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An ordered structure of rank two related to Dulac's Problem

by

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Abstract. For a vector field ξ on $\mathbb{R}^2$ we construct, under certain assumptions on ξ, an ordered model-theoretic structure associated to the flow of ξ. We do this in such a way that the set of all limit cycles of ξ is represented by a definable set. This allows us to give two restatements of Dulac's Problem for ξ—that is, the question whether ξ has finitely many limit cycles—in model-theoretic terms, one involving the recently developed notion of $\mathrm{U}^{\mathrm{þ}}$-rank and the other involving the notion of o-minimality.

Introduction. Let $\xi = a_1 \frac{\partial}{\partial x} + a_2 \frac{\partial}{\partial y}$ be a vector field on $\mathbb{R}^2$ of class C^1, and let

$$S(\xi) := \{(x,y) \in \mathbb{R}^2 : a_1(x,y) = a_2(x,y) = 0\}$$

be the set of singularities of ξ. By the existence and uniqueness theorems for ordinary differential equations (see Camacho and Lins Neto [2, p. 28] for details), ξ induces a C^1-foliation $\mathcal{F}^\xi$ on $\mathbb{R}^2 \setminus S(\xi)$ of dimension 1. Abusing terminology, we simply call a leaf of this foliation a *leaf* of ξ. A *cycle* of ξ is a compact leaf of ξ; a *limit cycle* of ξ is a cycle L of ξ for which there exists a noncompact leaf L' of ξ such that L is contained in the closure of L'.

Dulac's Problem is the following statement: "if ξ is polynomial, then ξ has finitely many limit cycles". It is a weakening of the second part of Hilbert's 16th problem, which states that "there is a function $H : \mathbb{N} \to \mathbb{N}$ such that for all $d \in \mathbb{N}$, if ξ is polynomial of degree d then ξ has at most $H(d)$ limit cycles". Both problems have an interesting history, and while Dulac's Problem was independently settled in the 1990s by Écalle [4] and Il'yashenko [6], Hilbert's 16th Problem remains open; see [6] for more details.

In this paper, we attempt to reformulate Dulac's Problem in model-theoretic terms. Our motivation to do so is twofold: we want to

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(i) find a model-theoretic structure naturally associated to ξ in which the flow of ξ and the set of limit cycles of ξ are represented by definable sets;
(ii) know to what extent the geometry of such a structure is determined by Dulac's Problem.

Our starting point for (i) is motivated by the piecewise triviality of Rolle foliations associated to analytic 1-forms as described by Chazal [3]. Let $U \subseteq \mathbb{R}^2$ be open; a leaf L of $\xi|_U$ is a *Rolle leaf* of $\xi|_U$ if for every C^1-curve $\delta : [0,1] \to U$ with $\delta(0) \in L$ and $\delta(1) \in L$, there is a $t \in [0,1]$ such that $\delta'(t)$ is tangent to $\xi(\delta(t))$. Based on Khovanskiĭ theory [7] over an o-minimal expansion of the real field [14], we establish (Proposition 1.5 and Theorem 3.4):

THEOREM A. *Assume that ξ is definable in an o-minimal expansion of the real field. Then there is a cell decomposition $\mathcal{C}$ of $\mathbb{R}^2$ compatible with $S(\xi)$ such that*, *with* $\mathcal{C}_{\mathrm{reg}} := \{C \in \mathcal{C} : C \cap S(\xi) = \emptyset\}$,

(1) *every* 1*-dimensional $C \in \mathcal{C}_{\mathrm{reg}}$ is either transverse to ξ or tangent to ξ;*
(2) *for every open $C \in \mathcal{C}_{\mathrm{reg}}$, every leaf of $\xi|_C$ is a Rolle leaf of $\xi|_C$;*
(3) *for every open $C \in \mathcal{C}_{\mathrm{reg}}$, the flow of ξ in C is represented by a lexicographic ordering of C.*

Part (3) of this theorem needs some explanation, as it represents our understanding of the "triviality" of the flow of ξ in C. Given an open $C \in \mathcal{C}_{\mathrm{reg}}$, it follows from part (2) that the direction of ξ induces a linear ordering $<_\Gamma$ on every leaf L of $\xi|_C$. We can furthermore define a relation on the set $\mathcal{L}(C)$ of all leaves of $\xi|_C$ as follows: given a leaf L of $\xi|_C$, the fact that L is a Rolle leaf of $\xi|_C$ implies (see Remark 1.2 below) that L separates $C \setminus L$ into two connected components $U_{L,1}$ and $U_{L,2}$ such that the vector $\xi^\perp(z) := (a_2(z), -a_1(z))$ points into $U_{L,2}$ for all $z \in L$. Thus, for a leaf L' of $\xi|_C$ different from L, we define $L \ll_C L'$ if $L' \subseteq U_{L,2}$ and $L' \ll_C L$ if $L' \in U_{L,1}$. In general, though, the relation $\ll_C$ does not always define an ordering, even if every leaf of $\xi|_C$ is Rolle; see Example 2.2 below.

Part (3) now means that the cell decomposition $\mathcal{C}$ may be chosen in such a way that for every open $C \in \mathcal{C}_{\mathrm{reg}}$, the ordering $\ll_C$ on $\mathcal{L}(C)$ is a linear ordering. (See Example 3.2 for such a decomposition in the situation of Example 2.2.) This leads to lexicographic orderings as follows: given $C \in \mathcal{C}_{\mathrm{reg}}$ and $z \in C$, we denote by L_z the leaf of $\xi|_C$ containing z. If $C \in \mathcal{C}_{\mathrm{reg}}$ is open, we define a linear ordering $<_C$ on C by $x <_C y$ if and only if either $L_x \ll_C L_y$, or $L_x = L_y$ and $x <_{L_x} y$. Letting E_C be a set of representatives of $\mathcal{L}(C)$, it is not hard to see that the structures $(C, <_C, E_C)$ and $(\mathbb{R}^2, <_{\mathrm{lex}}, \{y = 0\})$ are isomorphic, where $<_{\mathrm{lex}}$ is the usual lexicographic ordering of $\mathbb{R}^2$.

To complete the picture, we also define an ordering $<_C$ on each 1-dimensional $C \in \mathcal{C}_{\mathrm{reg}}$: if C is tangent to ξ, we let $<_C$ be the linear ordering induced on C by the direction of ξ, and if C is transverse to ξ, we let $<_C$ be the linear ordering induced on C by the direction of $\xi^\perp$. For each open $C \in \mathcal{C}_{\mathrm{reg}}$, we also let $<_{E_C}$ be the restriction of $<_C$ to E_C. Each of these orderings induces a topology on the corresponding set that makes it homeomorphic to the real line. Finally, for each 1-dimensional $C \in \mathcal{C}_{\mathrm{reg}}$ tangent to ξ, we fix an element $e_C \in C$.

In the situation of Theorem A, we reconnect the pieces of $\mathcal{C}$ according to the flow of ξ as follows: let B be the union of

- all 1-dimensional cells in $\mathcal{C}_{\mathrm{reg}}$ transverse to ξ,
- the sets E_C for all open cells $C \in \mathcal{C}_{\mathrm{reg}}$,
- all 0-dimensional cells in $\mathcal{C}_{\mathrm{reg}}$,
- the singletons $\{e_C\}$ for all 1-dimensional $C \in \mathcal{C}_{\mathrm{reg}}$ tangent to ξ.

We define the *forward progression map* $\mathfrak{f} : B \cup \{\infty\} \to B \cup \{\infty\}$ by (roughly speaking) putting $\mathfrak{f}(x)$ equal to the next point in B on the leaf of ξ through x if $x \neq \infty$ and if such a point exists; otherwise we put $\mathfrak{f}(x) := \infty$. In this situation, a point $x \in B$ belongs to a cycle of ξ if and only if there is a nonzero $n \in \mathbb{N}$ such that $\mathfrak{f}^n(x) = x$, where $\mathfrak{f}^n$ denotes the nth iterate of $\mathfrak{f}$.

In fact, only finitely many iterates of $\mathfrak{f}$ are necessary to capture all cycles of ξ (Proposition 5.3): since a cycle of ξ is a Jordan curve in $\mathbb{R}^2$, it is a Rolle leaf of ξ and therefore intersects each $C \in \mathcal{C}$ of dimension at most 1 in at most one connected component. Hence there is an $N \in \mathbb{N}$ such that for all $x \in B$, x belongs to a cycle of ξ if and only if $\mathfrak{f}^N(x) = x$.

To see how we can use this to detect limit cycles of certain ξ, we first define a cycle L of ξ to be a *boundary cycle* if, for every $x \in L$ and every neighborhood V of x, the set V intersects some noncompact leaf of ξ. Boundary cycles and limit cycles are the same if ξ is real-analytic, because of the following theorem of Poincaré's [12] (see also Perko [11, p. 217]):

Fact 1. *If ξ is real-analytic, then ξ cannot have an infinite number of limit cycles that accumulate on a cycle of ξ.*

On the other hand, it follows from the previous paragraph that for every $x \in B$, the point x belongs to a boundary cycle of ξ if and only if x is in the boundary (relative to B considered with the topology induced on it by the various orderings defined above) of the set of all fixed points of $\mathfrak{f}^N$.

Based on the observations mentioned in the preceding paragraphs (and a few related observations), we associate to each decomposition $\mathcal{C}$ as in Theorem A a *flow configuration* $\Phi_\xi = \Phi_\xi(\mathcal{C})$ of ξ, intended to code how the cells in $\mathcal{C}$ are linked together by the flow of ξ. To each flow configuration Φ, we associate in turn a unique first-order language $\mathcal{L}(\Phi)$ in such a way that the situation described in the preceding paragraphs naturally yields an $\mathcal{L}(\Phi_\xi)$-structure $\mathcal{M}_\xi$ in which the lexicographic orderings of Theorem A, the associated forward progression map $\mathfrak{f} : B \cup \{\infty\} \to B \cup \{\infty\}$ and the set of all $x \in B$ that belong to some boundary cycle of ξ are definable.

If, in the situation of Theorem A, there is an open $C \in \mathcal{C}_{\mathrm{reg}}$, then the induced structure on C in $\mathcal{M}_\xi$ is not o-minimal (because the structure $(C, <_C, E_C)$ described above is definable in $\mathcal{M}_\xi$). Thus, to answer (ii) we need to work with notions weaker than o-minimality. A weakening that includes lexicographic orderings is provided by the *rosy* theories introduced by Onshuus [9].

To recall this rather technical definition, we fix a complete first order theory T and a sufficiently saturated model $\mathcal{M}$ of T, and we work in $\mathcal{M}^{\mathrm{eq}}$. (For standard model-theoretic terminology, we refer the reader to Marker [8].) The definition of þ-forking is much like that of forking in the stable or simple context: A formula $\phi(x, a)$ *strongly divides* over a set A if $\mathrm{tp}(a/A)$ is nonalgebraic and the set $\{\phi(x, b) : b \models \mathrm{tp}(a/A)\}$ is k-inconsistent for some $k \in \mathbb{N}$. The formula $\phi(x, a)$ *þ-divides* over A if for some tuple c, $\phi(x, a)$ strongly divides over $A \cup \{c\}$. The formula $\phi(x, a)$ *þ-forks* over A if $\phi(x, a)$ implies a finite disjunction of formulas all of which þ-divide over A. A complete type $p(x)$ *þ-forks* over A if there is some formula $\phi(x)$ in $p(x)$ that þ-forks over A.

For a theory T to be rosy means, roughly speaking, that in models of T, þ-forking has many desirable properties, much like forking in the stable or simple contexts. For the formal definition we need only focus on a single one of these: T is *rosy* if for any complete type $p(x)$ over a parameter set B, there exists $B_0 \subseteq B$ with $\|B_0\| \le \|T\|$ such that $p(x)$ does not þ-fork over B_0.

The "degree of rosiness" of a theory is measured by the $\mathrm{U}^{\text{þ}}$-rank, defined analogously to the U-rank in stable theories. For an ordinal α and a complete type $p(x)$ with parameter set A, we define $\mathrm{U}^{\text{þ}}(p) \ge \alpha$ by ordinal induction:

(i) $\mathrm{U}^{\text{þ}}(p) \ge 0$ if p is consistent;
(ii) if α is a limit ordinal, then $\mathrm{U}^{\text{þ}}(p) \ge \alpha$ if $\mathrm{U}^{\text{þ}}(p) \ge \beta$ for all $\beta < \alpha$;
(iii) $\mathrm{U}^{\text{þ}}(p) \ge \alpha+1$ if there is a complete type $q(x)$ so that $p \subseteq q$, q þ-forks over A and $\mathrm{U}^{\text{þ}}(q) \ge \alpha$.

For an ordinal α, we say that $\mathrm{U}^{\text{þ}}(p) = \alpha$ if $\mathrm{U}^{\text{þ}}(p) \ge \alpha$ and $\mathrm{U}^{\text{þ}}(p) \not\ge \alpha + 1$. Finally, $\mathrm{U}^{\text{þ}}(T)$ is defined to be the supremum of $\mathrm{U}^{\text{þ}}(p)$ for all one-types p with parameters over the empty set. One of the fundamental facts about rosy theories is that T is rosy if $\mathrm{U}^{\text{þ}}(T)$ is an ordinal [9].

For example, every o-minimal theory is rosy of $\mathrm{U}^{\text{þ}}$-rank one. On the other hand, the theory T of the structure $(C, <_C, E_C)$ above has $\mathrm{U}^{\text{þ}}$-rank at least two. To see the latter, let $\mathcal{M} \models T$ be $\aleph_1$-saturated and write $C_z := \{x \in C : z_1 <_C x <_C z_2$ for all $z_1, z_2 \in E_C$ such that $z_1 <_C z <_C z_2\}$. Since $E_C^{\mathcal{M}}$ is a dense linear ordering without endpoints, there are infinitely many $a \in E_C^{\mathcal{M}}$ such that $a \notin \mathrm{acl}(\emptyset)$. For any two such $a, b \in E_C^{\mathcal{M}}$, the fibers $C_a^{\mathcal{M}}$ and $C_b^{\mathcal{M}}$ are disjoint, infinite definable sets. Hence $\mathrm{U}^{\text{þ}}(\mathcal{M}) \geq 2$.

In this paper, we use the argument of the previous example to establish lower bounds on $\mathrm{U}^{\text{þ}}$-rank for the theories we are interested in. For upper bounds, we need a special case of the Coordinatization Theorem [10, Theorem 2.2.2]:

Fact 2. *Assume that T defines a dense linear ordering without endpoints, and let $\mathcal{M} \models T$ be saturated. Let also $n \in \mathbb{N}$ and assume that for all $a \in M$, there are $a_1, \ldots, a_n \in M$ such that $a = a_n$ and for each $i \in \{1, \ldots, n\}$, the type of $(a_1, \ldots, a_i)$ over $(a_1, \ldots, a_{i-1})$ is implied in T by the order type of $(a_1, \ldots, a_i)$ over $(a_1, \ldots, a_{i-1})$. Then $\mathrm{U}^{\text{þ}}(T) \leq n$.*

Note that our discussion above and the previous example imply that $\mathrm{U}^{\text{þ}}(\mathcal{M}_\xi) \geq 2$. The main result of this paper is the following restatement of Dulac's Problem:

Theorem B. *Assume that ξ is definable in an o-minimal expansion of the real field, and let $\mathcal{M}_\xi$ be the $\mathcal{L}(\Phi_\xi)$-structure associated to some flow configuration Φ_ξ of ξ. Then:*

(1) *ξ has finitely many boundary cycles if and only if $\mathrm{U}^{\text{þ}}(\mathcal{M}_\xi) = 2$;*
(2) *if ξ is real-analytic, then ξ has finitely many limit cycles if and only if $\mathrm{U}^{\text{þ}}(\mathcal{M}_\xi) = 2$.*

The proof of Theorem B is lengthy, but straightforward: we prove that $\mathcal{M}_\xi$ admits quantifier elimination in a certain expanded language (Theorem 9.11). The main ingredient in this proof is a reduction—modulo the theory of $\mathcal{M}_\xi$ in the expanded language, roughly speaking—of general quantifier-free formulas to certain quantifier-free order formulas, which allows us to deduce the quantifier elimination for $\mathcal{M}_\xi$ from quantifier elimination of the theory of $(\mathbb{R}^2, <_{\mathrm{lex}}, \{y = 0\}, \pi)$, where $\pi : \mathbb{R}^2 \to \{y = 0\}$ is the canonical projection on the x-axis. Under the assumption of having only finitely many boundary cycles, the new predicates of the expanded language are easily seen to define subsets of the various cells obtained by Theorem A that are finite unions of points and intervals. Sufficiency in Theorem B then follows from Fact 2; necessity follows by general $\mathrm{U}^{\text{þ}}$-rank arguments.

