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Quasi-linear maps

by

D. J. Grubb (DeKalb, IL)

Abstract. A quasi-linear map from a continuous function space $C(X)$ is one which is linear on each singly generated subalgebra. We show that the collection of quasi-linear functionals has a Banach space pre-dual with a natural order. We then investigate quasi-linear maps between two continuous function spaces, classifying them in terms of generalized image transformations.

Let X be a compact Hausdorff space and $C(X)$ the space of real-valued continuous functions on X. If $f \in C(X)$, let $A(f)$ denote the closed subalgebra of $C(X)$ generated by 1 and f. In other words,

$$A(f) = \{\varphi \circ f : \varphi \in C(f(X))\}.$$

A map from $C(X)$ into a vector space is said to be *quasi-linear* if it is linear on $A(f)$ for each $f \in C(X)$. If B is a Banach space, a quasi-linear map $\varrho : C(X) \to B$ is is said to be *bounded* if there is an $M < \infty$ such that $\|\varrho(f)\| \le M\|f\|_\infty$ for each $f \in C(X)$. In this case define the norm $\|\varrho\|$ to be the infimum of all such M. In the case where $B = \mathbb{R}$, we say that ϱ is a quasi-linear *functional* on $C(X)$. The linear space of all bounded quasi-linear functionals on $C(X)$ will be denoted by $QL(X)$.

If B is an ordered Banach space, and $\varrho : C(X) \to B$ is quasi-linear, we say that ϱ is *positive* if $\varrho(f) \ge 0$ for each $f \ge 0$. The positive quasi-linear functionals on $C(X)$ were characterized by Johan Aarnes in [1] by associating a set function to each positive quasi-linear functional, generalizing the Riesz representation theorem. The primary difference between Aarnes' set functions and regular Borel measures is that subadditivity is no longer required and they are only defined on subsets which are either open or closed. These set functions are now called *topological measures*, a terminology which replaces the older one of *quasi-measures*. One consequence of these results is that all positive quasi-linear functionals are bounded. In [7], the current

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author extends Aarnes' result to the case of all bounded quasi-linear functionals by considering signed topological measures.

The purpose of this paper is to more fully describe the space of quasi-linear functionals $QL(X)$, and to investigate quasi-linear maps between $C(X)$ and other Banach spaces.

1. The pre-dual of $QL(X)$. The first remarkable thing about $QL(X)$ is that it has a Banach space pre-dual. This has been previously noted in [13, 14], but we give a more thorough treatment.

THEOREM 1. *Let X be a compact Hausdorff space. Then there is a Banach space $D(X)$ and a bounded quasi-linear map $\Phi : C(X) \to D(X)$ with the following properties*:

(i) *Φ is an isometry.*
(ii) *The span of the image of Φ is dense in $D(X)$.*
(iii) *If $L \in D(X)^*$, then $L \circ \Phi$ is a bounded quasi-linear functional on $C(X)$ with $\|L\| = \|L \circ \Phi\|$. Furthermore, every bounded quasi-linear functional on $C(X)$ is of this form.*
(iv) *If $\varrho : C(X) \to B$ is any bounded quasi-linear map into a Banach space B, then there is a unique bounded* linear *map $\widehat{\varrho} : D(X) \to B$ such that $\varrho = \widehat{\varrho} \circ \Phi$. Also, $\|\widehat{\varrho}\| = \|\varrho\|$.*
(v) *The space $D(X)$ is characterized by the above properties as follows*: *Suppose D is any Banach space with a bounded quasi-linear isometry $\Phi' : C(X) \to D$ and that whenever $\varrho : C(X) \to B$ is a bounded quasi-linear map, there exists a unique $\widehat{\varrho} : D \to B$ which is bounded and linear with $\varrho = \widehat{\varrho} \circ \Phi'$ and $\|\varrho\| = \|\widehat{\varrho}\|$. Then there is an isometric isomorphism $J : D(X) \to D$ such that $\Phi' \circ J = \Phi$.*

Finally, the Banach space dual of $D(X)$ is naturally isomorphic to $QL(X)$.

Proof. Let

$$A = \bigoplus_{f \in C(X)} C(f(X))$$

be the l_1 direct sum of the spaces $C(f(X))$ where f ranges over $C(X)$. A typical element of A is an indexed family (φ_f) where f ranges over $C(X)$, $\varphi_f \in C(f(X))$ for all f and

$$\|(\varphi_f)\| = \sum_f \|\varphi_f\|_\infty < \infty.$$

In particular, at most countably many φ_f are non-zero. For any fixed $f \in C(X)$, let I_f denote the injection of $C(f(X))$ to the f coordinate of A.

Let $N \subseteq A$ be the closed linear span of elements of the form $I_{\varphi \circ f}(\mathrm{id}) - I_f(\varphi)$ where f ranges over $C(X)$ and φ over $C(f(X))$ and id represents the identity function from $\varphi(f(X))$ into $\mathbb{R}$. The notation will not distinguish between these identity functions for different f. Finally, let $D(X) = A/N$ with the quotient norm and define $\Phi(f) = I_f(\mathrm{id}) + N \in D(X)$ for $f \in C(X)$.

Then $\Phi(\varphi \circ f) = I_{\varphi\circ f}(\mathrm{id}) + N = I_f(\varphi) + N$ for all $\varphi \in C(f(X))$. Thus, for every φ, ψ in $C(f(X))$, we have $\Phi(\varphi \circ f + \psi \circ f) = I_f(\varphi + \psi) + N = I_f(\varphi) + I_f(\psi) + N = \Phi(\varphi \circ f) + \Phi(\psi \circ f)$. Hence Φ is linear on $A(f)$. Also, $\|\Phi(f)\| = \|I_f(\mathrm{id}) + N\| \le \|I_f(\mathrm{id})\| = \|f\|_\infty$, so Φ is bounded and quasi-linear. Now notice that for any (φ_f) in A, $(\varphi_f) + N = \sum_f \Phi(\varphi_f \circ f)$, so that the span of the image of Φ is dense in $D(X)$. Even more, this sum is countable, so every element of $D(X)$ can be written in the form $\sum_i \Phi(f_i)$ for some countable collection of $f_i \in C(X)$.

