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Lusin sequences under CH and under Martin's Axiom

by

Uri Abraham (Beer-Sheva) and **Saharon Shelah** (Jerusalem)

Abstract. Assuming the continuum hypothesis there is an inseparable sequence of length ω_1 that contains no Lusin subsequence, while if Martin's Axiom and $\neg$CH are assumed then every inseparable sequence (of length ω_1) is a union of countably many Lusin subsequences.

1. Introduction. We first fix some notations and definitions. The set of natural numbers is denoted by ω, and for $A, B \subseteq \omega$ we write $A \subseteq^* B$ iff $A \setminus B$ is finite, and $A \perp B$ iff $A \cap B$ is finite (almost inclusion, almost disjointness). Let $\mathcal{A} = \langle A_\zeta \mid \zeta \in \omega_1 \rangle$ be a sequence of pairwise almost disjoint, infinite subsets of ω. So $A_\zeta \subset \omega$ and $A_{\zeta_1} \perp A_{\zeta_2}$ for $\zeta_1 \neq \zeta_2$. We say that $B \subseteq \omega$ *separates* $\mathcal{A}$ if $\{\xi \in \omega_1 \mid A_\xi \subseteq^* B\}$ and $\{\xi \in \omega_1 \mid A_\xi \subseteq^* \omega \setminus B\}$ are both uncountable. If no B separates $\mathcal{A}$ then $\mathcal{A}$ is said to be *inseparable*. That is, $\mathcal{A}$ is inseparable if it is an almost disjoint family of infinite subsets of ω such that there is no $B \subset \omega$ for which

$$(\exists^{\aleph_1} A \in \mathcal{A})(A \subseteq^* B) \;\&\; (\exists^{\aleph_1} A \in \mathcal{A})(A \subseteq^* \omega \setminus B).$$

An inseparable family of size $\aleph_1$ can be constructed in ZFC alone (Lusin [1], cited by [2]). We say that $\mathcal{A}$ is a *Lusin* sequence if for every $i < \omega_1$ and $n \in \omega$,

$$\{j < i \mid A_i \cap A_j \subseteq n\} \text{ is finite.}$$

A seemingly stronger property is the following. We say that $\mathcal{A}$ is a *Lusin** family if for every $i < \omega_1$ and $n \in \omega$,

$$\{j < i \mid |A_i \cap A_j| < n\}$$

is finite.

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It is not difficult to prove that every Lusin sequence is inseparable, and Lusin constructed a Lusin sequence in ZFC. Is this the only way to build inseparable families? The answer depends on set-theoretical assumptions as the following two results show (obtained by the first and second author respectively).

THEOREM 1.1. (1) *CH implies that there is an inseparable family which contains no Lusin subsequence.* (2) *"Martin's Axiom + ¬CH" implies that every inseparable sequence is a countable union of Lusin* sequences.*

2. Proofs

2.1. *CH gives an inseparable non-Lusin sequence.* Assume the continuum hypothesis (CH) throughout this subsection. We shall define a sequence $\mathcal{A} = \langle A_\alpha \mid \alpha \in \omega_1 \rangle$ of almost disjoint subsets of ω which is inseparable by virtue of the following property **P**.

For every infinite $X \subseteq \omega$ one of the following three possibilities holds:

P1 X is finitely covered by $\mathcal{A}$ (which means that for some finite set $u \subset \omega_1$, $X \subseteq^* \bigcup\{A_\alpha \mid \alpha \in u\}$).

P2 $\omega \setminus X$ is finitely covered by $\mathcal{A}$.

P3 For some $\alpha_0 < \omega_1$ for all $\alpha_0 \leq \alpha < \omega_1$, X splits A_α (which means that both $X \cap A_\alpha$ and $A_\alpha \setminus X$ are infinite).

It is quite obvious that if $\mathcal{A}$ has this property then it is inseparable, and so we describe the construction, assuming CH, of a sequence that has property **P**, but does not contain any Lusin subsequence.

Let $\langle X_\xi \mid \xi \in \omega_1 \rangle$ be an enumeration of all infinite subsets of ω, and let $\langle e_i \mid i \in \omega_1 \rangle$ be an enumeration of all countable subsets of ω_1 of order-type a limit ordinal. The sequence $\mathcal{A} = \langle A_\alpha \mid \alpha \in \omega_1 \rangle$ is defined by induction on α. First $\langle A_i \mid i \in \omega \rangle$ are defined as some almost disjoint family of infinite subsets of ω. Each A_α, for $\alpha \geq \omega$, is required to satisfy the following three conditions.

