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Almost-free $E(R)$-algebras and $E(A,R)$-modules

by

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Abstract. Let R be a unital commutative ring and A a unital R-algebra. We introduce the category of $E(A,R)$-modules which is a natural extension of the category of E-modules. The properties of $E(A,R)$-modules are studied; in particular we consider the subclass of $E(R)$-algebras. This subclass is of special interest since it coincides with the class of E-rings in the case $R=\mathbb{Z}$. Assuming diamond $\diamondsuit$, almost-free $E(R)$-algebras of cardinality κ are constructed for any regular non-weakly compact cardinal $\kappa > \aleph_0$ and suitable R. The set-theoretic hypothesis can be weakened.

1. Introduction. In 1958 Fuchs [F2, Problem 45] raised the problem to characterize those rings R for which $\mathrm{End}_{\mathbb{Z}}(R^+) \cong R$, where R^+ is the additive group of R. Introducing the class of E-rings Schultz [S] gave a partial solution. Recall that a ring R is an E-ring if the evaluation map $\varepsilon : \mathrm{End}_{\mathbb{Z}}(R^+) \to R$ given by $\varphi \mapsto \varphi(1)$ is a bijection. First examples are subrings of $\mathbb{Q}$ and pure subrings of the ring of p-adic integers. Schultz characterized E-rings of finite rank. The books by Feigelstock [Fe1], [Fe2] and the article [PV] survey the results obtained in the eighties (see also [Re], [F]). In a natural way the notion of E-rings extends to modules by calling a left R-module M an $E(R)$-module or just E-module if $\mathrm{Hom}_{\mathbb{Z}}(R,M) = \mathrm{Hom}_R(R,M)$ (see [BS]). It turned out that a unital ring R is an E-ring if and only if it is an E-module.

E-rings and E-modules have played an important role in the theory of torsion-free abelian groups of finite rank. For example Niedzwecki and Reid [NR] proved that a torsion-free abelian group G of finite rank is cyclic projective over its endomorphism ring if and only if $G = R \oplus A$, where R is an E-ring and A is an $E(R)$-module. Moreover, Casacuberta and Rodríguez

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[R], [CRT] noticed the role of E-rings in homotopy theory and further results on E-modules are in [DG2], [MV] and [P].

We want to consider these objects in a more general context of unital algebras A over commutative unital rings R with the same 1, different from 0, which we keep fixed throughout.

A left A-module M is called an $E(A,R)$-module if $\mathrm{Hom}_R(A,M) = \mathrm{Hom}_A(A,M)$. If $R = \mathbb{Z}$ then this category $E(A,R)$-Mod becomes $E(A)$-Mod. In particular, A is an $E(R)$-algebra if ${}_RA$ is in $E(A,R)$-Mod. This is equivalent to saying that the evaluation map $\varepsilon : \mathrm{End}_R(A) \to A$ given by $\varphi \mapsto \varphi(1)$ is an isomorphism (see Theorem 2.2). Therefore $E(R)$-algebras are natural generalizations of E-rings and we will extend results on E-rings to $E(R)$-algebras, e.g. any $E(R)$-algebra has to be commutative (see Theorem 3.3).

Often $E(R)$-algebras can be described by tensor products. This is the case for so-called $T(R)$-algebras which extend T-rings (see [Fe1, p. 85]). Recall that A is a $T(R)$-*algebra* if the multiplication map $m : A \otimes A \to A$ is bijective and note that any $T(R)$-algebra is an $E(R)$-algebra. The converse does not hold.

After having discussed the basic properties of $E(R)$-algebras and $E(A,R)$-modules and their relationship in Sections 2, 3 and 4 it is clear that large $E(R)$-algebras are far from being free as R-modules. Therefore it is natural to ask whether there exist $E(R)$-algebras A which are almost-free, i.e. for which every R-submodule of cardinality $< |A|$ can be embedded into a free R-submodule of A. A first step was already done by Dugas, Mader and Vinsonhaler. They proved in [DMV] that any torsion-free p-reduced p-cotorsion-free commutative ring S may be embedded into an E-ring of cardinality λ whenever λ is any cardinal such that $\lambda^{|S|} = \lambda$. It can be easily seen (see [St]) that the constructed E-rings are $\aleph_1$-free provided S is $\aleph_1$-free. Thus, assuming the continuum hypothesis, we derive the existence of $\aleph_1$-free E-rings of cardinality $\aleph_1$.

In general, we show that, assuming the diamond axiom, for any reduced countable domain R which is not a field and for any regular non-weakly compact cardinal $\kappa > \aleph_0$ there exist 2^κ non-isomorphic almost-free $E(R)$-algebras A of cardinality κ (see Theorem 5.6). Moreover, it is shown that any free R-module can be embedded into an $E(A,R)$-module M of arbitrary large cardinality which is almost-free in the sense that any R-submodule $U \subseteq M$ with $|U| < |M|$ is a submodule of a free R-module $F \subseteq M$ (see Theorem 5.9). This proves that even almost-free $E(A,R)$-modules are quite complex and do not constitute a set, a result parallel to E-modules from [D].

2. The category of $E(A,R)$-modules. In this section we study the category of $E(A,R)$-modules as a natural extension of the category of E-

modules which has been studied extensively in [D] and [MV]. If A is an R-algebra then $\mathrm{Hom}_R(A,M)$ admits an A-module structure for any A-module M. This leads to the following definition:

DEFINITION 2.1. A left A-module M is called an *$E(A,R)$-module* if $\mathrm{Hom}_R(A,M)=\mathrm{Hom}_A(A,M)$.

Recall that an abelian group G is *p-local* for some prime p if $G\otimes\mathbb{Q}_{(p)}=G$ and $\mathbb{Q}_{(p)}=\{z/q \mid q,z\in\mathbb{Z},(q,p)=1\}$. Hence, by Definition 2.1 any p-local abelian group is an example of an $E(\mathbb{Q}_{(p)},\mathbb{Z})$-module. We want to show the abundance of almost-free $E(A,R)$-modules, in particular we will see that they form a proper class. Following [P], [MV] and [S] we first extend basic properties from E-modules to $E(A,R)$-modules.

THEOREM 2.2. *For a left A-module M the following statements are equivalent*:

(i) *M is an $E(A,R)$-module.*

(ii) *The evaluation map $\varepsilon:\mathrm{Hom}_R(A,M)\to M$ via $\varphi\mapsto\varphi(1)$ is a bijection.*

(iii) *For all $\varphi\in\mathrm{Hom}_R(A,M)$, $\varphi=0$ if and only if $\varphi(1)=0$.*

(iv) $\mathrm{Hom}_R(A/R1,M)=0$.

Proof. First we prove the equivalence of (i) and (ii). If M is an $E(A,R)$-module, then each R-homomorphism from A to M is uniquely determined by the image of 1. Hence the evaluation map ε is a bijection. Conversely, if ε is a bijection and $\varphi\in\mathrm{Hom}_R(A,M)$, then choose any $a\in A$ and define two R-homomorphisms φ_1,φ_2 from A to M by

$$\varphi_1(x)=x\varphi(a)\quad\text{and}\quad\varphi_2(x)=\varphi(xa)$$

for all $x\in A$. Hence $\varphi_1(1)=\varphi_2(1)$ and $\varphi_1=\varphi_2$. Since a was chosen arbitrary we obtain $\varphi(xa)=x\varphi(a)$ for all $a,x\in A$ and thus φ is A-linear and M is an $E(A,R)$-module.

