Generating varieties for the triple loop space of classical Lie groups

by

Yasuhiko Kamiyama (Okinawa)

Abstract. For $G = SU(n), Sp(n)$ or Spin$(n)$, let $C_G(SU(2))$ be the centralizer of a certain $SU(2)$ in $G$. We have a natural map $J : G/C_G(SU(2)) \to \Omega_0^3 G$. For a generator $\alpha$ of $H_* (G/C_G(SU(2)); \mathbb{Z}/2)$, we describe $J_*(\alpha)$. In particular, it is proved that $J_* : H_* (G/C_G(SU(2)); \mathbb{Z}/2) \to H_* (\Omega_0^3 G; \mathbb{Z}/2)$ is injective.

1. Introduction. The purpose of this paper is to study an analogue of Bott’s theorem on generating varieties when $S^1$ is replaced by $SU(2)$ and $G$ is a classical Lie group. We will explain the motivation for the study later. We recall Bott’s theorem. Let $G$ be a compact simply connected Lie group. For a circle $S^1$ on $G$, let $C_G(S^1)$ be the centralizer of $S^1$ in $G$. We associate with it a map $f : G/C_G(S^1) \to \Omega G$ defined by $f(gC_G(S^1))(x) = gxg^{-1}x^{-1}$, where $x \in S^1$. According to Bott [2], if $S^1$ is a suitable circle (more precisely, $S^1$ determines an element of $H_1(T; \mathbb{Z})$, where $T$ is a maximal torus of $G$ containing $S^1$, and if the element is dual to a long root), then ($S^1$ becomes a generating circle and) $f$ has the property that the image of $f_* : H_* (G/C_G(S^1); \mathbb{Z}) \to H_* (\Omega G; \mathbb{Z})$ generates the Pontryagin ring $H_* (\Omega G; \mathbb{Z})$. We call $G/C_G(S^1)$ a generating variety.

Let $SU(2)$ be a subgroup of $G$ and $C_G(SU(2))$ its centralizer in $G$. Hereafter we abbreviate $C_G(SU(2))$ to $C$ or $C_G$ and consider homology with $\mathbb{Z}/2$-coefficients. We have a map

$$J : G/C \to \Omega_0^3 G$$

defined by

$$J(gC)(x) = gxg^{-1}x^{-1},$$

where $x \in SU(2)$. For $G = SU(2)$, $J$ is essentially the well known $J$-homomorphism $J : SO(3) \to \Omega_1^3 S^3$. At present, the Pontryagin ring $H_* (\Omega_0^3 G; \mathbb{Z}/2)$

2000 Mathematics Subject Classification: Primary 58D27; Secondary 53C07, 55R40.

Key words and phrases: generating variety, instantons, triple loop space.
is known for all $G$. (See [3] for $G = SU(n)$, [4] for $G = \text{Spin}(n)$, [5] for $G = \text{Sp}(n)$ and [6] for exceptional Lie groups.) The ring is polynomial with infinitely many generators and the generators are constructed as follows. From a finite number of elements $b_{\mu_1}, \ldots, b_{\mu_r}$ of $H_*(\Omega^3_0 G; \mathbb{Z}/2)$, all the generators are constructed by applying the homology operations $Q_1$, $Q_2$ (and possibly $Q_3$). In this paper the subscript of an element means its degree, that is, $\text{deg}(b_\mu) = \mu$. We call these $b_\mu$ fundamental generators. Now we consider the following questions:

1. Is $J_* : H_*(G/C; \mathbb{Z}/2) \to H_*(\Omega^3_0 G; \mathbb{Z}/2)$ injective?
2. Does the image of $J_*$ contain all the fundamental generators?

The motivation for the questions is as follows. For a compact simple simply connected Lie group $G$, let $M(k, G)$ be the moduli space of based gauge equivalence classes of $G$-instantons over $S^4$ with instanton number $k$. Let $i_k : M(k, G) \to \Omega^3_0 G$ be the inclusion. In [3] Boyer, Mann and Waggoner constructed non-trivial classes in $H_*(M(k, SU(n)); \mathbb{Z}/2)$. For that purpose, they used a description of $M(1, G)$ in terms of a homogeneous space. More precisely, let $C = C_G(SU(2))$ be the centralizer of a certain $SU(2)$ in $G$. Then a result of [3] tells us that there exists a diffeomorphism $M(1, G) \cong \mathbb{R}^5 \times G/C$ such that the following diagram is homotopy commutative:

$$
\begin{array}{ccc}
M(1, G) & \xrightarrow{i_1} & \Omega^3_1 G \\
\downarrow \cong & & \downarrow \cong \\
G/C & \xrightarrow{J} & \Omega^3_0 G
\end{array}
$$

A crucial result of [3] determines $J_*(\alpha)$, where $\alpha \in H_*(SU(n)/C; \mathbb{Z}/2)$ is an even-dimensional generator (see Theorem 2.6). The motivation of this paper is to continue the study for $G = SU(n), \text{Sp}(n)$, and $\text{Spin}(n) (n \neq 4)$. Hence hereafter we consider an embedding of $SU(2)$ into $G$ so that $M(1, G) \cong \mathbb{R}^5 \times G/C$. (See [3], [9] or Sections 2–3.)

For $G/C$, we have the following examples:

**Example 1.1** ([3], [9]).

1. $SU(n)/C_{SU(n)}$ is diffeomorphic to the unit tangent bundle of $\mathbb{C}P^{n-1}$.
2. $\text{Sp}(n)/C_{\text{Sp}(n)}$ is diffeomorphic to $\mathbb{R}P^{4n-1}$.
3. Since $\text{Spin}(5) \cong \text{Sp}(2)$ and $\text{Spin}(6) \cong SU(4)$, we obtain examples of $\text{Spin}(n)/C_{\text{Spin}(n)}$ for $n = 5$ and 6 from (i) and (ii).

