In this paper the subscript of an element means its degree, that is, $\deg(b_\mu) = \mu$. We call these $b_\mu$ *fundamental generators*. Now we consider the following questions:

(1) Is $J_* : H_*(G/C; \mathbb{Z}/2) \to H_*(\Omega_0^3 G; \mathbb{Z}/2)$ injective?
(2) Does the image of $J_*$ contain all the fundamental generators?

The motivation for the questions is as follows. For a compact simple simply connected Lie group $G$, let $M(k, G)$ be the moduli space of based gauge equivalence classes of $G$-instantons over $S^4$ with instanton number $k$. Let $i_k : M(k, G) \to \Omega_k^3 G$ be the inclusion. In [3] Boyer, Mann and Waggoner constructed non-trivial classes in $H_*(M(k, SU(n)); \mathbb{Z}/2)$. For that purpose, they used a description of $M(1, G)$ in terms of a homogeneous space. More precisely, let $C = C_G(SU(2))$ be the centralizer of a certain $SU(2)$ in $G$. Then a result of [3] tells us that there exists a diffeomorphism $M(1, G) \cong \mathbb{R}^5 \times G/C$ such that the following diagram is homotopy commutative:

$$\begin{tikzcd} M(1,G) \arrow[r, "i_1"] \arrow[d, "\simeq"] & \Omega_1^3 G \arrow[d, "\simeq"] \\ G/C \arrow[r, "J"] & \Omega_0^3 G \end{tikzcd}$$

A crucial result of [3] determines $J_*(\alpha)$, where $\alpha \in H_*(SU(n)/C; \mathbb{Z}/2)$ is an *even*-dimensional generator (see Theorem 2.6). The motivation of this paper is to continue the study for $G = SU(n), Sp(n)$, and $\mathrm{Spin}(n)$ ($n \neq 4$). Hence hereafter we consider an embedding of $SU(2)$ into $G$ so that $M(1, G) \cong \mathbb{R}^5 \times G/C$. (See [3], [9] or Sections 2–3.)

For $G/C$, we have the following examples:

Example 1.1 ([3], [9]). (i) $SU(n)/C_{SU(n)}$ is diffeomorphic to the unit tangent bundle of $\mathbb{C}P^{n-1}$.

(ii) $Sp(n)/C_{Sp(n)}$ is diffeomorphic to $\mathbb{R}P^{4n-1}$.

(iii) Since $\mathrm{Spin}(5) \cong Sp(2)$ and $\mathrm{Spin}(6) \cong SU(4)$, we obtain examples of $\mathrm{Spin}(n)/C_{\mathrm{Spin}(n)}$ for $n = 5$ and 6 from (i) and (ii).

In the main theorems of this paper, we describe $J_*(\alpha)$ for $\alpha$ a generator of $H_*(G/C; \mathbb{Z}/2)$. (See Theorem 2.7 for $G = SU(n)$ and $\alpha$ an odd-dimensional generator, Theorem 2.8 for $G = Sp(n)$, and Theorem 3.4 for $G = \mathrm{Spin}(n)$.) In particular, we have the following:

THEOREM A. *Let $G$ be $SU(n), Sp(n)$, or $\mathrm{Spin}(n)$ $(n \neq 4)$. Embed $SU(2)$ into $G$ so that $M(1,G) \cong \mathbb{R}^5 \times G/C$. Then*

(i) $J_* : H_*(G/C;\mathbb{Z}/2) \to H_*(\Omega_0^3 G;\mathbb{Z}/2)$ *is injective.*

(ii) *Excluding a fundamental generator in $H_{2n-6}(\Omega_0^3 \mathrm{Spin}(n);\mathbb{Z}/2)$ for $n \equiv 3 \pmod 4$, there are choices of the fundamental generators $b_\mu$ of $H_*(\Omega_0^3 G;\mathbb{Z}/2)$ such that every $b_\mu$ is in the image of $J_*$.*

If $\alpha \in H_*(G/C_G;\mathbb{Z}/2)$ is a stable element, that is, $\alpha$ is non-trivial in $H_*(G(\infty)/C_{G(\infty)};\mathbb{Z}/2)$ (where $G(\infty)$ denotes $SU$, $Sp$ or Spin), then we can use the Bott periodicity to study $J_*(\alpha)$ in $H_*(\Omega_0^3 G;\mathbb{Z}/2)$ (see (5.4)). On the other hand, for $G = SU(n)$ or $\mathrm{Spin}(n)$, $H_*(G/C_G;\mathbb{Z}/2)$ contains a non-stable element. The following theorem is a key to determining $J_*(\alpha)$ for $\alpha$ non-stable. For a map $f : X \to \Omega_0^3 G$, let $\mathrm{Ad}(f) : \Sigma^3 X \to G$ be the adjoint map of $f$.

THEOREM B. (i) *For $G = SU(n)$ with $n$ even, let $i : S^{2n-3} \to SU(n)/C$ be the inclusion of the fiber of the unit tangent bundle of $\mathbb{C}P^{n-1}$. Then $\mathrm{Ad}(J \circ i)$ generates the 2-component of $\pi_{2n}(SU(n)) \cong \mathbb{Z}/n!$.*

(ii) *For $G = \mathrm{Spin}(n)$ with $n \neq 7$, there exists a spherical class in $H_{n-4}(\mathrm{Spin}(n)/C;\mathbb{Z})$ which is denoted by $\nu : S^{n-4} \to \mathrm{Spin}(n)/C$. Then $\mathrm{Ad}(J \circ \nu)$ is a generator of $\pi_{n-1}(\mathrm{Spin}(n))$. (More precisely; $\mathrm{Ad}(J \circ \nu)$ is one of the generators for $n \equiv 0,1,2,4 \pmod 8$; $\mathrm{Ad}(J \circ \nu)$ has infinite order for $n$ even and is of order 2 for $n$ odd with $n \neq 7$.)*

This paper is organized as follows. In Section 2 we study the cases $G = SU(n)$ and $Sp(n)$. Theorems 2.7 and 2.8 are the main results for $G = SU(n)$ and $G = Sp(n)$ respectively. In Section 3 we study the case $G = \mathrm{Spin}(n)$, where Theorem 3.4 is the main result. Theorems 2.7 and 2.8 are proved in Section 4 and Theorem 3.4 is proved in Section 5, where Theorem B is a key for the proofs.