Next suppose that $\varrho : C(X) \to B$ is a bounded quasi-linear map into a Banach space. Define $\varrho' : A \to B$ by $\varrho'((\varphi_f)) = \sum \varrho(\varphi_f \circ f)$. Since

$$\sum_f \|\varrho(\varphi_f \circ f)\|_\infty \le \sum_f \|\varrho\| \cdot \|\varphi_f \circ f\|_\infty = \|\varrho\| \cdot \sum_f \|\varphi_f\|_\infty = \|\varrho\| \cdot \|(\varphi_f)\|,$$

this map is defined on all of A and $\|\varrho'\| \le \|\varrho\|$. Moreover, since ϱ is quasi-linear, ϱ' is linear on A. Also, since $\varrho(f) = \varrho'(I_f(\mathrm{id}))$, it follows that $\|\varrho\| \le \|\varrho'\|$ as well.

We now have $\varrho'(I_f(\varphi) - I_{\varphi\circ f}(\mathrm{id})) = \varrho(\varphi \circ f) - \varrho(\mathrm{id} \circ \varphi \circ f) = 0$, so ϱ' factors through $D(X) = A/N$. Hence, there is a bounded linear map $\widehat{\varrho} : D(X) \to B$ with

$$\widehat{\varrho} \circ \Phi(f) = \widehat{\varrho}(I_f(\mathrm{id}) + N) = \varrho'(I_f(\mathrm{id})) = \varrho(\mathrm{id} \circ f) = \varrho(f).$$

Note that $\|\widehat{\varrho}\| = \|\varrho'\| = \|\varrho\|$. Since the span of the image of Φ is dense, the map $\widehat{\varrho}$ is unique. The statement about the duality of $D(X)$ and $QL(X)$ follows by taking $B = \mathbb{R}$.

The isometry of Φ follows from [13, 14], where it is shown that if ϱ is a quasi-linear functional on $C(X)$, then $|\varrho(f) - \varrho(g)| \le \|\varrho\| \cdot \|f - g\|_\infty$. By the Hahn–Banach theorem and the fact that $\|L \circ \Phi\| = \|L\|$ for $L \in QL(X)$, it follows that

$$\begin{aligned}\|\Phi(f) - \Phi(g)\| &= \sup\{|L \circ \Phi(f) - L \circ \Phi(g)| : L \in D(X)^*,\ \|L\| \le 1\} \\ &\le \sup\{\|L \circ \Phi\| \cdot \|f - g\|_\infty : L \in D(X)^*,\ \|L\| \le 1\} \\ &= \|f - g\|_\infty.\end{aligned}$$

For the reverse inequality, let $x \in X$ and L be point evaluation at x. Then L is quasi-linear on $C(X)$ and hence can be lifted to $\widehat{L}$ on $D(X)$ with $\|L\| = \|\widehat{L}\|$. Thus $|f(x) - g(x)| = |\widehat{L}(\Phi(f)) - \widehat{L}(\Phi(g))| \le \|\Phi(f) - \Phi(g)\|$. Hence $\|f - g\|_\infty \le \|\Phi(f) - \Phi(g)\|$. In particular, by taking $g = 0$, we see that $\|\Phi(f)\| = \|f\|_\infty$ for all $f \in C(X)$.

Finally, suppose D is a Banach space with a quasi-linear isometry $\Phi' : C(X) \to D$ as above. Thus there are unique bounded linear maps J :

$D(X) \to D$ and $J' : D \to D(X)$ with $J \circ \Phi = \Phi'$ and $J' \circ \Phi' = \Phi$. Then $J \circ J' \circ \Phi' = J \circ \Phi = \Phi'$, so by the uniqueness clause, $J \circ J'$ is the identity map. Similarly, $J' \circ J$ is the identity. Also, we have $\|J'\| = \|\Phi\| = 1$ and $\|J\| = \|\Phi'\| = 1$, so both J and J' are isometries. ∎

While it is not clear from the definition that quasi-linear maps are continuous, we see that a main result of [14] is true in slightly more generality. It is also possible to deduce this directly from [14] by use of the Hahn–Banach theorem.

COROLLARY 2. *Bounded quasi-linear maps are Lipschitz.*

COROLLARY 3. *Let $\varrho : C(X) \to C(Y)$ be a bounded quasi-linear map. Then there is a bounded linear $\widehat{\varrho} : D(X) \to D(Y)$ such that $\widehat{\varrho} \circ \Phi = \Phi \circ \varrho$ if and only if $L \circ \varrho \in QL(X)$ for every $L \in QL(Y)$. In this case $\eta \circ \varrho$ is quasi-linear for every bounded quasi-linear map $\eta : C(Y) \to B$.*

Proof. Suppose $\widehat{\varrho}$ exists. For each $L \in QL(X)$, there is an $\widehat{L} \in D(X)^*$ with $L = \widehat{L} \circ \Phi$. Then $L \circ \varrho = \widehat{L} \circ \Phi \circ \varrho = \widehat{L} \circ \widehat{\varrho} \circ \Phi$ is quasi-linear. For the converse, we need only show that $\Phi \circ \varrho$ is quasi-linear. If not, then there are $f \in C(X)$ and $\varphi, \psi \in C(f(X))$ such that

$$\Phi \circ \varrho(\varphi \circ f + \psi \circ f) \neq \Phi \circ \varrho(\varphi \circ f) + \Phi \circ \varrho(\psi \circ f).$$

But then the Hahn–Banach theorem gives an $L \in D(Y)^*$ such that $L \circ \Phi \circ \varrho(\varphi \circ f + \psi \circ f) \neq L \circ \Phi \circ \varrho(\varphi \circ f) + L \circ \Phi \circ \varrho(\psi \circ f)$. In other words, $L \circ \Phi \circ \varrho \notin QL(X)$. Since $L \circ \Phi \in QL(Y)$, this is a contradiction.