In the main theorems of this paper, we describe $J_*(\alpha)$ for $\alpha$ a generator of $H_*(G/C; \mathbb{Z}/2)$. (See Theorem 2.7 for $G = SU(n)$ and $\alpha$ an odd-dimensional generator, Theorem 2.8 for $G = \text{Sp}(n)$, and Theorem 3.4 for $G = \text{Spin}(n)$.) In particular, we have the following:
Theorem A. Let $G$ be $SU(n)$, $Sp(n)$, or $Spin(n)$ ($n \neq 4$). Embed $SU(2)$ into $G$ so that $M(1, G) \cong \mathbb{R}^5 \times G/C$. Then

(i) $J_* : H_*(G/C; \mathbb{Z}/2) \to H_*(\Omega^3_0G; \mathbb{Z}/2)$ is injective.

(ii) Excluding a fundamental generator in $H_{2n-6}(\Omega^3_0 Spin(n); \mathbb{Z}/2)$ for $n \equiv 3 \pmod{4}$, there are choices of the fundamental generators $b_\mu$ of $H_*(\Omega^3_0G; \mathbb{Z}/2)$ such that every $b_\mu$ is in the image of $J_*$. If $\alpha \in H_*(G/C_G; \mathbb{Z}/2)$ is a stable element, that is, $\alpha$ is non-trivial in $H_*(G(\infty)/C_{G(\infty)}; \mathbb{Z}/2)$ (where $G(\infty)$ denotes $SU$, $Sp$ or $Spin$), then we can use the Bott periodicity to study $J_*(\alpha)$ in $H_*(\Omega^3_0G; \mathbb{Z}/2)$ (see (5.4)). On the other hand, for $G = SU(n)$ or $Spin(n)$, $H_*(G/C_G; \mathbb{Z}/2)$ contains a non-stable element. The following theorem is a key to determining $J_*(\alpha)$ for $\alpha$ non-stable. For a map $f : X \to \Omega^3_0 G$, let $Ad(f) : \Sigma^3 X \to G$ be the adjoint map of $f$.

Theorem B. (i) For $G = SU(n)$ with $n$ even, let $i : S^{2n-3} \to SU(n)/C$ be the inclusion of the fiber of the unit tangent bundle of $\mathbb{C}P^{n-1}$. Then $Ad(J \circ i)$ generates the $2$-component of $\pi_{2n}(SU(n)) \cong \mathbb{Z}/n!$.

(ii) For $G = Spin(n)$ with $n \neq 7$, there exists a spherical class in $H_{n-4}(Spin(n)/C; \mathbb{Z})$ which is denoted by $\nu : S^{n-4} \to Spin(n)/C$. Then $Ad(J \circ \nu)$ is a generator of $\pi_{n-1}(Spin(n))$. (More precisely; $Ad(J \circ \nu)$ is one of the generators for $n \equiv 0, 1, 2, 4 \pmod{8}$; $Ad(J \circ \nu)$ has infinite order for $n$ even and is of order $2$ for $n$ odd with $n \neq 7$.)

This paper is organized as follows. In Section 2 we study the cases $G = SU(n)$ and $Sp(n)$. Theorems 2.7 and 2.8 are the main results for $G = SU(n)$ and $G = Sp(n)$ respectively. In Section 3 we study the case $G = Spin(n)$, where Theorem 3.4 is the main result. Theorems 2.7 and 2.8 are proved in Section 4 and Theorem 3.4 is proved in Section 5, where Theorem B is a key for the proofs.

2. Main results for $G = SU(n)$ and $Sp(n)$. First we study the case $G = SU(n)$. Since $SU(2) = Sp(1)$, we assume $n \geq 3$. (The (2.2) below holds for $n = 2$, but it is more natural to consider (2.4) for $n \geq 3$.) We embed $SU(2)$ into $SU(n)$ as the first $2 \times 2$ elements. Then $C = C_{SU(n)}$ is given by the following set of matrices:

$$C = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & A \end{pmatrix} : A \in U(n-2), \ a^2 \det A = 1 \right\}. \tag{2.1}$$

Therefore $SU(n)/C$ is diffeomorphic to the unit tangent bundle of $\mathbb{C}P^{n-1}$ (see [3], [9] or (4.1)) and $H^*(SU(n)/C; \mathbb{Z}/2)$ is given as follows:
(1) For $n$ even,
\[ H^*(SU(n)/C; \mathbb{Z}/2) \cong H^*(\mathbb{C}P^{n-1}; \mathbb{Z}/2) \otimes H^*(S^{2n-3}; \mathbb{Z}/2). \]

(2) For $n$ odd,
\[ H^*(SU(n)/C; \mathbb{Z}/2) \cong H^*(\mathbb{C}P^{n-2}; \mathbb{Z}/2) \otimes H^*(S^{2n-1}; \mathbb{Z}/2). \]

We write the generators of $H_*(SU(n)/C; \mathbb{Z}/2)$ as follows.

(1) For $n$ even,
\[
\begin{cases}
\alpha_{2i}, & 1 \leq i \leq n-1, \\
\beta_{2i+1}, & n-2 \leq i \leq 2n-3.
\end{cases}
\]

(2) For $n$ odd,
\[
\begin{cases}
\alpha_{2i}, & 1 \leq i \leq n-2, \\
\beta_{2i+1}, & n-1 \leq i \leq 2n-3.
\end{cases}
\]

The structure of $H_*(\Omega^3_0 SU(n); \mathbb{Z}/2)$ is known: First, the following elements are defined in $H_*(\Omega^3_0 SU(n); \mathbb{Z}/2)$:
\[
\begin{cases}
x_2, \\
y_{2i}, & 2 \leq i \leq n-2, \ i \equiv 0 \pmod{2}, \\
z_{4i+1}, & \left\lceil \frac{n-1}{2} \right\rceil \leq i \leq n-2, \ i \equiv 1 \pmod{2}.
\end{cases}
\]
Here $x_2$ is defined by $x_2 = Q_2[1] * [-2]$. (2.4) gives the set of fundamental generators of $H_*(\Omega^3_0 SU(n); \mathbb{Z}/2)$. The set of all ring generators is as follows:
\[
\begin{align*}
Q_2^a(x_2), & \quad a \geq 0, \\
Q_2^a(y_{2i}), & \quad a \geq 0, \ 2 \leq i \leq \left\lceil \frac{n-3}{2} \right\rceil, \ i \equiv 0 \pmod{2}, \\
Q_i^aQ_j^b(y_{2i}), & \quad a, b \geq 0, \ \left\lceil \frac{n-1}{2} \right\rceil \leq i \leq n-2, \ i \equiv 0 \pmod{2}, \\
Q_i^aQ_j^b(z_{4i+1}), & \quad a, b \geq 0, \ \left\lceil \frac{n-1}{2} \right\rceil \leq i \leq n-2, \ i \equiv 1 \pmod{2}.
\end{align*}
\]