As a corollary of Theorem B, Écalle's and Il'yashenko's solutions of Dulac's Problem imply the following:

Corollary. *Assume ξ is polynomial, and let $\mathcal{M}_\xi$ be the $\mathcal{L}(\Phi_\xi)$-structure associated to some flow configuration Φ_ξ of ξ. Then $\mathrm{U}^{\text{þ}}(\mathcal{M}_\xi) = 2$.*

It remains an open question whether, in the situation of the Corollary, the structures are definable in some o-minimal expansion of the real line.

An answer to this question, however, seems to go far beyond our current knowledge surrounding Dulac's Problem.

Finally, our proof of Theorem B gives rise to a second restatement of Dulac's Problem that does not involve $\mathrm{U}^{\flat}$-rank: Let G be the union of all 1-dimensional $C \in \mathcal{C}_{\mathrm{reg}}$ that are transverse to ξ, all 0-dimensional $C \in \mathcal{C}_{\mathrm{reg}}$ and $\{\infty\}$. Let $\mathcal{G}_\xi$ be the expansion of G by all corresponding orderings $<_C$ and by the map $\mathfrak{f}^2|_G$. (Note that $\mathfrak{f}^2|_G$ maps G into G.) We may view $\mathcal{G}_\xi$ as a graph whose vertices are the elements of G and whose edges are defined by $\mathfrak{f}^2$.

THEOREM C. *Assume that ξ is definable in an o-minimal expansion of the real field, and let $\mathcal{G}_\xi$ be as above. Then:*

(1) *ξ has finitely many boundary cycles if and only if the structure induced by $\mathcal{G}_\xi$ on each 1-dimensional $C \subseteq G$ is o-minimal;*
(2) *if ξ is real-analytic, then ξ has finitely many limit cycles if and only if the structure induced by $\mathcal{G}_\xi$ on each 1-dimensional $C \subseteq G$ is o-minimal.*

Our paper is organized as follows: in Sections 1–3, we establish Theorem A: In Section 1, we combine basic o-minimal calculus with Khovanskiĭ's Lemma to obtain a cell decomposition satisfying (1) and (2) of Theorem A. To refine this decomposition so that (3) holds, we need to study what sets we obtain as Hausdorff limits of a sequence of leaves of $\xi|_C$ (Proposition 2.5). The refinement is then given in Section 3, where (3) is established as Theorem 3.4. In Sections 4 and 5, we define the relevant orderings and progression maps associated to ξ as mentioned earlier. Inspired by the latter, we then introduce the notion of a flow configuration and the associated first-order language in Section 6, where we also give an axiomatization of the crucial properties satisfied by the models $\mathcal{M}_\xi$ above. Some basic facts about the iterates of the forward progression map are deduced from these axioms in Section 7. In Section 8, we extend our axioms to reflect the additional assumption that there are only finitely many boundary cycles, and we introduce additional predicates for certain definable sets related to the sets of fixed points of the iterates of the forward progression map. The quantifier elimination result is then given in Section 9, and we prove Theorems C and B in Section 10. We finish with a few questions and remarks in Section 11.

GLOBAL CONVENTIONS. We fix an o-minimal expansion $\mathcal{R}$ of the real field; "definable" means "definable in $\mathcal{R}$ with parameters".

For $1 \le m \le n$, we denote by $\Pi_m : \mathbb{R}^n \to \mathbb{R}^m$ the projection on the first m coordinates.

Given $(x, y) \in \mathbb{R}^2$, we put $(x, y)^\perp := (y, -x)$.

For a subset $A \subseteq \mathbb{R}^n$, we let $\mathrm{cl}(A)$, $\mathrm{int}(A)$, $\mathrm{bd}(A) := \mathrm{cl}(A) \setminus \mathrm{int}(A)$ and $\mathrm{fr}(A) := \mathrm{cl}(A) \setminus A$ denote the topological closure, interior, boundary and frontier, respectively.

For $n \in \mathbb{N}$, we define the analytic diffeomorphism $\phi_n : \mathbb{R}^n \to (-1,1)^n$ by $\phi_n(x_1, \ldots, x_n) := (x_1/\sqrt{1+x_1^2}, \ldots, x_n/\sqrt{1+x_n^2})$. Given $X \subseteq \mathbb{R}^n$, we write $X^* := \phi_n(X)$, and given a vector field η on $\mathbb{R}^n$ of class C^1, we write η^* for the push-forward $(\phi_n)_* \eta$ of η to $(-1,1)^n$.

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1. Rolle decomposition. Let $U \subseteq \mathbb{R}^2$ be open and $p \geq 1$ be an integer. Let $\xi = a_1 \frac{\partial}{\partial x} + a_2 \frac{\partial}{\partial y}$ be a definable vector field on U of class C^p (that is, the functions $a_1, a_2 : U \to \mathbb{R}$ are definable and of class C^p), and let

$$S(\xi) := \{z \in U : a_1(z) = a_2(z) = 0\}$$

be the set of singularities of ξ. By the existence and uniqueness theorems for ordinary differential equations [2, p. 28], ξ induces a C^p-foliation $\mathcal{F}^\xi$ on $U \setminus S(\xi)$ of dimension 1. Abusing terminology, we simply call a leaf of this foliation a *leaf of* ξ.

Remark. Put $\omega := a_2 dx - a_1 dy$; then $S(\xi)$ is the set of singularities of ω, and the foliation $\mathcal{F}^\xi$ is exactly the foliation on $U \setminus S(\xi)$ defined by the equation $\omega = 0$. Below, we will use this observation (mainly in connection with some citations) without further mention.

Definition 1.1. Let $\gamma : I \to U$ be of class C^p, where $I \subseteq \mathbb{R}$ is an interval. We call γ a *C^p-curve in U* and usually write $\Gamma := \gamma(I)$. If $t \in I$ is such that $\xi^\perp(\gamma(t)) \cdot \gamma'(t) \neq 0$, we say that γ is *transverse to ξ at t*; otherwise, γ is *tangent to ξ at t*. The curve γ is *transverse* (*tangent*) *to* ξ if γ is transverse (tangent) to ξ at every $t \in I$.

A leaf L of ξ is a *Rolle leaf of* ξ if for every C^1-curve $\gamma : [0,1] \to U$ with $\gamma(0) \in L$ and $\gamma(1) \in L$, there is a $t \in [0,1]$ such that $\xi^\perp(\gamma(t)) \cdot \gamma'(t) = 0$.

A *cycle* of ξ is a compact leaf of ξ. A cycle L of ξ is a *limit cycle* of ξ if there is a noncompact leaf L' of ξ such that $L \subseteq \mathrm{cl}(L')$. A cycle L of ξ is a *boundary cycle* of ξ if for every open set $V \subseteq \mathbb{R}^2$ with $V \cap L \neq \emptyset$, there is a noncompact leaf L' of ξ such that $V \cap L' \neq \emptyset$.

Remark 1.2. Since ξ is integrable in $U \setminus S(\xi)$, every Rolle leaf L of ξ is an embedded submanifold of $U \setminus S(\xi)$ that is closed in $U \setminus S(\xi)$. In particular, by Theorem 4.6 and Lemma 4.4 of Chapter 4 in [5], if $U \setminus S(\xi)$ is simply connected, then $U \setminus (S(\xi) \cup L)$ has exactly two connected components such that L is equal to the boundary in $U \setminus S(\xi)$ of each of these components.

LEMMA 1.3 (Khovanskiĭ [7]).

(1) *Assume that* $U \setminus S(\xi)$ *is simply connected, and let* $L \subseteq U \setminus S(\xi)$ *be an embedded leaf of* ξ *that is closed in* $U \setminus S(\xi)$. *Then* L *is a Rolle leaf of* ξ *in* U.

(2) *Let* L *be a cycle of* ξ. *Then* L *is a Rolle leaf of* ξ.

Sketch of proof. (1) Arguing as in the preceding remark, we see that the set $U \setminus S(\xi)$ has exactly two connected components U_1 and U_2, such that $\mathrm{bd}(U_i) \cap (U \setminus S(\xi)) = L$ for $i = 1, 2$. The argument of Example 1.3 in [14] now shows that L is a Rolle leaf of ξ.

(2) Since L is compact, L is an embedded and closed submanifold of $\mathbb{R}^2$. Now conclude as in part (1). ∎

DEFINITION 1.4. We call ξ *Rolle* if $S(\xi) = \emptyset$, ξ is of class C^1 and every leaf of ξ is a Rolle leaf of ξ.

We now let $\mathcal{C}$ be a C^p-cell decomposition of $\mathbb{R}^2$ compatible with U and $S(\xi)$, and we put $\mathcal{C}_U := \{C \in \mathcal{C} : C \subseteq U\}$. Refining $\mathcal{C}$, we may assume that $\xi|_C$ is of class C^p for every $C \in \mathcal{C}_U$, and that every $C \in \mathcal{C}_U$ of dimension 1 is either tangent or transverse to ξ. Refining $\mathcal{C}$ again, we also assume that

(I) a_1 and a_2 have constant sign on every $C \in \mathcal{C}_U$.

Such a decomposition $\mathcal{C}$ is called a *Rolle decomposition for* ξ, because of the following:

PROPOSITION 1.5. *Let* $C \in \mathcal{C}_U$ *be open such that* $C \cap S(\xi) = \emptyset$. *Then* $\xi|_C$ *is Rolle. Moreover, if both* a_1 *and* a_2 *have nonzero constant sign on* C, *then either every leaf of* $\xi|_C$ *is the graph of a strictly increasing* C^p*-function* $f : I \to \mathbb{R}$, *or every leaf of* $\xi|_C$ *is the graph of a strictly decreasing* C^p*-function* $f : I \to \mathbb{R}$, *where* $I \subseteq \mathbb{R}$ *is an open interval depending on* f.

Proof. If $a_1|_C = 0$ or $a_2|_C = 0$, the conclusion is obvious. So we assume that $a_1|_C$ and $a_2|_C$ have constant positive sign, say; the remaining three cases are handled similarly. Let L be a leaf of $\xi|_C$; we claim that L is the graph of a strictly increasing C^p-function $f : I \to \mathbb{R}$, where $I := \Pi_1(L)$.

To see this, assume first that there are $x, y_1, y_2 \in \mathbb{R}$ such that $(x, y_i) \in L$ for $i = 1, 2$ and $y_1 \neq y_2$. Since $\xi|_C$ is of class C^p, the leaf L is a C^p-curve, so by Rolle's Theorem, there is an $a \in L$ such that L is tangent at a to $\partial/\partial y$. But this means that $a_1(a) = 0$, a contradiction. Thus, L is the graph of a strictly increasing C^p-function $f : I \to \mathbb{R}$.

It follows from the claim that L is an embedded submanifold of C and, since $C \cap S(\xi) = \emptyset$, that L is a closed subset of C. Thus by Lemma 1.3(1), L is a Rolle leaf of $\xi|_C$. ∎

2. Rolle foliations and Hausdorff limits of Rolle leaves. We continue working with ξ as in Section 1, and we fix a Rolle decomposition $\mathcal{C}$ for ξ. We fix an open $C \in \mathcal{C}_U$ such that $C \cap S(\xi) = \emptyset$.

To simplify notation, we write ξ in place of $\xi|_C$ throughout this section.

Let L be a leaf of ξ. Since L is a Rolle leaf of ξ, $C \setminus L$ has two connected components $U_{L,1}$ and $U_{L,2}$, and L is the boundary of $U_{L,i}$ in C for $i = 1, 2$. Since $\xi^\perp(z) \neq (0,0)$ for all $z \in C$ and L is connected, there is an $i \in \{1,2\}$ such that $\xi^\perp(z)$ points inside $U_{L,i}$ for all $z \in L$; reindexing if necessary, we may assume that $\xi^\perp(z)$ points inside $U_{L,2}$ for every leaf L of ξ.

Definition 2.1. For a point $z \in C$, we let L_z^ξ be the unique leaf of ξ such that $z \in L_z^\xi$. For any subset $X \subseteq C$, we define

$$F^\xi(X) := \bigcup_{z \in X} L_z^\xi,$$

called the *ξ-saturation of* X, and we put

$$\mathcal{L}^\xi(X) := \{L_z^\xi : z \in X\}.$$

For $X \subseteq C$, we define a relation $\ll_X^\xi$ on the set $\mathcal{L}^\xi(X)$ as follows: $L \ll_X^\xi M$ if and only if $L \subseteq U_{M,1}$ (if and only if $M \subseteq U_{L,2}$).

Whenever ξ is clear from context, we omit "ξ" in the definitions and notations above.

Note that in general the relation $\ll_C$ may not define an order relation on $\mathcal{L}(C)$:

Example 2.2. Let $\zeta := -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$, and let $g : \mathbb{R}^2 \to \mathbb{R}$ be defined by $g(x,y) := (y - (x-2))^2$. Then $g\zeta$ is a real-analytic vector field on $\mathbb{R}^2$ and $S(g\zeta) = \{0\} \cup \{(x,y) : y = x - 1\}$. Let also C be the cell (α, β), where $\alpha, \beta : (0,1) \to \mathbb{R}$ are defined by $\alpha(x) := x - 2$ and $\beta(x) := x - 1$.

Then $C \cap S(g\zeta) = \emptyset$, and since every leaf of ζ is a Rolle leaf of ζ, the vector field $g\zeta|_C$ is Rolle. However, $\ll_C^{g\zeta}$ is not an ordering of $\mathcal{L}(C)$: Pick a leaf L of ξ (that is, a circle with center $(0,0)$) such that $L \cap \mathrm{gr}(\alpha)$ contains two points. Then $L \cap C$ consists of two distinct leaves L_1 and L_2 of $g\zeta|_C$. Since $\zeta^\perp(z)$ points outside the circle L for every $z \in L$, we get $L_1 \subseteq U_{L_2,1}$ and $L_2 \subseteq U_{L_1,1}$, that is, $L_1 \ll_C^{g\zeta} L_2$ and $L_2 \ll_C^{g\zeta} L_1$.

However, for certain X the relation $\ll_X$ is a linear ordering of $\mathcal{L}(X)$, as discussed in the following lemma. For a curve $\gamma : I \to C$, we write

$$L(t) := L_{\gamma(t)} \quad \text{for all } t \in I;$$

in this situation, we have $F(\Gamma) = \bigcup_{t \in I} L(t)$.

Lemma 2.3. *Let $\gamma : I \to C$ be a C^p-curve transverse to ξ, where $I \subseteq \mathbb{R}$ is an interval.*

(1) *If I is open, then $F(\Gamma)$ is open.*
(2) *The relation $\ll_\Gamma$ is a linear ordering of $\mathcal{L}(\Gamma)$, and the map $t \mapsto L(t) : I \to \mathcal{L}(\Gamma)$ is order-preserving if $\xi^\perp(\gamma(t)) \cdot \gamma'(t) > 0$ for all $t \in I$ and order-reversing if $\xi^\perp(\gamma(t)) \cdot \gamma'(t) < 0$ for all $t \in I$.*

Proof. (1) Assume that I is open, and let $t \in I$. Because ξ is C^p and nonsingular and γ is transverse to ξ, by a variant of Picard's Theorem (see Theorem 8-2 of [1]), there is an open set $B_t \subseteq C$ containing $\gamma(t)$ such that $B_t \subseteq F(\Gamma)$. Put $B := \bigcup_{t\in I} B_t$; then $\Gamma \subseteq B \subseteq F(\Gamma)$, so $F(\Gamma) = F(B)$. Since B is open, it follows from Theorem III.1 in [2] that $F(\Gamma)$ is open.