The equivalence of (i) and (iii) is easy to check and left to the reader.

It remains to show the equivalence of (i) and (iv). The exact sequence $0\to R1\to A\to A/R1\to 0$ induces the sequence

$$(*)\qquad 0\to\mathrm{Hom}_R(A/R1,M)\to\mathrm{Hom}_R(A,M)\to\mathrm{Hom}_R(R1,M)\to 0,$$

which is exact by (i) and (ii). The equivalence of (i) and (iv) is now clear. ∎

Theorem 2.2 is an easy test for being an $E(A,R)$-module. Additionally, it follows that the full subcategory of A-modules formed by $E(A,R)$-modules is closed under taking A-submodules. The next lemma shows that this category is also closed under taking direct sums and extensions.

LEMMA 2.3. *The full subcategory of A-modules formed by the $E(A,R)$-modules is closed under submodules, direct summands, arbitrary direct sums and extensions.*

Proof. By Theorem 2.2 and an easy projection argument it is easily seen that the $E(A,R)$-modules are closed under submodules, direct summands and arbitrary direct sums. Moreover, if $0 \to B \to C \to D \to 0$ is an exact sequence, where B and D are $E(A,R)$-modules, then it is almost obvious to see that also C is an $E(A,R)$-module by using the evaluation map or applying the five-lemma (see [F1, Lemma 2.3] or [CE]). ∎

By Lemma 2.3 the category of $E(A,R)$-modules is also closed under A-isomorphism but in general it is not closed under taking quotients as the following example shows.

EXAMPLE 2.4. Let $R = \mathbb{Z}[x]$ and choose any homomorphism $\varphi : R \to R$ which is not $\mathbb{Z}[x]$-linear. By Theorem 5.5 there exists an $E(R)$-algebra A containing R and we let D be the completion of $\mathbb{Z}[x] \otimes A$. Take any A-free resolution

$$0 \to B \to C \to D \to 0$$

of D. Then B and C are $E(A,R)$-modules by Lemma 4.1. However D is a pure injective abelian group and R is pure in A, hence φ lifts to $\widehat{\varphi} : A \to D$ which is not A-linear by choice of φ. Hence D is not an $E(A,R)$-module.

To get further insight into the category of $E(A,R)$-modules we first have to consider a proper subclass of the $E(A,R)$-modules.

3. The class of $E(R)$-algebras. The notion of E-ring (see [S] or [BS]) extends naturally to $E(R)$-algebras.

DEFINITION 3.1. An R-algebra A is called an $E(R)$-*algebra* if

$$\mathrm{End}_R(A) = \mathrm{End}_A(A) \cong A.$$

Note that an R-algebra A is an $E(R)$-algebra if and only if A is an $E(A,R)$-module, and $E(\mathbb{Z})$-algebras are E-rings. But obviously an $E(R)$-algebra need not be an E-ring. This is illustrated by

EXAMPLE 3.2. *The quotient field Q of the p-adic integers J_p for some prime p satisfies*

$$\mathrm{End}_{J_p}(Q) \cong Q \not\cong \mathrm{End}_{\mathbb{Z}}(Q)$$

and $|\mathrm{End}_{\mathbb{Z}}(Q)| > |Q|$. Hence Q is an example of an $E(J_p)$-algebra which is not an E-ring.

Our first result is a natural generalization from E-rings to $E(R)$-algebras (see also [R] or [CRT]).

THEOREM 3.3. *For an R-algebra A the following are equivalent*:

(i) *A is an $E(R)$-algebra.*

(ii) *The evaluation map $\varepsilon : \mathrm{End}_R(A) \to A$ $(\varphi \mapsto \varphi(1))$ is an R-algebra isomorphism.*

(iii) *The R-algebra* $\mathrm{End}_R(A)$ *is commutative.*

(iv) *The multiplication map* $\mu : A \to \mathrm{End}_R(A)$ *given by* $\mu(a)(x) = ax$ *is a bijection.*

Moreover, any $E(R)$-algebra is commutative.

Proof. The equivalence of (i) and (ii) follows easily from Theorem 2.2. Moreover, (i) and (iv) are equivalent since μ is a right inverse of the evaluation map ε. To prove that (i) and (iii) are equivalent we first show the last claim, i.e. any $E(R)$-algebra A is commutative. If $a \in A$, then we define two R-endomorphisms φ_1, φ_2 of A by

$$\varphi_1(x) = xa \quad \text{and} \quad \varphi_2(x) = ax$$

for each $x \in A$. Hence $\varphi_1(1) = \varphi_2(1)$ and $\varphi_1 = \varphi_2$ by (ii), which implies the commutativity of A. We are now able to prove the equivalence of (i) and (iii). By the above any $E(R)$-algebra is commutative and by (ii) the evaluation map ε is an R-algebra isomorphism. Hence $\mathrm{End}_R(A)$ is commutative. Conversely, let $\mathrm{End}_R(A)$ be commutative and define $m_a \in \mathrm{End}_R(A)$ for any $a \in A$ by $m_a(x) = xa$. We have to show that the evaluation map ε is a bijection and it is enough to show injectivity. If $\varepsilon(\psi_1) = \varepsilon(\psi_2)$, then $\psi_1(1) = \psi_2(1)$ and for any $x \in A$ we have

$$\psi_1(x) = (\psi_1 \circ m_x)(1) = (m_x \circ \psi_1)(1) = (m_x \circ \psi_2)(1) = (\psi_2 \circ m_x)(1) = \psi_2(x),$$

hence $\psi_1 = \psi_2$ and ε is injective. ■

It is important to know that R-summands of an $E(R)$-algebra are also A-summands. For this we state

Corollary 3.4. *Let A be an $E(R)$-algebra.*

(i) *If φ is an R-endomorphism of A, then $\varphi(A)$ is a principal ideal in A;*

(ii) *Any direct sum decomposition of A as an R-module is a decomposition as an A-module.*

(iii) *Let S be an R-algebra. If $A \cong S$ as R-modules, then S is an $E(R)$-algebra.*

Proof. All facts are easily checked by standard arguments. ■

Examples of $E(R)$-algebras follow more easily from the following

Remark 3.5. It is easy to see that if the multiplication map of A is surjective then A is an $E(R)$-algebra. These algebras are called *$T(R)$-algebras.*

Note that the (divisible) Prüfer group C_{p^∞} can be expressed as a quotient of two E-rings $\mathbb{Q}^{(p)}/\mathbb{Z}$, but $\mathrm{End}_{\mathbb{Z}}(C_{p^\infty}) = J_p$, hence C_{p^∞} is not an E-ring. This shows that the class of E-rings (in particular of $E(R)$-algebras) is not closed under taking quotients. However, the class of $T(R)$-algebras is closed under taking quotients (see [Fe1, Observation 4.7.27]). Moreover, the p-adic integers J_p form an E-ring but not a T-ring. Nevertheless, the classes coincide if we restrict to torsion rings (see [Fe1, Theorem 4.7.25]). It is still open whether $\mathrm{End}_R(A) \cong A$ implies that A is an $E(R)$-algebra. Partial results were obtained e.g. in [GS].

