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Homotopy decompositions of orbit spaces and the Webb conjecture

by

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Abstract. Let p be a prime number. We prove that if G is a compact Lie group with a non-trivial p-subgroup, then the orbit space $(B\mathcal{A}_p(G))/G$ of the classifying space of the category associated to the G-poset $\mathcal{A}_p(G)$ of all non-trivial elementary abelian p-subgroups of G is contractible. This gives, for every G-CW-complex X each of whose isotropy groups contains a non-trivial p-subgroup, a decomposition of X/G as a homotopy colimit of the functor $X^{E_n}/(NE_0\cap\ldots\cap NE_n)$ defined over the poset $(\mathrm{sd}\,\mathcal{A}_p(G))/G$, where sd is the barycentric subdivision. We also investigate some other equivariant homotopy and homology decompositions of X and prove that if G is a compact Lie group with a non-trivial p-subgroup, then the map $EG\times_G B\mathcal{A}_p(G)\to BG$ induced by the G-map $B\mathcal{A}_p(G)\to *$ is a mod p homology isomorphism.

Introduction. In this paper we will study homotopy and homology decompositions which are associated to the equivariant structure of a G-CW-complex X where G is a Lie group. We will try to generalize and streamline techniques of such decompositions.

Let $\mathcal{C}$ be a small topological category and let $F:\mathcal{C}\to G$-CW be a functor such that, for every $c\in\mathcal{C}$, $F(c)=G\times_{H(c)}X^{H'(c)}$ where $H(c),H'(c)$ are closed subgroups of G and $H(c)$ is a subgroup of the normalizer $NH'(c)=N_GH'(c)$ of $H'(c)$ in G. Suppose also that there is a natural transformation from F to the constant functor X induced by the inclusions $X^{H'(c)}\to X$. G-maps

$$u:\mathrm{hocolim}_{c\in\mathcal{C}}\,G\times_{H(c)}X^{H'(c)}\to X$$

induced by such natural transformations can be used in constructing different homotopy and homology decompositions. If u is a G-homotopy equivalence then it will be called a *G-homotopy decomposition* of X.

In Section 0 we will introduce a "universal" category $\mathcal{C}_G$ and, for every G-CW-complex X, a functor $\widehat{X}:\mathcal{C}_G\to G$-CW and a natural transformation

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of functors $\widehat{X} \to X$. We will study the decompositions induced by functors F which are compositions $\widehat{X}F'$, where $F' : \mathcal{C} \to \mathcal{C}_G$.

For a given G-CW-complex K, we will investigate homotopy decompositions of the orbit space $K \times_G X$, i.e. homotopy equivalences of the form

$$\mathrm{id} \times_G u : \operatorname{hocolim}_{c\in\mathcal{C}} K \times_{H(c)} X^{H'(c)} \simeq K \times_G X.$$

We will also study mod p homology decompositions. In this case the map $\mathrm{id} \times_G u$ is an F_p-equivalence. We will show how the known examples of decompositions of $K \times_G X$ can be described using $\mathcal{C}_G$.

The best known examples of homology decompositions are the cases where $K = EG$ is a universal free G-space and $X = *$ is a one-point space ([JM2], [JMO]).

Let p be a prime number and let $\mathcal{A}_p(G)$ be the G-poset of all elementary abelian non-trivial p-subgroups of G. If G does not contain a p-subgroup, then the set $\mathcal{A}_p(G)$ is empty. Let $A_p(G)$ be the category whose objects are elements of $\mathcal{A}_p(G)$ and whose morphisms are homomorphisms which are restrictions of inner automorphisms of G. Let $C_G(E)$ be the centralizer of E in G. There is a contravariant functor $F : A_p(G) \to G\text{-CW}$ such that $F(E) = G \times_{C_G E} X^E$. In the case where $X = *$ and G is a compact Lie group which contains a non-trivial p-subgroup, there is a mod p homology decomposition (Theorem 1.3 of [JM2])

$$\operatorname{hocolim}_{E\in A_p(G)} BC_G(E) \to BG.$$

Using this fact it is proved in [H1] that if the isotropy groups of X are compact and contain a non-trivial p-group, then the map

$$\operatorname{hocolim}_{E\in A_p(G)} EG \times_{C_G(E)} X^E \to EG \times_G X$$

is a mod p homology isomorphism.

We will prove that one can take instead of EG any F_p-acyclic complex K. We will also construct, for such K, another mod p homology decomposition

$$\operatorname{hocolim}_{[(E_0,\ldots,E_n)]\in(\operatorname{sd}\mathcal{A}_p(G))/G} K \times_{NE_0\cap\ldots\cap NE_n} X^{E_n} \to K \times_G X.$$

Here we take $\mathcal{C}$ equal to the poset $(\operatorname{sd}\mathcal{A}_p(G))/G$ of the orbits of the G-action on the barycentric subdivision of $\mathcal{A}_p(G)$. (Recall that the elements of $\operatorname{sd}\mathcal{A}_p(G)$ are the increasing sequences $(E_0 < \ldots < E_n)$ of elements of $\mathcal{A}_p(G)$.) If G is a compact Lie group, then in the special case when $X = *$ and $K = EG$, we obtain a mod p homology isomorphism

$$\operatorname{hocolim}_{[(E_0,\ldots,E_n)]\in(\operatorname{sd}\mathcal{A}_p(G))/G} B(NE_0 \cap \ldots \cap NE_n) \to BG,$$

which is in fact equal to the mod p isomorphism

$$EG \times_G B(\mathcal{A}_p(G)) \to BG.$$

This last fact is well known in the finite case and can be obtained using 1.3 of [JM2]. The compact case is more complicated because of the topological structure of $\mathcal{A}_p(G)$.

If $K = *$ then we obtain not only a homology but also a homotopy decomposition of X/G (Theorem 0.1). In the case when G is a compact Lie group and $X = *$ this means that the space $(B\mathcal{A}_p(G))/G$ is contractible. For finite groups this was conjectured in [We]. A combinatorial proof of this fact in the finite case was given in [Sy]. Our proof is a generalization of an equivariant approach described for finite groups in [S1].

We will also study h^*_G decompositions, where h^*_G is a generalized equivariant cohomology theory, i.e. maps u which induce isomorphisms

$$h^*_G(u) : h^*_G(X) \to h^*_G(\operatorname{hocolim}_{c\in\mathcal{C}} G \times_{H(c)} X^{H'(c)}).$$

We will use the fact that such a decomposition gives a spectral sequence

$$H^m(\mathcal{C}, h^n_{H(-)}(X^{H'(-)})) \Rightarrow h^{m+n}_G(X),$$

where $h^*_H(-) = h^*_G(G \times_H -)$ and $H^m(\mathcal{C}, -) = \lim^m_{\mathcal{C}}(-) = \operatorname{Ext}^m_{\mathcal{C}}(\mathbb{Z}, -)$ are the cohomology groups of the category $\mathcal{C}$ (Ch. XII of [BK], Section 5 of [DF1]).

0. The main results. Let G be a Lie group. Let $\mathcal{O}_G$ be the orbit category of G whose objects are the orbits G/H, where H is a closed subgroup of G. The morphisms of $\mathcal{O}_G$ are the equivariant continuous maps. Every morphism $f : G/H \to G/H_1$ corresponds to a class $[g] \in (G/H_1)^H$ such that $f([g']) = g'gH_1$. It follows from the definitions that $[g] \in (G/H_1)^H$ if and only if $H \subseteq gH_1g^{-1}$. The topology of the morphism space $\operatorname{Mor}_{\mathcal{O}_G}(G/H, G/H_1) = (G/H_1)^H$ is induced from G/H_1. The category $\mathcal{O}_G$ is a topological category in the sense of [HV], i.e. a small category $\mathcal{C}$ with topological morphism sets such that the composition is continuous and the structural map $\operatorname{Ob}\mathcal{C} \to \operatorname{Mor}\mathcal{C}$ is a closed cofibration. Similarly to [HV] we will work in the category Top of compactly generated spaces. We will consider $\mathcal{O}_G$ as a full subcategory of the category G-CW of all G-CW-complexes and equivariant cellular maps. This category is described, for example, in [Wi] and [JMO].

We introduce another topological category $\mathcal{C}_G$ which plays a crucial role in our considerations concerning equivariant decompositions. Its object set $\mathcal{W}(G)$ consists of all pairs (H, H') of closed subgroups of G such that H is a subgroup of NH'. The morphisms $(H, H') \to (H_1, H'_1)$ of $\mathcal{C}_G$ are all morphisms $f = [g] : G/H \to G/H_1$ of $\mathcal{O}_G$ such that $H'_1 \subseteq g^{-1}H'g$. If $f' = [g'] : (H_1, H'_1) \to (H_2, H'_2)$ is a morphism of $\mathcal{C}_G$, then the condition $H'_2 \subseteq g'^{-1}H'_1g'$ implies that $H'_2 \subseteq g'^{-1}g^{-1}H'gg'$ so $f'f = [gg']$ is a morphism of $\mathcal{C}_G$. The topology of the morphism spaces is induced from the morphism space topology in $\mathcal{O}_G$. There is an inclusion of categories $i : \mathcal{O}_G \to \mathcal{C}_G$ such that $i(H) = (H, e)$. The category $\mathcal{C}_G$ has a final object (G, e).

Let X be a G-CW-complex. Let $\widehat{X} : \mathcal{C}_G \to G$-CW be the functor defined by $\widehat{X}(H,H') = G \times_H X^{H'}$, $\widehat{X}([g])([g',x]) = [g'g, g^{-1}x]$. Hence $\widehat{X}(G,e) = G \times_G X = X$. The equivariant maps

$$\alpha(H,H') = \widehat{X}([e]) : G \times_H X^{H'} \to X$$

such that $\alpha(H,H')[g',x] = g'x$ form a natural transformation of functors $\alpha : \widehat{X} \to X$ where X is the constant functor. Let $\mathcal{C}$ be a topological category. Suppose that we have a functor $(H(-), H'(-)) : \mathcal{C} \to \mathcal{C}_G$. Then α induces a G-map

$$u : \operatorname{hocolim}_{c\in\mathcal{C}} G \times_{H(c)} X^{H'(c)} \to X.$$

Many examples of decompositions induced by such maps will be described and studied in Sections 3 and 4. For example, let V be a G-set of closed subgroups of G and let $\mathcal{O}_V$ be the full subcategory of $\mathcal{O}_G$ such that G/H is an object of $\mathcal{O}_V$ if and only if $H \in V$. Let $\mathcal{C}(V)$ be the full subcategory of $\mathcal{C}_G$ whose objects are pairs (H,H') where H is a subgroup of H' and $H' \in V$. We will prove the following result in Section 3.

0.0. Proposition. *Assume that all isotropy groups of X are in V. Then*:

(i) *The map*

$$u : \operatorname{hocolim}_{\mathcal{C}(V)} \widehat{X} \to X$$

is a G-homotopy decomposition.

(ii) *The map u/G gives a homotopy decomposition*

$$\operatorname{hocolim}_{G/H\in\mathcal{O}_V} X^H \simeq X/G.$$

The homotopy decomposition from (ii) is well known. It appears in [E] and [DF2].

In Sections 1 and 2 we will consider the case where $\mathcal{C}$ is the orbit category of the barycentric subdivision of a poset of subgroups of G. In order to describe this case we need the following notation. Let W be a topological G-poset. This means that W is a topological poset in the sense of [Ž] (i.e. the order relation is a closed subset of W^2) together with a continuous and order preserving action of G on W. Let d_nW denote the G-subspace of W^{n+1} consisting of all non-decreasing sequences $w. = (w_0,\ldots,w_n)$. The G-subspace of d_nW consisting of all $w.$ such that $w_i \neq w_{i+1}$ for all i will be denoted by $\operatorname{sd}_n W$. The disjoint union $\operatorname{sd} W = \coprod_{n\in\mathbb{N}} \operatorname{sd}_n W$ is a topological G-poset such that $(w_0,\ldots,w_n) \le (w'_0,\ldots,w'_m)$ if and only if $\{w'_0,\ldots,w'_m\} \subseteq \{w_0,\ldots,w_n\}$. There are two G-poset maps $p_0 : \operatorname{sd} W \to W$ and $p_1 : (\operatorname{sd} W)^{\mathrm{op}} \to W$ such that $p_0(w.) = w_0$, $p_1(w.) = w_n$. We will assume that as a topological space, W is equal to the disjoint union of its G-orbits $Gw = G/G_w$ with the topology induced from the topology of G. In this case the topological space W/G is discrete. If W satisfies the condition that $w \le gw$ implies that $w = gw$ then W/G is a poset such that $[w] \le [w']$

if and only if $w \leq gw'$ for some $g \in G$. The G-poset $\operatorname{sd} W$ satisfies this condition.

Let $\mathcal{S}(G)$ denote the poset of all closed subgroups of G. The group G acts on $\mathcal{S}(G)$ by conjugation. If $H \in \mathcal{S}(G)$, then the isotropy group of this action at H is equal to NH. We will assume that $\mathcal{S}(G)$ is a topological space equal to the disjoint union of its G-orbits Gx with topology induced from the topology of G. Let W be a G-subposet of $\mathcal{S}(G)$ satisfying the condition that $w \leq gw$ implies that $w = gw$. Suppose that $(\operatorname{sd} W)/G$ is also a discrete space. Then the space $\operatorname{sd} W$ is equal to the disjoint union of its G-orbits $G/(Nw_0 \cap \ldots \cap Nw_n)$. There is a functor $F : (\operatorname{sd} W)/G \to \mathcal{C}_G$ such that

$$F([w_0, \ldots, w_n]) = (Nw_0 \cap \ldots \cap Nw_n, w_n).$$

If $[w_0, \ldots, w_n] \leq [w'_0, \ldots, w'_m]$, then there exists exactly one element $[g]$ of $G/(Nw'_0 \cap \ldots \cap Nw'_m)$ such that $(w_0, \ldots, w_n) \leq (gw'_0 g^{-1}, \ldots, gw_m g^{-1})$. This implies that $gw'_m g^{-1} \subseteq w_n$ and $F([w_0, \ldots, w_n] \leq [w_0, \ldots, w'_m])$ is the morphism of $\mathcal{C}_G$ defined by $[g]$.

