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Descriptive set theoretical complexity of randomness notions

by

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Abstract. We study the descriptive set theoretical complexity of various randomness notions.

1. Introduction. The original motivation of this paper is to characterize weakly 2-random reals by prefix-free Kolmogorov complexity. Since Schnorr characterized Martin–Löf randomness by prefix-free Kolmogorov complexity, many people thought that every randomness notion should have a characterization by initial segment complexity. For example, Miller and others obtained a very successful characterization of 2-randomness.

Theorem 1.1 (Miller [8], [9]; Nies, Stephan and Terwijn [12]). *A real x is 2-random if and only if*

$$\exists c\, \forall n\, \exists m\, (C(x{\upharpoonright}m) \geq m - c)$$

if and only if

$$\exists c\, \forall n\, \exists m > n\, (K(x{\upharpoonright}m) \geq m + K(m) - c).$$

Recently, Miller and Yu [10] obtained the following result.

Theorem 1.2 (Miller and Yu [10]). *$x \oplus y$ is random if and only if*

$$\exists c\, \forall n\, (K(x{\upharpoonright}n) + C(y{\upharpoonright}n) \geq 2n - c).$$

This theorem gives almost all the "relativizable" randomness notions stronger than Martin–Löf randomness unrelativized Kolmogorov complexity characterizations. An important question remaining open is whether there is a Kolmogorov complexity characterization for weak 2-randomness. This question has been approached in many ways. For example, one way is to ask whether there is a sequence $\{f_n\}_{n\in\omega}$ of functions such that for every real x, x is weakly 2-random if and only if $\exists n\, \forall m\, \exists k \geq m\, (K(x{\upharpoonright}k) \geq k + f_n(k))$. Most of these attempts aimed at some kind of $\mathbf{\Sigma}^0_3$-characterizations for weak

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2-randomness. But all of the ways (of course) failed. So people suspected that the collection of weakly 2-random reals is not $\mathbf{\Sigma}^0_3$. We confirm this in this paper.

Then we also study the descriptive set theoretical complexity of some other classical randomness notions. Many results have been obtained in [5] by using Wadge reductions. Given two sets of reals A and B, A is *Wadge reducible* to B, written $A \leq_W B$, if there is a continuous function $f : 2^\omega \to 2^\omega$ such that for every x, $x \in A$ if and only if $f(x) \in B$. The authors of [5] prove, for example, that the collection of Schnorr random reals is $\mathbf{\Pi}^0_3$-complete (and so non-$\mathbf{\Sigma}^0_3$). Here we give another more direct way, by using forcing arguments, to prove that result. One might think that the results in [5] are stronger since it is proved that the collection of Schnorr random reals is $\mathbf{\Pi}^0_3$-complete. Actually they are not by the following well known descriptive set theory result.

THEOREM 1.3 (Folklore). *For any $\xi < \omega_1$ and each $\mathbf{\Sigma}^0_\xi$ (or $\mathbf{\Pi}^0_\xi$) set A, if A is not $\mathbf{\Pi}^0_\xi$ (or $\mathbf{\Sigma}^0_\xi$), then every $\mathbf{\Sigma}^0_\xi$ set is Wadge reducible to A.*

Theorem 1.3 is an immediate consequence of Borel determinacy. Moreover, our technique yields results of independent interest. For example, we prove that the forcing notion of Π^0_1-classes with computable positive measures does not produce a Martin–Löf random real.

We also study the complexity of the collection of Δ^1_1-random reals. Sacks [13] essentially proves that the collection of Δ^1_1-random reals is $\mathbf{\Pi}^0_3$. Hjorth and Nies [6] introduced Π^1_1-Martin–Löf randomness, which is an analog to the classical Martin–Löf randomness in higher recursion theory. But a difficult question was whether Π^1_1-Martin–Löf randomness is different from Δ^1_1-randomness. The separation of Π^1_1-Martin–Löf randomness from Δ^1_1-randomness was given in [2]. The proof in that paper was rather involved, and only a sketch was presented. Here we give a full proof by a simpler argument. Furthermore, we have a total characterization of where Δ^1_1-randomness is different from Π^1_1-Martin–Löf randomness.

The paper is organized as follows: In Section 2, we give some basic definitions. In Section 3, we present some easy facts about the descriptive set theoretical complexity of various randomness notions. Most of them are probably known. In Section 4, we prove that the collection of weakly 2-random reals is not $\mathbf{\Sigma}^0_3$. In Section 5, we prove that the collection of Schnorr random reals is not $\mathbf{\Sigma}^0_3$. In Section 6, we prove that the collection of Δ^1_1-random reals is not $\mathbf{\Sigma}^0_3$. In Section 7, we raise some questions.

2. Preliminaries. A real is *Kurtz random* if it does not belong to any Π^0_1 null set. Since every co-null open Σ^0_1 set is dense, every weakly 1-generic real is Kurtz random.

A *Schnorr test* is a uniformly c.e. sequence $\{U_n\}_{n\in\omega}$ of open sets such that $\mu(U_n) = 2^{-n}$ for every n. A real x is *Schnorr random* if for every Schnorr test $\{U_n\}_{n\in\omega}$, $x \notin \bigcap_{n\in\omega} U_n$. This is equivalent to saying that $x \notin \bigcap_{n\in\omega} U_n$ for any c.e. sequence $\{U_n\}_{n\in\omega}$ of open sets such that $\mu(U_n) = 2^{-f(n)}$ for every n where f is a computable function from ω to $[0,1]$ such that $\lim_{n\to\infty} f(n) = 0$.

