Homotopy decompositions of orbit spaces and the Webb conjecture

by

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Abstract. Let $p$ be a prime number. We prove that if $G$ is a compact Lie group with a non-trivial $p$-subgroup, then the orbit space $(B\mathcal{A}_p(G))/G$ of the classifying space of the category associated to the $G$-poset $\mathcal{A}_p(G)$ of all non-trivial elementary abelian $p$-subgroups of $G$ is contractible. This gives, for every $G$-CW-complex $X$ each of whose isotropy groups contains a non-trivial $p$-subgroup, a decomposition of $X$ as a homotopy colimit of the functor $X_{E^n}/(NE_0\cap\ldots\cap NE_n)$ defined over the poset $(\text{sd}\mathcal{A}_p(G))/G$, where sd is the barycentric subdivision. We also investigate some other equivariant homotopy and homology decompositions of $X$ and prove that if $G$ is a compact Lie group with a non-trivial $p$-subgroup, then the map $EG \times_G B\mathcal{A}_p(G) \to BG$ induced by the $G$-map $B\mathcal{A}_p(G) \to \ast$ is a mod $p$ homology isomorphism.

Introduction. In this paper we will study homotopy and homology decompositions which are associated to the equivariant structure of a $G$-CW-complex $X$ where $G$ is a Lie group. We will try to generalize and streamline techniques of such decompositions.

Let $\mathcal{C}$ be a small topological category and let $F : \mathcal{C} \to G$-CW be a functor such that, for every $c \in \mathcal{C}$, $F(c) = G \times_{H(c)} X^{H'(c)}$ where $H(c), H'(c)$ are closed subgroups of $G$ and $H(c)$ is a subgroup of the normalizer $NH'(c) = N_GH'(c)$ of $H'(c)$ in $G$. Suppose also that there is a natural transformation from $F$ to the constant functor induced by the inclusions $X^{H'(c)} \to X$. $G$-maps

$$u : \text{hocolim}_{c \in \mathcal{C}} G \times_{H(c)} X^{H'(c)} \to X$$

induced by such natural transformations can be used in constructing different homotopy and homology decompositions. If $u$ is a $G$-homotopy equivalence then it will be called a $G$-homotopy decomposition of $X$.

In Section 0 we will introduce a “universal” category $\mathcal{C}_G$ and, for every $G$-CW-complex $X$, a functor $\hat{X} : \mathcal{C}_G \to G$-CW and a natural transformation

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of functors \( \hat{X} \to X \). We will study the decompositions induced by functors \( F \) which are compositions \( \hat{X}F' \), where \( F' : C \to C_G \).

For a given \( G \)-CW-complex \( K \), we will investigate homotopy decompositions of the orbit space \( K \times_G X \), i.e. homotopy equivalences of the form

\[
\text{id} \times_G u : \text{hocolim}_{c \in C} K \times_{H(c)} X^{H'(c)} \simeq K \times_G X.
\]

We will also study mod \( p \) homology decompositions. In this case the map \( \text{id} \times_G u \) is an \( F_p \)-equivalence. We will show how the known examples of decompositions of \( K \times_G X \) can be described using \( C_G \).

The best known examples of homology decompositions are the cases where \( K = EG \) is a universal free \( G \)-space and \( X = * \) is a one-point space ([JM2], [JMO]).

Let \( p \) be a prime number and let \( \mathcal{A}_p(G) \) be the \( G \)-poset of all elementary abelian non-trivial \( p \)-subgroups of \( G \). If \( G \) does not contain a \( p \)-subgroup, then the set \( \mathcal{A}_p(G) \) is empty. Let \( \mathcal{A}_p(G) \) be the category whose objects are elements of \( \mathcal{A}_p(G) \) and whose morphisms are homomorphisms which are restrictions of inner automorphisms of \( G \). Let \( C_G(E) \) be the centralizer of \( E \) in \( G \). There is a contravariant functor \( F : \mathcal{A}_p(G) \to G \)-CW such that \( F(E) = G \times_{C_G(E)} X^E \). In the case where \( X = * \) and \( G \) is a compact Lie group which contains a non-trivial \( p \)-subgroup, there is a mod \( p \) homology decomposition (Theorem 1.3 of [JM2])

\[
\text{hocolim}_{E \in \mathcal{A}_p(G)} BC_G(E) \to BG.
\]

Using this fact it is proved in [H1] that if the isotropy groups of \( X \) are compact and contain a non-trivial \( p \)-group, then the map

\[
\text{hocolim}_{E \in \mathcal{A}_p(G)} EG \times_{C_G(E)} X^E \to EG \times_G X
\]

is a mod \( p \) homology isomorphism.

We will prove that one can take instead of \( EG \) any \( F_p \)-acyclic complex \( K \). We will also construct, for such \( K \), another mod \( p \) homology decomposition

\[
\text{hocolim}_{[(E_0, \ldots, E_n)] \in (\text{sd} \mathcal{A}_p(G))/G} K \times_{NE_0 \cap \ldots \cap NE_n} X^{E_n} \to K \times_G X.
\]

Here we take \( C \) equal to the poset \( (\text{sd} \mathcal{A}_p(G))/G \) of the orbits of the \( G \)-action on the barycentric subdivision of \( \mathcal{A}_p(G) \). (Recall that the elements of \( \text{sd} \mathcal{A}_p(G) \) are the increasing sequences \( (E_0 < \ldots < E_n) \) of elements of \( \mathcal{A}_p(G) \).) If \( G \) is a compact Lie group, then in the special case when \( X = * \) and \( K = EG \), we obtain a mod \( p \) homology isomorphism

\[
\text{hocolim}_{[(E_0, \ldots, E_n)] \in (\text{sd} \mathcal{A}_p(G))/G} B(NE_0 \cap \ldots \cap NE_n) \to BG,
\]

which is in fact equal to the mod \( p \) isomorphism

\[
EG \times_G B(\mathcal{A}_p(G)) \to BG.
\]
This last fact is well known in the finite case and can be obtained using 1.3 of [JM2]. The compact case is more complicated because of the topological structure of $A_p(G)$.

If $K = *$ then we obtain not only a homology but also a homotopy decomposition of $X/G$ (Theorem 0.1). In the case when $G$ is a compact Lie group and $X = *$ this means that the space $(B A_p(G))/G$ is contractible. For finite groups this was conjectured in [We]. A combinatorial proof of this fact in the finite case was given in [Sy]. Our proof is a generalization of an equivariant approach described for finite groups in [S1].

We will also study $h^*_G$ decompositions, where $h^*_G$ is a generalized equivariant cohomology theory, i.e. maps $u$ which induce isomorphisms

$$h^*_G(u) : h^*_G(X) \to h^*_G(\text{hocolim}_{c \in C} G \times H(c) X^{H^*(c)}) .$$

We will use the fact that such a decomposition gives a spectral sequence

$$H^m(C, h^*_G(X^{H^*(c)})) \Rightarrow h^m + n(X),$$

where $h^*_G(-) = h^*_G(G \times_h -)$ and $H^m(C, -) = \lim^m(C, -) = \text{Ext}^m_C(\mathbb{Z}, -)$ are the cohomology groups of the category $C$ (Ch. XII of [BK], Section 5 of [DF1]).

0. The main results. Let $G$ be a Lie group. Let $O_G$ be the orbit category of $G$ whose objects are the orbits $G/H$, where $H$ is a closed subgroup of $G$. The morphisms of $O_G$ are the equivariant continuous maps. Every morphism $f : G/H \to G/H_1$ corresponds to a class $[g] \in (G/H_1)^H$ such that $f([g']) = g'gH_1$. It follows from the definitions that $[g] \in (G/H_1)^H$ if and only if $H \subseteq gH_1 g^{-1}$. The topology of the morphism space $\text{Mor}_{O_G}(G/H, G/H_1) = (G/H_1)^H$ is induced from $G/H_1$. The category $O_G$ is a topological category in the sense of [HV], i.e. a small category $C$ with topological morphism sets such that the composition is continuous and the structural map $\text{Ob}C \to \text{Mor}C$ is a closed cofibration. Similarly to [HV] we will work in the category Top of compactly generated spaces. We will consider $O_G$ as a full subcategory of the category $G$-CW of all $G$-CW-complexes and equivariant cellular maps. This category is described, for example, in [Wi] and [JMO].

We introduce another topological category $C_G$ which plays a crucial role in our considerations concerning equivariant decompositions. Its object set $\mathcal{W}(G)$ consists of all pairs $(H, H')$ of closed subgroups of $G$ such that $H$ is a subgroup of $NH'$. The morphisms $(H, H') \to (H_1, H'_1)$ of $C_G$ are all morphisms $f = [g] : G/H \to G/H_1$ of $O_G$ such that $H_1 \subseteq g^{-1}H'g$. If $f' = [g'] : (H_1, H'_1) \to (H_2, H'_2)$ is a morphism of $C_G$, then the condition $H'_2 \subseteq g'^{-1}H_1'g'$ implies that $H_2' \subseteq g'^{-1}g^{-1}H'g'g$ so $f'f = [gg']$ is a morphism of $C_G$. The topology of the morphism spaces is induced from the morphism space topology in $O_G$. There is an inclusion of categories $i : O_G \to C_G$ such that $i(H) = (H, e)$. The category $C_G$ has a final object $(G, e)$. 

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Let $X$ be a $G$-CW-complex. Let $\hat{X} : \mathcal{C}_G \to G$-CW be the functor defined by $\hat{X}(H, H') = G \times_H X^{H'}$, $\hat{X}([g])([g', x]) = [g'g, g^{-1}x]$. Hence $\hat{X}(G, e) = G \times_G X = X$. The equivariant maps $\alpha(H, H') = \hat{X}([e]) : G \times_H X^{H'} \to X$ such that $\alpha(H, H')[g', x] = g'x$ form a natural transformation of functors $\alpha : \hat{X} \to X$ where $X$ is the constant functor. Let $\mathcal{C}$ be a topological category. Suppose that we have a functor $(H(-), H'(\cdot)) : \mathcal{C} \to \mathcal{C}_G$. Then $\alpha$ induces a $G$-map

$$u : \text{hocolim}_{c \in \mathcal{C}} G \times_{H(c)} X^{H'(c)} \to X.$$  

Many examples of decompositions induced by such maps will be described and studied in Sections 3 and 4. For example, let $V$ be a $G$-set of closed subgroups of $G$ and let $\mathcal{O}_V$ be the full subcategory of $\mathcal{O}_G$ such that $G/H$ is an object of $\mathcal{O}_V$ if and only if $H \in V$. Let $\mathcal{C}(V)$ be the full subcategory of $\mathcal{C}_G$ whose objects are pairs $(H, H')$ where $H$ is a subgroup of $H'$ and $H' \in V$. We will prove the following result in Section 3.

0.0. Proposition. Assume that all isotropy groups of $X$ are in $V$. Then:

(i) The map $u : \text{hocolim}_{c \in \mathcal{C}(V)} \hat{X} \to X$ is a $G$-homotopy decomposition.

(ii) The map $u/G$ gives a homotopy decomposition

$$\text{hocolim}_{G/H \in \mathcal{O}_V} X^H \simeq X/G.$$  

The homotopy decomposition from (ii) is well known. It appears in [E] and [DF2].

In Sections 1 and 2 we will consider the case where $\mathcal{C}$ is the orbit category of the barycentric subdivision of a poset of subgroups of $G$. In order to describe this case we need the following notation. Let $W$ be a topological $G$-poset. This means that $W$ is a topological poset in the sense of [Z] (i.e. the order relation is a closed subset of $W^2$) together with a continuous and order preserving action of $G$ on $W$. Let $d_n W$ denote the $G$-subspace of $W^{n+1}$ consisting of all non-decreasing sequences $w. = (w_0, \ldots, w_n)$. The $G$-subspace of $d_n W$ consisting of all $w.$ such that $w_i \neq w_{i+1}$ for all $i$ will be denoted by $sd_n W$. The disjoint union $sd W = \bigsqcup_{n \in \mathbb{N}} sd_n W$ is a topological $G$-poset such that $(w_0, \ldots, w_n) \leq (w'_0, \ldots, w'_m)$ if and only if $\{w'_0, \ldots, w'_m\} \subseteq \{w_0, \ldots, w_n\}$. There are two $G$-poset maps $p_0 : sd W \to W$ and $p_1 : (sd W)^{\text{op}} \to W$ such that $p_0(w.) = w_0$, $p_1(w.) = w_n$. We will assume that as a topological space, $W$ is equal to the disjoint union of its $G$-orbits $Gw = G/Gw$ with the topology induced from the topology of $G$. In this case the topological space $W/G$ is discrete. If $W$ satisfies the condition that $w \leq gw$ implies that $w = gw$ then $W/G$ is a poset such that $[w] \leq [w']$.
if and only if \( w \leq gw' \) for some \( g \in G \). The \( G \)-poset \( sdW \) satisfies this condition.

Let \( S(G) \) denote the poset of all closed subgroups of \( G \). The group \( G \) acts on \( S(G) \) by conjugation. If \( H \in S(G) \), then the isotropy group of this action at \( H \) is equal to \( NH \). We will assume that \( S(G) \) is a topological space equal to the disjoint union of its \( G \)-orbits \( Gx \) with topology induced from the topology of \( G \). Let \( W \) be a \( G \)-subposet of \( S(G) \) satisfying the condition that \( w \leq gw \) implies that \( w = gw \). Suppose that \((sdW)/G\) is also a discrete space. Then the space \( sdW \) is equal to the disjoint union of its \( G \)-orbits \( G/(Nw_0 \cap \ldots \cap Nw_n) \). There is a functor \( F: (sdW)/G \to C_G \) such that

\[
F([w_0, \ldots, w_n]) = (Nw_0 \cap \ldots \cap Nw_n, w_n).
\]

If \([w_0, \ldots, w_n] \leq [w_0', \ldots, w_m']\), then there exists exactly one element \([g] \) of \( G/(Nw'_0 \cap \ldots \cap Nw'_m) \) such that \((w_0, \ldots, w_n) \leq (gw'_0g^{-1}, \ldots, gw'_mg^{-1})\). This implies that \( gw'_m g^{-1} \subseteq w_n \) and \( F([w_0, \ldots, w_n]) \leq [w_0', \ldots, w_m']\) is the morphism of \( C_G \) defined by \([g] \).