If X is a G-CW-complex then there is a functor $\widetilde{X} : (\operatorname{sd} W)/G \to G\text{-CW}$ such that

$$\widetilde{X}([w_0, \ldots, w_n]) = G \times_{Nw_0 \cap \ldots \cap Nw_n} X^{w_n}.$$

In Section 2 of this paper we will prove the following result which in the case when G is a finite group was proved in [S1] (2.10.iv and 2.11).

0.1. Theorem. *Let X be a G-CW-complex such that all its isotropy groups are compact and contain a non-trivial p-subgroup. Then there is a homotopy equivalence*

$$\operatorname{hocolim}_{[(E_0, \ldots, E_n)] \in (\operatorname{sd} \mathcal{A}_p(G))/G} X^{E_n}/(NE_0 \cap \ldots \cap NE_n) \simeq X/G.$$

If $X = *$ is a one-point G-CW-complex, then

$$\widetilde{*}([w_0, \ldots, w_n]) = G/(Nw_0 \cap \ldots \cap Nw_n)$$

and 0.1 specializes to the fact that, in the case when G is a compact Lie group, the classifying space $B((\operatorname{sd} \mathcal{A}_p(G))/G)$ of the category associated to the poset $(\operatorname{sd} \mathcal{A}_p(G))/G$ is contractible.

If W is a poset (discrete as topological space), then the geometrical realization $|W|$ of the simplicial complex associated to W is equal to the classifying space BW of the category associated to W. An action of G on W induces a G-action on $|W|$. Then there are homotopy equivalences $|\operatorname{sd} W|/G \simeq |W|/G$ and $|\operatorname{sd} W|/G \simeq |(\operatorname{sd} W)/G|$. Let G be a finite group. Let $\mathcal{S}_p(G)$ be the G-poset of all non-trivial p-subgroups of G. Then the spaces $|\mathcal{S}_p(G)|$ and $|\mathcal{A}_p(G)|$ are G-homotopy equivalent (Theorem 2 of [TW]). It is proved in [We] (2.6.1) that $|\mathcal{S}_p(G)|/G$ is F_p-acyclic and conjectured that $|\mathcal{S}_p(G)|/G$ is contractible. It is also proved in [We] (2.1.2) that $|\mathcal{S}_p(G)|^H$ is contractible whenever H is a subgroup of G which contains a normal non-trivial p-subgroup. In [S1] a proof of the Webb conjecture was presented which uses this fact and methods introduced in [O1]. We will generalize this proof to the case of a compact Lie group.

If W is a topological poset then the morphism space of the topological category associated to the poset W has topology induced from the topology of $W \times W$ and the classifying space BW of this category is equal to $\coprod_{n\in\mathbb{N}} \Delta_n \times d_n W/{\sim}$ where Δ_n is the standard n-dimensional simplex and $\sim$ is an appropriate equivalence relation (3.6 of [Ž]).

Let W be a topological G-poset such that the condition that $w \le gw$ implies that $w = gw$. Then W/G is a topological poset. Suppose that the topological space W/G is discrete and that, for every $n \in \mathbb{N}$, $(d_n W)/G$ is discrete. (This holds for example if W is a subposet of $\mathcal{S}(G)$ and all subgroups in W are finite. Indeed, let $p : (d_n W)/G \to W/G$ be the projection such that $p([w_0,\ldots,w_n]) = [w_n]$. Then, for every $[w] \in W/G$, the preimage $p^{-1}([w])$ is a finite space.) The topological space $\operatorname{sd} W/G = (\operatorname{sd} W)/G$ is also discrete in this case and $BW = \coprod_{n\in\mathbb{N}} \Delta_n \times \operatorname{sd}_n W/{\sim}$. There is a natural G-CW-complex structure on BW such that the poset $\operatorname{sd} W/G$ is equal to the poset of the G-cells of BW. We will show in Section 2 (cf. the proof of 2.3) that

$$(BW)/G = \coprod_{n\in\mathbb{N}} \Delta_n \times (\operatorname{sd}_n W)/G/{\sim}$$

is a classifying space $B((\operatorname{sd} W)/G)$ of the category associated to the poset $\operatorname{sd} W/G$. We will also show that there are G-homotopy equivalences

$$\operatorname{hocolim}_{[(w_0,\ldots,w_n)]\in \operatorname{sd} W/G} G/(Nw_0 \cap \ldots \cap Nw_n) \simeq B \operatorname{sd} W \simeq BW.$$

In Section 1 we will prove that if G is a compact Lie group and contains a non-trivial p-subgroup, then the space $B\mathcal{A}_p(G)/G$ is contractible. The proof consists of several steps which will be described below. Recall that P is a *p-toral group* if its identity component P_0 is a torus and $\pi_0(P) = P/P_0$ is a finite p-group. The following result is an immediate consequence of 0.1 but in the proof of 0.1 we will use 0.2 in the case when X has finitely many orbit types. We will prove this fact in Section 1.

0.2. THEOREM. *Let G be a compact Lie group. Let X be a G-CW-complex such that all its isotropy groups contain a non-trivial p-subgroup. Suppose that X^P/H is contractible whenever P is a non-trivial p-toral subgroup of G and H is a closed subgroup of the normalizer NP of P in G. Then X/G is contractible.*

To prove 0.1 we will also need the following result.

0.3. Proposition. *Let R be a commutative ring. Let X and Y be G-*CW*-complexes such that all their isotropy groups are compact and contain a non-trivial p-subgroup. Let $f : X \to Y$ be a cellular G-map of G-*CW*-complexes. Then*:

(i) *If*, *for every compact subgroup H of G containing a non-trivial normal p-toral subgroup*, *$f^H : X^H \to Y^H$ is a homotopy equivalence*, *then so is $f/G : X/G \to Y/G$.*

(ii) *If*, *for every compact subgroup H of G containing a non-trivial normal p-toral subgroup*, *$f^H : X^H \to Y^H$ is an R-equivalence*, *then so is $f/G : X/G \to Y/G$.*

If G is a compact Lie group and $Y = *$ then 0.3 is a consequence of 0.2 and the well known decomposition described in 0.0(ii). This result will be proved in Section 1 in the case when X has finitely many orbit types. We will show that the map $B\mathcal{A}_p(G) \to *$ satisfies the assumptions of 0.3(i). Hence $B\mathcal{A}_p(G)/G$ is contractible and using this we will infer 0.1. We will also prove 0.3 for an arbitrary Lie group G.

Let W be a poset of closed subgroups of G. In Section 4 we will describe a condition on W which ensures that $h^*_G(Y) \to h^*_G(X)$ is an isomorphism if $X^H \to Y^H$ is an R-homology isomorphism for all $H \in W$. As an example we will consider the case when

$$h^*_G(X) = H^*(K \times_G X, R).$$

In particular, we will show how 0.3(ii) and the results of [JMO] and [JO] concerning the mod p decomposition

$$\operatorname{hocolim}_{G/P \in \mathcal{O}_{R_p(G)}} BP \to BG,$$

where $R_p(G)$ is a certain poset of p-toral subgroups of G, imply the following result.

0.4. Proposition. *Let X and Y be G-*CW*-complexes such that all their isotropy groups are compact and contain non-trivial p-subgroups. Let K be an F_p-acyclic G-*CW*-complex. If*, *for every non-trivial p-toral subgroup H of G*, *$f^H : X^H \to Y^H$ is an F_p-equivalence*, *then so is $\mathrm{id}_K \times_G f : K \times_G X \to K \times_G Y$.*

If G is a compact Lie group with a non-trivial p-subgroup, then from the fact (cf. the proof of 1.5) that all isotropy groups of $B\mathcal{A}_p(G)$ contain non-trivial normal p-subgroups and that, for every subgroup H of G containing a non-trivial normal p-subgroup, the space $B\mathcal{A}_p(G)^H$ is contractible, we obtain the following result.

0.5. Corollary. *Let G be a compact Lie group with a non-trivial p-subgroup. Then the map $EG \times_G B\mathcal{A}_p(G) \to BG$ induced by the G-map $B\mathcal{A}_p(G) \to *$ is an F_p-equivalence.*

The following posets of subgroups will be defined and used in the paper.

List of posets of subgroups of G

- $\mathcal{A}_p(G)$ — the set of all elementary abelian non-trivial p-subgroups,
- $\mathcal{A}'_p(G)$ — the set of all elementary abelian p-subgroups,
- $\mathcal{K}_p(G)$ — the set of all compact subgroups H such that, for every $P \in \mathcal{M}_p(G)$, $H \cap Z(P)$ contains a non-trivial p-subgroup,
- $\mathcal{M}_p(G)$ — the set of all maximal non-trivial p-toral subgroups,
- $\mathcal{N}_p(G)$ — the set of all compact subgroups containing a non-trivial normal p-toral subgroup,
- $\mathcal{S}(G)$ — the set of all closed subgroups,
- $\mathcal{S}(G,X)$ — the set of all isotropy groups of X,
- $\mathcal{S}_0(G,X) = \mathcal{S}(G,X) \cup \mathcal{S}(G,*)$,
- $\mathcal{S}'_{\mathrm{c}}(G)$ — the set of all compact subgroups,
- $\mathcal{S}_{\mathrm{c}}(G)$ — the set of all compact subgroups which contain a non-trivial p-subgroup,
- $\mathcal{S}'_p(G)$ — the set of all subtoral p-subgroups,
- $\mathcal{S}_p(G)$ — the set of all subtoral p-subgroups which contain a non-trivial p-subgroup,
- $\mathcal{T}'_p(G)$ — the set of all p-toral subgroups,
- $\mathcal{T}_p(G)$ — the set of all non-trivial p-toral subgroups,
- $\mathcal{T}_p(G,X)$ — the set of all maximal p-toral subgroups of isotropy groups of X,
- $\mathcal{Z}_p(G)$ — the set of all compact subgroups containing a non-trivial central p-subgroup.

1. Orbit spaces of compact Lie group actions. Let G be a Lie group. The set of all compact subgroups of G will be denoted by $\mathcal{S}'_{\mathrm{c}}(G)$. The set of all elements of $\mathcal{S}'_{\mathrm{c}}(G)$ which contain a non-trivial p-subgroup will be denoted by $\mathcal{S}_{\mathrm{c}}(G)$. The set of all closed p-toral subgroups of G will be denoted by $\mathcal{T}'_p(G)$. The set of all non-trivial p-toral subgroups of G will be denoted by $\mathcal{T}_p(G)$. The set of all compact subgroups of G containing a non-trivial normal p-toral subgroup will be denoted by $\mathcal{N}_p(G)$.

If G is a compact Lie group, T is a maximal torus of G and N_pT/T is a Sylow p-subgroup of NT/T, then N_pT is a maximal p-toral subgroup of G. All maximal p-toral subgroups of G are conjugate to N_pT (Lemma A.1 of [JMO]). The set of all maximal p-toral subgroups of G will be denoted by $\mathcal{M}_p(G)$ and the set of all maximal p-toral subgroups of isotropy groups of X by $\mathcal{T}_p(G,X)$.

Let $\mathcal{S}$ be a subset of the set of compact subgroups of G. We will use the notation

$$\mathcal{W}_{\mathcal{S}} = \{(H,H') : H' \subseteq H \subseteq NH',\ H' \in \mathcal{S},\ H \in \mathcal{S}'_{\mathrm{c}}(G).\}$$

A non-empty G-poset $\mathcal{P}$ of p-toral subgroups of G will be called *concave* if, for any p-toral subgroups P and P' the condition that $P \subseteq P'$ and $P \in \mathcal{P}$ implies that $P' \in \mathcal{P}$. If G is a compact Lie group and $\mathcal{P}$ is concave, then $\mathcal{M}_p(G) \subseteq \mathcal{P}$ because all maximal p-toral subgroups are conjugate by elements of G.

Let $\mathcal{CW}$ denote the category of spaces having the homotopy type of CW-complexes and let $\mathcal{CW}_0$ be the subcategory of $\mathcal{CW}$ consisting of the connected spaces. We will say that a class $\mathcal{A}$ of objects of $\mathcal{CW}$ is *thick* if it is closed under homotopy equivalences and taking homotopy pushouts.

In this section we will assume that G is a compact Lie group with a non-trivial p-subgroup and that X is a G-CW-complex with finitely many orbit types.

1.1. Theorem. *Let $\mathcal{A}$ be thick. Let $\mathcal{P}$ be a concave G-poset of p-toral subgroups of G containing all maximal p-toral subgroups of the isotropy groups of X. If $X^P/H \in \mathcal{A}$ whenever $P \in \mathcal{P}$ and $P \subseteq H \subseteq NP$, then $X/G \in \mathcal{A}$.*

Proof. If $(e) \in \mathcal{P}$, then the assumptions imply that $X/G \in \mathcal{A}$. Let $k(G,X)$ denote the number of elements of $\mathcal{T}_p(G,X)/G$.

If $k(G,X) = 1$, then $\mathcal{T}_p(G,X) = (P) = \{gPg^{-1} : g \in G\}$, where P is, up to conjugacy, the unique maximal p-toral group of an isotropy group of X. Hence $X = X^{(P)} = \bigcup_{P' \in (P)} X^{P'}$. It is proved in [O1] (in the proof of Proposition 3) that the map $X^P/NP \to X^{(P)}/G$ is a homeomorphism. (This is a consequence of the fact that, if G' is a closed subgroup of G and P is a maximal p-toral subgroup of G', then NP acts transitively on $(G/G')^P$. Indeed, let $aG', bG' \in (G/G')^P$. Then $a^{-1}Pa, b^{-1}Pb$ are maximal p-toral subgroups of G' so they are conjugate in G' and there is $c \in G'$ such that $bca^{-1} \in NP$.) If the assumptions hold, then P is a maximal toral p-subgroup of G. Hence, in this case, $X/G = X^P/NP \in \mathcal{A}$.

