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On finite groups acting on acyclic low-dimensional manifolds

by

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Abstract. We consider finite groups which admit a faithful, smooth action on an acyclic manifold of dimension three, four or five (e.g. Euclidean space). Our first main result states that a finite group acting on an acyclic 3- or 4-manifold is isomorphic to a subgroup of the orthogonal group O(3) or O(4), respectively. The analogous statement remains open in dimension five (where it is not true for arbitrary continuous actions, however). We prove that the only finite nonabelian simple groups admitting a smooth action on an acyclic 5-manifold are the alternating groups $\mathbb{A}_5$ and $\mathbb{A}_6$, and deduce from this a short list of finite groups, closely related to the finite subgroups of SO(5), which are the candidates for orientation-preserving actions on acyclic 5-manifolds.

1. Introduction. All finite group actions considered in the present paper will be faithful and smooth (or locally linear).

By the recent geometrization of finite group actions on 3-manifolds, every finite group action on the 3-sphere is conjugate to an orthogonal action; in particular, the finite groups which occur are exactly the well-known finite subgroups of the orthogonal group O(4). Finite groups acting on arbitrary homology 3-spheres are considered in [MeZ1, MeZ2] and [Z]; here some other finite groups occur and the situation is not completely understood yet (see [Z] and Section 7).

In dimension four, it is no longer true that a finite group action on the 4-sphere is conjugate to an orthogonal action (e.g. the Smith conjecture does not remain true for the 4-sphere, that is, the fixed point set of a periodic diffeomorphism of S^4 may be a knotted 2-sphere). However it has been shown in [MeZ3, MeZ4] that a finite group which admits an orientation-preserving action on the 4-sphere, and more generally on any homology 4-sphere, is isomorphic to a subgroup of the orthogonal group SO(5) (up to 2-fold extensions in the case of solvable groups).

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In the present paper we consider finite groups acting on acyclic (compact or non-compact) low-dimensional manifolds, i.e. manifolds with trivial reduced integer homology (e.g. Euclidean spaces). Our first main result is the following.

THEOREM 1. *A finite group which admits a faithful, smooth action on an acyclic* 3- *or* 4-*manifold is isomorphic to a subgroup of* O(3) *or* O(4), *respectively, and to a subgroup of* SO(3) *or* SO(4) *if the action is orientation-preserving. In particular, the only finite nonabelian simple group admitting such an action is the alternating group* $\mathbb{A}_5$.

See [DV] for the finite subgroups of O(3) and O(4). Theorem 1 answers [E, Problem 11] on finite groups acting on Euclidean space $\mathbb{R}^4$, for the case of smooth actions.

In the situation of Theorem 1, whereas each solvable group admitting an action has a global fixed point, this does not remain true in general for nonsolvable groups. As a classical example, the Poincaré homology 3-sphere admits an action of $\mathbb{A}_5$ with a single fixed point, and the complement of this fixed point is an acyclic 3-manifold with a fixed point free $\mathbb{A}_5$-action. In dimension five, we do not know whether there exists a fixed point even for solvable groups. On the other hand, for nonsolvable groups, one of the main technical problems in the proof of such classification results is to get hold of the finite simple groups which may occur; in dimension five, the following is true.

THEOREM 2. *The only finite nonabelian simple groups admitting a faithful, smooth action on an acyclic* 5-*manifold are the alternating groups* $\mathbb{A}_5$ *and* $\mathbb{A}_6$ (*for actions of quasisimple groups there occurs, in addition, the binary dodecahedral group* $\mathbb{A}_5^*$).

So these are exactly the simple (or quasisimple) subgroups of the orthogonal group SO(5); we recall that a *quasisimple group* is a perfect, central extension of a simple group. From Theorem 2 we deduce the following result for arbitrary finite groups acting on acyclic 5-manifolds.

THEOREM 3. *Let G be a finite group admitting a smooth, faithful, orientation-preserving action on an acyclic* 5-*manifold. Then one of the following cases occurs.*

(i) *G is a subgroup of* SO(5);
(ii) *G contains a normal subgroup N which is cyclic or a central product of a cyclic group with $\mathbb{A}_5$, $\mathbb{A}_5^*$ or $\mathbb{A}_6$, and the factor group G/N is an elementary abelian* 2-*group of rank at most four.*

See [MeZ4, Corollary 2] for a characterization of the finite subgroups of group SO(5). At present, we do not know an example of a group which admits a faithful, smooth, orientation-preserving action on an acyclic 5-manifold but is not isomorphic to a subgroup of SO(5); however for continuous actions such examples in fact do exist. Specifically, among the groups G described in Theorem 3(ii) there are the Milnor groups $Q(8a,b,c)$ ([Mn]) which are extensions of a cyclic group by the elementary abelian 2-group $(\mathbb{Z}_2)^2$. Some of the Milnor groups admit a faithful, continuous, orientation-preserving action on $\mathbb{R}^5$ but none of them is isomorphic to a subgroup of SO(5); see Section 7.

The paper is organized as follows. Section 2 contains some preliminary results about finite group actions on acyclic manifolds. In Section 3 we present the proof of Theorem 1 in the 3-dimensional case which is much shorter than that in four dimensions (the main ingredient in dimension three is the Gorenstein–Walter classification of the finite simple groups with dihedral Sylow 2-subgroups, while in dimensions four and five this is replaced by the more involved Gorenstein–Harada classification of the finite simple groups of sectional 2-rank at most four); so this gives the reader a shortcut to the basic methods of the present paper, without the technical problems on finite simple groups arising in dimensions four and five. In Section 4 we prove Theorem 2 concerning simple groups acting on acyclic 5-manifolds; this also implies the analogue for simple groups acting on acyclic 4-manifolds, needed for the proof of the 4-dimensional case of Theorem 1 given in Section 5 (a direct proof in dimension four would be somewhat shorter since the case of quasisimple groups can be avoided; on the other hand, the main technical difficulties related to the Gorestein–Harada list remain the same, so we prefer to give the proof only in dimension five). In Section 6 we prove Theorem 3, and in the last section we discuss continuous actions of the Milnor groups on acyclic 5-manifolds.