Now, if B is a Banach space and $\eta : C(Y) \to B$ is a bounded quasi-linear map, there is a bounded linear map $\widehat{\eta} : D(Y) \to B$ such that $\eta = \widehat{\eta} \circ \Phi$. Then $\eta \circ \varrho = \widehat{\eta} \circ \Phi \circ \varrho = \widehat{\eta} \circ \widehat{\varrho} \circ \Phi$ is quasi-linear. ∎

As an example, suppose $\varrho : C(X) \to C(Y)$ is an *algebra* homomorphism on each singly generated subalgebra $A(f)$, i.e., ϱ is a quasi-homomorphism. It was shown in [4] that $\varrho(\varphi \circ f) = \varphi \circ \varrho(f)$ for all $\varphi \in C(f(X))$ and $f \in C(X)$. Hence $L \circ \varrho$ is again quasi-linear whenever $L \in QL(Y)$. The previous result then shows that ϱ lifts to a map $\widehat{\varrho} : D(X) \to D(Y)$.

On the other hand, if ϱ is not a quasi-homomorphism, then there may be no such lift, as the following example shows.

EXAMPLE 1. Let $\varrho \in QL(X)$ and define $T : C(X \times Y) \to C(Y)$ by $T(f)(y) = \varrho(f^y)$, where $f^y \in C(X)$ is defined by $f^y(x) = f(x, y)$. It is shown in [8] that T is a quasi-linear map, but that $\eta \circ T$ is not quasi-linear if $\eta \in QL(Y)$ is non-linear, and ϱ is not a quasi-homomorphism. Hence, T does not lift to a map $D(X \times Y) \to D(Y)$ unless ϱ is a quasi-homomorphism or every element of $QL(Y)$ is linear.

Returning to the general case, let $f \in C(X)$ and let $B(f)$ denote the collection of those $g \in C(X)$ that are constant on components of level sets of f. Hence, $B(f)$ is the analytic subalgebra generated by the single function $f \in C(X)$. In general, $B(f)$ is much larger than the subalgebra $A(f)$. For example, if $X = S^1 \subseteq \mathbb{R}^2$ and f is projection to the x-axis, $B(f) = C(X)$.

If $f \in C(X)$, let X/f denote the space obtained from X by identifying all components of level sets of f to points. Let $\pi_f : X \to X/f$ be the quotient map. Then π_f is a monotone map in the sense that inverse images of points are connected. Also, $B(f)$ consists of all functions of the form $g \circ \pi_f$ for $g \in C(X/f)$. By Katětov's theorem (see [5]), the covering dimension of X/f is at most 1.

PROPOSITION 4. *For any $f \in C(X)$, the map Φ is linear on $B(f)$.*

Proof. Let $\widehat{\pi}_f : C(X/f) \to C(X)$ be defined by $\widehat{\pi}_f(g) = g \circ \pi_f$. If $L \in D(X)^*$, then the quasi-linear map $L \circ \Phi \circ \widehat{\pi}_f : C(X/f) \to \mathbb{R}$ is linear since *every* quasi-linear functional on $C(Y)$ is linear when $\dim Y \le 1$ (see [7]). Hence, Φ must be linear on the image of $\widehat{\pi}_f$, i.e. on $B(f)$. ∎

COROLLARY 5. *If $\dim X \le 1$, then $D(X) = C(X)$.*

Proof. This follows since, by Katětov's theorem, any two elements of $C(X)$ are in a common singly generated analytic subalgebra, $B(f)$. By the last result, Φ is linear. Hence, $C(X)$ is isometrically imbedded in $D(X)$ as a dense linear subspace. This subspace must be closed, giving the result. ∎

Now we put an order structure on $D(X)$ so that $\Phi : C(X) \to D(X)$ is order preserving and so that the positive, bounded, linear functionals on $D(X)$ correspond to the positive, bounded, quasi-linear functionals on $C(X)$. To this end, let $C^+(X)$ denote the non-negative functions in $C(X)$, and let $D(X)^+$ be the closed cone in $D(X)$ generated by the image of $C^+(X)$ under Φ. We use the terminology of ordered Banach spaces from [11].

THEOREM 6. *Suppose that whenever $\varphi \in D(X)$ with $\varphi \neq 0$, there is a positive quasi-linear functional $L \in QL(X)$ with $\widehat{L}(\varphi) \neq 0$. Then the space $D(X)$ with positive cone $D(X)^+$ is an ordered Banach space with a regular order. The positive linear functionals on $D(X)$ correspond to the positive quasi-linear functionals on $C(X)$. In addition, $\Phi(1)$ is a strong order unit for $D(X)$ and the map Φ is order preserving in the sense that $\Phi(f) \le \Phi(g)$ whenever $f \le g$ in $C(X)$.*

Proof. First notice that if L is positive and $\varphi = \sum \Phi(f_i)$ where each $f_i \ge 0$, then $\widehat{L}(\varphi) \ge 0$. Hence if $\varphi \in D(X)^+ \cap -D(X)^+$, then $\widehat{L}(\varphi) = 0$ for all L positive. By our assumption, $\varphi = 0$.

Now, if $\varphi = \sum \Phi(f_i)$ is in $D(X)$, we see, upon noticing $f_i^+, f_i^- \in A(f_i)$, that $\varphi = \sum \Phi(f_i^+) - \sum \Phi(f_i^-) \in D(X)^+ - D(X)^+$. Thus, $D^+(X)$ generates $D(X)$. Similarly, setting $\psi = \sum \Phi(|f_i|)$, we see that $\varphi, -\varphi \le \psi$. Now, $\|\psi\| \le$

$\sum \|f_i\|_\infty$, so $\|\psi\| < 1$ if $\|\varphi\| < 1$. Hence, $D(X)$ is an ordered Banach space with a regular order.