We use induction on the dimension of G and then on the order of $\pi_0(G) = G/G_0$, where G_0 is the identity component of G. Assume that the result is true for all proper closed Lie subgroups of G. Now we use induction on $k(G,X)$. Let $k(G,X) = k+1 > 1$. Suppose that the result is true for all G-CW-complexes X' such that $k(G,X') \leq k$. Let P be a minimal element of $\mathcal{T}_p(G,X)$. As P is not a maximal p-toral group, it follows that NP/P contains a non-trivial p-toral subgroup (cf. [O1], Lemma 2). Let X' be a G-CW-subcomplex of X such that $x \in X \setminus X'$ if and only if maximal p-toral subgroups of the isotropy group G_x are conjugate to P. The induction assumption implies that $X'/G \in \mathcal{A}$ because $k(G,X') \leq k$. Indeed, let $\mathcal{P}_o = \mathcal{P} \setminus (P)$. Then, for every $(H,P') \in \mathcal{W}_{\mathcal{P}_o}$, $X'^{P'}/H = X^{P'}/H$.

It follows from the definition that $X = X' \cup X^{(P)}$ and that X/G is equal to the pushout of the diagram

$$X^{(P)}/G \leftarrow X'^{(P)}/G \to X'/G.$$

If $x \in X \setminus X'$, then $\mathcal{M}_p(G_x)$ is a subset of (P) and NP acts transitively on $(Gx)^P = (G/G_x)^P$. Hence X/G is the pushout of the diagram

$$X^P/NP \leftarrow X'^P/NP \to X'/G.$$

Since $X'^P/NP \to X^P/NP$ is a cofibration, X/G is the homotopy pushout of this diagram.

The space X'^P, which has the structure of an NP-CW complex, satisfies the assumptions of the proposition. It is of finite orbit type because, for every closed subgroup G' of G, $(G/G')^P/NP$ is finite (II.5.7 of [Br1]). Let $\mathcal{P}' = \{P' \in \mathcal{P} : P \subset P' \subseteq NP, P' \neq P\}$. From the fact that, for every $x \in X'^P$, $P \subseteq G_x \cap NP$ and P is not a maximal p-toral subgroup of G_x, it follows that P is not a maximal p-toral subgroup of $G_x \cap NP$ (Lemma 2 of [O1]). Hence

$$\mathcal{T}_p(NP, X'^P) = \bigcup_{x \in X'} \mathcal{M}_p(G_x \cap NP) \subseteq \mathcal{P}'$$

and $X'^{P'}/H = X^{P'}/H \in \mathcal{A}$ whenever $(H, P') \in \mathcal{W}_{\mathcal{P}'}$.

If P is a normal subgroup of G, then $NP = G$ but $k(X'^P, G) \leq k$, because $P \notin \mathcal{T}_p(G, X'^P) \subseteq \mathcal{T}_p(G, X)$. If P is not a normal subgroup of G, then $NP < G$ and we can use the induction assumption. In both cases we find that $X'^P/NP \in \mathcal{A}$. Hence $X/G \in \mathcal{A}$.

In particular, if $\mathcal{P} = \mathcal{T}_p(G)$ and $\mathcal{A}$ is the class of all contractible objects of $\mathcal{CW}_0$ then 1.1 specializes to 0.2.

In what follows let $\mathcal{A}$ be a thick category. We now define three conditions for thick categories.

A1: For every compact Lie group H and for every H-CW-complex X, if $X^{H'} \in \mathcal{A}$ for every closed subgroup H' of H, then $X/H \in \mathcal{A}$.

A2: For every compact Lie group H and for every H-CW-complex X, if $\dim X < \infty$ and $X \in \mathcal{A}$, then $X/H \in \mathcal{A}$.

A3: For every compact Lie group H and for every H-CW-complex X, if $X/P \in \mathcal{A}$ for every $P \in \mathcal{M}_p(H)$, then $X/H \in \mathcal{A}$.

Let H' be a closed subgroup of G and let $\mathcal{P}$ be a set of subgroups of G. We use the notation

$$\mathcal{N}_{H'} = \{H \in \mathcal{S}(G) : H' \subseteq H \subseteq NH'\},$$
$$\mathcal{N}_{\mathcal{P}} = \{H \in \mathcal{S}(G) : H' \subseteq H \subseteq NH',\ H' \in \mathcal{P}\},$$
$$\mathcal{S}_{\mathcal{P}} = \bigcup_{P,P' \in \mathcal{P}} \{H \in \mathcal{S}(G) : P \subseteq H \subseteq P' \subseteq NP\},$$
$$\mathcal{S}'_p(G) = \mathcal{S}_{\mathcal{T}'_p(G)}, \qquad \mathcal{S}_p(G) = \mathcal{S}_{\mathcal{T}_p(G)}.$$

1.2. Corollary. *Let $\mathcal{P}$ be a concave G-poset of p-toral subgroups of G. Let X be a G-CW-complex such that maximal p-toral subgroups of isotropy groups of X are in $\mathcal{P}$. Suppose that $\mathcal{A}$ is thick and that one of the following conditions holds*:

(i) *$\mathcal{A}$ satisfies* **A3** *and $X^P/P' \in \mathcal{A}$ whenever $(P', P) \in \mathcal{W}_{\mathcal{P}}$ and $P' \in \mathcal{P}$.*
(ii) *$\mathcal{A}$ satisfies* **A1** *and $X^H \in \mathcal{A}$ whenever $H \in \mathcal{N}_{\mathcal{P}}$.*
(iii) *$\mathcal{A}$ satisfies* **A1** *and* **A3** *and $X^H \in \mathcal{A}$ whenever $H \in \mathcal{S}_{\mathcal{P}}$.*
(iv) *$\mathcal{A}$ satisfies* **A2**, $\dim X < \infty$ *and $X^H \in \mathcal{A}$ whenever $H \in \mathcal{P}$.*

Then $X/G \in \mathcal{A}$.

Proof. The result is a consequence of 1.1. Suppose that $(H, P') \in \mathcal{W}_{\mathcal{P}}$.

If (i) holds, then $X^{P'}/P'' \in \mathcal{A}$ whenever $P'' \in \mathcal{M}_p(H)$. Since $\mathcal{A}$ satisfies **A3**, it follows that $X^{P'}/H \in \mathcal{A}$.

Assume that (ii) holds. Let $H' = H/P'$ and let $Y = X^{P'}$. We can consider Y as an H'-CW-complex. If H'_0 is a subgroup of H', then $H'_0 = H_0/P'$, where $P' \subseteq H_0 \subseteq H$, and $Y^{H'_0} = X^{H_0} \in \mathcal{A}$ because $H_0 \in \mathcal{N}_{\mathcal{P}}$. Hence $X^{P'}/H = Y/H' \in \mathcal{A}$.

If $\mathcal{A}$ satisfies **A1** and $X^{G'} \in \mathcal{A}$ whenever $G' \in \mathcal{S}_{\mathcal{P}}$ then $X^P/P' \in \mathcal{A}$ whenever $P, P' \in \mathcal{P}$, $P' \in \mathcal{N}_P$. Now we can use part (ii) of this result to obtain (iii).

If (iv) holds, then $X^{P'}/H \in \mathcal{A}$ by the definitions.

1.3. Examples. Let

$$\mathcal{C} = \{X \in \mathcal{CW}_0 : X \text{ is contractible}\},$$
$$\mathcal{D}(R) = \{X \in \mathcal{CW}_0 : X \text{ is } R\text{-acyclic}\},$$
$$B_k(R) = \{X \in \mathcal{CW}_0 : H^i(X, R) = 0 \text{ for } i = 1, \ldots, k\}.$$

(i) The well known decomposition from 0.0(ii) implies that all these classes satisfy **A1**.

(ii) The classes $\mathcal{D}(F_p)$ and $B_k(F_p)$ satisfy **A3**. This is a consequence of the existence of an appropriate transfer. Let H be a closed subgroup of G and let $\pi_X : X/H \to X/G$ be the projection to the orbit space. It is proved in [O2], [LMM], [LMS] that there exists a natural transfer map

$$t_X : H^*(X/H, R) \to H^*(X/G, R)$$

such that the composition $H^*(\pi_X)t_X$ is the multiplication by the Euler characteristic $\chi(G/H)$ of G/H. If H is a maximal p-toral subgroup of G, then $\chi(G/H)$ is prime to p. Hence, if $H^n(X/H, F_p) = 0$, then $H^n(X/G, F_p) = 0$.

(iii) The classes $\mathcal{D}(\mathbb{Z})$ and $\mathcal{D}(F_p)$ satisfy **A2**. This follows from Theorems 1 and 2 of [O1].

The next result describes the case when $\mathcal{P} = \mathcal{T}_p(G)$ and $\mathcal{A}$ is one of the classes from 1.3. The statement (i) is a special case of 0.3. For a finite group G, this result is proved in 2.11 of [S1]. The statement (iii), for finite groups, finite G-CW-complexes and F_p-acyclic spaces, is proved in [We].

1.4. Proposition. *Let X be a G-CW-complex such that all its isotropy groups contain a non-trivial p-subgroup. Then*:

(i) *X/G is contractible (resp. R-acyclic) if X^H is contractible (resp. R-acyclic) for all closed subgroups H containing a non-trivial normal p-toral subgroup.*

(ii) *X/G is F_p-acyclic if, for every $H \in \mathcal{S}_p(G)$, X^H is F_p-acyclic.*

(iii) *If $\dim X < \infty$ and, for every non-trivial p-toral subgroup H of G, X^H is $\mathbb{Z}$-acyclic (resp. F_p-acyclic), then X/G is $\mathbb{Z}$-acyclic (resp. F_p-acyclic).*

Proof. $\mathcal{T}_p(G)$ is a concave set of p-subgroups of G. By 1.2(ii), $\mathcal{N}_{\mathcal{T}_p(G)} = \mathcal{N}_p(G)$ so (i) follows. The statement (ii) is a consequence of 1.2(iii) because $\mathcal{S}_{\mathcal{T}_p(G)} = \mathcal{S}_p(G)$, and (iii) follows from 1.2(iv).

1.5. Corollary. *If G is a compact Lie group with a non-trivial p-subgroup, then the space $B\mathcal{A}_p(G)/G$ is contractible.*

Proof. It is proved in 6.1 of [JM2] that there are only finitely many conjugacy classes of elementary abelian p-subgroups in G. If $x \in B\mathcal{A}_p(G)$, then $G_x = NE_0 \cap \ldots \cap NE_k$, where $E_i \in \mathcal{A}_p(G)$ and $E_0 < \ldots < E_k$, so $E_0 \subseteq G_x \subseteq NE_0$. For every $H \in \mathcal{N}_p(G)$, the space $(B\mathcal{A}_p(G))^H = B(\mathcal{A}_p(G)^H)$ is contractible. For G finite this follows from 2.1.2 of [We]. The proof for any compact Lie group is similar. The space $\mathcal{A}_p(G)^H$ is a disjoint union of its NH/H-orbits. Let

$$\mathcal{A}_p(G)_{\geq E} = \{E' \in \mathcal{A}_p(G) : E \subseteq E'\}.$$

There exists a non-trivial normal p-toral subgroup P of H such that NH is a subgroup of NP. Indeed, let Q be the intersection of all maximal p-toral subgroups of H. Then NH is a subgroup of NQ. Let Q_0 be the component of the identity of Q. We can take $P = Q_0$ if Q_0 is non-trivial. If $Q_0 = e$, then we can take as P the intersection of all Sylow p-subgroups of Q. In this case P is the maximal normal p-toral subgroup of H. It follows from A3 of [JMO] and 7.6 of [JM1] that if $P' \in \mathcal{T}_p(G)$, then the center $Z(P')$ of P' is also in $\mathcal{T}_p(G)$. Let E be the maximal elementary abelian p-subgroup of $Z(P)$. Then $E \subset H \subset NH \subset NE$, $NH \subset NCE$ and, for every $E' \in \mathcal{A}_p(G)^H$,

$E' \cap CE = E'^E$ is a non-trivial group. The poset map $h_E : \mathcal{A}_p(G)^H \to (\mathcal{A}_p(G)^H \cap \mathcal{A}_p(CE))_{\geq E}$ such that $h_E(E') = (E' \cap CE)E$ whenever $E' \in \mathcal{A}_p(G)^H$, is continuous because it is an NH/H-poset map. The map Bh_E is the composition of the homotopy equivalences $B\mathcal{A}_p(G)^H \to B(\mathcal{A}_p(G)^H \cap \mathcal{A}_p(CE))$ and $B(\mathcal{A}_p(G)^H \cap \mathcal{A}_p(CE)) \to B((\mathcal{A}_p(G)^H \cap \mathcal{A}_p(CE))_{\geq E})$. The space $B((\mathcal{A}_p(G)^H \cap \mathcal{A}_p(CE))_{\geq E})$ is contractible because $\mathcal{A}_p(G)^H_{\geq E}$ has the final object E. Now we can apply 1.4(i).

If G is finite and a normal subgroup H of G contains a non-trivial p-subgroup then it was proved in [Dw] that the space $B\mathcal{A}_p(G)/H$ is F_p-acyclic. In 1.6 we will prove that this space is contractible.

Let $\mathcal{K}_p(G)$ denote the set of all subgroups H of G satisfying the condition that, for every maximal p-toral subgroup P in G, $H \cap Z(P)$ contains a non-trivial p-subgroup. If $H \in \mathcal{K}_p(G)$ and $H \subseteq H'$, then $H' \in \mathcal{K}_p(G)$. If H is a normal subgroup of G which, for every maximal p-toral subgroup P of G, contains a non-trivial normal p-toral subgroup P' of P, then $H \in \mathcal{K}_p(G)$. Indeed, $H \cap ZP$ contains P'^P, hence it contains a non-trivial p-group. If G is finite and a normal subgroup H of G contains a non-trivial p-subgroup, then H belongs to $\mathcal{K}_p(G)$. It was proved in [Dw] that in this case $H \cap P$ is a normal subgroup of P and a Sylow p-subgroup of H so $H \cap Z(P)$ contains a non-trivial p-subgroup. The following result is a generalization of 1.5.