(2) Since γ is transverse to ξ and each $L(t)$ is Rolle, the map $t \mapsto L(t) : I \to \mathcal{L}(\Gamma)$ is injective. It therefore suffices to show that either

$$s < t \Leftrightarrow L(s) \ll_\Gamma L(t) \quad \text{for all } s, t \in I,$$

or

$$s < t \Leftrightarrow L(t) \ll_\Gamma L(s) \quad \text{for all } s, t \in I.$$

Since γ is transverse to ξ, the continuous map $t \mapsto \xi^\perp(\gamma(t)) \cdot \gamma'(t) : I \to \mathbb{R}$ has constant positive or negative sign. Assume it has constant positive sign; the case of constant negative sign is handled similarly. Then for every $t \in I$, the set

$$\Gamma_{<t} := \{\gamma(s) : s \in I,\ s < t\}$$

is contained in $U_{L(t),1}$. Hence $L(s) \subseteq U_{L(t),1}$ for all $s \in I$ with $s < t$, that is, $L(s) \ll_\Gamma L(t)$ for all $s \in I$ with $s < t$. Similarly, $L(t) \ll_\Gamma L(s)$ for all $s \in I$ with $s > t$, and since $t \in I$ was arbitrary, the lemma follows. ∎

We assume for the rest of this section that C is *bounded*. Let ξ_C be the 1-form on C defined by

$$\xi_C := \frac{\xi|_C}{\|\xi|_C\|}.$$

Then ξ_C is a bounded, definable C^p-map on C, so by o-minimality, there is a finite set $F_C \subseteq \mathrm{fr}(C)$ such that ξ_C extends continuously to $\mathrm{cl}(C) \setminus F_C$; we denote this continuous extension by ξ_C as well.

Let $c, d \in \mathbb{R}$ and $\alpha, \beta : (c, d) \to \mathbb{R}$ be definable and C^p such that $C = (\alpha, \beta)$. Because C is bounded, the limits $\alpha(c) := \lim_{x\to c} \alpha(x)$, $\alpha(d) := \lim_{x\to d} \alpha(x)$, $\beta(c) := \lim_{x\to c} \beta(x)$ and $\beta(d) := \lim_{x\to d} \beta(x)$ exist in $\mathbb{R}$. The points of the set

$$V_C := \{(c, \alpha(c)), (d, \alpha(d)), (c, \beta(c)), (d, \beta(d))\}$$

are called the *corners* of C.

EXAMPLE 2.4. In Example 2.2, we have $F_C \subseteq V_C$ and both $g\zeta \cdot (\partial/\partial x)$ and $g\zeta \cdot (\partial/\partial y)$ have constant nonzero sign. The next proposition shows that under the latter assumptions, the situation of Example 2.2 is as bad as it gets.

Proposition 2.5. *Suppose that $F_C \subseteq V_C$, $a_1|_C \neq 0$ and $a_2|_C \neq 0$. Let $\gamma : [0,1] \to C$ be a C^p-curve transverse to ξ, and let $t_i \in (0,1)$ be such that $t_0 < t_1 < t_2 < \cdots$ and $t_i \to 1$. Then the sequence $(\mathrm{cl}(L(t_i)))$ converges in the Hausdorff metric to a compact set $K := \lim \mathrm{cl}(L(t_i)) \subseteq \mathrm{cl}(C)$ such that*

(i) *$\Pi_1(K) = [a,b]$ with $c \leq a < b \leq d$;*
(ii) *each component of $K \cap C$ is a leaf of ξ;*
(iii) *$K \cap \Pi_1^{-1}(a,b) = \mathrm{gr}(f)$ for some continuous function $f : (a,b) \to \mathbb{R}$.*

Proof. By Proposition 1.5, we may assume that for every $t \in [0,1]$, the leaf $L(t)$ is the graph of a strictly increasing C^p-function $f_t : (a(t), b(t)) \to \mathbb{R}$ (the other cases are handled similarly). Since C is bounded, the limits $f_t(a(t)) := \lim_{x \to a(t)} f_t(x)$ and $f_t(b(t)) := \lim_{x \to b(t)} f_t(x)$ exist, and we also denote by $f_t : [a(t), b(t)] \to \mathbb{R}$ the corresponding continuous extension of f_t. Then $\mathrm{cl}(L(t)) = \mathrm{gr}(f_t)$. By Lemma 2.3, we may also assume that the map $t \mapsto L(t) : [0,1] \to \mathcal{L}(\Gamma)$ is order-preserving (again, the other case is handled similarly). Finally, since each f_t is strictly increasing and the map $t \mapsto L(t) : [0,1] \to \mathcal{L}(\Gamma)$ is order-preserving, it follows that $f_s(x) > f_t(x)$ for all $s, t \in [0,1]$ such that $s < t$ and $x \in (a(s), b(s)) \cap (a(t), b(t))$.

Since each $\mathrm{cl}(L(t_i))$ is connected, the set K is connected, so $\Pi_1(K)$ is an interval $[a,b]$, which proves (i). It follows in particular that for every $x \in (a,b)$, there is an open interval $I_x \subseteq (a,b)$ containing x such that $I_x \subseteq (a(t_i), b(t_i))$ for all sufficiently large i. Thus by our assumptions,

($*$) for every $x \in (a,b)$ we have $f_{t_i}|_{I_x} > f_{t_{i+1}}|_{I_x}$ for sufficiently large i.

Next, we show that $K \cap C$ is an integral manifold of ξ. Fix a point $(x,y) \in K \cap C$; it suffices to show that there is an open box $B \subseteq C$ containing (x,y) such that $K \cap B$ is an integral manifold of ξ. Let $B = I \times J$ be an open box containing (x,y) such that $I \subseteq I_x$. Since $a_1(x,y) \neq 0$, we may also assume (after shrinking B) that there is an $\varepsilon > 0$ such that $|a_1(x', y')| \geq \varepsilon$ for all $(x', y') \in B$; in particular, there is an $M > 0$ such that $f_{t_i}|_I$ is M-Lipschitz for all sufficiently large i. Hence by ($*$), the function $f : I \to \mathbb{R}$ defined by $f(x') := \lim_{i \to \infty} f_{t_i}(x')$ is Lipschitz and satisfies $K \cap (I \times \mathbb{R}) = K \cap B = \mathrm{gr}(f)$. Finally, shrinking B again if necessary, we see that $\mathcal{F}^\xi$ being a foliation implies that $K \cap B$ is an integral manifold of ξ, as required.

Since K is compact and $K \cap C$ is an integral manifold of ξ, every component of $K \cap C$ is a leaf of ξ. It also follows from the previous paragraph that $K \cap C$ is the graph of a continuous function $g : \Pi_1(K \cap C) \to \mathbb{R}$, which proves (ii).

Let now $x \in (a,b)$ be such that $x \notin \Pi_1(K \cap C)$. Then $(x, \alpha(x))$ or $(x, \beta(x))$ belongs to K, because $(a,b) \subseteq \Pi_1(K)$; by ($*$) we have $(x, \beta(x)) \notin K$, so $(x, \alpha(x)) \in K$. If $\left(\xi_C \cdot \frac{\partial}{\partial x}\right)(x, \alpha(x)) \neq 0$, then by the same arguments as used for (ii), we conclude that there are open intervals $I, J \subseteq \mathbb{R}$ such that

$(x, \alpha(x)) \in I \times J$ and $K \cap (I \times J)$ is the graph of a continuous function defined on I. Therefore, part (iii) is proved once we show that $\left(\xi_C \cdot \frac{\partial}{\partial x}\right)(x, \alpha(x)) \neq 0$ for all $x \in (a, b) \setminus \Pi_1(K \cap C)$.

Assume for a contradiction that there is an $x \in (a, b) \setminus \Pi_1(K \cap C)$ such that $\left(\xi_C \cdot \frac{\partial}{\partial x}\right)(x, \alpha(x)) = 0$. Let $M > |\alpha'(x)|$, and let $I, J \subseteq \mathbb{R}$ be open intervals such that $I \subseteq I_x$ and $|a_2/a_1| > M$ on $B := I \times J$. Since $f_{t_i}(x) \to \alpha(x)$, it follows from the fundamental theorem of calculus for all sufficiently large i that $f_{t_i}(x_i) = \alpha(x_i)$ for some $x_i \in I$, a contradiction. ■

3. Piecewise trivial decomposition. We continue working with ξ as in Section 1, and we adopt the notations used there. Note that ξ^* (as defined at the end of the Introduction) is a definable vector field on U^* of class C^p, and that $\mathcal{C}$ is a Rolle decomposition of $\mathbb{R}^2$ for ξ if and only if $\mathcal{C}^* := \{C^* : C \in \mathcal{C}\}$ is a Rolle decomposition of $(-1, 1)^2$ for ξ^*.

Let $C \subseteq U$ be a bounded, open, definable C^p-cell such that $\xi|_C$ is Rolle. To detect situations like the one described in Example 2.2, we associate the following notations to such a C. There are real numbers $c < d$ and definable C^p-functions $\alpha, \beta : (c, d) \to \mathbb{R}$ such that $C = (\alpha, \beta)$. Given a C^1-function $\delta : (c, d) \to \mathbb{R}$ such that $\alpha(x) \leq \delta(x) \leq \beta(x)$ for all $x \in (c, d)$, we define $\sigma_\delta : C \to \mathbb{R}$ by

$$\sigma_\delta(x, y) := \xi^\perp(x, y) \cdot \begin{pmatrix} 1 \\ \delta'(x) \end{pmatrix}.$$

Note that for each $x \in (c, d)$, there are by o-minimality a maximal $\alpha_0^C(x) \in (\alpha(x), \beta(x)]$ and a minimal $\beta_0^C(x) \in [\alpha(x), \beta(x))$ such that the function σ_α has constant sign on $\{x\} \times (\alpha(x), \alpha_0^C(x))$ and the function σ_β has constant sign on $\{x\} \times (\beta_0^C(x), \beta(x))$; we omit the superscript "C" whenever C is clear from context. Note that $\alpha_0, \beta_0 : (c, d) \to \mathbb{R}$ are definable.

Definition 3.1. A C^p-cell decomposition of $\mathbb{R}^2$ compatible with U, $\mathrm{bd}(U)$ and $S(\xi)$ is called *almost piecewise trivial for* ξ if

(I) every $C \in \mathcal{C}_U$ of dimension 1 is either tangent or transverse to ξ;
(II) the components of ξ have constant sign on every $C \in \mathcal{C}_U$;

and for every open, bounded $C \in \mathcal{C}_U$ such that $C \cap S(\xi) = \emptyset$, the following hold:

(III) $F_C \subseteq V_C$;
(IV) the maps $\alpha_0, \beta_0 : (c, d) \to \mathbb{R}$ are continuous;
(V) the map σ_α has constant sign on the cell (α, α_0), and the map σ_β has constant sign on the cell (β_0, β).

We call $\mathcal{C}$ *piecewise trivial for* ξ if $\mathcal{C}^*$ is almost piecewise trivial for ξ^*.

EXAMPLE 3.2. Let $\zeta := -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$, and let $\mathcal{C}$ be the cell decomposition of $\mathbb{R}^2$ consisting of the sets of the form $\{(x,y) : x * 0,\ y \star 0\}$ with $*, \star \in \{=,<,>\}$. Then $\mathcal{C}$ is piecewise trivial for ζ.

REMARKS 3.3.

(1) Any piecewise trivial decomposition for ξ is a Rolle decomposition for ξ.
(2) If U is bounded, then $\mathcal{C}$ is almost piecewise trivial for ξ if and only if $\mathcal{C}$ is piecewise trivial for ξ.
(3) We obtain a piecewise trivial decomposition for ξ in the following way: First, obtain a C^p-cell decomposition $\mathcal{C}$ compatible with U, $\mathrm{bd}(U)$ and $S(\xi)$ satisfying (I) and (II). Then, to satisfy (III)–(V), we only need to refine $\Pi_1(\mathcal{C}) := \{\Pi_1(C) : C \in \mathcal{C}\}$.

We now fix a piecewise trivial decomposition $\mathcal{C}$ of $\mathbb{R}^2$ for ξ. The name "piecewise trivial" is justified by:

THEOREM 3.4. *Let $C \in \mathcal{C}_U$ be open such that $C \cap S(\xi) = \emptyset$. Then the relation $\ll_C$ on $\mathcal{L}(C)$ is a linear ordering.*

To prove the theorem, we fix a *bounded*, open $C \in \mathcal{C}_U$ such that $C \cap S(\xi) = \emptyset$. Establishing the theorem for this C suffices: if the theorem holds for every bounded, open $D \in \mathcal{C}$ such that $D \cap S(\xi) = \emptyset$, then the theorem holds with $\mathcal{C}^*$ and ξ^* in place of $\mathcal{C}$ and ξ (because every $D \in \mathcal{C}^*$ is bounded). Since ϕ_2 is an analytic diffeomorphism, it follows that the theorem holds for every open $D \in \mathcal{C}$ such that $D \cap S(\xi) = \emptyset$.

We need quite a bit of preliminary work (see the end of this section for the proof of the theorem). For Lemma 3.5 and Corollary 3.6 below, we fix a C^p-curve $\gamma : [0,1] \to C$ transverse to ξ.

LEMMA 3.5. *Let $t_i \in (0,1)$ for $i \in \mathbb{N}$ be such that $t_i \to t \in [0,1]$. Then $C \cap \lim \mathrm{cl}(L(t_i)) = L(t)$.*

Proof. From Proposition 2.5 we know that $C \cap K$ is a union of leaves of $\xi|_C$, where $K := \lim \mathrm{cl}(L(t_i))$. Thus, since $\gamma(t_i) \to \gamma(t)$ and $\gamma(t) \in L(t)$, it follows that $L(t) \subseteq C \cap K$. To prove the opposite inclusion, we may assume by Proposition 1.5 that every leaf of $\xi|_C$ is the graph of a strictly increasing function (the other case is handled similarly). By Proposition 2.5 again, $\Pi_1(K) = [a,b]$ with $c \le a < b \le d$, and there is a continuous function $f : (a,b) \to \mathbb{R}$ such that $K \cap \big((a,b) \times \mathbb{R}\big) = \mathrm{gr}(f)$.

Assume for a contradiction that there is a leaf M of $\xi|_C$ such that $M \neq L(t)$ and $M \subseteq C \cap K$. Then $L(t)$ and M are disjoint subsets of $\mathrm{gr}(f)$; say $L(t) = \mathrm{gr}(f_t)$, where $f_t : (a(t), b(t)) \to \mathbb{R}$, and $M = \mathrm{gr}(g)$, where $g : (a', b') \to \mathbb{R}$. We assume here that $a' < b' \le a(t) < b(t)$; the other case is again handled similarly. By our assumption, $c < a(t)$ and hence $\lim_{x \to a(t)^+} f_t(x) \in$

$\{\alpha(a(t)), \beta(a(t))\}$. We assume here $\lim_{x\to a(t)^+} f_t(x) = \alpha(a(t))$, the other case being handled similarly. Then by the Mean Value Theorem, for every $\varepsilon > 0$ there is an $x \in (a(t), a(t)+\varepsilon)$ such that $f_t'(x) > \alpha'(x)$, that is, $\sigma_\alpha(x, f_t(x)) < 0$. It follows from (V) that

(∗) the map σ_α has constant negative sign on (α, α_0).

On the other hand, $b' < d$, and we may assume that $\lim_{x\to b'^-} g(x) = \alpha(b')$: otherwise, $\lim_{x\to b'^-} g(x) = \beta(b')$, and since

$$\lim_{x\to a(t)} f(x) = \lim_{x\to a(t)^+} f_t(x) = \alpha(a(t)),$$

we can replace M by a leaf of $\xi|_C$ that is contained in $\mathrm{gr}(f)$ and has the desired property. But $\lim_{x\to b'^-} g(x) = \alpha(b')$ means (as above) that for every $\varepsilon > 0$ there is an $x \in (b'-\varepsilon, b')$ such that $g'(x) < \alpha'(x)$, that is, $\sigma_\alpha(x, g(x)) > 0$. This contradicts (∗), so the lemma is proved. ∎

Put $F := F(\gamma((0,1)))$; note that F is open by Lemma 2.3(1).