If \( X \) is a \( G \)-CW-complex then there is a functor \( \tilde{X}: (sdW)/G \to G \)-CW such that

\[
\tilde{X}([w_0, \ldots, w_n]) = G \times_{Nw_0 \cap \ldots \cap Nw_n} X^{w_n}.
\]

In Section 2 of this paper we will prove the following result which in the case when \( G \) is a finite group was proved in [S1] (2.10.iv and 2.11).

0.1. THEOREM. Let \( X \) be a \( G \)-CW-complex such that all its isotropy groups are compact and contain a non-trivial \( p \)-subgroup. Then there is a homotopy equivalence

\[
\text{holim}_{E \subseteq (sd A_p(G))/G} X^E_n/(NE_0 \cap \ldots \cap NE_n) \simeq X/G.
\]

If \( X = \ast \) is a one-point \( G \)-CW-complex, then

\[
\tilde{\ast}([w_0, \ldots, w_n]) = G/(Nw_0 \cap \ldots \cap Nw_n)
\]

and 0.1 specializes to the fact that, in the case when \( G \) is a compact Lie group, the classifying space \( B((sd A_p(G))/G) \) of the category associated to the poset \((sd A_p(G))/G\) is contractible.

If \( W \) is a poset (discrete as topological space), then the geometrical realization \( |W| \) of the simplicial complex associated to \( W \) is equal to the classifying space \( BW \) of the category associated to \( W \). An action of \( G \) on \( W \) induces a \( G \)-action on \( |W| \). Then there are homotopy equivalences \(|sdW|/G \simeq \ |W|/\ G \) and \(|sdW|/G \simeq \ |(sdW)/G| \). Let \( G \) be a finite group. Let \( S_p(G) \) be the \( G \)-poset of all non-trivial \( p \)-subgroups of \( G \). Then the spaces \( |S_p(G)| \) and \(|A_p(G)|\) are \( G \)-homotopy equivalent (Theorem 2 of [TW]). It is proved in [We] (2.6.1) that \(|S_p(G)|/G \) is \( F_p \)-acyclic and conjectured that \(|S_p(G)|/G \) is contractible. It is also proved in [We] (2.1.2) that \(|S_p(G)|^H \) is
contractible whenever \( H \) is a subgroup of \( G \) which contains a normal non-trivial \( p \)-subgroup. In [S1] a proof of the Webb conjecture was presented which uses this fact and methods introduced in [O1]. We will generalize this proof to the case of a compact Lie group.

If \( W \) is a topological poset then the morphism space of the topological category associated to the poset \( W \) has topology induced from the topology of \( W \times W \) and the classifying space \( BW \) of this category is equal to \( \bigcup_{n \in \mathbb{N}} \Delta_n \times d_n W / \sim \) where \( \Delta_n \) is the standard \( n \)-dimensional simplex and \( \sim \) is an appropriate equivalence relation (3.6 of [Z]).

Let \( W \) be a topological \( G \)-poset such that the condition that \( w \leq gw \) implies that \( w = gw \). Then \( W/G \) is a topological poset. Suppose that the topological space \( W/G \) is discrete and that, for every \( n \in \mathbb{N} \), \( (d_n W)/G \) is discrete. (This holds for example if \( W \) is a subposet of \( S(G) \) and all subgroups in \( W \) are finite. Indeed, let \( p : (d_n W)/G \to W/G \) be the projection such that \( p([w_0, \ldots, w_n]) = [w_n] \). Then, for every \( [w] \in W/G \), the preimage \( p^{-1}([w]) \) is a finite space.) The topological space \( sd W/G = (sd W)/G \) is also discrete in this case and \( BW = \bigcup_{n \in \mathbb{N}} \Delta_n \times sd_n W / \sim \). There is a natural \( G \)-CW-complex structure on \( BW \) such that the poset \( sd W/G \) is equal to the poset of the \( G \)-cells of \( BW \). We will show in Section 2 (cf. the proof of 2.3) that

\[
(BW)/G = \bigcup_{n \in \mathbb{N}} \Delta_n \times (sd_n W)/G / \sim
\]

is a classifying space \( B((sd W)/G) \) of the category associated to the poset \( sd W/G \). We will also show that there are \( G \)-homotopy equivalences

\[
hocolim_{([w_0, \ldots, w_n]) \in sd W/G} G/(Nw_0 \cap \ldots \cap Nw_n) \simeq Bsd W \simeq BW.
\]

In Section 1 we will prove that if \( G \) is a compact Lie group and contains a non-trivial \( p \)-subgroup, then the space \( BA_p(G)/G \) is contractible. The proof consists of several steps which will be described below. Recall that \( P \) is a \( p \)-toral group if its identity component \( P_0 \) is a torus and \( \pi_0(P) = P/P_0 \) is a finite \( p \)-group. The following result is an immediate consequence of 0.1 but in the proof of 0.1 we will use 0.2 in the case when \( X \) has finitely many orbit types. We will prove this fact in Section 1.

0.2. THEOREM. Let \( G \) be a compact Lie group. Let \( X \) be a \( G \)-CW-complex such that all its isotropy groups contain a non-trivial \( p \)-subgroup. Suppose that \( XP/H \) is contractible whenever \( P \) is a non-trivial \( p \)-toral subgroup of \( G \) and \( H \) is a closed subgroup of the normalizer \( NP \) of \( P \) in \( G \). Then \( X/G \) is contractible.

To prove 0.1 we will also need the following result.
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0.3. Proposition. Let $R$ be a commutative ring. Let $X$ and $Y$ be $G$-CW-complexes such that all their isotropy groups are compact and contain a non-trivial $p$-subgroup. Let $f : X \to Y$ be a cellular $G$-map of $G$-CW-complexes. Then:

(i) If, for every compact subgroup $H$ of $G$ containing a non-trivial normal $p$-toral subgroup, $f^H : X^H \to Y^H$ is a homotopy equivalence, then so is $f/G : X/G \to Y/G$.

(ii) If, for every compact subgroup $H$ of $G$ containing a non-trivial normal $p$-toral subgroup, $f^H : X^H \to Y^H$ is an $R$-equivalence, then so is $f/G : X/G \to Y/G$.

If $G$ is a compact Lie group and $Y = \ast$ then 0.3 is a consequence of 0.2 and the well known decomposition described in 0.0(ii). This result will be proved in Section 1 in the case when $X$ has finitely many orbit types. We will show that the map $BA_p(G) \to \ast$ satisfies the assumptions of 0.3(i). Hence $BA_p(G)/G$ is contractible and using this we will infer 0.1. We will also prove 0.3 for an arbitrary Lie group $G$.

Let $W$ be a poset of closed subgroups of $G$. In Section 4 we will describe a condition on $W$ which ensures that $h_G^*(X) \to h_G^*(Y)$ is an isomorphism if $X^H \to Y^H$ is an $R$-homology isomorphism for all $H \in W$. As an example we will consider the case when

$$h_G^*(X) = H^*(K \times_G X, R).$$

In particular, we will show how 0.3(ii) and the results of [JMO] and [JO] concerning the mod $p$ decomposition

$$\hocolim_{G/P \in \mathcal{O}_{R_p(G)}} BP \to BG,$$

where $R_p(G)$ is a certain poset of $p$-toral subgroups of $G$, imply the following result.

0.4. Proposition. Let $X$ and $Y$ be $G$-CW-complexes such that all their isotropy groups are compact and contain non-trivial $p$-subgroups. Let $K$ be an $F_p$-acyclic $G$-CW-complex. If, for every non-trivial $p$-toral subgroup $H$ of $G$, $f^H : X^H \to Y^H$ is an $F_p$-equivalence, then so is $\text{id}_K \times_G f : K \times_G X \to K \times_G Y$.

If $G$ is a compact Lie group with a non-trivial $p$-subgroup, then from the fact (cf. the proof of 1.5) that all isotropy groups of $BA_p(G)$ contain non-trivial normal $p$-subgroups and that, for every subgroup $H$ of $G$ containing a non-trivial normal $p$-subgroup, the space $BA_p(G)^H$ is contractible, we obtain the following result.

0.5. Corollary. Let $G$ be a compact Lie group with a non-trivial $p$-subgroup. Then the map $EG \times_G BA_p(G) \to BG$ induced by the $G$-map $BA_p(G) \to \ast$ is an $F_p$-equivalence.
The following posets of subgroups will be defined and used in the paper.

**List of posets of subgroups of \( G \)**

- \( \mathcal{A}_p(G) \) — the set of all elementary abelian non-trivial \( p \)-subgroups,
- \( \mathcal{A}'_p(G) \) — the set of all elementary abelian \( p \)-subgroups,
- \( \mathcal{K}_p(G) \) — the set of all compact subgroups \( H \) such that, for every \( P \in \mathcal{M}_p(G) \), \( H \cap Z(P) \) contains a non-trivial \( p \)-subgroup,
- \( \mathcal{M}_p(G) \) — the set of all maximal non-trivial \( p \)-toral subgroups,
- \( \mathcal{N}_p(G) \) — the set of all compact subgroups containing a non-trivial normal \( p \)-toral subgroup,
- \( \mathcal{S}(G) \) — the set of all closed subgroups,
- \( \mathcal{S}(G, X) \) — the set of all isotropy groups of \( X \),
- \( \mathcal{S}_0(G, X) = \mathcal{S}(G, X) \cup \mathcal{S}(G, *) \),
- \( \mathcal{S}'_c(G) \) — the set of all compact subgroups,
- \( \mathcal{S}_c(G) \) — the set of all compact subgroups which contain a non-trivial \( p \)-subgroup,
- \( \mathcal{S}'_p(G) \) — the set of all subtoral \( p \)-subgroups,
- \( \mathcal{S}_p(G) \) — the set of all subtoral \( p \)-subgroups which contain a non-trivial \( p \)-subgroup,
- \( \mathcal{T}'_p(G) \) — the set of all \( p \)-toral subgroups,
- \( \mathcal{T}_p(G) \) — the set of all non-trivial \( p \)-toral subgroups,
- \( \mathcal{T}_p(G, X) \) — the set of all maximal \( p \)-toral subgroups of isotropy groups of \( X \),
- \( \mathcal{Z}_p(G) \) — the set of all compact subgroups containing a non-trivial central \( p \)-subgroup.

1. **Orbit spaces of compact Lie group actions.** Let \( G \) be a Lie group. The set of all compact subgroups of \( G \) will be denoted by \( \mathcal{S}_c(G) \). The set of all elements of \( \mathcal{S}'_c(G) \) which contain a non-trivial \( p \)-subgroup will be denoted by \( \mathcal{S}_c(G) \). The set of all closed \( p \)-toral subgroups of \( G \) will be denoted by \( \mathcal{T}'_p(G) \). The set of all non-trivial \( p \)-toral subgroups of \( G \) will be denoted by \( \mathcal{T}_p(G) \). The set of all compact subgroups of \( G \) containing a non-trivial normal \( p \)-toral subgroup will be denoted by \( \mathcal{N}_p(G) \).

If \( G \) is a compact Lie group, \( T \) is a maximal torus of \( G \) and \( N_pT/T \) is a Sylow \( p \)-subgroup of \( NT/T \), then \( N_pT \) is a maximal \( p \)-toral subgroup of \( G \). All maximal \( p \)-toral subgroups of \( G \) are conjugate to \( N_pT \) (Lemma A.1 of [JMO]). The set of all maximal \( p \)-toral subgroups of \( G \) will be denoted by \( \mathcal{M}_p(G) \) and the set of all maximal \( p \)-toral subgroups of isotropy groups of \( X \) by \( \mathcal{T}_p(G, X) \).

Let \( S \) be a subset of the set of compact subgroups of \( G \). We will use the notation

\[
\mathcal{W}_S = \{ (H, H') : H' \subseteq H \subseteq NH', \ H' \in S, \ H \in \mathcal{S}'_c(G) \}.
\]
A non-empty $G$-poset $\mathcal{P}$ of $p$-toral subgroups of $G$ will be called \textit{concave} if, for any $p$-toral subgroups $P$ and $P'$ the condition that $P \subseteq P'$ and $P \in \mathcal{P}$ implies that $P' \in \mathcal{P}$. If $G$ is a compact Lie group and $\mathcal{P}$ is concave, then $\mathcal{M}_p(G) \subseteq \mathcal{P}$ because all maximal $p$-toral subgroups are conjugate by elements of $G$.

Let $\mathcal{CW}$ denote the category of spaces having the homotopy type of CW-complexes and let $\mathcal{CW}_0$ be the subcategory of $\mathcal{CW}$ consisting of the connected spaces. We will say that a class $\mathcal{A}$ of objects of $\mathcal{CW}$ is \textit{thick} if it is closed under homotopy equivalences and taking homotopy pushouts.

In this section we will assume that $G$ is a compact Lie group with a non-trivial $p$-subgroup and that $X$ is a $G$-CW-complex with finitely many orbit types.

1.1. THEOREM. Let $\mathcal{A}$ be thick. Let $\mathcal{P}$ be a concave $G$-poset of $p$-toral subgroups of $G$ containing all maximal $p$-toral subgroups of the isotropy groups of $X$. If $X^P/H \in \mathcal{A}$ whenever $P \in \mathcal{P}$ and $P \subseteq H \subseteq NP$, then $X/G \in \mathcal{A}$.

Proof. If $(e) \in \mathcal{P}$, then the assumptions imply that $X/G \in \mathcal{A}$. Let $k(G, X)$ denote the number of elements of $\mathcal{T}_p(G, X)/G$.

