FUNDAMENTA
MATHEMATICAE
194 (2007)

Generic absoluteness under projective forcing

by

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Abstract. We study the preservation of the property of $L(\mathbb{R})$ being a Solovay model under projective ccc forcing extensions. We compute the exact consistency strength of the generic absoluteness of $L(\mathbb{R})$ under forcing with projective ccc partial orderings and, as an application, we build models in which Martin's Axiom holds for $\utilde{\Sigma}^1_n$ partial orderings, but it fails for the $\utilde{\Sigma}^1_{n+1}$.

1. Introduction. In this paper we continue the systematic study of the preservation of the property of $L(\mathbb{R})$ being a Solovay model under various classes of forcing notions. This work started in [2], where we considered the class of projective absolutely-ccc forcing notions and obtained an exact consistency result for the preservation of the property of $L(\mathbb{R})$ being a Solovay model under this class of forcing extensions. It turned out that the large cardinals involved were the definably Mahlo cardinals, a weak form of Mahlo cardinals that satisfy some definability conditions. As a corollary we obtained the equiconsistency of: (1) there exists a definably-Mahlo cardinal; and (2) $L(\mathbb{R})$-absoluteness for projective absolutely ccc posets.

In [3] we showed that every projective strongly proper forcing notion preserves the property of $L(\mathbb{R})$ being a definably Mahlo Solovay model. Hence, the consistency of $L(\mathbb{R})$-absoluteness under projective strongly proper forcing notions has the existence of a definably Mahlo cardinal as an upper bound. We also proved in [3] that the consistency strength of the preservation of $L(\mathbb{R})$ being a Solovay model under σ-linked forcing notions is exactly

2000 *Mathematics Subject Classification*: 03E15, 03E35, 03E50, 03E55.

Key words and phrases: Solovay model, projective ccc forcing extension, generic absoluteness.

Research partially supported by the MEC-FEDER grants BFM2002-03236 and MTM 2005-01025, and the grants 2002SGR-00126 and 2005SGR-00738 of the Generalitat de Catalunya, and PR-01-GE-10 of the Principado de Asturias.

that of a Mahlo cardinal, in contrast with the general ccc case, for which a weakly compact cardinal is required.

Recall that a *Solovay model* over V is the $L(\mathbb{R})$ of a model $M \supseteq V$ which has the following properties:

(1) For every $x \in \mathbb{R}$, ω_1 is an inaccessible cardinal in $V[x]$.
(2) Every $x \in \mathbb{R}$ is *small-generic* over V. That is, for some forcing notion $\mathbb{P}$ in V that is countable in M, there is, in M, a $\mathbb{P}$-generic filter g over V such that $x \in V[g]$.

The reason we call a model with properties (1) and (2) above a *Solovay model* is the following result of Woodin (see [2]), which says that it is elementarily equivalent to Solovay's model from [10].

LEMMA 1.1. *Suppose that $V \subseteq M$ are models of (a fragment of) ZFC and M satisfies (1) and (2) above. Then there is a forcing notion $\mathbb{W}$ in M which does not add new reals and creates a generic filter C for the Levy collapse of ω_1^M over V such that M and $V[C]$ have the same reals.*

Our interest in the preservation of the property of $L(\mathbb{R})$ being a Solovay model under forcing extensions that do not collapse ω_1 lies mainly in the fact (Lemma 1.3 below) that it implies a strong form of generic absoluteness for the theory of the reals (see [2]).

DEFINITION 1.2. Let V be a model of ZF. Let $\mathbb{P} \in V$ be a forcing notion and let φ be a formula (possibly with parameters in V). V is *φ-absolute for* $\mathbb{P}$ iff

$$V \models \varphi \quad \text{iff} \quad V^{\mathbb{P}} \models \varphi.$$

If Σ is a set of formulas, V is *Σ-absolute for* $\mathbb{P}$ iff for every $\varphi \in \Sigma$, V is φ-absolute for $\mathbb{P}$. Given a class Γ of posets, V is *Σ-absolute for Γ* iff for every $\mathbb{P} \in \Gamma$, V is Σ-absolute for $\mathbb{P}$ in V.

V is *$L(\mathbb{R})$-absolute for* $\mathbb{P}$ iff there exists an elementary embedding

$$j : L(\mathbb{R})^V \to L(\mathbb{R})^{V^{\mathbb{P}}}$$

that fixes all the ordinals (and therefore all the reals). For Γ a class of posets, V is *$L(\mathbb{R})$-absolute for Γ* if it is $L(\mathbb{R})$-absolute for every $\mathbb{P}$ in Γ.

The following lemma is proved in [2].

LEMMA 1.3. *Suppose that $L(\mathbb{R})^M$ and $L(\mathbb{R})^N$ are Solovay models over V such that $\mathbb{R}^M \subseteq \mathbb{R}^N$ and $\omega_1^M = \omega_1^N$. Then there exists an elementary embedding $j : L(\mathbb{R})^M \to L(\mathbb{R})^N$ which fixes all the ordinals.*

Recall that for Γ a point-class, a *Γ-poset* is a triple $\mathbb{P} = \langle P, \leq_P, \perp_P \rangle$, where $\leq_P$ is a Γ-subset of $\omega^\omega \times \omega^\omega$, $P = \text{field}(\leq_P)$, $\langle P, \leq_P \rangle$ is a partial order, and $\perp_P$ is a Γ-subset of $\omega^\omega \times \omega^\omega$ contained in $P \times P$ such that for every $x, y \in P$, $x \perp_P y$ iff x, y are incompatible. $\mathbb{P}$ is a *projective poset* iff it is (isomorphic to) a Γ-poset for some projective point-class Γ. Notice that a poset $\mathbb{P}$ is projective iff it is (isomorphic to a poset that is) first-order definable in $H(\omega_1)$, with parameters.

In this paper we consider the class of projective ccc forcing notions. We show that the property of $L(\mathbb{R})$ being a Σ_n-weakly compact Solovay model (see definitions below) is preserved by forcing with $\underset{\sim}{\Sigma}^1_{n+1}$ ccc posets, and that the property of $L(\mathbb{R})$ being a definably weakly compact Solovay model is preserved by all projective ccc posets. We give an example of a Δ^1_3 poset $\mathbb{P}$ with the property K, hence ccc, such that Σ^1_4 generic absoluteness under forcing with $\mathbb{P}$ implies that ω_1 is Σ_1-weakly compact in L. A generalization of this example to higher projective levels shows that the consistency strength of $L(\mathbb{R})$-absoluteness under $\underset{\sim}{\Sigma}^1_{n+1}$ ccc forcing is exactly the existence of a Σ_n-weakly compact cardinal. Further, the consistency strength of $L(\mathbb{R})$-absoluteness under projective ccc forcing extensions is exactly that of the existence of a definably weakly compact cardinal. In the last section, and as an application of the previous results, we build models in which Martin's axiom holds for $\underset{\sim}{\Sigma}^1_n$ partial orderings but not for the $\underset{\sim}{\Sigma}^1_{n+1}$.

2. Projective ccc forcing extensions. We will address the question of the preservation of the property of $L(\mathbb{R})$ being a Solovay model under arbitrary projective ccc forcing notions. As we will see, we need to consider a definable form of weakly compact cardinals.

2.1. Σ_n-*weakly compact cardinals.* Recall that a Π^1_1 *sentence* of the language of set theory is a sentence of the form $\forall X\ \varphi(X)$, where $\varphi(X)$ is a first-order formula of the language of set theory expanded with the predicate symbol X.

Definition 2.1. Let κ be a cardinal and $n \in \omega$. Then κ is Σ_n-*weakly compact* (Σ_n-w.c., for short) iff κ is inaccessible and for every $R \subseteq V_\kappa$ which is definable by a Σ_n formula (with parameters) over V_κ and every Π^1_1 sentence Φ, if

$$\langle V_\kappa, \in, R\rangle \models \Phi$$

then there is $\alpha < \kappa$ (equivalently, unboundedly many $\alpha < \kappa$) such that

$$\langle V_\alpha, \in, R \cap V_\alpha\rangle \models \Phi.$$

That is, κ *reflects* Π^1_1 *sentences with* Σ_n *predicates.* Moreover, κ being Π_n-*weakly compact* (Π_n-w.c., for short) is defined analogously by substituting Π_n for Σ_n in the definition above. Thus, an inaccessible cardinal κ is Π_n-w.c. iff it reflects Π^1_1 sentences with Π_n predicates. An inaccessible cardinal is Δ_n-*weakly compact* (Δ_n-w.c., for short) iff it reflects Π^1_1 sentences with Δ_n predicates.

DEFINITION 2.2 (A. Leshem, [9]). A cardinal κ is *Σ_ω-weakly compact* (Σ_ω-w.c., for short) iff κ is Σ_n-w.c. for every $n \in \omega$.

PROPOSITION 2.3. *For κ an inaccessible cardinal, the following are equivalent*:

(1) *κ is Σ_n-w.c.*
(2) *κ is Π_n-w.c.*
(3) *κ is Δ_{n+1}-w.c.*
(4) *For every Π^1_1 formula $\Phi(x_0, \ldots, x_k)$ in the language of set theory and every $a_0, \ldots, a_k \in V_\kappa$, if $V_\kappa \models \Phi(a_0, \ldots, a_k)$, then there is $\lambda \in I_n := \{\lambda < \kappa : \lambda$ is inaccessible and $V_\lambda \preccurlyeq_n V_\kappa\}$ such that $V_\lambda \models \Phi(a_0, \ldots, a_k)$.*

Proof. (3)⇒(1) and (3)⇒(2) are trivial.

(1)⇒(2): Suppose that $R \subseteq V_\kappa$. For every Π^1_1 formula Ψ where R appears as a predicate, let $\widetilde{\Psi}$ be the formula obtained from Ψ by substituting every occurrence of the subformula Rx, where x is a first order variable, by $\neg Rx$. Note that $\widetilde{\Psi}$ is also Π^1_1.

It is easily shown, by induction on the complexity of formulas, that for every formula Ψ and every α,

$$\langle V_\alpha, \in, R \cap V_\alpha\rangle \models \Psi \quad \text{iff} \quad \langle V_\alpha, \in, V_\alpha \setminus R\rangle \models \widetilde{\Psi}.$$

Suppose now that $R \subseteq V_\kappa$ is definable by means of a Π_n formula over V_κ and Φ is a Π^1_1 sentence. If $\langle V_\kappa, \in, R\rangle \models \Phi$, then $\langle V_\kappa, \in, V_\kappa \setminus R\rangle \models \widetilde{\Phi}$. Since κ is Σ_n-w.c., there is $\alpha < \kappa$ such that $\langle V_\alpha, \in, (V_\kappa \setminus R) \cap V_\alpha\rangle = \langle V_\alpha, \in, V_\alpha \setminus R\rangle \models \widetilde{\Phi}$, and therefore $\langle V_\alpha, \in, R \cap V_\alpha\rangle \models \Phi$.

(2)⇒(4): Suppose that $\Phi(x_0, \ldots, x_k) = \forall X\ \varphi(X, x_0, \ldots, x_k)$ is a Π^1_1 formula and $a_0, \ldots, a_k \in V_\kappa$ are such that $V_\kappa \models \Phi(a_0, \ldots, a_k)$.

Let Ψ be the Π^1_1 sentence expressing that κ is inaccessible, and let σ be the first order sentence saying that the Π_n-club $C_n := \{\alpha < \kappa : V_\alpha \preccurlyeq_n V_\kappa\}$ is unbounded. Then

$$\langle V_\kappa, \in, C_n\rangle \models \Phi(a_0, \ldots, a_k) \wedge \Psi \wedge \sigma.$$

Since κ is Π_n-w.c., there is $\lambda < \kappa$ such that

$$\langle V_\lambda, \in, C_n \cap V_\lambda\rangle \models \Phi(a_0, \ldots, a_k) \wedge \Psi \wedge \sigma.$$

But then λ is inaccessible, and since $C_n \cap \lambda$ is unbounded, $\lambda \in I_n$.

(4)⇒(3): Suppose that R is a Δ_{n+1} subset of V_κ and Φ is a Π^1_1 sentence such that

$$\langle V_\kappa, \in, R\rangle \models \Phi.$$

Let $\varphi(x, y_0, \ldots, y_k)$ be a Σ_{n+1} formula and $\psi(x, z_0, \ldots, z_l)$ a Π_{n+1} formula that define R in V_κ with parameters $a_0, \ldots, a_k$ and $b_0, \ldots, b_l$, respectively.