A *Martin–Löf test* is a uniformly c.e. sequence $\{U_n\}_{n\in\omega}$ of open sets such that $\mu(U_n) < 2^{-n}$ for every n. A real x is *Martin–Löf random* (or 1-*random*) if for every Martin–Löf test $\{U_n\}_{n\in\omega}$, $x \notin \bigcap_{n\in\omega} U_n$. There exists a universal Martin–Löf test, i.e. a Martin–Löf test covering all the Martin–Löf tests.

A *generalized Martin–Löf test* is a uniformly c.e. sequence $\{U_n\}_{n\in\omega}$ of open sets such that $\lim_{n\to\infty} \mu(U_n) = 0$. A real x is *weakly* 2-*random* if for every generalized Martin–Löf test $\{U_n\}_{n\in\omega}$, $x \notin \bigcap_{n\in\omega} U_n$. There is no universal Martin–Löf test. We have the following nice result.

Theorem 2.1 (Downey, Nies, Weber and Yu [4]; Hirschfeldt and Miller [4]). *A real x is weakly 2-random if and only if x is 1-random and does not Turing-compute any non-computable Δ^0_2-real.*

For some information about higher randomness, see [13], [6] and [2]. A real is Δ^1_1-random if and only if it does not belong to any Δ^1_1 null set. It is essentially due to Sacks [13] that a real x is Δ^1_1-random if and only if for any Δ^1_1-sequence of Δ^1_1 open sets $\{U_n\}_{n\in\omega}$ for which $\lim_{n\to\infty} \mu(U_n) = 0$, $x \notin \bigcap_n U_n$. So the collection of Δ^1_1-random reals is $\mathbf{\Pi}^0_3$.

A Π^1_1-*Martin–Löf test* is a Π^1_1-sequence of Π^1_1-coded open sets $\{U_n\}_{n\in\omega}$ (i.e. the set $\{(n,\sigma) \mid \sigma \in U_n\}$ is Π^1_1) so that $\mu(U_n) < 2^{-n}$ for every n. Hjorth and Nies [6] proved that there is a universal Π^1_1-Martin–Löf test. A real is Π^1_1-*Martin–Löf random* if it does not belong to any Π^1_1-Martin–Löf test. We have the following result.

Theorem 2.2 (Chong, Nies and Yu [2]). *If $\omega_1^x = \omega_1^{\mathrm{CK}}$, then x is Δ^1_1-random if and only if x is Π^1_1-Martin–Löf random.*

We identify an open set U with a set of finite strings. For any finite string $\sigma \in 2^{<\omega}$, we use $[\sigma]$ to denote the open set $\{x \mid x \succ \sigma\}$. For any tree T, we write $[T]$ for the closed set $\{x \mid \forall n\ (x{\upharpoonright}n \in T)\}$.

For more information about randomness and computability theory, see [11] and [3].

3. Some basic facts. The following facts are immediate and probably known. Many of them can be found in [5].

Proposition 3.1.

(1) *The collection of Kurtz random reals is $\mathbf{\Pi}^0_2$ but not Π^0_2.*
(2) *The collection of Schnorr random reals is Π^0_3.*

(3) *The collection of* 1*-random reals is* Σ^0_2.
(4) *The collection of weakly* 2*-random reals is* $\mathbf{\Pi}^0_3$ *but not* Π^0_3.
(5) *The collection of* Δ^1_1*-random reals is* $\mathbf{\Pi}^0_3$.

Proof. (1) Obviously the collection K of Kurtz random reals is $\mathbf{\Pi}^0_2$. Suppose that K is Π^0_2. Then there is a recursive set $R \subseteq \omega \times \omega \times 2^{<\omega}$ so that $x \in K$ if and only if $\forall n\, \exists m\, R(n, x{\upharpoonright}m)$. For each n, let $K_n = \{x \mid \exists m\, R(n, x{\upharpoonright}m)\}$. Then K_n is Σ^0_1, co-null and $K \subseteq K_n$ for every n. Hence it would be easy to computably construct a sequence of finite strings $\sigma_0 \prec \sigma_1 \prec \cdots$ so that $[\sigma_n] \subseteq K_n$ for every n. Then the computable real $x = \bigcup_{n\in\omega} \sigma_n \in \bigcap_{n\in\omega} K_n = K$ would be Kurtz random, a contradiction.

(2) Obvious (see [5]).

(3) Obvious.

(4) Obviously the collection of weakly 2-random reals W is $\mathbf{\Pi}^0_3$. Suppose that K is Π^0_3. Then there is a computable set $R \subseteq \omega \times \omega \times \omega \times 2^{<\omega}$ such that $x \in W$ if and only if $\forall n\, \exists m\, \forall j\, R(n, m, x{\upharpoonright}j)$. For each n, let $W_n = \{x \mid \exists m\, \forall j\, R(n, m, x{\upharpoonright}j)\}$ and $W_{n,m} = \{x \mid \forall j\, R(n, m, x{\upharpoonright}j)\}$. Then K_n is Σ^0_2, co-null and $W \subseteq W_n$ for every n. We $\emptyset'$-computably construct a sequence of finite strings $\sigma_0 \prec \sigma_1 \prec \cdots$ and Π^0_1 positive measure sets $T_0 \supseteq T_1 \supseteq \cdots$ so that $\sigma_n \in T_n$ as follows: $\sigma_0 = \emptyset$ and $W_0 = 2^\omega$. Given σ_n and R_n, since W_{n+1} is co-null, we may $\emptyset'$-computably find the least m such that $T_n \cap W_{n,m} \cap [\sigma_n] = \{x \succ \sigma_n \mid x \in [T_n] \wedge \forall j\, R(n, m, x{\upharpoonright}j)\}$ has positive measure. Let $T_{n+1} = T_n \cap W_{n,m} \cap [\sigma_n]$ and σ_{n+1} be a finite string in T_{n+1} extending σ_n. Then the $\emptyset'$-computable real $x = \bigcup_{n\in\omega} \sigma_n \in \bigcap_{n\in\omega} W_n = W$ is weakly 2-random, a contradiction to Theorem 2.1.