1.6. Proposition. *Let G be compact Lie group with a non-trivial p-subgroup. If $H \in \mathcal{K}_p(G)$ then the space $B\mathcal{A}_p(G)/H$ is contractible.*

Proof. The result is a consequence of 1.4(i). It follows from the definition that $\mathcal{N}_p(H) \subseteq \mathcal{N}_p(G)$, hence, as in the proof of 1.5, for every $H_0 \in \mathcal{N}_p(H)$, $B\mathcal{A}_p(G)^{H_0}$ is contractible. If $x \in B\mathcal{A}_p(G)$, then H_x contains a non-trivial p-subgroup. Indeed, let $G_x = NE_0 \cap \ldots \cap NE_k$, where $E_i \in \mathcal{A}_p(G)$ and $E_0 < \ldots < E_k$. Let P be a maximal p-toral subgroup of G such that $E_k \subseteq P$. It follows from the definitions that $H \cap ZP \subseteq H \cap NE_0 \cap \ldots \cap NE_k = H_x$. The assumption that $H \in \mathcal{K}_p(G)$ now implies that H_x contains a non-trivial p-subgroup.

2. Homotopy decompositions over $(\operatorname{sd} W)/G$**.** Let $\mathcal{C}$ be a topological category. For any two functors $Y : \mathcal{C} \to \mathrm{Top}$ and $Y' : \mathcal{C}^{\mathrm{op}} \to \mathrm{Top}$, the topological space $Y' \times_{\mathcal{C}} Y$ is the coequalizer of the two natural maps

$$p_0, p_1 : \coprod_{\alpha : c \to c'} Y(c) \times Y'(c') \to \coprod_{c \in C} Y(c) \times Y'(c)$$

induced by the maps

$$p_0(\alpha)(y, y') = (Y(\alpha)y, y'), \quad p_1(\alpha)(y, y') = (y, Y'(\alpha)y').$$

In particular $\operatorname{hocolim}_{\mathcal{C}} Y = B(- \downarrow \mathcal{C}) \times_{\mathcal{C}} Y$, where $c \downarrow \mathcal{C}$ is the "under" category of the morphisms $c \to c'$ of $\mathcal{C}$.

If G_1 and G_2 are groups and $Y : \mathcal{C} \to G_1$-Top, $Y' : \mathcal{C}^{\mathrm{op}} \to G_2$-Top, then $G_1 \times G_2$ acts in a natural way on $Y' \times_{\mathcal{C}} Y$. If $G_1 = e$ then we obtain a G_2-action.

Let G be a Lie group and let X be a G-CW-complex. Let $\mathcal{S}(G, X)$ denote the set of isotropy groups of X and $\mathcal{S}_0(G, X) = \mathcal{S}(G, X) \cup \{G\}$. The full subcategory of $\mathcal{O}_G$ whose objects are the orbit spaces G/H, where $H \in \mathcal{S}(G, X)$, is denoted by $\mathcal{O}(G, X)$. The G-map spaces will be denoted by $\mathrm{Map}_G(-, -)$.

Let $F_1, F_2 : G\text{-CW} \to G\text{-CW}$ be functors such that

$$F_i(X) = \mathrm{Map}_G(-, X) \times_{\mathcal{O}_G} F_i, \qquad F_i(f) = \mathrm{Map}_G(-, f) \times_{\mathcal{O}_G} F_i$$

whenever $f : X \to X'$. In the formulas above the restriction of F_i to the subcategory $\mathcal{O}_G$ of G-CW is denoted by the same letter. We will need the following fact.

2.1. Proposition. *Let $\tau : F_1 \to F_2$ be a natural transformation of functors induced by its restriction to $\mathcal{O}_G$. If, for every $G/H \in \mathcal{O}(G, X)$, $\tau(G/H)$ is a G-homotopy equivalence, then so is $\tau(X) : F_1(X) \to F_2(X)$.*

Proof. Since the $\mathcal{O}_G$-orbits of the functor $\mathrm{Map}_G(-, X)$ have the form $\mathrm{Map}_G(-, G/G_x)$, where $x \in X$, the restriction of $\mathrm{Map}_G(-, X)$ to $\mathcal{O}(G, X)$ is a free functor in the sense of [DF1] and

$$F_i(X) = \mathrm{Map}_G(-, X) \times_{\mathcal{O}(G,X)} F_i.$$

This can be proved by induction on the dimension of X. Assume that the n-skeleton of X, denoted by X_n, is equal to the pushout

$$D^n \times T_n \leftarrow S^{n-1} \times T_n \to X_{n-1}$$

where T_n is a disjoint union of G-orbits from $\mathcal{O}(G, X)$ and the left arrow is the cofibration induced by the natural inclusion $S^{n-1} \to D^n$. Then $F_i(X_n)$ is equal to the homotopy pushout

$$D^n \times F_i(T_n) \leftarrow S^n \times F_i(T_n) \to F_i(X_{n-1}).$$

This implies that, if $\tau(X_{n-1})$ is a homotopy equivalence then so is $\tau(X_n)$. Now one can use the fact that $\tau(X) = \mathrm{hocolim}_{n \in \mathbb{N}} \tau(X_n)$.

2.2. Examples. (i) Let K be a G-CW-complex. Let $F = (H(-), H'(-)) : \mathcal{C} \to \mathcal{C}_G$ be a functor such that, for every isotropy group G' of X, the map

$$\mathrm{hocolim}_{c \in \mathcal{C}} K \times_{H(c)} (G/G')^{H'(c)} \to K/G'$$

is a homotopy equivalence. Then so is the map

$$\mathrm{hocolim}_{c \in \mathcal{C}} K \times_{H(c)} X^{H'(c)} \to K \times_G X.$$

(ii) Let $f : K_1 \to K_2$ be a cellular map of G-CW-complexes. If, for every isotropy group H of X, $f/H : K_1/H \to K_2/H$ is a homotopy equivalence, then so is $f \times_G X : K_1 \times_G X \to K_2 \times_G X$.

(iii) Let V be a G-subposet of $\mathcal{S}(G)$. Using the fact that for every $G' \in V$,

$$\operatorname{hocolim}_{G/H \in \mathcal{O}_V} (G/G')^H = B(G/G' \downarrow \mathcal{O}_V) \simeq *,$$

we obtain the decomposition described in 0.0(ii).

(iv) Let $F = (H(-), H'(-)) : \mathcal{C} \to \mathcal{C}_G$ be a functor such that, for every isotropy group G' of X, the map

$$\operatorname{hocolim}_{c \in \mathcal{C}} G \times_{H(c)} (G/G')^{H'(c)} \to G/G'$$

is a G-homotopy equivalence. Then so is the map

$$\operatorname{hocolim}_{c \in \mathcal{C}} G \times_{H(c)} X^{H'(c)} \to X.$$

In this section we will assume that W is a topological G-subposet of $\mathcal{S}(G)$ and that all elements of W are finite subgroups of G. This implies that the orbit spaces $d_n W/G$ are discrete and that W satisfies the condition that $w \le gw$, where $g \in G$, implies that $w = gw$.

Let H be a closed subgroup of G. We will use the notation

$$W_H = \{H' \in W : H' \subseteq H\}.$$

If H is a compact Lie group then the topology on W_H induced from W is equal to the topology induced from $\mathcal{S}(H)$. This follows from the fact that $(G/H)^{H'}/NH'$ is discrete (cf. the proof of II.5.7 in [Br2]).

2.3. Proposition. *Let X be a G-*CW*-complex such that all its isotropy groups are compact.*

(i) *If, for every $x \in X$, the map $K \times_{G_x} B(W_{G_x}) \to K/G_x$ is a homotopy equivalence, then there is a homotopy decomposition*

$$\operatorname{hocolim}_{[(H_0,\ldots,H_n)] \in \operatorname{sd} W/G} K \times_{NH_0 \cap \ldots \cap NH_n} X^{H_n} \simeq K \times_G X.$$

(ii) *If, for every $x \in X$, the map $G \times_{G_x} B(W_{G_x}) \to G/G_x$ is a G-homotopy equivalence, then there is a G-homotopy decomposition*

$$\operatorname{hocolim}_{[(H_0,\ldots,H_n)] \in \operatorname{sd} W/G} G \times_{NH_0 \cap \ldots \cap NH_n} X^{H_n} \simeq X.$$

Proof. Let

$$F'_K(X) = \operatorname{hocolim}_{[(H_0,\ldots,H_n)] \in \operatorname{sd} W/G} K \times_{NH_0 \cap \ldots \cap NH_n} X^{H_n}.$$

It follows from the definitions that $F'_K(X) = K \times_G F'_G(X)$.

If $X = * = G/G$, then there is a G-homotopy equivalence

$$F'_G(*) = \operatorname{hocolim}_{[(H_0,\ldots,H_n)] \in \operatorname{sd} W/G} G/(NH_0 \cap \ldots \cap NH_n) \simeq BW.$$

Indeed, $F'_G(*)$ is the classifying space of the category $W[G]$ whose objects are the pairs $([w.], [g])$, where $[w.] \in \operatorname{sd} W/G$, $[g] \in G/(NH_0 \cap \ldots \cap NH_n)$, $w. = (H_0, \ldots, H_n)$. The category $W[G]$ is a topological poset with an action of G defined by the action of G on $G/G_{w.}$ and there is an equivariant isomorphism of topological G-posets $F : W[G] \to \operatorname{sd} W$ such that $F([w.], [g]) = gw.$. Hence we have equivariant homotopy equivalences

$F'_G(*) \simeq BW[G] \simeq B\,\mathrm{sd}\,W$. Let $\mathcal{N}$ be the category whose objects are finite posets $[n] = \{0 \le 1 \le \ldots \le n\}$ and whose morphisms are the injective poset maps. Let $F_W : \mathcal{N} \to G\text{-}\mathrm{Top}$ be the functor such that $F_W([n]) = \mathrm{sd}_n\, W$ consists of all injective poset maps $[n] \to W$. Let Δ_n be the standard n-dimensional simplex. Then $\Delta_{(-)}$ is a free functor on the category $\mathcal{N}$. This implies that there are equivariant homotopy equivalences

$$B\,\mathrm{sd}\,W \simeq \mathrm{hocolim}_{\mathcal{N}}\, F_W \simeq \Delta_{(-)} \times_{\mathcal{N}} \mathrm{sd}_{(-)}\, W \simeq BW.$$

There is a natural G-CW-complex structure on BW such that the poset $\mathrm{sd}\,W/G = (\mathrm{sd}\,W)/G$ is equal to the poset of the G-cells of BW. For $K = *$ we obtain homotopy equivalences

$$B((\mathrm{sd}\,W)/G) = F'_*(*) = F'_G(*)/G \simeq B(\mathrm{sd}\,W)/G \simeq (BW)/G.$$

The inclusions $X^{H_n} \to X$ induce a map $p_K(X) : F'_K(X) \to K \times_G X$. The map $p_G(X)$ is a G-map and $p_K(X) = K \times_G p_G(X)$. Let $\pi_X : F'_G(X) \to F'_G(*) \simeq BW$ be the natural G-projection. To obtain the result it is sufficient to prove that, for every $x \in X$, the map $p_K(G/G_x)$ is a homotopy equivalence. This follows from the fact that, for every closed subgroup H of G, π_X induces an H-homotopy equivalence $p_G^{-1}(G/H)(H) \to BW_H$. Indeed, consider the natural projection $f_{w.} : G \times_{NH_0 \cap \ldots \cap NH_n} (G/H)^{H_n} \to G/H$. Then $G \times_{NH_0 \cap \ldots \cap NH_n} (G/H)^{H_n} = G \times_H f_{w.}^{-1}(H)$. Let

$$\begin{aligned} Y(w., H) &= \{g \in G : gH_n g^{-1} \subseteq H\}/(NH_0 \cap \ldots \cap NH_n) \\ &\subseteq G/(NH_0 \cap \ldots \cap NH_n). \end{aligned}$$

Then there is an H-isomorphism $\mu : Y(w., H) \to f_{w.}^{-1}(H)$ such that $\mu([g]) = [g, g^{-1}H]$. The space

$$p_G^{-1}(G/H)(H) \simeq \mathrm{hocolim}_{[w.] \in \mathrm{sd}\,W/G}\, Y(w., H)$$

is the classifying space of the category $W[H]$ whose objects are the pairs $([w.], [g])$, where $[w.] \in \mathrm{sd}\,W/G$, $[g] \in Y(w., H)$. $W[H]$ is a topological subposet of $W[G]$ and the restriction of F_W gives us an H-poset isomorphism $W[H] \to \mathrm{sd}\,W_H$. Now we can use the H-homotopy equivalence $B\,\mathrm{sd}\,W_H \simeq BW_H$ to conclude that $p_K(G/H)$ is homotopy equivalent to the projection $K \times_H BW_H \to K/H$ (which implies (i)) and that $p_G(G/H)$ is G-homotopy equivalent to the projection $G \times_H BW_H \to G/H$ (which implies (ii)).

The following result is an immediate consequence of 2.3.

2.4. Corollary. *Let X be a G-CW-complex such that all its isotropy groups are compact. Let W be a G-poset of finite subgroups of G such that the space $B\,\mathrm{sd}\,W/G$ is contractible. Suppose that $\mathcal{A}$ is thick and satisfies the condition* **A1**.

(i) *Suppose that, for every $x \in X$, the map $K \times_{G_x} B(W_{G_x}) \to K/G_x$ is a homotopy equivalence and that, for every $(H_0, \ldots, H_n) \in \operatorname{sd} W$, we have $K \times_{NH_0 \cap \ldots \cap NH_n} X^{H_n} \in \mathcal{A}$. Then $K \times_G X \in \mathcal{A}$.*

(ii) *Suppose that, for every $x \in X$, the space*

$$B(W_{G_x})/G_x = B \operatorname{sd} W_{G_x}/G_x$$

is contractible and that $X^H \in \mathcal{A}$ whenever

$$H \in \{NH_0 \cap \ldots \cap NH_n \cap G' : (H_0, \ldots, H_n) \in \operatorname{sd} W, \\ G' \in \mathcal{S}_0(G, X),\ H_n \subseteq G'\}.$$

Then $X/G \in \mathcal{A}$.

2.5. Examples. (i) Let X be a G-CW-complex such that all its isotropy groups are finite. Then there exists a G-homotopy decomposition

$$\operatorname{hocolim}_{[(H_0,\ldots,H_n)] \in \operatorname{sd} \mathcal{S}(G,X)/G} G \times_{NH_0 \cap \ldots \cap NH_n} X^{H_n} \simeq X$$

because, for every $x \in X$, the space $B\mathcal{S}(G, X)_{G_x}$ is G_x-contractible.