Consider the following diagram:
\[
\begin{array}{c}
SU(n)/C_{SU(n)} \xrightarrow{J} \Omega^3_0 SU(n) \\
\downarrow i \quad \downarrow j \\
SU(n+1)/C_{SU(n+1)} \xrightarrow{J} \Omega^3_0 SU(n+1)
\end{array}
\]
where $i$ and $j$ are the inclusions. If $\alpha_{2i}$ is defined both in $H_{2i}(SU(n)/C_{SU(n)}; \mathbb{Z}/2)$ and $H_{2i}(SU(n+1)/C_{SU(n+1)}; \mathbb{Z}/2)$, then $\alpha_{2i}$ in the top row is mapped by $i_*$ to $\alpha_{2i}$ in the bottom row. Similar remarks hold for $\beta_{2i+1}$ with respect to $i_*$ and for $x_2$, $y_{2i}$, and $z_{4i+1}$ with respect to $j_*$. $J_*(\alpha_{2i})$ is known:

**Theorem 2.6 ([3]).** There are choices of the fundamental generators $y_{2i}$ such that:
(i) For $i$ even, $J_*(\alpha_{2i}) = y_{2i}$.
(ii) For $i$ odd, $J_*(\alpha_{2i})$ contains the term $x_2 \ast y_{2i-2}$.

Our main result for $G = SU(n)$ is as follows.

**Theorem 2.7.** There are choices of the fundamental generators $z_{4i+1}$ (where $i \equiv 1 \pmod{2}$) such that

(i) $J_*(\beta_{8k+1})$ contains the term $Q_1(y_{4k})$.
(ii) $J_*(\beta_{8k+3})$ contains the term $x_2 \ast Q_1(y_{4k})$.
(iii) $J_*(\beta_{8k+5}) = z_{8k+5}$.
(iv) $J_*(\beta_{8k+7})$ contains the term $x_2 \ast z_{8k+5}$.

**Remark.** In [3], $J_*(\beta_5)$ is also studied.

From Theorems 2.6, 2.7 and the structure of $H^*(SU(n)/C; \mathbb{Z}/2)$, we see that $J^*: H^*(\Omega^3_0 SU(n); \mathbb{Z}/2) \to H^*(SU(n)/C; \mathbb{Z}/2)$ is surjective, hence Theorem A holds for $G = SU(n)$.

Next we study the case $G = Sp(n)$. We embed $Sp(1)$ into $Sp(n)$ as the first $1 \times 1$ element. Then $C = C_{Sp(n)} = \mathbb{Z}/2 \times Sp(n-1)$, hence $Sp(n)/C$ is diffeomorphic to $\mathbb{R}P^{4n-1}$. We write the generators of $H_*(Sp(n)/C; \mathbb{Z}/2)$ as $\alpha_\mu$ ($1 \leq \mu \leq 4n-1$). The structure of $H_*(\Omega^3_0 Sp(n); \mathbb{Z}/2)$ is given in [5]. The result is

$$H_*(\Omega^3_0 Sp(n); \mathbb{Z}/2) \cong \mathbb{Z}/2[Q^a_1 Q^b_2 [1] \ast [-2^{a+b}] : a, b \geq 0]$$

$$\otimes \mathbb{Z}/2[Q^a_1 Q^b_2 (z_{4i}) : a, b \geq 0, 1 \leq i \leq n-1].$$

Setting $x_1 = Q_1[1] \ast [-2]$ and $y_2 = Q_2[1] \ast [-2]$, we have the set of fundamental generators $x_1, y_2$ and $z_{4i}$ ($1 \leq i \leq n-1$).

Our result for $G = Sp(n)$ is as follows.

**Theorem 2.8.** There are choices of the fundamental generators $z_{4i}$ such that

(i) $J_*(\alpha_1) = x_1$, $J_*(\alpha_2) = y_2$ and $J_*(\alpha_{4i}) = z_{4i}$ ($1 \leq i \leq n-1$).
(ii) In general, for $1 \leq \mu \leq 4n-1$, set $\mu = \varepsilon_1 + 2\varepsilon_2 + 4i$, where $\varepsilon_j = 0$ or 1. Then $J_*(\alpha_\mu)$ contains the term $x_1^{\varepsilon_1} \ast y_2^{\varepsilon_2} \ast z_{4i}$.

Theorem A for $G = Sp(n)$ follows from Theorem 2.8 and the structure of $H^*(Sp(n)/C; \mathbb{Z}/2)$.

We prove Theorems 2.7 and 2.8 in Section 4.

**3. Main result for $G = \text{Spin}(n)$.** Since $\text{Spin}(3) \cong SU(2)$ and $\text{Spin}(4)$ is not simple, we assume $n \geq 5$. Recall that $\text{Spin}(4) \cong S^3 \times S^3$, where $S^3$ and $S^3$ are two copies of $\text{Spin}(3) \cong S^3$. (In Section 5 we write $S^3$ in terms of the Clifford algebra.) We embed $\text{Spin}(3)$ into $\text{Spin}(n)$ as $S^3$. Then $C = C_{\text{Spin}(n)} = \text{Spin}(n-4) \times S^3$. We recall the structure of $H^*(\text{Spin}(n)/C; \mathbb{Z}/2)$.
PROPOSITION 3.1 ([7]). Let \( n = 4t + l \) with \( 0 \leq l \leq 3 \). Then we have the following isomorphism of modules:

\[
H^*(\text{Spin}(n)/C; \mathbb{Z}/2) \cong \mathbb{Z}/2[c_2]/(c_2^4) \otimes \Delta(u_{i_1}, u_{i_2}, u_{i_3}),
\]

where \( \deg c_2 = 4 \) and \( \deg u_{i_j} = i_j \) with

\[
(i_1, i_2, i_3) = \begin{cases} 
(4t - 4, 4t - 3, 4t - 2), & n = 4t, \\
(4t - 3, 4t - 2, 4t), & n = 4t + 1, \\
(4t - 2, 4t, 4t + 1), & n = 4t + 2, \\
(4t, 4t + 1, 4t + 2), & n = 4t + 3.
\end{cases}
\]

Here \( \Delta(x_1, \ldots, x_m) \) denotes the graded algebra over \( \mathbb{Z}/2 \) with \( \mathbb{Z}/2 \)-basis \( \{x_{i_1} \cdots x_{i_r} : 1 \leq i_1 < \ldots < i_r \leq m\} \). (If we add the relations \( x_i^2 = 0 \), \( \Delta(x_1, \ldots, x_m) \) becomes the exterior algebra \( \Lambda(x_1, \ldots, x_m) \).)