(5) Obvious. ∎

The results above about descriptive complexity of the collections of Kurtz random and 1-random reals are rigid.

Proposition 3.2.

(1) *The collection of Kurtz random reals is not* $\mathbf{\Sigma}^0_2$.
(2) *The collection of* 1*-random reals is not* $\mathbf{\Pi}^0_2$.

Proof. (1) Otherwise, there is a sequence $\{P_n\}_{n\in\omega}$ of closed sets such that $\bigcup_n P_n$ contains exactly all the Kurtz random reals. Since all the generic reals are Kurtz random, $\bigcup_n P_n$ is comeager. Hence there must be some n such that P_n is not meager. Then P_n must contain an interval and so contain a computable real, a contradiction.

(2) Otherwise, there is a sequence $\{U_n\}_{n\in\omega}$ of open sets such that $\bigcap_n U_n$ contains exactly all the 1-random reals. Then for every n, $\mu(U_n) = 1$. So every U_n is dense. Hence every sufficiently generic real would belong to $\bigcap_n U_n$. But no 1-generic real can be random, a contradiction. ∎

The second result above can be found in [5].

4. Weak 2-randomness. In this section, we prove that the collection of weakly 2-random reals is not $\mathbf{\Sigma}^0_3$. We apply a forcing argument.

DEFINITION 4.1. Define a forcing notion $\mathbb{P} = (\boldsymbol{P}, \leq)$ as follows:

(1) $P \in \boldsymbol{P}$ if and only if P is a Π^0_1-class with positive measure.
(2) For $P, Q \in \boldsymbol{P}$, $P \leq Q$ if and only if $P \subseteq Q$.

Let $\{F_m\}_{m\in\omega}$ be an increasing sequence of Π^0_1 sets such that $\bigcup_{m\in\omega} F_m$ is of measure 1. Set $C = \bigcup_{m\in\omega} F_m$. Let $\mathcal{D}_C = \{P \mid P \in \boldsymbol{P} \wedge P \subseteq C\}$.

LEMMA 4.2. *$\mathcal{D}_C$ is dense.*

Proof. Suppose that $\{F_m\}_{m\in\omega}$ is an increasing sequence of Π^0_1 sets such that $\bigcup_{m\in\omega} F_m$ is of measure 1 and $C = \bigcup_{m\in\omega} F_m$. Let $P \in \boldsymbol{P}$. Then there is some large enough m such that $\mu(F_m) > 1 - \mu(P)/2$. So

$$\mu(F_m \cap P) = \mu(F_m) + \mu(P) - \mu(F_m \cup P) > 1 - \mu(P)/2 + \mu(P) - 1 = \mu(P)/2.$$

Thus $F_m \cap P \in \mathcal{D}_C$. ∎

The following lemma is a stronger version of Lemma 2.2 in [1].

LEMMA 4.3. *For every computable tree T, there is a generalized Martin–Löf test $\{V_n\}_{n\in\omega}$ such that for any σ, if $[\sigma] \cap [T]$ is not empty, then $[\sigma] \cap [T] \cap \bigcap_n V_n$ is not empty.*

Proof. The idea is to build a uniformly c.e. sequence $\{V_n\}_{n\in\omega}$ of open sets densely meeting $[T]$. The method is just like building a null comeager set. But we may make some mistakes since there is no effective way to predict whether $[\sigma] \cap [T]$ is not empty. So, at every step, we need to "correct" the construction of the previous steps. But the measure of mistakes will become very small whenever the step is large enough. This is the reason we can ensure that $\{V_n\}_{n\in\omega}$ is a generalized Martin–Löf test.

Fix a computable tree T. So there is a computable approximation to T by computable trees $\{T_s\}_{s\in\omega}$ such that

(1) $T_0 = T$;
(2) $T_{s+1} = \{\sigma \mid \sigma \in T \wedge \exists \tau \in 2^{s+1} \cap T\ (\tau \text{ is compatible with } \sigma)\}$.

Then $T_{s+1} \subseteq T_s$ for every s.

Fix a computable enumeration $\{\sigma_i\}_{i\in\omega}$ of $2^{<\omega}$ and an enumeration $\{\sigma^{s+1}_i\}_{i\leq 2^{s+1}}$ of 2^{s+1} for each s.

We construct V_n for every n step by step.

STEP 0: We put the empty string λ into V_0. So the open set V_0 is 2^ω.

STEP $s+1$:

SUBSTEP 1: We correct $\{V_k\}_{k\leq s}$ step by step.

SUBSTEP 1.0: Check whether there is a $\sigma \in T_{s+1} \cap 2^{s+1}$. If so, do nothing. Otherwise, stop the construction.