4. Connecting the E-structure of algebras and modules. From Definition 3.1 it follows that any algebra which is an $E(A,R)$-module is an $E(R)$-algebra as well. We want to strengthen this implication and establish some converse. From this point of view we consider first A-modules over an $E(R)$-algebra A. Since any projective A-module is a summand of a free A-module (see [EM, Lemma 2.3]), we may apply Theorem 2.2 to obtain the following

LEMMA 4.1. *Let A be an $E(R)$-algebra and M a projective left A-module. Then M is an $E(A,R)$-module.*

This result can be applied to almost-free A-modules.

DEFINITION 4.2. Let M be any R'-module over some ring R'. If κ is any cardinal, then M is called *κ-free* if every submodule $N \subseteq M$ of cardinality $|N| < \kappa$ can be embedded into a free submodule of N. In particular, M is called *almost-free* if M is $|M|$-free.

THEOREM 4.3. *Let A be an $E(R)$-algebra of cardinality κ. Any κ^+-free left A-module is also an $E(A,R)$-module.*

Proof. The proof is easy and left to the reader. ∎

Next we will show that it is no restriction to assume for an $E(A,R)$-module that the underlying algebra is already an $E(R)$-algebra. Therefore let $\mathfrak{J}(A)$ be the set of all two-sided ideals I of A such that A/I is an $E(A,R)$-module.

LEMMA 4.4. *Let M be an $E(A,R)$-module. Then the annihilator $\mathrm{Ann}_A(M)$ is an element of $\mathfrak{J}(A)$.*

Proof. If $S := A/\mathrm{Ann}_A(M)$ then choose $\varphi \in \mathrm{Hom}_R(A,S)$. We will show that φ is A-linear. Fix any $m \in M$ and for any $s \in S$ let $a_s \in A$ be such that $s = [a_s] = a_s + \mathrm{Ann}_A(M) \in S$. We define

$$\varphi_m : A \to M, \qquad a \mapsto m a_{\varphi(a)}.$$

First we show that $\varphi_m \in \mathrm{Hom}_R(A,M)$ is well defined. If $s = [a_s] = [\widetilde{a}_s]$ then $a_s - \widetilde{a}_s \in \mathrm{Ann}_A(M)$ and hence $m a_s = m\widetilde{a}_s$. Thus the definition of φ_m is independent of the choice of the representative $a_{\varphi(a)}$ and therefore φ_m is well defined. Obviously, φ_m is an R-homomorphism. Hence φ_m is A-linear by assumption for all $m \in M$. We obtain

$$m a_{\varphi(a\widetilde{a})} = \varphi_m(a\widetilde{a}) = \varphi_m(a)\widetilde{a} = m a_{\varphi(a)}\widetilde{a}$$

and thus $m(a_{\varphi(a\widetilde{a})} - a_{\varphi(a)}\widetilde{a}) = 0$ for all $m \in M$, hence

$$(a_{\varphi(a\widetilde{a})} - a_{\varphi(a)}\widetilde{a}) \in \mathrm{Ann}_A(M)$$

for all $a, \widetilde{a} \in A$. Therefore

$$\varphi(a\widetilde{a}) = [a_{\varphi(a\widetilde{a})}] = [a_{\varphi(a)}\widetilde{a})] = [a_{\varphi(a)}]\widetilde{a} = \varphi(a)\widetilde{a},$$

which proves that φ is A-linear and hence S is an $E(A,R)$-module. ∎

LEMMA 4.5. *If $I \in \mathfrak{J}(A)$ and M is an $E(A,R)$-module as well as an A/I-module, then M is an $E(A/I,R)$-module.*

Proof. If $I \in \mathfrak{J}(A)$ and $\pi : A \to A/I$ is the canonical homomorphism, then π induces a surjection $A/R1 \to (A/I)/R(1+I)$. Let $A_0 = A/R1$ and $I_0 = (A/I)/R(1+I)$. Now, if M is an $E(A,R)$-module we obtain, by Theorem 2.2, the sequence

$$0 \to \mathrm{Hom}_R(I_0, M) \to \mathrm{Hom}_R(A_0, M) = 0.$$

Thus $\mathrm{Hom}_R(I_0, M) = 0$ and M is an $E(A/I,R)$-module by Theorem 2.2. ∎

COROLLARY 4.6. *If $I \in \mathfrak{J}(A)$, then A/I is an $E(R)$-algebra.*

Proof. Follows from Lemma 4.5. ∎

COROLLARY 4.7. *If M is an $E(A,R)$-module then $A/\mathrm{Ann}_A(M)$ is an $E(R)$-algebra.*

Proof. Follows from Lemma 4.4 and Corollary 4.6. ∎

Note that by Corollary 4.7 the algebra A is an $E(R)$-algebra if a faithful $E(A,R)$-module exists. Moreover, in view of Lemma 4.5 and Corollary 4.7 any $E(A,R)$-module can be considered as an $E(A/\mathrm{Ann}_A(M),R)$-module over the $E(R)$-algebra $A/\mathrm{Ann}_A(M)$. So by a change of the algebra argument we may assume that A is an $E(R)$-algebra.

5. Almost-free $E(R)$-algebras and $E(A,R)$-modules. In the previous sections we have seen that $E(R)$-algebras must have commutative endomorphism ring, which shows non-freeness in a strong sense. Hence it is interesting to find almost-free $E(R)$-algebras. This question cannot be decided in ZFC as there are models of ZFC and Martin's axiom in which $\aleph_2$-free modules of cardinality $\aleph_2$ are free (see [GS, Theorem 5.1]). Assuming the continuum hypothesis the existence of $\aleph_1$-free E-rings of cardinality $\aleph_1$ follows immediately from [DMV] and [St, Theorem 3.3]. Next we want to find almost-free $E(R)$-algebras of larger cardinality under a suitable set-theoretic assumption. As in [DG1] we want to apply a weak version of the diamond principle which will be explained first. For standard notations we refer to [EM]. Recall that a subset $S \subset \kappa$ is *sparse* if $S \cap \alpha$ is not stationary in α for all limit ordinals $\alpha < \kappa$. A *κ-filtration* of a set A of cardinality κ is a set $\{A_\alpha \mid \alpha < \kappa\}$ of subsets of A such that $A = \bigcup_{\alpha<\kappa} A_\alpha$ and

(i) $\{A_\alpha \mid \alpha < \kappa\}$ is a smooth chain, i.e. $A_\lambda = \bigcup_{\nu<\lambda} A_\nu$ for all limit ordinals $\lambda < \kappa$;

(ii) $|A_\alpha| < \kappa$ for all $\alpha < \kappa$.

Now let $E \subseteq \kappa$ and $\{A_\alpha \mid \alpha < \kappa\}$ be a κ-filtration of A. Then we consider two prediction principles.

$\diamondsuit_\kappa(E)$ (*diamond*): there is a family $\{S_\alpha \mid \alpha < \kappa\}$ such that $S_\alpha \subseteq A_\alpha$ and, for all $X \subseteq A$, the set $\{\alpha \in E \mid X \cap A_\alpha = S_\alpha\}$ is stationary in κ.