The mod 2 cohomology ring with squaring operations is also determined in [7]. By Proposition 3.1, we can define elements of \( H_*(\text{Spin}(n)/C; \mathbb{Z}/2) \) as follows. Note that two of \( u_{i_j} \) are even-dimensional and one of them has dimension congruent to 1 mod 4. We write them as \( u_{2a_j} \) \( (j = 1, 2) \) and \( u_{4b+1} \). Hence \( a_2 = a_1 + 1 \). We set

\[
\begin{align*}
\alpha_{4i} &= (c_2^i)^*, & 1 \leq i \leq t - 1, \\
\beta_{4b+4i+1} &= (c_2^i u_{4b+1})^*, & 0 \leq i \leq t - 1, \\
\gamma_{2a_j+4i} &= (c_2^i u_{2a_j})^*, & 0 \leq i \leq t - 1, \ j = 1, 2,
\end{align*}
\]

where \( (\cdot)^* \) denotes the dual element with respect to the monomial basis.

The structure of \( H_*(\Omega_0^3 \text{Spin}(n); \mathbb{Z}/2) \) is given in [4]. It depends on the congruence class of \( n \) mod 8. For example, for \( n = 8g + 2 \), we have the following set of fundamental generators of \( H_*(\Omega_0^3 \text{Spin}(8g + 2); \mathbb{Z}/2) \):

\[
\begin{align*}
x_{4k}, & \quad 1 \leq k \leq 2g - 1, \\
y_{8g+8k+5}, & \quad 0 \leq k \leq g - 1, \\
z_{8g+2k}, & \quad -1 \leq k \leq 4g - 2, \ k \not\equiv 1 \ (\text{mod} \ 4).
\end{align*}
\]

Hence \( y_\mu \) is defined only when \( \mu \equiv 5 \ (\text{mod} \ 8) \) and \( z_\mu \) is defined only when \( \mu \equiv 0, 4 \) or 6 \ (\text{mod} \ 8) \). We define \( y_\mu \) for \( \mu \equiv 1 \ (\text{mod} \ 8) \) and \( z_\mu \) for \( \mu \equiv 2 \ (\text{mod} \ 8) \) by setting

\[
\begin{align*}
y_{8g+8k+1} &= Q_1(x_{4g+4k}), & 0 \leq k \leq g - 1, \\
z_{8g+2k} &= Q_2(x_{4g+k-1}), & 1 \leq k \leq 4g - 3, \ k \equiv 1 \ (\text{mod} \ 4).
\end{align*}
\]

The elements of (3.3) are not fundamental generators. We do the same procedure for all \( n \). Then we obtain the following elements of \( H_*(\Omega_0^3 \text{Spin}(n); \mathbb{Z}/2) \), among which the elements of the form (3.3) are not fundamental generators.
(1) For \( n = 4t \),
\[
\begin{align*}
x_{4i}, & \quad 1 \leq i \leq t - 1, \\
y_{4t+4i-3}, & \quad 0 \leq i \leq t - 1, \\
z_{4t+2i-4}, & \quad 0 \leq i \leq 2t - 1.
\end{align*}
\]

(2) For \( n = 4t + 1 \),
\[
\begin{align*}
x_{4i}, & \quad 1 \leq i \leq t - 1, \\
y_{4t+4i-3}, & \quad 0 \leq i \leq t - 1, \\
z_{4t+2i-2}, & \quad 0 \leq i \leq 2t - 1.
\end{align*}
\]

(3) For \( n = 4t + 2 \),
\[
\begin{align*}
x_{4i}, & \quad 1 \leq i \leq t - 1, \\
y_{4t+4i+1}, & \quad 0 \leq i \leq t - 1, \\
z_{4t+2i-2}, & \quad 0 \leq i \leq 2t - 1.
\end{align*}
\]

(4) For \( n = 4t + 3 \),
\[
\begin{align*}
x_{4i}, & \quad 1 \leq i \leq t - 1, \\
y_{4t+4i+1}, & \quad 0 \leq i \leq t - 1, \\
z_{4t+2i}, & \quad 0 \leq i \leq 2t.
\end{align*}
\]

For \( \alpha, \beta, \gamma \) in (3.2) and \( x, y, z \), a similar naturality to (2.5) holds. Note that \( \alpha, \beta, \gamma \) correspond to \( x, y, z \) respectively except the fundamental generator \( z_{8t} \) in \( H_{8t}(\Omega_0^3 \text{Spin}(n); \mathbb{Z}/2) \) for \( n = 4t + 3 \). The following main theorem asserts that the correspondence is realized through \( J_* \):

**Theorem 3.4.** There are choices of the fundamental generators \( x, y, z \) for \( \mu \equiv 0 \pmod{4}, \mu \equiv 5 \pmod{8} \) and \( z \) for \( \mu \equiv 0, 4, 6 \pmod{8} \) such that

(i) \( J_*(\alpha) = x \).

(ii) (a) When \( \mu \equiv 5 \pmod{8} \), \( J_*(\beta) = y \).

(b) When \( \mu \equiv 1 \pmod{8} \), \( J_*(\beta) \) contains the term \( y \).

(iii) (a) When \( \mu \equiv 0, 4 \text{ or } 6 \pmod{8} \), \( J_*(\gamma) = z \).

(b) When \( \mu \equiv 2 \pmod{8} \), \( J_*(\gamma) \) contains the term \( z \).