SUBSTEP 1.k: Check whether there is some $\tau \in V_k$ such that there is no $\nu \in T_{s+1} \cap 2^{s+1}$ with $\nu \succ \tau$. If so, check whether there is some $\tau' \succ \tau\restriction k$ in $2^{|\tau|}$ such that there is a $\nu \in T_{s+1} \cap 2^{s+1}$ with that $\nu \succ \tau'$. If so, put τ' into V_j for all $j \leq k$; otherwise, do nothing.

SUBSTEP 2: For every i, check whether there is some $\tau \in T_{s+1}$ extending σ_i^{s+1}. If not, go to $i+1$; otherwise, check whether there is some $\tau \in V_s$ such that $\tau \succ \sigma_i^{s+1}$. If yes, put τ into V_{s+1}; otherwise, check whether there is some very long $\tau \succ \sigma_i^{s+1}$ in T_{s+1} that is longer than any finite strings mentioned before. If yes, pick such a τ and put it into V_{s+1}; otherwise, do nothing. Now for every $k \leq s$, check whether there is some $\tau' \in V_k$ compatible with τ. If yes, do nothing; otherwise, put τ into V_k.

This finishes the construction.

By the construction, $V_{n+1} \subseteq V_n$ for any n.

If $\sigma \in T$ and $[\sigma] \cap [T] \neq \emptyset$, then there is some stage $s_0 \geq |\sigma|$ at which we find some $\sigma_0 \succ \sigma$ such that $\sigma_0 \in T$ and $[\sigma_0] \cap [T] \neq \emptyset$ and put it into $V_{|\sigma|}$. Then there is some larger stage $s_1 \geq |\sigma_0|$ at which we find some $\sigma_1 \succ \sigma_0$ such that $\sigma_1 \in T$ and $[\sigma_1] \cap [T] \neq \emptyset$ and put it into $V_{|\sigma_0|}$, etc. Since $\bigcap_{n\in\omega} V_n = \bigcap_{i\in\omega} V_{|\sigma_i|}$, the real $x = \bigcup_{i\in\omega} \sigma_i$ is in $(\bigcap_{n\in\omega} V_n) \cap T$. In other words, $[\sigma] \cap [T] \cap \bigcap_n V_n$ is not empty.

To see that $\{V_n\}_{n\in\omega}$ is a generalized Martin–Löf test, it is sufficient to show $\lim_{n\to\infty} \mu(V_n) = 0$. For any i, there is a large enough $s > i+1$ such that the open set $E_s = \{\sigma \in 2^s \mid \sigma \in T_s\}$ has measure less than $\mu([T]) + 2^{-i-1}$. Then from step s of the construction, except the correction substep, we only put a prefix-free set of finite strings into V_s. Moreover, except those strings put in at the correction substep, for different strings in V_s, they have different lengths greater than or equal to s. But at the correction substep, by the assumption on E_s, we put into V_s a set of finite strings of measure at most 2^{-i-1} . So

$$\mu(V_s) \leq \sum_{t\geq s} 2^{-t} + 2^{-i-1} = 2^{-s+1} + 2^{-i-1} \leq 2^{-i-1} + 2^{-i-1} = 2^{-i}.$$

Thus $\lim_{n\to\infty} \mu(V_n) = 0$. ∎

For any $\mathbf{\Pi}_2^0$ set G, let $\mathcal{D}_G = \{P \mid P \in \boldsymbol{P} \wedge P \cap G = \emptyset\}$.

LEMMA 4.4. *If G is a $\mathbf{\Pi}_2^0$ set only containing weakly 2-random reals, then $\mathcal{D}_G$ is dense in $\mathbb{P}$.*

Proof. Suppose that G is $\mathbf{\Pi}_2^0$ only containing weakly 2-random reals. Let $\{U_n\}_{n\in\omega}$ be a sequence of open sets such that $G = \bigcap_n U_n$. Let $P \in \boldsymbol{P}$. Without loss of generality, we may assume that for any σ, if $[\sigma] \cap P \neq \emptyset$, then $\mu([\sigma] \cap P) > 0$ (since we may assume that P only contains 1-random reals). Then we claim that there is some σ such that $P \cap [\sigma] \cap G = \emptyset$ but $P \cap [\sigma] \neq \emptyset$.

Suppose not. By Lemma 4.3, there is a generalized Martin–Löf test $\{V_n\}_{n\in\omega}$ such that for any σ, if $[\sigma]\cap P$ is not empty, then $[\sigma]\cap P\cap\bigcap_n V_n$ is not empty. Then we build a sequence of strings $\sigma_0\prec\sigma_1\prec\cdots$ as follows.

Let $\sigma_0=\emptyset$. Now suppose $[\sigma_i]\cap P\neq\emptyset$. Let $\tau\succ\sigma_i$ with $[\tau]\cap P\neq\emptyset$ and $[\tau]\cap P\subseteq V_i$. By the property of $\{V_n\}_n$, there exists such a τ. Then by assumption, let $\sigma_{i+1}\succ\tau$ be such that $[\sigma_{i+1}]\cap P\subseteq U_i$.

Let $x=\bigcup_{i\in\omega}\sigma_i$. Then $x\in P\cap(\bigcap_{n\in\omega}U_n)\cap(\bigcap_{n\in\omega}V_n)$. Since $x\in\bigcap_{n\in\omega}V_n$, x is not weakly 2-random, which contradicts the fact that G only contains weakly 2-random reals.