Thus,

$$\langle V_\kappa, \in, R\rangle \models \forall x\ (Rx \leftrightarrow \varphi(x, a_0, \ldots, a_k) \leftrightarrow \psi(x, b_0, \ldots, b_l)).$$

Let $\Phi'(y_0, \ldots, y_k)$ be the Π^1_1 formula (with $y_0, \ldots, y_k$ as the only free individual variables) obtained by substituting every occurrence of the formula Rx in Φ by the formula $\varphi(x, y_0, \ldots, y_k)$. Then, clearly, $V_\kappa \models \Phi'(a_0, \ldots, a_k)$.

Hence, there is $\lambda \in I_n$ such that

$$V_\lambda \models \Phi'(a_0, \ldots, a_k) \wedge \forall x\ (\varphi(x, a_0, \ldots, a_k) \leftrightarrow \psi(x, b_0, \ldots, b_l)).$$

But since $V_\lambda \preccurlyeq_n V_\kappa$, $R \cap V_\lambda = \{x : V_\lambda \models \varphi(x, a_0, \ldots, a_k)\}$. Therefore,

$$\langle V_\lambda, \in, R \cap V_\lambda\rangle \models \Phi. \ \blacksquare$$

Notice that in the proof of (4)⇒(3) above, we have not made use of the fact that λ was inaccessible. Thus an inaccessible cardinal κ is Σ_n-w.c. iff κ reflects Π^1_1 sentences (in the language with $\in$ only) to some $\lambda < \kappa$ such that $V_\lambda \preccurlyeq_n V_\kappa$.

Leshem [9] has proved that if κ is Mahlo, then the set of Σ_ω-w.c. cardinals below κ is stationary. So, all these cardinals are, consistency-wise, below a Mahlo cardinal.

Let us recall from [2] that a subset C of a cardinal κ is a $\underset{\sim}{\Pi}_n$*-club* iff C is a club subset of κ that is definable over V_κ by means of a Π_n formula, possibly with parameters. A subset $S \subseteq \kappa$ is $\underset{\sim}{\Pi}_n$*-stationary* iff for every $\underset{\sim}{\Pi}_n$-club subset C of κ, $S \cap C \neq \emptyset$. (Notice that we do not require that S itself be Π_n-definable.) Finally, κ is a $\underset{\sim}{\Pi}_n$*-Mahlo* cardinal iff it is inaccessible and the set of all inaccessible cardinals below κ is $\underset{\sim}{\Pi}_n$-stationary. For more information about $\underset{\sim}{\Pi}_n$-Mahlo cardinals see [2] and [4]. The next fact shows that Σ_n-w.c. cardinals are $\underset{\sim}{\Pi}_n$-Mahlo, and that the least $\underset{\sim}{\Pi}_n$-Mahlo cardinal is not Σ_n-w.c.

FACT 2.4. *Every Σ_n-w.c. cardinal κ is $\underset{\sim}{\Pi}_n$-Mahlo, and the set of $\underset{\sim}{\Pi}_n$-Mahlo cardinals below κ is $\underset{\sim}{\Pi}_n$-stationary.*

Proof. Suppose that κ is Σ_n-w.c. Let C be a $\underset{\sim}{\Pi}_n$-club of κ, i.e., C is a club on κ which is definable over V_κ by means of a Π_n formula with parameters. Let Φ the Π^1_1 sentence expressing that κ is inaccessible. Let ϱ be the first-order sentence expressing that C is unbounded. Then

$$\langle V_\kappa, \in, C\rangle \models \Phi \wedge \varrho.$$

So, there is $\alpha < \kappa$ such that

$$\langle V_\alpha, \in, C \cap V_\alpha\rangle \models \Phi \wedge \varrho.$$

Therefore α is inaccessible, and since $C \cap V_\alpha = C \cap \alpha$ is unbounded in α, $\alpha \in C$.

Note that "every $\utilde{\Pi}_n$-club of κ contains an inaccessible cardinal" is expressible by a first-order sentence. Therefore, the above argument shows that there is a $\utilde{\Pi}_n$-stationary set of $\utilde{\Pi}_n$-Mahlo cardinals below κ. ■

Recall κ is a Σ_ω-*Mahlo cardinal* iff it is $\utilde{\Pi}_n$-Mahlo for every $n \in \omega$. In [4] it is shown that every Σ_ω-w.c. cardinal is $\utilde{\Sigma}_\omega$-Mahlo, and that the set of Σ_ω-Mahlo cardinals below a Σ_ω-w.c. cardinal is Σ_ω-stationary. However, also from [4], if κ is Π_{n+1}-Mahlo, then the set of Σ_n-w.c. cardinals below κ is Π_{n+1}-stationary.

2.1.1. *The tree property*

Definition 2.5. Let κ be a cardinal and $n \in \omega$. A tree $T = \langle T, \leq_T \rangle$ with $T \subseteq V_\kappa$ is a Σ_n-*tree* (over V_κ) iff there are Σ_n formulas $\varphi_T(x)$, $\varphi_{\leq_T}(x,y)$ and $\varphi_{\mathrm{ht}_T}(x,y)$, possibly with parameters in V_κ, such that for every $t, t' \in V_\kappa$ and every $\alpha < \kappa$,

$$\begin{aligned} t \in T &\quad \text{iff} \quad V_\kappa \models \varphi_T(t), \\ t \leq_T t' &\quad \text{iff} \quad V_\kappa \models \varphi_{\leq_T}(t, t'), \\ t \in T_\alpha &\quad \text{iff} \quad V_\kappa \models \varphi_{\mathrm{ht}_T}(t, \alpha), \end{aligned}$$

where T_α denotes the αth level of the tree T. Similarly, we define the notion of Π_n-*tree* by substituting Π_n for Σ_n in the above definition. Moreover, T is a Δ_n-*tree* iff T is both a Σ_n-tree and a Π_n-tree. Finally, T is a Σ_ω-*tree* iff T is a Σ_n-tree for some $n \in \omega$.

Definition 2.6. Let κ be a cardinal and $n \in \omega$. κ has the Σ_n-*tree property* iff κ is inaccessible and every κ-tree which is a Σ_n-tree has a cofinal branch. The Π_n-*tree property*, Δ_n-*tree property*, and Σ_ω-*tree property* are defined analogously.

Lemma 2.7. *For every $n \in \omega$, if κ is Σ_n-w.c., then κ has the Σ_n-tree property.*

Proof. Suppose that κ is a Σ_n-w.c. cardinal and let T be a κ-tree which is a Σ_n-tree over V_κ. Suppose that T does not have a branch of length κ. So, since κ is regular, every branch of T belongs to V_κ.

Let Φ be the Π^1_1 sentence expressing that κ is inaccessible.

Let Ψ be the following Π^1_1 sentence:

$$\forall B \ (B \text{ is a branch of } T \rightarrow \exists x \ B = x).$$

Let F be the function with domain κ such that $F(\alpha) = T_\alpha$, the αth level of T. Since $t \in T_\alpha$ is a Σ_n fact over V_κ, F is Δ_{n+1}-definable over V_κ. Let φ be the following first-order sentence:

$$\forall \alpha \ (\alpha \text{ is an ordinal} \rightarrow \exists x \ F(\alpha) = x).$$

Thus,

$$\langle V_\kappa, \in, T, F\rangle \models \Phi \wedge \Psi \wedge \varphi.$$

Hence, there is $\lambda < \kappa$ such that

$$\langle V_\lambda, \in, T \cap V_\lambda, F \cap V_\lambda\rangle \models \Phi \wedge \Psi \wedge \varphi.$$

Fix some $t \in T_\lambda$. Let $\text{pred}(t) = \{t' \in T : t' <_T t\}$. It is clear that $\text{pred}(t)$ is a branch through $T \cap V_\lambda$. So, $\text{pred}(t) \in V_\lambda$, and hence, since λ is inaccessible, $|\text{pred}(t)| < \lambda$. A contradiction. ∎

COROLLARY 2.8. *If κ is Σ_ω-w.c., then κ has the Σ_ω-tree property.*

2.1.2. *The partition property.* Recall that if κ is a cardinal and $n > 0$ is a natural number, $[\kappa]^n$ is the set of all subsets of κ with exactly n elements.

Given a cardinal κ, natural numbers n, m ($n > 0$), and a function $f : [\kappa]^n \to m$, a set $H \subseteq \kappa$ is said to be *f-homogeneous* iff $f``[H]^n = \{i\}$ for some $i \in m$.

DEFINITION 2.9. Let κ be a cardinal. Then κ has the *Σ_n-partition property* iff κ is an inaccessible cardinal and for every function $f : [\kappa]^2 \to \{0,1\}$ that is Σ_n-definable over V_κ there exists an f-homogeneous set of cardinality κ. We write $\kappa \xrightarrow{\Sigma_n} (\kappa)^2$ to indicate that κ has the Σ_n-partition property. The *Σ_ω-partition property* is defined analogously, and we write $\kappa \xrightarrow{\Sigma_\omega} (\kappa)^2$.

LEMMA 2.10. *For every $n \in \omega$, $n > 0$, if κ has the Σ_n-tree property, then $\kappa \xrightarrow{\Sigma_n} (\kappa)^2$.*

Proof. Let $F : [\kappa]^2 \to \{0,1\}$ be Σ_n-definable over V_κ. Let $\varphi(x, y, z)$ be a Σ_n formula, possibly with parameters in V_κ, that defines it.

For every $\beta < \kappa$, let $f_\beta : \beta \to \{0,1\}$ be such that for all $\alpha < \beta$, $f_\beta(\alpha) = F(\{\alpha, \beta\})$. Let $T = \{f_\beta \restriction \gamma : \gamma \leq \beta < \kappa\}$ be ordered by extension. Note that T is Σ_n-definable over V_κ:

$$t \in T \text{ iff } V_\kappa \models \exists \beta, \gamma(\gamma \leq \beta \wedge \text{dom}(t) = \gamma \wedge (\forall \alpha < \gamma)(\exists i \in \{0,1\})(\varphi(\alpha, \beta, i))).$$

It is clear that for every $\beta < \kappa$, we have: $t \in T_\beta$ iff $t \in T$ and $\text{dom}(t) = \beta$. So, T is a Σ_n-tree. Moreover, $\text{ht}(T) = \kappa$, and since for every $\beta < \kappa$, $T_\beta \subseteq 2^\beta$, and κ is inaccessible, $|T_\beta| < \kappa$. Therefore T is a κ-tree.

Since κ has the Σ_n-tree property, there is a cofinal branch B through T. Let $\{t_\xi : \xi < \kappa\}$ be an increasing enumeration of B so that $\text{dom}(t_\xi) = \xi$ for all $\xi < \kappa$. For every $i \in \{0,1\}$, let

$$H_i = \{\xi < \kappa : t_\xi{}^\frown\langle \xi, i\rangle \in B\}.$$

We claim that for every $i \in \{0,1\}$, H_i is a homogeneous subset of κ for F. Fix $\alpha, \beta, \gamma \in H_i$ with $\alpha < \beta < \gamma$. Since $t_\alpha{}^\frown\langle \alpha, i\rangle \subseteq t_\beta$ and $t_\beta{}^\frown\langle \beta, i\rangle \subseteq t_\gamma$,

$$F(\{\alpha, \beta\}) = t_\beta(\alpha) = i = t_\gamma(\beta) = F(\{\beta, \gamma\}).$$

So, the H_i are homogeneous for $i \in \{0,1\}$. Since $|B| = \kappa$, either $|H_0| = \kappa$ or $|H_1| = \kappa$. Therefore, $\kappa \xrightarrow{\Sigma_n} (\kappa)^2$. ∎

Corollary 2.11. *If κ has the Σ_ω-tree property, then $\kappa \xrightarrow{\Sigma_\omega} (\kappa)^2$.*

Lemma 2.12 (E. Kranakis, [8]). *Assume $V = L$. For every $n > 0$, $\kappa \xrightarrow{\Sigma_n} (\kappa)^2$ implies that for every Π^1_1 formula $\Phi(x_0, \ldots, x_k)$ and $a_0, \ldots, a_k \in L_\kappa$ such that $L_\kappa \models \Phi(a_0, \ldots, a_k)$, there is $\lambda < \kappa$ with $L_\lambda \preccurlyeq_n L_\kappa$ such that $L_\kappa \models \Phi(a_0, \ldots, a_k)$.*

Finally, we have:

Theorem 2.13. $(V = L)$ *Let κ be a cardinal. Then for every $n \geq 1$ the following are equivalent*:

(1) *κ is a Σ_n-w.c. cardinal.*
(2) *κ has the Σ_n-tree property.*
(3) *$\kappa \xrightarrow{\Sigma_n} (\kappa)^2$.*

Proof. (1)⇒(2) follows from Lemma 2.7.
(2)⇒(3) follows from Lemma 2.10.