**2. Main results for $G = SU(n)$ and $Sp(n)$.** First we study the case $G = SU(n)$. Since $SU(2) = Sp(1)$, we assume $n \geq 3$. (The (2.2) below holds for $n = 2$, but it is more natural to consider (2.4) for $n \geq 3$.) We embed $SU(2)$ into $SU(n)$ as the first $2 \times 2$ elements. Then $C = C_{SU(n)}$ is given by the following set of matrices:

$$C = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & A \end{pmatrix} : A \in U(n-2),\ a^2 \det A = 1 \right\}. \tag{2.1}$$

Therefore $SU(n)/C$ is diffeomorphic to the unit tangent bundle of $\mathbb{C}P^{n-1}$ (see [3], [9] or (4.1)) and $H^*(SU(n)/C;\mathbb{Z}/2)$ is given as follows:

(1) For $n$ even,

$$H^*(SU(n)/C;\mathbb{Z}/2) \cong H^*(\mathbb{C}P^{n-1};\mathbb{Z}/2) \otimes H^*(S^{2n-3};\mathbb{Z}/2).$$

(2) For $n$ odd,

$$H^*(SU(n)/C;\mathbb{Z}/2) \cong H^*(\mathbb{C}P^{n-2};\mathbb{Z}/2) \otimes H^*(S^{2n-1};\mathbb{Z}/2).$$

We write the generators of $H_*(SU(n)/C;\mathbb{Z}/2)$ as follows.

(1) For $n$ even,

$$\begin{cases} \alpha_{2i}, & 1 \le i \le n-1, \\ \beta_{2i+1}, & n-2 \le i \le 2n-3. \end{cases} \tag{2.2}$$

(2) For $n$ odd,

$$\begin{cases} \alpha_{2i}, & 1 \le i \le n-2, \\ \beta_{2i+1}, & n-1 \le i \le 2n-3. \end{cases} \tag{2.3}$$

The structure of $H_*(\Omega_0^3 SU(n);\mathbb{Z}/2)$ is given in [3]. First, the following elements are defined in $H_*(\Omega_0^3 SU(n);\mathbb{Z}/2)$:

$$\begin{cases} x_2, & \\ y_{2i}, & 2 \le i \le n-2,\ i \equiv 0 \pmod 2, \\ z_{4i+1}, & \left[\frac{n-1}{2}\right] \le i \le n-2,\ i \equiv 1 \pmod 2. \end{cases} \tag{2.4}$$

Here $x_2$ is defined by $x_2 = Q_2[1] * [-2]$. (2.4) gives the set of fundamental generators of $H_*(\Omega_0^3 SU(n);\mathbb{Z}/2)$. The set of all ring generators is as follows:

$$\begin{cases} Q_2^a(x_2), & a \ge 0, \\ Q_2^a(y_{2i}), & a \ge 0,\ 2 \le i \le \left[\frac{n-3}{2}\right],\ i \equiv 0 \pmod 2, \\ Q_1^a Q_2^b(y_{2i}), & a, b \ge 0,\ \left[\frac{n-1}{2}\right] \le i \le n-2,\ i \equiv 0 \pmod 2, \\ Q_1^a Q_3^b(z_{4i+1}), & a, b \ge 0,\ \left[\frac{n-1}{2}\right] \le i \le n-2,\ i \equiv 1 \pmod 2. \end{cases}$$

Consider the following diagram:

$$\begin{tikzcd} SU(n)/C_{SU(n)} \arrow[r, "J"] \arrow[d, "i"] & \Omega_0^3 SU(n) \arrow[d, "j"] \\ SU(n+1)/C_{SU(n+1)} \arrow[r, "J"] & \Omega_0^3 SU(n+1) \end{tikzcd} \tag{2.5}$$

where $i$ and $j$ are the inclusions. If $\alpha_{2i}$ is defined both in $H_{2i}(SU(n)/C_{SU(n)};\mathbb{Z}/2)$ and $H_{2i}(SU(n+1)/C_{SU(n+1)};\mathbb{Z}/2)$, then $\alpha_{2i}$ in the top row is mapped by $i_*$ to $\alpha_{2i}$ in the bottom row. Similar remarks hold for $\beta_{2i+1}$ with respect to $i_*$ and for $x_2$, $y_{2i}$ and $z_{4i+1}$ with respect to $j_*$.

$J_*(\alpha_{2i})$ is known:

Theorem 2.6 ([3]). *There are choices of the fundamental generators $y_{2i}$ such that*:

(i) *For $i$ even*, $J_*(\alpha_{2i}) = y_{2i}$.
(ii) *For $i$ odd*, $J_*(\alpha_{2i})$ *contains the term* $x_2 * y_{2i-2}$.

Our main result for $G = SU(n)$ is as follows.

THEOREM 2.7. *There are choices of the fundamental generators* $z_{4i+1}$ (*where* $i \equiv 1 \pmod 2$) *such that*

(i) $J_*(\beta_{8k+1})$ *contains the term* $Q_1(y_{4k})$.
(ii) $J_*(\beta_{8k+3})$ *contains the term* $x_2 * Q_1(y_{4k})$.
(iii) $J_*(\beta_{8k+5}) = z_{8k+5}$.
(iv) $J_*(\beta_{8k+7})$ *contains the term* $x_2 * z_{8k+5}$.

REMARK. In [3], $J_*(\beta_5)$ is also studied.