(ii) Suppose that, for every $x \in X$, $y \in K$, $G_x \in \mathcal{S}_{\mathrm{c}}(G)$ and $G_x \cap G_y \in \mathcal{K}_p(G_x)$. Then there is a homotopy equivalence

$$\operatorname{hocolim}_{[(E_0,\ldots,E_n)] \in \operatorname{sd} \mathcal{A}_p(G)/G} K \times_{NE_0 \cap \ldots \cap NE_n} X^{E_n} \simeq K \times_G X.$$

This is a consequence of 2.3, 2.2(ii) and 1.6. In particular, for $K = *$ we obtain 0.1.

(iii) Let G be compact Lie group with a non-trivial p-subgroup. Let $\mathcal{P}$ be the poset of all non-trivial finite p-subgroups of G. Then the space $(B\mathcal{P})/G$ is contractible. This follows from (ii) and from the fact that, for every $(E_0, \ldots, E_n) \in \operatorname{sd} \mathcal{A}_p(G)$, the space $B(\mathcal{P})^H$ is contractible whenever $E_n \leq H \leq NE_0 \cap \ldots \cap NE_n$ because $P'E_n \in \mathcal{P}^H$ if $P' \in \mathcal{P}^H$.

(iv) Let X be a G-CW-complex such that all its isotropy groups are compact and contain a non-trivial normal p-subgroup. Then there exists a G-homotopy decomposition

$$\operatorname{hocolim}_{[(E_0,\ldots,E_n)] \in \operatorname{sd} \mathcal{A}_p(G)/G} G \times_{NE_0 \cap \ldots \cap NE_n} X^{E_n} \simeq X$$

because, for every $x \in X$, the space $B\mathcal{A}_p(G_x)$ is G_x contractible. This follows from the fact that the poset $\mathcal{A}_p(G_x)^{G_x}$ is non-empty (cf. the proof of 1.5), and that, for every isotropy group H of $B\mathcal{A}_p(G_x)$, the map $B\mathcal{A}_p(G_x)^H \to *$ is a homotopy equivalence because all isotropy groups of $B\mathcal{A}_p(G_x)$ contain non-trivial normal p-subgroups.

One can prove this fact using similar methods to those in 1.5. Let E be a non-trivial, normal, elementary abelian p-subgroup of G_x. Let W be the G-poset of all subgroups of G_x of the form $E'E''$ where $E' \in \mathcal{A}_p(G_x)$ and E'' is a subgroup of E. Then BW is G_x-contractible.

Let G be a finite group. If $\mathcal{P}$ is a concave G-poset of p-subgroups of G, then $\mathcal{P}^o$ is the G-subposet of $\mathcal{P}$ such that $P \in \mathcal{P}^o$ if and only if $P \in \mathcal{P}$ and $\Phi(P) \notin \mathcal{P}$. Here $\Phi(P)$ denotes the Frattini subgroup of P. If $\mathcal{P} = \mathcal{T}_p(G)$, then $\mathcal{P}^o = \mathcal{A}_p(G)$.

2.6. Proposition. *Let G be a finite group. Let $\mathcal{P}'$ be a concave G-poset of p-subgroups of G. Let X be a G-CW-complex such that all Sylow p-subgroups of its isotropy groups are in $\mathcal{P}'$. Suppose that $\mathcal{P}$ is a G-poset of p-subgroups of G such that $\mathcal{P}'^o \subseteq \mathcal{P} \subseteq \mathcal{P}'$. Then there is a homotopy equivalence*

$$\operatorname{hocolim}_{[(P_0,\ldots,P_n)]\in \operatorname{sd}\mathcal{P}/G} X^{P_n}/(NP_0 \cap \ldots \cap NP_n) \simeq X/G.$$

Proof. The space $B(\mathcal{P}')/G$ is contractible. (This is a generalization of Corollary 2.6.1 of [We], which states that $B(\mathcal{P}')/G$ is F_p-acyclic.) Indeed, if $x \in B(\mathcal{P}')$, then $G_x = NP_0 \cap \ldots \cap NP_k$, where $P_i \in \mathcal{P}'$ and $P_0 < \ldots < P_k$, so Sylow p-subgroups of G_x are in $\mathcal{P}'$. It is proved in [We] (2.1.2) that, for every $H \in \mathcal{N}_{\mathcal{P}'}$, the space $B(\mathcal{P}')^H$ is contractible. Thus we can apply 1.2(ii) to the class $\mathcal{C}$. Proposition 1.7 of [TW] implies that the H-map $B(\mathcal{P}_H) \to B(\mathcal{P}'_H)$, induced by the inclusion of H-posets of subgroups, is an H-homotopy equivalence. The proof of this fact is similar to the proof of 2.1(i) of [TW]. Hence $B(\mathcal{P}_H)/H \simeq B(\mathcal{P}'_H)/H$ and the space $B(\mathcal{P}_H)/H$ is contractible. Now we can use 2.1.

The following result is an immediate consequence of 2.6. It is stronger than 1.2.

2.7. Corollary. *Let G be a finite group. Let $\mathcal{P}$ and X satisfy the assumptions of* 2.6. *Suppose that $\mathcal{A}$ is thick and satisfies the condition* **A1** *and that one of the following conditions holds*:

(i) $X^{P_n}/(NP_0 \cap \ldots \cap NP_n) \in \mathcal{A}$ *whenever* $(P_0,\ldots,P_n) \in \operatorname{sd}\mathcal{P}$,
(ii) $X^H \in \mathcal{A}$ *whenever*

$$H \in \{NP_0 \cap \ldots \cap NP_n \cap G' : (P_0,\ldots,P_n) \in \operatorname{sd}\mathcal{P},\ G' \in \mathcal{S}_0(G,X),\ P_n \subseteq G'\}.$$

Then $X/G \in \mathcal{A}$.

2.8. Corollary. *Let G be a finite group. Let $\mathcal{P}$ be a G-poset of p-subgroups of G such that $\mathcal{A}_p(G) \subseteq \mathcal{P}$. If, for every $x \in X$ and $y \in K$, G_x contains a non-trivial p-subgroup and $G_x \cap G_y \in \mathcal{K}_p(G_x)$, then there is a homotopy equivalence*

$$\operatorname{hocolim}_{[(P_0,\ldots,P_n)]\in \operatorname{sd}\mathcal{P}/G}\ K \times_{NP_0\cap\ldots\cap NP_n} X^{P_n} \simeq K \times_G X.$$

Proof. This result is a consequence of 2.5(ii). Let P be a non-trivial p-subgroup of G. It follows from [TW], 1.7 and 2.1, that there is an H-

homotopy equivalence $B(\mathcal{A}_p(G)_H) \to B(\mathcal{P}_H)$ whenever H is a subgroup of G and contains a non-trivial p-subgroup. Now we can use 2.1 and 2.3.

3. Categories associated to G-posets. Let K be a G-CW-complex. Every equivariant cellular map $f : X_1 \to X_2$ of G-CW-complexes induces maps $f(H, H') : K \times_H X_1^{H'} \to K \times_H X_2^{H'}$ where $(H, H') \in \mathcal{W}(G)$, i.e. $H, H' \in \mathcal{S}(G)$ and $H \subseteq NH'$.

For every functor $F = (H(-), H'(-)) : \mathcal{C} \to \mathcal{C}_G$ we have maps

$$\phi_i : \mathrm{hocolim}_{c\in\mathcal{C}}\, K \times_{H(c)} X_i^{H'(c)} \to K \times_G X_i,$$

$$f_F : \mathrm{hocolim}_{c\in\mathcal{C}}\, K \times_{H(c)} X_1^{H'(c)} \to \mathrm{hocolim}_{c\in\mathcal{C}}\, K \times_{H(c)} X_2^{H'(c)}$$

such that $f_F = \mathrm{hocolim}_{c\in\mathcal{C}}\, f(H(c), H'(c))$ and $f(G, e)\phi_1 = \phi_2 f_F$. It follows from general homotopy colimit theory that, if $f(H(c), H'(c))$ are homotopy equivalences for all $c \in \mathcal{C}$, then the map $f(G, e) : K \times_G X_1 \to K \times_G X_2$ is a homotopy equivalence. This motivates the following definition.

3.0. Definition. Let $\mathcal{S}$ be a G-poset of closed subgroups of G. A G-subposet W of $\mathcal{W}(G)$ is $(\mathcal{S}, K)$-*essential* if, for every equivariant cellular map $f : X \to Y$ of G-CW-complexes with all isotropy groups in $\mathcal{S}$, the condition that $K \times_H X^{H'} \to K \times_H Y^{H'}$ is a homotopy equivalence for every $(H, H') \in W$ implies that $K \times_G X \to K \times_G Y$ is a homotopy equivalence.

In particular, if W is $(\mathcal{S}_0(G, X), K)$-essential and $K \times_H X^{H'} \to K/H$ is a homotopy equivalence whenever $(H, H') \in W$, then $K \times_G X \to K/G$ is a homotopy equivalence.

The results of previous sections enable us to exhibit many non-trivial examples of essential posets. Our main tool will be the following consequence of 2.2(i).

3.1. Proposition. *Suppose that*

$$F = (H(-), H'(-)) : \mathcal{C} \to \mathcal{C}_G(W, d_{\mathcal{W}(G)})$$

is a functor such that for every $G' \in \mathcal{S}$, *the map*

$$\mathrm{hocolim}_{c\in\mathcal{C}}\, K \times_{H(c)} (G/G')^{H'(c)} \to K/G'$$

is a homotopy equivalence. Then the poset W *is* $(\mathcal{S}, K)$-*essential.*

3.2. Examples. (i) Let $\mathcal{P}$ be a concave G-subposet of p-toral subgroups of G such that all maximal p-toral subgroups of elements of $\mathcal{S}$ are in $\mathcal{P}$. Then it follows from 1.1 that the poset $\mathcal{W}_{\mathcal{P}} = \{(H, P) : P \subseteq H \subseteq NP,\ P \in \mathcal{P},\ H \in \mathcal{S}'_{\mathrm{c}}(G)\}$ is $(\mathcal{S}, *)$-essential.

(ii) The poset $\mathcal{W}_{\mathcal{A}_p(G)} = \{(H, E) : E \subseteq H \subseteq NE,\ E \in \mathcal{S},\ H \in \mathcal{S}'_{\mathrm{c}}(G)\}$ is $(\mathcal{S}_{\mathrm{c}}(G), *)$-essential. Let $\mathcal{S}_K(G)$ be the poset of all compact subgroups H of G with non-trivial p-subgroups and such that $H \cap G_k \in \mathcal{K}_p(H)$ for every $k \in K$. Then the poset $\mathcal{W}_{\mathcal{A}_p(G)}$ is also $(\mathcal{S}_K(G), K)$-essential. This is a consequence of 2.5(ii).

(iii) Let $f : X \to Y$ be an equivariant cellular map of G-CW-complexes such that, for every compact subgroup H of G with a non-trivial normal p-toral subgroup, the map $f^H : X^H \to Y^H$ is a homotopy equivalence. This implies that, for every $(H, H') \in \mathcal{W}_{\mathcal{A}_p(G)}$, the map $f^{H'}$ is an H-homotopy equivalence so the map $K \times_H X^{H'} \to K \times_H Y^{H'}$ is a homotopy equivalence. If all isotropy groups of points of X and Y are in $\mathcal{S}_K(G)$, then, by (ii), the map $F_K(f) : K \times_G X \to K \times_G Y$ is also a homotopy equivalence. In the case when $K = *$ we obtain 0.3(i).

Now we describe a construction of topological categories $\mathcal{C}$ associated to topological G-posets and some examples of functors $\mathcal{C} \to \mathcal{C}_G$ defined on such categories. We show that the known homotopy and homology decompositions can be obtained using this construction.

Let W be a topological G-poset such that W/G is a discrete topological space. Let $d : W \to \mathcal{S}(G)$ be a G-poset map such that, for every $w \in W$, dw is a subgroup of G_w. It follows that dw is a closed normal subgroup of G_w. The G-poset maps with the above property will be called *admissible maps*. Let $\mathcal{C}_G(W, d)$ be the topological category whose objects are the elements of W and whose morphism spaces are defined by

$$\mathrm{Mor}_{\mathcal{C}_G(W,d)}(w, w') = \{g \in G : w \le gw'\}/dw' \subseteq G/dw'.$$

The composition of $[g] : w \to w'$ and $[g'] : w' \to w''$ is $[gg'] : w \to w''$. The categories $\mathcal{C}_G(W, d)$, for discrete groups G, are studied in [S1-3], [JS].

3.3. Examples. (i) Let $W(G)$ denote the G-subposet of $\mathcal{S}(G) \times W$ whose elements are all pairs (H, w) where $w \in W$ and $H \subseteq G_w$. Let $d_{W(G)}$ be the admissible map $W(G) \to \mathcal{S}(G)$ such that $d_{W(G)}(H, w) = H$. Let $\mathcal{C}_G(W(G), d_{W(G)}) = \mathcal{C}_G(W)$. It follows from the definitions that $\mathcal{C}_G(*) = \mathcal{O}_G$. If $p_W : W(G)/G \to \mathcal{S}(G)/G$ is the map induced by the natural projection, then, for every closed subgroup H of G, $p_W^{-1}([H]) = W^H/NH$. (In the notation of [T], $\mathcal{C}_G(W) = \int_{H \in \mathcal{O}_G} W^H$.) The space $W(G)/G$ is discrete if, for every $H \in \mathcal{S}(G)$, W^H/NH is discrete. Hence if, for every $w \in W$, $(G/G_w)^H/NH$ is discrete then $W(G)/G$ is a discrete space. This is, in particular, the case when, for every $w \in W$, G_w is compact (cf. II.5.7 of [Br2]).

(ii) Let $d : W \to \mathcal{S}(G)$ be an arbitrary admissible function. Then there exists an inclusion $j_d : \mathcal{C}_G(W, d) \to \mathcal{C}_G(W)$ such that $j_d(w) = (dw, w)$ and the image of j_d is a full subcategory of $\mathcal{C}_G(W)$.