Notice that $L \in D(X)^*$ is a positive linear functional if and only if $L \circ \Phi(f) \geq 0$ for all $f \geq 0$ in $C(X)$. Hence $L \circ \Phi$ is a positive quasi-linear functional on $C(X)$ if and only if L is a positive linear functional on $D(X)$.

Now, since $D(X)^+$ is closed, we see by use of a Hahn–Banach separation argument that $\sum \Phi(f_i) \geq 0$ if and only if $\sum \varrho(f_i) \geq 0$ for all positive quasi-linear functionals $\varrho \in QL(X)$. In particular, since $\varrho(\|f\|_\infty \cdot 1 - f) \geq 0$ for any $f \in C(X)$, we see that $\Phi(f) \leq \|f\|_\infty \cdot \Phi(1)$. Also, if $\varphi = \sum \Phi(f_i)$, we have $\varphi \leq \Phi(1) \sum \|f_i\|_\infty$. From the definition of the norm on $D(X)$, this shows that $\varphi \leq \Phi(1) \cdot \|\varphi\|$. Similarly, $-\Phi(1) \cdot \|\varphi\| \leq \varphi$. Thus $\Phi(1)$ is a strong order unit for $D(X)$.

Also, if $f \leq g$ in $C(X)$ and ϱ is a positive quasi-linear functional, it is known that $\varrho(f) \leq \varrho(g)$ [1, Lemma 4.1]. From the previous paragraph, this implies that $\Phi(f) \leq \Phi(g)$, so Φ is order preserving. ∎

In particular, if every $L \in QL(X)$ can be written as $L = L_1 - L_2$ where L_1 and L_2 are positive, then $D(X)$ is an ordered Banach space. It is not known if this decomposition is always possible. Even if $D(X)$ is not an ordered Banach space, we can still define the cone $D(X)^+$. We would then find that $\phi \leq \psi$ and $\psi \leq \phi$ exactly when $\widehat{L}(\phi) = \widehat{L}(\psi)$ for all positive quasi-linear functionals $L \in QL(X)$. The map $\Phi : C(X) \to D(X)$ will still be order preserving with this modification.

Proposition 7. *If X is simply connected* (*more generally*, *if $g(X) = 0$*, *see below*), *then $D(X)$ is an ordered Banach space and the order on $D(X)$ is monotone. That is*, *$-\varphi \leq \psi \leq \varphi$ in $D(X)$ implies that $\|\psi\| \leq \|\varphi\|$.*

Proof. The results from [13, 14] show that if X is of this form, the order on $QL(X)$ is 1-generating in the sense that every $L \in QL(X)$ can be written in the form $L = L_1 - L_2$ with L_1, L_2 positive and $\|L\| = \|L_1\| + \|L_2\|$. Hence, the cone of $D(X)$ is 1-normal. It follows that the order on $D(X)$ is monotone. ∎

It is natural to ask whether $D(X)$ has the Riesz decomposition property. Unfortunately, this is not the case even when $g(X) = 0$. In fact, if $X = [0,1]^2$, then $QL(X)$ is not a Banach lattice, so this is precluded (see [7]).

2. Generalized image transformations. At this point, we would like to recall the representation of quasi-linear functionals on $C(X)$ in terms of integration with respect to signed topological measures. For X a compact Hausdorff space, let $\mathcal{O}(X)$ denote the collection of open sets in X and $\mathcal{C}(X)$ denote the collection of closed sets. Also, let $\mathcal{A}(X) = \mathcal{O}(X) \cup \mathcal{C}(X)$. A *signed topological measure* on X is a map $\mu : \mathcal{A}(X) \to \mathbb{R}$ such that

(a) If $A_1 \cap A_2 = \emptyset$ and $A_1 \cup A_2 \in \mathcal{A}(X)$, then $\mu(A_1 \cup A_2) = \mu(A_1) + \mu(A_2)$.
(b) There is an $M < \infty$ such that whenever $\{A_i\}$ are disjoint in $\mathcal{A}(X)$, then $\sum |\mu(A_i)| \leq M$.
(c) For $U \in \mathcal{O}(X)$, $\mu(U) = \lim \mu(C)$ where C ranges over the directed set of $C \in \mathcal{C}(X)$ with $C \subseteq U$.

If, in addition, $\mu(A) \geq 0$ for all $A \in \mathcal{A}(X)$, we say that μ is a *positive topological measure* on X. Notice that for this case the second property above follows from the other two with $M = \mu(X)$.

If μ is a signed topological measure on X and $f \in C(X)$, it is possible to define a *measure* on $\mathbb{R}$ via $\mu_f(A) = \mu(f^{-1}(A))$ for open sets $A \subseteq \mathbb{R}$ and then define, for $f \in C(X)$,

$$\varrho_\mu(f) = \int_X f \, d\mu = \int_{\mathbb{R}} x \, d\mu_f(x).$$

For positive quasi-linear functionals, the following results is due to Aarnes [1]. The general case is due to the current author [6].

THEOREM 8. *If μ is a signed topological measure, then ϱ_μ is a bounded quasi-linear functional on $C(X)$. Conversely, every bounded quasi-linear functional on $C(X)$ is of the form ϱ_μ for a unique signed topological measure μ. The positive topological measures correspond to the positive quasi-linear functionals in this way.*

Since $QL(X)$ is the dual of $D(X)$, it has a natural weak-$*$ topology induced from this duality, i.e. $\sigma(QL(X), D(X))$. Similarly, there is a weak-$*$ topology induced from the action on just $C(X)$, i.e. $\sigma(QL(X), C(X))$. The topologies induced on bounded subsets of $QL(X)$ are the same. In particular, let $QP(X)$ denote the collection of quasi-probabilities on X, that is, the positive topological measures μ with $\mu(X) = 1$. Then it is known that $QP(X)$ is a compact Hausdorff space in the unique weak-$*$ topology (see [10]). The topology on $QP(X)$ has an alternative description which is found in [10]: the sets of the form

$$\widehat{U}_\alpha = \{\mu \in QP(X) : \mu(U) > \alpha\}$$

where $U \in \mathcal{O}(X)$ and $\alpha \in \mathbb{R}$ form a subbase for this topology.