$\Phi_\kappa(E)$ (*weak-diamond*): If $P_\alpha : \mathcal{P}(A_\alpha) \to \{0,1\}$ $(\alpha \in E)$ is a partition, then there is a function $\varphi : E \to \{0,1\}$ such that, for all $X \subseteq A$, the set $\{\alpha \in E \mid P_\alpha(X \cap A_\alpha) = \varphi(\alpha)\}$ is stationary in κ.

Sets E which satisfy $\Phi_\kappa(E)$ are called *non-small* and in particular $\Phi_{\aleph_1}(\aleph_1)$ is equivalent to $2^{\aleph_0} < 2^{\aleph_1}$ (see Devlin and Shelah [DS]). Also recall from Jensen [J] that $\diamondsuit_\kappa(E)$ holds in $V = L$ for all non-weakly compact cardinals κ and all stationary sets E. We combine these results with some from [Sh] and define

$\triangledown_\kappa(S)$ (*half diamond*): S is non-small and sparse if $\kappa > \aleph_1$ and $\mathrm{cf}(\lambda) = \omega$ for any $\lambda \in S$.

Moreover, $\triangledown_\kappa$ will mean that there exists a subset $S \subseteq \kappa$ such that $\triangledown_\kappa(S)$ holds. Hence we summarize the results on $\triangledown_\kappa$ as follows (see also [DG1]).

LEMMA 5.1. *The following hold*:

(i) $(ZFC + V = L)$ $\triangledown_\kappa$ *holds for all uncountable regular non-weakly compact cardinals* $\kappa > \aleph_0$.

(ii) $(ZFC + 2^{\aleph_0} < 2^{\aleph_1})$ $\triangledown_{\aleph_1}$ *holds.*

(iii) $(ZFC + \triangledown_\kappa)$ *There are* κ *disjoint subsets* S_β $(\beta < \kappa)$ *such that* $\triangledown_\kappa(S_\beta)$ *holds for all* $\beta < \kappa$ *and* $\bigcup_{\beta<\kappa} S_\beta$ *is sparse in* κ.

By Lemma 5.1 the construction of almost-free $E(R)$-algebras reduces to a Step Lemma which we will prove next. It is based on the S-topology of a free R-module. For the rest of this paper we restrict ourselves to countable torsion-free domains which are not fields. They are cotorsion-free as explained shortly. Let S be a countable multiplicatively closed subset S of R such that $1 \in S$. An R-module M is *reduced* if $\bigcap_{s\in S} Ms = 0$, and M is *torsion-free* if $ms = 0$ implies $m = 0$ for $m \in M$ and $0 \neq s \in S$. We assume that R_R is reduced and torsion-free, hence S induces a Hausdorff S-topology on M by enumerating $S = \{s_n \mid n \in \omega\}$ and putting

$$q_0 = 1 \quad \text{and} \quad q_{n+1} = q_n s_n \quad \text{for all } n \in \omega.$$

The system $q_n M$ $(n \in \omega)$ generates the S-topology on M and M is naturally a submodule of its S-adic completion $\widehat{M}$. Recall that an R-module M is *cotorsion-free* if $\mathrm{Hom}(\widehat{R}, M) = 0$ (cf. [EM, p. 134]). A submodule U of M is *S-pure* if $U \cap sM = sU$ for all $s \in S$, hence the S-topology on M induces the S-topology on U. If U is any submodule of M, then U_* denotes the smallest (in this case unique) pure submodule of M containing U. Similarly U is *S-divisible* if $sU = U$ for all $s \in S$. In Section 2 we discussed almost-free modules. However, we will use a stronger version of almost freeness and say that an R-algebra is *polynomial-almost-free* if all its subalgebras of smaller cardinality are contained in a polynomial ring over R. Note that polynomial-almost-free implies almost-free. The following is the first step of the final Step Lemma.

LEMMA 5.2. *Assume that R is a countable torsion-free domain which is not a field. Let $F = R[X]$ be the polynomial ring over R in a countable set $X = \{x_i \mid i \in \omega\}$ of commuting variables, and let $b \in {}_R F$ be a basic element. Moreover, let $X_n = \{x_0, \ldots, x_n\}$ and $F_n := R[X_n] \subseteq F$ canonically. Then there exist two ring extensions F^ε of F with the following properties for $\varepsilon = 0, 1$:*

(i) *$F \subset F^\varepsilon$ and F^ε/F is S-divisible.*

(ii) *F^ε is a polynomial ring over the ring F_n for each $n \in \omega$.*

(iii) *F^ε is a polynomial ring over R.*

(iv) *If $\varphi \in \mathrm{End}_R(F)$ extends to both $\varphi^\varepsilon \in \mathrm{End}_R(F^\varepsilon)$ for $\varepsilon = 0, 1$, then $\varphi(b) = \varphi(1)b$.*

Proof. By topology any element $x \in \widehat{F}$ has a unique representation

$$x = \sum_{m \in T} s_m m,$$

where T is a countable set of monomials in X and $s_m \in \widehat{R}$ are such that, for all $n \in \omega$, $s_m \in q_n \widehat{R}$ for almost all $M \in T$. The *support* $[x]$ of x is defined to be

$$[x] = \{m \in T \mid s_m \neq 0\}.$$

Note that $x = 0$ if and only if $[x] = \emptyset$. If x_n is some variable and $x \in m$, then we write $x_n \in_* [x]$ if there is a monomial $m \in [x]$ such that x_n divides m. If there is no such monomial in $[x]$ we write $x_n \notin_* [x]$. Furthermore, if we restrict some equation to a monomial that is divisible by x_n, then we say for short that we restrict to x_n. By [GM] we can find an S-adic integer $\pi \in \widehat{R}$ which is algebraically independent over R. Since $F = R[X]$ is a free R-module we see that π is also algebraically independent over F, i.e.

$$\text{whenever} \quad \sum_{i=0}^{n} f_i \pi^i = 0 \text{ where } f_i \in F \quad \text{then} \quad f_i = 0 \text{ for all } i \leq n.$$

Now let n_0 be the least integer such that $x_{n_0+n} \notin_* [b]$. We define a “branch” element

$$e := \sum_{k \in \omega} q_k x_{n_0+k} \tag{1}$$

in the S-adic completion $\widehat{F}$ of F. Obviously $[e] \cap [b] = \emptyset$. We want to show that the following two pure subrings of $\widehat{F}$ satisfy our claims (here $*$ denotes purification):

$$F^0 := F[e]_* \subseteq \widehat{F} \quad \text{and} \quad F^1 := F[e + \pi b + \pi^2 1]_* \subseteq \widehat{F}.$$

First we prove (iv). If $\varphi \in \mathrm{End}_R(F)$ extends to both $\varphi^\varepsilon \in \mathrm{End}_R(F^\varepsilon)$ for $\varepsilon = 0, 1$, then we have representations

$$q_k \varphi^0(e) = \sum_{i=0}^{n} f_i e^i, \tag{2}$$

$$q_l \varphi^1(e + \pi b + \pi^2 1) = \sum_{i=0}^{m} g_i (e + \pi b + \pi^2 1)^i \tag{3}$$

for some $k, l, m, n \in \mathbb{Z}$. Absorbing multiples we may assume $k = l$. Subtracting (2) and (3) we get

$$q_k \varphi^1(\pi b + \pi^2 1) = \sum_{i=0}^{m} g_i (e + \pi b + \pi^2 1)^i - \sum_{i=0}^{n} f_i e^i. \tag{4}$$

If $T := \bigcup_{i=0}^{n} [f_i] \cup \bigcup_{i=0}^{m} [g_i]$, then T is finite and hence we can choose $x_j \in_* [e]$ such that $x_j^l \notin_* T$ and $x_j^l \notin_* [b] \cup [\varphi^1(\pi b + \pi^2 1)]$ for all $l \in \omega$.