If $k(G, X) = 1$, then $\mathcal{T}_p(G, X) = (P) = \{gP : g \in G\}$, where $P$ is, up to conjugacy, the unique maximal $p$-toral group of an isotropy group of $X$. Hence $X = X^{(P)} = \bigcup_{P' \in (P)} X^{P'}$. It is proved in [O1] (in the proof of Proposition 3) that the map $X^P/NP \to X^{(P)}/G$ is a homeomorphism. (This is a consequence of the fact that, if $G'$ is a closed subgroup of $G$ and $P$ is a maximal $p$-toral subgroup of $G'$, then $NP$ acts transitively on $(G/G')^P$. Indeed, let $aG', bG' \in (G/G')^P$. Then $a^{-1}Pa, b^{-1}Pb$ are maximal $p$-toral subgroups of $G'$ so they are conjugate in $G'$ and there is $c \in G'$ such that $bca^{-1} \in NP$.) If the assumptions hold, then $P$ is a maximal toral $p$-subgroup of $G$. Hence, in this case, $X/G = X^P/NP \in \mathcal{A}$.

We use induction on the dimension of $G$ and then on the order of $\pi_0(G) = G/G_0$, where $G_0$ is the identity component of $G$. Assume that the result is true for all proper closed Lie subgroups of $G$. Now we use induction on $k(G, X)$. Let $k(G, X) = k + 1 > 1$. Suppose that the result is true for all $G$-CW-complexes $X'$ such that $k(G, X') \leq k$. Let $P$ be a minimal element of $\mathcal{T}_p(G, X)$. As $P$ is not a maximal $p$-toral group, it follows that $NP/P$ contains a non-trivial $p$-toral subgroup (cf. [O1], Lemma 2). Let $X'$ be a $G$-CW-subcomplex of $X$ such that $x \in X \setminus X'$ if and only if maximal $p$-toral subgroups of the isotropy group $G_x$ are conjugate to $P$. The induction assumption implies that $X'^H/G \in \mathcal{A}$ because $k(G, X') \leq k$. Indeed, let $\mathcal{P}_o = \mathcal{P} \setminus (P)$. Then, for every $(H, P') \in \mathcal{W}_{\mathcal{P}_o}$, $X'^{P'}/H = X^{P'}/H$. 

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It follows from the definition that \( X = X' \cup X^{(P)} \) and that \( X/G \) is equal to the pushout of the diagram

\[
X^{(P)}/G \leftarrow X'^{(P)}/G \rightarrow X'/G.
\]

If \( x \in X \setminus X' \), then \( \mathcal{M}_p(G_x) \) is a subset of \( (P) \) and \( NP \) acts transitively on \( (Gx)^P = (G/G_x)^P \). Hence \( X/G \) is the pushout of the diagram

\[
X^P/NP \leftarrow X'^P/NP \rightarrow X'/G.
\]

Since \( X'^P/NP \rightarrow X^P/NP \) is a cofibration, \( X/G \) is the homotopy pushout of this diagram.

The space \( X'^P \), which has the structure of an \( NP \)-CW complex, satisfies the assumptions of the proposition. It is of finite orbit type because, for every closed subgroup \( G' \) of \( G \), \( (G'/G')^P/NP \) is finite (II.5.7 of [Br1]). Let \( \mathcal{P}' = \{ P' \in \mathcal{P} : P \subset P' \subseteq NP, P' \neq P \} \). From the fact that, for every \( x \in X'^P \), \( P \subseteq G_x \cap NP \) and \( P \) is not a maximal \( p \)-toral subgroup of \( G_x \), it follows that \( P \) is not a maximal \( p \)-toral subgroup of \( G_x \cap NP \) (Lemma 2 of [O1]). Hence

\[
\mathcal{T}_p(NP, X'^P) = \bigcup_{x \in X'} \mathcal{M}_p(G_x \cap NP) \subseteq \mathcal{P}'
\]

and \( X'^P/H = X^P/H \in \mathcal{A} \) whenever \( (H, P') \in \mathcal{W}_{P'} \).

If \( P \) is a normal subgroup of \( G \), then \( NP = G \) but \( k(X'^P, G) \leq k \), because \( P \notin \mathcal{T}_p(G, X'^P) \subseteq \mathcal{T}_p(G, X) \). If \( P \) is not a normal subgroup of \( G \), then \( NP < G \) and we can use the induction assumption. In both cases we find that \( X'^P/NP \in \mathcal{A} \). Hence \( X/G \in \mathcal{A} \).

In particular, if \( \mathcal{P} = \mathcal{T}_p(G) \) and \( \mathcal{A} \) is the class of all contractible objects of \( CW_0 \) then 1.1 specializes to 0.2.

In what follows let \( \mathcal{A} \) be a thick category. We now define three conditions for thick categories.

\[ \mathbf{A1} \text{: For every compact Lie group } H \text{ and for every } H\text{-CW-complex } X, \text{ if } X^{H'} \in \mathcal{A} \text{ for every closed subgroup } H' \text{ of } H, \text{ then } X/H \in \mathcal{A}. \]

\[ \mathbf{A2} \text{: For every compact Lie group } H \text{ and for every } H\text{-CW-complex } X, \text{ if } \dim X < \infty \text{ and } X \in \mathcal{A}, \text{ then } X/H \in \mathcal{A}. \]

\[ \mathbf{A3} \text{: For every compact Lie group } H \text{ and for every } H\text{-CW-complex } X, \text{ if } X/P \in \mathcal{A} \text{ for every } P \in \mathcal{M}_p(H), \text{ then } X/H \in \mathcal{A}. \]

Let \( H' \) be a closed subgroup of \( G \) and let \( \mathcal{P} \) be a set of subgroups of \( G \). We use the notation
\[ N_{H'} = \{ H \in S(G) : H' \subseteq H \subseteq NH' \}, \]
\[ N_P = \{ H \in S(G) : H' \subseteq H \subseteq NH', \ H' \in P \}, \]
\[ S_P = \bigcup_{P, P' \in P} \{ H \in S(G) : P \subseteq H \subseteq P' \subseteq NP \}, \]
\[ S'_p(G) = S_{T'_p(G)}, \quad S_p(G) = S_{T_p(G)}. \]

1.2. Corollary. Let \( \mathcal{P} \) be a concave \( G \)-poset of \( p \)-toral subgroups of \( G \). Let \( X \) be a \( G \)-CW-complex such that maximal \( p \)-toral subgroups of isotropy groups of \( X \) are in \( \mathcal{P} \). Suppose that \( \mathcal{A} \) is thick and that one of the following conditions holds:

(i) \( \mathcal{A} \) satisfies \( \text{A3} \) and \( X^{P'}/P' \in \mathcal{A} \) whenever \( (P', P) \in \mathcal{W}_P \) and \( P' \in \mathcal{P} \).

(ii) \( \mathcal{A} \) satisfies \( \text{A1} \) and \( X^H \in \mathcal{A} \) whenever \( H \in N_P \).

(iii) \( \mathcal{A} \) satisfies \( \text{A1} \) and \( \text{A3} \) and \( X^H \in \mathcal{A} \) whenever \( H \in S_P \).

(iv) \( \mathcal{A} \) satisfies \( \text{A2} \), \( \dim X < \infty \) and \( X^H \in \mathcal{A} \) whenever \( H \in \mathcal{P} \).

Then \( X/G \in \mathcal{A} \).

Proof. The result is a consequence of 1.1. Suppose that \( (H, P') \in \mathcal{W}_P \). If (i) holds, then \( X^{P'}/P'' \in \mathcal{A} \) whenever \( P'' \in M_P(H) \). Since \( \mathcal{A} \) satisfies \( \text{A3} \), it follows that \( X^{P'}/H \in \mathcal{A} \).

Assume that (ii) holds. Let \( H' = H/P' \) and let \( Y = X^{P'} \). We can consider \( Y \) as an \( H'-\text{CW-complex} \). If \( H'_0 \) is a subgroup of \( H' \), then \( H'_0 = H_0/P' \), where \( P' \subseteq H_0 \subseteq H \), and \( Y^{H'_0} = X^{H_0} \in \mathcal{A} \) because \( H_0 \in N_P \). Hence \( X^{P'}/H = Y/H' \in \mathcal{A} \).

If \( \mathcal{A} \) satisfies \( \text{A1} \) and \( X^{G'} \in \mathcal{A} \) whenever \( G' \in S_P \) then \( X^{P'}/P' \in \mathcal{A} \) whenever \( P, P' \in \mathcal{P}, P' \in N_P \). Now we can use part (ii) of this result to obtain (iii).

If (iv) holds, then \( X^{P'}/H \in \mathcal{A} \) by the definitions.

1.3. Examples. Let

\[ \mathcal{C} = \{ X \in CW_0 : X \text{ is contractible} \}, \]
\[ \mathcal{D}(R) = \{ X \in CW_0 : X \text{ is } R\text{-acyclic} \}, \]
\[ B_k(R) = \{ X \in CW_0 : H^i(X, R) = 0 \text{ for } i = 1, \ldots, k \}. \]

(i) The well known decomposition from 0.0(ii) implies that all these classes satisfy \( \text{A1} \).

(ii) The classes \( \mathcal{D}(F_p) \) and \( B_k(F_p) \) satisfy \( \text{A3} \). This is a consequence of the existence of an appropriate transfer. Let \( H \) be a closed subgroup of \( G \) and let \( \pi_X : X/H \to X/G \) be the projection to the orbit space. It is proved in [O2], [LMM], [LMS] that there exists a natural transfer map

\[ t_X : H^*(X/H, R) \to H^*(X/G, R) \]
such that the composition $H^*(\pi_X)t_X$ is the multiplication by the Euler
characteristic $\chi(G/H)$ of $G/H$. If $H$ is a maximal $p$-toral subgroup of $G$, then
$\chi(G/H)$ is prime to $p$. Hence, if $H^n(X/H, F_p) = 0$, then $H^n(X/G, F_p) = 0$.

(iii) The classes $D(Z)$ and $D(F_p)$ satisfy A2. This follows from Theorems 1 and 2 of [O1].

The next result describes the case when $\mathcal{P} = \mathcal{T}_p(G)$ and $\mathcal{A}$ is one of the
classes from 1.3. The statement (i) is a special case of 0.3. For a finite group
$G$, this result is proved in 2.11 of [S1]. The statement (iii), for finite groups,
finite $G$-CW-complexes and $F_p$-acyclic spaces, is proved in [We].

1.4. PROPOSITION. Let $X$ be a $G$-CW-complex such that all its isotropy
groups contain a non-trivial $p$-subgroup. Then:

(i) $X/G$ is contractible (resp. $R$-acyclic) if $X^H$ is contractible (resp. $R$-
acyclic) for all closed subgroups $H$ containing a non-trivial normal $p$-toral
subgroup.

(ii) $X/G$ is $F_p$-acyclic if, for every $H \in \mathcal{S}_p(G)$, $X^H$ is $F_p$-acyclic.

(iii) If dim $X < \infty$ and, for every non-trivial $p$-toral subgroup $H$ of $G$,
$X^H$ is $\mathbb{Z}$-acyclic (resp. $F_p$-acyclic), then $X/G$ is $\mathbb{Z}$-acyclic (resp. $F_p$-acyclic).

Proof. $\mathcal{T}_p(G)$ is a concave set of $p$-subgroups of $G$. By 1.2(ii), $\mathcal{N}_{\mathcal{T}_p(G)} =
\mathcal{N}_p(G)$ so (i) follows. The statement (ii) is a consequence of 1.2(iii) because
$\mathcal{S}_{\mathcal{T}_p(G)} = \mathcal{S}_p(G)$, and (iii) follows from 1.2(iv).

1.5. COROLLARY. If $G$ is a compact Lie group with a non-trivial $p$
subgroup, then the space $BA_p(G)/G$ is contractible.

Proof. It is proved in 6.1 of [JM2] that there are only finitely many
conjugacy classes of elementary abelian $p$-subgroups in $G$. If $x \in BA_p(G),
then G_x = NE_0 \cap \ldots \cap NE_k$, where $E_i \in A_p(G)$ and $E_0 < \ldots < E_k$, so $E_0 \subseteq
G_x \subseteq NE_0$. For every $H \in \mathcal{N}_p(G)$, the space $(BA_p(G))^H = B(A_p(G))^H$ is
contractible. For $G$ finite this follows from 2.1.2 of [We]. The proof for any
compact Lie group is similar. The space $A_p(G)^H$ is a disjoint union of its
$NH/H$-orbits. Let

$$A_p(G)^{\geq E} = \{E' \in A_p(G) : E \subseteq E'\}.$$ 

There exists a non-trivial normal $p$-toral subgroup $P$ of $H$ such that $NH$ is
a subgroup of $NP$. Indeed, let $Q$ be the intersection of all maximal $p$-toral
subgroups of $H$. Then $NH$ is a subgroup of $NQ$. Let $Q_0$ be the component
of the identity of $Q$. We can take $P = Q_0$ if $Q_0$ is non-trivial. If $Q_0 = e$,
then we can take as $P$ the intersection of all Sylow $p$-subgroups of $Q$. In this
case $P$ is the maximal normal $p$-toral subgroup of $H$. It follows from A3 of
[JM0] and 7.6 of [JM1] that if $P' \in \mathcal{T}_p(G)$, then the center $Z(P')$ of $P'$ is
also in $\mathcal{T}_p(G)$. Let $E$ be the maximal elementary abelian $p$-subgroup of $Z(P)$.
Then $E \subset H \subset NH \subset NE$, $NH \subset NCE$ and, for every $E' \in A_p(G)^H,$
$E' \cap CE = E'^E$ is a non-trivial group. The poset map $h_E : A_p(G)^H \rightarrow (A_p(G)^H \cap A_p(CE))_{\geq E}$ such that $h_E(E') = (E' \cap CE)E$ whenever $E' \in A_p(G)^H$, is continuous because it is an $NH/H$-poset map. The map $ Bh_E$ is the composition of the homotopy equivalences $B A_p(G)^H \rightarrow B(A_p(G)^H \cap A_p(CE))$ and $B(A_p(G)^H \cap A_p(CE)) \rightarrow B((A_p(G)^H \cap A_p(CE))_{\geq E})$. The space $B((A_p(G)^H \cap A_p(CE))_{\geq E})$ is contractible because $A_p(G)^H_{\geq E}$ has the final object $E$. Now we can apply 1.4(i).

If $G$ is finite and a normal subgroup $H$ of $G$ contains a non-trivial $p$-subgroup then it was proved in [Dw] that the space $BA_p(G)/H$ is $F_p$-acyclic. In 1.6 we will prove that this space is contractible.