Since $L \models \kappa \xrightarrow{\Sigma_n} (\kappa)^2$, by definition, κ is inaccessible in L. The rest of implication (3)⇒(1) follows from Lemma 2.12 (this is the only place where $V = L$ is used) and Proposition 2.3. ∎

Corollary 2.14. $(V = L)$ *Let κ be a cardinal. Then the following are equivalent*:

(1) *κ is Σ_ω-w.c.*
(2) *κ has the Σ_ω-tree property.*
(3) *$\kappa \xrightarrow{\Sigma_\omega} (\kappa)^2$.*

2.2. *Generic absoluteness for projective ccc posets*

Definition 2.15. $L(\mathbb{R})^M$ is a Σ_n-*w.c.* (resp. Σ_ω-*w.c.*) *Solovay model over $V \subseteq M$* iff M satisfies:

(1) For every $x \in \mathbb{R}$, ω_1 is a Σ_n-w.c. (resp. Σ_ω-w.c.) cardinal in $V[x]$.
(2) Every $x \in \mathbb{R}$ is small-generic over V.

Notice that since every Σ_n-w.c. (resp. Σ_ω-w.c.) cardinal is inaccessible, Lemma 1.1 also holds for Σ_n-w.c. (resp. Σ_ω-w.c.) Solovay models.

We will make use of the following property of Σ_n-w.c. cardinals:

Lemma 2.16. *Let $n \geq 1$. Suppose that κ is a Σ_n-w.c. cardinal and $\mathbb{P}$ is a κ-cc poset that is Σ_n-definable (with parameters) over V_κ. If $X \subseteq \mathbb{P}$ has cardinality less than κ, then there is a complete subposet $\mathbb{Q}$ of $\mathbb{P}$, also of cardinality less than κ, such that $X \subseteq \mathbb{Q}$.*

Proof. Let $X \subseteq \mathbb{P}$ with $|X| < \kappa$. Since κ is inaccessible, there is a cardinal $\lambda < \kappa$ with $X \subseteq V_\lambda$.

Let $R = \{D : D \text{ is a maximal antichain of } \mathbb{P}\}$. Since $\mathbb{P}$ is κ-cc, $R \subseteq V_\kappa$. For all $D \in V_\kappa$, $D \in R$ iff V_κ satisfies:

$$D \subseteq \mathbb{P} \wedge \forall x, y \in D\ (x \neq y \rightarrow x \perp_{\mathbb{P}} y) \wedge \forall z\ (z \in \mathbb{P} \rightarrow \exists y \in D\ (\neg z \perp_{\mathbb{P}} y)).$$

Note that the formula above is the conjunction of a Σ_n formula and a Π_n formula. Hence, R is a Δ_{n+1} predicate in V_κ.

Let Φ be the conjunction of the following sentences of the second-order language of type $\{\in, \mathbb{P}, \leq_{\mathbb{P}}, \perp_{\mathbb{P}}, R\}$:

(1) $\leq_{\mathbb{P}}$ is a partial order with $\text{field}(\leq_{\mathbb{P}}) = \mathbb{P}$.
(2) $\perp_{\mathbb{P}}$ is the incompatibility relation of $\langle \mathbb{P}, \leq_{\mathbb{P}} \rangle$.
(3) $\forall Y\ (Y \subseteq \mathbb{P} \wedge \forall xy\ (Yx \wedge Yy \wedge x \neq y \rightarrow x \perp_{\mathbb{P}} y)$ $\wedge\ \forall z\ (\mathbb{P}z \rightarrow \exists y\ (Yy \wedge \neg y \perp_{\mathbb{P}} z)) \rightarrow \exists x\ (Rx \wedge Y = x))$, i.e, every maximal antichain of $\mathbb{P}$ belongs to R.

Notice that (1) and (2) are first-order, and (3) is Π^1_1.

We have

$$\langle V_\kappa, \in, \mathbb{P}, \leq_{\mathbb{P}}, \perp_{\mathbb{P}}, R \rangle \models \Phi.$$

So, since κ is Σ_n-w.c., there is $\alpha < \kappa$ with $\lambda < \alpha$ such that

$$\langle V_\alpha, \in, \mathbb{P} \cap V_\alpha, \leq_{\mathbb{P}} \cap V_\alpha, \perp_{\mathbb{P}} \cap V_\alpha, R \cap V_\alpha \rangle \models \Phi.$$

Let $\mathbb{Q} = \langle \mathbb{P} \cap V_\alpha, \leq_{\mathbb{P}} \cap V_\alpha, \perp_{\mathbb{P}} \cap V_\alpha \rangle$. So, $|\mathbb{Q}| < \kappa$. By (1) and (2), $\mathbb{Q}$ is a subposet of $\mathbb{P}$ that preserves the incompatibility relation of $\mathbb{P}$. Since $\lambda < \alpha$, we have $X \subseteq \mathbb{P} \cap V_\alpha$. Finally, let D be a maximal antichain of $\mathbb{Q}$. Then, by (3), $D \in R \cap V_\alpha$. So since $D \in R$, it follows that D is a maximal antichain of $\mathbb{P}$. This shows that $\mathbb{Q}$ is a complete subposet of $\mathbb{P}$ of cardinality less than κ which includes X. ■

For α an ordinal, we shall write Coll_α for the Levy collapse below α, instead of the usual and more cumbersome $\text{Coll}(\omega, {<}\alpha)$.

THEOREM 2.17. *Let $n \geq 1$. Suppose $L(\mathbb{R})^M$ is a Σ_n-w.c. Solovay model over V and $\mathbb{P}$ is a ccc poset which is, in M, Σ_n-definable* (*with parameters*) *over $H(\omega_1)$. Then the $L(\mathbb{R})$ of any $\mathbb{P}$-extension of M is also a Σ_n-w.c. Solovay model over V.*

Proof. Let $\kappa = \omega_1^M$. Force over M with Woodin's partial ordering $\mathbb{W}$ (see Lemma 1.1) to obtain a Coll_κ-generic C over V so that $\mathbb{R}^M = \mathbb{R}^{V[C]}$. Notice that for a generic filter $G \subseteq \mathbb{P}$, G is $\mathbb{P}$-generic over M iff it is $\mathbb{P}$-generic over $V[C]$ and, moreover, $\mathbb{R}^{M[G]} = \mathbb{R}^{V[C][G]}$. Thus, to prove the theorem it will be enough to show that every real in $V[C][G]$ is generic over V for some forcing notion $\mathbb{P}$ in V that is countable in $V[C][G]$.

Let $\dot{\mathbb{P}}$ be a Coll_κ-name for $\mathbb{P}$ in V. By the Factor Lemma for the Levy collapse, we may assume that the parameters of the definition of $\mathbb{P}$ are in V. Further, since the Levy collapse is homogeneous, we may assume that $\Vdash_{\mathrm{Coll}_\kappa}$ "$\dot{\mathbb{P}}$ is a poset". Notice that Coll_κ is definable by means of a Σ_1 and a Π_1 formula without parameters over V_κ (see [2]). Hence, for $n \geq 1$, $\mathrm{Coll}_\kappa * \dot{\mathbb{P}}$ is a poset which is Σ_n-definable over V_κ, possibly with parameters.

Let x be a real in $V[C][G]$. Let $\dot{x}$ be a simple $\mathrm{Coll}_\kappa * \dot{\mathbb{P}}$-name for x in V, and let X be the set of all conditions of $\mathrm{Coll}_\kappa * \dot{\mathbb{P}}$ in $TC(\dot{x})$. Since $\mathrm{Coll}_\kappa * \dot{\mathbb{P}}$ is κ-cc, $|X| < \kappa$. So, by Lemma 2.16, there is a complete subposet $\mathbb{Q}$ of $\mathrm{Coll}_\kappa * \dot{\mathbb{P}}$ such that $X \subseteq \mathbb{Q}$ and $\mathbb{Q}$ has cardinality less than κ. Let $H = (C * G) \cap \mathbb{Q}$. Then H is $\mathbb{Q}$-generic over V and $\dot{x}[H] = \dot{x}[C * G] = x$. This completes the proof of the theorem since it shows that x is generic over V for the countable poset $\mathbb{Q}$. ∎

Corollary 2.18.

(1) *For every* $n \geq 1$, Con(ZFC + *there exists a* Σ_n*-w.c. cardinal*) *implies* Con(ZFC + $L(\mathbb{R})$*-absoluteness for* $\undertilde{\Sigma}^1_{n+1}$ *ccc posets*).

(2) Con(ZFC + *there exists a* Σ_ω*-w.c. cardinal*) *implies* Con(ZFC+$L(\mathbb{R})$-*absoluteness for projective ccc posets*).

Proof. (1): Suppose κ is Σ_n-w.c. Force with Coll_κ so that the $L(\mathbb{R})$ of the generic extension M is a Σ_n-w.c. Solovay model over V. By Theorem 2.17 and Lemma 1.3, $L(\mathbb{R})$-absoluteness holds in M for ccc posets that are Σ_n definable, with parameters, in $H(\omega_1)$, and hence, for $\undertilde{\Sigma}^1_{n+1}$ ccc posets. ∎

Recall that for Γ a class of posets, a poset $\mathbb{P}$ is *Γ-productive-ccc* iff it is ccc and for every ccc poset $\mathbb{Q}$ in Γ, $\mathbb{P} \times \mathbb{Q}$ is ccc.

Let Γ_n be the class of all $\undertilde{\Sigma}^1_{n+1}$ ccc posets, and let Γ_ω be the class of all projective ccc forcing notions. Then, as in [2], we can show:

Theorem 2.19.

(1) *If* $L(\mathbb{R})^M$ *is a* Σ_n*-w.c. Solovay model, then in* $L(\mathbb{R})^M$ *every ccc poset is* Γ_n*-productive-ccc.*

(2) *If* $L(\mathbb{R})^M$ *is a* Σ_ω*-w.c. Solovay model, then in* $L(\mathbb{R})^M$ *every ccc poset is* Γ_ω*-productive-ccc.*

Proof. (1): Suppose $L(\mathbb{R})^M$ is a Σ_n-w.c. Solovay model over V, and in $L(\mathbb{R})^M$, $\mathbb{P}$ is a ccc poset and $\mathbb{Q}$ is a poset in the class Γ_n.

It is known (see [7]) that there is a ccc poset $\mathbb{Q}^*$ in Γ_n such that $\mathbb{Q}$ completely embeds into $\mathbb{Q}^*$, and if G is $\mathbb{Q}^*$-generic over some model M, then $M[G]$ is of the form $M[g]$ for some real g.

Let $\mathbb{Q}^*$ be as above, and suppose τ is a $\mathbb{Q}^*$-name for an uncountable antichain of $\mathbb{P}$, $\tau \in L(\mathbb{R})^M$. Let $\varphi_{\mathbb{P}}(x)$, $\varphi_{\leq_{\mathbb{P}}}(x,y)$ and $\varphi_{\perp_{\mathbb{P}}}(x,y)$ be formulas with only reals and ordinals as parameters that define, respectively, $\mathbb{P}$,

$\leq_{\mathbb{P}}$, and $\perp_{\mathbb{P}}$ in $L(\mathbb{R})^M$, and let $\varphi_{\mathbb{Q}^*}(x)$, $\varphi_{\leq_{\mathbb{Q}^*}}(x,y)$, and $\varphi_{\perp_{\mathbb{Q}^*}}(x,y)$ be Σ^1_{n+1} formulas with real parameters that define, respectively, $\mathbb{Q}^*$, $\leq_{\mathbb{Q}^*}$, and $\perp_{\mathbb{Q}^*}$. Thus, there is a formula $\varphi(x,y)$ with only reals and ordinals as parameters such that the following holds in $L(\mathbb{R})^M$:

(i) For all p,a, if $\varphi(p,a)$, then $\varphi_{\mathbb{Q}^*}(p)$ and $\varphi_{\mathbb{P}}(a)$.
(ii) For all p,q,a,b, if $\varphi(p,a)$, $\varphi(q,b)$, and not $\varphi_{\perp_{\mathbb{Q}^*}}(p,q)$, then $\varphi_{\perp_{\mathbb{P}}}(a,b)$.
(iii) For all p,a, $\varphi(p,a)$ iff $\langle p,\check{a}\rangle\in\tau$.