If $n > m$ then $x_j^n \in_* [e^n]$ but it does not appear in the support of any other element in (4) by the choice of x_j. Restricting to x_j^n shows $f_n = 0$. If $m > n$ we argue similarly and $n = m$ follows. It is easy to see that in this case $f_n = g_n$ and restricting to x_j^{n-1} shows

$$g_n(\pi b + \pi^2 1) + g_{n-1} - f_{n-1} = 0 \quad \text{if } n > 1. \tag{5}$$

By algebraic independence of π over F we obtain $g_{n-1} = f_{n-1}$and $f_n = g_n$ $= 0$. Inductively $f_i = g_i = 0$ for all $i > 1$ and $f_1 = g_1$. Hence (4) reduces to

$$q_k \varphi^1(\pi b + \pi^2 1) = g_1(e + \pi b + \pi^2 1) + g_0 - g_1 e - f_0.$$

Since φ^1 viewed as a homomorphism from $\widehat{F}$ to $\widehat{F}$ is $\widehat{R}$-linear we get

$$\pi q_k \varphi^1(b) + \pi^2 q_k \varphi^1(1) = g_1 \pi b + g_1 \pi^2 + g_0 - f_0.$$

Using algebraic independence of π over F also $g_0 = f_0$, $g_1 = q_k \varphi^1(1) = q_k \varphi(1)$, and $q_k \varphi^1(b) = q_k \varphi(b) = g_1 b$. Therefore $\varphi(b) = \varphi(1) b$ by the torsion-freeness of $\widehat{F}$ and thus (iv) holds.

Next we show (ii) and (iii). By definition

$$e = \sum_{n \in \omega} q_n x_{n_0+n} \quad \text{and} \quad a := e + h = \sum_{n \in \omega} q_n x_{n_0+n} + h$$

where $h = \pi b + \pi^2 1$. Write $h = \sum_{i\in\omega} q_i \widetilde{h}_i$ as an S-adic limit. Then let $e_k := \sum_{i\geq k}(q_i/q_k)x_{n_0+i}$ and $a_k := e_k + (1/q_k)h_k$ where $h_k := \sum_{i\geq k}\widetilde{h}_i$ is the q_k-divisible part of h. It follows that

$$q_k e_k + \sum_{n<k} q_n x_{n_0+n} = e, \tag{6}$$

$$q_k a_k + \sum_{n<k} q_n x_{n_0+n} + (h - h_k) = a. \tag{7}$$

Note that $h - h_k \in F$ for all $k \in \omega$ and hence it is easy to check that $F[e]_* = \bigcup_{k\in\omega} F[e_k]$ and $F[a]_* = \bigcup_{k\in\omega} F[a_k]$.

We claim that

$$F[e]_* = \bigcup_{k\in\omega} F[e_k] = R[x_0, \ldots, x_{n_0-1}, e_0, e_1, \ldots], \tag{8}$$

$$F[a]_* = \bigcup_{k\in\omega} F[a_k] = R[x_0, \ldots, x_{n_0-1}, a_0, a_1, \ldots]. \tag{9}$$

By (6), (7) and $h - h_k \in F$ it is clear that $R[x_0, \ldots, x_{n_0-1}, e_0, e_1, \ldots] \subseteq F[e]_*$ and also $R[x_0, \ldots, x_{n_0-1}, a_0, a_1, \ldots] \subseteq F[a]_*$. To prove the converse it remains to show that $F \subset R[x_0, \ldots, x_{n_0-1}, e_0, e_1, \ldots]$ and $F \subset R[x_0, \ldots, x_{n_0-1}, a_0, a_1, \ldots.]$. By easy calculations

$$x_{n_0+n} = e_n - s_n e_{n+1} \in R[x_0, \ldots, x_{n_0-1}, e_0, e_1, \ldots]$$

for all $n \in \omega$ and thus $F \subseteq R[x_0, \ldots x_{n_0-1}, e_0, e_1, \ldots]$. Hence (8) holds. Similarly,

$$a_n - s_n a_{n+1} = x_{n_0+n} + \frac{1}{q_n}(h_n - h_{n+1}).$$

As $h_n - h_{n+1} \in F$ and $[h_n - h_{n+1}] \subseteq X_{n_0}$, we now obtain $x_{n_0+n} \in R[x_0, \ldots \ldots, x_{n_0-1}, a_0, a_1, \ldots]$, which implies $F \subseteq R[x_0, \ldots, x_{n_0-1}, a_0, a_1, \ldots]$ and thus (9) holds.

It remains to show that $R[x_0, \ldots, x_{n_0-1}, e_0, e_1, \ldots]$ and $R[x_0, \ldots, x_{n_0-1}, a_0, a_1, \ldots]$ are polynomial rings. Assume that

$$\sum_{i=0}^{n} r_i \mu_i = 0 \tag{$*$}$$

where $\mu_i \neq \mu_j$ $(i \neq j)$ are monomials in the variables $\{x_0, \ldots, x_{n_0-1}, e_0, e_1, \ldots\}$ and $r_i \in R$ such that each $r_i \neq 0$. Write μ_i as

$$\mu_i = \prod_{j=0}^{n_0-1} x_j^{n_{j,i}} \prod_{j\in\omega} e_j^{m_{j,i}},$$

where $m_{j,i} = 0$ for almost all j. If $\prod_{j\in\omega} e_j^{m_{j,i}} = 1$ for all $i \leq n$, then $(*)$ is a non-trivial linear combination in $R[X_{n_0-1}]$—a contradiction. Therefore assume the existence of an i such that $\prod_{j\in\omega} e_j^{m_{j,i}} \neq 1$. Let j_0 be the least integer such that $m_{j_0,i} \neq 0$ for at least one $i \leq n$. Then a power of $x_{n_0+j_0}$ does not appear in those $[e_j^{m_{j,i}}]$ with $j \neq j_0$ but it appears in $[e_{j_0}^{m_{j_0,i}}]$ for all i with $m_{j_0,i} \neq 0$. If there exists a unique i_0 such that

$$m_{j_0,i_0} = \max\{m_{j_0,i} \mid i \leq n\} =: \max_0,$$

then restricting $(*)$ to $\prod_{j=0}^{n_0-1} x_j^{n_{j,i_0}} x_{j_0}^{m_{j_0,i_0}} t$, where t is any element in the support of $\prod_{j\neq j_0} e_j^{m_{j,i_0}}$, forces $r_{i_0} = 0$—a contradiction.