2. Preliminary results. If G is a finite group acting on an n-manifold, the fixed point set of G is a submanifold and by Newman's Theorem has dimension strictly smaller than n (see [B, Chapter III]). If G has non-empty global fixed point set, the group G leaves invariant a tubular neighborhood of the fixed point set (see [B, Chapter VI]). Hence, once we have found that G fixes pointwise a submanifold of dimension d, we automatically deduce that G is a subgroup of $\mathrm{O}(n-d)$, and of $\mathrm{SO}(n-d)$ if the action is orientation-preserving.

Suppose that G is a p-group acting smoothly on a $\mathbb{Z}_p$-acyclic n-manifold (a manifold with trivial homology with coefficients in $\mathbb{Z}_p$, the integers mod p). By Smith theory (see [B, Chapter III, Section 5]) the fixed point set of G is again a $\mathbb{Z}_p$-acyclic manifold (of even codimension if p is odd) and, in particular, it is nonempty; thus, G is a subgroup of $\mathrm{O}(n)$.

By the above discussion we can state the following:

Lemma 1. *A finite p-group acting on an acyclic n-manifold is isomorphic to a subgroup of* $\mathrm{O}(n)$, *and to a subgroup of* $\mathrm{SO}(n)$ *if the action is orientation-preserving.*

We note that every action of a finite group G on an acyclic 1- or 2-manifold has a global fixed point, so that, in particular, an analogue of Theorem 1 also holds in the 1- and 2-dimensional case. The 1-dimensional case is obvious: an acyclic 1-manifold (even $\mathbb{Z}_p$-acyclic) is diffeomorphic to $\mathbb{R}$, $[0,\infty)$ or $[0,1]$, hence a finite group acting on it is either trivial or generated by an orientation-reversing involution with one fixed point. The 2-dimensional case is considered in the following lemma.

Lemma 2. *Any faithful and smooth action of a finite group G on an acyclic 2-manifold X admits a global fixed point, and G is therefore either dihedral with an orientation-reversing involution, or cyclic.*

Proof. Suppose first that G is a nonabelian simple group; then the action is orientation-preserving (since otherwise G would have a subgroup of index two). Let S be a Sylow 2-subgroup of G. By Lemma 1, S is a finite subgroup of $\mathrm{SO}(2)$, so it is cyclic; this is a contradiction since a simple group cannot have a cyclic Sylow 2-subgroup (see [Su2, Corollary 2 of Theorem 2.2.10, p. 144]).

Hence we can suppose that G admits no nonabelian simple subgroups and, if N is a minimal nontrivial normal subgroup in G, it is an elementary abelian p-group by [Su1, Corollary 3 of (2.4.14), p. 137]. Let X^N be the submanifold of points fixed by N; it has dimension at most 1, is invariant under the action of G and $\mathbb{Z}_p$-acyclic. So either X^N is a point and we are done, or X^N is an acyclic 1-manifold. In the latter case, let T be the normal subgroup of G fixing all the points in X^N. Then the factor group G/T acts faithfully on X^N, with at least one global fixed point as noted above, and also G fixes that point. Hence G is a finite subgroup of $\mathrm{O}(2)$, and of $\mathrm{SO}(2)$ if the action is orientation-preserving.

This concludes the proof of Lemma 1.

Note that every orientable $\mathbb{Z}_p$-acyclic 2-manifold is acyclic. This can easily be seen, using simplicial homology and considering a cycle α that is not an integral boundary (note that α can always be chosen to have a connected and simple geometric realization), but that is a boundary modulo p of a 2-chain β and reaching the contradiction that α is an integral boundary. The orientability hypothesis cannot be omitted, as it is needed to induce a coherent orientation on all 2-simplices appearing with nonzero coefficients in β. The projective plane is an example of a nonorientable $\mathbb{Z}_p$-acyclic 2-manifold (with p odd) that is not acyclic.

This remark, together with Lemma 2, leads us directly to a crucial lemma.

Lemma 3. *Let G be a finite group acting on an acyclic n-manifold X, with a nontrivial normal p-subgroup N. Suppose that one of the following conditions holds:*

(i) *the submanifold of points fixed by N has dimension $d \leq 2$;*
(ii) $n = 3$*;*
(iii) $n = 4$ *and the action of G is orientation-preserving;*
(iv) $n = 5$*, the action of G is orientation-preserving and N is not cyclic.*

Then G has at least one global fixed point. Hence G acts orthogonally on the boundary of some regular neighborhood of the fixed point and it is isomorphic to a subgroup of $\mathrm{O}(n)$ (*and of* $\mathrm{SO}(n)$ *if the action is orientation-preserving*).

Proof. Suppose that condition (i) holds. If $d \leq 1$, we can proceed as in the proof of Lemma 2. Suppose therefore that N fixes pointwise X^N, a $\mathbb{Z}_p$-acyclic 2-manifold. Then X^N is also orientable. In fact, if X^N is a $\mathbb{Z}_2$-acyclic manifold, it is orientable since otherwise the first homology $H_1(X^N, \mathbb{Z})$ would surject onto $\mathbb{Z}_2$; if p is odd instead, the order of N is odd and X^N is orientable by [B, Chapter IV, Theorem 2.1, p. 175]. Therefore, X^N is an acyclic 2-manifold.