DEFINITION 9. For $f \in C(X)$, define $\widetilde{f} : QP(X) \to \mathbb{R}$ by $\widetilde{f}(\mu) = \varrho_\mu(f)$. We also define, for $A \in \mathcal{A}(X)$, $e_A : QP(X) \to \mathbb{R}$ by $e_A(\mu) = \mu(A)$.

PROPOSITION 10. *The function $\widetilde{f}$ is continuous for each $f \in C(X)$. Also, for $U \subseteq X$ open, the function e_U is lower semicontinuous. We have the following*:

(a) $e_\emptyset = 0$, *and* $e_X = 1$.
(b) *If $A = A_1 \cup A_2$ is a disjoint union in $\mathcal{A}(X)$, then $e_A = e_{A_1} + e_{A_2}$.*

(c) *If* $U \in \mathcal{O}(X)$, *then* e_U *is lower semicontinuous and* $e_U = \sup\{e_C : C \subseteq U, C \in \mathcal{C}(X)\}$.

Proof. The continuity of $\widetilde{f}$ is clear by definition of the weak-$*$ topology. The listed properties follow directly from the properties of the quasi-probabilities μ. Finally, if $U \subseteq X$ is open, then

$$e_U(\mu) = \mu(U) = \sup\{\varrho_\mu(f) : 0 \le f \le U\} = \sup\{\widetilde{f} : 0 \le f \le U\},$$

so that e_U is the supremum of a collection of continuous functions, and so is lower semicontinuous. ∎

This result inspires the following definition.

DEFINITION 11. Suppose that X and Y are compact Hausdorff spaces and for each $A \in \mathcal{A}(X)$, there is a non-negative semicontinuous function k_A on Y such that:

(a) $k_\emptyset = 0$, $k_X = 1$.
(b) If $A = A_1 \cup A_2$ is a disjoint union in $\mathcal{A}(X)$, then $k_A = k_{A_1} + k_{A_2}$.
(c) If $U \in \mathcal{O}(X)$, then k_U is lower semicontinuous and $k_U = \sup\{k_C : C \subseteq U, C \in \mathcal{C}(X)\}$.

We then call the collection $\{k_A\}$ a *generalized image transformation* from X to Y.

We note in particular that $k_{X \setminus A} = 1 - k_A$, so for $C \in \mathcal{C}(X)$, $k_C = \inf\{k_U : C \subseteq U \in \mathcal{O}(X)\}$ is upper semicontinuous.

THEOREM 12. *There are correspondences between normalized positive quasi-linear maps* $\varrho : C(X) \to C(Y)$, *weak-$*$ continuous functions* $w : Y \to QP(X)$, *and generalized image transformations* $\{k_A\}$ *from* X *to* Y *such that for corresponding entities*, $\varrho(f)(y) = \int_X f\, dw(y)$, *and* $k_A = e_A \circ w$.

Proof. Given a normalized quasi-linear map ϱ and a point $y \in Y$, the map sending f to $\varrho(f)(y)$ is a normalized quasi-linear functional, so can be represented by integration with respect to a topological measure $w(y)$. Since each $\varrho(f)$ is continuous on Y by assumption and since $QP(X)$ has the weak-$*$ topology, the map w is continuous.

On the other hand, if $w : Y \to QP(X)$ is continuous, we may define $\varrho(f)(y) = \int f\, dw(y)$ and $k_A(y) = w(y)(A)$ for $f \in C(X)$ and $A \in \mathcal{A}(X)$. Clearly, then, $\varrho(f)$ is continuous for $f \in C(X)$ and ϱ is then a normalized quasi-linear map from $C(X)$ to $C(Y)$. On the other hand, we see that $k_A = e_A \circ w$. Since each e_A is semicontinuous, the same is true for each k_A. Also the collection $\{k_A\}$ inherits the defining properties of a generalized image transformation from the transformation $\{e_A\}$.

Now suppose that $\{k_A\}$ is a generalized image transformation and $y \in Y$. Define a set function $w(y)$ on $\mathcal{A}(X)$ by $w(y)(A) = k_A(y)$. The properties of generalized image transformations show that $w(y)$ is a positive topological measure on X with $w(y)(X) = 1$. Thus $w(y) \in QP(X)$. To show that w defines a continuous map, we consider the inverse image of the subbasic open set $\widehat{U}_\alpha$. We have

$$w^{-1}(\widehat{U}_\alpha) = \{y \in Y : w(y)(U) > \alpha\} = \{y \in Y : k_U(y) > \alpha\}.$$

Since k_U is lower semicontinuous on Y for U open in X, this shows $w^{-1}(\widehat{U}_\alpha)$ is open, and so w is continuous. Finally, $k_A(y) = w(y)(A) = e_A \circ w(y)$. ■

This result should be compared with the characterization of quasi-homomorphisms in terms of image transformations given in [4]. In particular, the map ϱ is a quasi-homomorphism if and only if the topological measures $w(y)$ are all $\{0,1\}$-valued.

If we consider the situation from Example 1 above and let $U \subseteq X \times Y$ be open and μ the topological measure associated with ϱ, then the generalized image transformation corresponding to T will be given by $k_U(y) = \mu(U^y)$. This is shown in [8].