Corollary 3.6. *$C \cap \mathrm{bd}(F) = L(0) \cup L(1)$; in particular, there are distinct $j_0, j_1 \in \{1,2\}$ such that $C \setminus \mathrm{cl}(F) = U_{L(0),j_0} \cup U_{L(1),j_1}$.*

Proof. Let $z \in \mathrm{cl}(F) \cap C$, and let $z_i \in F$ be such that $z_i \to z$. Let $t_i \in (0,1)$ be such that $z_i \in L(t_i)$; passing to a subsequence if necessary, we may assume that $t_i \to t \in [0,1]$. Then $z \in C \cap \lim \mathrm{cl}(L(t_i))$, so $z \in L(t)$ by Lemma 3.5. Since F is open by Lemma 2.3(1), it follows that $C \cap \mathrm{bd}(F) \subseteq L(0) \cup L(1)$. On the other hand, by Lemma 2.3(2), there is a $j \in \{1,2\}$ such that $L(t) \subseteq U_{L(0),j}$ for all $t \in (0,1]$ and $L(t) \subseteq U_{1,j'}$ for all $t \in [0,1)$, where $j' \in \{1,2\} \setminus \{j\}$. Hence $L(0) \cup L(1) \subseteq C \cap \mathrm{bd}(F(\Gamma))$, and the corollary is proved. ∎

Definition 3.7. Let $\tau : [0,1] \to U$ be continuous. We call τ *piecewise C^p-monotone in ξ* if there are $t_0 := 0 < t_1 < t_2 < \cdots < t_k < t_{k+1} := 1$ and $* \in \{<,>\}$ such that for all $i = 0, \ldots, k$, the restriction $\tau|_{(t_i,t_{i+1})}$ is C^p, and either $\xi^\perp(\tau(t)) \cdot \tau'(t) = 0$ for all $t \in (t_i, t_{i+1})$ or $\xi^\perp(\tau(t)) \cdot \tau'(t) * 0$ for all $t \in (t_i, t_{i+1})$. In this situation, we also say that τ is *$*$-piecewise C^p-monotone in ξ*. We call such a τ *tangent to ξ* if each $\tau|_{(t_i,t_{i+1})}$ is tangent to ξ.

Lemma 3.8. *Let $v, w \in C$. Then there is a curve $\tau : [0,1] \to C$ that is piecewise C^p-monotone in ξ and satisfies $\tau(0) = v$ and $\tau(1) = w$.*

Proof. If $L_v = L_w$, then there is a C^p-curve $\tau : [0,1] \to L_v$ such that $\tau(0) = v$ and $\tau(1) = w$, and we are done. So we assume from now on that $L_v \neq L_w$. Let $j_{vw} \in \{1,2\}$ be such that $w \in U_{L_v, j_{vw}}$, and put

$$*_{vw} := \begin{cases} < & \text{if } j_{vw} = 1, \\ > & \text{if } j_{vw} = 2. \end{cases}$$

By o-minimality, there is a definable C^p-curve $\tau : [0,1] \to C$ such that

(I) $\tau(0) = v$ and $\tau(1) = w$.

Again by o-minimality, there are $t_0 := 0 < t_1 < \cdots < t_k < t_{k+1} := 1$ such that for each $i = 0, \ldots, k$,

(II) the map $t \mapsto \xi^\perp(\tau(t)) \cdot \tau'(t)$ has constant sign on (t_i, t_{i+1}).

By Khovanskiĭ theory [14], we may also assume that for every $i = 0, \ldots, k$,

(III) either $\tau((t_i, t_{i+1})) \cap (L_v \cup L_w) = \emptyset$ or $\tau((t_i, t_{i+1})) \subseteq L_v \cup L_w$.

We now proceed by induction on k, simultaneously for all $v, w \in C$ and τ satisfying (I)–(III), to prove that τ can be changed into a curve that is $*_{vw}$-piecewise C^p-monotone in ξ. If $k = 0$, then τ is $*_{vw}$-piecewise C^p-monotone in ξ, so we are done. Therefore, we assume that $k > 0$ and that the claim holds for lower values of k.

Since $\tau(1) = w \notin L_v$ and L_v is closed in C, there is a maximal $t \in [0, 1)$ such that $\tau(t) \in L_v$, and by our choice of $t_1, \ldots, t_k$, we have $t = t_i$ for some $i \in \{0, \ldots, k\}$. If $i > 1$, we replace $\tau|_{[0,t_i]}$ by a C^p-curve $\tau_1 : [0, t_i] \to L_v$ such that $\tau_1(0) = v$ and $\tau_1(t_i) = \tau(t_i)$, and we reindex $t_i, \ldots, t_{k+1}$ as $t_1, \ldots, t_{k-i+2}$. Hence by the inductive hypothesis, we may assume that $i \leq 1$ and $\tau([0, 1]) \subseteq L_v \cup U_{L_v, j_{vw}}$. Put $v' := \tau(t_1)$; we now distinguish two cases:

Case 1: $v' \in L_v$. Then $*_{v'w} = *_{vw}$, so by the inductive hypothesis (and rescaling), there is a curve $\tau_1 : [t_1, 1] \to C$ that is $*_{vw}$-piecewise C^p-monotone in ξ and satisfies $\tau_1(t_1) = v'$ and $\tau_1(1) = w$. Now replace $\tau|_{[t_1,1]}$ by τ_1.

Case 2: $v' \notin L_v$. Then we must have $\xi^\perp(\tau(t)) \cdot \tau(t) *_{vw} 0$ for all $t \in (0, t_1)$. If $v' \in L_w$, the lemma follows by a similar argument to that in Case 1, so we assume that $v' \notin L_w$. We claim again that $*_{v'w} = *_{vw}$ in this situation, from which the lemma then follows by the inductive hypothesis as in Case 1.

To see the claim, we note that by Corollary 3.6 the complement of $F(\tau([0, t_1]))$ in C has two connected components $U_{L_v, j}$ and $U_{L_{v'}, j'}$, where $j, j' \in \{1, 2\}$ are distinct. By the above, j must be different from j_{vw}, so $w \in U_{L_{v'}, j'}$, that is, $j' = j_{v'w}$, which implies $j_{vw} = j_{v'w}$ as required. ∎

Lemma 3.9. *Let $\tau : [0, 1] \to C$ be piecewise C^p-monotone in ξ such that τ is not tangent to ξ. Then there is a C^p-curve $\gamma : [0, 1] \to C$ such that γ is transverse to C, $\gamma(0) = \tau(0)$ and $\gamma(1) = \tau(1)$.*

Proof. Let $t_0 := 0 < t_1 < t_2 < \cdots < t_k < t_{k+1} := 1$ be as in Definition 3.7. We work by induction on k. If $k = 0$, then by hypothesis τ is transverse to ξ, and we take $\gamma := \tau$. So we assume that $k > 0$; for the inductive step, it suffices to consider the case $k = 1$. The hypothesis on τ then implies that at least one of $\tau|_{(0,t_1)}$ and $\tau|_{(t_1,1)}$ is transverse to ξ; so we distinguish three cases:

Case 1: Both $\tau|_{(0,t_1)}$ and $\tau|_{(t_1,1)}$ are transverse to ξ. By Picard's Theorem, there are an open neighborhood $W \subseteq C$ of $\tau(t_1)$ and a C^p-diffeomorphism $f : \mathbb{R}^2 \to W$ such that $f(0) = \tau(t_1)$ and $f^*\xi = \partial/\partial x$, where $f^*\xi$ is the pull-back of ξ via f. Then for some $\varepsilon > 0$, the continuous curve $f^{-1} \circ \tau|_{(t_1-\varepsilon,t_1+\varepsilon)}$ is C^p and transverse to $\partial/\partial x$ on $(t_1 - \varepsilon, t_1) \cup (t_1, t_1 + \varepsilon)$. Using standard smoothing arguments from analysis, we can now find a C^p-curve $\eta : (t_1 - \varepsilon, t_1 + \varepsilon) \to \mathbb{R}^2$ that is transverse to $\partial/\partial x$ and satisfies $\eta(t) = f^{-1}(\tau(t))$ for all $t \in (t_1 - \varepsilon, t_1 - \varepsilon/2) \cup (t_1 + \varepsilon/2, t_1 + \varepsilon)$. Now define $\gamma : [0, 1] \to C$ by

$$\gamma(t) := \begin{cases} \tau(t) & \text{if } 0 \le t < t_1 - \varepsilon \text{ or } t_1 + \varepsilon < t \le 1, \\ f(\eta(t)) & \text{if } t_1 - \varepsilon \le t \le t_1 + \varepsilon. \end{cases}$$

Case 2: $\tau|_{(0,t_1)}$ is transverse to ξ and $\tau|_{(t_1,1)}$ is tangent to ξ. Since $\tau([t_1, 1])$ is compact, there are (by Picard's theorem again) $s_0 := t_1 < s_1 < \cdots < s_l < s_{l+1} := 1$, open neighborhoods $W_i \subseteq U$ of $\tau(s_i)$ and C^p-diffeomorphisms $f_i : \mathbb{R}^2 \to W_i$ for $i = 0, \ldots, l + 1$ such that $\tau([t_1, 1]) \subseteq W_0 \cup \cdots \cup W_{l+1}$, $f_i(0) = \tau(s_i)$ and $f_i^*\xi = \partial/\partial x$ for each i. We assume that $l = 0$, so that $s_0 = t_1$ and $s_1 = 1$; the general case then follows by induction on l.

Let $u \in (t_1, 1)$ be such that $\tau(u) \in W_0 \cap W_1$. Working with f_0 as in Case 1, we can replace $\tau|_{[0,u]}$ by a C^p-curve $\eta : [0, u] \to C$ transverse to ξ such that $\eta(0) = \tau(0)$ and $\eta(u) = \tau(u)$. Define $\eta(t) := \tau(t)$ for $t \in (u, 1]$; repeating the procedure with η and f_1 in place of τ and f_0, we obtain a C^p-curve $\gamma : [0, 1] \to C$ that is transverse to ξ and satisfies $\gamma(0) = \tau(0)$ and $\gamma(1) = \tau(1)$, as desired.

Case 3: $\tau|_{(0,t_1)}$ is tangent to ξ and $\tau|_{(t_1,1)}$ is transverse to ξ. This case is similar to Case 2. ∎

Combining Lemmas 3.8 and 3.9, we obtain:

Corollary 3.10. *Let $u, v \in C$ be such that $L_u \neq L_v$. Then there is a C^p-curve $\gamma : [0, 1] \to C$ such that $\gamma(0) = u$, $\gamma(1) = v$ and γ is transverse to ξ.* ∎

Proof of Theorem 3.4. Let $M, L \in \mathcal{L}(C)$ be distinct and choose $v \in M$ and $w \in L$. By Corollary 3.10, there is a C^p-curve $\gamma : [0, 1] \to C$ such that $\gamma(0) = v$, $\gamma(1) = w$ and γ is transverse to ξ. Hence $t \mapsto \xi^\perp(\gamma(t)) \cdot \gamma'(t)$ has constant nonzero sign on $[0, 1]$; this shows that $\ll_C$ is irreflexive. Transitivity follows by a similar argument. ∎

4. Foliation orderings. Let $\xi = a_1\frac{\partial}{\partial x} + a_2\frac{\partial}{\partial y}$ be a definable vector field of class C^1 on $\mathbb{R}^2$. We fix a piecewise trivial decomposition $\mathcal{C}$ of $\mathbb{R}^2$ for ξ; refining $\mathcal{C}$ if necessary, we may assume that $\mathcal{C}$ is a stratification. To simplify statements, we put

$$\mathcal{C}_{\mathrm{reg}} := \{C \in \mathcal{C} : C \cap S(\xi) = \emptyset\}.$$

For instance, in Example 3.2, the piecewise trivial decomposition $\mathcal{C}$ is a stratification and $\mathcal{C}_{\mathrm{reg}} = \mathcal{C} \setminus \{0\}$.

REMARK 4.1. $\mathcal{C}$ being a stratification has the following consequence: for every 1-dimensional $C \in \mathcal{C}$, there are exactly two distinct open $D \in \mathcal{C}$ such that $C \cap \mathrm{fr}(D) \neq \emptyset$, and for each of these D we have $C \subseteq \mathrm{fr}(D)$.

Let $V \subseteq \mathbb{R}^2 \setminus S(\xi)$ be an integral manifold of ξ, that is, a 1-dimensional manifold tangent to ξ. Given $u, v \in V$, we define $u <_V^\xi v$ if and only if there is a C^1-path $\gamma : [0,1] \to V$ such that $\gamma(0) = u$, $\gamma(1) = v$ and $\xi(\gamma(t)) \cdot \gamma'(t) > 0$ for all $t \in [0,1]$.

LEMMA 4.2. *Assume that V is connected and not a compact leaf. Then the relation $<_V^\xi$ defines a dense linear ordering of V without endpoints.*

Proof. Let $u, v \in V$ be such that $u \neq v$. Since V is connected, we get $u <_V^\xi v$ or $v <_V^\xi u$. On the other hand, if there are C^1-paths $\gamma, \delta : [0,1] \to V$ such that $\gamma(0) = \delta(1) = u$, $\gamma(1) = \delta(0) = v$ and $\xi(\gamma(t)) \cdot \gamma'(t) > 0$ and $\xi(\delta(t)) \cdot \delta'(t) > 0$ for all $t \in [0,1]$, then $\gamma([0,1]) \cup \delta([0,1])$ is a compact leaf of ξ contained in V; since V is connected, it follows that V is a compact leaf, a contradiction. ■

We now fix a $C \in \mathcal{C}_{\mathrm{reg}}$ such that $\dim(C) > 0$.

DEFINITION 4.3. The foliation of ξ induces an ordering $<_C^\xi$ on C as follows:

- Suppose that C is open, and let $u, v \in C$. Then every leaf of $\xi|_C$ is noncompact by Proposition 1.5. Thus, we define $u <_C^\xi v$ if and only if $L_u \ll_C^\xi L_v$ or $L_u = L_v$ and $u <_{L_u}^\xi v$.
- Suppose that $\dim(C) = 1$ and C is tangent to ξ. Then C is a connected, noncompact integral manifold of ξ, so we define $<_C^\xi$ as before Lemma 4.2.
- Suppose that $\dim(C) = 1$ and C is transverse to ξ. Let $u, v \in C$; we define $u <_C^\xi v$ if and only if there is a C^1-curve $\gamma : [0,1] \to C$ such that $\xi^\perp(\gamma(t)) \cdot \gamma'(t) > 0$ for all $t \in [0,1]$.

As before, we omit the superscript ξ whenever it is clear from context.

A $<_C$*-interval* is a set A of the form $(a,b) := \{c \in C : a *_1 c *_2 b\}$ with $a, b \in C$, or $(a, \infty) := \{c \in C : a * c\}$ with $a \in C$, or $(-\infty, b) := \{c \in C : c * c\}$ with $b \in C$, where $*, *_1, *_2 \in \{<_C, \leq_C\}$; we call A *open* if $* = *_1 = *_2 = <_C$.

LEMMA 4.4. *The ordering $<_C$ is a dense linear ordering on C without endpoints. Moreover, if* $\dim(C) = 1$, *then every $<_C$-bounded subset of C has a least upper bound.*

Proof. It is clear from the definition that C has no endpoints with respect to $<_C$. Density and linearity follow from Lemmas 2.3 and 4.2 if $\dim(C) = 1$, and if C is open, they follow from Lemma 4.2 and Theorem 3.4.

For the second statement, assume that $\dim(C) = 1$ and let $\alpha : (0,1) \to \mathbb{R}^2$ be C^1 and injective such that $C = \alpha((0,1))$. If C is tangent to ξ, then the map $t \mapsto \xi(\alpha(t)) \cdot \alpha'(t)$ has constant nonzero sign, and if C is transverse to ξ, then the map $t \mapsto \xi^{\perp}(\alpha(t)) \cdot \alpha'(t)$ has constant nonzero sign. Thus in both cases, the map $\alpha : ((0,1),<) \to (C,<_C)$ is either order-preserving or order-reversing; the second statement follows. ∎

We assume for the remainder of this section that either C is open, or C is 1-dimensional and tangent to ξ.