Now, as in the proof of Lemma 2, we consider the normal subgroup T of G fixing each point of X^N. Then G/T acts on X^N and has a global fixed point, hence also G has a global fixed point. This concludes the proof in the first case.

Note that conditions (ii) and (iii) each easily imply (i).

If $n = 5$ and the action of G is orientation-preserving, the fixed point set of N may also have dimension three. In this case, by the discussion at the beginning of this section, the subgroup N is isomorphic to a finite subgroup of SO(2), hence it is cyclic. This concludes the proof of Lemma 3.

3. Proof of Theorem 1 for acyclic 3-manifolds. Let G be a finite group with a smooth and faithful action on an acyclic 3-manifold.

Proposition 1. *Suppose that G is a nonabelian simple group acting on an acyclic* 3*-manifold. Then G is isomorphic to the alternating group $\mathbb{A}_5$.*

Proof. The action of G is orientation-preserving as otherwise it would have a subgroup of index two. By Lemma 1 a Sylow 2-subgroup of G is isomorphic to a subgroup of SO(3) and hence is dihedral (since a Sylow 2-subgroup of a nonabelian simple group cannot be cyclic by [Su2, Corollary 2 of Theorem 2.2.10, p. 144]). By the Gorenstein–Walter characterization of the finite simple groups with dihedral Sylow 2-subgroups ([G1, Theorem 1.4.7], [Su2, Theorem 6.8.6, p. 505]), G is isomorphic to a linear fractional group $\mathrm{PSL}(2,q)$ for an odd prime power $q = p^n$, or to the alternating group $\mathbb{A}_7$.

Suppose first that G is $\mathrm{PSL}(2,q)$ for an odd prime power $q = p^n$. If $n > 1$ then $\mathrm{PSL}(2,q)$ has a noncyclic elementary abelian p-subgroup $(\mathbb{Z}_p)^n$ (represented by the upper triangular matrices with both diagonal entries equal to one, isomorphic to the additive group of the finite field with q elements). By Lemma 1 the group $(\mathbb{Z}_p)^n$ does not admit an action on an acyclic 3-manifold. Hence $n = 1$; now $\mathrm{PSL}(2,p)$ has a semidirect product $\mathbb{Z}_p \rtimes \mathbb{Z}_{(p-1)/2}$ as a subgroup, with an effective action of $\mathbb{Z}_{(p-1)/2}$ (represented by diagonal matrices) on the normal subgroup $\mathbb{Z}_p$. By Lemma 3, this is possible only for $p = 5$, so we are left with the group $\mathrm{PSL}(2,5)$ isomorphic to $\mathbb{A}_5$.

Finally, $\mathbb{A}_7$ has a subgroup $\mathbb{Z}_3 \times \mathbb{Z}_3$ which is again excluded by Lemma 1.

This completes the proof of Proposition 1.

Recall that a *quasisimple* group is a perfect central extension of a simple group, i.e. it is perfect and the factor group by its center is a nonabelian simple group. A *semisimple* group is a central product of quasisimple groups, i.e. the factor group by its center is a direct product of nonabelian simple groups (see [Su2, Chapter 6.6]). Any finite group G contains a unique maximal semisimple normal group $E(G)$ (which may be trivial); the subgroup $E(G)$ is characteristic in G, and the quasisimple factors of $E(G)$ are called the *components* of G. The *generalized Fitting subgroup* $F^*(G)$ of G is defined as the (central) product of $E(G)$ and the Fitting subgroup $F(G)$ (the maximal nilpotent normal subgroup of G) which is characteristic in G. The generalized Fitting subgroup of a nontrivial group is never trivial and its centralizer in G coincides with its center, i.e. $C_G(F^*(G)) = Z(F^*(G)) = Z(F(G))$ (see [Su2, Chapter 6.6, p. 452]).

Proof of Theorem 1 for 3-dimensional manifolds. We divide the proof into two subcases.

If the Fitting subgroup $F(G)$ is not trivial then G has a nontrivial normal p-subgroup, and by Lemma 3 the action of G has a global fixed point.

If $F(G)$ is trivial, $E(G)$ coincides with the generalized Fitting subgroup $F^*(G)$, hence $C_G(E(G)) = Z(E(G)) = Z(F(G))$ is trivial. Therefore, G acts faithfully by conjugation on its normal subgroup $E(G)$, so G is a subgroup of $\mathrm{Aut}(E(G))$, up to isomorphism. Also, by the definition of a semisimple group, $E(G) \cong E(G)/Z(E(G))$ is a direct product of simple groups acting on an acyclic 3-manifold. Therefore, by Proposition 1, $E(G) \cong (\mathbb{A}_5)^k$.

Suppose that $k \geq 2$. Then $E(G)$ would contain an elementary abelian 5-subgroup of rank 2, which is not a subgroup of $\mathrm{O}(3)$; a contradiction. Therefore $E(G)$ is isomorphic to $\mathbb{A}_5$ and G is a subgroup of $\mathrm{Aut}(\mathbb{A}_5) \cong \mathbb{S}_5$, hence either $G \cong \mathbb{S}_5$ or $G \cong \mathbb{A}_5$. Suppose that $G \cong \mathbb{S}_5$. Then the subgroup generated by $(4532), (12345)$ is a semidirect product $\mathbb{Z}_5 \rtimes \mathbb{Z}_4$, which is not a subgroup of $\mathrm{O}(3)$. Thus, we have a contradiction by Lemma 3.