(iii) For $W = \mathcal{S}(G)^{\mathrm{op}}$, $W(G) = \mathcal{W}(G)$ and $\mathcal{C}_G(W) = \mathcal{C}_G$. Let V be a G-set of subgroups of G. Denote by $\mathcal{W}(V)$ the G-subposet of $\mathcal{W}(G)$ such that $(H, H') \in \mathcal{W}(V)$ if and only if $H, H' \in V$ and $H \subseteq H'$. The full subcategory of $\mathcal{C}_G$ whose object set is $\mathcal{W}(V)$ will be denoted by $\mathcal{C}(V)$. If

$p : \mathcal{W}(V)/G \to \mathcal{S}(G)/G$ is induced by the natural projection, then, for every closed subgroup H of G, $p^{-1}([H]) = V(\geq H)/NH$, where $V(\geq H)$ is the set of all elements of V which contain H. (That is, $\mathcal{C}(V) = \int_{H\in\mathcal{O}_V} V(\geq H)$.) Hence the space $\mathcal{W}(V)$ is discrete if, for every $H, H' \in V$, $H \subseteq H'$ implies that $(NH'\backslash(G/H')^H)/NH$ is discrete. In particular, if V is a G-poset of compact subgroups of G, then $\mathcal{W}(V)/G$ is discrete (II.5.7 of [Br2]).

(iv) Let U be a G-space and let W be a G-poset of non-empty finite subsets of U. There exists an admissible function d_U such that, for every $w \in W$, $d_U w = \bigcap_{u\in w} G_u$.

There exists a functor $O_d : \mathcal{C}_G(W,d) \to \mathcal{O}_G$ such that $O_d(w) = G/dw$ for every $w \in W$, and $O_d([g])(g'dw) = g'gdw'$ for every morphism $[g] : w \to w'$ of $\mathcal{C}_G(W,d)$. We will use the notation

$$E_G(W,d) = \operatorname{hocolim}_{w\in\mathcal{C}_G(W,d)} G/dw.$$

Let $d' : W^{\mathrm{op}} \to \mathcal{S}(G)$ be a G-poset map. Then, for every $w \in W$, $dw \subseteq G_w \subseteq Nd'w$. Hence there exists a functor $(d,d') : \mathcal{C}_G(W,d) \to \mathcal{C}_G$ such that $(d,d')(w) = (dw, d'w)$.

Let G' be a subgroup of G. We will use the notation

$$W_{d',G'} = \{w \in W : d'w \subseteq G'\}.$$

$W_{d',G'}$ will be considered as a G'-poset. The admissible function $d_{G'} : W_{d',G'} \to \mathcal{S}(G')$ will be defined in such a way that, for every $w \in W_{d',G'}$, $d_{G'}w = G' \cap dw$.

3.4. Lemma. *Let G' be a closed subgroup of G such that $W_{d',G'}/G'$ is a discrete space. Then there exists a G-homotopy equivalence*

$$\operatorname{hocolim}_{w\in\mathcal{C}_G(W,d)} G \times_{dw} (G/G')^{d'w} \simeq G \times_{G'} E_{G'}(W_{d',G'}, d_{G'}).$$

Proof. Let

$$R_w = \operatorname{Mor}_{\mathcal{C}_G(W,d)}(-,w) = \coprod_{[g]\in G/dw} \operatorname{Mor}_W(-,gw) = G \times_{dw} \operatorname{Mor}_W(-,w),$$

where $\coprod_{[g]\in G/dw} \operatorname{Mor}_W(-,gw)$ is topologized as a subspace of G/dw. Then for every functor $T : \mathcal{C}_G(W,d) \to G\text{-CW}$, $R_w \times_{\mathcal{C}_G(W,d)} T = T(w)$.

Hence

$$\begin{aligned} R_w \times_{\mathcal{C}_G(W,d)} G \times_{d(-)} (G/G')^{d'(-)} &= G \times_{dw} (G/G')^{d'w} \\ &= G \times_{dw} (\{g \in G : d'gw \subseteq G'\}/G') = G \times_{G'} Y \end{aligned}$$

where $Y = \{g : d'gw \subseteq G'\}/dw$ is a G'-subspace of G/dw.

We will consider $\mathcal{C}_{G'}(W_{d',G'}, d_{G'})$ as a subcategory of $\mathcal{C}_G(W,d)$. Then

$$Y = R_w \times_{\mathcal{C}_{G'}(W_{d',G'},d_{G'})} G'/d_{G'}(-)$$

and R_w after restriction to $\mathcal{C}_{G'}(W_{d',G'}, d_{G'})$ is equal to

$$\coprod_{[gdw]\in Y/G'} \mathrm{Mor}_{\mathcal{C}_{G'}(W_{d',G'},d_{G'})}(-, gw).$$

Let $E_d = B(- \downarrow \mathcal{C}_G(W,d))$. Then E_d is a $\mathcal{C}_G(W,d)$-CW-complex whose orbits have the form R_w. Hence,

$$\begin{aligned}\mathrm{hocolim}_{w\in\mathcal{C}_G(W,d)}\, G\times_{dw} (G/G')^{d'w} &= E_d \times_{\mathcal{C}_G(W,d)} G\times_{d(-)} (G/G')^{d'(-)}\\ &= G\times_{G'} (E_d \times_{\mathcal{C}_{G'}(W_{d',G'},d_{G'})} G'/d_{G'}(-)).\end{aligned}$$

The functor E_d after restriction to the category $\mathcal{C}_{G'}(W_{d',G'}, d_{G'})$ remains free in the sense of [DF1]. Hence there exists a G'-homotopy equivalence

$$E_d \times_{\mathcal{C}_{G'}(W_{d',G'},d'_G)} G'/d_{G'}(-) \simeq \mathrm{hocolim}_{w\in\mathcal{C}_{G'}(W_{d',G'},d_{G'})}\, G'/d_{G'}w.$$

3.5. Proposition. *Suppose that, for every $G'\in\mathcal{S}$, $W_{d',G'}/G'$ is a discrete space and the map*

$$\mathrm{hocolim}_{w\in\mathcal{C}_{G'}(W_{d',G'},d_{G'})}\, K/d_{G'}w \to K/G'$$

is a homotopy equivalence. Then:

(i) *The map*

$$\mathrm{hocolim}_{w\in\mathcal{C}_G(W,d)}\, K\times_{dw} X^{d'w} \to K\times_G X$$

*is a homotopy equivalence if X is a G-*CW*-complex and the isotropy groups of X are in $\mathcal{S}$.*

(ii) *The G-poset $\{(dw, d'w) : w\in W\}$ is $(\mathcal{S}, K)$-essential.*

Proof. Let $F_{d'} : G\text{-CW} \to G\text{-CW}$ be a functor such that

$$F_{d'}(X) = \mathrm{hocolim}_{w\in\mathcal{C}_G(W,d)}\, G\times_{dw} X^{d'w}.$$

It follows from 3.4 that, for every $G'\in\mathcal{S}$, there are homotopy equivalences

$$\begin{aligned}K\times_G F_{d'}(G/G') &\simeq K\times_{G'} \mathrm{hocolim}_{w\in\mathcal{C}_{G'}(W_{d',G'},d_{G'})}\, G'/dw\cap G'\\ &= \mathrm{hocolim}_{w\in\mathcal{C}_{G'}(W_{d',G'},d_{G'})}\, K/d_{G'}w \simeq K/G' = K\times_G G/G'.\end{aligned}$$

Now, it is sufficient to apply 2.2(i) and 3.1.

We now describe some special cases of 3.5.

3.6. Examples. (i) Let W be a topological G-poset satisfying the condition that $w\le gw$ implies $w = gw$. Assume that the spaces d_nW/G are discrete. Let $d_s : \mathrm{sd}\, W \to \mathcal{S}(G)$ be an admissible function such that

$$d_s w. = G_{w.} = G_{w_0}\cap\ldots\cap G_{w_n}.$$

The natural projection $\mathrm{sd}\, W \to (\mathrm{sd}\, W)/G$ induces a natural equivalence of categories $\mathcal{C}_G(\mathrm{sd}\, W, d_s) \to (\mathrm{sd}\, W)/G$. It follows from the definitions that there is a G-homotopy equivalence

$$\operatorname{hocolim}_{\mathcal{C}_G(\operatorname{sd} W, d_s)} G/d_s(-) \to BW.$$

If W is a G-subset of $\mathcal{S}(G)$, then $d_s(H_0,\ldots,H_n) = NH_0 \cap \ldots \cap NH_n$ and $d'(H_0,\ldots,H_n) = H_n$. Hence 3.5 can be considered as a generalization of 2.3.

(ii) Let W be a G-subposet of $\mathcal{S}(G)$. Then $G_w = N_G w = Nw$. If $d : W^{\mathrm{op}} \to \mathcal{S}(G)$ is an arbitrary admissible function, then we can take $d'w = w$ whenever $w \in W$. Let $d_c : W^{\mathrm{op}} \to \mathcal{S}(G)$ be an admissible map such that, for every $w \in W$, $d_c w = C_G w = Cw$. Then $\mathcal{C}_G(W^{\mathrm{op}}, d_c) = \mathcal{C}_W$ is the category whose objects are elements of W and whose morphisms are the group homomorphisms which are restrictions of inner automorphisms of G. Let X be a G-CW-complex such that all its isotropy groups are compact. If the space $\operatorname{hocolim}_{w\in\mathcal{C}_{W_H}} H/C_H w$ is H-contractible whenever H is an isotropy group of X, then the map

$$\operatorname{hocolim}_{w\in\mathcal{C}_W} G \times_{C_G w} X^w \to X$$

is a G-homotopy equivalence. If the map

$$\operatorname{hocolim}_{w\in\mathcal{C}_{W_H}} K/C_H w \to K/H$$

is a homotopy equivalence whenever H is an isotropy group of X, then the map

$$\operatorname{hocolim}_{w\in\mathcal{C}_W} K \times_{C_G w} X^w \to K \times_G X$$

is also a homotopy equivalence.

(iii) Let $W = \mathcal{A}_p(G)$. Then $\mathcal{C}_G(\mathcal{A}_p(G)^{\mathrm{op}}, d_c) = A_p(G)$. If H is a compact Lie group with a non-trivial p-subgroup, then there is an H-homotopy equivalence

$$\operatorname{hocolim}_{E\in A_p(H)} H/C_H E \simeq \mathcal{E}\mathcal{O}_{\mathcal{Z}_p(H)}$$

where $\mathcal{Z}_p(G)$ is the poset of all compact subgroups of G with a non-trivial central p-subgroup and

$$\mathcal{E}\mathcal{O}_{\mathcal{Z}_p(H)} = E_H(\mathcal{Z}_p(H), \mathrm{id}) = \operatorname{hocolim}_{H/H'\in\mathcal{O}_{\mathcal{Z}_p(H)}} H/H'.$$

Indeed, for every $H' \in \mathcal{Z}_p(H)$, the space $(\operatorname{hocolim}_{E\in A_p(H)} H/C_H E)^{H'} = B(H/H' \downarrow O_{d_c})$ is homotopy equivalent to $B(H' \downarrow d_c) = B(\mathcal{A}_p(C_H H'))$ and hence is contractible. This implies that there is a G-homotopy equivalence

$$\operatorname{hocolim}_{E\in A_p(G)} G \times_{C_G E} X^E \simeq X$$

whenever all isotropy groups of X are in $\mathcal{Z}_p(G)$.

3.7. Example. Let V be a G-subset of $\mathcal{S}(G)$ such that $\mathcal{W}(V)/G$ is discrete. Let

$$r_V(X) = \operatorname{hocolim}_{(H,H')\in\mathcal{C}(V)} G \times_H X^{H'}.$$

This construction is natural in X and $\mathcal{S}(G, r_V(X)) \subseteq V$. The G-maps $G \times_H X^{H'} \to X$ define a natural transformation of functors $p_V : r_V \to \mathrm{Id}_{G\text{-CW}}$.

There exists a G-homotopy equivalence (natural in X)

$$r_V(X) \to B(\mathrm{Map}_G(G/e,-),\mathcal{O}_V,\mathrm{Map}_G(-,X))$$

where $B(-,-,-)$ is the bar construction described in Section 3 of [HV] and in Section 4 of [Dw].

If $G' \in V$, then the map $p_V(X)^{G'} : r_V(X)^{G'} \to X^{G'}$ is a homotopy equivalence. Indeed, in this case we have homotopy equivalences

$$(\mathrm{hocolim}_{(H,H')\in\mathcal{C}(V)} G\times_H X^{H'})^{G'} \simeq \mathrm{hocolim}_{(H,H')\in\mathcal{W}(V(\ge G'))} X^{H'} \simeq X^{G'}.$$

Suppose that all isotropy groups of X are in V. Then $p_V(X) : r_V(X) \to X$ is a G-homotopy equivalence and gives us a G-homotopy decomposition of X

$$\mathrm{hocolim}_{(H,H')\in\mathcal{C}(V)} G\times_H X^{H'} \simeq X$$

from 0.0(i). If $f : X_1 \to X_2$ is an equivariant map of G-CW-complexes and, for every $H \in V$, $f^H : X_1^H \to X_2^H$ is a homotopy equivalence, then $r_V(f)$ is a G-homotopy equivalence because, for every $(H,H') \in \mathcal{W}(V)$, H acts trivially on $X^{H'}$. Hence, for every K, $\mathcal{W}(V)$ is (V,K)-essential.

It follows from the definitions that $p_V(X)/G$ gives us a homotopy decomposition of X/G from 0.0(ii):

$$\mathrm{hocolim}_{G/H'\in\mathcal{O}_V} X^{H'} \simeq \mathrm{hocolim}_{(H,H')\in\mathcal{C}(V)} X^{H'} \simeq X/G$$

and that

$$\begin{aligned}\mathcal{E}\mathcal{O}_V = E_G(V,\mathrm{id}) &= \mathrm{hocolim}_{G/H\in\mathcal{O}_V} G/H\\ &= \mathrm{hocolim}_{(H,H')\in\mathcal{C}(V)} G/H = r_V(*).\end{aligned}$$

Let G' be a closed subgroup of G and let V be a G-subposet of $\mathcal{S}(G)$ such that the spaces $\mathcal{W}(V)/G$ and $\mathcal{W}(V_{G'})/G'$ are discrete. The following two results are consequences of 3.5 and the fact that $\mathcal{C}(V) = \mathcal{C}_G(\mathcal{W}(V), d_{\mathcal{W}(G)})$ and $r_V(*) = E_G(\mathcal{W}(V_G), d_{\mathcal{W}(G)})$.