Suppose G is $\mathbb{Q}^*$-generic over $L(\mathbb{R})^M$. So, G is also generic over M. Let N be the $L(\mathbb{R})$ of $L(\mathbb{R})^M[G]$. Clearly, since $M[G]$ and $L(\mathbb{R})^M[G]$ have the same reals, $N=L(\mathbb{R})^{M[G]}$. Thus, by Lemma 1.3 and Theorem 2.17, (i) and (ii) above hold in N. Since G is easily coded by a real, $G\in N$. In N, let $A=\{a:\exists p\in G\ \varphi(p,a)\}$. Notice that, by (iii) above, $\tau[G]\subseteq A$, and so A is an uncountable set in N. Also, for every $a\in A$, $N\models\varphi_{\mathbb{P}}(a)$. Let $\mathbb{P}^N$ and $\leq_{\mathbb{P}}^N$ be the sets defined in N by the formulas $\varphi_{\mathbb{P}}(x)$ and $\varphi_{\leq_{\mathbb{P}}}(x,y)$, respectively. Then $N\models$ "$\langle\mathbb{P}^N,\leq_{\mathbb{P}}{}^N\rangle$ is a ccc poset". So, since

$$N\models \text{“}A \text{ is an uncountable subset of } \mathbb{P}^N\text{”},$$

we have

$$N\models \text{“}\exists p,q,a,b\ (\varphi(p,a)\wedge\varphi(q,b)\wedge\neg\varphi_{\perp_{\mathbb{Q}^*}}(p,q)\wedge\neg\varphi_{\perp_{\mathbb{P}}}(a,b))\text{”}.$$

Therefore, by 1.3 and 2.17,

$$L(\mathbb{R})^M\models \text{“}\exists p,q,a,b\ (\varphi(p,a)\wedge\varphi(q,b)\wedge\neg\varphi_{\perp_{\mathbb{Q}^*}}(p,q)\wedge\neg\varphi_{\perp_{\mathbb{P}}}(a,b))\text{”},$$

which contradicts (ii) above.

Now suppose H is $\mathbb{Q}$-generic over $L(\mathbb{R})^M$. Let G be $\mathbb{Q}^*$-generic over $L(\mathbb{R})^M$ such that

$$L(\mathbb{R})^M[H]\subseteq L(\mathbb{R})^M[G].$$

Since $\mathbb{P}$ is ccc in $L(\mathbb{R})^M[G]$, it is also ccc in $L(\mathbb{R})^M[H]$. ∎

COROLLARY 2.20. *If $L(\mathbb{R})^M$ is a Σ_n-w.c. Solovay model over V, then in M there are no $\underset{\sim}{\Sigma}^1_{n+1}$ Suslin trees. And if $L(\mathbb{R})^M$ is a Σ_ω-w.c. Solovay model over V, then in M there are no projective Suslin trees.*

Proof. If T is a $\underset{\sim}{\Sigma}^1_{n+1}$ Suslin tree, then $T\times T$ with the product ordering is a $\underset{\sim}{\Sigma}^1_{n+1}$ poset which is not ccc (see [6]). ∎

3. The strength of generic absoluteness under projective ccc forcing notions. In this section we shall prove the following:

THEOREM 3.1. *If Σ^1_4-absoluteness holds for Δ^1_3 ccc forcing notions, then ω_1 is a Σ_1-w.c. cardinal in L.*

Proof. Suppose towards a contradiction that ω_1 is not Σ_1-w.c. in L. We know (see [2]) that ω_1 is inaccessible in L and, in fact, ω_1 is inaccessible to reals, i.e., $\omega_1^{L[x]}$ is countable for every real x. Hence, by Theorem 2.13, there is, in L, an Aronszajn tree $T = \langle T, \leq_T \rangle$ whose nodes are elements of $2^{<\omega_1}$ and which is a Σ_1-tree over L_{ω_1}.

We need the following version of the Silver tree S_T for T (See [5]): For every set M and every $X \subseteq M$, let $H^M(X)$ denote the Skolem hull of X in M. Then the Silver tree S_T for T is defined as follows:

(1) $\langle \alpha, \beta, a \rangle \in S_T$ iff

- (a) $\alpha < \beta < \omega_1$,
- (b) $a \in L_\beta$ is a function with $\alpha \subseteq \mathrm{dom}(a)$,
- (c) $L_\beta = H^{L_\beta}(\alpha \cup \{a\})$,
- (d) $a \restriction \alpha \in T$.

(2) $\langle \alpha, \beta, a \rangle \leq_{S_T} \langle \gamma, \delta, c \rangle$ iff

- (a) $\alpha \leq \gamma$,
- (b) $L_\beta = \mu\text{"} H^{L_\delta}(\alpha \cup \{c\})$, where μ is Mostowski's transitive collapse function, and $\mu(c) = a$.

Note that if $\langle \alpha, \beta, a \rangle \in S_T$, then $\langle \alpha, \beta, a \rangle$ is a node of height α.

Lemma 3.2 (J. H. Silver, see [5]). *S_T is an Aronszajn tree in L such that in any model of* ZFC (*extending L*), *if there is a branch of length ω_1 through S_T, then* $\mathrm{cf}(\omega_1) = \omega$.

An important fact for our purposes is that the complexity of S_T is the same as that of T. That is:

Lemma 3.3. *For all $n \geq 1$, if $T \subseteq 2^{<\omega_1}$ is a Σ_n-tree* (*resp. Π_n-tree*) *over L_{ω_1}, then S_T is also a Σ_n-tree* (*resp. Π_n-tree*) *over L_{ω_1}.*

Proof. Fix some recursive enumeration $\langle \varphi_i : i \in \omega \rangle$ of all formulas of the language of set theory of the form $\exists \overline{x}\ \varphi(\overline{y}, \overline{z}, \overline{x})$, where $\overline{y}, \overline{z}, \overline{x}$ are finite sequences of variables and $\overline{x}$ is non-empty. We use the following notational conventions: given a formula φ_i, we denote by φ_i' the formula resulting from the removal of the first block of existential quantifiers of φ_i. Also, $\exists \overline{y}\ \varphi_i$ denotes the formula resulting by adding the block of existential quantifiers $\exists \overline{y}$ to the formula φ_i. Note that the maps $\varphi_i \mapsto \varphi_i'$ and $\varphi_i \mapsto \exists \overline{y}\ \varphi_i$ are recursive.

If x is an ordered pair, then let $(x)_0$ and $(x)_1$ denote, respectively, the first and second coordinates of x.

For every set $M \in L$, we define the function r^M from $\omega \times M^{<\omega}$ to $M^{<\omega} \times M^{<\omega}$ as follows: for all $i \in \omega$ and every $b \in M^{<\omega}$,

$$r^M(i, b) = \begin{cases} \text{the } <_L \text{-least } a \in M^{<\omega} \times M^{<\omega} \text{ such that} & \\ M \models \varphi_i'((a)_0, b, (a)_1) & \text{if } M \models \exists y\ \varphi_i(b), \\ \langle \emptyset, \emptyset \rangle & \text{if } M \not\models \exists y\ \varphi_i(b). \end{cases}$$

Let Sk^M be the function from $\omega \times M^{<\omega}$ into $M^{<\omega}$ defined by $\mathrm{Sk}^M(i,b) = (r^M(i,b))_0$ for every $i \in \omega$ and $b \in M^{<\omega}$.

CLAIM 3.4. ($V = L$) *For every set M, the functions r^M and Sk^M are Δ_1 with M as a parameter.*

Proof. We only need to show that r^M is Δ_1. Let $\mathrm{Sat}(x,y,z)$ denote the satisfaction relation for sets, i.e., $\mathrm{Sat}(x,y,z)$ iff the set x satisfies the formula y with the sequence z of elements of x. Notice that this is a Δ_1 relation.

For every $i \in \omega$, and every $b \in M^{<\omega}$, $r^M(i,b) = a$ iff

(1) a is an ordered pair, and $(a)_0, (a)_1 \in M^{<\omega}$.
(2) Either $\mathrm{Sat}(M, \exists y\ \varphi_i, b)$ and
  (a) $\mathrm{Sat}(M, \varphi_i', (a)_0^\frown b^\frown (a)_1)$,
  (b) $(\forall c, d \in M)(\mathrm{Sat}(M, \varphi_i', c^\frown b^\frown d) \to a <_L \langle c, d\rangle)$,
(3) or $\neg\mathrm{Sat}(M, \exists \overline{y}\ \varphi_i, b)$ and $(\forall c, d \in M)(a \le_L \langle c, d\rangle)$.

Since $<_L$ is a Δ_1 relation, (1), (2), and (3) can be written as both Σ_n and Π_n sentences. Hence, r^M is a Δ_1 function. ∎

Therefore, the functions $M \mapsto r^M$ and $M \mapsto \mathrm{Sk}^M$ are Δ_1 definable in L without parameters.

CLAIM 3.5. ($V = L$) *For every set M and every $X \subseteq M$, $H^M(X)$ is a Δ_1 definable set with M and X as parameters.*

Proof. Given M and $X \subseteq M$, define a sequence $(H^M(X,n))_{n<\omega}$ recursively by:

$$H^M(X,0) = \mathrm{Sk}^M``(\omega \times X^{<\omega}),$$
$$H^M(X,n+1) = \mathrm{Sk}^M``(\omega \times H^M(X,n)^{<\omega}).$$

Since Sk^M is Δ_1 definable, with M as parameter, the map $n \mapsto H^M(X,n)$ is also Δ_1 definable with parameters M and X. Note that $H^M(X) = \bigcup_{n\in\omega} H^M(X,n)$. Thus, for all a,

$$a \in H^M(X) \quad \text{iff} \quad (\exists n \in \omega)(a \in H^M(x,n)),$$

and so $H^M(X)$ is Δ_1-definable with M and X as parameters. ∎

We continue with the proof of Lemma 3.3. Recall that T is a tree which is definable over L_{ω_1} with Σ_n formulas $\varphi_T(x)$ and $\varphi_{\le_T}(x,y)$, possibly with parameters. Then, for all $\alpha, \beta < \omega_1$ and every $b \in L_{\omega_1}$, $\langle \alpha, \beta, b\rangle \in S_T$ iff L_{ω_1} satisfies:

(1) α and β are ordinals and $\alpha < \beta$.
(2) b is a function such that $(\forall \gamma \in \alpha)(\gamma \in \mathrm{dom}(b))$ and $b \in L_\beta$.
(3) $(\forall x \in L_\beta)(x \in H^{L_\beta}(\alpha \cup \{b\}))$ and $(\forall x \in H^{L_\beta}(\alpha \cup \{b\}))(x \in L_\beta)$.
(4) $\varphi_T(b{\restriction}\alpha)$.

(1) is Δ_0. Since the maps $\beta \mapsto L_\beta$, and $(X, M) \mapsto H^M(X)$ are Δ_1, (2) and (3) are Δ_1. Finally, it is clear that (4) is Σ_n.

Note that μ, the Mostowski collapsing map, is Δ_1. So, for all $\alpha, \beta, \gamma, \delta < \omega_1$ and every $b, d \in L_{\omega_1}$, $\langle \alpha, \beta, b\rangle \leq_{S_T} \langle \gamma, \delta, d\rangle$ iff L_{ω_1} satisfies:

(1) $\langle \alpha, \beta, b\rangle, \langle \gamma, \delta, d\rangle \in S_T$.
(2) $\alpha \leq \gamma$.
(3) $(\forall x \in L_\beta)(x \in \mu(H^{L_\delta}(\alpha \cup \{d\})))$ and $(\forall x \in \mu(H^{L_\delta}(\alpha \cup \{d\}))(x \in L_\beta)$.
(4) $\mu(d) = b$.

(1) is Σ_n in L_{ω_1}, (2) is Δ_0, and (3) and (4) are Δ_1 in L_{ω_1}.

Therefore $\langle S_T, \leq_{S_T}\rangle$ is a tree which is Σ_n-definable over L_{ω_1}.

It only remains to show that the relation $t \in (S_T)_\alpha$ is Σ_n over L_{ω_1}. But this is clear, since $t \in (S_T)_\alpha$ iff $t \in S_T$ and $t_0 = \alpha$. This finishes the proof of Lemma 3.3. ∎

Remark 3.6. Notice that the arguments above show that in L, if $(T, \leq_T)$ is a tree where both T and $\leq_T$ are Σ_n-definable over L_κ and, possibly, the levels of T are not Σ_n-definable over L_κ, where κ is an uncountable cardinal, then S_T is a Σ_n-tree over L_κ. Thus, if $V = L$, then the conclusion of Lemma 2.7 can be strengthened to: every κ-tree that is Σ_n-definable over L_κ has a cofinal branch. Hence, in Theorem 2.13 we can add the following as a further equivalence: κ is inaccessible and every κ-tree that is Σ_n-definable over V_κ has a cofinal branch.

Continuing now with the proof of Theorem 3.1, recall that WO is the Π^1_1 set of elements of the Baire space ω^ω that code well-orderings of ω. If $a \in \mathrm{WO}$, let $\|a\|$ be the order-type of the well-ordering coded by a (see [6]). For $x \subseteq \omega$, let $\overline{x}$ be the element of ω^ω coded by x, via some recursive bijection between $\mathcal{P}(\omega)$ and ω^ω.