We conclude that every finite group with trivial Fitting subgroup acting smoothly on an acyclic 3-manifold is isomorphic to $\mathbb{A}_5$, a subgroup of $\mathrm{SO}(3)$, and this completes the proof of Theorem 1 for acyclic 3-manifolds.

4. Proof of Theorem 2. To prove Theorem 2 we need some preliminary results.

LEMMA 4. *A finite group G which admits a faithful, orientation-preserving action on an acyclic 5-manifold has sectional 2-rank at most four.*

Proof. By Lemma 1, every 2-subgroup of G is in particular a subgroup of $\mathrm{SO}(5)$ and it is therefore generated by at most four elements (e.g. by [MeZ3, Proposition 3.1.]).

LEMMA 5. *For a prime p and a positive integer r, let $\mathbb{Z}_p \rtimes \mathbb{Z}_r$ be a metacyclic group (semidirect product), with normal subgroup $\mathbb{Z}_p$ and factor group $\mathbb{Z}_r$, which admits a faithful and orientation-preserving action on an acyclic 5-manifold. Then, by conjugation, the square of each element of $\mathbb{Z}_r$ acts trivially or dihedrally on $\mathbb{Z}_p$.*

Proof. If the fixed point set of $\mathbb{Z}_p$ is a 3-manifold, then $\mathbb{Z}_p$ locally acts as a group of rotations around it and an element in $\mathbb{Z}_r$ conjugates a rotation of minimal angle to a rotation of minimal angle, so it acts trivially or dihedrally on $\mathbb{Z}_p$. If instead $\mathbb{Z}_p$ fixes pointwise a submanifold of dimension at most 2, then, by Lemma 2, the group $\mathbb{Z}_p \rtimes \mathbb{Z}_r$ is a subgroup of $\mathrm{SO}(5)$ and the claim follows from [MeZ3, Lemma 2.2].

We also state the following algebraic lemma that will frequently be used in the proof of Theorem 2 and that is a simple consequence of [Su1, Theorem 2.9.18, p. 257].

LEMMA 6. *Let S be a simple group. If H is a simple subgroup of S then any central perfect extension of S contains a central perfect extension of H.*

Proof of Theorem 2. Let G be a finite nonabelian quasisimple group acting on an acyclic 5-manifold. Since G is perfect and has no subgroup of index two, the action of G is orientation-preserving. By Lemma 4, the Sylow 2-subgroup S has sectional 2-rank at most four. By the Gorenstein–Harada classification of the simple groups of sectional 2-rank at most four ([G1, p. 6], [Su2, Theorem 6.8.12, p. 513]), the factor of G by its center is one of the groups in the following list (q denotes an odd prime power):

$$\begin{gathered}\mathrm{PSL}(m,q), \quad \mathrm{PSU}(m,q), \quad m \leq 5,\\ \mathrm{G}_2(q), \quad {}^3\mathrm{D}_4(q), \quad \mathrm{PSp}(4,q), \quad {}^2\mathrm{G}_2(3^{2m+1}) \quad (m \geq 1),\\ \mathrm{PSL}(2,8), \quad \mathrm{PSL}(2,16), \quad \mathrm{PSL}(3,4), \quad \mathrm{PSU}(3,4), \quad \mathrm{Sz}(8),\\ \mathbb{A}_m \quad (7 \leq m \leq 11), \quad \mathrm{M}_i \quad (i \leq 23), \quad \mathrm{J}_i \quad (i \leq 3), \quad \mathrm{McL}, \quad \mathrm{Ly}.\end{gathered}$$

In the following, we will exclude all the central perfect extensions of these groups except $\mathbb{A}_5$, $\mathbb{A}_5^*$, and $\mathbb{A}_6$.

We suppose first that G is isomorphic to $\mathrm{SL}(2,p)$ or to $\mathrm{PSL}(2,p)$, for a prime $p \geq 5$. The group $\mathrm{SL}(2,p)$ has a metacyclic subgroup $\mathbb{Z}_p \rtimes \mathbb{Z}_{p-1}$ (represented by all upper triangular matrices): the normal subgroup $\mathbb{Z}_p$ consists of the matrices having both entries on the diagonal equal to one and the subgroup $\mathbb{Z}_{p-1}$ consists of the diagonal matrices. The projection of $\mathbb{Z}_p \rtimes \mathbb{Z}_{p-1}$ to $\mathrm{PSL}(2,p)$ is a metacyclic subgroup $\mathbb{Z}_p \rtimes \mathbb{Z}_{(p-1)/2}$, and the action of $\mathbb{Z}_{(p-1)/2}$ on the normal subgroup $\mathbb{Z}_p$ is effective. By Lemma 5 we conclude that $p = 5$ and G can be isomorphic to $\mathrm{PSL}(2,5) \cong \mathbb{A}_5$ or to $\mathrm{SL}(2,5) \cong \mathbb{A}_5^*$.