From Theorems 2.6, 2.7 and the structure of $H^*(SU(n)/C; \mathbb{Z}/2)$, we see that $J^* : H^*(\Omega_0^3 SU(n); \mathbb{Z}/2) \to H^*(SU(n)/C; \mathbb{Z}/2)$ is surjective, hence Theorem A holds for $G = SU(n)$.

Next we study the case $G = Sp(n)$. We embed $Sp(1)$ into $Sp(n)$ as the first $1 \times 1$ element. Then $C = C_{Sp(n)} = \mathbb{Z}/2 \times Sp(n-1)$, hence $Sp(n)/C$ is diffeomorphic to $\mathbb{R}P^{4n-1}$. We write the generators of $H_*(Sp(n)/C; \mathbb{Z}/2)$ as $\alpha_\mu$ ($1 \le \mu \le 4n-1$). The structure of $H_*(\Omega_0^3 Sp(n); \mathbb{Z}/2)$ is given in [5]. The result is

$$H_*(\Omega_0^3 Sp(n); \mathbb{Z}/2) \cong \mathbb{Z}/2[Q_1^a Q_2^b[1] * [-2^{a+b}] : a, b \ge 0] \otimes \mathbb{Z}/2[Q_1^a Q_2^b(z_{4i}) : a, b \ge 0, 1 \le i \le n-1].$$

Setting $x_1 = Q_1[1] * [-2]$ and $y_2 = Q_2[1] * [-2]$, we have the set of fundamental generators $x_1, y_2$ and $z_{4i}$ ($1 \le i \le n-1$).

Our result for $G = Sp(n)$ is as follows.

THEOREM 2.8. *There are choices of the fundamental generators* $z_{4i}$ *such that*

(i) $J_*(\alpha_1) = x_1$, $J_*(\alpha_2) = y_2$ *and* $J_*(\alpha_{4i}) = z_{4i}$ ($1 \le i \le n-1$).
(ii) *In general*, *for* $1 \le \mu \le 4n-1$, *set* $\mu = \varepsilon_1 + 2\varepsilon_2 + 4i$, *where* $\varepsilon_j = 0$ *or* 1. *Then* $J_*(\alpha_\mu)$ *contains the term* $x_1^{\varepsilon_1} * y_2^{\varepsilon_2} * z_{4i}$.

Theorem A for $G = Sp(n)$ follows from Theorem 2.8 and the structure of $H^*(Sp(n)/C; \mathbb{Z}/2)$.

We prove Theorems 2.7 and 2.8 in Section 4.

**3. Main result for $G = \mathrm{Spin}(n)$.** Since $\mathrm{Spin}(3) \cong SU(2)$ and $\mathrm{Spin}(4)$ is not simple, we assume $n \ge 5$. Recall that $\mathrm{Spin}(4) \cong S_1^3 \times S_2^3$, where $S_1^3$ and $S_2^3$ are two copies of $\mathrm{Spin}(3) \cong S^3$. (In Section 5 we write $S_i^3$ in terms of the Clifford algebra.) We embed $\mathrm{Spin}(3)$ into $\mathrm{Spin}(n)$ as $S_1^3$. Then $C = C_{\mathrm{Spin}(n)} = \mathrm{Spin}(n-4) \times S_2^3$. We recall the structure of $H^*(\mathrm{Spin}(n)/C; \mathbb{Z}/2)$.

PROPOSITION 3.1 ([7]). *Let* $n = 4t + l$ *with* $0 \le l \le 3$. *Then we have the following isomorphism of modules*:

$$H^*(\mathrm{Spin}(n)/C; \mathbb{Z}/2) \cong \mathbb{Z}/2[c_2]/(c_2^t) \otimes \Delta(u_{i_1}, u_{i_2}, u_{i_3}),$$

*where* $\deg c_2 = 4$ *and* $\deg u_{i_j} = i_j$ *with*

$$(i_1, i_2, i_3) = \begin{cases} (4t-4, 4t-3, 4t-2), & n = 4t, \\ (4t-3, 4t-2, 4t), & n = 4t+1, \\ (4t-2, 4t, 4t+1), & n = 4t+2, \\ (4t, 4t+1, 4t+2), & n = 4t+3. \end{cases}$$

*Here* $\Delta(x_1, \dots, x_m)$ *denotes the graded algebra over* $\mathbb{Z}/2$ *with* $\mathbb{Z}/2$*-basis* $\{x_{i_1} \dots x_{i_r} : 1 \le i_1 < \dots < i_r \le m\}$. (*If we add the relations* $x_i^2 = 0$, $\Delta(x_1, \dots, x_m)$ *becomes the exterior algebra* $\Lambda(x_1, \dots, x_m)$.)

The mod 2 cohomology ring with squaring operations is also determined in [7]. By Proposition 3.1, we can define elements of $H_*(\mathrm{Spin}(n)/C; \mathbb{Z}/2)$ as follows. Note that two of $u_{i_j}$ are even-dimensional and one of them has dimension congruent to 1 mod 4. We write them as $u_{2a_j}$ ($j = 1, 2$) and $u_{4b+1}$. Hence $a_2 = a_1 + 1$. We set

$$(3.2) \qquad \begin{cases} \alpha_{4i} = (c_2^i)^*, & 1 \le i \le t-1, \\ \beta_{4b+4i+1} = (c_2^i u_{4b+1})^*, & 0 \le i \le t-1, \\ \gamma_{2a_j+4i} = (c_2^i u_{2a_j})^*, & 0 \le i \le t-1,\ j = 1, 2, \end{cases}$$

where $(\,\cdot\,)^*$ denotes the dual element with respect to the monomial basis.