Let $K_p(G)$ denote the set of all subgroups $H$ of $G$ satisfying the condition that, for every maximal $p$-toral subgroup $P$ in $G$, $H \cap Z(P)$ contains a non-trivial $p$-subgroup. If $H \in K_p(G)$ and $H \subseteq H'$, then $H' \in K_p(G)$. If $H$ is a normal subgroup of $G$ which, for every maximal $p$-toral subgroup $P$ of $G$, contains a non-trivial normal $p$-toral subgroup $P'$ of $P$, then $H \in K_p(G)$. Indeed, $H \cap ZP$ contains $P^{pP}$, hence it contains a non-trivial $p$-group. If $G$ is finite and a normal subgroup $H$ of $G$ contains a non-trivial $p$-subgroup, then $H$ belongs to $K_p(G)$. It was proved in [Dw] that in this case $H \cap P$ is a normal subgroup of $P$ and a Sylow $p$-subgroup of $H$ so $H \cap Z(P)$ contains a non-trivial $p$-subgroup. The following result is a generalization of 1.5.

1.6. PROPOSITION. Let $G$ be compact Lie group with a non-trivial $p$-subgroup. If $H \in K_p(G)$ then the space $BA_p(G)/H$ is contractible.

Proof. The result is a consequence of 1.4(i). It follows from the definition that $\mathcal{N}_p(H) \subseteq Q_p(G)$, hence, as in the proof of 1.5, for every $H_0 \in Q_p(H)$, $BA_p(G)^{H_0}$ is contractible. If $x \in BA_p(G)$, then $H_x$ contains a non-trivial $p$-subgroup. Indeed, let $G_x = NE_0 \cap \ldots \cap NE_k$, where $E_i \in A_p(G)$ and $E_0 < \ldots < E_k$. Let $P$ be a maximal $p$-toral subgroup of $G$ such that $E_k \subseteq P$. It follows from the definitions that $H \cap ZP \subseteq H \cap NE_0 \cap \ldots \cap NE_k = H_x$. The assumption that $H \in K_p(G)$ now implies that $H_x$ contains a non-trivial $p$-subgroup.

2. Homotopy decompositions over $(sd W)/G$. Let $\mathcal{C}$ be a topological category. For any two functors $Y : \mathcal{C} \rightarrow \text{Top}$ and $Y' : \mathcal{C}^{op} \rightarrow \text{Top}$, the topological space $Y' \times_\mathcal{C} Y$ is the coequalizer of the two natural maps

$$p_0, p_1 : \coprod_{\alpha : c \rightarrow c'} Y(c) \times Y'(c') \rightarrow \coprod_{c \in \mathcal{C}} Y(c) \times Y'(c)$$

induced by the maps

$$p_0(\alpha)(y, y') = (Y(\alpha)y, y'), \quad p_1(\alpha)(y, y') = (y, Y'(\alpha)y').$$

In particular $\text{hocolim}_{\mathcal{C}} Y = B(- \downarrow \mathcal{C}) \times_\mathcal{C} Y$, where $c \downarrow \mathcal{C}$ is the “under” category of the morphisms $c \rightarrow c'$ of $\mathcal{C}$. 
If \( G_1 \) and \( G_2 \) are groups and \( Y : C \to G_1 \text{-Top}, \ Y' : C^{\text{op}} \to G_2 \text{-Top} \), then \( G_1 \times G_2 \) acts in a natural way on \( Y' \times_C Y \). If \( G_1 = e \) then we obtain a \( G_2 \text{-action.} \)

Let \( G \) be a Lie group and let \( X \) be a \( G \text{-CW-complex.} \) Let \( S(G, X) \) denote the set of isotropy groups of \( X \) and \( S_0(G, X) = S(G, X) \cup \{G\} \). The full subcategory of \( O_G \) whose objects are the orbit spaces \( G/H \), where \( H \in S(G, X) \), is denoted by \( O(G, X) \). The \( G \)-map spaces will be denoted by \( \text{Map}_G(-, -) \).

Let \( F_1, F_2 : G \text{-CW} \to G \text{-CW} \) be functors such that

\[
F_i(X) = \text{Map}_G(-, X) \times_{O_G} F_i, \quad F_i(f) = \text{Map}_G(-, f) \times_{O_G} F_i
\]

whenever \( f : X \to X' \). In the formulas above the restriction of \( F_i \) to the subcategory \( O_G \) of \( G \text{-CW} \) is denoted by the same letter. We will need the following fact.

2.1. PROPOSITION. Let \( \tau : F_1 \to F_2 \) be a natural transformation of functors induced by its restriction to \( O_G \). If, for every \( G/H \in O(G, X) \), \( \tau(G/H) \) is a \( G \)-homotopy equivalence, then so is \( \tau(X) : F_1(X) \to F_2(X) \).

Proof. Since the \( O_G \)-orbits of the functor \( \text{Map}_G(-, X) \) have the form \( \text{Map}_G(-, G/G_x) \), where \( x \in X \), the restriction of \( \text{Map}_G(-, X) \) to \( O(G, X) \) is a free functor in the sense of \([DF1]\) and

\[
F_i(X) = \text{Map}_G(-, X) \times_{O(G, X)} F_i.
\]

This can be proved by induction on the dimension of \( X \). Assume that the \( n \)-skeleton of \( X \), denoted by \( X_n \), is equal to the pushout

\[
D^n \times T_n \leftarrow S^{n-1} \times T_n \to X_{n-1}
\]

where \( T_n \) is a disjoint union of \( G \)-orbits from \( O(G, X) \) and the left arrow is the cofibration induced by the natural inclusion \( S^{n-1} \to D^n \). Then \( F_i(X_n) \) is equal to the homotopy pushout

\[
D^n \times F_i(T_n) \leftarrow S^n \times F_i(T_n) \to F_i(X_{n-1}).
\]

This implies that, if \( \tau(X_{n-1}) \) is a homotopy equivalence then so is \( \tau(X_n) \). Now one can use the fact that \( \tau(X) = \text{hocolim}_{n \in \mathbb{N}} \tau(X_n) \).

2.2. EXAMPLES. (i) Let \( K \) be a \( G \text{-CW-complex.} \) Let \( F = (H(-), H'(-)) : C \to C_G \) be a functor such that, for every isotropy group \( G' \) of \( X \), the map

\[
\text{hocolim}_{c \in \mathcal{C}} K \times_{H(c)} (G/G')^{H'(c)} \to K/G'
\]

is a homotopy equivalence. Then so is the map

\[
\text{hocolim}_{c \in \mathcal{C}} K \times_{H(c)} X^{H'(c)} \to K \times_G X.
\]

(ii) Let \( f : K_1 \to K_2 \) be a cellular map of \( G \text{-CW-complexes.} \) If, for every isotropy group \( H \) of \( X \), \( f/H : K_1/H \to K_2/H \) is a homotopy equivalence, then so is \( f \times_G X : K_1 \times_G X \to K_2 \times_G X \).
(iii) Let $V$ be a $G$-subposet of $S(G)$. Using the fact that for every $G' \in V$, 
\[
\text{hocolim}_{G/H \in \mathcal{O}_V} (G/G')^H = B(G/G' \downarrow \mathcal{O}_V) \simeq * ,
\]
we obtain the decomposition described in 0.0(ii).

(iv) Let $F = (H(-), H'(-)) : \mathcal{C} \to C_G$ be a functor such that, for every isotropy group $G'$ of $X$, the map
\[
\text{hocolim}_{c \in \mathcal{C}} G \times_{H'(c)} (G/G')^{H'(c)} \to G/G'
\]
is a $G$-homotopy equivalence. Then so is the map
\[
\text{hocolim}_{c \in \mathcal{C}} G \times_{H'(c)} X^{H'(c)} \to X.
\]

In this section we will assume that $W$ is a topological $G$-subposet of $S(G)$ and that all elements of $W$ are finite subgroups of $G$. This implies that the orbit spaces $d_nW/G$ are discrete and that $W$ satisfies the condition that $w \leq gw$, where $g \in G$, implies that $w = gw$.

Let $H$ be a closed subgroup of $G$. We will use the notation
\[
W_H = \{ H' \in W : H' \subseteq H \}.
\]
If $H$ is a compact Lie group then the topology on $W_H$ induced from $W$ is equal to the topology induced from $S(H)$. This follows from the fact that $(G/H)^{H'}/NH'$ is discrete (cf. the proof of II.5.7 in [Br2]).

2.3. PROPOSITION. Let $X$ be a $G$-CW-complex such that all its isotropy groups are compact.

(i) If, for every $x \in X$, the map $K \times_{G_x} B(W_{G_x}) \to K/G_x$ is a homotopy equivalence, then there is a homotopy decomposition
\[
\text{hocolim}_{[(H_0, \ldots, H_n)] \in \text{sd} W/G} K \times_{\text{NH}_0 \cap \ldots \cap \text{NH}_n} X^H_n \simeq K \times_G X.
\]

(ii) If, for every $x \in X$, the map $G \times_{G_x} B(W_{G_x}) \to G/G_x$ is a $G$-homotopy equivalence, then there is a $G$-homotopy decomposition
\[
\text{hocolim}_{[(H_0, \ldots, H_n)] \in \text{sd} W/G} G \times_{\text{NH}_0 \cap \ldots \cap \text{NH}_n} X^H_n \simeq X.
\]

Proof. Let
\[
F'_{K}(X) = \text{hocolim}_{[(H_0, \ldots, H_n)] \in \text{sd} W/G} K \times_{\text{NH}_0 \cap \ldots \cap \text{NH}_n} X^H_n.
\]

It follows from the definitions that $F'_{K}(X) = K \times_G F'_G(X)$.

If $X = * = G/G$, then there is a $G$-homotopy equivalence
\[
F'_G(*) = \text{hocolim}_{[(H_0, \ldots, H_n)] \in \text{sd} W/G} G/(\text{NH}_0 \cap \ldots \cap \text{NH}_n) \simeq BW.
\]

Indeed, $F'_G(*)$ is the classifying space of the category $W[G]$ whose objects are the pairs $([w], [g])$, where $[w] \in \text{sd} W/G$, $[g] \in G/(\text{NH}_0 \cap \ldots \cap \text{NH}_n)$, $w = (H_0, \ldots, H_n)$. The category $W[G]$ is a topological poset with an action of $G$ defined by the action of $G$ on $G/G/\text{sd}$ and there is an equivariant isomorphism of topological $G$-poses $F : W[G] \to \text{sd} W$ such that $F([w], [g]) = gw$. Hence we have equivariant homotopy equivalences.
$F_G^\prime(*) \simeq BW[G] \simeq B \text{sd} W$. Let $\mathcal{N}$ be the category whose objects are finite posets $[n] = \{0 \leq 1 \leq \ldots \leq n\}$ and whose morphisms are the injective poset maps. Let $F_W : \mathcal{N} \to G$-Top be the functor such that $F_W([n]) = \text{sd}_n W$ consists of all injective poset maps $[n] \to W$. Let $\Delta_n$ be the standard $n$-dimensional simplex. Then $\Delta \sim$ is a free functor on the category $\mathcal{N}$. This implies that there are equivariant homotopy equivalences

$$B \text{sd} W \simeq \text{hocolim}_N F_W \simeq \Delta \times N \text{sd}(-) W \simeq BW.$$ There is a natural $G$-CW-complex structure on $BW$ such that the poset $\text{sd} W/G = (\text{sd} W)/G$ is equal to the poset of the $G$-cells of $BW$. For $K = \ast$ we obtain homotopy equivalences

$$B((\text{sd} W)/G) = F_G^\prime(*) = F_G^\prime G(*) / G \simeq B(\text{sd} W)/G \simeq (BW)/G.$$ The inclusions $X^{H_n} \to X$ induce a map $p_K(X) : F_K^\prime(X) \to K \times_G X$. The map $p_G(X)$ is a $G$-map and $p_K(X) = K \times_G p_G(X)$. Let $\pi_X : F_G^\prime(X) \to F_G^\prime G(*) \simeq BW$ be the natural $G$-projection. To obtain the result it is sufficient to prove that, for every $x \in X$, the map $p_K(G/G_x)$ is a homotopy equivalence. This follows from the fact that, for every closed subgroup $H$ of $G$, $\pi_X$ induces an $H$-homotopy equivalence $p_G^{-1}(G/H)(H) \to BW$. Indeed, consider the natural projection $f_w : G \times N H_0 \cap \ldots \cap N H_n (G/H)^{H_n} \to G/H$. Then $G \times N H_0 \cap \ldots \cap N H_n (G/H)^{H_n} = G \times_H f_w^{-1}(H)$. Let

$$Y(w, H) = \{g \in G : g H_n g^{-1} \subseteq H\}/(N H_0 \cap \ldots \cap N H_n) \subseteq G/(N H_0 \cap \ldots \cap N H_n).$$

Then there is an $H$-isomorphism $\mu : Y(w, H) \to f_w^{-1}(H)$ such that $\mu([g]) = [g, g^{-1}H]$. The space $p_G^{-1}(G/H)(H) \simeq \text{hocolim}_{[w] \in \text{sd} W/G} Y(w, H)$ is the classifying space of the category $W[H]$ whose objects are the pairs $([w], [g])$, where $[w] \in \text{sd} W/G$, $[g] \in Y(w, H)$. $W[H]$ is a topological subposet of $W[G]$ and the restriction of $F_W$ gives us an $H$-poset isomorphism $W[H] \to \text{sd} W_H$. Now we can use the $H$-homotopy equivalence $B \text{sd} W_H \simeq BW_H$ to conclude that $p_K(G/H)$ is homotopy equivalent to the projection $K \times_H BW_H \to K/H$ (which implies (i)) and that $p_G(G/H)$ is $G$-homotopy equivalent to the projection $G \times_H BW_H \to G/H$ (which implies (ii)).

The following result is an immediate consequence of 2.3.