C1 $A_\alpha \subseteq \omega$ is infinite and $A_\beta \perp A_\alpha$ for all $\beta < \alpha$.

C2 For every $\xi < \alpha$ one of the following possibilities holds:

- **p1** X_ξ is finitely covered by $\langle A_\beta \mid \beta < \alpha \rangle$, or
- **p2** $\omega \setminus X_\xi$ is finitely covered by $\langle A_\beta \mid \beta < \alpha \rangle$, or else
- **p3** X_ξ splits A_α (i.e. both $X_\xi \cap A_\alpha$ and $A_\alpha \setminus X_\xi$ are infinite).

C3 For every $i < \alpha$ such that $e_i \subseteq \alpha$ there are two possibilities:

1. For some $m \in \omega$, $A_\alpha \cap A_\xi \subseteq m + 1$ for an infinite number of indices $\xi \in e_i$. (This is the "good" possibility.)

2. For some $m \in A_\alpha$ there is $\xi_0 < \sup(e_i)$ such that for every ξ with $\xi_0 < \xi \in e_i$,

$$\min(A_\xi \setminus m+1) < \min(A_\alpha \setminus m+1).$$

Or, equivalently, if n is the first member of A_α above m then $A_\xi \cap (m,n) \neq \emptyset$.

If we succeed then **C1** and **C2** clearly imply that $\mathcal{A}$ is pairwise almost disjoint and inseparable. We are going to show that **C3** implies that $\mathcal{A}$ contains no Lusin subsequences. Suppose that $L = \langle A_i \mid i \in I \rangle$ is a subsequence of $\mathcal{A}$, where $I \subseteq \omega_1$ is uncountable. We want to find some $\alpha \in I$ and $m \in \omega$ for which $\{\xi \in I \cap \alpha \mid A_\alpha \cap A_\xi \subseteq m\}$ is infinite.

Consider the structure on $\omega \cup I$ with predicates for ω, I, $\in$, and the binary relation $m \in A_i$ (for $m \in \omega$, $i \in I$). Let $e \subseteq I$ be the universe of a countable elementary substructure. Then $e = e_i$ for some $i \in \omega_1$. Let $\alpha \in I$ be any ordinal such that $\alpha > i$ and $\alpha > \sup(e)$. We want to prove that possibility **C3**(1) holds for α. This shows that L is not a Lusin sequence. Suppose instead that **C3**(2) holds. Then there are $m \in A_\alpha$ and $\xi_0 < \sup(e)$ as in **C3**(2). Namely, if n is the first member of A_α above m then

$$A_\xi \cap (m,n) \neq \emptyset \tag{1}$$

for every $\xi > \xi_0$ in e. However, since e_i is an elementary substructure, we actually have (1) for every $\xi_0 < \xi$ in I. But this is clearly impossible for $\xi = \alpha$ itself!

Having shown the usefulness of the three conditions **C1**–**C3**, we now return to the inductive construction. At the αth stage of this construction, to construct A_α, it is convenient to define a poset $P = (P, \leq)$ and a countable collection of dense subsets of P, and then to define a filter $G \subseteq P$ such that G intersects each of the dense sets in the countable collection. With this we shall define $A_\alpha = \bigcup\{a \mid (\exists E)((a,E) \in G)\}$, and A_α will satisfy all three conditions because of the choice of the dense sets. In this fashion one does not have to over-specify the construction.

A condition $p = (a, E) \in P$ consists of:

1. A finite set $a \subseteq \omega$ (which will grow to become A_α).
2. A finite set $E \subseteq \alpha$ (p promises that $A_\beta \cap A_\alpha = A_\beta \cap a$ for all $\beta \in E$).

Following the tradition that $p_1 \leq p_2$ means that p_2 gives more information than p_1, the partial order on P is defined by

$$(a_1, E_1) \geq (a_0, E_0) \quad \text{iff} \quad E_0 \subseteq E_1 \text{ and } a_1 \text{ is an end-extension of } a_0 \text{ such that, for every } \xi \in E_0,\ (a_1 \setminus a_0) \cap A_\xi = \emptyset.$$

We say that an end-extension a_1 of a_0 *respects* E (where $E \subseteq \omega_1$ is finite) if $(a_1 \setminus a_0) \cap A_\beta = \emptyset$ for every $\beta \in E$. So (a_1, E_1) extends (a_0, E_0) if and only if $E_0 \subseteq E_1$ and a_1 is an end-extension of a_0 that respects E_0.