The structure of $H_*(\Omega_0^3 \mathrm{Spin}(n); \mathbb{Z}/2)$ is given in [4]. It depends on the congruence class of $n$ mod 8. For example, for $n = 8g + 2$, we have the following set of fundamental generators of $H_*(\Omega_0^3 \mathrm{Spin}(8g+2); \mathbb{Z}/2)$:

$$\begin{cases} x_{4k}, & 1 \le k \le 2g-1, \\ y_{8g+8k+5}, & 0 \le k \le g-1, \\ z_{8g+2k}, & -1 \le k \le 4g-2,\ k \not\equiv 1 \pmod 4. \end{cases}$$

Hence $y_\mu$ is defined only when $\mu \equiv 5 \pmod 8$ and $z_\mu$ is defined only when $\mu \equiv 0, 4$ or $6 \pmod 8$. We define $y_\mu$ for $\mu \equiv 1 \pmod 8$ and $z_\mu$ for $\mu \equiv 2 \pmod 8$ by setting

$$(3.3) \qquad \begin{cases} y_{8g+8k+1} = Q_1(x_{4g+4k}), & 0 \le k \le g-1, \\ z_{8g+2k} = Q_2(x_{4g+k-1}), & 1 \le k \le 4g-3,\ k \equiv 1 \pmod 4. \end{cases}$$

The elements of (3.3) are not fundamental generators. We do the same procedure for all $n$. Then we obtain the following elements of $H_*(\Omega_0^3 \mathrm{Spin}(n); \mathbb{Z}/2)$, among which the elements of the form (3.3) are not fundamental generators.

(1) For $n = 4t$,

$$\begin{cases} x_{4i}, & 1 \leq i \leq t-1, \\ y_{4t+4i-3}, & 0 \leq i \leq t-1, \\ z_{4t+2i-4}, & 0 \leq i \leq 2t-1. \end{cases}$$

(2) For $n = 4t+1$,

$$\begin{cases} x_{4i}, & 1 \leq i \leq t-1, \\ y_{4t+4i-3}, & 0 \leq i \leq t-1, \\ z_{4t+2i-2}, & 0 \leq i \leq 2t-1. \end{cases}$$

(3) For $n = 4t+2$,

$$\begin{cases} x_{4i}, & 1 \leq i \leq t-1, \\ y_{4t+4i+1}, & 0 \leq i \leq t-1, \\ z_{4t+2i-2}, & 0 \leq i \leq 2t-1. \end{cases}$$

(4) For $n = 4t+3$,

$$\begin{cases} x_{4i}, & 1 \leq i \leq t-1, \\ y_{4t+4i+1}, & 0 \leq i \leq t-1, \\ z_{4t+2i}, & 0 \leq i \leq 2t. \end{cases}$$

For $\alpha_\mu, \beta_\mu, \gamma_\mu$ in (3.2) and $x_\mu, y_\mu, z_\mu$, a similar naturality to (2.5) holds. Note that $\alpha_\mu, \beta_\mu, \gamma_\mu$ correspond to $x_\mu, y_\mu, z_\mu$ respectively except the fundamental generator $z_{8t}$ in $H_{8t}(\Omega_0^3 \operatorname{Spin}(n); \mathbb{Z}/2)$ for $n = 4t+3$. The following main theorem asserts that the correspondence is realized through $J_*$:

Theorem 3.4. *There are choices of the fundamental generators $x_\mu$ for $\mu \equiv 0 \pmod 4$, $y_\mu$ for $\mu \equiv 5 \pmod 8$ and $z_\mu$ for $\mu \equiv 0, 4, 6 \pmod 8$ such that*

(i) $J_*(\alpha_\mu) = x_\mu$.

(ii) (a) *When $\mu \equiv 5 \pmod 8$, $J_*(\beta_\mu) = y_\mu$.*

(b) *When $\mu \equiv 1 \pmod 8$, $J_*(\beta_\mu)$ contains the term $y_\mu$.*

(iii) (a) *When $\mu \equiv 0, 4$ or $6 \pmod 8$, $J_*(\gamma_\mu) = z_\mu$.*

(b) *When $\mu \equiv 2 \pmod 8$, $J_*(\gamma_\mu)$ contains the term $z_\mu$.*

Note that $y_\mu$ in Theorem 3.4(ii)(b) and $z_\mu$ in (iii)(b) are not fundamental generators. (Compare (3.3).)

Theorem A for $G = \operatorname{Spin}(n)$ follows from Theorem 3.4 and the structure of $H^*(\operatorname{Spin}(n)/C; \mathbb{Z}/2)$ (see Proposition 3.1).

We prove Theorem 3.4 in Section 5.

**4. Proofs of Theorems 2.7 and 2.8.** First we prove Theorem B(i). Recall that $C = C_{SU(n)}$ is given in (2.1). We set

$$SU(n-2) = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & A \end{pmatrix} : A \in SU(n-2) \right\},$$

$$SU(n-1) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix} : B \in SU(n-1) \right\},$$

$$U(n-1) = \left\{ \begin{pmatrix} a & 0 \\ 0 & B \end{pmatrix} : B \in U(n-1),\ a \det B = 1 \right\}.$$

We have the following fiber bundle which realizes $SU(n)/C$ as the unit tangent bundle of $\mathbb{C}P^{n-1}$:

$$SU(n-1)/SU(n-2) \xrightarrow{i} SU(n)/C \to SU(n)/U(n-1), \tag{4.1}$$

where $i$ is the inclusion. We obtain a map $\mathrm{Ad}(J \circ i) : S^{2n-3} \wedge S^3 \to SU(n)$. Consider the homotopy sequence of a principal bundle

$$SU(n-1) \to SU(n) \xrightarrow{p} S^{2n-1}.$$

Since

$$\pi_{2n-1}(SU(n-1)) \cong \begin{cases} 0, & n \text{ even}, \\ \mathbb{Z}/2, & n \text{ odd}, \end{cases}$$

(see, for example, [8]), $p_* : \pi_{2n}(SU(n)) \to \pi_{2n}(S^{2n-1})$ is surjective for $n$ even. Hence in order to prove Theorem B(i), it suffices to prove the following:

Proposition 4.2. *For $n$ even, set $\phi = p \circ \mathrm{Ad}(J \circ i)$. Then $\phi$ is a generator of $\pi_{2n}(S^{2n-1})$.*

*Proof.* First we calculate $\phi$ explicitly. Let $(z_1, \dots, z_{n-1}) \wedge (a, b) \in S^{2n-3} \wedge S^3$. We use the following notations.