2.4. Corollary. Let $X$ be a $G$-CW-complex such that all its isotropy groups are compact. Let $W$ be a $G$-poset of finite subgroups of $G$ such that the space $B \text{sd} W/G$ is contractible. Suppose that $A$ is thick and satisfies the condition A1.
(i) Suppose that, for every \( x \in X \), the map \( K \times_{G_x} B(W_{G_x}) \to K/G_x \) is a homotopy equivalence and that, for every \( (H_0, \ldots, H_n) \in \text{sd} W \), we have \( K \times_{NH_0 \cap \cdots \cap NH_n} X^{H_n} \in \mathcal{A} \). Then \( K \times G X \in \mathcal{A} \).

(ii) Suppose that, for every \( x \in X \), the space 
\[
B(W_{G_x})/G_x = B \text{sd} W_{G_x}/G_x
\]
is contractible and that \( X^H \in \mathcal{A} \) whenever 
\[
H \in \{NH_0 \cap \cdots \cap NH_n \cap G' : (H_0, \ldots, H_n) \in \text{sd} W, G' \in S_0(G, X), H_n \subseteq G'\}.
\]

Then \( X/G \in \mathcal{A} \).

2.5. Examples. (i) Let \( X \) be a \( G \)-CW-complex such that all its isotropy groups are finite. Then there exists a \( G \)-homotopy decomposition
\[
\text{hocolim}_{[(H_0, \ldots, H_n)]} \in \text{sd} S(G, X)/G \ G \times_{NH_0 \cap \cdots \cap NH_n} X^{H_n} \simeq X
\]
because, for every \( x \in X \), the space \( BS(G, X)_{G_x} \) is \( G_x \)-contractible.

(ii) Suppose that, for every \( x \in X \), \( y \in K \), \( G_x \in \mathcal{S}_c(G) \) and \( G_x \cap G_y \in \mathcal{K}_p(G_x) \). Then there is a homotopy equivalence 
\[
\text{hocolim}_{[(E_0, \ldots, E_n)]} \in \text{sd} A_p(G)/G \ K \times_{NE_0 \cap \cdots \cap NE_n} X^{E_n} \simeq K \times G X
\]
This is a consequence of 2.3, 2.2(ii) and 1.6. In particular, for \( K = \ast \) we obtain 0.1.

(iii) Let \( G \) be compact Lie group with a non-trivial \( p \)-subgroup. Let \( \mathcal{P} \) be the poset of all non-trivial finite \( p \)-subgroups of \( G \). Then the space \( (BP)/G \) is contractible. This follows from (ii) and from the fact that, for every \( (E_0, \ldots, E_n) \in \text{sd} A_p(G) \), the space \( B(\mathcal{P})^H \) is contractible whenever \( E_n \leq H \leq NE_0 \cap \cdots \cap NE_n \) because \( P'E_n \in \mathcal{P}^H \) if \( P' \in \mathcal{P}^H \).

(iv) Let \( X \) be a \( G \)-CW-complex such that all its isotropy groups are compact and contain a non-trivial normal \( p \)-subgroup. Then there exists a \( G \)-homotopy decomposition 
\[
\text{hocolim}_{(E_0, \ldots, E_n)]} \in \text{sd} A_p(G)/G \ G \times_{NE_0 \cap \cdots \cap NE_n} X^{E_n} \simeq X
\]
because, for every \( x \in X \), the space \( B\mathcal{A}_p(G_x) \) is \( G_x \)-contractible. This follows from the fact that the poset \( A_p(G_x)^{G_x} \) is non-empty (cf. the proof of 1.5), and that, for every isotropy group \( H \) of \( B\mathcal{A}_p(G_x) \), the map \( B\mathcal{A}_p(G_x)^H \to \ast \) is a homotopy equivalence because all isotropy groups of \( B\mathcal{A}_p(G_x) \) contain non-trivial normal \( p \)-subgroups.

One can prove this fact using similar methods to those in 1.5. Let \( E \) be a non-trivial, normal, elementary abelian \( p \)-subgroup of \( G_x \). Let \( W \) be the \( G \)-poset of all subgroups of \( G_x \) of the form \( E'E'' \) where \( E' \in A_p(G_x) \) and \( E'' \) is a subgroup of \( E \). Then \( BW \) is \( G_x \)-contractible.
Let $G$ be a finite group. If $\mathcal{P}$ is a concave $G$-poset of $p$-subgroups of $G$, then $\mathcal{P}^o$ is the $G$-subposet of $\mathcal{P}$ such that $P \in \mathcal{P}^o$ if and only if $P \in \mathcal{P}$ and $\Phi(P) \not\in \mathcal{P}$. Here $\Phi(P)$ denotes the Frattini subgroup of $P$. If $\mathcal{P} = T_p(G)$, then $\mathcal{P}^o = \mathcal{A}_p(G)$.

### 2.6. Proposition

Let $G$ be a finite group. Let $\mathcal{P}'$ be a concave $G$-poset of $p$-subgroups of $G$. Let $X$ be a $G$-CW-complex such that all Sylow $p$-subgroups of its isotropy groups are in $\mathcal{P}'$. Suppose that $\mathcal{P}$ is a $G$-poset of $p$-subgroups of $G$ such that $\mathcal{P}^o \subseteq \mathcal{P} \subseteq \mathcal{P}'$. Then there is a homotopy equivalence

$$\hocolim_{[(P_0,\ldots,P_n)] \in \text{sd} \mathcal{P}/G} X^{P_n}/(NP_0 \cap \ldots \cap NP_n) \simeq X/G.$$ 

**Proof.** The space $B(\mathcal{P}')/G$ is contractible. (This is a generalization of Corollary 2.6.1 of [We], which states that $B(\mathcal{P}')/G$ is $F_p$-acyclic.) Indeed, if $x \in B(\mathcal{P}')$, then $G_x = NP_0 \cap \ldots \cap NP_k$, where $P_i \in \mathcal{P}'$ and $P_0 < \ldots < P_k$, so Sylow $p$-subgroups of $G_x$ are in $\mathcal{P}'$. It is proved in [We] (2.1.2) that, for every $H \in \mathcal{N}_{\mathcal{P}'}$, the space $B(\mathcal{P}')^H$ is contractible. Thus we can apply 1.2(ii) to the class $\mathcal{C}$. Proposition 1.7 of [TW] implies that the $H$-map $B(\mathcal{P}_H) \to B(\mathcal{P}_H')$, induced by the inclusion of $H$-posets of subgroups, is an $H$-homotopy equivalence. The proof of this fact is similar to the proof of 2.1(i) of [TW]. Hence $B(\mathcal{P}_H)/H \simeq B(\mathcal{P}_H')/H$ and the space $B(\mathcal{P}_H)/H$ is contractible. Now we can use 2.1.

The following result is an immediate consequence of 2.6. It is stronger than 1.2.

### 2.7. Corollary

Let $G$ be a finite group. Let $\mathcal{P}$ and $X$ satisfy the assumptions of 2.6. Suppose that $\mathcal{A}$ is thick and satisfies the condition $\mathbf{A1}$ and that one of the following conditions holds:

(i) $X^{P_n}/(NP_0 \cap \ldots \cap NP_n) \in \mathcal{A}$ whenever $(P_0,\ldots,P_n) \in \text{sd} \mathcal{P},$

(ii) $X^H \in \mathcal{A}$ whenever $H \in \{NP_0 \cap \ldots \cap NP_n \cap G' : (P_0,\ldots,P_n) \in \text{sd} \mathcal{P}, G' \in S_0(G,X), P_n \subseteq G'\}.$

Then $X/G \in \mathcal{A}$.

### 2.8. Corollary

Let $G$ be a finite group. Let $\mathcal{P}$ be a $G$-poset of $p$-subgroups of $G$ such that $\mathcal{A}_p(G) \subseteq \mathcal{P}$. If, for every $x \in X$ and $y \in K$, $G_x$ contains a non-trivial $p$-subgroup and $G_x \cap G_y \in K_p(G_x)$, then there is a homotopy equivalence

$$\hocolim_{[(P_0,\ldots,P_n)] \in \text{sd} \mathcal{P}/G} K \times_{NP_0 \cap \ldots \cap NP_n} X^{P_n} \simeq K \times_G X.$$ 

**Proof.** This result is a consequence of 2.5(ii). Let $P$ be a non-trivial $p$-subgroup of $G$. It follows from [TW], 1.7 and 2.1, that there is an $H$-
homotopy equivalence $B(A_p(G)_H) \to B(P_H)$ whenever $H$ is a subgroup of $G$ and contains a non-trivial $p$-subgroup. Now we can use 2.1 and 2.3.

3. Categories associated to $G$-posets. Let $K$ be a $G$-CW-complex. Every equivariant cellular map $f : X_1 \to X_2$ of $G$-CW-complexes induces maps $f(H, H') : K \times_H X_1^{H'} \to K \times_H X_2^{H'}$ where $(H, H') \in \mathcal{W}(G)$, i.e. $H, H' \in \mathcal{S}(G)$ and $H \subseteq NH'$.

Then it follows from 1.1 that the poset $\mathcal{P}$ is a homotopy equivalence. Then the poset $\mathcal{P}$ is a homotopy equivalence whenever $(H, H') \in \mathcal{W}(G)$. This motivates the following definition.

### 3.0. Definition. Let $\mathcal{S}$ be a $G$-poset of closed subgroups of $G$. A $G$-subposet $W$ of $\mathcal{W}(G)$ is $(\mathcal{S}, K)$-essential if, for every equivariant cellular map $f : X \to Y$ of $G$-CW-complexes with all isotropy groups in $\mathcal{S}$, the condition that $K \times_H X^{H'} \to K \times_H Y^{H'}$ is a homotopy equivalence for every $(H, H') \in W$ implies that $K \times_G X \to K \times_G Y$ is a homotopy equivalence.

In particular, if $W$ is $(\mathcal{S}_0(G, X), K)$-essential and $K \times_H X^{H'} \to K/H$ is a homotopy equivalence whenever $(H, H') \in W$, then $K \times_G X \to K/G$ is a homotopy equivalence.

The results of previous sections enable us to exhibit many non-trivial examples of essential posets. Our main tool will be the following consequence of 2.2(i).

### 3.1. Proposition. Suppose that $F = (H(-), H'(-)) : \mathcal{C} \to \mathcal{C}_G(W, d_{\mathcal{W}(G)})$ is a functor such that for every $G' \in \mathcal{S}$, the map $\hocolim_{c \in \mathcal{C}} K \times_{H(c)} (G/G')^{H'(c)} \to K/G'$ is a homotopy equivalence. Then the poset $W$ is $(\mathcal{S}, K)$-essential.

### 3.2. Examples. (i) Let $\mathcal{P}$ be a concave $G$-subposet of $p$-toral subgroups of $G$ such that all maximal $p$-toral subgroups of elements of $\mathcal{S}$ are in $\mathcal{P}$. Then it follows from 1.1 that the poset $\mathcal{W}_{\mathcal{P}} = \{ (H, P) : P \subseteq H \subseteq NP, P \in \mathcal{P}, H \in S'_c(G) \}$ is $(\mathcal{S}_c, *)$-essential. Let $S_K(G)$ be the poset of all compact subgroups $H$ of $G$ with non-trivial $p$-subgroups and such that $H \cap G_k \in \mathcal{K}_p(H)$ for
every $k \in K$. Then the poset $\mathcal{W}_{\mathcal{A}p(G)}$ is also $(\mathcal{S}_K(G), K)$-essential. This is a consequence of 2.5(ii).

(iii) Let $f : X \to Y$ be an equivariant cellular map of $G$-CW-complexes such that, for every compact subgroup $H$ of $G$ with a non-trivial normal $p$-toral subgroup, the map $f^H : X^H \to Y^H$ is a homotopy equivalence. This implies that, for every $(H, H') \in \mathcal{W}_{\mathcal{A}p(G)}$, the map $f^{H'}$ is an $H$-homotopy equivalence so the map $K \times_H X^{H'} \to K \times_H Y^{H'}$ is a homotopy equivalence. If all isotropy groups of points of $X$ and $Y$ are in $\mathcal{S}_K(G)$, then, by (ii), the map $F_K(f) : K \times_G X \to K \times_G Y$ is also a homotopy equivalence. In the case when $K = *$ we obtain 0.3(i).

Now we describe a construction of topological categories $\mathcal{C}$ associated to topological $G$-posets and some examples of functors $\mathcal{C} \to \mathcal{C}_G$ defined on such categories. We show that the known homotopy and homology decompositions can be obtained using this construction.

Let $W$ be a topological $G$-poset such that $W/G$ is a discrete topological space. Let $d : W \to \mathcal{S}(G)$ be a $G$-poset map such that, for every $w \in W$, $dw$ is a subgroup of $G_w$. The $G$-poset maps with the above property will be called admissible maps. Let $\mathcal{C}_G(W, d)$ be the topological category whose objects are the elements of $W$ and whose morphism spaces are defined by

$$\text{Mor}_{\mathcal{C}_G(W, d)}(w, w') = \{g \in G : w \leq gw'\}/dw' \subseteq G/dw'.$$

The composition of $[g] : w \to w'$ and $[g'] : w' \to w''$ is $[gg'] : w \to w''$. The categories $\mathcal{C}_G(W, d)$, for discrete groups $G$, are studied in [S1-3], [JS].

3.3. EXAMPLES. (i) Let $W(G)$ denote the $G$-subposet of $\mathcal{S}(G) \times W$ whose elements are all pairs $(H, w)$ where $w \in W$ and $H \subseteq G_w$. Let $d_{W(G)}$ be the admissible map $W(G) \to \mathcal{S}(G)$ such that $d_{W(G)}(H, w) = H$. Let $\mathcal{C}_G(W(G), d_{W(G)}) = \mathcal{C}_G(W)$. It follows from the definitions that $\mathcal{C}_G(*) = \mathcal{O}_G$. If $p_W : W(G)/G \to \mathcal{S}(G)/G$ is the map induced by the natural projection, then, for every closed subgroup $H$ of $G$, $p_W^{-1}([H]) = W^H/NH$. (In the notation of $[T]$, $\mathcal{C}_G(W) = \bigcup_{H \in \mathcal{O}_G} W^H$.) The space $W(G)/G$ is discrete if, for every $H \in \mathcal{S}(G)$, $W^H/NH$ is discrete. Hence if, for every $w \in W$, $(G/G_w)^H/NH$ is discrete then $W(G)/G$ is a discrete space. This is, in particular, the case when, for every $w \in W$, $G_w$ is compact (cf. II.5.7 of [Br2]).