Next we consider G isomorphic to $\mathrm{PSL}(2,q)$ or to $\mathrm{SL}(2,q)$ for $q = p^n$ with p an odd prime and $n > 1$. In $\mathrm{SL}(2,q)$ the subgroup of upper triangular matrices is a semidirect product $(\mathbb{Z}_p)^n \rtimes \mathbb{Z}_{q-1}$. The projection of the subgroup to $\mathrm{PSL}(2,q)$ is a semidirect product $(\mathbb{Z}_p)^n \rtimes \mathbb{Z}_{(q-1)/2}$, and the action of $\mathbb{Z}_{(q-1)/2}$ on the normal subgroup is effective. In any case we can suppose that the group contains a subgroup isomorphic to $(\mathbb{Z}_p)^n \rtimes \mathbb{Z}_r$ where r depends on the group we consider. The fixed point set of the subgroup $(\mathbb{Z}_p)^n$ is an acyclic 1-manifold M; in fact, it cannot be a $\mathbb{Z}_p$-acyclic 3-manifold since $(\mathbb{Z}_p)^n$ would act faithfully on an orthogonal 1-sphere around M. Now $(\mathbb{Z}_p)^n \rtimes \mathbb{Z}_r$ has a global fixed point, acts faithfully on a 3-sphere around M which is the boundary of a 4-disk orthogonal to M at a fixed point and is a subgroup of $\mathrm{O}(4)$. This is possible only for $n = 2$ and $p = 3$ (e.g. $r = 2^k$ with $k \leq 3$ by [MeZ1, Proposition 3 and 4]). The group $\mathrm{SL}(2,9)$ can be excluded since it contains a subgroup $(\mathbb{Z}_3)^2 \rtimes \mathbb{Z}_4$ acting faithfully and orientation-preservingly on a 3-sphere around M but the elements of order four in $\mathbb{Z}_4$ act dihedrally on the subgroup $(\mathbb{Z}_3)^2$ and this is impossible (e.g. by [MeZ1, Proposition 3]). In this case G is isomorphic to $\mathrm{PSL}(2,9) \cong \mathbb{A}_6$.

In general the unique central perfect extension of $\mathrm{PSL}(2,q)$ is $\mathrm{SL}(2,q)$; the only exception is $\mathrm{PSL}(2,9)$ that has two other central extensions, one with center of order three and the other with center of order six (see [Co, Table 5]). By [Co] we deduce that neither of these extensions contains an any element of order nine. In both cases the Sylow 3-subgroup of order 27 contains a normal subgroup isomorphic to $(\mathbb{Z}_3)^2$ and by the same argument used above, we can conclude that these groups cannot act on an acyclic 5-manifold.

Now we consider $\mathrm{PSL}(m,q)$, $\mathrm{SL}(m,q)$, $\mathrm{PSU}(m,q)$, $\mathrm{SU}(m,q)$ with q odd and $3 \le m \le 5$ (for $m = 2$ we have $\mathrm{PSL}(2,q) \cong \mathrm{PSU}(2,q)$). The groups $\mathrm{PSL}(m,q)$ and $\mathrm{SL}(m,q)$ contain a subgroup isomorphic to $\mathrm{SL}(m-1,q)$, and the groups $\mathrm{PSU}(m,q)$ and $\mathrm{SU}(m,q)$ contain a subgroup isomorphic to $\mathrm{SU}(m-1,q)$. Using these subgroups we can eliminate inductively most of the groups. This process ends up with a case by case analysis of the simple groups with $m = 3$ and $q = 3, 5$ (these groups do not admit any central perfect extension with nontrivial center). To eliminate these groups we use [Co] to find subgroups (metacyclic or simple) that we have already excluded.

Also in most cases the unique central extension of $\mathrm{PSL}(m,q)$ (resp. $\mathrm{PSU}(m,q)$) is $\mathrm{SL}(m,q)$ (resp. $\mathrm{SU}(m,q)$). We can have an intermediate extension between $\mathrm{PSL}(4,q)$ and $\mathrm{SL}(4,q)$ and between $\mathrm{PSU}(4,q)$ and $\mathrm{SU}(4,q)$ but the above argument works again. The only simple group of this type that admits further central extensions is $\mathrm{PSU}(4,3)$ (see [Co, Table 5]); in this case Lemma 6 and the inclusion $\mathrm{PSL}(2,7) \subset \mathbb{A}_7 \subset \mathrm{PSU}(4,3)$ exclude directly all the central perfect extensions.

The group $\mathrm{PSp}(4,q)$ contains a subgroup isomorphic to $\mathrm{PSL}(2,q)$. This inclusion excludes automatically most of the simple groups and their central perfect extensions (by Lemma 6); only a few groups have to be checked case by case. The group $\mathrm{PSp}(4,3)$ and its central extension contain a subgroup isomorphic to $(\mathbb{Z}_3)^3$ that can be excluded by the same argument used for $\mathrm{PSL}(2,p^n)$ with $n \ge 3$. The group $\mathrm{PSp}(4,5)$ contains $\mathrm{PSL}(2,25)$ (see [Co]), while $\mathrm{PSp}(4,9)$ is excluded as it has a subgroup isomorphic to $\mathrm{PSp}(4,3)$.

Up to central extension we have ${}^3\mathrm{D}_4(q) \supset \mathrm{G}_2(q) \supset \mathrm{PSL}(3,q)$ (see [St, Table 0A8], [GL, Table 4-1]). These inclusions and Lemma 6 exclude ${}^3\mathrm{D}_4(q)$, $\mathrm{G}_2(q)$ and their central extensions.

The Ree groups ${}^2\mathrm{G}_2(3^{2m+1})$ have one conjugacy class of involutions, the centralizer of an involution is $\mathbb{Z}_2^2 \times \mathrm{PSL}(2,3^{2m+1})$ ([G2, p. 164]), so for $m \ge 1$ they do not act (the group ${}^2\mathrm{G}_2(3)$ is not simple).