Definition 4.5. For each leaf L of $\xi|_C$, it follows from Proposition 1.5 that $\mathrm{fr}(L)$ consists of exactly two points $P_L^>, P_L^< \in \mathrm{fr}(C) \cup \{\infty\}$, where, for $* \in \{>,<\}$, P_L^* is the unique of these two points with the property that for every C^1-curve $\gamma : [0,1) \to L$ satisfying $\gamma(0) \in L$ and $\lim_{t\to 1} \gamma(t) = P_L^*$, we have $\xi(\gamma(t)) \cdot \gamma'(t) * 0$ for all $t \in [0,1)$. In this situation, we define the *forward projection* $\mathfrak{f}_C : C \to \mathrm{fr}(C) \cup \{\infty\}$ and the *backward projection* $\mathfrak{b}_C : C \to \mathrm{fr}(C) \cup \{\infty\}$ as

$$\mathfrak{f}_C(z) := P_{L_z}^> \quad \text{and} \quad \mathfrak{b}_C(z) := P_{L_z}^< \quad \text{for all } z \in C.$$

From now on we assume that C is open, and we let $D \in \mathcal{C}_{\mathrm{reg}}$ be of dimension 1 and contained in $\mathrm{fr}(C)$ such that D is transverse to ξ.

Lemma 4.6. *Either $D \subseteq \mathfrak{f}_C(C)$ and $D \cap \mathfrak{b}_C(C) = \emptyset$, or $D \subseteq \mathfrak{b}_C(C)$ and $D \cap \mathfrak{f}_C(C) = \emptyset$.*

Proof. Let $\alpha : (0,1) \to \mathbb{R}^2$ be a definable C^1-map such that $D = \alpha((0,1))$ and $\xi^{\perp}(\alpha(t)) \cdot \alpha'(t) > 0$ for all $t \in (0,1)$. Thus, either $\xi(\alpha(t))$ points into C for all t, or $\xi(\alpha(t))$ points out of C for all t. In the first case, we have $\mathfrak{f}_C(C) \cap D = \emptyset$, and in the second case $\mathfrak{b}_C(C) \cap D = \emptyset$. Moreover, by Picard's Theorem, for every $w \in D$ there is an integral manifold $V \subseteq \mathbb{R}^2$ of ξ such that $V \cap D = \{w\}$; hence, either $w \in \mathfrak{f}_C(C)$ or $w \in \mathfrak{b}_C(C)$. ∎

Lemma 4.7. *The maps $\mathfrak{f}_C|_{\mathfrak{f}_C^{-1}(D)}$ and $\mathfrak{b}_C|_{\mathfrak{b}_C^{-1}(D)}$ are increasing.*

Proof. We prove the lemma for $\mathfrak{f}_C$. Let $u, v \in C$ with $u <_C v$ be such that $\mathfrak{f}_C(u), \mathfrak{f}_C(v) \in D$; we may clearly assume that $L_u \ll_C L_v$, and hence (by Picard's Theorem) that $\mathfrak{f}_C(u) \neq \mathfrak{f}_C(v)$.

We assume here that $D = \mathrm{gr}(\alpha)$, where $\alpha : (a,b) \to \mathbb{R}$ is a definable C^1-function; the case $D = \{a\} \times (b,c)$ is handled similarly. Let also $\beta : (a,b) \to \mathbb{R}$ be a definable C^1-function such that $C = (\alpha, \beta)$ or $C = (\beta, \alpha)$; we assume here the former, the latter being handled similarly. For $s \in [0,1]$, we put

$$\alpha_s(t) := (1-s)\alpha(t) + s\beta(t), \qquad a < t < b.$$

Then for every $t \in (a,b)$, we have $\lim_{s\to 0} \alpha_s(t) = \alpha(t)$ and $\lim_{s\to 0} \alpha_s'(t) = \alpha'(t)$.

Let now $a < a' < b' < b$ be such that $\mathfrak{f}_C(u), \mathfrak{f}_C(v) \in \operatorname{gr} \alpha|_{(a',b')}$. Since D is transverse to ξ, there is an $\varepsilon > 0$ such that $\operatorname{gr} \alpha_s|_{(a',b')}$ is transverse to ξ for all $s \in [0,\varepsilon)$. It follows from the previous paragraph that the map $t \mapsto \sigma_\alpha(t,\alpha(t))$ has the same constant nonzero sign as the map $t \mapsto \sigma_{\alpha_s}(t,\alpha_s(t))$ for all $s \in (0,\varepsilon)$. Therefore by Lemma 2.3(2) and the definition of $<_D$, we have $\mathfrak{f}_C(u) <_D \mathfrak{f}_C(v)$, as required. ∎

COROLLARY 4.8. *Let $I \subseteq C$ be a $<_C$-interval. Then each of $\mathfrak{f}_C(I) \cap D$ and $\mathfrak{b}_C(I) \cap D$ is either empty, a point or an open $<_D$-interval.*

Proof. Assume that $a, b \in \mathfrak{f}_C(I) \cap D$ are such that $a <_D b$, and let $c \in D$ be such that $a <_D c <_D b$; it suffices to show that $c \in \mathfrak{f}_C(I)$. By Lemma 4.6, $c \in \mathfrak{f}_C(C)$. Let $u, v, w \in C$ be such that $a = \mathfrak{f}_C(u)$, $b = \mathfrak{f}_C(v)$, $c = \mathfrak{f}_C(w)$ and $u, v \in I$. Then $u <_C w <_C v$ by Lemma 4.7, as required. ∎

We fix a set $E_C \subseteq C$ such that $|E_C \cap L| = 1$ for every $L \in \mathcal{L}(C)$ and put $<_{E_C} := <_C|_{E_C}$, and we denote by e_L the unique element of $E \cap L$, for every $L \in \mathcal{L}(C)$.

REMARK. The map $L \mapsto L \cap E_C : (\mathcal{L}(C), \ll_C) \to (E_C, <_{E_C})$ is an isomorphism of ordered structures.

PROPOSITION 4.9. *Let $\mathfrak{g} \in \{\mathfrak{f}, \mathfrak{b}\}$. If $D \subseteq \mathfrak{g}_C(C)$, then $D_\mathfrak{g} := \mathfrak{g}_C^{-1}(D) \cap E_C$ is an $<_{E_C}$-interval, and the map $\mathfrak{g}_C|_{D_\mathfrak{g}} : (D_\mathfrak{g}, <_{E_C}|_{D_\mathfrak{g}}) \to (D, <_D)$ is an isomorphism of ordered structures.*

Proof. The transversality of D to ξ implies that if $u \in D$ and $L_1, L_2 \in \mathcal{L}(C)$ are such that $u = P^>_{L_1} = P^>_{L_2}$ or $u = P^<_{L_1} = P^<_{L_2}$, then $L_1 = L_2$. Thus by Lemma 4.7, the map $\mathfrak{g}_C|_{D_\mathfrak{f}}$ is strictly increasing, so the lemma follows. ∎

5. Progression map. We continue working with ξ and $\mathcal{C}$ as in Section 4, and we adopt all corresponding notations. We let

(i) $\mathcal{C}_{\text{open}}$ be the collection of all open cells in $\mathcal{C}_{\text{reg}}$;
(ii) $\mathcal{C}_{\tan}$ be the collection of all cells in $\mathcal{C}_{\text{reg}}$ that are of dimension 1 and tangent to ξ;
(iii) $\mathcal{C}_{\text{trans}}$ be the collection of all cells in $\mathcal{C}_{\text{reg}}$ that are of dimension 1 and transverse to ξ;
(iv) $\mathcal{C}_{\text{single}}$ be the collection of all $p \in \mathbb{R}^2$ such that $\{p\} \in \mathcal{C}_{\text{reg}}$.

By Lemma 4.6 and since $\mathcal{C}$ is a stratification, there are, for each $C \in \mathcal{C}_{\text{trans}}$, distinct and unique cells $C^\mathfrak{b}, C^\mathfrak{f} \in \mathcal{C}_{\text{open}}$ such that $C \cap \operatorname{cl}(C^\mathfrak{b}) \neq \emptyset$, $C \cap \operatorname{cl}(C^\mathfrak{f}) \neq \emptyset$ and

$$C \subseteq \mathfrak{f}_{C^\mathfrak{b}}(C^\mathfrak{b}) \quad \text{and} \quad C \subseteq \mathfrak{b}_{C^\mathfrak{f}}(C^\mathfrak{f}).$$

Similarly, there are, for each $p \in \mathcal{C}_{\text{single}}$, distinct and unique cells $p^{\mathfrak{b}}, p^{\mathfrak{f}} \in \mathcal{C}_{\text{open}} \cup \mathcal{C}_{\tan}$ such that $p \in \text{cl}(p^{\mathfrak{b}})$, $p \in \text{cl}(p^{\mathfrak{f}})$ and

$$p \in \mathfrak{f}_{p^{\mathfrak{b}}}(p^{\mathfrak{b}}) \quad \text{and} \quad p \in \mathfrak{b}_{p^{\mathfrak{f}}}(p^{\mathfrak{f}}).$$

(For $p \in \mathcal{C}_{\text{single}}$, we use the fact that there is an open box B containing p such that the leaf of $\xi|_B$ passing through p is a Rolle leaf.) For each $C \in \mathcal{C}_{\tan}$, we fix an arbitrary element $e_C \in C$; note that for each $z \in C$, C is the unique leaf L_z of $\xi|_C$ containing z.

We now define $\mathfrak{f}', \mathfrak{b}' : \mathbb{R}^2 \to \mathbb{R}^2 \cup \{\infty\}$ by

$$\mathfrak{f}'(z) := \begin{cases} \mathfrak{f}_C(z) & \text{if } z \in C \in \mathcal{C}_{\text{open}} \cup \mathcal{C}_{\tan} \text{ and } e_{L_z} \leq_{L_z} z, \\ e_{L_z} & \text{if } z \in C \in \mathcal{C}_{\text{open}} \cup \mathcal{C}_{\tan} \text{ and } z <_{L_z} e_{L_z}, \\ (\mathfrak{b}_{C^{\mathfrak{f}}}|_{E_{C^{\mathfrak{f}}}})^{-1}(z) & \text{if } z \in C \in \mathcal{C}_{\text{trans}} \cup \mathcal{C}_{\text{single}}, \\ z & \text{if } z \in S(\xi) \end{cases}$$

and

$$\mathfrak{b}'(z) := \begin{cases} \mathfrak{b}_C(z) & \text{if } z \in C \in \mathcal{C}_{\text{open}} \cup \mathcal{C}_{\tan} \text{ and } z \leq_{L_z} e_{L_z}, \\ e_{L_z} & \text{if } z \in C \in \mathcal{C}_{\text{open}} \cup \mathcal{C}_{\tan} \text{ and } e_{L_z} <_{L_z} z, \\ (\mathfrak{f}_{C^{\mathfrak{b}}}|_{E_{C^{\mathfrak{b}}}})^{-1}(z) & \text{if } z \in C \in \mathcal{C}_{\text{trans}} \cup \mathcal{C}_{\text{single}}, \\ z & \text{if } z \in S(\xi). \end{cases}$$

Definition 5.1. We define $\mathfrak{f}, \mathfrak{b} : \mathbb{R}^2 \cup \{\infty\} \to \mathbb{R}^2 \cup \{\infty\}$ by

$$\mathfrak{f}(z) := \begin{cases} \mathfrak{f}'(z) & \text{if } z \in \mathbb{R}^2 \text{ and } \mathfrak{f}'(z) \notin S(\xi), \\ \infty & \text{otherwise} \end{cases}$$

and

$$\mathfrak{b}(z) := \begin{cases} \mathfrak{b}'(z) & \text{if } z \in \mathbb{R}^2 \text{ and } \mathfrak{b}'(z) \notin S(\xi), \\ \infty & \text{otherwise.} \end{cases}$$

We call $\mathfrak{f}$ a *progression map* associated to ξ and $\mathfrak{b}$ a *reverse progression map* associated to ξ. We put

$$\mathcal{C}_1 = \mathcal{C}_{\text{trans}} \cup \mathcal{C}_{\text{single}} \cup \bigcup \{E_C : C \in \mathcal{C}_{\text{open}}\} \cup \{\{e_C\} : C \in \mathcal{C}_{\tan}\}$$

and let $B := \bigcup \mathcal{C}_1$; note that $\mathfrak{f}(\mathbb{R}^2) \subseteq B \cup \{\infty\}$ and $\mathfrak{b}(\mathbb{R}^2) \subseteq B \cup \{\infty\}$. Finally, we define $\mathfrak{f}^0 : \mathbb{R}^2 \cup \{\infty\} \to \mathbb{R}^2 \cup \{\infty\}$ by $\mathfrak{f}^0(x) := x$, and for $k > 0$ we define $\mathfrak{f}^k : \mathbb{R}^2 \cup \{\infty\} \to \mathbb{R}^2 \cup \{\infty\}$ inductively on k by $\mathfrak{f}^k(x) := \mathfrak{f}(\mathfrak{f}^{k-1}(x))$.

Proposition 5.2. *Let $X \in \mathcal{C}_1$ and L be a compact leaf of ξ. Then $|X \cap L| \leq 1$.*

Proof. If $X \in \mathcal{C}_{\text{single}}$ or $X = \{e_C\}$ for some $C \in \mathcal{C}_{\tan}$, the conclusion is trivial. By Lemma 1.3(2), L is a Rolle leaf of ξ; in particular, $|X \cap L| \leq 1$ if $X \in \mathcal{C}_{\text{trans}}$. So we may assume that $X = E_C$ for some $C \in \mathcal{C}_{\text{open}}$. Then there is at most one $L' \in \mathcal{L}(C)$ contained in L: otherwise by Corollary 3.10, there is a C^1-curve $\gamma : [0,1] \to C$ transverse to ξ such that $\gamma(0), \gamma(1) \in L$, a contradiction. It follows again that $|X \cap L| \leq 1$. ∎

PROPOSITION 5.3. *There is an $N \in \mathbb{N}$ such that for every $x \in B$, the leaf of ξ through x is compact if and only if $\mathfrak{f}^N(x) = x$.*

Proof. Let $x \in B$; if $\mathfrak{f}^k(x) = x$ for some $k > 0$, then the leaf of ξ through x is compact. For the converse, we assume that the leaf L of ξ through x is compact. Since L is compact, we have $L \cap S(\xi) = \emptyset$, that is, $\mathfrak{f}^k(x) \in B$ for every $k > 0$. Thus with $n := |\mathcal{C}_{\mathrm{reg}}| + 1$, there are a $C \in \mathcal{C}_{\mathrm{reg}}$ and $0 \le k_1 < k_2 \le n$ such that $\mathfrak{f}^{k_1}(x), \mathfrak{f}^{k_2}(x) \in C$. It follows from Proposition 5.2 that $\mathfrak{f}^{k_1}(x) = \mathfrak{f}^{k_2}(x)$, and hence that

$$x = \mathfrak{b}^{k_1} \circ \mathfrak{f}^{k_1}(x) = \mathfrak{b}^{k_1} \circ \mathfrak{f}^{k_2}(x) = \mathfrak{f}^{k_2 - k_1}(x).$$

Since n is independent of $x \in B$, the number $N := n!$ will do. ∎

6. Flow configuration theories. Inspired by the previous sections, we now define a first-order theory as described in the introduction. Our main goal, reached in Section 9, is to show that this theory admits quantifier elimination in a language suitable to our purposes.