(1) A matrix in $SU(n)$ which represents $(z_1, \dots, z_{n-1})$ is given by

$$g = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & z_1 & 0 & \dots & 0 \\ \vdots & \vdots & & \mathrm{A} & \\ 0 & z_{n-1} & & & \end{pmatrix}$$

where

$$\begin{pmatrix} z_1 & 0 & \dots & 0 \\ \vdots & & \mathrm{A} & \\ z_{n-1} & & & \end{pmatrix} \in SU(n-1).$$

(2) A matrix in $SU(n)$ which corresponds to $(a,b)$ is given by

$$x = \begin{pmatrix} a & -\overline{b} & 0 & \\ b & \overline{a} & & \\ & 0 & & E_{n-2} \end{pmatrix}$$

where $E_{n-2}$ denotes the unit matrix.

(3) We define the projection $p : SU(n) \to S^{2n-1}$ by mapping an element of $SU(n)$ to its first row.

Now it is easy to calculate the first row of $gxg^{-1}x^{-1}$. The result is

$$\phi((z_1,\dots,z_{n-1}) \wedge (a,b)) = (|b|^2\overline{z}_1 + |a|^2, a\overline{b}(1-\overline{z}_1), -\overline{b}\overline{z}_2, \dots, -\overline{b}\overline{z}_{n-1}).$$

Define $\psi : S^{2n-3} \wedge S^3 \to S^{2n-1}$ by

$$\psi((z_1,\dots,z_{n-1}) \wedge (a,b)) = (|b|^2 z_1 + |a|^2, bz_2, \dots, bz_{n-1}, ab(1-z_1)). \tag{4.3}$$

It is clear that $\phi \simeq \psi$.

Next we calculate the generator of $\pi_{2n}(S^{2n-1})$ explicitly. The following lemma is well known (see, for example, [11, p. 107]).

LEMMA 4.4. *The map $j : S^m \wedge S^p \to S^{m+p}$ defined by*

$$j((u_1,\dots,u_{m+1}) \wedge (x_1,\dots,x_{p+1})) = \Bigg(\frac{(1-x_1)u_1 + (1+x_1)}{2}, \frac{(1-x_1)u_2}{2}, \dots, \frac{(1-x_1)u_{m+1}}{2}, \sqrt{\frac{1-u_1}{2}}\, x_2, \dots, \sqrt{\frac{1-u_1}{2}}\, x_{p+1}\Bigg)$$

*is a homeomorphism.*

Let $\eta : S^3 \to S^2$ be the Hopf map. It is given by $\eta(a,b) = (1-2|b|^2, 2ab) \in S^2 \subset \mathbb{R} \times \mathbb{C}$. Then $\Sigma^{2n-3}\eta$ is the generator of $\pi_{2n}(S^{2n-1})$, and using Lemma 4.4, we have

$$\begin{aligned} &\Sigma^{2n-3}\eta((z_1,\dots,z_{n-1}) \wedge (a,b)) \\ &\qquad = (|b|^2 z_1 + |a|^2, |b|^2 z_2, \dots, |b|^2 z_{n-1}, ab\sqrt{2(1-\operatorname{Re} z_1)}). \end{aligned} \tag{4.5}$$

Now we have two maps, $\psi$ in (4.3) and $\Sigma^{2n-3}\eta$ in (4.5). We prove that $\psi \simeq \Sigma^{2n-3}\eta$ when $n$ is even. We define a map $h : S^{2n-1} \to S^{2n-2}$ by

$$h(w_1,\dots,w_n) = \frac{(w_1, 2\overline{w}_2 w_3, |w_2|^2 - |w_3|^2, w_4, \dots, w_n)}{\sqrt{|w_1|^2 + (|w_2|^2 + |w_3|^2)^2 + \sum_{i=4}^n |w_i|^2}}.$$

It is easy to see that $h \simeq \Sigma^{2n-4}\eta$, hence the composition $h\circ : \pi_{2n}(S^{2n-1}) \to \pi_{2n}(S^{2n-2})$ is an isomorphism. We define a map $\psi_1 : S^{2n-3} \wedge S^3 \to S^{2n-1}$ by

$$\begin{aligned} &\psi_1((z_1,\dots,z_{n-1}) \wedge (a,b)) \\ &\qquad = (|b|^2 z_1 + |a|^2, |b|z_2, |b|z_3, bz_4, \dots, bz_{n-1}, ab(1-z_1)). \end{aligned}$$

It is clear that $h \circ \psi = h \circ \psi_1$, hence $\psi \simeq \psi_1$. This process implies that two coordinates of $\psi$ can be changed simultaneously from $bz_i$ to $|b|z_i$. Since $n$ is even, we see that $\psi$ is homotopic to the map $\widetilde{\psi} : S^{2n-3} \wedge S^3 \to S^{2n-1}$ defined by

$$\widetilde{\psi}((z_1, \ldots, z_{n-1}) \wedge (a, b)) = (|b|^2 z_1 + |a|^2, |b|z_2, \ldots, |b|z_{n-1}, ab(1 - z_1)).$$

Finally, using (4.5), it is easy to construct a homotopy $\widetilde{\psi} \simeq \Sigma^{2n-3}\eta$. This completes the proof of Proposition 4.2, and hence also that of Theorem B(i). ■

For $n$ even, let $\sigma_{2n-3} \in H_{2n-3}(S^{2n-3}; \mathbb{Z}/2)$ be the generator, where $S^{2n-3}$ is the fiber of (4.1). By (2.2), we have $i_*(\sigma_{2n-3}) = \beta_{2n-3}$. Recall that we have a map $J \circ i : S^{2n-3} \to \Omega_0^3 SU(n)$.