Note that \( y \) in Theorem 3.4(ii)(b) and \( z \) in (iii)(b) are not fundamental generators. (Compare (3.3).)

Theorem A for \( G = \text{Spin}(n) \) follows from Theorem 3.4 and the structure of \( H^*(\text{Spin}(n)/C; \mathbb{Z}/2) \) (see Proposition 3.1).

We prove Theorem 3.4 in Section 5.
4. Proofs of Theorems 2.7 and 2.8. First we prove Theorem B(i). Recall that \( C = C_{SU(n)} \) is given in (2.1). We set

\[
SU(n - 2) = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & A \end{pmatrix} : A \in SU(n - 2) \right\},
\]

\[
SU(n - 1) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix} : B \in SU(n - 1) \right\},
\]

\[
U(n - 1) = \left\{ \begin{pmatrix} a & 0 \\ 0 & B \end{pmatrix} : B \in U(n - 1), a \det B = 1 \right\}.
\]

We have the following fiber bundle which realizes \( SU(n)/C \) as the unit tangent bundle of \( \mathbb{C}P^{n-1} \):

\[
SU(n - 1)/SU(n - 2) \overset{i}{\to} SU(n)/C \to SU(n)/U(n - 1),
\]

where \( i \) is the inclusion. We obtain a map \( \text{Ad}(J \circ i) : S^{2n-3} \wedge S^3 \to SU(n) \). Consider the homotopy sequence of a principal bundle

\[
SU(n - 1) \to SU(n) \to \mathbb{S}^{2n-1}.
\]

Since

\[
\pi_{2n-1}(SU(n - 1)) \cong \begin{cases} 0, & n \text{ even,} \\ \mathbb{Z}/2, & n \text{ odd,} \end{cases}
\]

(see, for example, [8]), \( p_* : \pi_{2n}(SU(n)) \to \pi_{2n}(S^{2n-1}) \) is surjective for \( n \) even. Hence in order to prove Theorem B(i), it suffices to prove the following:

**Proposition 4.2.** For \( n \) even, set \( \phi = p \circ \text{Ad}(J \circ i) \). Then \( \phi \) is a generator of \( \pi_{2n}(S^{2n-1}) \).

**Proof.** First we calculate \( \phi \) explicitly. Let \((z_1, \ldots, z_{n-1}) \wedge (a, b) \in S^{2n-3} \wedge S^3 \). We use the following notations.

(1) A matrix in \( SU(n) \) which represents \((z_1, \ldots, z_{n-1}) \) is given by

\[
g = \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 \\
0 & z_1 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & \cdots & & A & 0 \\
z_{n-1} & & & & A
\end{pmatrix}
\]

where

\[
\begin{pmatrix}
z_1 & 0 & \cdots & 0 \\
\vdots & A \\
z_{n-1}
\end{pmatrix} \in SU(n - 1).
\]
(2) A matrix in $SU(n)$ which corresponds to $(a, b)$ is given by

$$x = \begin{pmatrix} a & -\bar{b} & 0 \\ b & \bar{a} & E_{n-2} \end{pmatrix}$$

where $E_{n-2}$ denotes the unit matrix.

(3) We define the projection $p : SU(n) \to S^{2n-1}$ by mapping an element of $SU(n)$ to its first row.

Now it is easy to calculate the first row of $gxg^{-1}x^{-1}$. The result is

$$\phi((z_1, \ldots, z_{n-1}) \land (a, b)) = (|b|^2z_1 + |a|^2, ab(1 - \bar{z}_1), -\bar{b}z_2, \ldots, -\bar{b}z_{n-1}).$$

Define $\psi : S^{2n-3} \land S^3 \to S^{2n-1}$ by

$$(4.3) \quad \psi((z_1, \ldots, z_{n-1}) \land (a, b)) = (|b|^2z_1 + |a|^2, bz_2, \ldots, bz_{n-1}, ab(1 - z_1)).$$

It is clear that $\phi \simeq \psi$.

Next we calculate the generator of $\pi_2n(S^{2n-1})$ explicitly. The following lemma is well known (see, for example, [11, p. 107]).

**Lemma 4.4.** The map $j : S^m \land S^p \to S^{m+p}$ defined by

$$j((u_1, \ldots, u_{m+1}) \land (x_1, \ldots, x_{p+1})) = \left( \frac{(1 - x_1)u_1 + (1 + x_1)}{2}, \frac{(1 - x_1)u_2}{2}, \ldots, \frac{(1 - x_1)u_{m+1}}{2}, \sqrt{\frac{1 - u_1}{2}}x_2, \ldots, \sqrt{\frac{1 - u_1}{2}}x_{p+1} \right)$$

is a homeomorphism.

Let $\eta : S^3 \to S^2$ be the Hopf map. It is given by $\eta(a, b) = (1 - 2|b|^2, 2ab) \in S^2 \subset \mathbb{R} \times \mathbb{C}$. Then $\Sigma^{2n-3}\eta$ is the generator of $\pi_2n(S^{2n-1})$, and using Lemma 4.4, we have

$$(4.5) \quad \Sigma^{2n-3}\eta((z_1, \ldots, z_{n-1}) \land (a, b))$$

$$= (|b|^2z_1 + |a|^2, |b|^2z_2, \ldots, |b|^2z_{n-1}, ab\sqrt{2(1 - \text{Re}z_1)}).$$

Now we have two maps, $\psi$ in (4.3) and $\Sigma^{2n-3}\eta$ in (4.5). We prove that $\psi \simeq \Sigma^{2n-3}\eta$ when $n$ is even. We define a map $h : S^{2n-1} \to S^{2n-2}$ by

$$h(w_1, \ldots, w_n) = \frac{(w_1, 2\bar{w}_2w_3, |w_2|^2 - |w_3|^2, w_4, \ldots, w_n)}{\sqrt{|w_1|^2 + (|w_2|^2 + |w_3|^2)^2 + \sum_{i=4}^{n} |w_i|^2}}.$$ 