We now consider the simple groups of Lie type and even characteristic. The group $\mathrm{PSL}(2,2^n)$ with $n = 3,4$ contains a semidirect product $(\mathbb{Z}_2)^n \ltimes \mathbb{Z}_r$. If $\mathrm{PSL}(2,2^n)$ acted on an acyclic 5-manifold, by Lemma 3 the semidirect product $(\mathbb{Z}_2)^n \ltimes \mathbb{Z}_r$ would be a subgroup of $\mathrm{SO}(5)$, and this is impossible as $\mathbb{Z}_r$ acts transitively by conjugation on $(\mathbb{Z}_2)^n$ (see [MeZ4, Lemma 1]).

The group $\mathrm{PSU}(3,4)$ contains a subgroup isomorphic to $\mathbb{Z}_{13} \rtimes \mathbb{Z}_3$ (see [Co]) and this group cannot act by Lemma 5. The group $\mathrm{PSU}(3,4)$ does not admit any central extension with nontrivial center.

The group $\mathrm{PSL}(3,4)$ contains a subgroup isomorphic to $\mathrm{PSL}(2,7)$ that we have already excluded. To eliminate the central extensions we apply Lemma 6.

Concerning the Suzuki group Sz(8), its Sylow 2-subgroup has order 64 with a normal subgroup $(\mathbb{Z}_2)^3$ and it has a unique conjugacy class of involutions (see [Co]). If Sz(8) admitted an action, the subgroup $(\mathbb{Z}_2)^3$ would have a global fixed point and would act on the 4-sphere that is the boundary of some regular neighborhood of the fixed point. This is impossible as the involutions in $(\mathbb{Z}_2)^3$ are all conjugate (see [MeZ4, Lemma 1]). Suppose now that G is a central perfect extension of Sz(8); the center of G is an elementary abelian 2-group of rank one or two. In any case the center fixes pointwise a $\mathbb{Z}_2$-acyclic manifold M of dimension at most three. Since Sz(8) is simple, the normal subgroup of the elements of G leaving invariant each point of M coincides with the center of G. Hence, the quotient of G by its center, and thus $(\mathbb{Z}_2)^3$, acts faithfully and orientation-preservingly on M, which is impossible by Lemma 1.

The alternating group $\mathbb{A}_7$ has a subgroup $\mathrm{PSL}(2,7)$, which excludes all alternating groups $\mathbb{A}_n$ for $n \geq 7$. For any of the remaining simple groups it is possible to find a simple subgroup already excluded. We will not give further details and refer to [Co] and its references for the maximal subgroups.

This concludes the proof of Theorem 2.

5. Proof of Theorem 1 for acyclic 4-manifolds. In dimension 4, Lemma 3 still applies to groups with nontrivial Fitting subgroup if we suppose that the action is orientation-preserving. Hence, let us first suppose that G is a finite group admitting a smooth, faithful and orientation-preserving action on an acyclic 4-manifold. We will then extend our results to any smooth and faithful action.

Proposition 2. *A finite group G which admits a smooth and orientation-preserving action on an acyclic 4-manifold X is isomorphic to a subgroup of* $\mathrm{SO}(4)$. *In particular if $F(G)$ is trivial, then G is isomorphic to the alternating group $\mathbb{A}_5$.*

Proof. Suppose first that G is simple and nonabelian. Then G acts also on $X \times \mathbb{R}$, an acyclic 5-manifold, and, by Theorem 2, G is isomorphic to either $\mathbb{A}_5$ or $\mathbb{A}_6$. But $\mathbb{A}_6 \cong \mathrm{PSL}(2,9)$ contains a solvable subgroup of the form $(\mathbb{Z}_3)^2 \rtimes \mathbb{Z}_4$, which is not a subgroup of $\mathrm{SO}(4)$ (it is shown in [MeZ1, proof of Theorem 2] that it does not even act on a homology 3-sphere); a contradiction, by Lemma 3. Hence, $\mathbb{A}_5$ is the only nonabelian simple group which can act on an acyclic 4-manifold.

Suppose now that the Fitting subgroup of G is trivial. Then $E(G)$ coincides with the generalized Fitting group $F^*(G)$, and hence $C_G(E(G)) = Z(E(G)) = Z(F^*(G))$. Therefore G is isomorphic to a subgroup of $\mathrm{Aut}(E(G))$, and $E(G) = E(G)/Z(E(G))$ is a (nontrivial) direct sum of simple nonabelian groups, acting on an acyclic 4-manifold; thus $E(G) \cong (\mathbb{A}_5)^k$. For $k \geq 2$,

$(\mathbb{A}_5)^k$ contains an elementary abelian 2-subgroup of rank at least 4, which is not a subgroup of SO(4) (e.g. by [MeZ1, Proposition 3]). Hence, $\mathbb{A}_5 \cong E(G) \subset G \subset \mathrm{Aut}(E(G)) \cong \mathbb{S}_5$ and we deduce that either $G \cong \mathbb{A}_5$ or $G \cong \mathbb{S}_5$. Finally, $\mathbb{S}_5$ contains a subgroup isomorphic to $\mathbb{Z}_5 \rtimes \mathbb{Z}_4$ (with faithful action of $\mathbb{Z}_4$ on $\mathbb{Z}_5$) which is not a subgroup of SO(4) (by [Z, Proposition 3] it does not even act on a homology 3-sphere). We conclude that $G \cong \mathbb{A}_5$.

If $F(G)$ is not trivial we can apply Lemma 3 and the conclusion follows. This concludes the proof of Proposition 2.

Proof of Theorem 1 for acyclic 4-manifolds. Let G be a group acting on an acyclic 4-manifold X. We divide the proof into two subcases.