Lemma 4.6. *$(J \circ i)_*(\sigma_{2n-3})$ is non-zero in $H_{2n-3}(\Omega_0^3 SU(n); \mathbb{Z}/2)$.*

*Proof.* Let $W$ be the homotopy-theoretic fiber of the inclusion $\Omega_0^3 SU(n) \hookrightarrow \Omega_0^3 SU \simeq BU$. Consider the following diagram:

$$\begin{tikzcd} W \arrow[r] & \Omega_0^3 SU(n) \arrow[r] & \Omega_0^3 SU \simeq BU \\ S^{2n-3} \arrow[u, "\widetilde{J\circ i}"] \arrow[ur, "J\circ i"'] & & \end{tikzcd}$$

where $\widetilde{J \circ i}$ is the lift of $J \circ i$. The first non-vanishing homotopy of $W$ is $\pi_{2n-3}(W) \cong \mathbb{Z}$ and $\widetilde{J \circ i}$ is not divisible by 2 by Theorem B(i). For $n$ even, $H_{2n-3}(W; \mathbb{Z}/2) \to H_{2n-3}(\Omega_0^3 SU(n); \mathbb{Z}/2)$ is an isomorphism by (2.4). Hence $(J \circ i)_*(\sigma_{2n-3}) \neq 0$. ■

Proposition 4.7. *In $H_{2n-3}(\Omega_0^3 SU(n); \mathbb{Z}/2)$,*

$$J_*(\beta_{2n-3}) = \begin{cases} Q_1(y_{4k}), & n = 4k+2, \\ z_{8k+5}, & n = 4k+4. \end{cases}$$

*Proof.* By Lemma 4.6, $J_*(\beta_{2n-3}) \neq 0$ in $H_{2n-3}(\Omega_0^3 SU(n); \mathbb{Z}/2)$. By (2.4), $H_{2n-3}(\Omega_0^3 SU(n); \mathbb{Z}/2) \cong \mathbb{Z}/2$ and the generator is as on the right-hand side of the conclusion. Hence the result follows. ■

*Proof of Theorem 2.7.* If $\beta_{2n-3}$ is defined in $H_{2n-3}(SU(d)/C_{SU(d)}; \mathbb{Z}/2)$, then $d \leq n$. Using (2.5) we can regard Proposition 4.7 as relations in $H_{2n-3}(SU(d)/C_{SU(d)}; \mathbb{Z}/2)$ for each $d$ for which $\beta_{2n-3}$ is defined. For $\beta_{2n-3}$ with $n = 4k+2$, the relation is up to the kernel of $i_* : H_{2n-3}(\Omega_0^3 SU(d); \mathbb{Z}/2) \to H_{2n-3}(\Omega_0^3 SU(n); \mathbb{Z}/2)$, since $Q_1(y_{4k})$ is a fixed element. On the other hand, for $\beta_{2n-3}$ with $n = 4k+4$, we can define $z_{8k+5}$ to be $J_*(\beta_{2n-3})$. Therefore (i) and (iii) of Theorem 2.7 hold.

Consider $\beta_{2n-1} \in H_{2n-1}(SU(n)/C; \mathbb{Z}/2)$ for $n$ even. Since $\Delta_*(\beta_{2n-1})$ contains the term $\alpha_2 \otimes \beta_{2n-3}$ (where $\Delta_*$ denotes the coproduct), Proposition 4.7 tells us that $J_*(\beta_{2n-1}) \neq 0$ in $H_{2n-1}(\Omega_0^3 SU(n); \mathbb{Z}/2)$. By (2.4), $H_{2n-1}(\Omega_0^3 SU(n); \mathbb{Z}/2) \cong \mathbb{Z}/2$, hence

$$J_*(\beta_{2n-1}) = \begin{cases} x_2 * Q_1(y_{4k}), & n = 4k+2, \\ x_2 * z_{8k+5}, & n = 4k+4. \end{cases} \tag{4.8}$$

Then using (2.5) we can regard (4.8) as relations in $H_{2n-1}(SU(d)/C_{SU(d)}; \mathbb{Z}/2)$ for each $d$ for which $\beta_{2n-1}$ is defined. Therefore (ii) and (iv) of Theorem 2.7 hold. This completes the proof of Theorem 2.7, and hence also that of Theorem A for $G = SU(n)$. ∎

*Proof of Theorem 2.8.* If $G = Sp(1)$, then by the property of the classical $J$-homomorphism (see, for example, [3]), we have

$$\begin{cases} J_*(\alpha_1) = x_1, \\ J_*(\alpha_2) = y_2. \end{cases} \tag{4.9}$$

Then a similar diagram to (2.5) shows that (4.9) holds for all $n$.

Since $\Delta_* J_*(\alpha_3)$ contains the term $x_1 \otimes y_2$, $J_*(\alpha_3)$ must contain the term $x_1 * y_2$. (Actually, by [3], we have $J_*(\alpha_3) = x_1 * y_2 + x_1^3 + Q_1(x_1)$.) Similarly, since $\Delta_* J_*(\alpha_4)$ contains the term $y_2 \otimes y_2$, we can define $z_4$ to be $J_*(\alpha_4)$.

In general, we can prove the following results.

(i) When $\mu = 4i$, $\Delta_* J_*(\alpha_\mu)$ contains the term $J_*(\alpha_{2i}) \otimes J_*(\alpha_{2i})$. Hence we can define $z_{4i}$ to be $J_*(\alpha_{4i})$.

(ii) When $\mu = 4i+1$, $4i+2$ or $4i+3$, $\Delta_* J_*(\alpha_\mu)$ contains the term $x_1 \otimes z_{4i}$, $y_2 \otimes z_{4i}$ or $(x_1 * y_2) \otimes z_{4i}$. Hence $J_*(\alpha_\mu)$ must contain $x_1 * z_{4i}, y_2 * z_{4i}$ or $x_1 * y_2 * z_{4i}$.