Now we shall define a countable collection of dense subsets of P. First, to ensure that A_α is infinite, for every $k \in \omega$ and $p \in P$ observe that there is an extension (a', E') of p with $k < \sup(a')$. Then to ensure that $A_\alpha \perp A_\beta$ for all $\beta < \alpha$ observe that $(a, E \cup \{\beta\})$ extends (a, E). These dense sets take care of **C1**.

For every $X \subseteq \omega$ such that neither X nor $\omega \setminus X$ are finitely covered by $\langle A_\beta \mid \beta < \alpha \rangle$, and for every $k \in \omega$, define $D_{X,k} \subset P$ by

$$D_{X,k} = \{(a, E) \in P \mid \text{both } a \cap X \text{ and } a \setminus X \text{ contain } \geq k \text{ members}\}.$$

Claim 2.1. *$D_{X,k}$ is dense (open) in P.*

Proof. Take any $(a_0, E_0) \in P$. Since neither X nor its complement are $\subseteq^*$-included in $A = \bigcup\{A_\beta \mid \beta \in E_0\}$, both $X \setminus A$ and $(\omega \setminus X) \setminus A$ are infinite. We can find an end-extension a_1 of a_0 such that $(a_1 \setminus a_0) \cap A = \emptyset$ and both $a_1 \cap X$ and $a_1 \setminus X$ contain $\geq k$ members. Thus $(a_0, E_0) < (a_1, E_0) \in D_{X,k}$. ∎

So add to the countable list of dense sets all sets $D_{X_\xi,k}$ for $k \in \omega$ and $\xi < \alpha$ such that neither X_ξ nor $\omega \setminus X_\xi$ are finitely covered by $\langle A_\beta \mid \beta < \alpha \rangle$. This ensures **C2**.

The main issue of the proof is to take care of **C3**. What dense sets will do the job? Fix $e = e_i$ for any $i < \alpha$ such that $e \subseteq \alpha$. We say that a condition $p = (a, E) \in P$ is of *type* (a) for e if for some $m \in a$ the following holds:

(2) For every end-extension a' of a that respects E and for every $\xi_0 \in e$ there is some $\xi \in e$, $\xi_0 \leq \xi$, such that $A_\xi \cap a' \subseteq m + 1$.

If p is of type (a) then the least $m \in a$ that satisfies (2) is denoted by m_p. Observe that if p is of type (a) then any extension of p is also of type (a) (and with the same m).

We say that $p = (a, E) \in P$ is of *type* (b) for e if there are two adjacent members of a, m and n (i.e. $m, n \in a$ and $(m, n) \cap a = \emptyset$), such that for some $\xi_0 \in e$ for every $\xi_0 \leq \xi \in e$, $A_\xi \cap (m, n) \neq \emptyset$.

Claim 2.2. *Any condition in P has an extension of type* (a) *or an extension of type* (b)*.*

Proof. Given $p = (a, E)$ let $m = \max(a)$. Is p of type (a) by virtue of m? If yes, we are done, and if not then there are

1. a' an end-extension of a, respecting E, and
2. $\xi_0 \in e$,

such that for every $\xi \in e$ with $\xi_0 \leq \xi$, $A_\xi \cap a' \setminus m + 1 \neq \emptyset$. Let $n > \max a'$ be such that $n \notin \bigcup\{A_\beta \mid \beta \in E\}$ and consider the condition $p' = (a \cup \{n\}, E)$ extending p. Then for every $\xi_0 \leq \xi \in e$, $A_\xi \cap (m, n) \neq \emptyset$. That is, p' is of type (b). ∎

For every $\xi_0 \in e$ define $D_{\xi_0,e}$ by $p = (a,E) \in D_{\xi_0,e}$ iff either p is of type (b), or p is of type (a) and there exists some $\xi \in e \cap E$ above ξ_0 with $A_\xi \cap a \subseteq m+1$ (where $m = m_p$).