So there is some σ such that $P\cap[\sigma]\cap G=\emptyset$ but $P\cap[\sigma]\neq\emptyset$. Let $Q=P\cap[\sigma]$. Then $Q\in\boldsymbol{P}$ and $Q\leq P$. ∎

Theorem 4.5. *The collection of weakly 2-random reals is not $\boldsymbol{\Sigma}^0_3$.*

Proof. Suppose otherwise. Then there is a countable sequence $\{G_n\}_n$ of $\boldsymbol{\Pi}^0_2$ sets such that $\bigcup_n G_n$ contains exactly all the weakly 2-random reals. So G_n only contains weakly 2-random reals for every n. Then by Lemma 4.4, for any sufficiently generic real g over $\mathbb{P}$, $g\notin G_n$ for any n. By Lemma 4.2, for any sufficiently generic real g over $\mathbb{P}$, g is weakly 2-random, a contradiction. ∎

5. Schnorr randomness. In this section, we give another proof that the collection of Schnorr random reals is not $\boldsymbol{\Sigma}^0_3$. We use a similar method to the previous section with some modifications.

Definition 5.1. Define a forcing notion $\mathbb{Q}=(\boldsymbol{Q},\leq)$ as follows:

(1) $Q\in\boldsymbol{Q}$ if and only if Q is a Π^0_1-class with some computable positive measure.
(2) For $P,Q\in\boldsymbol{Q}$, $P\leq Q$ if and only if $P\subseteq Q$.

For any Schnorr test $\{U_n\}_{n\in\omega}$ with $\mu(U_n)=2^{-n}$ for every n, set $U=\bigcap_n U_n$. Let $\mathcal{D}_U=\{P\mid P\in\boldsymbol{Q}\wedge P\cap U=\emptyset\}$.

Lemma 5.2. *$\mathcal{D}_U$ is dense.*

Proof. Suppose that $\{U_n\}_{n\in\omega}$ is a Schnorr test with $\mu(U_n)=2^{-n}$ for every n, $U=\bigcap_n U_n$ and $P\in\boldsymbol{Q}$. Then there is some large enough n such that $\mu(U_n)<\mu(P)/2$. Hence the complement $P_0=2^\omega-U_n$ has measure $\geq 1-\mu(P)/2$. So $P_0\cap P$ has measure $\geq\mu(P)/2$. We show that $\mu(P_0\cap P)$ is a computable real. Both P and P_0 can be represented by computable trees T and T^0 respectively. Since both P and P_0 belong to $\boldsymbol{Q}$, for any i we may computably find some large enough s_i such that $\mu((\bigcup_{\sigma\in E_{s_i}}[\sigma])-P)<2^{-i-1}$ and $\mu((\bigcup_{\sigma\in E^0_{s_i}}[\sigma])-P_0)<2^{-i-1}$ where $E_{s_i}=\{\sigma\in 2^{s_i}\mid\sigma\in T\}$ and

$E^0_{s_i} = \{\sigma \in 2^{s_i} \mid \sigma \in T^0\}$. Then

$$\mu\Big(\Big(\bigcup_{\sigma\in E_{s_i}\cap E^0_{s_i}}[\sigma]\Big)-(P\cap P_0)\Big)=\mu\Big(\Big(\Big(\bigcup_{\sigma\in E_{s_i}\cap E^0_{s_i}}[\sigma]\Big)-P\Big)\cup\Big(\Big(\bigcup_{\sigma\in E_{s_i}\cap E^0_{s_i}}[\sigma]\Big)-P_0\Big)\Big)$$
$$\le \mu\Big(\Big(\bigcup_{\sigma\in E_{s_i}\cap E^0_{s_i}}[\sigma]\Big)-P\Big)+\mu\Big(\Big(\bigcup_{\sigma\in E_{s_i}\cap E^0_{s_i}}[\sigma]\Big)-P_0\Big)\le 2^{-i-1}+2^{-i-1}=2^{-i}.$$

So

$$\mu\Big(\bigcup_{\sigma\in E_{s_i}\cap E^0_{s_i}}[\sigma]\Big)-2^{-i}\le\mu(P\cap P_0)\le\mu\Big(\bigcup_{\sigma\in E_{s_i}\cap E^0_{s_i}}[\sigma]\Big).$$

Thus $\mu(P\cap P_0)$ is computable. In other words, $P\cap P_0\in\boldsymbol{Q}$. ∎

Now we want to mimic the proof of Lemma 4.4. But there is a problem: in that proof we can ensure that, for any condition $P\in\boldsymbol{P}$, $\mu([\sigma]\cap P)>0$ whenever $[\sigma]\cap P$ is not empty. The reason is that we can ensure that P only contains 1-random reals. But every condition $Q\in\boldsymbol{Q}$ contains a computable real. So we have to be more careful.

Lemma 5.3. *For ever computable tree T for which $\mu([T])>0$ is computable, there is a Schnorr test $\{V_n\}_{n\in\omega}$ such that for any σ, if $\mu([\sigma]\cap[T])>0$, then $\mu([\sigma]\cap[T]\cap V_n)>0$ for each n.*

Proof. Suppose that T is a computable tree such that $\mu([T])>0$ is computable. Then there is a computable function $f:\omega\to\omega$ such that for every s, $|E_{f(s)}|/2^{f(s)}-\mu(T)<2^{-s}$ where $E_t=\{\sigma\in 2^t\mid\sigma\in T\}$. Fix a computable enumeration $\{\sigma_i\}_{i\in\omega}$ of $2^{<\omega}$ and an enumeration $\{\sigma^{s+1}_i\}_{i\le 2^{s+1}}$ of 2^{s+1} for each s. We define $U_0=\bigcup_s U_0[s]$ as follows:

At step 0, do nothing.