It is easy to see that $h \simeq \Sigma^{2n-4}\eta$, hence the composition $h \circ \pi_2n(S^{2n-1}) \to \pi_2n(S^{2n-2})$ is an isomorphism. We define a map $\psi_1 : S^{2n-3} \land S^3 \to S^{2n-1}$ by

$$\psi_1((z_1, \ldots, z_{n-1}) \land (a, b))$$

$$= (|b|^2z_1 + |a|^2, |b|z_2, |b|z_3, bz_4, \ldots, bz_{n-1}, ab(1 - z_1)).$$
It is clear that \( h \circ \psi = h \circ \psi_1 \), hence \( \psi \simeq \psi_1 \). This process implies that two coordinates of \( \psi \) can be changed simultaneously from \( bz_i \) to \( |b|z_i \). Since \( n \) is even, we see that \( \psi \) is homotopic to the map \( \tilde{\psi} : S^{2n-3} \wedge S^{3} \to S^{2n-1} \) defined by
\[
\tilde{\psi}((z_1, \ldots, z_{n-1}) \wedge (a, b)) = (|b|^2z_1 + |a|^2, |b|z_2, \ldots, |b|z_{n-1}, ab(1 - z_1)).
\]
Finally, using (4.5), it is easy to construct a homotopy \( \tilde{\psi} \simeq \Sigma^{2n-3} \eta \). This completes the proof of Proposition 4.2, and hence also that of Theorem B(i).

For \( n \) even, let \( \sigma_{2n-3} \in H_{2n-3}(S^{2n-3}; \mathbb{Z}/2) \) be the generator, where \( S^{2n-3} \) is the fiber of (4.1). By (2.2), we have \( i_*(\sigma_{2n-3}) = \beta_{2n-3} \). Recall that we have a map \( J \circ i : S^{2n-3} \to \Omega_0^3 SU(n) \).

**Lemma 4.6.** \((J \circ i)_*(\sigma_{2n-3})\) is non-zero in \( H_{2n-3}(\Omega_0^3 SU(n); \mathbb{Z}/2) \).

**Proof.** Let \( W \) be the homotopy-theoretic fiber of the inclusion \( \Omega_0^3 SU(n) \hookrightarrow \Omega_0^3 SU \simeq BU \). Consider the following diagram:

\[
\begin{array}{ccc}
W & \longrightarrow & \Omega_0^3 SU(n) \\
\downarrow {\widetilde{J} \circ i} & & \downarrow {J \circ i} \\
S^{2n-3} & \longrightarrow & \Omega_0^3 SU \simeq BU
\end{array}
\]

where \( \widetilde{J} \circ i \) is the lift of \( J \circ i \). The first non-vanishing homotopy of \( W \) is \( \pi_{2n-3}(W) \cong \mathbb{Z} \) and \( \widetilde{J} \circ i \) is not divisible by 2 by Theorem B(i). For \( n \) even, \( H_{2n-3}(W; \mathbb{Z}/2) \to H_{2n-3}(\Omega_0^3 SU(n); \mathbb{Z}/2) \) is an isomorphism by (2.4). Hence \((J \circ i)_*(\sigma_{2n-3}) \neq 0 \).

**Proposition 4.7.** In \( H_{2n-3}(\Omega_0^3 SU(n); \mathbb{Z}/2) \),

\[
J_*(\beta_{2n-3}) = \begin{cases} 
Q_1(y_{4k}), & n = 4k + 2, \\
z_{8k+5}, & n = 4k + 4.
\end{cases}
\]

**Proof.** By Lemma 4.6, \( J_*(\beta_{2n-3}) \neq 0 \) in \( H_{2n-3}(\Omega_0^3 SU(n); \mathbb{Z}/2) \). By (2.4), \( H_{2n-3}(\Omega_0^3 SU(n); \mathbb{Z}/2) \cong \mathbb{Z}/2 \) and the generator is as on the right-hand side of the conclusion. Hence the result follows.

**Proof of Theorem 2.7.** If \( \beta_{2n-3} \) is defined in \( H_{2n-3}(SU(d)/CSU(d); \mathbb{Z}/2) \), then \( d \leq n \). Using (2.5) we can regard Proposition 4.7 as relations in \( H_{2n-3}(SU(d)/CSU(d); \mathbb{Z}/2) \) for each \( d \) for which \( \beta_{2n-3} \) is defined. For \( \beta_{2n-3} \) with \( n = 4k+2 \), the relation is up to the kernel of \( i_* : H_{2n-3}(\Omega_0^3 SU(d); \mathbb{Z}/2) \to H_{2n-3}(\Omega_0^3 SU(n); \mathbb{Z}/2) \), since \( Q_1(y_{4k}) \) is a fixed element. On the other hand, for \( \beta_{2n-3} \) with \( n = 4k + 4 \), we can define \( z_{8k+5} \) to be \( J_*(\beta_{2n-3}) \). Therefore (i) and (iii) of Theorem 2.7 hold.

Consider \( \beta_{2n-1} \in H_{2n-1}(SU(n)/C; \mathbb{Z}/2) \) for \( n \) even. Since \( \Delta_*(\beta_{2n-1}) \) contains the term \( \alpha_2 \otimes \beta_{2n-3} \) (where \( \Delta_* \) denotes the coproduct), Propo-
Then a similar diagram to (2.5) shows that (4.9) holds for all $n$ since
\[
J_*(\beta_{2n-1}) = \begin{cases} 
  x_2 \ast Q_1(y_{4k}), & n = 4k + 2, \\
  x_2 \ast z_{8k+5}, & n = 4k + 4.
\end{cases}
\]
Then using (2.5) we can regard (4.8) as relations in $\mathbb{Z}$ rem $A$ for
(4 of Theorem A for
rem 2.7 hold. This completes the proof of Theorem 2.7, and hence also that of Theorem A for $G = SU(n)$. ■

Proof of Theorem 2.8. If $G = Sp(1)$, then by the property of the classical $J$-homomorphism (see, for example, [3]), we have
\[
\begin{align*}
  J_*(\alpha_1) &= x_1, \\
  J_*(\alpha_2) &= y_2.
\end{align*}
\]
Then a similar diagram to (2.5) shows that (4.9) holds for all $n$.

Since $\Delta_*, J_*(\alpha_3)$ contains the term $x_1 \otimes y_2$, $J_*(\alpha_3)$ must contain the term $x_1 \ast y_2$. (Actually, by [3], we have $J_*(\alpha_3) = x_1 \ast y_2 + x_1^3 + Q_1(x_1).$) Similarly, since $\Delta_*, J_*(\alpha_4)$ contains the term $y_2 \otimes y_2$, we can define $z_4$ to be $J_*(\alpha_4)$.