(ii) Let $d : W \to \mathcal{S}(G)$ be an arbitrary admissible function. Then there exists an inclusion $j_d : \mathcal{C}_G(W, d) \to \mathcal{C}_G(W)$ such that $j_d(w) = (dw, w)$ and the image of $j_d$ is a full subcategory of $\mathcal{C}_G(W)$.

(iii) For $W = \mathcal{S}(G)^{\text{op}}$, $W(G) = \mathcal{W}(G)$ and $\mathcal{C}_G(W) = \mathcal{C}_G$. Let $V$ be a $G$-set of subgroups of $G$. Denote by $\mathcal{W}(V)$ the $G$-subposet of $\mathcal{W}(G)$ such that $(H, H') \in \mathcal{W}(V)$ if and only if $H, H' \in V$ and $H \subseteq H'$. The full subcategory of $\mathcal{C}_G$ whose object set is $\mathcal{W}(V)$ will be denoted by $\mathcal{C}(V)$. If
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Let $G$ be a closed subgroup of $G$, let $H$ be the space of all elements of $G$ which contain $H$. (That is, $C(V) = \bigcap_{H \in \mathcal{O}_V} V(\geq H)$.) Hence the space $W(V)$ is discrete if, for every $H, H' \in \mathcal{H}$, $H \subseteq H'$ implies that $(N_{H'} \setminus (G/H')^{\mathcal{H}})/NH$ is discrete. In particular, if $V$ is a $G$-poset of compact subgroups of $G$, then $W(V)/G$ is discrete (II.5.7 of [Br2]).

(iv) Let $U$ be a $G$-space and let $W$ be a $G$-poset of non-empty finite subsets of $U$. There exists an admissible function $d_U$ such that, for every $w \in W$, $d_Uw = \bigcap_{u \in w} G_u$.

There exists a functor $O_d : \mathcal{C}_G(W, d) \to \mathcal{O}_G$ such that $O_d(w) = G/dw$ for every $w \in W$, and $O_d([g])(g'dw) = g'gdw'$ for every morphism $[g] : w \to w'$ of $\mathcal{C}_G(W, d)$. We will use the notation

$$E_G(W, d) = \operatorname{hocolim}_{w \in \mathcal{C}_G(W, d)} G/dw.$$ 

Let $d' : W^{\text{op}} \to S(G)$ be a $G$-poset map. Then, for every $w \in W$, $dw \subseteq G_w \subseteq Nd^{'w}$. Hence there exists a functor $(d, d') : \mathcal{C}_G(W, d) \to \mathcal{C}_G$ such that $(d, d')(w) = (dw, d^{'w})$.

Let $G'$ be a subgroup of $G$. We will use the notation

$$W_{d', G'} = \{w \in W : d^{'w} \subseteq G'\}.$$ 

$W_{d', G'}$ will be considered as a $G'$-poset. The admissible function $d_{G'} : W_{d', G'} \to S(G')$ will be defined in such a way that, for every $w \in W_{d', G'}$, $d_{G'}w = G' \cap dw$.

3.4. Lemma. Let $G'$ be a closed subgroup of $G$ such that $W_{d', G'}/G'$ is a discrete space. Then there exists a $G$-homotopy equivalence

$$\operatorname{hocolim}_{w \in \mathcal{C}_G(W, d)} G \times_{dw} (G/G')^{d^{'w}} \simeq G \times_{d_{G'}} \operatorname{E}_{G'}(W_{d', G'}, d_{G'}).$$

Proof. Let

$$R_w = \operatorname{Mor}_{\mathcal{C}_G(W, d)}(-, w) = \bigsqcup_{[g] \in G/dw} \operatorname{Mor}_W(-, gw) = G \times_{dw} \operatorname{Mor}_W(-, w),$$

where $\bigsqcup_{[g] \in G/dw} \operatorname{Mor}_W(-, gw)$ is topologized as a subspace of $G/dw$. Then for every functor $T : \mathcal{C}_G(W, d) \to G$-CW, $R_w \times_{\mathcal{C}_G(W, d)} T = T(w)$.

Hence

$$R_w \times_{\mathcal{C}_G(W, d)} G \times_{d(-)} (G/G')^{d(-)} = G \times_{dw} (G/G')^{d^{'w}}$$

$$= G \times_{dw} (\{g \in G : d^{'gw} \subseteq G'\}/G') = G \times_{G'} Y,$$

where $Y = \{g : d^{'gw} \subseteq G'\}/dw$ is a $G'$-subspace of $G/dw$.

We will consider $\mathcal{C}_{G'}(W_{d', G'}, d_{G'})$ as a subcategory of $\mathcal{C}_G(W, d)$. Then

$$Y = R_w \times_{\mathcal{C}_{G'}(W_{d', G'}, d_{G'})} G'/d_{G'}(-).$$
and $R_w$ after restriction to $C_{G'}(W_d', G', d_{G'})$ is equal to

\[
\bigcup_{[gdw] \in Y/G'} \text{Mor}_{C_{G'}(W_d', G', d_{G'})}(-, gw).
\]

Let $E_d = B(- \downarrow C_G(W, d))$. Then $E_d$ is a $C_G(W, d)$-CW-complex whose orbits have the form $R_w$. Hence,

\[
\text{hocolim}_{w \in C_G(W, d)} G \times_{d w} (G/G')^{d' w} \cong E_d \times_{C_G(W, d)} G \times_d (-) (G/G')^{d'}(-) = G \times_{G'} (E_d \times_{C_G(W_d', G', d_{G'})} G'/d_{G'}(-)).
\]

The functor $E_d$ after restriction to the category $C_{G'}(W_d', G', d_{G'})$ remains free in the sense of [DF1]. Hence there exists a $G'$-homotopy equivalence

\[
E_d \times_{C_G(W_d', G', d_{G'})} G'/d_{G'}(-) \cong \text{hocolim}_{w \in C_{G'}(W_d', G', d_{G'})} G'/d_{G'}w.
\]

3.5. **Proposition.** Suppose that, for every $G' \in S$, $W_d', G'/G'$ is a discrete space and the map

\[
\text{hocolim}_{w \in C_{G'}(W_d', G', d_{G'})} K/d_{G'}w \to K/G'
\]

is a homotopy equivalence. Then:

(i) The map

\[
\text{hocolim}_{w \in C_G(W, d)} K \times_{d w} X^{d' w} \to K \times_G X
\]

is a homotopy equivalence if $X$ is a $G$-CW-complex and the isotropy groups of $X$ are in $S$.

(ii) The $G$-poset $\{ (dw, d' w) : w \in W \}$ is $(S, K)$-essential.

**Proof.** Let $F_{d'} : G$-CW $\to G$-CW be a functor such that

\[
F_{d'}(X) = \text{hocolim}_{w \in C_G(W, d)} G \times_{d w} X^{d' w}.
\]

It follows from 3.4 that, for every $G' \in S$, there are homotopy equivalences

\[
K \times_{G'} F_{d'}(G/G') \cong K \times_{G'} \text{hocolim}_{w \in C_{G'}(W_d', G', d_{G'})} G'/dw \cap G'
\]

\[
= \text{hocolim}_{w \in C_{G'}(W_d', G', d_{G'})} K/d_{G'}w \cong K/G' = K \times_{G'} G/G'.
\]

Now, it is sufficient to apply 2.2(i) and 3.1.

We now describe some special cases of 3.5.

3.6. **Examples.** (i) Let $W$ be a topological $G$-poset satisfying the condition that $w \leq gw$ implies $w = gw$. Assume that the spaces $d_n W/G$ are discrete. Let $d_s : \text{sd} W \to S(G)$ be an admissible function such that

\[
d_s w = G_w = G_{w_0} \cap \ldots \cap G_{w_n}.
\]

The natural projection $\text{sd} W \to (\text{sd} W)/G$ induces a natural equivalence of categories $C_G(\text{sd} W, d_s) \to (\text{sd} W)/G$. It follows from the definitions that
there is a $G$-homotopy equivalence
\[
\hocolim_{\mathcal{C}_G(sd W, d_s)} G/d_s(-) \to BW.
\]
If $W$ is a $G$-subset of $\mathcal{S}(G)$, then $d_s(H_0, \ldots, H_n) = NH_0 \cap \ldots \cap NH_n$ and $d'(H_0, \ldots, H_n) = H_n$. Hence 3.5 can be considered as a generalization of 2.3.

(ii) Let $W$ be a $G$-subposet of $\mathcal{S}(G)$. Then $G_w = N_G w = Nw$. If $d : W^{\text{op}} \to \mathcal{S}(G)$ is an arbitrary admissible function, then we can take $d'w = w$ whenever $w \in W$. Let $d_c : W^{\text{op}} \to \mathcal{S}(G)$ be an admissible map such that, for every $w \in W$, $d_c w = C_G w = Cw$. Then $C_G(W^{\text{op}}, d_c) = \mathcal{C}_W$ is the category whose objects are elements of $W$ and whose morphisms are the group homomorphisms which are restrictions of inner automorphisms of $G$. Let $X$ be a $G$-CW-complex such that all its isotropy groups are compact. If the space $\hocolim_{w \in \mathcal{C}_W} H/C_H w$ is $H$-contractible whenever $H$ is an isotropy group of $X$, then the map
\[
\hocolim_{w \in \mathcal{C}_W} G \times_{C_G w} X^w \to X
\]
is a $G$-homotopy equivalence. If the map
\[
\hocolim_{w \in \mathcal{C}_W} K/C_H w \to K/H
\]
is a homotopy equivalence whenever $H$ is an isotropy group of $X$, then the map
\[
\hocolim_{w \in \mathcal{C}_W} K \times_{C_G w} X^w \to K \times_G X
\]
is also a homotopy equivalence.

(iii) Let $W = \mathcal{A}_p(G)$. Then $C_G(\mathcal{A}_p(G)^{\text{op}}, d_c) = \mathcal{A}_p(G)$. If $H$ is a compact Lie group with a non-trivial $p$-subgroup, then there is an $H$-homotopy equivalence
\[
\hocolim_{E \in \mathcal{A}_p(H)} H/C_H E \simeq \mathcal{E}O_{Z_p(H)}
\]
where $Z_p(G)$ is the poset of all compact subgroups of $G$ with a non-trivial central $p$-subgroup and
\[
\mathcal{E}O_{Z_p(H)} = E_H(Z_p(H), \text{id}) = \hocolim_{H/H' \in \mathcal{O}_{Z_p(H)}} H/H'.
\]
Indeed, for every $H' \in Z_p(H)$, the space $(\hocolim_{E \in \mathcal{A}_p(H)} H/C_H E)^{H'} = B(H/H' \downarrow d_c)$ is homotopy equivalent to $B(H' \downarrow d_c) = B(\mathcal{A}_p(C_H H'))$ and hence is contractible. This implies that there is a $G$-homotopy equivalence
\[
\hocolim_{E \in \mathcal{A}_p(G)} G \times_{C_G E} X^E \simeq X
\]
whenever all isotropy groups of $X$ are in $Z_p(G)$.

3.7. Example. Let $V$ be a $G$-subset of $\mathcal{S}(G)$ such that $\mathcal{W}(V)/G$ is discrete. Let
\[
r_V(X) = \hocolim_{(H, H') \in \mathcal{C}(V)} G \times_H X^{H'}.
\]
This construction is natural in $X$ and $\mathcal{S}(G, r_V(X)) \subseteq V$. The $G$-maps $G \times_H X^{H'} \to X$ define a natural transformation of functors $p_V : r_V \to \text{Id}_{G\text{-CW}}$. 

\textit{Homotopy decompositions of orbit spaces}
There exists a $G$-homotopy equivalence (natural in $X$)
\[ r_V(X) \to B(\text{Map}_G(G/e, -), \mathcal{O}_V, \text{Map}_G(-, X)) \]
where $B(-, -)$ is the bar construction described in Section 3 of [HV] and in Section 4 of [Dw].

If $G' \in V$, then the map $p_V(X)G' : r_V(X)G' \to XG'$ is a homotopy equivalence. Indeed, in this case we have homotopy equivalences
\[
(\text{hocolim}_{(H,H')} \in \mathcal{C}(V)) G \times_H X^{H'} \simeq \text{hocolim}_{(H,H') \in \mathcal{W}(V(G'))} X^{H'} \simeq X^{G'}.
\]
Suppose that all isotropy groups of $X$ are in $V$. Then $p_V(X) : r_V(X) \to X$ is a $G$-homotopy equivalence and gives us a $G$-homotopy decomposition of $X$
\[
\text{hocolim}_{(H,H') \in \mathcal{C}(V)} G \times_H X^{H'} \simeq X
\]
from 0.0(i). If $f : X_1 \to X_2$ is an equivariant map of $G$-CW-complexes and, for every $H \in V$, $f^H : X_1^H \to X_2^H$ is a homotopy equivalence, then $r_V(f)$ is a $G$-homotopy equivalence because, for every $(H,H') \in \mathcal{W}(V), H$ acts trivially on $X^{H'}$. Hence, for every $K$, $\mathcal{W}(V)$ is $(V,K)$-essential.

It follows from the definitions that $p_V(X)/G$ gives us a homotopy decomposition of $X/G$ from 0.0(ii):
\[
\text{hocolim}_{(H,H') \in \mathcal{O}_V} X^{H'} \simeq \text{hocolim}_{(H,H') \in \mathcal{C}(V)} X^{H'} \simeq X/G
\]
and that
\[
\mathcal{E}\mathcal{O}_V = E_G(V, \text{id}) = \text{hocolim}_{(H,H') \in \mathcal{O}_V} G/H = \text{hocolim}_{(H,H') \in \mathcal{C}(V)} G/H = r_V(\ast).
\]

Let $G'$ be a closed subgroup of $G$ and let $V$ be a $G$-subposet of $S(G)$ such that the spaces $\mathcal{W}(V)/G$ and $\mathcal{W}(V_{G'})/G'$ are discrete. The following two results are consequences of 3.5 and the fact that $\mathcal{C}(V) = \mathcal{C}_G(\mathcal{W}(V), d_{\mathcal{W}(G)})$ and $r_V(\ast) = E_G(\mathcal{W}(V), d_{\mathcal{W}(G)})$.