At step $s+1$, select the least index i such that

(1) There is no $\tau\succeq\sigma_i$ belonging to $U_0[s]$.
(2) $|[\sigma_i]\cap E_{f(s)}|>2^{f(s)-s+1}$.

Then pick any $2^{f(s)-s+1}$ finite strings in $[\sigma_i]\cap E_{f(s)}$ and put them into $U_0[s+1]$.

Then by the definition of f, $U_0[s+1]\cap[\sigma_i]\cap[T]\ne\emptyset$. Obviously at any stage $s+1$, $\mu(U_0[s+1]-U_0[s])<2^{-s+2}$. So $\mu(U_0)$ is computable. Moreover, for any σ, if $\mu([\sigma]\cap[T])>0$, then $\mu([\sigma]\cap[T]\cap U_0)>0$. If not, pick the least index i such that $\mu([\sigma_i]\cap[T])>0$ but $\mu([\sigma]\cap[T]\cap U_0)=0$. Then there is a large enough stage s_0 such that for each $j<i$, if $\mu([\sigma_j]\cap[T])>0$, then $\mu([\sigma_j]\cap[T]\cap U_0[s_0])>0$. Suppose that $\mu([\sigma_i]\cap[T])>2^{-k}$; then at any stage $t>s_0+k$, $|[\sigma_i]\cap E_{f(t)}|>2^{f(t)-k}>2^{f(t)-t+1}$. Then we pick any $2^{f(t)-t+1}$ finite strings in $[\sigma_i]\cap E_{f(t)}$ and put them into $U_0[t]$. Then $\mu([\sigma_i]\cap[T]\cap U_0[t])>2^{-t}$, a contradiction.

Generally, for each n, we define $U_n = \bigcup_s U_n[s]$ as follows:
At step 0, do nothing.
At step $s+1$, select the least index i such that

(1) There is no $\tau \succeq \sigma_i$ belonging to $U_0[s]$.
(2) $|[\sigma_i] \cap E_{f(s+n)}| > 2^{f(s+n)-s-n+1}$.

Then pick any $2^{f(s+n)-s-n+1}$ finite strings in $[\sigma_i] \cap E_{f(s+n)}$ and put them into $U_n[s+1]$.

By the same argument as above, for every s, $\mu(U_n[s+1] - U_n[s]) < 2^{-s-n+2}$. So for any n, $\mu(U_n) < 2^{-n+3}$ is computable. Moreover, for any σ, if $\mu([\sigma] \cap [T]) > 0$, then $\mu([\sigma] \cap [T] \cap U_n) > 0$.

Now define $V_n = \bigcup_{m \geq n} U_m$. Hence $\mu(V_n) < 2^{-n+4}$ for each n. Hence by an easy calculation, $\{\mu(V_n)\}_{n\in\omega}$ is uniformly computable. Thus $\{V_n\}_{n\in\omega}$ is a Schnorr test. By the property of $\{U_n\}_{n\in\omega}$, for any σ and n, if $\mu([\sigma] \cap [T]) > 0$, then $\mu([\sigma] \cap [T] \cap V_n) > 0$. ∎

For any $\mathbf{\Pi}^0_2$ set G, let $\mathcal{D}_G = \{P \mid P \in \boldsymbol{Q} \wedge P \cap G = \emptyset\}$.

Lemma 5.4. *If G is a $\mathbf{\Pi}^0_2$ set only containing Schnorr random reals, then $\mathcal{D}_G$ is dense in $\mathbb{Q}$.*

Proof. Suppose that G is $\mathbf{\Pi}^0_2$ only containing Schnorr random reals. Let $\{U_n\}_{n\in\omega}$ be a sequence of open sets such that $G = \bigcap_n U_n$. Let $P \in \boldsymbol{Q}$. We claim that there is some σ such that $P \cap [\sigma] \cap G = \emptyset$ but $\mu(P \cap [\sigma]) > 0$.

Suppose not. By Lemma 5.3, there is a Schnorr test $\{V_n\}_{n\in\omega}$ such that for any σ, if $\mu([\sigma] \cap P) > 0$, then $\mu([\sigma] \cap P \cap V_n) > 0$ for each n. Then we build a sequence of strings $\sigma_0 \prec \sigma_1 \prec \cdots$ as follows.

Let $\sigma_0 = \emptyset$. Now suppose $\mu([\sigma_i] \cap P) > 0$. Let $\tau \succ \sigma_i$ be such that $\mu([\tau] \cap P) > 0$ and $[\tau] \cap P \subseteq V_i$. By the property of $\{V_n\}_n$, there exists such a τ. Then by assumption, let $\sigma_{i+1} \succ \tau$ be such that $[\sigma_{i+1}] \cap P \cap G \neq \emptyset$. Since G only contains Schnorr random reals, $\mu([\sigma_{i+1}] \cap P \cap U_i) > 0$. Then we may assume that $[\sigma_{i+1}] \cap P \subseteq U_i$ and $\mu([\sigma_{i+1} \cap P]) > 0$.