Claim 2.3. *$D_{\xi_0,e}$ is dense in P.*

Proof. Suppose $p_0 \in P$ is given. If p_0 is extendible into a condition of type (b) then we are done. Otherwise there is $p_1 = (a_1, E_1) \geq p_0$ of type (a). By the definition of type (a), there is some $\xi \in e$ with $\xi_0 \leq \xi$ such that $A_\xi \cap a_1 \subseteq m+1$. Hence $(a_1, E_1 \cup \{\xi\}) \in D_{\xi_0,e}$ is as required. ∎

Add to the countable list of dense sets all sets $D_{\xi_0,e}$ where $e = e_i$ for some $i < \alpha$ such that $e_i \subseteq \alpha$ and $\xi_0 \in e_i$. We claim that if A_α is defined from a filter G that intersects all the above dense sets, then condition **C3** is ensured. Given $i < \alpha$ such that $e_i = e \subseteq \alpha$, we ask if there is $(a,E) \in G$ of type (b) for e. If yes, then possibility **C3**(2) holds for A_α.

So we assume that G contains no condition of type (b) for e. Since any two conditions in G are compatible, it follows that if $p, q \in G$ are of type (a), then $m_p = m_q$. Let m denote this common value. We claim that there is an unbounded set of $\xi \in e$ such that $A_\xi \cap A_\alpha \subseteq m+1$. To see this, consider any $\xi_0 \in e$ and pick $p = (a,E) \in D_{\xi_0,e} \cap G$. Then p is of type (a) and there is $\xi \in E \cap e$ above ξ_0 with $A_\xi \cap a \subseteq m+1$. But then $A_\xi \cap A_\alpha \subseteq m+1$ follows.

2.2. *Martin's Axiom: Inseparable ⇒ contains a Lusin* subsequence.* Assume Martin's Axiom + $2^{\aleph_0} > \aleph_0$. Let $\mathcal{A} = \langle A_\zeta \mid \zeta \in \omega_1 \rangle$ be an inseparable sequence of length ω_1 (any length below the continuum works). Define the following poset:

$$Q = \{(u,n) \mid u \subseteq \omega_1 \text{ is finite and } n < \omega\}$$

ordered by

$$\begin{aligned}(u_1,n_1) \leq (u_2,n_2) \quad \text{iff} \quad & u_1 \subseteq u_2 \ \&\ n_1 \leq n_2 \ \&\\ & (\forall i \in u_1)(\forall j \in u_2 \setminus u_1)(j < i \Rightarrow |A_i \cap A_j| > n_1).\end{aligned}$$

This relation is easily shown to be transitive. We intend to prove that Q is a c.c.c. poset, and that for every $\alpha < \omega_1$ and $k < \omega$ the set $D_{\alpha,k}$ of (u,n) in Q for which $\sup(u) > \alpha$ and $n > k$ is dense. So if $G \subset Q$ is a filter provided by Martin's Axiom which intersects each of these dense sets, then $U = \bigcup\{u \mid (\exists n)((u,n) \in G)\}$ is uncountable and $\langle A_\alpha \mid \alpha \in U \rangle$ is a Lusin* sequence. Because if $i \in U$ and $k < \omega$ then

$$\{A_j \mid j \in U \cap i \text{ and } |A_i \cap A_j| \leq k\}$$

is finite by the following argument. For some $(u,n) \in G$, $i \in u$ and $n \geq k$. This implies that $|A_i \cap A_j| > k$ for every $j < i$ such that $j \in U \setminus u$.

The full result, concerning the decomposition of $\mathcal{A}$ into countably many Lusin* subsequences, follows from the fact that (under Martin's Axiom) if

Q is a c.c.c. poset, then Q is a countable union of filters (each intersects the required dense sets). (Consider the finite support product of ω copies of Q, and remember that $|Q| = \aleph_1$.)

It is easy to see that if $(u, n) \in Q$ and v is any end-extension of u then $(u, n) \le (v, n)$. Also, if $n \le m$ then $(u, n) \le (u, m)$. This shows that the required sets $D_{\alpha,k}$ are indeed dense in Q, and so the main point of the proof is to show that Q satisfies the countable chain condition.