In general, we can prove the following results.

(i) When $\mu = 4i$, $\Delta_*, J_*(\alpha_\mu)$ contains the term $J_*(\alpha_{2i}) \otimes J_*(\alpha_{2i})$. Hence we can define $z_{4i}$ to be $J_*(\alpha_{4i})$.

(ii) When $\mu = 4i + 1$, $4i + 2$ or $4i + 3$, $\Delta_*, J_*(\alpha_\mu)$ contains the term $x_1 \otimes z_{4i}$, $y_2 \otimes z_{4i}$ or $(x_1 \ast y_2) \otimes z_{4i}$. Hence $J_*(\alpha_\mu)$ must contain $x_1 \ast z_{4i}$, $y_2 \ast z_{4i}$ or $x_1 \ast y_2 \ast z_{4i}$.

This completes the proof of Theorem 2.8, and hence also that of Theorem A for $G = Sp(n)$. ■

5. Proof of Theorem 3.4. Let $\nu : S^{n-4} \to \text{Spin}(n)/C$ be the composition of the following maps:
\[
S^{n-4} = \text{Spin}(n-3)/\text{Spin}(n-4) \xrightarrow{i} \text{Spin}(n)/\text{Spin}(n-4) \xrightarrow{\pi} \text{Spin}(n)/C,
\]
where $i$ is the inclusion and $\pi$ is the projection. First we prove Theorem B(ii). We recall some notations and results about the Clifford algebra from [1].

Let $e_1, \ldots, e_n$ be a basis of $\mathbb{R}^n$ and $C_n$ the Clifford algebra. Thus $e_i^2 = -1$ ($1 \leq i \leq n$) and $e_i e_j + e_j e_i = 0$ ($i \neq j$). For $g = e_{i_1} \ldots e_{i_k} \in C_n$, we set $g^t = e_{i_k} \ldots e_{i_1}$.

(1) Let $\text{Spin}(n-4)$ be defined from $\mathbb{R}^{n-4}$ with a basis $e_1, \ldots, e_{n-4}$. Then in the principal bundle
\[
\text{Spin}(n-4) \to \text{Spin}(n-3) \xrightarrow{p} S^{n-4},
\]
the projection $p$ is given by $p(g) = ge_{n-3}g^t$.  

(2) Let \( e_i \ (n - 3 \leq i \leq n) \) be a basis of \( \mathbb{R}^4 \). Recall that \( \text{Spin}(4) \cong S^3_1 \times S^3_2 \) (see Section 3). We write \( S^3_1 \) in terms of the Clifford algebra as follows:

\[
S^3_1 = \left\{ \frac{x_1}{2} (1 - e_{n-3}e_{n-2}e_{n-1}e_n) + \frac{x_2}{2} (e_{n-3}e_n + e_{n-2}e_{n-1}) + \frac{x_3}{2} (-e_{n-3}e_{n-1} + e_{n-2}e_n) + \frac{x_4}{2} (e_{n-3}e_{n-2} + e_{n-1}e_n) + \frac{1}{2} (1 + e_{n-3}e_{n-2}e_{n-1}e_n) : \sum_{i=1}^{4} x_i^2 = 1 \right\}
\]

and

\[
S^3_2 = \left\{ \frac{x_1}{2} (1 + e_{n-3}e_{n-2}e_{n-1}e_n) + \frac{x_2}{2} (e_{n-3}e_n - e_{n-2}e_{n-1}) + \frac{x_3}{2} (e_{n-3}e_{n-1} + e_{n-2}e_n) + \frac{x_4}{2} (e_{n-3}e_{n-2} - e_{n-1}e_n) + \frac{1}{2} (1 - e_{n-3}e_{n-2}e_{n-1}e_n) : \sum_{i=1}^{4} x_i^2 = 1 \right\}
\]

As in Section 3 we embed \( \text{Spin}(3) \) into \( \text{Spin}(n) \) as \( S^3_1 \), hence \( C = C_{\text{Spin}(n)} = \text{Spin}(n-4) \times S^3_2 \).

We calculate \( \text{Ad}(J \circ \nu) : S^{n-4} \times S^3 \to \text{Spin}(n) \) explicitly. Let \( u \wedge x \in S^{n-4} \times S^3 \), where \( u = \sum_{i=1}^{n-3} u_i e_i \) with \( \sum_{i=1}^{n-3} u_i^2 = 1 \) and \( x \in S^3_1 \). We choose \( g \in \text{Spin}(n-3) \) so as to satisfy \( ge_n g^t = u \). (Compare (1).) Then \( \text{Ad}(J \circ \nu) (u \wedge x) = gxg^t x^t \). It is easy to see that

\[
\text{Ad}(J \circ \nu) (u \wedge x) = \frac{-1 + x_1}{2} u e_{n-3} + \frac{1 + x_1}{2} e_n
\]

\[
+ \frac{1}{2} (-x_4 e_{n-2} + x_3 e_{n-1} - x_2 e_n) (u - e_{n-3}).
\]

We set

\[
\text{Pin}^1(n) = \text{Pin}(n) - \text{Spin}(n).
\]

We multiply the formula for \( \text{Ad}(J \circ \nu) (u \wedge x) \) by \( e_{n-3} \) from the right, then we exchange the coefficients of \( e_{n-2} \) and \( e_n \). Thus we obtain a map \( \psi : S^{n-4} \times S^3 \to \text{Pin}^1(n) \) defined by

\[
(5.1) \quad \psi (u \wedge x) = \frac{1 - x_1}{2} u + \frac{1 + x_1}{2} e_{n-3}
\]

\[
+ \frac{1}{2} (x_2 e_{n-2} + x_3 e_{n-1} + x_4 e_n) (1 + u e_{n-3}).
\]

In order to prove Theorem B(ii), it suffices to prove that \( \psi \) represents a generator of \( \pi_{n-1}(\text{Pin}^1(n)) \).