Let $x = \bigcup_{i\in\omega} \sigma_i$. Then $x \in P \cap (\bigcap_{n\in\omega} U_n) \cap (\bigcap_{n\in\omega} V_n)$. Since $x \in \bigcap_{n\in\omega} V_n$, x is not Schnorr random, which contradicts the assumption that G only contains Schnorr random reals.

So there is some σ such that $P \cap [\sigma] \cap G = \emptyset$ but $\mu(P \cap [\sigma]) > 0$. Let $Q = P \cap [\sigma]$. Then $Q \in \boldsymbol{Q}$ and $Q \leq P$. ∎

Theorem 5.5 (Hitchcock, Lutz and Terwijn [5]). *The collection of Schnorr random reals is not $\mathbf{\Sigma}^0_3$.*

Proof. Suppose otherwise. Then there is a countable sequence $\{G_n\}_n$ of $\mathbf{\Pi}^0_2$ sets such that $\bigcup_n G_n$ contains exactly the Schnorr random reals. Then by Lemmas 5.2 and 5.4, for any sufficiently generic real g over $\mathbb{Q}$, g is Schnorr random but $g \notin G_n$ for any n, a contradiction. ∎

We want to point out that the forcing $\mathbb{Q}$ does not produce a 1-random real. To see this, fix a universal Martin–Löf test $\{U_n\}_{n\in\omega}$. For each n, let $\mathcal{D}_n = \{P \in \mathbb{Q} \mid P \subseteq U_n\}$.

Corollary 5.6. *For each n, $\mathcal{D}_n$ is dense.*

Proof. Let $P \in \boldsymbol{Q}$ and $G = 2^\omega - U_n$. Then G is a Π^0_1 class only containing 1-random reals. Then by Lemma 5.4, there is some $Q \leq P$ such that $Q \in \mathcal{D}_n$. ∎

So if g is sufficiently generic over $\mathbb{Q}$, then g is Schnorr random but not 1-random.

6. Δ^1_1-randomness. In this section, we prove that the collection of Δ^1_1-random reals is not $\boldsymbol{\Sigma}^0_3$. Some basic facts in higher randomness theory can be found in [13], [6] and [2].

Definition 6.1. Define a forcing notion $\mathbb{D} = (\boldsymbol{D}, \leq)$ as follows:

(1) $P \in \boldsymbol{D}$ if and only if P is a Δ^1_1, closed set of reals with positive measure.
(2) For $P, Q \in \boldsymbol{D}$, $P \leq Q$ if and only if $P \subseteq Q$.

For any Δ^1_1-sequence of Δ^1_1-open sets $\{U_n\}_{n\in\omega}$ with $\lim_{n\to\infty} \mu(U_n) = 0$, set $U = \bigcap_n U_n$. Let $\mathcal{D}_U = \{P \mid P \in \boldsymbol{D} \wedge P \cap U = \emptyset\}$.

Lemma 6.2. *$\mathcal{D}_U$ is dense.*

Proof. Suppose that $\{U_n\}_{n\in\omega}$ is a Δ^1_1-sequence of Δ^1_1-open sets with $\lim_{n\to\infty} \mu(U_n) = 0$, $U = \bigcap_n U_n$ and $P \in \boldsymbol{D}$. Then there is some large enough n such that $\mu(U_n) < \mu(P)/2$. Hence the complement $P_0 = 2^\omega - U_n$ has measure $\geq 1 - \mu(P)/2$. So $P_0 \cap P$ is a Δ^1_1, closed set and has measure $\geq \mu(P)/2$. Thus $P \cap P_0 \in \boldsymbol{D}$. ∎

For any $\boldsymbol{\Pi}^0_2$ set G, let $\mathcal{D}_G = \{P \mid P \in \boldsymbol{D} \wedge P \cap G = \emptyset\}$.

Lemma 6.3. *If G is a $\boldsymbol{\Pi}^0_2$ set only containing Δ^1_1-random reals, then $\mathcal{D}_G$ is dense in $\mathbb{D}$.*

Proof. Suppose that G is $\boldsymbol{\Pi}^0_2$ only containing Δ^1_1-random reals. Let $\{U_n\}_{n\in\omega}$ be a sequence of open sets such that $G = \bigcap_n U_n$. Let $P \in \boldsymbol{D}$. Then there is a hyperarithmetic real x such that P is $\Pi^0_1(x)$. Without loss of generality, we may assume that for any σ, if $[\sigma] \cap P \neq \emptyset$, then $\mu([\sigma] \cap P) > 0$ (since we may assume that P only contains 1-x-random reals). We claim that there is some σ such that $P \cap [\sigma] \cap G = \emptyset$ but $P \cap [\sigma] \neq \emptyset$.

Suppose not. By Lemma 4.3 relativized to x, there is a generalized x-Martin–Löf test $\{V_n\}_{n\in\omega}$ such that for any σ, if $[\sigma] \cap P$ is not empty, then $[\sigma] \cap P \cap \bigcap_n V_n$ is not empty. Then we build a sequence of strings $\sigma_0 \prec \sigma_1 \prec \cdots$ as follows.

Let $\sigma_0 = \emptyset$. Now suppose $[\sigma_i] \cap P \neq \emptyset$. Let $\tau \succ \sigma_i$ be such that $[\tau] \cap P \neq \emptyset$ and $[\tau] \cap P \subseteq V_i$. By the property of $\{V_n\}_n$, there exists such a τ. Then by assumption, pick $\sigma_{i+1} \succ \tau$ such that $[\sigma_{i+1}] \cap P \subseteq U_i$.