Suppose that the Fitting subgroup $F(G)$ is not trivial. If G contains a nontrivial normal p-subgroup P such that the submanifold of points fixed by P has dimension at most 2, then Lemma 3 applies and the claim is proven. Otherwise, $F(G)$ being nontrivial, G admits a normal p-subgroup P which fixes pointwise a 3-submanifold and is therefore generated by an orientation-reversing involution t.

Let G_0 be the index two subgroup of orientation-preserving elements in G. As t is an orientation-reversing element, $G_0 \cap P = 1$ and both P and G_0 are normal in G. We find that $G = G_0 \times P \cong G_0 \times \mathbb{Z}_2$. Note that $F(G_0)$ is trivial, otherwise G_0 (and hence $G \cong G_0 \times P$) would admit a normal p-subgroup which, acting orientation-preservingly on X, fixes a submanifold of dimension at most 2; a contradiction. Therefore G_0 is a group acting orientation-preservingly on an acyclic 4-manifold with trivial Fitting subgroup.

If G_0 is trivial, then $G \cong \mathbb{Z}_2$; otherwise $G_0 \cong \mathbb{A}_5$, by Proposition 2. In the latter case, $G \cong \mathbb{A}_5 \times \mathbb{Z}_2$, which is a subgroup of O(3) and hence of O(4).

If instead $F(G)$ is trivial, $E(G) = F^*(G)$ is not trivial. Then $E(G)$, being semisimple, is in particular perfect and hence its action on X is orientation-preserving; also, $F(E(G))$ is trivial, thus $E(G) \cong \mathbb{A}_5$, by Proposition 2. Since the centralizer $C_G(E(G)) = C_G(F^*(G)) = Z(F^*(G)) = Z(F(G))$ is trivial, G is isomorphic to a subgroup of $\mathrm{Aut}(E(G)) \cong \mathbb{S}_5$ containing $\mathbb{A}_5$. Therefore either $G \cong \mathbb{A}_5$ or $G = \mathbb{S}_5$, both subgroups of O(4).

For an orthogonal action of $\mathbb{S}_5$ on $\mathbb{R}^4$, just consider the action of $\mathbb{S}_5$ on $\mathbb{R}^5$ permuting the standard orthonormal base and restrict the action to the hyperplane described by the equation $x_1 + \cdots + x_5 = 0$, which is invariant under the action of $\mathbb{S}_5$ on $\mathbb{R}^5$.

This concludes the proof of Theorem 1.

6. Proof of Theorem 3. Suppose that a group G acts preserving the orientation on an acyclic 5-manifold and that it has nontrivial Fitting subgroup $F(G)$. By Lemma 3 either G has a global fixed point and is a subgroup of $\mathrm{SO}(5)$, or $F(G)$ is a direct product of cyclic groups of coprime orders fixing 3-manifolds. In the latter case, in particular, $F(G)$ is cyclic.

This implies that if an abelian (or even nilpotent) group acts orientation-preservingly and with no global fixed point on an acyclic 5-manifold, it is cyclic. As for cyclic groups which are not p-groups, it is still an open question whether or not they can act with no global fixed points, depending on how many primes divide their order (see [HKMS]). Note that if the acyclic manifold is homeomorphic to a closed disk, this is not possible, by Brouwer's Theorem.

PROPOSITION 3. *Suppose that G is a* (*nontrivial*) *semisimple group acting on an acyclic* 5*-manifold. Then G is isomorphic to one of the following groups:*

$$\mathbb{A}_5, \quad \mathbb{A}_6, \quad \mathbb{A}_5^*, \quad \mathbb{A}_5^* \times_{\mathbb{Z}_2} \mathbb{A}_5^*.$$

In particular, a semisimple group can act on an acyclic 5*-manifold if and only if it is a subgroup of* $\mathrm{SO}(5)$.

Proof. Since G is perfect and nonabelian the action is orientation-preserving. By Theorem 2, the quasisimple components of G are isomorphic to $\mathbb{A}_5$, $\mathbb{A}_6$ or $\mathbb{A}_5^*$. Since by Lemma 4 the sectional 2-rank of G is at most four, G is the central product of at most two of these quasisimple groups, so it remains to analyze the case of groups with two quasisimple components. Suppose first that $G = Q_1 \times Q_2$, where Q_1 is isomorphic to either $\mathbb{A}_5$ or $\mathbb{A}_6$. In this case G has a subgroup isomorphic to $\mathbb{A}_4 \times Q_2$. The subgroup $\mathbb{A}_4 \times Q_2$ contains a normal elementary 2-group of rank two; by Lemma 3 the group $\mathbb{A}_4 \times Q_2$ would be a subgroup of $\mathrm{SO}(5)$, which is not the case (e.g. by [MeZ4]).

Next, the group $\mathbb{A}_5^* \times \mathbb{A}_5^*$ has a normal elementary 2-subgroup of rank two and again is not a subgroup of $\mathrm{SO}(5)$. So the only remaining semisimple group with two components is $\mathbb{A}_5^* \times_{\mathbb{Z}_2} \mathbb{A}_5^*$.

This finishes the proof of Proposition 3.

Proof of Theorem 3. By Lemma 3 we can suppose that either the Fitting subgroup $F(G)$ is trivial, or $F(G)$ is cyclic and the fixed point set of each p-subgroup of $F(G)$ is a 3-dimensional manifold (since otherwise G is a subgroup of $\mathrm{SO}(5)$ and we are done). In the latter case each p-subgroup of $F(G)$ acts as a rotation group around its 3-dimensional fixed point set. Each element of G acts by conjugation on each p-subgroup of $F(G)$; this action may be either trivial or dihedral since a rotation of minimal angle around the fixed point set is conjugate to a rotation of minimal angle. In any case the square of each element of G acts by conjugation trivially on $F(G)$.