LEMMA 2.4. *Q satisfies the c.c.c.*

Proof. Let $\langle (u_\zeta, n_\zeta) \mid \zeta \in \omega_1 \rangle$ be an ω_1-sequence of conditions in Q. We may assume that for some fixed n and k, $n = n_\zeta$ and $k = |u_\zeta|$ for all $\zeta \in \omega_1$, and that the sets u_ζ form a Δ-system. That is, for some finite $c_0 \subset \omega_1$, $c_0 = u_{\zeta_1} \cap u_{\zeta_2}$ for all $\zeta_1 \ne \zeta_2$ and $\max(u_{\zeta_1}) < \min(u_{\zeta_2} \setminus c_0)$ for $\zeta_1 < \zeta_2$.

We want to find $\zeta_1 < \zeta_2$ such that $(u_{\zeta_1} \cup u_{\zeta_2}, n)$ extends both (u_{ζ_1}, n) and (u_{ζ_2}, n). It is evident that $(u_{\zeta_1} \cup u_{\zeta_2}, n)$ extends (u_{ζ_1}, n) (the lower part) but the problem is the possibility that $|A_i \cap A_j| \le n$ for some $i \in u_{\zeta_2}$ and $j \in u_{\zeta_1} \setminus c_0$.

We shall find two uncountable sets $K, L \subseteq \omega_1$ such that for every $\zeta_1 \in K$ and $\zeta_2 \in L$, (u_{ζ_1}, n) and (u_{ζ_2}, n) are compatible. We start with $K_0 = L_0 = \omega_1$, and define $K_{i+1} \subseteq K_i$ and $L_{i+1} \subseteq L_i$ by induction, considering in turn each pair $0 \le i, j \le |u_\zeta \setminus c_0|$ (any ζ can be taken, as these sets have all the same size). The definition of K_i and L_i depends on a finite parameter set, and it is convenient to have a countable model in which the definition is carried out. So let $M \prec \langle H_{\aleph_1}, \mathcal{A}, Q, \{(u_\zeta, n_\zeta) : \zeta \in \omega_1\} \rangle$ be a countable elementary submodel (where $H_{\aleph_1}$ is the collection of all sets that are hereditarily countable). The following lemma is used.

LEMMA 2.5. *Let $U, V \in M$ be two uncountable subsets of ω_1 and $n < \omega$. There are uncountable subsets $U_1 \subseteq U$ and $V_1 \subseteq V$ (definable in M) such that $|A_\zeta \cap A_\xi| > n$ for every $\zeta \in U_1$ and $\xi \in V_1$ (and hence $(\{\zeta\}, n)$ and $(\{\xi\}, n)$ are compatible in Q).*

It should be obvious how successive applications of the lemma yield the c.c.c., and so we turn to the proof of the lemma. Let $\delta = \omega_1 \cap M$ be the set of countable ordinals in our countable structure M.

CASE 1: $|A_\zeta \cap A_\xi| > n$ *for some* $\zeta \in U \setminus \delta$ *and* $\xi \in V \setminus \delta$. In this case pick a finite $X \subset A_\zeta \cap A_\xi$ with $|X| > n$, and let $U_1 = \{i \in U \mid X \subset A_i\}$ and $V_1 = \{j \in V \mid X \subset A_j\}$. Both U_1 and V_1 are uncountable (for if U_1 is countable then it would be included in M, but A_ζ shows that this is not the case).

CASE 2: *not Case 1.* So $|A_\zeta \cap A_\xi| \le n$ for every $\zeta \in U \setminus \delta$ and $\xi \in V \setminus \delta$. Let $0 \le m_0 \le n$ be the maximal size of some intersection $F = A_\zeta \cap A_\xi$ for indices ζ and ξ as above. Then $U_1 = \{i \in U \mid F \subset A_i\}$ and $V_1 = \{j \in V \mid F \subset A_j\}$

are uncountable and $A_i \cap A_j = F$ for $i \in U_1 \setminus \delta$ and $j \in V_1 \setminus \delta$ (by maximality of $|F|$). So the set $B = \bigcup\{A_i \mid i \in U_1\}$ separates $\mathcal{A}$ (as $B \cap A_j = F$ for every $j \in V_1$), which is a contradiction.

References

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Departments of Mathematics and Computer Science
Ben-Gurion University
Beer-Sheva, Israel
E-mail: abraham@cs.bgu.ac.il

Institute of Mathematics
The Hebrew University
91904 Jerusalem, Israel
E-mail: shelah@math.huji.ac.il

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