3.8. COROLLARY. There exists a $G$-homotopy equivalence
\[
\text{hocolim}_{(H,H') \in \mathcal{C}(V)} G \times_H (G/G')^{H'} \simeq G \times_G' E_G'(\mathcal{W}(V_{G'}), d_{\mathcal{W}(G')}).
\]

3.9. COROLLARY. Let $f : X_1 \to X_2$ be a $G$-cellular map such that, for every $H \in V$, $f^H$ is a homotopy equivalence.

(i) If, for every isotropy group $G'$ of $X_i$, the map $r_{V_{G'}}(\ast) \to \ast$ is a $G'$-homotopy equivalence, then the maps
\[
\text{hocolim}_{(H,H') \in \mathcal{C}(V)} G \times_H X_i^{H'} \to X_i
\]
and $f$ are $G$-homotopy equivalences.

(ii) If, for every isotropy group $G'$ of $X_i$, the map $K \times_G r_{V_{G'}}(\ast) \to K/G'$ is a homotopy equivalence, then the maps
\[
\text{hocolim}_{(H,H') \in \mathcal{C}(V)} K \times_H X_i^{H'} \to K \times_G X_i
\]
and $\text{id}_K \times_G f$ are also homotopy equivalences.
3.10. Example. Let $X$ be a $G$-CW-complex. It follows from 3.6(iii) and 2.5(iv) that there are $G$-homotopy equivalences
\[ r_{Z_p(G)}(X) \simeq \text{hocolim}_{E \in A_p(G)} G \times_{C_{GE}} X^E, \]
\[ r_{N_p(G)}(X) \simeq \text{hocolim}_{((E_0,\ldots,E_n)) \in \text{sd} A_p(G)/G} G \times_{N E_0 \cap \ldots \cap N E_n} X^{E_n}. \]

3.11. Example. Let $G$ be a discrete group. Let $V$ be a $G$-poset of subgroups of $G$ satisfying the condition that $v \leq g v$ implies $v = g v$. Let $d : V^{\text{op}} \to \mathcal{S}(G)$ be an admissible function. It is proved in [JS] that, for every admissible function $d'' : W \to \mathcal{S}(G)$, there exists a natural $G$-map $E_G(W,d'') \to BW$ which is a homotopy equivalence. This implies that if, for every isotropy group $G_0$ of $X$, the space $BV_{G_0}$ is contractible, then the $G$-maps
\[ \text{hocolim}_{(H,H') \in \mathcal{C}(V)} G \times_{H} X^{H'} \to X, \]
\[ \text{hocolim}_{H \in \mathcal{C}_G(V^{\text{op}},d)} G \times_{dH} X^{H} \to X \]
are homotopy equivalences and that, for every free $G$-CW complex $K$, we have homotopy decompositions
\[ \text{hocolim}_{(H,H') \in \mathcal{C}(V)} K \times_{H} X^{H'} \simeq K \times_{G} X, \]
\[ \text{hocolim}_{H \in \mathcal{C}_G(V^{\text{op}},d)} K \times_{dH} X^{H} \simeq K \times_{G} X. \]
Here $V_{\leq G'} = \{ H \in V : H \leq G' \}$.

3.12. Remark. One can generalize the above result of [JS] and construct $G$-maps (natural in $X$)
\[ \text{hocolim}_{(H,H') \in \mathcal{C}(V)} G \times_{H} X^{H'} \to Y, \]
\[ \text{hocolim}_{H \in \mathcal{C}_G(V^{\text{op}},d)} G \times_{dH} X^{H} \to Y, \]
where
\[ Y = \text{hocolim}_{[(H_0,\ldots,H_n)] \in \text{sd} V/G} G \times_{N H_0 \cap \ldots \cap N H_n} X^{H_n}, \]
which are homotopy equivalences. Hence, for every free $G$-CW-complex $K$, we have homotopy equivalences
\[ K \times_{G} r_V(X) \simeq \text{hocolim}_{[(H_0,\ldots,H_n)] \in \text{sd} V/G} K \times_{N H_0 \cap \ldots \cap N H_n} X^{H_n}, \]
\[ \text{hocolim}_{H \in \mathcal{C}_G(V^{\text{op}},d)} K \times_{dH} X^{H} \]
\[ \simeq \text{hocolim}_{[(H_0,\ldots,H_n)] \in \text{sd} V/G} K \times_{N H_0 \cap \ldots \cap N H_n} X^{H_n}. \]

4. $h_G^*$-decompositions of $G$-CW-complexes. Let $G$ be a Lie group and let $h_G^*$ be a generalized $G$-cohomology theory. Let $h\mathcal{O}_G$ be the category whose objects are the same as the objects of $\mathcal{O}_G$ and whose morphisms are the $G$-homotopy classes of the morphisms of $\mathcal{O}_G$. Let $M$ be a functor from the category $h\mathcal{O}_G^{\text{op}}$ to the category $\text{Ab}$ of abelian groups. The ordinary equivariant cohomology of a $G$-CW-complex $Y$ with coefficients in $M$ will
be denoted by $H^*_G(Y,M)$. These cohomology groups, in the case when $G$ is a finite group, was defined in [Br1]. The case of a Lie group is described in [Wi] and in the appendix of [JMO]. For any generalized $G$-cohomology theory $h^*_G$ on $G$-CW, there is a spectral sequence

$$H^m_G(Y,h^n_G(-)) \Rightarrow h^{m+n}_G(Y).$$

For every closed subgroup $H$ of $G$, the $H$-cohomology theory such that $h^*_H(X') = h^*_G(G \times_H X')$ whenever $X'$ is an $H$-CW-complex will be denoted by $h^*_H$. This gives us a functor $h^*_H(-)(X^{H'}(-))$ defined on the homotopy category $\mathcal{C}$ associated to $\mathcal{C}$. This functor can be considered as coefficients of the generalized cohomology theory $h^*_G(- \times_{\mathcal{C}} (G \times_{H(-)} X^{H'}(-)))$ defined on the category of free $\mathcal{C}$-CW-complexes in the sense of [DF1], i.e. $\mathcal{C}$-CW-complexes with orbits of the form $\text{Mor}_{\mathcal{C}}(-,c)$. For every contravariant functor $M : h\mathcal{C} \to Ab$, $h^*(\mathcal{C}, M) = \text{Tor}^*_{\mathcal{C}}(\mathbb{Z}, M)$ is equal to the Bredon cohomology groups $H^*_\mathcal{C}(B(- \downarrow \mathcal{C}), M)$ (Sections 4 and 5 of [DF1]). Recall that

$$\text{hocolim}_{c \in \mathcal{C}}(G \times_{H(-)} X^{H'}(-)) = B(- \downarrow \mathcal{C}) \times_{\mathcal{C}} G \times_{H(-)} X^{H'}(-).$$

Let $W$ be a $G$-subposet of $\mathcal{W}(G)$. Let $F = (H(-), H'(-)) : \mathcal{C} \to \mathcal{C}_G(W, d_{\mathcal{W}(G)})$ be a functor such that the map

$$p_F(X) : \text{hocolim}_{c \in \mathcal{C}} G \times_{H(c)} X^{H'(c)} \to X$$

is an $h^*_G$-decomposition of $X$, i.e. the map

$$h^*_G(X) \to h^*_G(\text{hocolim}_{c \in \mathcal{C}} G \times_{H(c)} X^{H'(c)})$$

is an isomorphism. It follows from 5.3 of [DF1] that there exists a spectral sequence

$$H^m(\mathcal{C}, h^n_{H(-)}(X^{H'(-)})) \Rightarrow h^{m+n}_G(X).$$

The results of this section describe and use this spectral sequence in many examples.

We remark that if $X = *$ and $F = G/H(-) : \mathcal{C} \to \mathcal{O}_G$, then we obtain the spectral sequence of the generalized cohomology theory $h^*_G$ on $Y = \text{hocolim}_{c \in \mathcal{C}} F(c)$.

Let $f : X_1 \to X_2$ be a $G$-CW-complex map and let $p_F(X_i)$ be an $h^*_G$-decomposition of $X_i$ for $i = 1, 2$. If, for every $c \in \mathcal{C}$, $h^*_{H(c)}(X_2^{H'(c)}) \to h^*_{H(c)}(X_1^{H'(c)})$ is an isomorphism then the map $h^*(f) : h^*_G(X_2) \to h^*_G(X_1)$ is an isomorphism. This motivates the following definition.

4.0. DEFINITION. Let $\mathcal{S}$ be a $G$-subposet of $\mathcal{S}(G)$. Let $W$ be a $G$-subposet of $\mathcal{W}(G)$. We will say that $W$ is $(\mathcal{S}, h^*_G)$-essential if, for every equivariant cellular map $f : X \to Y$ of $G$-CW-complexes whose isotropy groups
are all in \( S \), the condition that \( h^*_H(Y^{H'}) \to h^*_H(X^{H'}) \) is an isomorphism whenever \((H, H') \in W\) implies that \( h^*_G(Y) \to h^*_G(X) \) is an isomorphism.

In particular, if \( W \) is \((S_0(G, X), h^*_G)\)-essential and \( h^*_H(*) \to h^*_H(X^{H'}) \) is an isomorphism whenever \((H', H) \in W\), then \( h^*_G(*) \to h^*_G(X) \) is an isomorphism.

The following result can be used to construct many non-trivial examples of \( h^*_G\)-essential posets.

4.1. Proposition. Let \( F = (H(-), H'(-)) : \mathcal{C} \to \mathcal{C}_G(W, d_{W(G)}) \) be a functor such that for every \( G' \in S \), the map

\[
\text{hocolim}_{c \in \mathcal{C}} G \times_{H(c)} (G/G')^{H'(c)} \to G/G'
\]

is an \( h^*_G\)-equivalence. Then:

(i) The map \( p_F(X) \) is an \( h^*_G\)-decomposition of \( X \) if all isotropy groups of \( X \) are in \( S \).

(ii) The poset \( W \) is \((S, h^*_G)\)-essential.

Proof. Let

\[
h^*_G(X) = h^*_G(\text{hocolim}_{c \in \mathcal{C}} G \times_{H(c)} X^{H'(c)}).
\]

Then \( p_F \) induces a natural transformation \( p^* : h^*_G \to h^*_G \) of \( G\)-cohomology theories. If the assumption of the proposition holds, then \( p^*(X) \) is an isomorphism. Hence \( p_F(X) \) is an \( h^*_G\)-equivalence.

Let \( R \) be a commutative ring. The generalized \( G\)-cohomology theories from the category \( G\)-CW to the category \( R^*\text{-Mod} \) of graded \( R\)-modules will be called \( R\)-cohomology theories.

Let \( V \) be a \( G\)-poset of compact subgroups of \( G \). Recall that \( \mathcal{C}(V) \) is a full subcategory of \( \mathcal{C}_G \) whose objects are the elements of the poset \( W(V) \) of pairs \((H, H')\) such that \( H \) is a subgroup of \( H' \) and \( H' \in V \).

4.2. Proposition. Let \( h^*_G = \{h^n \}_{n \in \mathbb{N}} \) be an \( R\)-cohomology theory. Let \( S \) and \( V \) be \( G\)-posets of compact subgroups of \( G \) such that, for every \( H \in S \), \( h^*_H(*) \to h^*_H(r_{VH}(*)) \) is an isomorphism. Then:

(i) The \( G\)-poset \( W(V) \) is \((S, h^*_G)\)-essential.

(ii) Let \( f : X \to Y \) be a map of \( G\)-CW-complexes whose isotropy groups are all in \( S \). If, for every \( H \in V \), the map \( X^H \to Y^H \) is an \( R\)-equivalence, then \( h^*_G(Y) \to h^*_G(X) \) is an isomorphism.

Proof. (i) is a consequence of 3.8 and 4.1(ii).

(ii) Propositions 4.1(i) and 3.8 imply that, for every \( G\)-CW-complex \( X \) whose isotropy groups are in \( S \), there exists a spectral sequence

\[
H^m(\mathcal{C}(V), h^*_H(-)(X^{H'}(-))) \Rightarrow h^{m+n}_G(X).
\]
This spectral sequence is natural in $X$. The assumption implies that, for every $(H, H') \in \mathcal{C}(V)$, the map $h_H^*(Y^{H'}) \to h_H^*(X^{H'})$ is an isomorphism because $H \subseteq H'$. Hence the map $X \to Y$ is an $h_G^*$-equivalence.


(i) Let $K = \ast, R = F_p$. It is proved in [JMO] (1.2, 2.2, 2.12) that, if $H$ is a compact Lie group and $\dim H > 0$, then the space $\mathcal{E}O_{T_p}(H) = r_{T_p}(H)(\ast)$ is $F_p$-acyclic. Let $S_d(G)$ denote the set of all compact subgroups $H$ of $G$ such that $\dim H > 0$. Let $f : X \to Y$ be a map of $G$-CW-complexes whose isotropy groups are all in $S_d(G)$. If, for every non-trivial $p$-toral subgroup $H$ of $G$, the map $f^H : X^H \to Y^H$ is an $F_p$-homology isomorphism, then so is $f$. In particular, let $G$ be a compact Lie group. If all isotropy groups of $X$ are in $S_d(G)$ and, for every non-trivial $p$-toral subgroup $H$ of $G$, $X^H$ is $F_p$-acyclic, then $X$ is $F_p$-acyclic.

(ii) Let $A_p'(G) = A_p(G) \cup \{e\}$. If $H \in \mathbb{Z}_p(G)$ and $E \in A_p'(H)$, then the space $\mathcal{E}O_{A_p}(H)/E = \mathcal{E}O_{A_p'}(H)/E$ is contractible. Let $f : X \to Y$ be a map of $G$-CW-complexes whose isotropy groups are all in $\mathbb{Z}_p(G)$. Suppose that, for every $E \in A_p(G)$, $f^E$ is an $R$-homology isomorphism and that, for every $k \in K$ and $x \in X \cup Y$, $G_x \cap G_k$ is an elementary abelian $p$-subgroup of $G_x$. This implies that, for every $x \in X \cup Y$, the map $K \times_{G_x} \mathcal{E}O_{A_p}(G_x) \to K/G_x$ is an $R$-homology isomorphism. Hence the map $K \times_G X \to K \times_G Y$ is an $R$-homology isomorphism.