Suppose first that $E(G)$ is trivial. Then $F(G)$ coincides with the generalized Fitting subgroup $F^*(G)$ of G (the product of the Fitting subgroup $F(G)$

and the maximal normal semisimple subgroup $E(G)$), and $F^*(G) = F(G)$ contains its centralizer in G ([Su2, Theorem 6.6.11, p. 452]). By the preceding paragraph, this implies that each element in $G/F(G)$ has order at most two, so $G/F(G)$ is an elementary abelian 2-group, of rank at most four by Lemma 4.

Suppose now that $E(G)$ is nontrivial, and hence isomorphic to one of the groups in Proposition 3. By [Su2, Theorem 6.6.11, p. 452], the factor group of G by the center of $F(G)$ is isomorphic to a subgroup of the automorphism group $\mathrm{Aut}(F^*(G))$ of the generalized Fitting subgroup. In our situation, $F(G)$ is cyclic and hence $G/F(G)$ is isomorphic to a subgroup of $\mathrm{Aut}(F^*(G))$, which in turn is isomorphic to a subgroup of $\mathrm{Aut}(F(G)) \times \mathrm{Aut}(E(G))$ (since $F(G)$ and $E(G)$ are characteristic in G).

If $E(G)$ is isomorphic to $\mathbb{A}_5$ or $\mathbb{A}_5^*$, the outer automorphism of $E(G)$ has order two; if $E(G)$ is isomorphic to $\mathbb{A}_6$, the outer automorphism group is elementary abelian of order four (see [Co]). We have supposed that the square of each element in G acts trivially on $F(G)$ and, if $E(G)$ has only one component, such a square acts as an inner automorphism on $E(G)$; this implies easily that the square of each element in G is contained in $F^*(G)$, and hence the factor group $G/F^*(G)$ is an elementary abelian 2-group, of rank at most four.

Suppose that $E(G)$ is isomorphic to $\mathbb{A}_5^* \times_{\mathbb{Z}_2} \mathbb{A}_5^*$. Then the center $\mathbb{Z}_2$ of $E(G)$ is normal in G. By Lemma 3, we can assume that the fixed point set of this normal subgroup $\mathbb{Z}_2$ has dimension three and hence, by Smith theory, is a $\mathbb{Z}_2$-acyclic 3-manifold M. The factor group $E(G)/\mathbb{Z}_2$ is isomorphic to $\mathbb{A}_5 \times \mathbb{A}_5$ and admits a faithful, orientation-preserving action on M. Now $\mathbb{A}_5 \times \mathbb{A}_5$ has a subgroup $(\mathbb{Z}_2)^2 \times (\mathbb{Z}_2)^2 = (\mathbb{Z}_2)^4$; however, again by Smith theory, the group $(\mathbb{Z}_2)^4$ does not admit a faithful, orientation-preserving action on a $\mathbb{Z}_2$-homology 3-sphere. So, if $E(G) \cong \mathbb{A}_5^* \times_{\mathbb{Z}_2} \mathbb{A}_5^*$, we have shown that G is a subgroup of $\mathrm{SO}(5)$.

This finishes the proof of Theorem 3.

7. The Milnor groups $Q(8a, b, c)$. It is observed in [Mn] that the groups $Q(8a, b, c)$ have periodic cohomology of period four but do not admit faithful, linear, free actions on S^3 (in fact, they are not isomorphic to subgroups of $\mathrm{O}(4)$). We will assume in the following that $a > b > c \geq 1$ are odd coprime integers. Then $Q(8a, b, c)$ is a semidirect product $\mathbb{Z}_{abc} \rtimes Q_8$ of a normal cyclic subgroup $\mathbb{Z}_a \times \mathbb{Z}_b \times \mathbb{Z}_c \cong \mathbb{Z}_{abc}$ by the quaternion group $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ of order eight, where i, j and k act trivially on $\mathbb{Z}_a, \mathbb{Z}_b$ and $\mathbb{Z}_c$, respectively, and in a dihedral way on the other two. Note that $Q(8a, b, c)$ also has a normal subgroup $\mathbb{Z}_{2abc}$, with factor group the elementary abelian 2-group $\mathbb{Z}_2^2$, so it is one of the groups described in Theorem 3.

It has been shown by Milgram ([Mg]; see also the comments in [Ki, p. 173, Update A to Problem 3.37]) that some of the groups $Q(8a, b, c)$ admit a faithful, free action on a homology 3-sphere; let Q be one of these groups which admits such an action on a homology 3-sphere M. By the double suspension theorem (see e.g. [Ca]), the double suspension $M * S^1$ of M (the join of M with the 1-sphere) is homeomorphic to S^5. Letting Q act trivially on S^1, the actions of Q on M and S^1 induce a faithful, continuous, orientation-preserving action of Q on S^5 with fixed point set S^1, and hence also on $\mathbb{R}^5$ (the complement of a fixed point). Now it is not difficult to show that none of the groups $Q(8a, b, c)$ is isomorphic to a subgroup of SO(5). At present, we do not know if some Milnor group $Q(8a, b, c)$ admits a faithful, smooth, orientation-preserving action on an acyclic 5-manifold.

NOTE. The referee provided two additional references ([BKS] and [KS]) on related work; in particular [KS] considers and completes the geometrization of finite group actions on Euclidean space $\mathbb{R}^3$.

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