Suppose $I := \{i \leq n \mid m_{j_0,i} = \max_0\}$ is a set of at least two elements. Then choose the least integer $j_1 > j_0$ such that $m_{j_1,i} \neq 0$ for some $i \in I$. If there is a unique $i_1 \in I$ such that

$$m_{j_1,i_1} = \max\{m_{j_1,i} \mid i \in I\} =: \max_1,$$

then we restrict $(*)$ to $\prod_{j=0}^{n_0-1} x_j^{n_{j,i_1}} x_{j_0}^{m_{j_0,i_1}} x_{j_1}^{m_{j_1,i_1}} t$, where t is any element in the support of $\prod_{j\neq j_0, j\neq j_1} e_j^{m_{j_1,i_1}}$, leading to $r_{i_1} = 0$, again a contradiction.

If i_1 is not unique we repeat the above process and since all μ_i are different we always end up with a contradiction. Therefore $R[x_0, \ldots, x_{n_0-1}, e_0, e_1, \ldots]$ is a polynomial ring and similarly $R[x_0, \ldots, x_{n_0-1}, a_0, a_1, \ldots]$.

By the same arguments as above, we see that

$$F[e]_* = F_n[X_{n_0} \setminus X_n, e_i : i \geq m_n],$$
$$F[a]_* = F_n[X_{n_0} \setminus X_n, a_i : i \geq m_n],$$

where $m_n := \max\{0, n - n_0\}$ and thus (ii) holds.

By (6) and (7),

$$e \equiv q_k e_k \quad \text{and} \quad a \equiv q_k a_k \text{ modulo } F$$

for all $k \in \omega$. Thus $F[e]_*/F$ and $F[a]_*/F$ are S-divisible, hence (i) holds. ∎

We bring Lemma 5.2 into a form suitable for immediate application. Here the *rank* of a countable torsion-free domain is the rank of its additive group.

Reduction Lemma 5.3. *Assume that R is a countable torsion-free domain which is not a field. Let $F = R[V]$ be a polynomial ring over R of rank $\kappa \geq |R|$ and V be a set of commuting variables. Furthermore, let $\varphi \in \mathrm{End}_R(F) \setminus F$, i.e. $\varphi - \varphi(1)\,\mathrm{id}_F \neq 0$. Then there exists a subring G of F with the following properties*:

(i) *G is a polynomial ring over R.*
(ii) *F is a polynomial ring over G.*
(iii) *The rank of G is less than or equal to $|R|$.*
(iv) $\varphi\restriction G \in \mathrm{End}_R(G) \setminus G$.

Proof. Let H be a subring of F such that H is a polynomial ring over R and F is a polynomial ring over H. We define the *φ-closure* of H as follows: Let $H_0 := H$ and denote by H_1 the ring $\mathrm{pol}(H_0\varphi)$ which is the smallest polynomial ring T over R containing $H_0\varphi$ such that F is a polynomial ring over T. Inductively, we define $H_{i+1} := \mathrm{pol}(H_i\varphi)$. Moreover, we can write each H_i as a polynomial ring over R in the form $H_i = R[V_i]$ where V_i is a subset of V. Let $I_0 := V_0$ and $I_{i+1} := I_i \cup V_{i+1}$. Then the φ-closure of H is the polynomial ring

$$H^{\mathrm{cl}(\varphi)} := R[I], \quad \text{where} \quad I := \bigcup_{i\in\omega} I_i.$$

Clearly $H^{\mathrm{cl}(\varphi)}$ is a polynomial ring over R and F is a polynomial ring over $H^{\mathrm{cl}(\varphi)}$ in the variables $V \setminus I$ which contains H. Moreover, $H^{\mathrm{cl}(\varphi)}$ is invariant under φ and hence $\varphi\restriction H^{\mathrm{cl}(\varphi)} \in \mathrm{End}_R(H^{\mathrm{cl}(\varphi)})$. If the lemma does not hold and G_0 is any polynomial ring over R such that $\mathrm{rk}(G_0) \le |R|$ and F is a polynomial ring over G_0, then let G_0^{c} be the φ-closure of G_0; hence (i)–(iii) hold for G_0^{c} and $\varphi\restriction G_0^{\mathrm{c}} \in \mathrm{End}_R(G_0^{\mathrm{c}})$. By assumption, $\varphi\restriction G_0^{\mathrm{c}} = g$ for some $g \in G_0^{\mathrm{c}}$. But, since $\varphi \in \mathrm{End}_R(F) \setminus F$, there exists an element $f \in F$ such that $(\varphi - g)(f) \neq 0$. Let $G_1 = \mathrm{pol}(\langle G_0, f\rangle)$ and $G_1^{\mathrm{c}} = G_1^{\mathrm{cl}(\varphi)}$, a summand of F which is again a polynomial ring over R such that $\mathrm{rk}(G_1^{\mathrm{c}}) \le |R|$ and $\varphi\restriction G_1^{\mathrm{c}} \in \mathrm{End}(G_1^{\mathrm{c}})$. By the same arguments $\varphi\restriction G_1^{\mathrm{c}} \in G_1^{\mathrm{c}}$ and, since $G_0^{\mathrm{c}} \subset G_1^{\mathrm{c}}$, we conclude $\varphi\restriction G_1^{\mathrm{c}} = g$. But then $(\varphi - g)(f) = 0$—a contradiction. ∎

We combine the Reduction Lemma 5.3 and Lemma 5.2 to get the desired

Step Lemma 5.4. *Assume that R is a countable torsion-free domain which is not a field. Let $F = \bigcup_{n\in\omega} F_n$ be the union of a chain of polynomial rings F_n over R of rank $\kappa > \aleph_0$ such that F is a polynomial ring over R and each F_{n+1} is a polynomial ring over F_n. If $b \in {}_R F$ is a basic element, then there exist two ring extensions F^ε of F with the following properties for $\varepsilon = 0, 1$:*

(i) *$F \subseteq F^\varepsilon$ and F^ε/F is S-divisible.*
(ii) *F^ε is a polynomial ring over the ring F_n for each $n \in \omega$.*
(iii) *F^ε is a polynomial ring over R.*
(iv) *If $\varphi \in \mathrm{End}_R(F)$ extends to both $\varphi \in \mathrm{End}_R(F^\varepsilon)$ for $\varepsilon = 0, 1$, then $\varphi(b) = \varphi(1)b$.*

5.1. *The polynomial-almost-free $E(R)$-algebras.* Using Step Lemma 5.4 and $\triangledown_\kappa$ we will prove the existence of polynomial-almost-free $E(R)$-algebras of cardinality κ for every regular non-weakly compact cardinal $\kappa > \aleph_0$.

Theorem 5.5. ($ZFC + \triangledown_\kappa$) *Assume that R is a countable torsion-free domain which is not a field. For any regular non-weakly compact cardinal $\kappa > \aleph_0$ there exists a polynomial-almost-free $E(R)$-algebra A of cardinality κ.*

Proof. We apply Lemma 5.1 to find a set $E \subseteq \kappa$ satisfying $\triangledown_\kappa(E)$. Moreover, E decomposes into $E = \bigcup_{\beta<\kappa} E_\beta$, where each E_β is sparse and satisfies $\triangledown_\kappa(E_\beta)$.