This completes the proof of Theorem 2.8, and hence also that of Theorem A for $G = Sp(n)$. ∎

**5. Proof of Theorem 3.4.** Let $\nu : S^{n-4} \to \mathrm{Spin}(n)/C$ be the composition of the following maps:

$$S^{n-4} = \mathrm{Spin}(n-3)/\mathrm{Spin}(n-4) \xrightarrow{i} \mathrm{Spin}(n)/\mathrm{Spin}(n-4) \xrightarrow{\pi} \mathrm{Spin}(n)/C,$$

where $i$ is the inclusion and $\pi$ is the projection. First we prove Theorem B(ii). We recall some notations and results about the Clifford algebra from [1].

Let $e_1, \ldots, e_n$ be a basis of $\mathbb{R}^n$ and $C_n$ the Clifford algebra. Thus $e_i^2 = -1$ $(1 \le i \le n)$ and $e_i e_j + e_j e_i = 0$ $(i \neq j)$. For $g = e_{i_1} \ldots e_{i_k} \in C_n$, we set $g^t = e_{i_k} \ldots e_{i_1}$.

(1) Let $\mathrm{Spin}(n-4)$ be defined from $\mathbb{R}^{n-4}$ with a basis $e_1, \ldots, e_{n-4}$. Then in the principal bundle

$$\mathrm{Spin}(n-4) \to \mathrm{Spin}(n-3) \xrightarrow{p} S^{n-4},$$

the projection $p$ is given by $p(g) = g e_{n-3} g^t$.

(2) Let $e_i$ $(n-3 \leq i \leq n)$ be a basis of $\mathbb{R}^4$. Recall that $\mathrm{Spin}(4) \cong S_1^3 \times S_2^3$ (see Section 3). We write $S_i^3$ in terms of the Clifford algebra as follows:

$$\begin{aligned} S_1^3 = \Big\{ & \frac{x_1}{2}\,(1 - e_{n-3}e_{n-2}e_{n-1}e_n) + \frac{x_2}{2}\,(e_{n-3}e_n + e_{n-2}e_{n-1}) \\ & + \frac{x_3}{2}\,(-e_{n-3}e_{n-1} + e_{n-2}e_n) + \frac{x_4}{2}\,(e_{n-3}e_{n-2} + e_{n-1}e_n) \\ & + \frac{1}{2}(1 + e_{n-3}e_{n-2}e_{n-1}e_n) : \sum_{i=1}^{4} x_i^2 = 1 \Big\} \end{aligned}$$

and

$$\begin{aligned} S_2^3 = \Big\{ & \frac{x_1}{2}\,(1 + e_{n-3}e_{n-2}e_{n-1}e_n) + \frac{x_2}{2}\,(e_{n-3}e_n - e_{n-2}e_{n-1}) \\ & + \frac{x_3}{2}\,(e_{n-3}e_{n-1} + e_{n-2}e_n) + \frac{x_4}{2}\,(e_{n-3}e_{n-2} - e_{n-1}e_n) \\ & + \frac{1}{2}\,(1 - e_{n-3}e_{n-2}e_{n-1}e_n) : \sum_{i=1}^{4} x_i^2 = 1 \Big\}. \end{aligned}$$

As in Section 3 we embed $\mathrm{Spin}(3)$ into $\mathrm{Spin}(n)$ as $S_1^3$, hence $C = C_{\mathrm{Spin}(n)} = \mathrm{Spin}(n-4) \times S_2^3$.

We calculate $\mathrm{Ad}(J \circ \nu) : S^{n-4} \wedge S^3 \to \mathrm{Spin}(n)$ explicitly. Let $u \wedge x \in S^{n-4} \wedge S^3$, where $u = \sum_{i=1}^{n-3} u_i e_i$ with $\sum_{i=1}^{n-3} u_i^2 = 1$ and $x \in S_1^3$. We choose $g \in \mathrm{Spin}(n-3)$ so as to satisfy $g e_n g^t = u$. (Compare (1).) Then $\mathrm{Ad}(J \circ \nu)(u \wedge x) = g x g^t x^t$. It is easy to see that

$$\begin{aligned} \mathrm{Ad}(J \circ \nu)(u \wedge x) = & \frac{-1 + x_1}{2}\, u e_{n-3} + \frac{1 + x_1}{2} \\ & + \frac{1}{2}\,(-x_4 e_{n-2} + x_3 e_{n-1} - x_2 e_n)(u - e_{n-3}). \end{aligned}$$

We set

$$\mathrm{Pin}^1(n) = \mathrm{Pin}(n) - \mathrm{Spin}(n).$$

We multiply the formula for $\mathrm{Ad}(J \circ \nu)(u \wedge x)$ by $e_{n-3}$ from the right, then we exchange the coefficients of $e_{n-2}$ and $e_n$. Thus we obtain a map $\psi : S^{n-4} \wedge S^3 \to \mathrm{Pin}^1(n)$ defined by

$$\begin{aligned} \text{(5.1)} \qquad \psi(u \wedge x) = & \frac{1 - x_1}{2}\, u + \frac{1 + x_1}{2}\, e_{n-3} \\ & + \frac{1}{2}\,(x_2 e_{n-2} + x_3 e_{n-1} + x_4 e_n)(1 + u e_{n-3}). \end{aligned}$$

In order to prove Theorem B(ii), it suffices to prove that $\psi$ represents a generator of $\pi_{n-1}(\mathrm{Pin}^1(n))$.

Recall that for $n \neq 3, 7$, the characteristic map $T : S^{n-1} \to SO(n)$ of the principal bundle $SO(n) \to SO(n+1) \to S^n$ represents a generator of $\pi_{n-1}(SO(n))$, the first non-stable homotopy group (see, for example, [10]).