Lemma 3.7. *If C is a Coll_{ω_1}-generic filter over V, then there is a function $\pi \in V[C]$ from* WO *into* WO *such that*:

(1) *For every $x \in \mathrm{WO}$, $\pi(x)$ is a code for the ordinal $\|x\|$.*
(2) *For every $x, y \in \mathrm{WO}$, if $\|x\| = \|y\|$, then $\pi(x) = \pi(y)$.*
(3) *π has a Coll_{ω_1}-name that can be coded by a Δ^1_3 subset of ω^ω.*

Proof. Let $\dot{\mathrm{WO}}$ be the set of all simple Coll_{ω_1}-names σ for a subset of ω such that $\Vdash_{\mathrm{Coll}_{\omega_1}}$ "$\overline{\sigma} \in \dot{\mathrm{WO}}$".

Note that, since $\mathrm{Coll}_{\omega_1} \in L$, every Coll_{ω_1}-generic filter over V is also generic over L. So, for every $\gamma < \omega_1$ let τ_γ be the $<_L$-least simple Coll_{ω_1}-name for a subset of ω such that $\Vdash_{\mathrm{Coll}_{\omega_1}}$ "$\|\overline{\tau}_\gamma\| = \check{\gamma}$". Let $B_{\omega_1} = \{\tau_\gamma : \gamma < \omega_1\}$ and let $\dot{B} = \mathrm{Coll}_{\omega_1} \times B_{\omega_1}$.

Define the function π_{ω_1} from $\dot{\mathrm{WO}}$ into B_{ω_1} as follows: for every $\sigma \in \dot{\mathrm{WO}}$, $\pi_{\omega_1}(\sigma) = \tau$ iff

(1) $\tau \in B_{\omega_1}$,
(2) $\Vdash_{\mathrm{Coll}_{\omega_1}}$ "$\|\overline{\sigma}\| = \|\overline{\tau}\|$.

Let $\dot{\pi} = \mathrm{Coll}_{\omega_1} \times \pi_{\omega_1}$.

We can now easily check that if C is Coll_{ω_1}-generic over V, then in $V[C]$, $\pi := \dot{\pi}[C]$ is a function satisfying: if $\pi(a) = b$, then $\|\overline{a}\| = \|\overline{b}\|$ and b is the unique code in $\dot{B}[C]$ coding the ordinal $\|\overline{a}\|$. Thus π satisfies (1) and (2) of the lemma, modulo a recursive coding of elements of the Baire space ω^ω by subsets of ω.

To prove (3) we need to compute the complexity of the sets and names involved in the definition of π.

First observe that Coll_{ω_1} is a Δ^1_2 poset (see [2]).

Let WO^* be the set of codes of elements of $\dot{\mathrm{WO}}$. Then WO^* is a Δ^1_2 set of reals (cf. [1]).

CLAIM 3.8. *Let B^* be the set of all codes of elements of B_{ω_1}. Then B^* is a Δ^1_3 set of reals.*

Proof. Let $<^*_L$ be the following relation: for every $x, y \in \omega^\omega$, $x <^*_L y$ iff x, y code simple Coll_{ω_1}-names in L for subsets of ω and the name coded by x is $<_L$-less than the name coded by y. Since every simple Coll_{ω_1}-name for a subset of ω is hereditarily countable, the predicate "x codes a simple Coll_{ω_1}-name in L for a subset of ω" is Σ_1 in $H(\omega_1)$. Hence, as $<_L$ is also Σ_1 over $H(\omega_1)$, $<^*_L$ is a Σ^1_2 relation.

Recall that B_{ω_1} is the range of a function that assigns to each $\gamma < \omega_1$ the $<_L$-least Coll_{ω_1}-name for a subset of ω that is forced by Coll_{ω_1} to be a code for γ. Thus, $x \in B^*$ iff

(1) x codes a simple Coll_{ω_1}-name in L for a subset of ω and $\Vdash_{\mathrm{Coll}_{\omega_1}}$ "$x \in \dot{\mathrm{WO}}$",
(2) for every w, if w codes a simple Coll_{ω_1}-name for a subset of ω, and $w <^*_L x$, then $\nVdash_{\mathrm{Coll}_{\omega_1}}$ "$\|\overline{w}\| = \|\overline{x}\|$".

Since (1) is a Σ^1_2 sentence and (2) is Π^1_2, B^* is a Δ^1_3 set.

Let π^* be the relation given by: $\pi^*(x, y)$ iff x and y code simple Coll_{ω_1}-names σ and τ, respectively, for subsets of ω, and $\pi_{\omega_1}(\sigma) = \tau$.

We will finish the proof of (3) of Lemma 3.7 by showing that π^* is a Δ^1_3 relation.

Let $S(v, x, y)$ iff v codes a condition $p \in \mathrm{Coll}_{\omega_1}$, x and y code simple Coll_{ω_1}-names σ and τ, respectively, for subsets of ω, and $p \Vdash_{\mathrm{Coll}_{\omega_1}}$ "$\|\overline{\sigma}\| = \|\overline{\tau}\|$". Since the relation $\|\overline{\sigma}\| = \|\overline{\tau}\|$ is Σ^1_1, and Coll_{ω_1} is a Δ^1_2 ccc poset, S is a Δ^1_2 relation.

So, for every $x, y \in \omega^\omega$, $\pi^*(x, y)$ iff

(1) $x \in \mathrm{WO}^*$,

(2) $y \in B^*$,
(3) $\forall v\ S(v, x, y)$.

Since (1) is Δ^1_2, (2) is Δ^1_3 and (3) is Π^1_2, we see that π^* is Δ^1_3. This concludes the proof of Lemma 3.7. ∎

Recall that WF denotes the Π^1_1 set of all reals that code a well-founded relation on ω (see [6]). Every set in $H(\omega_1)$ can be coded by some $x \in \mathrm{WF}$ as follows: $x \in \omega^\omega$ codes $a \in H(\omega_1)$ iff $\langle \omega, E_x\rangle \cong \langle \mathrm{TC}(a), \in\rangle$, where for $n, m \in \omega$, nE_xm iff $x(J(n,m)) = 0$, where J is some recursive one-to-one pairing function from $\omega \times \omega$ onto ω. Moreover, every $x \in \mathrm{WF}$ codes one and only one set in $H(\omega_1)$. So, given $x \in \mathrm{WF}$, denote by $[x]$ the set coded by x. Note that the map $x \mapsto [x]$ is Δ_1 over $H(\omega_1)$. Let $[x] \sim [y]$ iff $x \notin \mathrm{WF}$ or $y \notin \mathrm{WF}$ or $\langle \omega, E_x\rangle \cong \langle \omega, E_y\rangle$. Thus, $[x] \sim [y]$ is a Σ^1_1 relation on the reals. Hence, we may code every function $f \in H(\omega_1)$ by a real so that the set F of all such codes is a Δ^1_2 set of reals: for every $x \in \omega^\omega$, $x \in F$ iff

(1) x codes $\langle x_n : n \in \omega\rangle$,
(2) $\forall n\ (x_n$ codes $\langle x^0_n, x^1_n\rangle \wedge x^0_n,\ x^1_n \in \mathrm{WF})$,
(3) $\forall n, m\ ([x^0_n] \sim [x^0_m] \to [x^1_n] \sim [x^1_m])$.

Back to the proof of Theorem 3.1, recall that we have a tree T whose nodes are functions in $2^{<\omega_1}$ and which is Σ_1-definable in L_{ω_1}. By Lemma 3.3, S_T is also Σ_1-definable in L_{ω_1}. And by Lemma 3.2, S_T is still an Aronszajn tree in V, and in any generic extension of V that preserves ω_1. Force with Coll_{ω_1} over V. In the generic extension $V[C]$, and using the function π from Lemma 3.7, we may code the nodes of S_T by reals to obtain an isomorphic tree S^*_T on the reals. Namely: for all $x, y, z \in \omega^\omega$, $\langle x, y, z\rangle \in S^*_T$ iff

(1) $x, y \in \mathrm{WO}$,
(2) $\pi(x) = x \wedge \pi(y) = y$,
(3) $\exists f\ (\langle \|x\|, \|y\|, f\rangle \in S_T \wedge z$ codes the $<_L$-least Coll_{ω_1}-name σ for a real such that $\sigma[C]$ codes $f)$.

Thus, S^*_T is Σ_1-definable in $H(\omega_1)$ with π and C as additional predicates.

We will now define a version of the specializing forcing of Harrington–Shelah ([5]) which will code, using S^*_T, any given ω_1-sequence of reals into a single real. So, let X be a fixed sequence of reals of length ω_1, and let X_α denote the αth element of X.

Let the forcing notion $\mathbb{P}(S^*_T, X)$ be defined as follows:

- $q \in \mathbb{P}(S^*_T, X)$ iff q is a finite function from S^*_T into $\mathbb{Q}$ such that
  (1) $(\forall s, t \in \mathrm{dom}(q))(s <_{S^*_T} t \to q(s) < q(t))$,
  (2) $(\forall s = \langle x, y, z\rangle \in \mathrm{dom}(q))((z$ codes $\sigma\ \wedge\ \sigma[C]$ codes $f\ \wedge$ $\mathrm{dom}(f) = \omega \cdot \alpha\ \wedge\ q(s) \in \omega) \to q(s) \in X_\alpha)$.
- $q \le q'$ iff $q' \subseteq q$.

It is clear that $\mathbb{P}(S_T^*, X)$ is Σ_1-definable in $H(\omega_1)$ with π, C, and X as additional predicates. And as in [5] one can show that $\mathbb{P}(S_T^*, X)$ has the property K, i.e., every uncountable subset contains an uncountable subset of pairwise compatible conditions. Hence it is ccc. Forcing with $\mathbb{P}(S_T^*, X)$ adds an order-preserving and continuous function $F_X : S_T^* \to \mathbb{Q}$, with the property that for every $n \in \omega$, $n \in X_\alpha$ iff $F(t) = n$ for some $t \in S_T^*$ of height $\omega \cdot \alpha$. Moreover, F_X specializes S_T^*, i.e., for every $a \in \mathbb{Q}$, $F_X^{-1}(a)$ is an antichain of S_T^*.

Now let $X^0 = \text{range}(\pi) = \{x \in \omega^\omega : \exists y\ (y \in \text{WO} \wedge \pi(y) = x)\}$, ordered by $x \leq_{X^0} x'$ iff $x, x' \in X_0 \wedge \|x\| \leq \|x'\|$. Clearly, $(X^0, \leq_{X^0})$ is a well-ordering of reals of order-type ω_1. By using some fixed recursive coding of elements of ω^ω by subsets of ω, we may assume that $X^0_\alpha \in \mathcal{P}(\omega)$ for all $\alpha < \omega_1$.

We next describe a finite-support iteration of length ω, Δ_2-definable over $H(\omega_1)$, with π, C, and X^0 as additional predicates. Let $\mathbb{P}_0 = \mathbb{P}(S_T^*, X^0)$. Given $\mathbb{P}_n$, which is Δ_2-definable over $H(\omega_1)$, with π, C, and X^0 as additional predicates, we define $\mathbb{P}_{n+1}$:

For $\beta < \omega_1$, let $(S_T^*)_{<\beta}$ denote the set of nodes of S_T^* of height $< \beta$. Notice that the predicate $x \in (S_T^*)_{<\beta}$ is Σ_1 in the parameter β over $H(\omega_1)$. Let $\dot{F}_{X^n}$ be the $\mathbb{P}_n$-name for the generic specializing function F_{X^n}. Thus,

$$\dot{F}_{X^n} \restriction (S_T^*)_{<\omega\cdot(\alpha+1)} = \{\langle p, \langle t, r\rangle\rangle : p \in \mathbb{P}_n,\ \langle t, r\rangle \in p,\ t \in (S_T^*)_{<\omega\cdot(\alpha+1)}\}.$$

Since $\mathbb{P}_n$ is Δ_2-definable over $H(\omega_1)$, with π, C, and X^0 as additional predicates, so is the set displayed above, with α as a parameter. Let $\dot{X}^{n+1}$ be a $\mathbb{P}_n$-name for a code for $\dot{F}_{X^n}$. i.e., $\dot{X}^{n+1} = \langle \dot{X}^{n+1}_\alpha : \alpha < \omega_1\rangle$, where for every $\alpha < \omega_1$,

$$\Vdash_{\mathbb{P}_n} \text{“}\dot{X}^{n+1}_\alpha \subseteq \omega \text{ codes } \dot{F}_{X^n} \restriction (S_T^*)_{<\omega\cdot(\alpha+1)}\text{”}.$$

So, $\mathbb{P}_n$ forces that $\dot{X}^{n+1}_\alpha$ codes $\langle x, \dot{y}\rangle$, where $x = \langle x_k : k \in \omega\rangle$ codes $(S_T^*)_{<\omega\cdot(\alpha+1)}$, $\dot{y} = \langle \dot{y}_k : k \in \omega\rangle$, and $\dot{y}_k = \{\langle p, r\rangle : \langle x_k, r\rangle \in p\}$. Notice that the sentence “x codes $(S_T^*)_{<\omega\cdot(\alpha+1)}$” is Δ_2.