It is of some interest to determine the exact correspondence between quasi-linear maps ϱ and generalized image transformations above in more specific terms. To accomplish this, let $f \in C(X)$, $y \in Y$, $\alpha \in \mathbb{R}$, and define $k_f(y,\alpha) = k_U(y)$ where $U = f^{-1}(\alpha,\infty)$. Then, if $f \geq 0$ is continuous, we have

$$\varrho(f)(y) = \int_{\mathbb{R}} x\, dw(y)_f(x) = \int_0^\infty \int_0^x 1\, dt\, dw(y)_f(x) = \int_0^\infty w(y)_f(\alpha,\infty)\, d\alpha = \int_0^\infty k_f(y,\alpha)\, d\alpha.$$

This equality provides a way to extend the domain of ϱ to include positive lower semicontinuous functions.

DEFINITION 13. Given a normalized, positive quasi-linear map $\varrho : C(X) \to C(Y)$ and a lower semicontinuous function $f : X \to [0,\infty]$, we define $k_f(y,\alpha) = k_U(y)$ where $U = f^{-1}(\alpha,\infty]$. We also define

$$\varrho(f) = \int_0^\infty k_f(y,\alpha)\, d\alpha.$$

Notice that $k_f(y,\alpha)$ is lower semicontinuous in α for fixed $y \in Y$ by inner regularity of topological measures. In particular, for the characteristic function $f = \xi_U$ of an open set, $\varrho(\xi_U) = k_U$ since in that case $k_f(y,\alpha) = k_U(y)$ for $\alpha < 1$ and is 0 otherwise. Notice also that if $0 \leq f \leq M$, then $0 \leq \varrho(f) \leq M$.

Proposition 14. *Suppose that $\mathcal{D}$ is a directed family of positive lower semicontinuous functions on X and $f = \sup\{g : g \in \mathcal{D}\}$ pointwise. If $\varrho : C(X) \to C(Y)$ is quasi-linear, then $\varrho(f) = \sup\{\varrho(g) : g \in \mathcal{D}\}$. In particular, $k_U = \sup\{\varrho(f) : 0 \le f \le \xi_U,\ f \in C(X)\}$ for U open in X.*

Proof. For each $\alpha \in \mathbb{R}$, $f^{-1}(\alpha, \infty] = \bigcup\{g^{-1}(\alpha, \infty] : g \in \mathcal{D}\}$ is a directed union of open sets. Hence

$$\begin{aligned} k_f(y, \alpha) = w(y)(f^{-1}(\alpha, \infty)) &= \sup\{w(y)(g^{-1}(\alpha, \infty)) : g \in \mathcal{D}\} \\ &= \sup\{k_g(y, \alpha) : g \in \mathcal{D}\}. \end{aligned}$$

Again we have used the inner regularity of the topological measures $w(y)$. Since all functions in this equality are lower semicontinuous, the result follows by standard results on measures. ∎

It is clear from this that $\varrho(f)$ is lower semicontinuous when $f \ge 0$ is lower semicontinuous. Another easy consequence is the following.

Proposition 15. *Suppose that $\varrho : C(X) \to C(Y)$ and $\eta : C(Y) \to \mathbb{R}$ are positive quasi-linear maps such that $\eta \circ \varrho$ is a quasi-linear functional. Let $\{k_A\}$ be the generalized image transformation associated with ϱ and let ν and τ be the topological measures associated with η and $\eta \circ \varrho$, respectively. Then, for $U \subseteq X$ open, we have $\tau(U) = \eta(k_U)$.*

In particular, since η may not even be linear on continuous functions, it may well happen that $\eta(k_{U \cup V}) = \eta(k_U) + \eta(k_V)$ fails for some disjoint open sets U, V in X. In this case, $\eta \circ \varrho$ cannot be quasi-linear since the corresponding set function is not a topological measure.

3. Examples. We now turn to methods of constructing examples of quasi-linear maps. We assume for the rest of this paper that X is a connected, locally connected, compact Hausdorff space. A subset A of X is called *solid* if both A and its complement $X \setminus A$ are connected. We denote the solid sets in any collection of sets by adding a subscript "s". Thus, $\mathcal{C}_s(X)$ denotes the collection of closed solid subsets of X and $\mathcal{U}_s(X)$ that of open solid subsets. A map $\mu : \mathcal{A}_s(X) \to [0, 1]$ is called a *solid set function* if

(i) whenever $\{C_n\}$ is a finite family of disjoint sets in $\mathcal{C}_s$, and $\bigcup C_n \subseteq C \in \mathcal{C}_s$, then $\sum \mu(C_n) \le \mu(C)$,
(ii) if $U \in \mathcal{U}_s$, then $\mu(U) = \sup\{\mu(C) : C \subseteq U,\ C \in \mathcal{C}_s\}$,
(iii) if $\{A_n\}$ is a finite partition of X into sets from $\mathcal{A}_s$, then $\sum \mu(A_n) = \mu(X) = 1$.

A fundamental result due to Aarnes [2] is that any solid set function extends uniquely to a normalized positive topological measure on X. See also [13, 14] for extensions of this result to signed topological measures.

Definition 16. We define the *genus*, $g(X)$, to be one less than the maximum number of components of $U \cap V$ where $U, V \in \mathcal{U}_s(X)$ and $U \cup V = X$.

It is known [12] that the fundamental group $\pi_1(X)$ of a locally simply connected space X is infinite if there are connected open sets U and V with $U \cap V$ disconnected. Hence, $g(X) = 0$ for any locally simply connected space X with finite fundamental group. In particular, this holds if X is simply connected.

In [9], the present author proves a number of results about $g(X)$, and shows how to construct examples of non-trivial solid set functions for certain spaces with $g(X) = 1$. Unfortunately, there are very few non-trivial examples of solid set functions known when $g(X) \geq 2$. This is one of the major open problems in the study of topological measures.

For the next result, recall that a *monotone* map $h : X \to Y$ is one where inverse images of points are connected.