In the Clifford algebra, the generator is given by the inclusion $\iota : S^{n-1} \to \mathrm{Pin}^1(n)$. In Lemma 4.4 for $m = n-4$ and $p = 3$, we exchange $u_1$ and $u_{n-3}$, and change the order of the coordinates of the right-hand side. Thus we obtain a map $\zeta : S^{n-4} \wedge S^3 \to \mathrm{Pin}^1(n)$ defined by

$$\begin{aligned}\zeta(u \wedge x) = {} & \frac{1-x_1}{2}\, u + \frac{1+x_1}{2}\, e_{n-3} \\ & + \sqrt{\frac{1-u_{n-3}}{2}}\,(x_2 e_{n-2} + x_3 e_{n-1} + x_4 e_n).\end{aligned} \tag{5.2}$$

Now we have two maps, $\psi$ in (5.1) and $\zeta$ in (5.2). Making use of the homotopy

$$u e_{n-3} \mapsto \Big\{ t\Big(\sum_{i=1}^{n-4} u_i e_i\Big) + u_{n-3} e_{n-3} \Big\} e_{n-3},$$

where $t \in I$, it is easy to construct a homotopy $\psi \simeq \zeta$. This completes the proof of Theorem B(ii). ∎

Now a result which corresponds to Lemma 4.6 and Proposition 4.7 is as follows.

PROPOSITION 5.3. *There are choices of the generators $y_\mu$ and $z_\mu$ such that the following relations hold in $H_{n-4}(\Omega_0^3 \mathrm{Spin}(n); \mathbb{Z}/2)$:*

(i) $J_*(\gamma_{n-4}) = z_{n-4}$ *for* $n \equiv 0, 2 \pmod 4$.
(ii) $J_*(\beta_{n-4}) = y_{n-4}$ *for* $n \equiv 1 \pmod 4$.

*Proof.* Let $W$ be the homotopy-theoretic fiber of the inclusion $\Omega_0^3 \mathrm{Spin}(n) \hookrightarrow \Omega_0^3 \mathrm{Spin} \simeq BSp$. Consider the following diagram:

$$\begin{tikzcd} W \arrow[r] & \Omega_0^3 \mathrm{Spin}(n) \arrow[r] & \Omega_0^3 \mathrm{Spin} \simeq BSp \\ S^{n-4} \arrow[u, "\widetilde{J\circ\nu}"] \arrow[ur, "J\circ\nu"'] & & \end{tikzcd}$$

where $\widetilde{J \circ \nu}$ is the lift of $J \circ \nu$. The first non-vanishing homotopy of $W$ is $\pi_{n-4}(W)$ and $\widetilde{J \circ \nu}$ is a generator by Theorem B(ii). If $n \not\equiv 3 \pmod 4$, then for dimensional reasons, $H_{n-4}(W; \mathbb{Z}/2) \to H_{n-4}(\Omega_0^3 \mathrm{Spin}(n); \mathbb{Z}/2)$ is injective. Hence $(J \circ \nu)_*(\sigma_{n-4}) \neq 0$, where $\sigma_{n-4}$ is a generator of $H_{n-4}(S^{n-4}; \mathbb{Z}/2)$. By (3.2), we have

$$\nu_*(\sigma_{n-4}) = \begin{cases} \gamma_{n-4}, & n \equiv 0, 2 \pmod 4, \\ \beta_{n-4}, & n \equiv 1 \pmod 4. \end{cases}$$

Then we can define $y_{n-4}$ or $z_{n-4}$ to be $J_*(\gamma_{n-4})$ or $J_*(\beta_{n-4})$. (More precisely, $J_*(\gamma_{n-4})$ for $n \equiv 6 \pmod 8$ and $J_*(\beta_{n-4})$ for $n \equiv 5 \pmod 8$ must be the right-hand side of (3.3), since $H_{n-4}(\Omega_0^3 \mathrm{Spin}(n); \mathbb{Z}/2) \cong \mathbb{Z}/2$.) ∎

*Proof of Theorem 3.4.* First, Theorem 3.4(i) follows from the homotopy commutative diagram

$$\begin{tikzcd}
\mathrm{Spin}(n)/C_{\mathrm{Spin}(n)} \arrow[r, "J"] \arrow[d, "i"] & \Omega_0^3\,\mathrm{Spin}(n) \arrow[d, "j"] \\
\mathrm{Spin}/C_{\mathrm{Spin}} \arrow[r, "J"] \arrow[d, "\simeq"] & \Omega_0^3\,\mathrm{Spin} \arrow[d, "\simeq"] \\
BSp(1) \arrow[r, "k"] & BSp
\end{tikzcd} \tag{5.4}$$

where $i, j$ and $k$ are the inclusions. (The homotopy commutativity of the bottom square is proved in the same way as in the third diagram of [9, p. 4054] for $k = 1$ and $l = \infty$.)

Next, (ii) and (iii) of Theorem 3.4 are proved using Proposition 5.3 and a diagram similar to (2.5). (Compare the proof of Theorem 2.7 in Section 4.) This completes the proof of Theorem 3.4, and hence also that of Theorem A for $G = \mathrm{Spin}(n)$. ■

Remark. Using Theorem 3.4, we can obtain information on $J_*(\alpha)$ for each generator $\alpha$ of $H_*(\mathrm{Spin}(n)/C; \mathbb{Z}/2)$. For example, consider an element of the form $\alpha = (u_{i_1}^{\varepsilon_1} u_{i_2}^{\varepsilon_2} u_{i_3}^{\varepsilon_3})^*$, where $\varepsilon_j = 0$ or 1. (For the generators $u_{i_j}$, see Proposition 3.1, and for the notation $(\,\cdot\,)^*$, see (3.2).) We write $\xi_\mu$ for $y_\mu$ or $z_\mu$. Then we can prove that $J_*(\alpha)$ contains the term $\xi_{i_1}^{\varepsilon_1} * \xi_{i_2}^{\varepsilon_2} * \xi_{i_3}^{\varepsilon_3}$. For the proof, consider the coproduct $\Delta_*$.

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*Received 20 January 2003*