Now let $A = \bigcup_{\nu\in\kappa} A_\nu$ be a κ-filtration of a set A of cardinality κ. Inductively we must define a ring structure on A_ν for all $\nu \in \kappa$ such that any endomorphism is ring multiplication on many layers. We enumerate $A = \{a_\nu \mid \nu \in \kappa\}$ so that $a_\beta \in A_\beta$ for all $\beta \in \kappa$; we may assume that $|A_\nu| = |\nu| + |R| = |A_{\nu+1} \setminus A_\nu|$ for all $\nu \in \kappa$. Let $\nu \in E$. Then $\mathrm{cf}(\nu) = \omega$ and hence there exists an increasing sequence $\nu_n < \nu$ such that $\sup_{n\in\omega} \nu_n = \nu$ and each ν_n is a successor ordinal, i.e. $\nu_n \notin E$.

The definition of the ring structure is standard and can be found in [DG1]. Hence we restrict to $\varphi \in \mathrm{End}(A_\nu)$. We define $P_\nu^\beta(\varphi) \in \{0, 1\}$ and let $P_\nu^\beta(\varphi) = 0$ if the following hold:

(1) A_ν is a polynomial ring over R of rank $> \omega$.

(2) A_{ν_n} is a polynomial ring over R, A_ν is a polynomial ring over A_{ν_n} for all n and $A_{\nu_n}/a_\beta R$ is a free R-module for almost all n.

(3) φ does not extend to F^0 if we apply the Step Lemma to $F_n = A_{\nu_n}$, $b = a_\beta$ and φ.

Otherwise we let $P_\nu^\beta(\varphi) = 1$.

Since all E_β are non-small we derive, by $\bigtriangledown_\kappa(E_\beta)$, functions $\chi_\beta : E_\beta \to 2$ such that

$$\chi_\beta(\varphi) := \{\nu \in E_\beta \mid P_\nu^\beta(\varphi\restriction A_\nu) = \chi_\beta(\nu)\}$$

is stationary in κ for all φ and $\beta < \kappa$.

Following a routine construction we define inductively a ring structure on A_ν such that

(i) A_ν is a polynomial ring over R;

(ii) if $\varrho \leq \nu$ and $\nu \notin E$ then A_ν is a polynomial ring over A_ϱ;

(iii) if $\varrho \in E_\beta$, $\sup_{n\in\omega} \varrho_n = \varrho$, and $A_{\varrho_n}/a_\beta R$ is a free R-module for some $n \in \omega$ then we apply the Step Lemma for $F_n = A_{\varrho_n}$, $b = a_\beta$ and let $A_{\varrho+1} = F^{\chi_\beta(\varrho)}$.

If τ is a limit ordinal, then $A_\tau = \bigcup_{\nu\in\tau} A_\nu$. Since E is sparse there are ordinals $\tau_\nu \in \tau \setminus E$ such that $A_\tau = \bigcup_{\nu<\mathrm{cf}(\tau)} A_{\tau_\nu}$. By (ii) we conclude that A_{τ_μ} is a polynomial ring over A_{τ_ν} for all $\nu < \mu < \mathrm{cf}(\tau)$. Therefore A_τ is a polynomial ring over A_{τ_ν} for all $\nu < \mathrm{cf}(\tau)$ and thus A_τ is a polynomial ring over R since (i) implies that A_{τ_ν} is a polynomial ring over R.

It remains to show (ii) for a limit ordinal τ. For $\varrho \leq \tau \notin E$ there is τ_ν such that $A_\varrho \subseteq A_{\tau_\nu}$. Hence A_{τ_ν} is a polynomial ring over A_ϱ by (ii) and, as we have seen above, A_τ is a polynomial ring over A_{τ_ν}, which implies that A_τ is also a polynomial ring over A_ϱ.

If $\tau = \mu + 1$ is a successor ordinal and $\mu \notin E_\beta$ for all $\beta < \tau$ then choose a set V_μ of new commuting variables of cardinality μ and define

$$A_\tau = A_\mu[V_\mu].$$

If $\mu \in E_\beta$ for some $\beta \in \tau$ then $\mathrm{cf}(\mu) = \omega$. If $A_{\mu_n}/a_\beta R$ is not a free R-module for all $n \in \omega$ then again set $A_\tau = A_\mu[V_\mu]$. Now conditions (i) to (iii) hold trivially. Therefore assume $A_{\mu_n}/a_\beta R$ is a free R-module for some $n \in \omega$ and hence for almost all $n \in \omega$. In this case we apply the Step Lemma to $F_n = A_{\mu_n}$ and $b = a_\beta$ and define $A_\tau = F^{\chi_\beta(\mu)}$. We have to verify (ii). Take $\varrho \in \tau \setminus E$; then $\varrho < \mu_n < \mu$ for almost all $n \in \omega$. By induction hypothesis A_{μ_n} is a polynomial ring over A_ϱ and the Step Lemma ensures that A_τ is a polynomial ring over A_{μ_n}. Therefore A_τ is a polynomial ring over A_ϱ.

Clearly $A = \bigcup_{\nu\in\kappa} A_\nu$ is a polynomial-almost-free R-algebra of cardinality κ by (i) to (iii). It remains to show that $\mathrm{End}_R(A) = \mathrm{End}_A(A)$. Otherwise there is $\varphi \in \mathrm{End}_R(A) \setminus A$. The set

$$C := \{\nu \in \kappa \mid \varphi{\restriction}A_\nu \in \mathrm{End}_R(A_\nu) \setminus A_\nu\}$$

is a cub. Furthermore, $\varphi(b) \neq \varphi(1)b$ for some fixed basic element $b = a_\beta \in A_\nu$ ($\nu \in C$). Now let $\nu \in C \cap \chi_\beta(\varphi{\restriction}A_\nu)$ and observe that $\varphi{\restriction}A_\nu$ obviously extends to $A_{\nu+1}$.

By (iii), $A_{\nu+1} = F^{\chi_\beta(\nu)}$ (as in the Step Lemma) and (3) tells us that $\chi_\beta(\nu) = 1$ and that $\varphi{\restriction}A_\nu$ also extends to F^0. The Step Lemma now shows that $\varphi(b) = \varphi(1)b$—a contradiction, and A is an $E(R)$-algebra. ∎

By an obvious modification of the proof of Theorem 5.5 (see [E] for details) we derive the following result:

Theorem 5.6. ($ZFC + \triangledown_\kappa$) *Assume that R is a countable torsion-free domain which is not a field. For any uncountable regular non-weakly compact cardinal κ there exist 2^κ non-isomorphic polynomial-almost-free $E(R)$-algebras A of cardinality κ.*

Remark 5.7. Theorem 5.6 shows that for any regular non-weakly compact cardinal $\kappa > \aleph_0$ there exist 2^κ non-isomorphic polynomial-almost-free E-rings.

5.2. *Almost-free $E(A,R)$-modules.* Next we will construct almost-free $E(A,R)$-modules which extend a given free R-module M. We must improve the Step Lemma 5.4.