Recall that for \( n \neq 3, 7 \), the characteristic map \( T : S^{n-1} \to SO(n) \) of the principal bundle \( SO(n) \to SO(n + 1) \to S^n \) represents a generator of \( \pi_{n-1}(SO(n)) \), the first non-stable homotopy group (see, for example, [10]).
In the Clifford algebra, the generator is given by the inclusion \( \iota : S^{n-1} \to \text{Pin}^1(n) \). In Lemma 4.4 for \( m = n - 4 \) and \( p = 3 \), we exchange \( u_1 \) and \( u_{n-3} \), and change the order of the coordinates of the right-hand side. Thus we obtain a map \( \zeta : S^{n-4} \land S^3 \to \text{Pin}^1(n) \) defined by

\[
(5.2) \quad \zeta(u \land x) = \frac{1 - x_1}{2} u + \frac{1 + x_1}{2} e_{n-3} + \sqrt{\frac{1 - u_{n-3}}{2}}(x_2 e_{n-2} + x_3 e_{n-1} + x_4 e_n).
\]

Now we have two maps, \( \psi \) in (5.1) and \( \zeta \) in (5.2). Making use of the homotopy \( \psi \) in (5.1) and \( \zeta \) in (5.2), we can construct a homotopy \( \psi \simeq \zeta \). This completes the proof of Theorem B(ii).

**Proposition 5.3.** There are choices of the generators \( y_{\mu} \) and \( z_{\mu} \) such that the following relations hold in \( H_{n-4}(\Omega^3_0 \text{Spin}(n); \mathbb{Z}/2) \):

(i) \( J_*(\gamma_{n-4}) = z_{n-4} \) for \( n \equiv 0, 2 \) (mod 4).

(ii) \( J_*(\beta_{n-4}) = y_{n-4} \) for \( n \equiv 1 \) (mod 4).

**Proof.** Let \( W \) be the homotopy-theoretic fiber of the inclusion \( \Omega^3_0 \text{Spin}(n) \hookrightarrow \Omega^3_0 \text{Spin} \simeq BSp \). Consider the following diagram:

\[
\begin{array}{ccc}
W & \longrightarrow & \Omega^3_0 \text{Spin}(n) \\
\downarrow J_{\circ \nu} & & \downarrow J_{\circ \nu} \\
S^{n-4} & \longrightarrow & \Omega^3_0 \text{Spin} \simeq BSp
\end{array}
\]

where \( J_{\circ \nu} \) is the lift of \( J \circ \nu \). The first non-vanishing homotopy of \( W \) is \( \pi_{n-4}(W) \) and \( J_{\circ \nu} \) is a generator by Theorem B(ii). If \( n \not\equiv 3 \) (mod 4), then for dimensional reasons, \( H_{n-4}(W; \mathbb{Z}/2) \to H_{n-4}(\Omega^3_0 \text{Spin}(n); \mathbb{Z}/2) \) is injective. Hence \( (J_{\circ \nu})_* (\sigma_{n-4}) \neq 0 \), where \( \sigma_{n-4} \) is a generator of \( H_{n-4}(S^{n-4}; \mathbb{Z}/2) \). By (3.2), we have

\[
\nu_*(\sigma_{n-4}) = \begin{cases} 
\gamma_{n-4}, & n \equiv 0, 2 \text{ (mod 4)}, \\
\beta_{n-4}, & n \equiv 1 \text{ (mod 4)}.
\end{cases}
\]

Then we can define \( y_{n-4} \) or \( z_{n-4} \) to be \( J_*(\gamma_{n-4}) \) or \( J_*(\beta_{n-4}) \). (More precisely, \( J_*(\gamma_{n-4}) \) for \( n \equiv 6 \) (mod 8) and \( J_*(\beta_{n-4}) \) for \( n \equiv 5 \) (mod 8) must be the right-hand side of (3.3), since \( H_{n-4}(\Omega^3_0 \text{Spin}(n); \mathbb{Z}/2) \cong \mathbb{Z}/2 \).)
Proof of Theorem 3.4. First, Theorem 3.4(i) follows from the homotopy commutative diagram

\[
\begin{array}{ccc}
\text{Spin}(n)/C_{\text{Spin}(n)} & \xrightarrow{J} & \Omega^3_0 \text{Spin}(n) \\
\downarrow i & & \downarrow j \\
\text{Spin}/C_{\text{Spin}} & \xrightarrow{J} & \Omega^3_0 \text{Spin} \\
\downarrow \approx & & \downarrow \approx \\
BSp(1) & \xrightarrow{k} & BSp
\end{array}
\]

(5.4)

where \(i, j\) and \(k\) are the inclusions. (The homotopy commutativity of the bottom square is proved in the same way as in the third diagram of [9, p. 4054] for \(k = 1\) and \(l = \infty\).)

Next, (ii) and (iii) of Theorem 3.4 are proved using Proposition 5.3 and a diagram similar to (2.5). (Compare the proof of Theorem 2.7 in Section 4.) This completes the proof of Theorem 3.4, and hence also that of Theorem A for \(G = \text{Spin}(n)\). 

**Remark.** Using Theorem 3.4, we can obtain information on \(J\_\ast(\alpha)\) for each generator \(\alpha\) of \(H\_\ast(\text{Spin}(n)/C; \mathbb{Z}/2)\). For example, consider an element of the form \(\alpha = (u_{i_1}^{\varepsilon_{i_1}} u_{i_2}^{\varepsilon_{i_2}} u_{i_3}^{\varepsilon_{i_3}})^\ast\), where \(\varepsilon_j = 0\) or \(1\). (For the generators \(u_{i_j}\), see Proposition 3.1, and for the notation \((\cdot)^\ast\), see (3.2).) We write \(\xi_{i_\mu}\) for \(y_{i_\mu}\) or \(z_{i_\mu}\). Then we can prove that \(J\_\ast(\alpha)\) contains the term \(\xi_{i_1}^{\varepsilon_{i_1}} \xi_{i_2}^{\varepsilon_{i_2}} \xi_{i_3}^{\varepsilon_{i_3}}\). For the proof, consider the coproduct \(\Delta\). 

**References**


Department of Mathematics
University of the Ryukyus
Okinawa 903-0213, Japan
E-mail: kamiyama@sci.u-ryukyu.ac.jp

Received 20 January 2003