Let $z = \bigcup_{i\in\omega} \sigma_i$. Then $z \in P \cap (\bigcap_{n\in\omega} U_n) \cap (\bigcap_{n\in\omega} V_n)$. Since $z \in \bigcap_{n\in\omega} V_n$, z is not weakly 2-x-random. But x is hyperarithmetic, so z is not Δ^1_1-random, which contradicts the assumption that G only contains Δ^1_1-random reals.

So there is some σ such that $P \cap [\sigma] \cap G = \emptyset$ but $P \cap [\sigma] \neq \emptyset$. Let $Q = P \cap [\sigma]$. Then $Q \in \boldsymbol{D}$ and $Q \leq P$. ∎

By the same proof as in the previous sections, we have the following result.

Proposition 6.4. *The collection of Δ^1_1-random reals is not $\boldsymbol{\Sigma}^0_3$.*

We give an application of Proposition 6.4.

It is difficult to separate Π^1_1-Martin–Löf randomness from Δ^1_1-randomness. The proof in [2] is rather involved and only sketched. Now we may apply the previous results to give a simpler proof (and even a stronger result).

Since the collection of Π^1_1-Martin–Löf random reals is $\boldsymbol{\Sigma}^0_3$, an immediate consequence of Proposition 6.4 is:

Corollary 6.5 (Chong, Nies and Yu [2]). *There is a Δ^1_1-random real z which is not Π^1_1-Martin–Löf random.*

By analyzing the proof of Proposition 6.4, we can obtain a characterization of where these notions differ.

Theorem 6.6. *For each $x \geq_h \mathscr{O}$, there is a Δ^1_1-random real $z \equiv_h x$ which is not Π^1_1-Martin–Löf random.*

Proof. The collection of Π^1_1-Martin–Löf random reals is a $\Sigma^0_2(\mathscr{O})$-set. Moreover, there is an $\mathscr{O}$-computable enumeration of the conditions in $\boldsymbol{D}$ (see Sacks [13]). Then hyperarithmetically in $\mathscr{O}$, by a finite extension argument, it is not difficult to construct a $\Delta^1_1(\mathscr{O})$-perfect tree T such that every infinite path in T is Δ^1_1-random but not Π^1_1-Martin–Löf random. By Theorem 2.2, every real $x \in [T]$ is hyperarithmetically above $\mathscr{O}$. So for each $x \geq_h \mathscr{O}$, there is a Δ^1_1-random real $z \equiv_h x$ which is not Π^1_1-Martin–Löf random. ∎

We want to make an observation here. In [13], Sacks does not use a forcing argument to study measure theoretic uniformity. Instead, he uses a model $\mathscr{M}(\omega_1^{\mathrm{CK}}, x)$. The advantage of his method is to show that $\mathscr{M}(\omega_1^{\mathrm{CK}}, x)$ satisfies Δ^1_1-CA (and so $\omega_1^x = \omega_1^{\mathrm{CK}}$) for almost all reals x. Now the reason that a forcing argument is avoided seems clear since the forcing notion with Δ^1_1 sets with positive measures does not produce a generic real x with $\omega_1^x = \omega_1^{\mathrm{CK}}$.

7. Some remarks. We do not know the exact complexity of the collection of Π^1_1-random reals. We conjecture that it cannot be $\mathbf{\Sigma}^0_{<\omega_1^{\mathrm{CK}}}$ $(= \bigcup_{\alpha<\omega_1^{\mathrm{CK}}} \mathbf{\Sigma}^0_\alpha)$.

For any cardinal κ and number n, we use κ-$\mathbf{\Sigma}^0_{n+1}$ to denote the class of sets which can be a union of less than κ-many $\mathbf{\Pi}^0_n$-sets. For example, $\aleph_1$-$\mathbf{\Sigma}^0_{n+1}$ is exactly the same as $\mathbf{\Sigma}^0_{n+1}$. We can also define κ-$\mathbf{\Pi}^0_{n+1}$ in a similar way. Then the following is true.

THEOREM 7.1. *Assume ZFC + Martin's axiom. Then:*

(1) *The collection of Kurtz random reals is not* $2^{\aleph_0}$-$\mathbf{\Sigma}^0_2$.
(2) *The collection of Schnorr random reals is not* $2^{\aleph_0}$-$\mathbf{\Sigma}^0_3$.
(3) *The collection of* 1*-random reals is not* $2^{\aleph_0}$-$\mathbf{\Pi}^0_2$.
(4) *The collection of weakly* 2*-random reals is not* $2^{\aleph_0}$-$\mathbf{\Sigma}^0_3$.
(5) *The collection of* Δ^1_1*-random reals is not* $2^{\aleph_0}$-$\mathbf{\Sigma}^0_3$.

Proof. All the negative results in the previous sections were proved by c.c.c. forcings except (1) and (3). But it is a theorem under $ZFC+$Martin's axiom that any set which is a union of less than $2^{\aleph_0}$-many meager sets is meager (see [7]). So under $ZFC+$ Martin's axiom, (1)–(5) are all true. ∎

We do not know whether the conclusions of Theorem 7.1 can be proved under ZFC. We do not know either whether the following is known.

QUESTION 7.2. *Is it consistent with* $ZFC + \neg CH$ *that every* $\mathbf{\Pi}^1_1$ *set is a union of* $\aleph_1$*-many closed sets?*

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