Now let $\langle p, \dot{q}\rangle \in \mathbb{P}_{n+1}$ iff $p \in \mathbb{P}_n$ and $p \Vdash_{\mathbb{P}_n}$ “$\dot{q} \in \mathbb{P}(S_T^*, \dot{X}^{n+1})$”. Let us check that $\mathbb{P}_{n+1}$ is Δ_2-definable over $H(\omega_1)$, with π, C, and X^0 as additional predicates.

First notice that the predicate “$N(\dot{q})$ iff $\dot{q}$ is a $\mathbb{P}_n$-name for a finite function from S_T^* into $\mathbb{Q}$” is Δ_2. Indeed, $N(\dot{q})$ iff $\dot{q}$ is a finite set of triples $\langle q, s, r\rangle$, where $q \in \mathbb{P}_n$, $s \in S_T^*$, and $r \in \mathbb{Q}$, and for every $\langle q_0, s_0, r_0\rangle, \langle q_1, s_1, r_1\rangle \in \dot{q}$, if $s_0 = s_1$ and $r_0 \neq r_1$, then $q_0 \perp q_1$.

Thus, we have: $p \Vdash_{\mathbb{P}_n}$ “$\dot{q} \in \mathbb{P}(S_T^*, \dot{X}^{n+1})$” iff $p \in \mathbb{P}_n$, $N(\dot{q})$, and

(1) $\forall \langle q_0, s_0, r_0\rangle, \langle q_1, s_1, r_1\rangle \in \dot{q}(s_0 <_{S_T^*} s_1 \wedge r_1 \geq r_0 \to q_0 \perp q_1)$,
(2) $\forall \langle q_0, s_0, r_0\rangle \in \dot{q}\ (s_0 = \langle x, y, z\rangle \wedge z$ codes $\sigma \wedge \sigma[C]$ codes $f \wedge$ $\text{dom}(f) = \omega\cdot\alpha \wedge p \leq q_0 \wedge r_0 \in \omega \to q_0 \Vdash_{\mathbb{P}_n}$ “$r_0 \in \dot{X}^{n+1}_\alpha$”$)$.

But $q_0 \Vdash_{\mathbb{P}_n}$ "$r_0 \in \dot{X}^{n+1}_\alpha$" iff $r_0 = \langle k, r\rangle$ and there exists $q_1 \leq q_0$ such that $\langle x_k, r\rangle \in q_1$, where $x = \langle x_k : k \in \omega\rangle$ is the code for $(S^*_T)_{<\omega\cdot(\alpha+1)}$.

This shows that $\mathbb{P}_{n+1}$ is also Δ_2 over $H(\omega_1)$, with π, C, and X^0 as additional predicates.

Let $\mathbb{P}$ be the direct limit of the iteration $\langle \mathbb{P}_n : n < \omega\rangle$. Since the support of the iteration is finite, it is easily seen that $\mathbb{P}$ is Δ_2-definable over $H(\omega_1)$ with π, C, and X^0 as additional predicates (see Lemma 4.1 below). Moreover, every $\mathbb{P}$-generic filter G over $V[C]$ adds a real c such that $X^0 \in L[c]$ (see [5]), and so $V[C][G] \models$ "$\exists x$ ($L[x]$ has uncountably many reals)".

It is interesting to observe that $\mathbb{P}$ (and, in fact, $\mathbb{P}(S^*_T, X)$) is not projective in $V[C]$, as there are no uncountable projective sequences of reals in $V[C]$. However, we claim that the two-step iteration $\mathrm{Coll}_{\omega_1} * \mathbb{P}$ is Δ^1_3.

It will be enough to show that the relation $R(x, y)$ given by:

$$\text{“}x \in \mathrm{Coll}_{\omega_1},\ y \text{ is a } \mathrm{Coll}_{\omega_1}\text{-name for a real, and } x \Vdash_{\mathrm{Coll}_{\omega_1}} y \in \dot{\mathbb{P}}\text{”}$$

is Δ_2 in $H(\omega_1)$, without parameters.

But since Coll_{ω_1} is a Δ^1_2 forcing notion, it will be enough to see that the formula "$x \Vdash_{\mathrm{Coll}_{\omega_1}} y \in \dot{\mathbb{P}}$" is equivalent both to a Σ_2 and a Π_2 formula in $H(\omega_1)$. For this, it is sufficient to show that the formula $y \in \dot{\mathbb{P}}$ is equivalent both to a Σ_2 and a Π_2 formula in $H(\omega_1)$. This is clearly so in the Coll_{ω_1}-name for π as a parameter. But since by Lemma 3.7, π has a Coll_{ω_1}-name that is Δ_2-definable in $H(\omega_1)$ without parameters, we are done.

Since "$\exists x$ ($L[x]$ has uncountably many reals)" is a Σ^1_4 sentence, and it holds in a $\mathrm{Coll}_{\omega_1} * \mathbb{P}$-generic extension of V, by Σ^1_4-absoluteness for Δ^1_3 ccc posets, it holds in V. Therefore, there exists a real $x \in V$ such that $\omega_1^{L[x]} = \omega_1$, contradicting the fact that ω_1 is inaccessible to reals. This finishes the proof of 3.1. ∎

Theorem 3.1 can be easily generalized:

COROLLARY 3.9. *Let $n \geq 2$. If $\undertilde{\Sigma}^1_4$ absoluteness holds for $\undertilde{\Sigma}^1_{n+1}$ ccc forcing notions, then ω_1 is a Σ_n-w.c. cardinal in L.*

Proof. As in Theorem 3.1, if ω_1 is not a Σ_n-w.c. cardinal in L, then there exists an Aronszajn tree T on $2^{<\omega_1}$ which is a Σ_n-tree over L_{ω_1}. As in Lemmas 3.2 and 3.3, we can find S_T, a version of the Silver tree for T, which is an Aronszajn tree definable over L_{ω_1} and has the same complexity as T. Using S_T, we may define the poset $\mathbb{P}$ as in Theorem 3.1 in such a way that $\mathrm{Coll}_{\omega_1} * \mathbb{P}$ is a $\undertilde{\Sigma}^1_{n+1}$ and ccc poset that adds a real x such that $\omega_1 = \omega_1^{L[x]}$, yielding a contradiction. ∎

We finish with two corollaries that summarize our results:

COROLLARY 3.10. *For every $n \geq 2$, the following are equiconsistent*:

(1) *$L(\mathbb{R})$-absoluteness under $\underset{\sim}{\Sigma}^1_{n+1}$ ccc posets.*
(2) *There exists a Σ_n-w.c. cardinal.*

COROLLARY 3.11. *The following are equiconsistent*:

(1) *$L(\mathbb{R})$-absoluteness under projective ccc posets.*
(2) *There exists a Σ_ω-w.c. cardinal.*

4. On iterations of projective ccc posets. We will show that after the Levy collapse of a Σ_n-w.c. cardinal, the property of $L(\mathbb{R})$ being a Σ_n-w.c. Solovay model is preserved under finite-support iterations of $\underset{\sim}{\Sigma}^1_{n+1}$ ccc forcing notions.

Recall that if $\mathbb{P}$ is a forcing notion, a *simple* $\mathbb{P}$-name for a real, i.e., for a function from ω to ω, is a set τ of triples $\langle p, m, n\rangle$ such that $p \in \mathbb{P}$, $n, m \in \omega$, and for every m, the set of all p such that $\langle p, m, n\rangle \in \tau$ for some $n \in \omega$, is a maximal antichain of $\mathbb{P}$.

Observe that if $\mathbb{P}$ is ccc and its conditions are real numbers, then for every simple $\mathbb{P}$-name τ for a real, $|\mathrm{TC}(\tau)|$ is countable. Further, if $\mathbb{P}$ is a finite-support iteration of ccc forcing notions whose conditions are reals, then it can be easily shown, by induction on the length of the iteration, that every simple $\mathbb{P}$-name for a real has countable transitive closure.

LEMMA 4.1. *Let $n \geq 1$. Suppose $L(\mathbb{R})^M$ is a Σ_n-w.c. Solovay model over V and $\mathbb{P} \in M$ is the direct limit of an iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \lambda\rangle$ of countable length and with finite support such that for every $\alpha < \lambda$,*

$$\Vdash_{\mathbb{P}_\alpha} \text{“}\dot{\mathbb{Q}}_\alpha \textit{ is a } \underset{\sim}{\Sigma}^1_{n+1} \textit{ ccc poset”}.$$

Then the $L(\mathbb{R})$ of any $\mathbb{P}$-extension of M is also a Σ_n-w.c. Solovay model over V.

Proof. Let $\kappa = \omega_1^M$. Force over M to obtain a Coll_κ-generic C over V with $\mathbb{R}^M = \mathbb{R}^{V[C]}$ (see Lemma 1.1).

In M, for each $\alpha < \lambda$, fix a simple $\mathbb{P}_\alpha$-name τ_α for a real that codes the parameters in some fixed Σ^1_{n+1} definition of $\dot{\mathbb{Q}}_\alpha$.

Since the iteration is of countable length and ccc, all the τ_α, $\alpha < \lambda$, belong to $V[C]$ and $\mathbb{P} = \mathbb{P}^{V[C]}$, where $\mathbb{P}^{V[C]}$ is the iteration in $V[C]$ defined in the same way as $\mathbb{P}$ is defined in M. Moreover, a filter $G \subseteq \mathbb{P}$ is $\mathbb{P}$-generic over M iff it is $\mathbb{P}$-generic over $V[C]$, and $\mathbb{R}^{M[G]} = \mathbb{R}^{V[C][G]}$. Thus, it is enough to show that for every real x in $V[C][G]$ and every $X \subseteq \mathrm{Coll}_\kappa * \dot{\mathbb{P}}$ of size less than κ there is a complete subposet $\mathbb{Q}$ of $\mathrm{Coll}_\kappa * \dot{\mathbb{P}}$ such that $\mathbb{Q}$ is countable in $V[C][G]$, $X \subseteq \mathbb{Q}$ and x is $\mathbb{Q}$-generic over V.

We proceed by induction on λ. So we assume that for every $\alpha < \lambda$ and every $X \subseteq \mathrm{Coll}_\kappa * \mathbb{P}_\alpha$ of size less than κ, there is a complete subposet $\mathbb{Q}$ of $\mathrm{Coll}_\kappa * \mathbb{P}_\alpha$, also of size less than κ, such that $X \subseteq \mathbb{Q}$.

We may assume that λ is a limit ordinal, since the successor case follows directly from the proof of Theorem 2.17.

Now fix a subset X of $\mathrm{Coll}_\kappa * \mathbb{P}$ of size less than κ, and fix a real x in $V[C][G]$. Let $\dot{x} \in V$ be a simple $\mathrm{Coll}_\kappa * \dot{\mathbb{P}}$-name for x, and let $Y = \mathrm{Coll}_\kappa * \dot{\mathbb{P}} \cap \mathrm{TC}(\dot{x})$. Since $\mathrm{Coll}_\kappa * \dot{\mathbb{P}}$ is κ-cc, Y has cardinality less than κ. Let $Z = X \cup Y$.

For every $\alpha < \lambda$, let $Z_\alpha = Z \cap \mathrm{Coll}_\kappa * \mathbb{P}_\alpha$. By inductive hypothesis, we can find a $\subseteq$-increasing chain $\langle \mathbb{Q}_\alpha : \alpha < \lambda \rangle$ such that $\mathbb{Q}_\alpha$ is a complete subposet of $\mathrm{Coll}_\kappa * \mathbb{P}_\alpha$, hence also a complete subposet of $\mathrm{Coll}_\kappa * \mathbb{P}$, such that $Z_\alpha \subseteq \mathbb{Q}_\alpha$ for all $\alpha < \lambda$. Let $\mathbb{Q} = \bigcup_{\alpha<\lambda} \mathbb{Q}_\alpha$. Since the iteration has finite support, $\mathbb{Q}$ is a complete subposet of $\mathrm{Coll}_\kappa * \mathbb{P}$. Moreover, $\mathbb{Q}$ has size less than κ and $Z \subseteq \mathbb{Q}$. Furthermore, letting $H = C * G \cap \mathbb{Q}$, we have $\dot{x}[H] = \dot{x}[C * G] = x$, and so x is $\mathbb{Q}$-generic over V. ■

For conciseness, in what follows we will use the notation $\mathbb{P} \lessdot \mathbb{Q}$ to express that $\mathbb{P}$ is a complete subposet of $\mathbb{Q}$.