DEFINITION 6.1. A *flow configuration* is a tuple

$$\Phi = (\Phi_{\mathrm{open}}, \Phi_{\mathrm{tan}}, \Phi_{\mathrm{trans}}, \Phi_{\mathrm{single}}, \phi^{\mathfrak{b}}, \phi^{\mathfrak{f}}, \min, \max, N_\Phi)$$

such that Φ_{open}, Φ_{tan}, Φ_{trans} and Φ_{single} are pairwise disjoint, finite sets,

$$\phi^{\mathfrak{b}}, \phi^{\mathfrak{f}} : \Phi_{\mathrm{trans}} \cup \Phi_{\mathrm{single}} \to \Phi_{\mathrm{open}} \cup \Phi_{\mathrm{tan}},$$
$$\min, \max : \Phi_{\mathrm{open}} \cup \Phi_{\mathrm{tan}} \cup \Phi_{\mathrm{trans}} \to \Phi_{\mathrm{single}} \cup \{\infty\}$$

and $N_\Phi \in \mathbb{N}$. In this situation, we shall write $a^{\mathfrak{b}}$ and $a^{\mathfrak{f}}$ instead of $\phi^{\mathfrak{b}}(a)$ and $\phi^{\mathfrak{f}}(a)$ for $a \in \Phi_{\mathrm{trans}} \cup \Phi_{\mathrm{single}}$.

EXAMPLE 6.2. Let ξ be a vector field on $\mathbb{R}^2$ of class C^1 and definable in an o-minimal expansion of the real field, and let $\mathcal{C}$ be a piecewise trivial cell decomposition of $\mathbb{R}^2$ that is also a stratification. We define $\mathcal{C}_{\mathrm{open}}$, $\mathcal{C}_{\mathrm{tan}}$, $\mathcal{C}_{\mathrm{trans}}$, $\mathcal{C}_{\mathrm{single}}$ and ${}^{\mathfrak{b}}, {}^{\mathfrak{f}} : \mathcal{C}_{\mathrm{trans}} \cup \mathcal{C}_{\mathrm{single}} \to \mathcal{C}_{\mathrm{open}} \cup \mathcal{C}_{\mathrm{tan}}$ as in Section 5, and we let $N \in \mathbb{N}$ be as in Proposition 5.3.

Let $C \in \mathcal{C}_{\mathrm{open}} \cup \mathcal{C}_{\mathrm{tan}} \cup \mathcal{C}_{\mathrm{trans}}$. If there is a point in $\mathcal{C}_{\mathrm{single}}$ that is contained in the closure of every set $\{x \in C : x <_C^\xi a\}$ with $a \in C$, we let $\min(C)$ be any such point; otherwise, we put $\min(C) := \infty$. Similarly, if there is a point in $\mathcal{C}_{\mathrm{single}}$ that is contained in the closure of every set $\{x \in C : a <_C^\xi x\}$ with $a \in C$, we let $\max(C)$ be any such point; otherwise, we put $\max(C) := \infty$. Then the tuple

$$\Phi_\xi = \Phi_\xi(\mathcal{C}) := (\mathcal{C}_{\mathrm{open}}, \mathcal{C}_{\mathrm{tan}}, \mathcal{C}_{\mathrm{trans}}, \mathcal{C}_{\mathrm{single}}, {}^{\mathfrak{b}}, {}^{\mathfrak{f}}, \min, \max, N)$$

is a *flow configuration associated to ξ*.

For the remainder of this section, we fix a flow configuration Φ.

Definition 6.3. Let $\mathcal{L}(\Phi)$ be the first-order language consisting of

(i) a unary predicate C and a binary predicate $<_C$ for each $C \in \Phi_{\text{open}} \cup \Phi_{\tan} \cup \Phi_{\text{trans}}$;
(ii) a unary predicate E_C for each $C \in \Phi_{\text{open}}$ and a constant symbol e_C for each $C \in \Phi_{\tan}$;
(iii) a constant symbol s, and a constant symbol c for each $c \in \Phi_{\text{single}}$;
(iv) unary function symbols $\mathfrak{f}$ and $\mathfrak{b}$;
(v) constant symbols $r_C^{\mathfrak{g}}$ and $s_C^{\mathfrak{g}}$ for each $C \in \Phi_{\text{trans}}$ and $\mathfrak{g} \in \{\mathfrak{f}, \mathfrak{b}\}$.

Throughout the rest of this paper, for $m \in \mathbb{N}$ we write $\mathfrak{f}^m$ for the $\mathcal{L}(\Phi)$-word consisting of m repetitions of the symbol $\mathfrak{f}$, and similarly for $\mathfrak{b}^m$.

Example 6.4. Let ξ and $\mathcal{C}$ be as in Example 6.2; we adopt the notations used there. We associate to ξ a unique $\mathcal{L}(\Phi_\xi)$-structure $\mathcal{M}_\xi = \mathcal{M}_\xi(\mathcal{C})$ as follows:

(i) the universe M_ξ of $\mathcal{M}_\xi$ is $\mathbb{R}^2 \setminus S(\xi) \cup \{\infty\}$;
(ii) for each $C \in \mathcal{C}_{\text{open}} \cup \mathcal{C}_{\tan} \cup \mathcal{C}_{\text{trans}}$, the predicate C is interpreted by the corresponding cell in $\mathcal{C}$, and the predicate $<_C$ is interpreted by the union of $<_C^\xi$ with $\{(\min(C), a) : a \in C\}$ and $\{(a, \max(C)) : a \in C\}$;
(iii) for each $C \in \mathcal{C}_{\text{open}}$, the predicate E_C is interpreted by the set E_C described in Section 5, and for each $C \in \mathcal{C}_{\tan}$, the constant e_C is interpreted by the element $e_C \in C$ picked in Section 5;
(iv) the constant s is interpreted as ∞, and for each $c \in \mathcal{C}_{\text{single}}$, the constant c is interpreted as the corresponding element of $\mathcal{C}_{\text{single}}$;
(v) the functions $\mathfrak{f}$ and $\mathfrak{b}$ are interpreted by the corresponding forward progression and reverse progression maps;
(vi) for each $C \in \mathcal{C}_{\text{trans}}$ and $\mathfrak{g} \in \{\mathfrak{f}, \mathfrak{b}\}$, the constants $r_C^{\mathfrak{g}}$ and $s_C^{\mathfrak{g}}$ are interpreted as the lower and upper endpoints, respectively, of the interval $\mathfrak{g}(C)$ in $E_{C^{\mathfrak{g}}} \cup \{\min(C^{\mathfrak{g}}), \max(C^{\mathfrak{g}})\}$.

Definition 6.5. We put $\Phi_0 := \Phi_{\text{open}} \cup \Phi_{\tan} \cup \Phi_{\text{trans}}$; intending to capture the theory of the previous example, we let $T(\Phi)$ be the $\mathcal{L}(\Phi)$-theory consisting of the universal closures of the formulas in the axiom schemes (F1)–(F15) below.

(F1) The formulas

(a) $$\bigwedge_{c,d \in \Phi_{\text{single}},\, c \neq d} \neg c = d \wedge \bigwedge_{c \in \Phi_{\text{single}},\, C \in \Phi_0} \neg C(c),$$

(b) $$\bigwedge_{c \in \Phi_{\text{single}}} \neg c = s \wedge \bigwedge_{C \in \Phi_0} \neg C(s),$$

(c) $$x = s \vee \bigvee_{c \in \Phi_{\text{single}}} x = c \vee \bigvee_{C \in \Phi_0} \Big(C(x) \wedge \bigwedge_{D \in \Phi_0, D \neq C} \neg D(x) \Big).$$

(F2) For each $C \in \Phi_0$ the sentences stating that $<_C$ is a dense linear ordering of C, together with $C(x) \to (x <_C \max(C) \wedge \min(C) <_C x)$.

REMARK. We do not wish to state that $<_C$ is a linear order on all of $C \cup \{\min(C), \max(C)\}$, because it is possible that $\min(C) = \max(C)$. The axioms (F2) suffice for our purpose, which is to be able to refer to C as the $<_C$-interval between $\min(C)$ and $\max(C)$.

(F3) The formula $\bigwedge_{C\in\Phi_{\text{tan}}} C(e_C) \wedge \bigwedge_{C\in\Phi_{\text{open}}} E_C(x) \to C(x)$.

(F4) For each $C \in \Phi_{\text{open}}$ the sentences stating that the restriction of $<_C$ to E_C is a dense linear ordering.

(F5) For each $(\mathfrak{g}, \mathfrak{h}) \in \{(\mathfrak{f}, \mathfrak{b}), (\mathfrak{b}, \mathfrak{f})\}$ and $* \in \{\leq, \geq\}$ the formulas

  (a) $\mathfrak{g}(s) = s \wedge (\neg x = s \to \neg \mathfrak{g}(x) = x)$,

  (b) $\bigwedge_{c\in\Phi_{\text{single}}} (\neg \mathfrak{g}(c) = s \to \mathfrak{h}(\mathfrak{g}(c)) = c)$,

  (c) $\bigwedge_{C\in\Phi_{\text{open}}} C\mathfrak{g}(x) \to E_C(\mathfrak{g}(x)) \wedge \bigwedge_{C\in\Phi_{\text{tan}}} C(\mathfrak{g}(x)) \to \mathfrak{g}(x) = e_C$,

  (d) $\bigwedge_{C\in\Phi_{\text{tan}}} (C(x) \wedge e_C *_C x *_C \mathfrak{g}(e_C)) \to \mathfrak{g}(x) = \mathfrak{g}(e_C)$,

  (e) $\bigwedge_{C\in\Phi_{\text{tan}}} (C(x) \wedge e_C *_C x *_C \mathfrak{h}(e_C)) \to \mathfrak{g}(x) = e_C$.

(F6) For each $C \in \mathcal{C}_{\text{open}}$ and $\mathfrak{g} \in \{\mathfrak{f}, \mathfrak{b}\}$ the formula

$$(E_C(x) \wedge E_C(y) \wedge \mathfrak{g}(x) = \mathfrak{g}(y)) \to (\mathfrak{g}(x) = s \vee x = y).$$

(F7) For each $c \in \Phi_{\text{single}}$ and $\mathfrak{g} \in \{\mathfrak{f}, \mathfrak{b}\}$, the sentences $\mathfrak{g}(c) = e_{c^\mathfrak{g}}$ if $c^\mathfrak{g} \in \Phi_{\text{tan}}$ and $E_{c^\mathfrak{g}}(\mathfrak{g}(c))$ if $c^\mathfrak{g} \in \Phi_{\text{open}}$.

(F8) For each $C \in \Phi_{\text{trans}}$ and $(\mathfrak{g}, \mathfrak{h}) \in \{(\mathfrak{f}, \mathfrak{b}), (\mathfrak{b}, \mathfrak{f})\}$ the sentences stating that $\mathfrak{g}(C)$ is an interval I_1 in $E_{C^\mathfrak{g}}$ and $\mathfrak{g}|_C : C \to I_1$ is an order-isomorphism.

(F9) For each $C \in \Phi_{\text{open}}$ and $(\mathfrak{g}, \mathfrak{h}) \in \{(\mathfrak{f}, \mathfrak{b}), (\mathfrak{b}, \mathfrak{f})\}$ the formula

$$E_C(x) \to \Big(\mathfrak{g}(x) = s \vee \bigvee_{D\in\Phi_{\text{trans}},\, C=D^\mathfrak{h}} D(\mathfrak{g}(x)) \vee \bigvee_{d\in\Phi_{\text{single}},\, C=d^\mathfrak{h}} \mathfrak{g}(x) = d\Big).$$

We need more axioms describing the ordering $<_C$ and the behavior of $\mathfrak{f}$ and $\mathfrak{b}$ on C for $C \in \Phi_{\text{open}}$. For example, if $x \in C \setminus E_C$, we want that x has either a unique predecessor or a unique successor in E_C. Also, for any $y \in E_C$, the set of points x for which y is either the predecessor or the successor is infinite and densely ordered by $<_C$. For convenience, we let $\phi^\mathfrak{f}_C(x, y)$ be the formula

$$C(x) \wedge \neg E_C(x) \wedge E_C(y) \wedge x <_C y \wedge \neg\exists z(E_C(z) \wedge x <_C z <_C y)$$

and $\phi^{\mathfrak{b}}_C(x,y)$ be the formula

$$C(x) \wedge \neg E_C(x) \wedge E_C(y) \wedge y <_C x \wedge \neg\exists z(E_C(z) \wedge y <_C z <_C x).$$

(F10) For each $C \in \Phi_{\text{open}}$ the formulas

(a) $C(x) \wedge \neg E_C(x) \to \exists y(\phi^{\mathfrak{f}}_C(x,y) \vee \phi^{\mathfrak{b}}_C(x,y))$,
(b) $\exists y \phi^{\mathfrak{f}}_C(x,y) \to \neg\exists y \phi^{\mathfrak{b}}_C(x,z)$,
(c) $\exists y \phi^{\mathfrak{b}}_C(x,y) \to \neg\exists y \phi^{\mathfrak{f}}_C(x,y)$,

and the formula scheme $E_C(y) \to \exists^\infty x \phi^{\mathfrak{f}}_C(x,y) \wedge \exists^\infty x \phi^{\mathfrak{b}}_C(x,y)$.

(F11) For each $C \in \Phi_{\text{open}}$ the sentences stating that for every $y \in E_C$, the restriction of $<_C$ to the set $C_y := \{x : \phi^{\mathfrak{b}}_C(x,y) \vee \phi^{\mathfrak{f}}_C(x,y) \vee x = y\}$ is a dense linear ordering, together with $C_y(x) \to (x <_C \mathfrak{f}(y) \wedge \mathfrak{g}(y) <_C x)$.

(F12) For each $C \in \Phi_{\text{open}}$ and $(\mathfrak{g},\mathfrak{h}) \in \{(\mathfrak{f},\mathfrak{b}),(\mathfrak{b},\mathfrak{f})\}$ the formulas

(a) $C(x) \wedge \neg E_C(x) \wedge \exists y \phi^{\mathfrak{g}}_C(x,y) \to \forall z(\phi^{\mathfrak{g}}_C(x,z) \to \mathfrak{g}(x) = z)$,
(b) $C(x) \wedge \neg E_C(x) \wedge \exists y \phi^{\mathfrak{h}}_C(x,y) \to \forall z(\phi^{\mathfrak{h}}_C(x,z) \to \mathfrak{g}(x) = \mathfrak{g}(z))$.

(F13) For each $C \in \Phi_{\text{trans}}$ and $(\mathfrak{g},\mathfrak{h}) \in \{(\mathfrak{f},\mathfrak{b}),(\mathfrak{b},\mathfrak{f})\}$ the formulas

(a) $E_{C^{\mathfrak{g}}}(r^{\mathfrak{g}}_C) \vee r^{\mathfrak{g}}_C = \min(C^{\mathfrak{g}}) \vee r^{\mathfrak{g}}_C = \max(C^{\mathfrak{g}})$,
(b) $E_{C^{\mathfrak{g}}}(s^{\mathfrak{g}}_C) \vee s^{\mathfrak{g}}_C = \min(C^{\mathfrak{g}}) \vee s^{\mathfrak{g}}_C = \max(C^{\mathfrak{g}})$,
(c) $r^{\mathfrak{g}}_C \leq_{C^{\mathfrak{g}}} s^{\mathfrak{g}}_C$,
(d) $E_{C^{\mathfrak{g}}}(x) \to (C(\mathfrak{h}(x)) \leftrightarrow r^{\mathfrak{g}}_C <_{C^{\mathfrak{g}}} x <_{C^{\mathfrak{g}}} s^{\mathfrak{g}}_C)$.

(F14) For each $m,n \in \mathbb{N}$, $C \in \Phi_{\text{open}}$, $D \in \Phi_{\text{trans}}$ and $\mathfrak{g} \in \{\mathfrak{f},\mathfrak{b}\}$ the formulas

(a) $E_C(x) \wedge E_C(\mathfrak{g}^m(x)) \wedge \mathfrak{g}^n(x) = x \to \mathfrak{g}^m(x) = x$,
(b) $D(x) \wedge D(\mathfrak{g}^m(x)) \wedge \mathfrak{g}^n(x) = x \to \mathfrak{g}^m(x) = x$.

(F15) For each $m \in \mathbb{N}$ and $\mathfrak{g} \in \{\mathfrak{f},\mathfrak{b}\}$ the formula $\mathfrak{g}^m(x) = x \to \mathfrak{g}^{N_\Phi}(x) = x$.

This completes our list of axioms for $T(\Phi)$.