(iii) Let $K = \ast$. Then we obtain 0.3(ii) as a consequence of 4.2 and 1.4.

Let $h_G^*(X) = H^*(K \times_G X, F_p)$. In this case there is a spectral sequence

$$H_G^m(K, H^n(X \times_G (-), F_p)) \Rightarrow h_G^{n+m}(X).$$

Hence if, for all maximal $p$-toral subgroups $P$ of isotropy groups of $K$, $X/P$ is $F_p$-acyclic, then $h_G^*(X) = H^*(K/G, F_p) = h_G^*(\ast)$. We will use this fact in the following examples.


(i) Let $S'_p(G)$ be the poset of all subgroups of $p$-toral subgroups of $G$, and let $S_p(G)$ be the subposet of $S'_p(G)$ consisting of all subgroups which contain a non-trivial $p$-subgroup. Let $H$ be a compact subgroup of $G$. Then $h_H^*(\ast) = h_H^*(r_{S'_p}(H)(\ast))$ because, for every $p$-toral subgroup $P$ of $H$, $r_{S'_p}(H)(\ast)/H$ is $F_p$-acyclic. It follows from Section 3 of [JO] that the maps $H_H^m(\ast, h_H^n) \to H_H^m(r_{S'_p}(H)(\ast), h_H^n)$, where $m > 0$, are isomorphisms. Hence so are the maps $H_H^0(\ast, h_H^n) \to H_H^0(r_{S'_p}(H)(\ast), h_H^n)$. From 3.3 of [JO] and 1.2 and 2.2 of [JMO], it follows that the map $r_{T_p}(H)(\ast) \to r_{S'_p}(H)(\ast)$ induces isomorphisms in $h_H^*$ and $H_H^*(-, h_H^n)$. This implies that the maps $h_G^*(Y) \to h_G^*(X)$ and...
$H^*_G(Y,h^n_P) \rightarrow H^*_G(X,h^n_P)$ are isomorphisms if, for every $p$-toral subgroup $P$ of $G$, $f^P$ is a mod $p$ homology isomorphism.

(ii) Suppose that, for all $n > 0$, $H^n(K,F_p) = 0$. Let $n > 0$. In this case $h^n_H(H/e) = 0$ and (i) implies that the map $H^*_H(*) \rightarrow H^*_H(r_{T_p}(H)(*)/H)$ is an isomorphism. Hence $H^*_G(Y,h^n_G) \rightarrow H^*_G(X,h^n_G)$ is an isomorphism if, for every non-trivial $p$-toral subgroup $P$ of $G$, $f^P$ is a mod $p$ homology isomorphism.

(iii) Suppose that $K$ is $F_p$-acyclic. Then, for every $H/H' \in O_H$, $h^n_H(H/H') = F_p$ and $h^n_H(-)$ is the constant functor after restriction to $O_H$. It follows from 1.2 and 2.2 of [JMO] and Proposition 2 and Theorem 3 of [O1] that the map $H^*(r_{T_p}(H)(*)/H,F_p) \rightarrow H^*(r_{T_p}(H)(*)/H,F_p)$ is an isomorphism. By (ii), so is $h^n_H(r_{T_p}(H)) \rightarrow h^n_H(r_{T_p}(H)(*)).$ Suppose that all isotropy groups of $X$ and $Y$ contain non-trivial $p$-subgroups. If, for every non-trivial $p$-toral subgroup $P$ of $G$, $f^P$ is a mod $p$ homology isomorphism, then, for all natural $n$, the maps $H^*_G(Y,h^n_G) \rightarrow H^*_G(X,h^n_G)$ and $H^n(K \times_G Y,f^P) \rightarrow H^n(K \times_G X,f^P)$ are isomorphisms. In particular, we obtain 0.4. If $G$ is a compact Lie group, then we can take $K = BA_p(G)$, $Y = *$ (cf. the proof of 1.5) to obtain 0.5.

(iv) Let $K$ be a $G$-CW-complex such that, for every $k \in K$ and for every $p$-toral subgroup $P$ of $G$, $G_k \cap P$ is an elementary abelian $p$-group. Suppose that $K$ is $F_p$-acyclic. In particular, we can take $K = EG.$ If all isotropy groups of $X$ and $Y$ contain non-trivial $p$-subgroups and, for every $E \in A_p(G)$, $f^E$ is a mod $p$ homology isomorphism, then $K \times_G Y$ is a mod $p$ homology isomorphism. Indeed, it follows from 4.3(ii) that $K \times_G r_{T_p(G)}(Y) \rightarrow K \times_G r_{T_p(G)}(Y)$ is a mod $p$ homology isomorphism. Now we can use the fact that, by (iii), $K \times_G r_{T_p(G)}(X) \rightarrow K \times_G X$ is a mod $p$ homology isomorphism.

4.5. EXAMPLES. Let $G$ be a discrete group and $A$ a $\mathbb{Z}(G)$-module. We will consider the Bredon cohomology theory $H^*_G = H^*_G(-,M_A)$, where $M_A(-) = \text{Hom}_{\mathbb{Z}(G)}(\mathbb{Z}(-),A)$. Hence

$$H^*_G(X,M_A) = H^*_G(\text{Hom}_{\mathbb{Z}(G)}(C_*(X),A))$$

where $C_*(X)$ is the ordinary cellular chain complex of $X$. For every $G/H \in O_G$, we have $h^n_G(G/H) = M_A(G/H) = A^H$.

(i) Let $G$ be a finite group. Suppose that there is a non-trivial $p$-subgroup $P$ of $G$ such that every element of $P$ acts trivially on $A$. Then $H^*_G(|\mathcal{S}_p(G)|,M_A) = A^G = H^*_G(*,M_A)$.

Indeed, $M_A$ is a Hecke functor and it follows from the results of [Wa1] that if $A$ is an $R(G)$-module and, for every subgroup $H$ of $G$, $X/H$ is $R$-acyclic, then $H^*_G(X,M_A) = A^G$. Let $H$ be a normal subgroup of $G$ with
a non-trivial \( p \)-subgroup and let \( G' = G/H \). If \( A' \) is a \( \mathbb{Z}(G') \)-module, then \( H^*_G((S_p(G)) \mid H, M_{A'}) = A'^G \) because, by 2.8, \( |S_p(G)| \mid H' \) is contractible whenever \( H \subseteq H' \subseteq G \).

(ii) Let \( G \) be a discrete group. Let \( S_A(G) \) denote the set of all finite subgroups \( H \) of \( G \) with a non-trivial \( p \)-subgroup \( P \) such that every element of \( P \) acts as identity on \( A \). Suppose that all isotropy groups of \( X \) and \( Y \) are in \( S_A(G) \). The map \( H^*_G(Y, M_A) \to H^*_G(X, M_A) \) is an isomorphism if, for every compact subgroup \( H \) of \( G \) with a non-trivial normal \( p \)-toral subgroup, \( f^H \) is a homology isomorphism.

(iii) Let \( A \) be an \( F_p(G) \)-module. Let \( K \) be a \( G \)-CW-complex. Suppose that all isotropy groups of points of \( X \) and \( Y \) are finite. In this case the maps \( h^*_G(K \times Y) \to h^*_G(K \times X) \) and

\[
H^*_G(Y, h^n_G(K \times (-))) \to H^*_G(X, h^n_G(K \times (-)))
\]

are isomorphisms if, for every \( p \)-subgroup \( P \) of \( G \), \( f^P \) is a mod \( p \) homology isomorphism. This is a consequence of the fact that, for every Hecke functor \( M : O^\text{op}_G \to F_p\text{-Mod} \), \( M(G/G) = H^*_G(r_{T_p}(G), M) \) (1.29 of [S3]).

Let \( W \) be a topological \( G \)-poset satisfying the condition that \( w \leq gw \), where \( g \in G \), implies that \( w = gw \). Let \( d : W \to S(G) \) be an admissible function and let \( d' : W^\text{op} \to S(G) \) be a \( G \)-poset map. The next result follows immediately from 3.4 and 4.1.

4.6. PROPOSITION. Suppose that, for every isotropy group \( H \) of the action of \( G \) on \( X \), the space \( W_{d',H}/H \) is discrete and the map

\[
h^*_H(*) \to h^*_H(\text{hocolim}_{w \in C_H(W_{d',H}, dH)} H/H \cap dw)
\]

is an isomorphism. Then so is the map

\[
h^*_G(X) \to h^*_G(\text{hocolim}_{w \in C_G(W,d)} G \times dw X^{d'w})
\]

and there is a spectral sequence

\[
H^m(C_G(W,d), h^n_{d(-)}(X^{d'(-)}) \to h^m_n(X).
\]

4.7. EXAMPLE. Let \( K \) be a \( G \)-CW-complex. Suppose that, for every \( x \in X \), the map

\[
H^*(K/G_x, R) \to H^*(\text{hocolim}_{w \in C_{G_x}(W_{d',G_x}, d_{G_x})} K/G_x \cap dw, R)
\]

is an isomorphism. Then so is the map

\[
H^*(K \times_G X, R) \to H^*(\text{hocolim}_{w \in C_G(W,d)} K \times dw X^{d'w}, R)
\]

and there is a spectral sequence

\[
H^m(C_G(W,d), H^n(K \times_{d(-)} X^{d'(-)}, R) \Rightarrow H^{m+n}(K \times_G X, R).
\]
In particular, if, for every isotropy group $H$ of $X$, $BC_H(W_{d',H},d_H)$ is $R$-acyclic, then there is a spectral sequence

$$H^m(C_G(W,d), H^n(X^{d,w}/dw,R)) \Rightarrow H^{m+n}(X/G,R).$$

4.8. Examples. Let $W$ be a poset of closed subgroups of $G$ satisfying the condition that $w \leq gw$ implies $w = gw$ and such that the spaces $d_nW/G$ are discrete. Let $d : W^{\text{op}} \to S(G)$ be an admissible function.

(i) Suppose that the map

$$h^*_H(*) \to h^*_H(\text{hocolim}_{w \in C_H(W_{d',H})} H/dw)$$

is an isomorphism whenever $H$ is an isotropy group of $X$. Then the map

$$h^*_G(X) \to h^*_G(\text{hocolim}_{w \in C_G(W,d)} G \times_{dw} X^w)$$

is an isomorphism and there is a spectral sequence

$$H^m(C_G(W,d), h^n_{dw}(X^w)) \Rightarrow h^{m+n}_G(X).$$

(ii) Suppose that the map

$$h^*_H(*) \to h^*_H(BW_H)$$

is an isomorphism whenever $H$ is an isotropy group of $X$. Then the map

$$h^*_G(X) \to h^*_G(\text{hocolim}_{[w,]} \in \text{sd } W/G G \times_{G_w} X^{w,n}) = h^*_G(\text{hocolim}_{w, \in \text{sd } W} X^{w,n})$$

is an isomorphism and there is a spectral sequence

$$H^m(\text{sd } W/G, h^n_{G_w}(X^{w,n})) \Rightarrow h^{m+n}_G(X).$$

(iii) Let $G$ be a discrete group. Let $K$ be a free $G$-CW-complex. Suppose that the map

$$K \times_H BW_H \to K/H$$

is a mod $p$ homology isomorphism whenever $H$ is an isotropy group of $X$. Then, similarly to 3.11, the map

$$\text{hocolim}_{C_G(W^{\text{op}},d)} K \times_{d_w} X^w \to K \times G X$$

is a mod $p$ homology isomorphism.

4.9. Examples. Let $W = A_p(G)$. Let $X$ be a $G$-CW-complex such that all its isotropy groups are compact and contain non-trivial $p$-subgroups. Let $K$ be an $F_p$-acyclic $G$-CW-complex.

(i) Let $d = d_c$. Then $C_G(A_p(G)^{\text{op}}, d_c) = A_p(G)$. Suppose that, for every isotropy group $H$ of the action of $G$ on $X$, the map $h^*_H(*) \to h^*_H(\mathcal{E}O_{Z_p(H)})$ is an isomorphism. Then it follows from 3.6(iii) that the map

$$h^*_G(X) \to h^*_G(\text{hocolim}_{E \in A_p(G)} G \times_{C_G E} X^E)$$

is an isomorphism and there is a spectral sequence

$$H^m(A_p(G), h^n_{C_G E}(X^E)) \Rightarrow h^{m+n}_G(X).$$
Let $h^*_G = H^*_G(K \times_G -, F_p)$. It follows from 4.4(iii) that if $H$ is a compact subgroup of $G$ and contains a non-trivial $p$-subgroup, then $h^*_H(\mathcal{E}O_{Z_p}(H))$. Hence there is a mod $p$ homology isomorphism
\[ \text{hocolim}_{w \in A_p(G)} K \times_{C_Gw} X^w \to K \times_G X, \]
and there exists a spectral sequence
\[ H^n(A_p(G), H^m(K \times_{C_Gw} X^w, F_p)) \Rightarrow H^{n+m}(K \times_G X, F_p). \]
If $K = EG$, then we obtain the case investigated in [H1,2].

(ii) The map
\[ \text{hocolim}_{[E_n] \in \text{sd} A_p(G)/G} K \times_{G_E} X^{E_n} \to K \times_G X \]
is a mod $p$ homology isomorphism and there is a spectral sequence
\[ H^m(\text{sd} A_p(G)/G, H^n(K \times_{G_E} X^{E_n}, F_p)) \Rightarrow H^{m+n}(K \times_G X, F_p). \]

In particular, if $A$ is one of the classes $B_k(F_p)$ or $D(F_p)$ described in 1.3 and, for every $(E_0, \ldots, E_n) \in \text{sd} A_p(G)$, $K \times_{N(E_0 \cap \ldots \cap E_n)} X^{E_n} \in A$, then $K \times_G X \in A$.

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