Extended Step Lemma 5.8. *Assume that R is a countable torsion-free domain which is not a field. Let $F = R[X]$ be the polynomial ring over R in a set $X = \{x_i \mid i \in \omega\}$ of commuting variables, and let $b \in {}_RF$ be a basic element. If $X_n = \{x_0, \ldots, x_n\}$ then consider $F_n := R[X_n]$ as a canonical subring of F. Let $H = \bigcup_{n\in\omega} H_n$ be a chain of free F_n-modules H_n and H a free F-module of countable rank such that H/H_n is a free F_n-module for each $n \in \omega$. Then there exist two ring extensions F^ε of F and two module extensions H^ε of H with the following properties for $\varepsilon = 0, 1$:*

(i) *$F \subset F^\varepsilon$ and F^ε/F is S-divisible.*

(ii) *F^ε is a polynomial ring over F_n for each $n \in \omega$.*

(iii) *F^ε is a polynomial ring over R.*

(iv) *If $\varphi \in \mathrm{End}_R(F)$ extends to both $\varphi^\varepsilon \in \mathrm{End}_R(F^\varepsilon)$ for $\varepsilon = 0, 1$, then $\varphi(b) = \varphi(1)b$.*

(v) *H^ε is a free F^ε-module such that H^ε/H_n is a free F_n-module for all $n \in \omega$.*

(vi) *If $\psi \in \mathrm{Hom}_R(F, H)$ extends to $\psi^\varepsilon \in \mathrm{Hom}_R(F^\varepsilon, H^\varepsilon)$ for $\varepsilon = 0, 1$ then $\psi(b) = \psi(1)b$.*

Proof. The existence of the two ring extensions with (i) to (iv) follows from Lemma 5.2. Therefore it remains to construct H^ε as in the lemma. If $H^\varepsilon := H \otimes F^\varepsilon$, then H^ε is a free F^ε-module for $\varepsilon = 0, 1$. Moreover, H^ε/H_n is a free F_n-module by (ii) since H/H_n is a free F_n-module. This shows (v) and it remains to prove (vi).

Suppose $\psi \in \mathrm{Hom}_R(F, H)$ extends to both $\psi^\varepsilon \in \mathrm{Hom}_R(F^\varepsilon, H^\varepsilon)$ for $\varepsilon = 0, 1$. We can write $H = \bigoplus_{i\in\omega} h_i F$, and let $\pi_i : H \to F$ be the projection onto the ith summand. Then $\psi = \bigoplus_{i\in\omega} \pi_i\psi$ where each $\pi_i\psi \in \mathrm{End}_R(F)$. Hence $H^\varepsilon = \bigoplus_{i\in\omega} h_i F^\varepsilon$ and let π_i^ε be the corresponding projection with $\pi_i^\varepsilon\psi^\varepsilon \in \mathrm{End}_R(F^\varepsilon)$ which extends $\pi_i\psi$ for $\varepsilon = 0, 1$. By (iv) we derive $\pi_i\psi(b) = \pi_i\psi(1)b$, hence $\psi(b) = \bigoplus_{i\in\omega} \pi_i\psi(b) = \bigoplus_{i\in\omega} \pi_i\psi(1)b = \psi(1)b$, which proves (vi). ∎

The Extended Step Lemma 5.8 is used to improve Theorem 5.5.

THEOREM 5.9. $(ZFC + \triangledown_\kappa)$ *Assume that R is a countable torsion-free domain which is not a field. If H is a free R-module of rank $\lambda \geq \aleph_0$ and $\kappa > \lambda$ is a regular non-weakly compact cardinal, then there exist a polynomial-almost-free $E(R)$-algebra A of cardinality κ and an $E(A, R)$-module M of cardinality κ which is κ-free as an R-submodule and extends H.*

Proof. The existence of A follows from Theorem 5.5. Hence we must find M. However, due to the combinatorial setting it turns out that we must construct A and M simultaneously. Hence we begin with two κ-filtrations $A = \bigcup_{\beta\in\kappa} A_\beta$ and $M = \bigcup_{\beta\in\kappa} M_\beta$ with $|M_\nu| = |\nu| + |R| = |M_{\nu+1} \setminus M_\nu|$ for all $\nu \in \kappa$. As in the proof of Theorem 5.5, we will only concentrate on the mapping properties and not on prediction of algebra and module structures.

We adopt the notation on A from the proof of Theorem 5.5 and decompose each E_β into stationary disjoint subsets E_β^A, E_β^M. The pair (A, M) is constructed inductively on each (A_ν, M_ν) where A_ν is a polynomial ring as before and M_ν is a free A_ν-module. If $\varphi : A_\nu \to M_\nu$, then (as before) we want to define $\widehat{P}_\nu^\beta(\varphi) \in \{0, 1\}$ and let the value be 0 if the following holds (the only interesting case is when $\nu \in E_\beta^M$ for some β):

There is an increasing sequence ν_n with $\sup \nu_n = \nu$ such that

(1) M_{ν_n} is a free A_{ν_n}-module and M_ν/M_{ν_n} and $A_{\nu_n}/a_\beta R$ are free R-modules for almost all n;

(2) if we identify $F_n = A_{\nu_n}$, $H_n = M_{\nu_n}$, $b = a_\beta$ in the Extended Step Lemma, then $\varphi : A_\nu \to M_\nu$ does not extend to H^0.

We set $\widehat{P}^\beta_\nu(\varphi) = 1$ otherwise.

By $\triangledown_\kappa(E^H_\beta)$ we obtain choice functions $\chi^H_\beta : E^H_\beta \to 2$ such that

$$\chi^H(\varphi) := \{\nu \in E^H_\beta \mid \widehat{P}^\beta_\nu(\varphi \restriction A_\nu) = \chi^H_\beta(\nu)\}$$

is stationary in κ. Now define an R-algebra structure on A_ν and an A_ν-module structure on M_ν subject to the following conditions:

(i) A_ν is a polynomial ring over R.

(ii) If $\varrho \le \nu$ and $\nu \notin E^A := \bigcup_{\beta<\kappa} E^A_\beta$ then A_ν is a polynomial ring over A_ϱ.

(iii) If $\varrho \in E^A_\beta$, $\sup_{n\in\omega}(\varrho_n) = \varrho$ and $A_{\varrho_n}/a_\beta R$ is a free R-module for some n then we apply the Extended Step Lemma for $F_n = A_{\varrho_n}$, $H_n = M_{\varrho_n}$, $b = a_\beta$ and let $A_{\varrho+1} = F^{\chi^A_\beta}(\varrho)$, $M_{\varrho+1} = H^{\chi^A_\beta}(\varrho)$.

(iv) M_ν is a free A_ν-module.

(v) If $\varrho \le \nu$ and $\nu \notin E^H$ then M_ν/M_ϱ is A_ϱ-free.

(vi) If $\varrho \in E^H_\beta$, $\sup_{n\in\omega} \varrho_n = \varrho$, and $A_{\varrho_n}/a_\beta R$ is a free R-module for some n then we apply the Extended Step Lemma for $F_n = A_{\varrho_n}$, $H_n = M_{\varrho_n}$, $b = a_\beta$ and let $A_{\varrho+1} = F^{\chi^H_\beta}(\varrho)$, $M_{\varrho+1} = H^{\chi^H_\beta}(\varrho)$.

We obtain two κ-filtrations $A = \bigcup_{\beta\in\kappa} A_\beta$ and $M = \bigcup_{\beta\in\kappa} M_\beta$. A by now routine checking as in Theorem 5.5 shows that A is a polynomial-almost-free $E(R)$-algebra of cardinality κ and M is an almost-free (as R-module) $E(A,R)$-module of cardinality κ which extends H. ∎

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