Our choice of axioms above and Sections 4 and 5 imply the following:

Proposition 6.6. *Let ξ be a vector field on $\mathbb{R}^2$ of class C^1 and definable in an o-minimal expansion of the real field, and let $\mathcal{M}_\xi$ be an $\mathcal{L}(\Phi_\xi)$-structure associated to ξ as in Example* 6.4. *Then $\mathcal{M}_\xi \models T(\Phi_\xi)$.* ∎

Definition 6.7. We write

$$\Phi_1 := \Phi_{\text{trans}} \cup \{E_C : C \in \Phi_{\text{open}}\}.$$

The following $\mathcal{L}(\Phi)$-formulas are of particular interest: for $C \in \Phi_1$, we let $\mathrm{Fix}_C(x)$ be the formula $C(x) \wedge \mathfrak{f}^{N_\Phi}(x) = x$ and $\mathrm{Fix}_C(x,y)$ be the formula

$$\exists z((x \leq_C z \leq_C y \vee y \leq_C z \leq_C x) \wedge \mathrm{Fix}_C(z)).$$

Next, we let $\mathrm{Bd}_C(x)$ be the formula

$$\mathrm{Fix}_C(x) \wedge \forall y \forall z \big(y <_C x <_C z \to \exists w (y <_C w <_C z \wedge \neg \mathrm{Fix}_C(w))\big),$$

and let $\mathrm{Lim}_C(x)$ be the formula

$$\mathrm{Fix}_C(x) \wedge \exists y (C(y) \wedge y \neq x \wedge \neg \mathrm{Fix}_C(x,y)).$$

EXAMPLE 6.8. Let ξ be a vector field on $\mathbb{R}^2$ of class C^1 and definable in an o-minimal expansion of the real field, and let $\mathcal{M}_\xi$ be an $\mathcal{L}(\Phi_\xi)$-structure associated to ξ as in Example 6.4. Let also $C \in \mathcal{C}_1 := \mathcal{C}_{\mathrm{trans}} \cup \{E_F : F \in \mathcal{C}_{\mathrm{open}}\}$. Then $\mathrm{Fix}_C(M)$ is the set of points in C that belong to a cycle of ξ; $\mathrm{Bd}_C(M)$ is the set of points in C that belong to a boundary cycle of ξ; and $\mathrm{Lim}_C(M)$ is the set of points in C that belong to a limit cycle of ξ. Note that if ξ is analytic, then the set $\mathrm{Bd}_C(M)$ is discrete by Poincaré's Theorem [12] (see also [11, p. 217]); in particular, $\mathrm{Bd}_C(M) = \mathrm{Lim}_C(M)$ in this case.

In general, by Proposition 5.3, the cardinality of $\mathrm{Bd}_C(M)$ is equal to the number of boundary cycles of ξ that intersect C. Since every cycle of ξ intersects the set $\bigcup \mathcal{C}_{\mathrm{tan}} \cup \bigcup \mathcal{C}_{\mathrm{trans}} \cup \bigcup \mathcal{C}_{\mathrm{single}}$, it follows that, with $b(\xi)$ denoting the cardinality of the set of all boundary cycles of ξ, we have

$$|\mathrm{Bd}_C(M)| \le b(\xi) \le |\mathcal{C}_{\mathrm{tan}}| + |\mathcal{C}_{\mathrm{single}}| + \sum_{D \in \mathcal{C}_{\mathrm{trans}}} |\mathrm{Bd}_D(M)|.$$

7. Iterating the progression maps. We continue to work with a flow configuration Φ as in Definition 6.1. Throughout this section, we fix $(\mathfrak{g},\mathfrak{h}) \in \{(\mathfrak{f},\mathfrak{b}),(\mathfrak{b},\mathfrak{f})\}$.

For the next lemma, we denote by $\Theta_{(\mathfrak{g},\mathfrak{h})}$ the universal closure of the conjunction of the formulas $(\bigwedge_{C \in \Phi_0} \neg C(x)) \to \mathfrak{g}(\mathfrak{h}(x)) = x$,

$$(C(x) \wedge E_C(\mathfrak{h}(x))) \to \mathfrak{g}(\mathfrak{h}(x)) = \mathfrak{g}(x)$$

and

$$(E_C(x) \wedge \mathfrak{h}(x) \neq s) \to \mathfrak{g}(\mathfrak{h}(x)) = x$$

for each $C \in \Phi_{\mathrm{open}}$,

$$(C(x) \wedge \mathfrak{h}(x) = e_C) \to \mathfrak{g}(\mathfrak{h}(x)) = \mathfrak{g}(x)$$

and

$$(x = e_C \wedge \mathfrak{h}(x) \neq s) \to \mathfrak{g}(\mathfrak{h}(x)) = x$$

for each $C \in \Phi_{\mathrm{tan}}$, and $C(x) \to \mathfrak{g}(\mathfrak{h}(x)) = x$ for each $C \in \Phi_{\mathrm{trans}} \cup \Phi_{\mathrm{single}}$.

LEMMA 7.1. $T(\Phi) \vdash \Theta_{(\mathfrak{g},\mathfrak{h})}$.

Proof. Let $\mathcal{M} \models T(\Phi)$, and let $a \in M$ be such that $a \notin \bigcup_{C \in \Phi_0} C$. Then by (F1), either $a = c$ for some $c \in \Phi_{\mathrm{single}}$, or $a = s$. In the latter case, we have $\mathfrak{g}(\mathfrak{h}(a)) = \mathfrak{h}(\mathfrak{g}(a)) = a$ by (F5), so we may assume that $a = c$ for some $c \in \Phi_{\mathrm{single}}$. Then $\mathfrak{h}(\mathfrak{g}(a)) = \mathfrak{g}(\mathfrak{h}(a)) = a$ by (F7)–(F9).

The proofs of the other conjuncts are similar, using also (F12); we leave the details to the reader. ∎

COROLLARY 7.2. *Let ϕ be any quantifier-free $\mathcal{L}(\Phi)$-formula. Then ϕ is equivalent in $T(\Phi)$ to a quantifier-free formula ϕ' such that no term occurring in ϕ' contains both the symbols $\mathfrak{f}$ and $\mathfrak{b}$.*

Proof. By induction on $l := \max\{\text{length}(t) : t \text{ is a term occurring in } \phi\}$, using Lemma 7.1. ∎

For the remainder of this section, we fix an arbitrary model $\mathcal{M}$ of $T(\Phi)$. To simplify notation, we omit the superscript $\mathcal{M}$ below and write $\overline{C} := C \cup \{\min(C), \max(C)\}$ for $C \in \Phi_1$.

DEFINITION 7.3. Let $C \in \Phi_1$ and $k \in \mathbb{N}$. We define

$$G_C^k := \{\mathfrak{g}^l(z) : z \text{ is a constant, } 0 \le l \le k \text{ and } \mathfrak{g}^l(z) \in C\},$$

and we let $\mathcal{O}_C^k$ be the collection of all possible order types of pairs $(a, b) \in \overline{C}^2$ over G_C^k. In addition, for $\zeta_0, \zeta_1 \in \overline{C}$ and $D \in \Phi_1$, we put

$$\mathfrak{g}_D^{-k}(\zeta_0, \zeta_1) := \{x \in D : \zeta_0 <_C \mathfrak{g}^k(x) <_C \zeta_1\}$$

and

$$H_D^k(\zeta_0, \zeta_1) := \{\mathfrak{h}^l(z) : z \in \{\zeta_0, \zeta_1\} \text{ or } z \text{ is a constant, } 0 \le l \le k \text{ and } \mathfrak{h}^l(z) \in D\}.$$

Note that G_C^k and $H_D^k(\zeta_0, \zeta_1)$, and hence $\mathcal{O}_C^k$, are finite sets whose cardinality is bounded by a number depending only on the language and k, but independent of $\mathcal{M}$, C, D, ζ_0 or ζ_1.

PROPOSITION 7.4. *Let $C, D \in \Phi_1$, $\zeta_0, \zeta_1 \in \overline{C}$ and $k \in \mathbb{N}$.*

(1) *The set $\mathfrak{g}_D^{-k}(\zeta_0, \zeta_1)$ is a union of points in $H_D^k(\zeta_0, \zeta_1)$ and open intervals with endpoints in $H_D^k(\zeta_0, \zeta_1)$.*
(2) *For each $\vartheta \in \mathcal{O}_C^k$, there is a conjunction $\sigma_\vartheta(x, y_0, y_1)$ of atomic formulas with free variables x, y_0 and y_1 such that whenever (ζ_0, ζ_1) has order type ϑ over G_C^k, the set $\mathfrak{g}_D^{-k}(\zeta_0, \zeta_1)$ is defined by the formula $\sigma_\vartheta(x, \zeta_0, \zeta_1)$.*
(3) *$\mathfrak{g}^k$ restricted to $\mathfrak{g}_D^{-k}(\zeta_0, \zeta_1)$ is continuous.*

Proof. For every $x \in \mathfrak{g}_D^{-k}(\zeta_0, \zeta_1)$, there is a sequence $E = (E_0, \ldots, E_k)$ of elements of $\Phi_2 := \Phi_1 \cup \{\{c\} : c \in \Phi_{\text{single}}\} \cup \{\{e_C\} : C \in \Phi_{\tan}\}$ such that $E_0 = D$, $E_k = C$ and $\mathfrak{g}^i(x) \in E_i$ for $i = 0, \ldots, k$. Thus, we fix a sequence $E = (E_0, \ldots, E_k) \in \Phi_2^{k+1}$ with $E_k = C$, and we define the set

$$\mathfrak{g}_E^{-k}(\zeta_0, \zeta_1) := \{x \in M : \mathfrak{g}^i(x) \in E_i \text{ for } i = 0, \ldots, k,\ \zeta_0 <_C \mathfrak{g}^k(x) <_C \zeta_1\};$$

it suffices to prove the proposition with $\mathfrak{g}_E^{-k}(\zeta_0, \zeta_1)$ and $H_{E_0}^k(\zeta_0, \zeta_1)$ in place of $\mathfrak{g}_D^{-k}(\zeta_0, \zeta_1)$ and $H_D^k(\zeta_0, \zeta_1)$.

Next, we note that if $E_i \in \{\{c\} : c \in \Phi_{\mathrm{single}}\} \cup \{\{e_C\} : C \in \Phi_{\mathrm{tan}}\}$ for some $i \in \{1, \dots, k-1\}$, then $a \in \mathfrak{g}_E^{-k}(\zeta_0, \zeta_1)$ if and only if $\mathfrak{g}^i(a)$ is the unique constant in E_i and $\zeta_0 <_C \mathfrak{g}^k(a) <_C \zeta_1$, so the proposition follows in this case.

We therefore assume from now on that $E_i \in \Phi_1$ for each $i = 0, \dots, k$, and in this case we prove the proposition with part (1) replaced by

(1)′ The set $\mathfrak{g}_E^{-k}(\zeta_0, \zeta_1)$ is an open interval with endpoints in $H^k_{E_0}(\zeta_0, \zeta_1)$.

We proceed by induction on k. The case $k = 0$ is trivial, so we assume that $k > 1$. By axiom (F8), the set $\mathfrak{g}^{-1}_{(E_{k-1}, E_k)}(\zeta_0, \zeta_1)$ is an open interval whose endpoints η_0, η_1 belong to the set $H^1_{E_{k-1}}(\zeta_0, \zeta_1)$ and are determined by the order type of (ζ_0, ζ_1) over $G^1_{E_k}$. In fact, we claim that the order type of (η_0, η_1) over $G^{k-1}_{E_{k-1}}$ is determined by the order type of (ζ_0, ζ_1) over $G^k_{E_k}$; together with the inductive hypothesis applied to $\mathfrak{g}^{k-1}_{(E_0,\dots,E_{k-1})}(\eta_0, \eta_1)$, the proposition then follows, because $H^{k-1}_{E_0}(c, d)$ is contained in $H^k_{E_0}(\zeta_0, \zeta_1)$ for all $c, d \in H^1_{E_{k-1}}(\zeta_0, \zeta_1)$.

To see the claim, assume first that $E_k = E_C$ for some $C \in \Phi_{\mathrm{open}}$. Then by axiom (F8), the set $\{\mathfrak{g}(z) : z \in G^{k-1}_{E_{k-1}}\}$ is contained in $G^k_{E_k}$ and the claim follows in this case. So we assume that $E_k \in \Phi_{\mathrm{trans}}$. Then by axiom (F13), $E_{k-1} = E_C$ for some $C \in \Phi_{\mathrm{open}}$ and there are constants a and b such that

$$(\eta_0, \eta_1) \subseteq (a, b) = \mathfrak{g}^{-1}(E_k) = h(E_k) \quad \text{(as intervals)}.$$

Hence the order type of (η_0, η_1) over $G^{k-1}_{E_C}$ is determined by the order type of (η_0, η_1) over the set $G' := \{z \in G^{k-1}_{E_C} : a <_C z <_C b\}$. Then again by axiom (F8), the set $\{\mathfrak{g}(z) : z \in G'\}$ is contained in $G^k_{E_k}$ and the claim also follows in this case. ∎

Corollary 7.5. *Let $C \in \Phi_1$ and put $G := \mathfrak{g}_C^{-N}(\min(C), \max(C))$.*

(1) *The set $\mathrm{Bd}_C(M)$ is a closed and nowhere dense subset of G.*
(2) *Assume that $\Phi = \Phi_\xi$ and $\mathcal{M} \equiv \mathcal{M}_\xi$ for some definable vector field ξ of class C^1 on $\mathbb{R}^2$. Then for every $c \in G \setminus \mathrm{Bd}_C(M)$, there are $a, b \in \overline{C}$ such that*

$$a = \sup\{x \in \mathrm{Bd}_C(M) \cup (\overline{C} \setminus G) : x <_C c\},$$
$$b = \inf\{x \in \mathrm{Bd}_C(M) \cup (\overline{C} \setminus G) : c <_C x\}.$$

Proof. Part (1) follows from the continuity of $\mathfrak{g}^N|_G$ and the definition of the set $\mathrm{Bd}_C(M)$. Part (2) follows from part (1) and the fact that $C^{\mathcal{M}_\xi}$ is complete. ∎

Finally, for each $C \in \Phi_1$ we let $\overline{C}(x)$ abbreviate $C(x) \vee x = \min(C) \vee x = \max(C)$. We let G^k be the set of all $\mathcal{L}(\Phi)$-terms $\mathfrak{g}^j c$ such that $0 \le j \le k$

and c is a constant symbol, and we let $\mathcal{O}^k$ be the set of all formulas of the form

$$(\overline{C}(y_0) \wedge \overline{C}(y_1)) \wedge \bigwedge_{\{\tau,\varrho\} \subseteq G^k \cup \{y_0,y_1\}} (\tau *_{\{\tau,\varrho\}} \varrho),$$

where $C \in \Phi_1$ and $*_{\{\tau,\varrho\}} \in \{<_C, >_C, =, \neq\}$. The cardinalities of G^k and $\mathcal{O}^k$ are bounded by a number depending only on k (and on $\mathcal{L}(\Phi)$). Moreover, in $\mathcal{M}$, each formula $\vartheta \in \mathcal{O}^k$ determines an order type in $\mathcal{O}^k_C$ for some $C \in \Phi_1$; and conversely, every order type in $\mathcal{O}^k_C$ with $C \in \Phi_1$ is determined by some formula $\vartheta \in \mathcal{O}^k$. Thus we obtain the following from Proposition 7.4:

Corollary 7.6. *Let $k \in \mathbb{N}$. Then there are $l = l(k) \in \mathbb{N}$ and quantifier-free formulas $\vartheta^k_1(y_0, y_1), \ldots, \vartheta^k_l(y_0, y_1)$ with free variables y_0 and y_1 such that*

(1) $T(\Phi) \vdash \bigvee_{i=1}^{l} \vartheta^k_i(y_0, y_1) \leftrightarrow \bigvee_{C \in \Phi_1} (\overline{C}(y_0) \wedge \overline{C}(y_1))$;

(2) *for every $D \in \Phi_1$ there are quantifier-free formulas $\sigma^{D,k}_i(x, y_0, y_1)$ with free variables x, y_0 and y_1, $i = 1, \ldots, l$, such that if $\mathcal{M} \models \vartheta^k_i(\zeta_0, \zeta_1)$ for $\zeta_0, \zeta_1 \in M$ and some i, then the set $\mathfrak{g}^{-k}_D(\zeta_0, \zeta_1)$ is defined by the formula $\sigma^{D,k}_i(x, \zeta_0, \zeta_1)$.* ∎

Remark 7.7. We obtain analogous statements to Proposition 7.4 and Corollary 7.6 if we replace the open interval (ζ_0, ζ_1) by a half-open or closed interval.