3.8. Corollary. *There exists a G-homotopy equivalence*

$$\mathrm{hocolim}_{(H,H')\in\mathcal{C}(V)} G\times_H (G/G')^{H'} \simeq G\times_{G'} E_{G'}(\mathcal{W}(V_{G'}), d_{\mathcal{W}(G')}).$$

3.9. Corollary. *Let $f : X_1 \to X_2$ be a G-cellular map such that, for every $H \in V$, f^H is a homotopy equivalence.*

(i) *If, for every isotropy group G' of X_i, the map $r_{V_{G'}}(*) \to *$ is a G'-homotopy equivalence, then the maps*

$$\mathrm{hocolim}_{(H,H')\in\mathcal{C}(V)} G\times_H X_i^{H'} \to X_i$$

and f are G-homotopy equivalences.

(ii) *If, for every isotropy group G' of X_i, the map $K\times_{G'} r_{V_{G'}}(*) \to K/G'$ is a homotopy equivalence, then the maps*

$$\mathrm{hocolim}_{(H,H')\in\mathcal{C}(V)} K\times_H X_i^{H'} \to K\times_G X_i$$

and $\mathrm{id}_K \times_G f$ *are also homotopy equivalences.*

3.10. Example. Let X be a G-CW-complex. It follows from 3.6(iii) and 2.5(iv) that there are G-homotopy equivalences

$$r_{\mathcal{Z}_p(G)}(X) \simeq \operatorname{hocolim}_{E\in A_p(G)} G \times_{C_G E} X^E,$$
$$r_{\mathcal{N}_p(G)}(X) \simeq \operatorname{hocolim}_{[(E_0,\dots,E_n)]\in \operatorname{sd}\mathcal{A}_p(G)/G} G \times_{NE_0\cap\dots\cap NE_n} X^{E_n}.$$

3.11. Example. Let G be a discrete group. Let V be a G-poset of subgroups of G satisfying the condition that $v \le gv$ implies $v = gv$. Let $d : V^{\mathrm{op}} \to \mathcal{S}(G)$ be an admissible function. It is proved in [JS] that, for every admissible function $d'' : W \to \mathcal{S}(G)$, there exists a natural G-map $E_G(W, d'') \to BW$ which is a homotopy equivalence. This implies that if, for every isotropy group G' of X, the space $BV_{\le G'}$ is contractible, then the G-maps

$$\operatorname{hocolim}_{(H,H')\in\mathcal{C}(V)} G \times_H X^{H'} \to X,$$
$$\operatorname{hocolim}_{H\in\mathcal{C}_G(V^{\mathrm{op}},d)} G \times_{dH} X^H \to X$$

are homotopy equivalences and that, for every free G-CW complex K, we have homotopy decompositions

$$\operatorname{hocolim}_{(H,H')\in\mathcal{C}(V)} K \times_H X^{H'} \simeq K \times_G X,$$
$$\operatorname{hocolim}_{H\in\mathcal{C}_G(V^{\mathrm{op}},d)} K \times_{dH} X^H \simeq K \times_G X.$$

Here $V_{\le G'} = \{H \in V : H \le G'\}$.

3.12. Remark. One can generalize the above result of [JS] and construct G-maps (natural in X)

$$\operatorname{hocolim}_{(H,H')\in\mathcal{C}(V)} G \times_H X^{H'} \to Y,$$
$$\operatorname{hocolim}_{H\in\mathcal{C}_G(V^{\mathrm{op}},d)} G \times_{dH} X^H \to Y,$$

where

$$Y = \operatorname{hocolim}_{[(H_0,\dots,H_n)]\in \operatorname{sd} V/G} G \times_{NH_0\cap\dots\cap NH_n} X^{H_n},$$

which are homotopy equivalences. Hence, for every free G-CW-complex K, we have homotopy equivalences

$$K \times_G r_V(X) \simeq \operatorname{hocolim}_{[(H_0,\dots,H_n)]\in \operatorname{sd} V/G} K \times_{NH_0\cap\dots\cap NH_n} X^{H_n},$$
$$\operatorname{hocolim}_{H\in\mathcal{C}_G(V^{\mathrm{op}},d)} K \times_{dH} X^H \simeq \operatorname{hocolim}_{[(H_0,\dots,H_n)]\in \operatorname{sd} V/G} K \times_{NH_0\cap\dots\cap NH_n} X^{H_n}.$$

4. h^*_G-decompositions of G-CW-complexes. Let G be a Lie group and let h^*_G be a generalized G-cohomology theory. Let $h\mathcal{O}_G$ be the category whose objects are the same as the objects of $\mathcal{O}_G$ and whose morphisms are the G-homotopy classes of the morphisms of $\mathcal{O}_G$. Let M be a functor from the category $h\mathcal{O}_G^{\mathrm{op}}$ to the category Ab of abelian groups. The ordinary equivariant cohomology of a G-CW-complex Y with coefficients in M will be denoted by $H_G^*(Y,M)$. These cohomology groups, in the case when G is a finite group, was defined in [Br1]. The case of a Lie group is described in [Wi] and in the appendix of [JMO]. For any generalized G-cohomology theory h_G^* on G-CW, there is a spectral sequence

$$H_G^m(Y, h_G^n(-)) \Rightarrow h_G^{m+n}(Y).$$

For every closed subgroup H of G, the H-cohomology theory such that $h_H^*(X') = h_G^*(G \times_H X')$ whenever X' is an H-CW-complex will be denoted by h_H^*. This gives us a functor $h_{H(-)}^*(X^{H'(-)})$ defined on the homotopy category $h\mathcal{C}$ associated to $\mathcal{C}$. This functor can be considered as coefficients of the generalized cohomology theory $h_G^*(- \times_{\mathcal{C}} (G \times_{H(-)} X^{H'(-)}))$ defined on the category of free $\mathcal{C}$-CW-complexes in the sense of [DF1], i.e. $\mathcal{C}$-CW-complexes with orbits of the form $\mathrm{Mor}_{\mathcal{C}}(-,c)$. For every contravariant functor $M : h\mathcal{C} \to \mathrm{Ab}$, $H^*(\mathcal{C}, M) = \mathrm{Tor}_{\mathcal{C}}^*(\mathbb{Z}, M)$ is equal to the Bredon cohomology groups $H_{\mathcal{C}}^*(B(- \downarrow \mathcal{C}), M)$ (Sections 4 and 5 of [DF1]). Recall that

$$\mathrm{hocolim}_{c\in\mathcal{C}}(G \times_H (-) X^{H'(-)}) = B(- \downarrow \mathcal{C}) \times_{\mathcal{C}} G \times_{H(-)} X^{H'(-)}.$$

Let W be a G-subposet of $\mathcal{W}(G)$. Let $F = (H(-), H'(-)) : \mathcal{C} \to \mathcal{C}_G(W, d_{\mathcal{W}(G)})$ be a functor such that the map

$$p_F(X) : \mathrm{hocolim}_{c\in\mathcal{C}}\, G \times_{H(c)} X^{H'(c)} \to X$$

is an h_G^*-decomposition of X, i.e. the map

$$h_G^*(X) \to h_G^*(\mathrm{hocolim}_{c\in\mathcal{C}}\, G \times_{H(c)} X^{H'(c)})$$

is an isomorphism. It follows from 5.3 of [DF1] that there exists a spectral sequence

$$H^m(\mathcal{C}, h_{H(-)}^n(X^{H'(-)})) \Rightarrow h_G^{m+n}(X).$$

The results of this section describe and use this spectral sequence in many examples.

We remark that if $X = *$ and $F = G/H(-) : \mathcal{C} \to \mathcal{O}_G$, then we obtain the spectral sequence of the generalized cohomology theory h_G^* on $Y = \mathrm{hocolim}_{c\in\mathcal{C}} F(c)$.

Let $f : X_1 \to X_2$ be a G-CW-complex map and let $p_F(X_i)$ be an h_G^*-decomposition of X_i for $i = 1, 2$. If, for every $c \in \mathcal{C}$, $h_{H(c)}^*(X_2^{H'(c)}) \to h_{H(c)}^*(X_1^{H'(c)})$ is an isomorphism then the map $h^*(f) : h_G^*(X_2) \to h_G^*(X_1)$ is an isomorphism. This motivates the following definition.

4.0. Definition. Let $\mathcal{S}$ be a G-subposet of $\mathcal{S}(G)$. Let W be a G-subposet of $\mathcal{W}(G)$. We will say that W is $(\mathcal{S}, h_G^*)$-*essential* if, for every equivariant cellular map $f : X \to Y$ of G-CW-complexes whose isotropy groups are all in $\mathcal{S}$, the condition that $h^*_H(Y^{H'}) \to h^*_H(X^{H'})$ is an isomorphism whenever $(H, H') \in W$ implies that $h^*_G(Y) \to h^*_G(X)$ is an isomorphism.

In particular, if W is $(\mathcal{S}_0(G,X), h^*_G)$-essential and $h^*_H(*) \to h^*_H(X^{H'})$ is an isomorphism whenever $(H', H) \in W$, then $h^*_G(*) \to h^*_G(X)$ is an isomorphism.

The following result can be used to construct many non-trivial examples of h^*_G-essential posets.

4.1. Proposition. *Let $F = (H(-), H'(-)) : \mathcal{C} \to \mathcal{C}_G(W, d_{\mathcal{W}(G)})$ be a functor such that for every $G' \in \mathcal{S}$, the map*

$$\mathrm{hocolim}_{c\in\mathcal{C}}\, G \times_{H(c)} (G/G')^{H'(c)} \to G/G'$$

is an h^*_G-equivalence. Then:

(i) *The map $p_F(X)$ is an h^*_G-decomposition of X if all isotropy groups of X are in $\mathcal{S}$.*

(ii) *The poset W is $(\mathcal{S}, h^*_G)$-essential.*

Proof. Let

$$h'^*_G(X) = h^*_G(\mathrm{hocolim}_{c\in\mathcal{C}}\, G \times_{H(c)} X^{H'(c)}).$$

Then p_F induces a natural transformation $p^* : h^*_G \to h'^*_G$ of G-cohomology theories. If the assumption of the proposition holds, then $p^*(X)$ is an isomorphism. Hence $p_F(X)$ is an h^*_G-equivalence.

Let R be a commutative ring. The generalized G-cohomology theories from the category G-CW to the category R^*-Mod of graded R-modules will be called *R-G-cohomology theories.*

Let V be a G-poset of compact subgroups of G. Recall that $\mathcal{C}(V)$ is a full subcategory of $\mathcal{C}_G$ whose objects are the elements of the poset $\mathcal{W}(V)$ of pairs (H, H') such that H is a subgroup of H' and $H' \in V$.

4.2. Proposition. *Let $h^*_G = \{h^n\}_{n\in\mathbb{N}}$ be an R-G-cohomology theory. Let $\mathcal{S}$ and V be G-posets of compact subgroups of G such that*, *for every $H \in \mathcal{S}$, $h^*_H(*) \to h^*_H(r_{V_H}(*))$ is an isomorphism. Then*:

(i) *The G-poset $\mathcal{W}(V)$ is $(\mathcal{S}, h^*_G)$-essential.*

(ii) *Let $f : X \to Y$ be a map of G-CW-complexes whose isotropy groups are all in $\mathcal{S}$. If*, *for every $H \in V$, the map $X^H \to Y^H$ is an R-equivalence*, *then $h^*_G(Y) \to h^*_G(X)$ is an isomorphism.*

Proof. (i) is a consequence of 3.8 and 4.1(ii).

(ii) Propositions 4.1(i) and 3.8 imply that, for every G-CW-complex X whose isotropy groups are in $\mathcal{S}$, there exists a spectral sequence

$$H^m(\mathcal{C}(V), h^n_{H(-)}(X^{H'(-)})) \Rightarrow h^{m+n}_G(X).$$

This spectral sequence is natural in X. The assumption implies that, for every $(H,H')\in\mathcal{C}(V)$, the map $h^*_H(Y^{H'})\to h^*_H(X^{H'})$ is an isomorphism because $H\subseteq H'$. Hence the map $X\to Y$ is an h^*_G-equivalence.

4.3. Examples. Let $h^*_G(X)=H^*(K\times_G X,R)$. Then $h^*_G(G/H)=H^*(K/H,R)$.

(i) Let $K=*$, $R=F_p$. It is proved in [JMO] (1.2, 2.2, 2.12) that, if H is a compact Lie group and $\dim H>0$, then the space $\mathcal{EO}_{\mathcal{T}_p(H)}=r_{\mathcal{T}_p(H)}(*)$ is F_p-acyclic. Let $\mathcal{S}_d(G)$ denote the set of all compact subgroups H of G such that $\dim H>0$. Let $f:X\to Y$ be a map of G-CW-complexes whose isotropy groups are all in $\mathcal{S}_d(G)$. If, for every non-trivial p-toral subgroup H of G, the map $f^H:X^H\to Y^H$ is an F_p-homology isomorphism, then so is f. In particular, let G be a compact Lie group. If all isotropy groups of X are in $\mathcal{S}_d(G)$ and, for every non-trivial p-toral subgroup H of G, X^H is F_p-acyclic, then X is F_p-acyclic.

(ii) Let $\mathcal{A}'_p(G)=\mathcal{A}_p(G)\cup\{e\}$. If $H\in\mathcal{Z}_p(G)$ and $E\in\mathcal{A}'_p(H)$, then the space $\mathcal{EO}_{\mathcal{A}_p(H)}/E=\mathcal{EO}_{\mathcal{A}'_p(H)}/E$ is contractible. Let $f:X\to Y$ be a map of G-CW-complexes whose isotropy groups are all in $\mathcal{Z}_p(G)$. Suppose that, for every $E\in\mathcal{A}_p(G)$, f^E is an R-homology isomorphism and that, for every $k\in K$ and $x\in X\cup Y$, $G_x\cap G_k$ is an elementary abelian p-subgroup of G_x. This implies that, for every $x\in X\cup Y$, the map $K\times_{G_x}\mathcal{EO}_{\mathcal{A}_p(G_x)}\to K/G_x$ is an R-homology isomorphism. Hence the map $K\times_G X\to K\times_G Y$ is an R-homology isomorphism.