THEOREM 4.2. *Let κ be a Σ_n-w.c. cardinal, $n \geq 1$, and let $\lambda > 0$. Suppose that $\mathbb{P} = \mathbb{P}_\lambda \in V$ is the direct limit of an iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \lambda \rangle$ with finite support such that $\mathbb{P}_0 = \mathrm{Coll}_\kappa$ and for every $\alpha < \lambda$,*

$$\Vdash_{\mathbb{P}_\alpha} \text{“}\dot{\mathbb{Q}}_\alpha \textit{ is a } \underset{\sim}{\Sigma}^1_{n+1} \textit{ ccc poset”}.$$

Then the $L(\mathbb{R})$ of any $\mathbb{P}$-generic extension of V is a Σ_n-w.c. Solovay model over V.

Proof. Suppose G is a $\mathbb{P}$-generic filter over V. Notice that $\omega_1^{V[G]} = \kappa$, and so $\omega_1^{V[G]}$ is a Σ_n-w.c. cardinal in V. We only need to prove that every real in $V[G]$ is small-generic over V, for then it will clearly follow that for every real x in $V[G]$, $\omega_1^{V[G]}$ is a Σ_n-w.c. cardinal in $V[x]$.

The proof is by induction on λ. So, suppose that for every $\beta < \lambda$, writing $\mathbb{P}_\beta$ for the iteration up to β and letting $G_\beta = G \cap \mathbb{P}_\beta$, we find that $L(\mathbb{R})^{V[G_\beta]}$ is a Σ_n-w.c. Solovay model over V.

Let $\mathbb{P}^1 = \langle \mathbb{P}^1_\alpha, \dot{\mathbb{Q}}^1_\alpha : \alpha < \lambda \rangle \in V[G_0]$ be the remaining part of the iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \lambda \rangle$, i.e., $\mathbb{P}^1_0 = \dot{\mathbb{Q}}_0[G_0]$, $\mathbb{P}^1_{n+1} = \mathbb{P}^1_n * \dot{\mathbb{Q}}_{n+1}$ for $n < \omega$, and $\mathbb{P}^1_{\alpha+1} = \mathbb{P}^1_\alpha * \dot{\mathbb{Q}}_\alpha$ for $\alpha \geq \omega$. We may assume that for every α,

$$\Vdash_{\mathbb{P}^1_\alpha} \text{“}\dot{\mathbb{Q}}^1_\alpha \text{ has a largest element } \mathbf{1}\text{”},$$

and $\mathbf{1}$ is some fixed real that does not depend on α. Moreover, we may assume that for every $p \in \mathbb{P}^1$ and every $\alpha < \lambda$, $p(\alpha)$ is a simple $\mathbb{P}^1_\alpha$-name for a real.

In $V[G_0]$, for each $\alpha < \lambda$, $\alpha > 0$, fix a simple $\mathbb{P}^1_\alpha$-name τ_α for a real that codes the parameters in a fixed Σ^1_{n+1} definition of $\dot{\mathbb{Q}}^1_\alpha$, so that for some Σ^1_{n+1}

formulas $\varphi_\alpha(x,y)$, $\psi_\alpha(x,y,z)$, and $\theta_\alpha(x,y,z)$,

$$\Vdash_{\mathbb{P}_\alpha} \text{“}\dot{\mathbb{Q}}_\alpha = \{x : \varphi_\alpha(x,\tau_\alpha)\}\text{''},$$
$$\Vdash_{\mathbb{P}_\alpha} \text{“}\leq_{\dot{\mathbb{Q}}^1_\alpha} = \{\langle x,y\rangle : \psi_\alpha(x,y,\tau_\alpha)\}\text{”},$$
$$\Vdash_{\mathbb{P}_\alpha} \text{“}\perp_{\dot{\mathbb{Q}}^1_\alpha} = \{\langle x,y\rangle : \theta_\alpha(x,y,\tau_\alpha)\}\text{”}.$$

Let x be a real in $V[G]$ and let $\dot{x} \in V[G_0]$ be a simple $\mathbb{P}^1$-name for x.

Work in $V[G_0]$. Since $\mathbb{P}^1$ is ccc, $|\mathrm{TC}(\dot{x})|$ is countable. Let μ be a large enough regular cardinal, and let $N \preccurlyeq H(\mu)$ be such that:

(1) $\mathbb{P}^1, \langle \tau_\alpha : \alpha < \lambda\rangle, \dot{x} \in N$,
(2) $\mathrm{TC}(\dot{x}) \subseteq N$,
(3) $|N| = \aleph_0$.

Notice that if $\alpha \in \mathrm{OR} \cap N$, then $\tau_\alpha \in N$, and since $|\mathrm{TC}(\tau_\alpha)|$ is countable, $\mathrm{TC}(\tau_\alpha) \subseteq N$.

Now let $\mathbb{P}^*$ be the direct limit of the finite-support iteration $\langle \mathbb{P}^*_\alpha, \dot{\mathbb{Q}}^*_\alpha : \alpha < \lambda\rangle$ defined as follows: $\mathbb{P}^*_0 = \mathbb{P}^1_0$, and $\Vdash_{\mathbb{P}^*_\alpha}$ “$\dot{\mathbb{Q}}^*_\alpha = \{x : \varphi_\alpha(x,\tau_\alpha)\}$” if $\alpha \in \mathrm{OR} \cap N$, and $\Vdash_{\mathbb{P}^*_\alpha}$ “$\dot{\mathbb{Q}}^*_\alpha = \{\mathbf{1}\}$” otherwise, i.e., $\dot{\mathbb{Q}}^*_\alpha$ is the trivial poset.

We need to check that the iteration is well-defined, i.e., if $\Vdash_{\mathbb{P}^*_\alpha}$ “$\dot{\mathbb{Q}}^*_\alpha = \{x : \varphi_\alpha(x,\tau_\alpha)\}$”, then τ_α is a $\mathbb{P}^*_\alpha$-name. We will show much more:

Claim 4.3.

(1) *If* $p \in \mathbb{P}^*_\alpha$, *then* $p \in \mathbb{P}^1_\alpha$. *And if* $p \in N$, *then the converse also holds.*
(2) *If* σ *is a simple* $\mathbb{P}^*_\alpha$*-name for a real*, *then it is also a simple* $\mathbb{P}^1_\alpha$*-name for a real. And if* $\sigma \in N$, *then the converse also holds.*
(3) *If* $p \in \mathbb{P}^*_\alpha$ *and* $\sigma, \sigma', \tau_\alpha$ *are simple* $\mathbb{P}^*_\alpha$*-names for reals*, *then*:
   (a) *If* $p \Vdash_{\mathbb{P}^*_\alpha} \varphi_\alpha(\sigma,\tau_\alpha)$, *then* $p \Vdash_{\mathbb{P}^1_\alpha} \varphi_\alpha(\sigma,\tau_\alpha)$.
   (b) *If* $p \Vdash_{\mathbb{P}^*_\alpha} \psi_\alpha(\sigma,\sigma',\tau_\alpha)$, *then* $p \Vdash_{\mathbb{P}^1_\alpha} \psi_\alpha(\sigma,\sigma',\tau_\alpha)$.
   (c) *If* $p \Vdash_{\mathbb{P}^*_\alpha} \theta_\alpha(\sigma,\sigma',\tau_\alpha)$, *then* $p \Vdash_{\mathbb{P}^1_\alpha} \theta_\alpha(\sigma,\sigma',\tau_\alpha)$.

   And if $\alpha, p, \sigma, \sigma' \in N$, *then the converses of* (a), (b), *and* (c) *also hold.*
(4) $\mathbb{P}^*_\alpha \lessdot \mathbb{P}^1_\alpha$.

Proof. By induction on α. For $\alpha = 0$ it is clear. So, let $\alpha = \beta + 1$.

(1) Fix $p \in \mathbb{P}^*_\alpha$. Then $p = \langle p{\restriction}\beta, \sigma'\rangle$, where $p{\restriction}\beta \in \mathbb{P}^*_\beta$, σ' is a simple $\mathbb{P}^*_\beta$-name, and either $p{\restriction}\beta \Vdash_{\mathbb{P}^*_\beta}$ “$\sigma' = \mathbf{1}$”, or $p{\restriction}\beta \Vdash_{\mathbb{P}^*_\beta} \varphi_\beta(\sigma',\tau_\beta)$. So, by induction hypothesis on (1), (2), and (3)(a), we deduce that $p{\restriction}\beta \in \mathbb{P}^1_\beta$, σ' is a simple $\mathbb{P}^1_\beta$-name, and either $p{\restriction}\beta \Vdash_{\mathbb{P}^1_\beta}$ “$\sigma' = \mathbf{1}$”, or $p{\restriction}\beta \Vdash_{\mathbb{P}^1_\beta} \varphi_\beta(\sigma',\tau_\beta)$. This shows that $p \in \mathbb{P}^1_\alpha$.

Fix now $p = \langle p{\restriction}\beta, \sigma'\rangle \in \mathbb{P}^1_\alpha \cap N$. Thus, $p{\restriction}\beta \in \mathbb{P}^1_\beta$, σ' is a simple $\mathbb{P}^1_\beta$-name, and $p{\restriction}\beta \Vdash_{\mathbb{P}^1_\beta} \varphi_\beta(\sigma',\tau_\beta)$. Since $p \in N$, we also know that $p{\restriction}\beta, \sigma' \in N$. So, again by induction hypothesis on (1), (2), and (3)(a), we infer that $p\restriction\beta \in \mathbb{P}^*_\beta$, σ' is a simple $\mathbb{P}^*_\beta$-name, and $p\restriction\beta \Vdash_{\mathbb{P}^*_\beta} \varphi_\beta(\sigma', \tau_\beta)$, which shows that $p \in \mathbb{P}^*_\alpha$.

(2) Now suppose that σ is a simple $\mathbb{P}^*_\alpha$-name for a real. If $q \in \mathbb{P}^*_\alpha \cap \mathrm{TC}(\sigma)$, we can conclude as before in the case of p that $q \in \mathbb{P}^1_\alpha$. This implies that σ is a simple $\mathbb{P}^1_\alpha$-name.

If σ is a simple $\mathbb{P}^1_\alpha$-name for a real and $\sigma \in N$, then $\mathrm{TC}(\sigma) \subseteq N$. So, if $q \in \mathbb{P}^1_\alpha \cap \mathrm{TC}(\sigma)$, we can conclude as before in the case of p that $q \in \mathbb{P}^*_\alpha$. This implies that σ is a simple $\mathbb{P}^*_\alpha$-name. In particular, if $\alpha \in N$, then τ_α is a $\mathbb{P}^*_\alpha$-name.

(3) Suppose now that $p \in \mathbb{P}^*_\alpha$, σ, τ_α are simple $\mathbb{P}^*_\alpha$-names for reals, and $p \Vdash_{\mathbb{P}^*_\alpha} \varphi_\alpha(\sigma, \tau_\alpha)$. We have already shown that $p \in \mathbb{P}^1_\alpha$ and σ is a simple $\mathbb{P}^1_\alpha$-name. Since by the induction hypothesis of the theorem, $L(\mathbb{R})^{V[G_0]^{\mathbb{P}^*_\beta}}$ and $L(\mathbb{R})^{V[G_0]^{\mathbb{P}^1_\beta}}$ are both Σ_n-w.c. Solovay models over V, with the same ω_1, and since, by induction hypothesis on (4), $\mathbb{P}^*_\beta \lessdot \mathbb{P}^1_\beta$, we also have $\mathbb{R}^{V[G_0]^{\mathbb{P}^*_\beta}} \subseteq \mathbb{R}^{V[G_0]^{\mathbb{P}^1_\beta}}$. So, by Lemma 1.3, there exists a canonical embedding from $L(\mathbb{R})^{V[G_0]^{\mathbb{P}^*_\beta}}$ into $L(\mathbb{R})^{V[G_0]^{\mathbb{P}^1_\beta}}$. We claim that $p \Vdash_{\mathbb{P}^1_\alpha} \varphi_\alpha(\sigma, \tau_\alpha)$. Indeed, suppose $G^1_\alpha = G^1_\beta * \dot{H}$ is $\mathbb{P}^1_\alpha$-generic over $V[G_0]$, with $p = p\restriction\beta * \dot{q} \in G^1_\alpha$. Since $p \Vdash_{\mathbb{P}^*_\alpha} \varphi_\alpha(\sigma, \tau_\alpha)$, and $\mathbb{P}^*_\beta \lessdot \mathbb{P}^1_\beta$, we deduce that $G^*_\beta := G^1_\beta \cap \mathbb{P}^*_\beta$ is $\mathbb{P}^*_\beta$-generic over $V[G_0]$ with $p\restriction\beta \in G^*_\beta$. Hence,

$$V[G_0][G^*_\beta] \models \text{“}i_{G^*_\beta}(\dot{q}) \Vdash_{\mathbb{Q}^*_\beta} \varphi_\alpha(\sigma, \tau_\alpha)\text{”}.$$

Since we have $i_{G^1_\beta}(\dot{q}) = i_{G^*_\beta}(\dot{q})$, by the canonical elementary embedding of $L(\mathbb{R})^{V[G_0][G^*_\beta]}$ into $L(\mathbb{R})^{V[G_0][G^1_\beta]}$, we obtain

$$V[G_0][G^1_\beta] \models \text{“}i_{G^1_\beta}(\dot{q}) \Vdash_{\mathbb{Q}^1_\beta} \varphi_\alpha(\sigma, \tau_\alpha)\text{”}.$$

Hence, $V[G_0][G^1_\alpha] \models \varphi_\alpha(\sigma, \tau_\alpha)$. This proves (a), and similar arguments prove (b) and (c).