Proposition 17. *Let $h : X \to Y$ be an onto monotone map. Then*:

(i) *If $A \subseteq Y$ is solid, $h^{-1}(A) \subseteq X$ is solid.*
(ii) $g(Y) \leq g(X)$.
(iii) *If $\mu : \mathcal{A}_s(X) \to [0,1]$ is a solid set function, then $h^*\mu : \mathcal{A}_s(Y) \to [0,1]$ defined by $h^*\mu(A) = \mu(h^{-1}(A))$ is a solid set function.*
(iv) *If $\varrho : C(X) \to \mathbb{R}$ is a quasi-linear functional with corresponding topological measure μ, then $h^*\mu$ corresponds to the quasi-linear functional defined by $h^*\varrho(g) = \varrho(g \circ h)$.*

The verification of all parts of this proposition is easy since it is known that $h^{-1}(C)$ is connected when $C \subseteq Y$ is connected. It should be noted that the topological measure $h^*\mu$ can be defined for *any* function $h : X \to Y$ by setting $h^*\mu(A) = \nu(h^{-1}(A))$ for $A \in \mathcal{A}(X)$ where ν is the topological measure extending μ to all of $\mathcal{A}(X)$. It is not usually so easy to characterize $h^*\mu$ in terms of the solid set function μ. The difficulty in general is exactly that $h^{-1}(A)$ need not be solid when A is.

Now suppose that $f \in C(X)$ and μ is a solid set function on X. Recall that the space X/f is defined by collapsing components of level sets of f to points and that the corresponding quotient map is denoted by $\pi_f : X \to X/f$.

Proposition 18. *For a solid set function μ on X and $f \in C(X)$, there is a Borel measure ν on X/f that is defined via the solid set function $\nu(A) = \mu(\pi_f^{-1}A)$. For every $g \in C(X/f)$, we have*

$$\int_X g \circ \pi_f \, d\mu = \int_{X/f} g \, d\nu.$$

Proof. From the above, $\pi_f^*\mu$ is a solid set function on X/f. Moreover, the dimension $\dim X/f \leq 1$, so $\pi_f^*\mu$ extends to a *measure* on X/f. This measure is ν. ■

One of the unusual aspects of the subject of topological measures is the fact that it can be difficult to recognise a Borel measure when it is described via solid set functions. In particular, the measure ν in the previous result characterizes the quasi-linear functional associated with μ on the analytic subalgebra $B(f)$. In the special case that μ takes on only the values 0 and 1, ν must be a point mass. Locating that point mass in special cases will be one of our goals.

If we have a space X with $g(X) = 0$, the only partitions involved in the definition of solid set functions are those of the form $\{U, X \setminus U\}$. This drastically simplifies the construction of solid set functions, and hence the construction of topological measures. One technique for doing so is through the *supermeasures* [3]. Below, $\mathcal{P}(\{1, \ldots, n\})$ denotes the collection of all subsets of $\{1, \ldots, n\}$.

Definition 19. A supermeasure on the set $\{1, \ldots, n\}$ is a map $\mu : \mathcal{P}(\{1, \ldots, n\}) \to [0, 1]$ such that

(i) $\mu(A) + \mu(A^c) = 1$ for all A,
(ii) $\mu(A \cup B) \geq \mu(A) + \mu(B)$ if A and B are disjoint.

Notice the direction of the inequality in the last condition. Supermeasures are *super*additive on disjoint subsets of $\{1, \ldots, n\}$. Note also that $\mu(B) \leq \mu(A)$ whenever $B \subseteq A$.

Suppose that $g(X) = 0$ and $\mu : \mathcal{P}(\{1, \ldots, n\}) \to [0, 1]$ is a supermeasure. Let $\varphi : \{1, \ldots, n\} \to X$ be any function into X. Define the function $\varphi^*\mu$ on $\mathcal{A}_s(X)$ by setting $\varphi^*\mu(A) = \mu(\varphi^{-1}A)$ for $A \in \mathcal{A}_s(X)$. It is then easy to see that $\varphi^*\mu$ is a solid set function. By Aarnes' construction, this solid set function extends to a topological measure on X which we will also denote by $\varphi^*\mu$. The topological measures on X obtained from supermeasures in this way are said to be *finitely defined.* It is known that the collection of finitely defined topological measures is dense in $QP(X)$ when $g(X) = 0$ (see [3]).

Proposition 20. *Suppose that* $g(X) = 0$ *and* μ *is a* $\{0, 1\}$*-valued finitely defined topological measure determined by the points* $\{x_1, \ldots, x_n\}$*. Let* $f \in C(X)$*. Then* $\int f\,d\mu$ *is either the value of* f *at some* x_i *or the value of* f *at some component* D *of some level set of* f *such that* $X \setminus D$ *has at least three components.*

Proof. Write $\mu = \varphi^*\nu$ where ν is a supermeasure on $\{1, \ldots, n\}$ and $\varphi : \{1, \ldots, n\} \to X$. As above, with $\pi_f : X \to X/f$, the measure $\mu_f = \pi_f^*\mu = (\pi_f \circ \varphi)^*\nu$ is a point mass on X/f, say $\mu_f = \delta_p$ with $p \in X/f$. We claim that p is either one of the points in $\pi_f \circ \varphi(\{1,\ldots,n\})$ or a cut-point of X/f whose complement has at least three components.

In fact, suppose that p is not in the image of $\varphi_f = \pi_f \circ \varphi$ and that the complement of $\{p\}$ has at most two components, U and V. Then both U and V are open, solid sets with $\mu_f(U) = \delta_p(U) = 0 = \mu_f(V)$. Since $X \setminus V$ is solid with $\varphi_f^{-1}(U) = \varphi_f^{-1}(X \setminus V)$, we see from the solid set function that $\mu_f(X \setminus V) = \mu_f(U) = 0$. Since $\mu_f(V) = 0$, this shows that $\mu_f(X/f) = 0$, a contradiction. Finally, set $D = \pi_f^{-1}(p)$. ∎

As an immediate corollary, we have the following.