(iii) Let $K=*$. Then we obtain 0.3(ii) as a consequence of 4.2 and 1.4.

Let $h^*_G(X)=H^*(K\times_G X,F_p)$. In this case there is a spectral sequence

$$H^m_G(K,H^n(X\times_G(-),F_p))\Rightarrow h^{n+m}_G(X).$$

Hence if, for all maximal p-toral subgroups P of isotropy groups of K, X/P is F_p-acyclic, then $h^*_G(X)=H^*(K/G,F_p)=h^*_G(*)$. We will use this fact in the following examples.

4.4. Examples. Let $h^*_G=H^*(K\times_G -,F_p)$. Let $f:X\to Y$ be an equivariant cellular map of G-CW-complexes with compact isotropy groups.

(i) Let $S'_p(G)$ be the poset of all subgroups of p-toral subgroups of G, and let $S_p(G)$ be the subposet of $S'_p(G)$ consisting of all subgroups which contain a non-trivial p-subgroup. Let H be a compact subgroup of G. Then $h^*_H(*)=h^*_H(r_{S'_p(H)}(*))$ because, for every p-toral subgroup P of H, $r_{S'_p(H)}(*)/H$ is F_p-acyclic. It follows from Section 3 of [JO] that the maps $H^m_H(*,h^n_H)\to H^m_H(r_{S'_p(H)}(*),h^n_H)$, where $m>0$, are isomorphisms. Hence so are the maps $H^0_H(*,h^n_H)\to H^0_H(r_{S'_p(H)}(*),h^n_H)$. From 3.3 of [JO] and 1.2 and 2.2 of [JMO], it follows that the map $r_{\mathcal{T}'_p(H)}(*)\to r_{S'_p(H)}(*)$ induces isomorphisms in h^*_H and $H^*_H(-,h^n_H)$. This implies that the maps $h^*_G(Y)\to h^*_G(X)$ and

$H^*_G(Y, h^n_G) \to H^*_G(X, h^n_G)$ are isomorphisms if, for every p-toral subgroup P of G, f^P is a mod p homology isomorphism.

(ii) Suppose that, for all $n > 0$, $H^n(K, F_p) = 0$. Let $n > 0$. In this case $h^n_H(H/e) = 0$ and (i) implies that the map $H^*_H(*, h^n_H) \to H^*_H(r_{\mathcal{T}_p(H)}(*), h^n_H)$ is an isomorphism. Hence $H^*_G(Y, h^n_G) \to H^*_G(X, h^n_G)$ is an isomorphism if, for every non-trivial p-toral subgroup P of G, f^P is a mod p homology isomorphism.

(iii) Suppose that K is F_p-acyclic. Then, for every $H/H' \in \mathcal{O}_H$, $h^0_H(H/H') = F_p$ and $h^0_H(-)$ is the constant functor after restriction to $\mathcal{O}_H$. It follows from 1.2 and 2.2 of [JMO] and Proposition 2 and Theorem 3 of [O1] that the map

$$H^*(r_{\mathcal{T}_p}(H)(*)/H, F_p) \to H^*(r_{\mathcal{T}'_p(H)}(*)/H, F_p)$$

is an isomorphism. By (ii), so is $h^*_H(r_{\mathcal{T}_p(H)}) \to h^*_H(r_{\mathcal{T}'_p(H)}(*))$. Suppose that all isotropy groups of X and Y contain non-trivial p-subgroups. If, for every non-trivial p-toral subgroup P of G, f^P is a mod p homology isomorphism, then, for all natural n, the maps $H^*_G(Y, h^n_G) \to H^*_G(X, h^n_G)$ and $H^n(K \times_G Y, F_p) \to H^n(K \times_G X, F_p)$ are isomorphisms. In particular, we obtain 0.4. If G is a compact Lie group, then we can take $X = B\mathcal{A}_p(G)$, $Y = *$ (cf. the proof of 1.5) to obtain 0.5.

(iv) Let K be a G-CW-complex such that, for every $k \in K$ and for every p-toral subgroup P of G_x, $G_k \cap P$ is an elementary abelian p-group. Suppose that K is F_p-acyclic. (In particular, we can take $K = EG$.) If all isotropy groups of X and Y contain non-trivial p-subgroups and, for every $E \in \mathcal{A}_p(G)$, f^E is a mod p homology isomorphism, then $K \times_G X \to K \times_G Y$ is a mod p homology isomorphism. Indeed, it follows from 4.3(ii) that $K \times_G r_{\mathcal{T}_p(G)}(X) \to K \times_G r_{\mathcal{T}_p(G)}(Y)$ is a mod p homology isomorphism. Now we can use the fact that, by (iii), $K \times_G r_{\mathcal{T}_p(G)}(X) \to K \times_G X$ is a mod p homology isomorphism.

4.5. Examples. Let G be a discrete group and A a $\mathbb{Z}(G)$-module. We will consider the Bredon cohomology theory $h^*_G = H^*_G(-, M_A)$, where $M_A(-) = \mathrm{Hom}_{\mathbb{Z}(G)}(\mathbb{Z}(-), A)$. Hence

$$H^n_G(X, M_A) = H^n(\mathrm{Hom}_{\mathbb{Z}(G)}(C_*(X), A))$$

where $C_*(X)$ is the ordinary cellular chain complex of X. For every $G/H \in \mathcal{O}_G$, we have $h^*_G(G/H) = M_A(G/H) = A^H$.

(i) Let G be a finite group. Suppose that there is a non-trivial p-subgroup P of G such that every element of P acts trivially on A. Then

$$H^*_G(|\mathcal{S}_p(G)|, M_A) = A^G = H^*_G(*, M_A).$$

Indeed, M_A is a Hecke functor and it follows from the results of [Wa1] that if A is an $R(G)$-module and, for every subgroup H of G, X/H is R-acyclic, then $H^*_G(X, M_A) = A^G$. Let H be a normal subgroup of G with a non-trivial p-subgroup and let $G' = G/H$. If A' is a $\mathbb{Z}(G')$-module, then $H^*_{G'}(|\mathcal{S}_p(G)|/H, M_{A'}) = A'^{G'}$ because, by 2.8, $|\mathcal{S}_p(G)|/H'$ is contractible whenever $H \subseteq H' \subseteq G$.

(ii) Let G be a discrete group. Let $\mathcal{S}_A(G)$ denote the set of all finite subgroups H of G with a non-trivial p-subgroup P such that every element of P acts as identity on A. Suppose that all isotropy groups of X and Y are in $\mathcal{S}_A(G)$. The map $H^*_G(Y, M_A) \to H^*_G(X, M_A)$ is an isomorphism if, for every compact subgroup H of G with a non-trivial normal p-toral subgroup, f^H is a homology isomorphism.

(iii) Let A be an $F_p(G)$-module. Let K be a G-CW-complex. Suppose that all isotropy groups of points of X and Y are finite. In this case the maps $h^*_G(K \times Y) \to h^*_G(K \times X)$ and

$$H^*_G(Y, h^n_G(K \times (-))) \to H^*_G(X, h^n_G(K \times (-)))$$

are isomorphisms if, for every p-subgroup P of G, f^P is a mod p homology isomorphism. This is a consequence of the fact that, for every Hecke functor $M : \mathcal{O}_G^{\mathrm{op}} \to F_p\text{-Mod}$, $M(G/G) = H^*_G(r_{\mathcal{T}'_p(G)}, M)$ (1.29 of [S3]).

Let W be a topological G-poset satisfying the condition that $w \le gw$, where $g \in G$, implies that $w = gw$. Let $d : W \to \mathcal{S}(G)$ be an admissible function and let $d' : W^{\mathrm{op}} \to \mathcal{S}(G)$ be a G-poset map. The next result follows immediately from 3.4 and 4.1.

4.6. Proposition. *Suppose that, for every isotropy group H of the action of G on X, the space $W_{d',H}/H$ is discrete and the map*

$$h^*_H(*) \to h^*_H(\mathrm{hocolim}_{w \in \mathcal{C}_H(W_{d',H}, d_H)} H/H \cap dw)$$

is an isomorphism. Then so is the map

$$h^*_G(X) \to h^*_G(\mathrm{hocolim}_{w \in \mathcal{C}_G(W,d)} G \times_{dw} X^{d'w})$$

and there is a spectral sequence

$$H^m(\mathcal{C}_G(W,d), h^n_{d(-)}(X^{d'(-)})) \Rightarrow h^{m+n}_G(X).$$

4.7. Example. Let K be a G-CW-complex. Suppose that, for every $x \in X$, the map

$$H^*(K/G_x, R) \to H^*(\mathrm{hocolim}_{w \in \mathcal{C}_{G_x}(W_{d',G_x}, d_{G_x})} K/G_x \cap dw, R)$$

is an isomorphism. Then so is the map

$$H^*(K \times_G X, R) \to H^*(\mathrm{hocolim}_{w \in \mathcal{C}_G(W,d)} K \times_{dw} X^{d'w}, R)$$

and there is a spectral sequence

$$H^m(\mathcal{C}_G(W,d), H^n(K \times_{d(-)} X^{d'(-)}, R)) \Rightarrow H^{m+n}(K \times_G X, R).$$

In particular, if, for every isotropy group H of X, $B\mathcal{C}_H(W_{d',H}, d_H)$ is R-acyclic, then there is a spectral sequence

$$H^m(\mathcal{C}_G(W,d), H^n(X^{d'w}/dw, R)) \Rightarrow H^{m+n}(X/G, R).$$

4.8. Examples. Let W be a poset of closed subgroups of G satisfying the condition that $w \leq gw$ implies $w = gw$ and such that the spaces $d_n W/G$ are discrete. Let $d : W^{\mathrm{op}} \to \mathcal{S}(G)$ be an admissible function.

(i) Suppose that the map

$$h^*_H(*) \to h^*_H(\operatorname{hocolim}_{w\in\mathcal{C}_H(W_H^{\mathrm{op}}, d_H)} H/dw)$$

is an isomorphism whenever H is an isotropy group of X. Then the map

$$h^*_G(X) \to h^*_G(\operatorname{hocolim}_{w\in\mathcal{C}_G(W^{\mathrm{op}}, d)} G \times_{dw} X^w)$$

is an isomorphism and there is a spectral sequence

$$H^m(\mathcal{C}_G(W^{\mathrm{op}}, d), h^n_{dw}(X^w)) \Rightarrow h^{m+n}_G(X).$$

(ii) Suppose that the map

$$h^*_H(*) \to h^*_H(BW_H)$$

is an isomorphism whenever H is an isotropy group of X. Then the map

$$h^*_G(X) \to h^*_G(\operatorname{hocolim}_{[w.]\in \operatorname{sd} W/G} G \times_{G_{w.}} X^{w_n}) = h^*_G(\operatorname{hocolim}_{w.\in \operatorname{sd} W} X^{w_n})$$

is an isomorphism and there is a spectral sequence

$$H^m(\operatorname{sd} W/G, h^n_{G_{w.}}(X^{w_n})) \Rightarrow h^{m+n}_G(X).$$

(iii) Let G be a discrete group. Let K be a free G-CW-complex. Suppose that the map

$$K \times_H BW_H \to K/H$$

is a mod p homology isomorphism whenever H is an isotropy group of X. Then, similarly to 3.11, the map

$$\operatorname{hocolim}_{\mathcal{C}_G(W^{\mathrm{op}}, d)} K \times_{dw} X^w \to K \times_G X$$

is a mod p homology isomorphism.

4.9. Examples. Let $W = \mathcal{A}_p(G)$. Let X be a G-CW-complex such that all its isotropy groups are compact and contain non-trivial p-subgroups. Let K be an F_p-acyclic G-CW-complex.

(i) Let $d = d_c$. Then $\mathcal{C}_G(\mathcal{A}_p(G)^{\mathrm{op}}, d_c) = A_p(G)$. Suppose that, for every isotropy group H of the action of G on X, the map $h^*_H(*) \to h^*_H(\mathcal{E}\mathcal{O}_{\mathcal{Z}_p(H)})$ is an isomorphism. Then it follows from 3.6(iii) that the map

$$h^*_G(X) \to h^*_G(\operatorname{hocolim}_{E\in A_p(G)} G \times_{C_G E} X^E)$$

is an isomorphism and there is a spectral sequence

$$H^m(A_p(G), h^n_{C_G E}(X^E)) \Rightarrow h^{m+n}_G(X).$$

Let $h_G^* = H_G^*(K \times_G -, F_p)$. It follows from 4.4(iii) that if H is a compact subgroup of G and contains a non-trivial p-subgroup, then $h_H^*(*) = h_H^*(\mathcal{E}\mathcal{O}_{\mathcal{Z}_p(H)})$. Hence there is a mod p homology isomorphism

$$\mathrm{hocolim}_{w\in A_p(G)}\ K \times_{C_G w} X^w \to K \times_G X,$$

and there exists a spectral sequence

$$H^n(A_p(G), H^m(K \times_{C_G w} X^w, F_p)) \Rightarrow H^{n+m}(K \times_G X, F_p).$$

If $K = EG$, then we obtain the case investigated in [H1,2].

(ii) The map

$$\mathrm{hocolim}_{[E.]\in \mathrm{sd}\,\mathcal{A}_p(G)/G}\, K \times_{G_{E.}} X^{E_n} \to K \times_G X$$

is a mod p homology isomorphism and there is a spectral sequence

$$H^m(\mathrm{sd}\,\mathcal{A}_p(G)/G, H^n(K \times_{G_{E.}} X^{E_n}, F_p)) \Rightarrow H^{m+n}(K \times_G X, F_p).$$

In particular, if $\mathcal{A}$ is one of the classes $\mathcal{B}_k(F_p)$ or $\mathcal{D}(F_p)$ described in 1.3 and, for every $(E_0, \ldots, E_n) \in \mathrm{sd}\,\mathcal{A}_p(G)$, $K \times_{NE_0\cap\ldots\cap NE_n} X^{E_n} \in \mathcal{A}$, then $K \times_G X \in \mathcal{A}$.

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