Suppose now that $\alpha, p, \sigma \in N$, and $p \Vdash_{\mathbb{P}^1_\alpha} \varphi_\alpha(\sigma, \tau_\alpha)$. We have already shown that $p \in \mathbb{P}^*_\alpha$ and σ, τ_α are $\mathbb{P}^*_\alpha$-names. To see that $p \Vdash_{\mathbb{P}^*_\alpha} \varphi_\alpha(\sigma, \tau_\alpha)$, suppose $G^*_\alpha = G^*_\beta * \dot{H}$ is $\mathbb{P}^*_\alpha$-generic over $V[G_0]$ with $p = p\restriction\beta * \dot{q} \in G^*_\alpha$. Since $\mathbb{P}^*_\beta \lessdot \mathbb{P}^1_\beta$, we can extend G^*_β to a $\mathbb{P}^1_\beta$-generic filter G^1_β over $V[G_0]$ such that

$$V[G_0][G^1_\beta] \models \text{“}i_{G^1_\beta}(\dot{q}) \Vdash_{\mathbb{Q}^1_\beta} \varphi_\alpha(\sigma, \tau_\alpha)\text{”}.$$

Since $i_{G^1_\beta}(\dot{q}) = i_{G^*_\beta}(\dot{q})$ and since $\beta \in N$, by the canonical elementary embedding we have

$$V[G_0][G^*_\beta] \models \text{“}i_{G^*_\beta}(\dot{q}) \Vdash_{\mathbb{Q}^*_\beta} \varphi_\alpha(\sigma, \tau_\alpha)\text{”}.$$

Hence, $V[G_0][G^*_\alpha] \models \varphi_\alpha(\sigma, \tau_\alpha)$. This proves the converse of (a), and similar arguments prove the converses of (b) and (c).

(4) Finally, suppose $\mathbb{P}^*_\beta \lessdot \mathbb{P}^1_\beta$. By (3), $\mathbb{P}^*_\alpha$ is a subposet of $\mathbb{P}^1_\alpha$ and the incompatibility relation is preserved. Now suppose $A \in V[G_0]$ is a maximal antichain of $\mathbb{P}^*_\alpha$. Then $A{\restriction}\beta := \{p{\restriction}\beta : p \in A\}$ is a maximal antichain of $\mathbb{P}^*_\beta$ and, by induction hypothesis, it is also a maximal antichain of $\mathbb{P}^1_\beta$. If $\beta \notin N$, then clearly A is maximal in $\mathbb{P}^1_\alpha$. So, suppose $\beta \in N$. Then every $p \in A$ is of the form $\langle p{\restriction}\beta, \sigma\rangle$, where $p{\restriction}\beta \Vdash_{\mathbb{Q}^*_\beta} \varphi_\beta(\sigma, \tau_\beta)$. Let $A(\beta) := \{p(\beta) : p \in A\}$. Then $\Vdash_{\mathbb{P}^*_\beta}$ "$A(\beta)$ is a maximal antichain of $\dot{\mathbb{Q}}^*_\beta$". Notice that, since $\Vdash_{\mathbb{P}^*_\beta}$ "$A(\beta)$ is countable", $A(\beta) \in L(\mathbb{R})^{V[G_0]^{\mathbb{P}^*_\beta}}$. Thus, by the canonical embedding from $L(\mathbb{R})^{V[G_0]^{\mathbb{P}^*_\beta}}$ into $L(\mathbb{R})^{V[G_0]^{\mathbb{P}^1_\beta}}$, we conclude that $\Vdash_{\mathbb{P}^1_\beta}$ "$A(\beta)$ is a maximal antichain of $\dot{\mathbb{Q}}^1_\beta$".

If α is a limit ordinal, then the claim follows by induction, using the fact that the iterations have finite support. This finishes the proof of the claim. ∎

Since the iterations have finite support, it follows from the claim above that $\mathbb{P}^* \lessdot \mathbb{P}$. Moreover, since $\dot{x} \in N$, $\dot{x}$ is a $\mathbb{P}^*$-name. Notice that $\mathbb{P}^*$ is a ccc iteration.

Let $\overline{\mathbb{P}} = \langle \overline{\mathbb{P}}_\beta, \dot{\overline{\mathbb{Q}}}_\beta : \beta < \mathrm{ot}(\mathrm{On} \cap N)\rangle$ be the iteration consisting of all non-trivial iterands of $\mathbb{P}^*$, i.e., $\overline{\mathbb{P}}_0 = \mathbb{P}^*_0$ and for every $\beta < \mathrm{ot}(\mathrm{On} \cap N)$, $\Vdash_{\overline{\mathbb{P}}_\beta}$ "$\dot{\overline{\mathbb{Q}}}_\beta = \{x : \varphi_\alpha(x, \tau_\alpha)\}$", where $\alpha \in N$ and $\beta = \mathrm{ot}(\alpha \cap N)$. For each $p \in \mathbb{P}^*$, let $\overline{p} \in \overline{\mathbb{P}}$ be the result of deleting the coordinates of p that correspond to the trivial iterands of $\mathbb{P}^*$. Clearly, the map $e : \overline{p} \mapsto p$ is a dense complete embedding of $\overline{\mathbb{P}}$ into $\mathbb{P}^*$. Notice that $\dot{x}$ is a $\overline{\mathbb{P}}$-name.

Recall that G is $\mathbb{P}$-generic over V, and x is a real in $V[G]$. Let us write G as $G_0 * G^1$, where G_0 is $\mathbb{P}_0$-generic over V and G^1 is $\mathbb{P}^1$-generic over $V[G_0]$. Then $\dot{x}$ is a $\mathbb{P}^1$-name in $V[G_0]$ and $i_{G^1}(\dot{x}) = x$. Let $g = e^{-1}[G^1 \cap \mathbb{P}^*]$. Then g is $\overline{\mathbb{P}}$-generic over $V[G_0]$ and $i_g(\dot{x}) = x$. This shows that x belongs to a countable finite-support iteration over $V[G_0]$ of $\utilde{\Sigma}^1_{n+1}$ ccc forcing notions. So, by Lemma 4.1, x is small-generic over V. This proves the theorem. ∎

COROLLARY 4.4. *Suppose that $L(\mathbb{R})^M$ is a Σ_ω-w.c. Solovay model over V and $\mathbb{P} \in M$ is the direct limit of an iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \lambda\rangle$ with finite support such that for every $\alpha < \lambda$,*

$$\Vdash_\alpha \text{"}\dot{\mathbb{Q}}_\alpha \text{ is a projective ccc poset"}.$$

Then the $L(\mathbb{R})$ of any $\mathbb{P}$-generic extension of M is also a Σ_ω-w.c. Solovay model over V.

4.1. *Two applications to Martin's Axiom for projective posets.* The first application will show, modulo the consistency of definable weakly compact cardinals, that Martin's Axiom restricted to posets in a given projective point-class does not imply Martin's Axiom for posets in higher point-classes.

Definition 4.5. Let Γ be a class of posets. *Martin's Axiom for* Γ, henceforth denoted by $\mathrm{MA}(\Gamma)$, is the following statement:

For every ccc poset $\mathbb{P} \in \Gamma$ *and for every family* $\langle A_i : i < \kappa \rangle$, $\kappa < 2^{\aleph_0}$, *of maximal antichains of* $\mathbb{P}$, *there exists* $G \subseteq \mathbb{P}$ *directed such that for every* $i < \kappa$, $G \cap A_i \neq \emptyset$.

For every $n \geq 1$, $\mathrm{MA}(\utilde{\Sigma}^1_n)$ is *Martin's Axiom for* $\utilde{\Sigma}^1_n$ *posets.* MA(Proj) is *Martin's Axiom for projective posets.*

Theorem 4.6. *Let* $n \geq 1$, *and suppose that there exists a* Σ_n*-w.c. cardinal in* L. *Then there exists a poset* $\mathbb{P}$ *such that for every* $\mathbb{P}$*-generic filter* G *over* L,

$$L[G] \models \mathrm{MA}(\utilde{\Sigma}^1_{n+1}) \wedge \neg\mathrm{MA}(\Sigma^1_{n+2}).$$

Proof. Let κ be the least Σ_n-w.c. cardinal in L. Let $\mathbb{P}$ be the direct limit of an iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \kappa^+ \rangle$, with finite support, where $\mathbb{P}_0 = \mathrm{Coll}_\kappa$ and for every $\alpha < \kappa^+$,

$$\Vdash_\alpha \text{“}\dot{Q}_\alpha \text{ is a } \utilde{\Sigma}^1_{n+1} \text{ ccc forcing notion”},$$

so that for every $\mathbb{P}$-generic filter G over L,

$$L[G] \models \mathrm{MA}(\utilde{\Sigma}^1_{n+1}) \wedge 2^{\aleph_0} = \aleph_2$$

(see [1, Theorem 3.10]).

Now assume, towards a contradiction, that

$$L[G] \models \mathrm{MA}(\Sigma^1_{n+2}).$$

Then, since $\omega_1^{L[G]} = \kappa$ is not a Σ_{n+1}-w.c. cardinal in L, there is, in L, a κ-Aronszajn tree T which is Σ_{n+1}-definable over L_κ. As in the proof of Theorem 3.1 we may define a Σ^1_{n+2} ccc poset of the form $\mathrm{Coll}_{\omega_1} * \mathbb{P}$ such that $\mathrm{MA}(\mathrm{Coll}_{\omega_1} * \mathbb{P})$ implies that there exists a real x such that $\omega_1^{L[G]} = \omega_1^{L[x]}$. But then $L(\mathbb{R})^{L[G]}$ is not a Σ_n-w.c. Solovay model over V, in contradiction with Theorem 4.2. ∎

Corollary 4.7. *Let* $n \geq 1$ *and suppose that the existence of a* Σ_n*-w.c. cardinal is consistent with* ZFC. *Then* $\mathrm{ZFC} + \mathrm{MA}(\utilde{\Sigma}^1_{n+1})$ *does not imply* $\mathrm{MA}(\Sigma^1_{n+2})$.

It is known that if ZFC is consistent, then $\mathrm{ZFC}+\mathrm{MA}(\utilde{\Sigma}^1_1)$ does not imply $\mathrm{MA}(\Sigma^1_2)$ (see [1, Section 5]).

For the second application, let φ be the statement “Every set of reals in $L(\mathbb{R})$ is Lebesgue measurable, has the property of Baire, is Ramsey, and has the perfect set property”.

THEOREM 4.8. *Let $n \geq 1$, and suppose that there exists a Σ_n-w.c. cardinal. Then there exists a poset $\mathbb{P}$ such that for every $\mathbb{P}$-generic filter G over V,*

$$V[G] \models \mathrm{MA}(\undertilde{\Sigma}^1_{n+1}) \wedge \neg\mathrm{CH} + \varphi.$$

Proof. Let κ be a Σ_n-w.c. cardinal, and let $\mathbb{P}$ be the direct limit of a finite-support iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \kappa^+ \rangle$, where $\mathbb{P}_0 = \mathrm{Coll}_\kappa$ and for every $\alpha < \kappa^+$,

$$\Vdash_\alpha \text{“}\dot{Q}_\alpha \text{ is a } \undertilde{\Sigma}^1_{n+1} \text{ ccc forcing notion”},$$

so that for every $\mathbb{P}$-generic filter G over V,

$$V[G] \models \mathrm{MA}(\undertilde{\Sigma}^1_{n+1}) \wedge 2^{\aleph_0} = \aleph_2$$

(see [1, Theorem 3.10]). By Theorem 4.2, $L(\mathbb{R})^{V[G]}$ is a Σ_n-w.c. Solovay model over V. Thus,

$$V[G] \models \varphi. \ \blacksquare$$

COROLLARY 4.9.

(1) *For every $n \geq 1$,* Con(ZFC + *there exists a Σ_n-w.c. cardinal*) *implies* Con(ZFC + MA($\undertilde{\Sigma}^1_{n+1}$) + ¬CH + φ).
(2) Con(ZFC + *there exists a Σ_ω-w.c. cardinal*) *implies* Con(ZFC + MA(Proj) + ¬CH + φ).

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Received 5 November 2005;
in revised form 21 February 2007