PROPOSITION 21. *Suppose that X is a smooth manifold with $g(X) = 0$, μ is as above and f is a smooth function on X. Then $\int_X f\,d\mu$ is either the value of f at some x_i or the value of f at some critical point. In the latter case, if the critical point is regular, then it is not a local extremum of f.*

This follows since the only way for a component of a level set of a smooth function to have more than two co-components is for the value of the function to be a critical point which is either non-regular or a saddle point.

We can now use these ideas to construct examples of non-trivial quasi-linear maps. To do this, fix a supermeasure μ on $\{1,\ldots,n\}$ and a function $\varphi : \{1,\ldots,n-1\} \to X$ where $g(X) = 0$. For each $x \in X$, define

$$\varphi_x(k) = \begin{cases} \varphi(k) & \text{if } k \in \{1,\ldots,n-1\}, \\ x & \text{if } k = n. \end{cases}$$

Then $\varphi_x : \{1,\ldots,n\} \to X$, and we may define the topological measure $\mu_x = \varphi_x^*\mu$ on X. In the case where μ is a $\{0,1\}$-valued supermeasure, each μ_x will be a $\{0,1\}$-valued topological measure on X.

We claim that the map $w : x \mapsto \mu_x$ is continuous from X to $QP(X)$. In fact, if $\widehat{U}_\alpha$ is a subbasic open set of $QP(X)$, then

$$w^{-1}\widehat{U}_\alpha = \{x : \mu_x(U) > \alpha\} = \{x : \mu(\varphi_x^{-1}(U)) > \alpha\}.$$

This last set is either X, U, or the empty set depending on the values of $\mu(\varphi^{-1}(U))$ and $\mu(\varphi^{-1}(U) \cup \{n\})$. In particular, it is the empty set if both values of μ are less than or equal to α; X if both are greater than α; and U otherwise. In any case, $w^{-1}\widehat{U}_\alpha$ is open.

We may now define a quasi-linear map $\varrho : C(X) \to C(X)$ by setting

$$\varrho(f)(x) = \int f\,d\mu_x.$$

The corresponding generalized image transformation is easily seen to be given by

$$k_U = \mu(\varphi^{-1}(U))\xi_{X\setminus U} + \mu(\varphi^{-1}(U) \cup \{n\})\xi_U.$$

Proposition 22. *In the above situation, $\eta \circ \varrho$ is quasi-linear for every quasi-linear map $\eta : C(X) \to C(Y)$. Moreover, for each $f \in C(X)$, $\varrho(f) \in B(f)$, the analytic subalgebra generated by f.*

Proof. The first statement follows from the second since if $g, h \in B(f)$, we have $\varrho(g), \varrho(h) \in B(f)$. Since η is linear on $B(f)$, $\eta \circ \varrho$ is also linear on $B(f)$ which contains $A(f)$. To show that $\varrho(f) \in B(f)$, we need to show that $\varrho(f)$ is constant on components of level sets of f. But

$$\varrho(f)(x) = \int_X f\, d\mu_x = \int_{X/f} \widehat{f}\, d\pi_f^* \mu_x.$$

Here, $\widehat{f}$ is the function on X/f such that $f = \widehat{f} \circ \pi_f$. Now, $\pi_f^* \mu_x$ is obtained from the supermeasure μ through the action of the function $\pi_f \circ \varphi_x$ which is only dependent on the component of the level set of f in which x lies. This completes the proof. ∎

In particular, define μ on $\mathcal{P}(\{1, 2, 3, 4, 5\})$ via

$$\mu(A) = \begin{cases} 0 & \text{if } \operatorname{card} A = 0 \text{ or } 1, \\ 1/2 & \text{if } \operatorname{card} A = 2 \text{ or } 3, \\ 1 & \text{if } \operatorname{card} A = 4 \text{ or } 5. \end{cases}$$

It is known that the finitely determined topological measures defined by μ are extreme points of $QP(X)$ for many spaces X. For example, if $X = [0, 1]^2$ is the unit square and $x_1, \ldots, x_5$ are points in the open square, this will be the case. Related examples can be found in [3].

Now, let $g(X) = 0$ and $\varphi : \{1, 2, 3, 4\} \to X$ take distinct values in X and consider the generalized image transform and quasi-linear map ϱ given above. Then, for U solid,

$$k_U = \begin{cases} 0 & \text{if } \operatorname{card}(U \cap \operatorname{im} \varphi) = 0, \\ (1/2)\xi_U & \text{if } \operatorname{card}(U \cap \operatorname{im} \varphi) = 1, \\ 1/2 & \text{if } \operatorname{card}(U \cap \operatorname{im} \varphi) = 2, \\ 1/2 + (1/2)\xi_U & \text{if } \operatorname{card}(U \cap \operatorname{im} \varphi) = 3, \\ 1 & \text{if } \operatorname{card}(U \cap \operatorname{im} \varphi) = 4. \end{cases}$$

Since k_U can take on values other than 0 or 1, ϱ is not a quasi-homomorphism even though $\eta \circ \varrho$ is quasi-linear for all quasi-linear η. This should be contrasted to the situation for products of quasi-measures noted earlier.

Finally, suppose that $\eta : C(X) \to \mathbb{R}$ is a positive quasi-linear functional with corresponding topological measure ν. We see that the topological measure associated with $\eta \circ \varrho$ is given on solid sets A by

$$\nu(A) = \begin{cases} 0 & \text{if } \operatorname{card}(A \cap \operatorname{im}\varphi) = 0, \\ (1/2)\nu(A) & \text{if } \operatorname{card}(A \cap \operatorname{im}\varphi) = 1, \\ 1/2 & \text{if } \operatorname{card}(A \cap \operatorname{im}\varphi) = 2, \\ 1/2 + (1/2)\nu(A) & \text{if } \operatorname{card}(A \cap \operatorname{im}\varphi) = 3, \\ 1 & \text{if } \operatorname{card}(A \cap \operatorname{im}\varphi) = 4. \end{cases}$$

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Department of Mathematical Sciences
Northern Illinois University
DeKalb, IL 60115, U.S.A.
E-mail: grubb@math.niu.edu

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