8. Dulac flow configurations. It is clear from Example 6.8 that, for a vector field ξ on $\mathbb{R}^2$ definable in $\mathcal{R}$, the set of boundary cycles of ξ is represented in $\mathcal{M}_\xi$ by the definable sets $\mathrm{Bd}_C(M)$. The following example shows that the theory $T(\Phi)$ has hardly any implications for the nature of these sets.

Example 8.1. Consider the vector field ζ of Example 3.2, and let $\mathcal{C}$ be the piecewise trivial decomposition obtained there. We denote by Φ_ζ the flow configuration corresponding to this $\mathcal{C}$ and write

$$C_0 := \{(x,y) : x > 0,\ y = 0\} \in \mathcal{C}.$$

We show here how to define, given any closed and nowhere dense subset F of C_0, a vector field ζ' of class C^∞ for which Φ_ζ is still a flow configuration and such that $\mathrm{Bd}_{C_0}(M_{\zeta'}) = F$.

First, given $0 < a < b < \infty$, we let $d_{(a,b)} : \mathbb{R}^2 \to \mathbb{R}$ be the function $d_{(a,b)}(x,y) := (b^2 - (x^2+y^2))((x^2+y^2) - a^2)$, and we let $e_{(a,b)} : \mathbb{R}^2 \to \mathbb{R}$ be the C^∞-function defined by $e_{(a,b)}(x,y) := \exp(-1/d_{(a,b)}(x,y))$. We let $\zeta_{(a,b)}$ be the vector field of class C^∞ on the annulus $A_{(a,b)} := \{(x,y) : d_{(a,b)}(x,y) > 0\}$

defined by

$$\zeta_{(a,b)} := -(y + e_{(a,b)}(x,y)x)\,\frac{\partial}{\partial x} + (x - e_{(a,b)}(x,y)y)\,\frac{\partial}{\partial y}.$$

Second, let $F \subseteq C_0$ be an arbitrary closed and nowhere dense subset. Then $C_0 \setminus F$ is open in C_0 and hence the union of countably many disjoint open intervals $I_0, I_1, I_2, \dots$. We let ζ' be the vector field on $\mathbb{R}^2$ of class C^∞ defined by

$$\zeta'(x,y) := \begin{cases} \zeta_{I_j}(x,y) & \text{if } (x,y) \in A_{I_j} \text{ for some } j \in \mathbb{N}, \\ \zeta(x,y) & \text{otherwise.} \end{cases}$$

(Note that by Wilkie's Theorem [15], ζ' is definable in some o-minimal expansion of the real field if and only if F is finite.)

In view of the previous example, we now introduce a strengthening of the setting described in Section 6.

DEFINITION 8.2. A *Dulac flow configuration* Ψ is a pair (Φ, ν) such that Φ is a flow configuration and $\nu \in \mathbb{N}$.

EXAMPLE 8.3. Let ξ be a definable vector field on $\mathbb{R}^2$ of class C^1. Let $\Phi = \Phi_\xi$ be a flow configuration associated to ξ as in Example 6.2 and let $\mathcal{M}_\xi$ be the associated $\mathcal{L}(\Phi_\xi)$-structure described in Example 6.4. Assume that there is a $\nu \in \mathbb{N}$ such that for each $C \in \Phi_1$, the set $\mathrm{Bd}_C(M_\xi)$ has cardinality at most ν. Then $\Psi_\xi := (\Phi_\xi, \nu)$ is called a *Dulac flow configuration associated to* ξ.

For the remainder of this section, we fix a Dulac flow configuration $\Psi = (\Phi, \nu)$.

DEFINITION 8.4. The language $\mathcal{L}(\Psi)$ consists of the symbols of $\mathcal{L}(\Phi)$ together with the following symbols for each $C \in \Phi_1$:

(i) binary predicates R_C and $S^{\mathfrak{f}}_{m,C}$, $B^{\mathfrak{f}}_{m,C}$, $S^{\mathfrak{b}}_{m,C}$ and $B^{\mathfrak{b}}_{m,C}$ for each $m \in \mathbb{N}$;
(ii) constant symbols $\gamma^1_C, \dots, \gamma^\nu_C$.

We put $\Gamma = \Gamma(\Psi) := \{\gamma^j_C : C \in \Phi_1,\ j = 1, \dots, \nu\}$.

EXAMPLE 8.5. Let ξ be a definable vector field on $\mathbb{R}^2$ of class C^1, and let $\mathcal{M}_\xi$ be an $\mathcal{L}(\Phi_\xi)$-structure associated to ξ as in Example 6.4. Assume that there is a $\nu \in \mathbb{N}$ such that for each $C \in \mathcal{C}_{\mathrm{trans}} \cup \mathcal{C}_{\mathrm{open}}$, the set $\mathrm{Bd}_C(M_\xi)$ has cardinality at most ν, and let Ψ_ξ be a Dulac flow configuration associated to ξ as in Example 8.3. We expand $\mathcal{M}_\xi$ into an $\mathcal{L}(\Psi_\xi)$-structure $\mathcal{M}^D_\xi$ as follows: for each $C \in \Phi_1$,

(i) R_C is interpreted as the set

$$\{(x,y) \in \overline{C}^2 : \exists z(x <_C z <_C y \wedge \mathrm{Fix}_C(z)) \vee (x = y \wedge \mathrm{Fix}_C(x))\};$$

(ii) for $m \in \mathbb{N}$, $\mathfrak{g} \in \{\mathfrak{f}, \mathfrak{b}\}$ and $G \in \{S^{\mathfrak{g}}_{m,C}, B^{\mathfrak{g}}_{m,C}\}$, we put

$$* := \begin{cases} <_C & \text{if } G \text{ is } S^{\mathfrak{g}}_{m,C}, \\ >_C & \text{if } G \text{ is } B^{\mathfrak{g}}_{m,C}, \end{cases}$$

and we interpret G as the union of the sets

$$\{(x,y) \in \overline{C}^2 : \exists z\big(C(z) \wedge x <_C z <_C y \wedge C(\mathfrak{g}^m(z)) \wedge \mathfrak{g}^m(z) * z\big)\}$$

and the set $\{(x,x) : C(x) \wedge C(\mathfrak{g}^m(x)) \wedge \mathfrak{g}^m(x) * x\}$;

(iii) if $a_1 <_C \cdots <_C a_m$ are the points in C that lie on boundary cycles of ξ, we interpret γ^j_C as a_j if $1 \le j \le m$ and as $\max(C)$ if $m < j \le \nu$.

This completes the description of $\mathcal{M}^D_\xi$.

Definition 8.6. Inspired by the previous example, we let $T(\Psi)$ be the $\mathcal{L}(\Psi)$-theory consisting of $T(\Phi)$ and the universal closures of the formulas in the axiom schemes (D1)–(D6):

(D1) For each $C \in \Phi_1$, $m \in \mathbb{N}$ and $G \in \{R_C, S^{\mathfrak{f}}_{m,C}, B^{\mathfrak{f}}_{m,C}, S^{\mathfrak{b}}_{m,C}, B^{\mathfrak{b}}_{m,C}\}$, the formulas

(a) $G(x,y) \to (\overline{C}(x) \wedge \overline{C}(y))$,
(b) $G(x,y) \to (x \le_C y \vee (x = \min(C) \wedge y = \max(C)))$.

(D2) For each $C \in \Phi_1$ the formulas

(a) $R_C(x,y) \leftrightarrow \exists z(x <_C z <_C y \wedge \mathrm{Fix}_C(z))$, and
(b) $R_C(x,x) \leftrightarrow \mathrm{Fix}_C(x)$.

(D3) For each $m \in \mathbb{N}$, $C \in \Phi_1$ and $\mathfrak{g} \in \{\mathfrak{f}, \mathfrak{b}\}$ the formulas

(a) $S^{\mathfrak{g}}_{m,C}(x,y) \leftrightarrow \exists z(x <_C z <_C y \wedge \mathfrak{g}^m(z) <_C z)$,
(b) $S^{\mathfrak{g}}_{m,C}(x,x) \leftrightarrow (C(x) \wedge \mathfrak{g}^m(x) <_C x)$,
(c) $B^{\mathfrak{g}}_{m,C}(x,y) \leftrightarrow \exists z(x <_C z <_C y \wedge z <_C \mathfrak{g}^m(z))$,
(d) $B^{\mathfrak{g}}_{m,C}(x,x) \leftrightarrow (C(x) \wedge x <_C \mathfrak{g}^m(x))$.

(D4) For each $m \in \mathbb{N}$, $C \in \Phi_1$, $\mathfrak{g} \in \{\mathfrak{f}, \mathfrak{b}\}$ and $G \in \{R_C, B^{\mathfrak{g}}_{m,C}, S^{\mathfrak{g}}_{m,C}\}$ the formula

$$\begin{aligned}[G(x,y) \wedge \forall z(x <_C z <_C y \to \overline{C}(\mathfrak{g}^m(z))) \\ \wedge \neg\exists z(x <_C z <_C y \wedge \mathrm{Bd}_C(z))] \\ \to \forall z(x <_C z <_C y \to G(z,z)).\end{aligned}$$

$(\mathrm{D5})_\nu$ For each $C \in \Phi_1$ the formulas

(a) $\overline{C}(\gamma^j_C) \wedge (C(\gamma^j_C) \to \mathrm{Fix}_C(\gamma^j_C))$ for $j = 0, \ldots, \nu$,
(b) $\gamma^j_C \le_C \gamma^{j+1}_C \wedge (\gamma^j_C = \gamma^{j+1}_C \to \gamma^j_C = \max(C))$ for $j = 0, \ldots, \nu - 1$.

$(\mathrm{D6})_\nu$ For each $C \in \Phi_1$ the formula

$$(C(x) \wedge \mathrm{Bd}_C(x)) \leftrightarrow \bigvee_{j=1}^{\nu} (x = \gamma_C^j \wedge C(\gamma_C^j)).$$

This completes the description of the axioms.

PROPOSITION 8.7. *If ξ is a definable vector field on $\mathbb{R}^2$ of class C^1 with finitely many boundary cycles, then $\mathcal{M}_\xi^D \models T(\Psi_\xi)$.*

Proof. This is almost immediate from the definition of $\mathcal{M}_\xi^D$ and Proposition 6.6, except perhaps for axiom (D4), which follows from Proposition 7.4 and the fact that every bounded subset of $\mathbb{R}$ has an infimum. ∎

REMARK 8.8. Let $T(\Phi)'$ be the union of $T(\Phi)$ with axioms (D1)–(D4) only. Since (D1)–(D3) just extend $T(\Phi)$ by definitions in the sense of Section 4.6 in Shoenfield [13], the argument in the proof of the previous proposition shows that any $\mathcal{L}(\Phi_\xi)$-structure $\mathcal{M}_\xi$ as defined in Example 6.4 can be expanded to a model $\mathcal{M}'_\xi$ of $T(\Phi)'$.

9. Quantifier elimination for $T(\Psi)$. We fix a Dulac flow configuration $\Psi = (\Phi, \nu)$; our ultimate goal is to show that $T(\Psi)$ eliminates quantifiers. Most of the work in this section goes towards showing that, in order to eliminate quantifiers, we need only consider formulas of the form $\exists y \phi(x, y)$ where ϕ is of a special form.

TERMINOLOGY. Let $x = (x_1, \ldots, x_m)$ be a tuple of variables and y and z single variables. To simplify terminology, we write "term" and "formula" for "$\mathcal{L}(\Psi)$-term" and "$\mathcal{L}(\Psi)$-formula". For a formula ϕ, we write $\phi(x, y)$ to indicate that the free variables of ϕ are among $x_1, \ldots, x_m$ and y. A *binary atomic formula* is a formula of the form At_1t_2, where A is a binary relation symbol in $\mathcal{L}(\Psi)$ and t_1 and t_2 are terms.

For this section fix an arbitrary model $\mathcal{M}$ of $T(\Psi)$; again, we omit the superscript $\mathcal{M}$ when interpreting predicates in $\mathcal{M}$.

DEFINITION 9.1. An *order formula* is a quantifier-free $\mathcal{L}(\Phi) \cup \Gamma$-formula. A *$z$-order formula* is a quantifier-free formula ϕ such that every atomic subformula of ϕ containing z is an $\mathcal{L}(\Phi) \cup \Gamma$-formula.

A z-order formula ϕ is *minimal* if the only subterm of ϕ containing z is z itself and every binary atomic subformula At_1t_2 of ϕ is such that at most one of t_1 and t_2 contains z.

Our first goal is to show that we may, in order to prove quantifier elimination, restrict our attention to y-order formulas. This argument is based on the following lemma, which will also be of use later.

LEMMA 9.2. *Let $G \in \mathcal{L}(\Psi) \setminus \mathcal{L}(\Phi)$.*

(1) *The formula Gyy is equivalent in $T(\Psi)$ to a minimal y-order formula $\psi(y)$.*
(2) *The formula Gyz is equivalent in $T(\Psi)$ to a formula $\psi(y,z)$ that is both a minimal y-order formula and a minimal z-order formula.*

Proof. Let $C \in \Phi_1$, $m \in \mathbb{N}$ and $\mathfrak{g} \in \{\mathfrak{f}, \mathfrak{b}\}$ be such that G is one of R_C, $S^{\mathfrak{g}}_{m,C}$ or $B^{\mathfrak{g}}_{m,C}$. In this proof, we write $<$ instead of $<_C$; if G is R_C, we assume $m = N = N_\Phi$. By Corollary 7.6(1), any formula ϕ is equivalent in $T(\Psi)$ to the conjunction of the formulas $\vartheta_i \to \phi$, where $i \in \{1, \dots, l(m)\}$ and ϑ_i is the formula $\vartheta^m_i(\min(C), \max(C))$. Hence it suffices to prove the lemma with each $\vartheta_i \to G(y,y)$ in place of $G(y,y)$ and each $\vartheta_i \to G(y,z)$ in place of $G(y,z)$; so we also fix an i below and write ϑ in place of ϑ_i. Now by Corollary 7.6(2), there are finitely many terms α^0_j, α^1_j for $1 \le j \le r$, built up exclusively from constants, such that whenever $\mathcal{M} \models \vartheta$ the set $\{z \in C : \mathfrak{g}^m(z) \in C\}$ is the union of the open intervals $I_j = (\alpha^0_j, \alpha^1_j)$ and points $\alpha^0_j = \alpha^1_j$.

(1) We claim that the formula $\vartheta \to G(y,y)$ is equivalent to $\vartheta \to \psi^G$, where ψ^G is of the form

$$C(y) \wedge \Big(\bigvee_{1 \le j \le r} (\alpha^0_j < y < \alpha^1_j \vee \alpha^0_j = y = \alpha^1_j) \Big) \wedge \Big(\bigvee_{\beta \in Y} \psi^G_\beta \vee \bigvee_{\beta_0,\beta_1 \in Y} \psi^G_{\beta_0,\beta_1} \Big)$$

with $Y := \Gamma \cup \{\alpha^l_j : l \in \{0,1\} \text{ and } 1 \le j \le r\}$, and for each $\beta \in Y$, the formula ψ^G_β is $C(y) \wedge ((y = \beta \wedge G(\beta,\beta)) \vee y = t^G)$ with

$$t^G \text{ the term } \begin{cases} y & \text{if } G \text{ is } R_C, \\ \mathfrak{h}^m \min(C) & \text{if } G \text{ is } S^{\mathfrak{g}}_{m,C}, \\ \mathfrak{h}^m \max(C) & \text{if } G \text{ is } B^{\mathfrak{g}}_{m,C}, \end{cases}$$

and for each $\beta_0, \beta_1 \in Y$, the formula $\psi^G_{\beta_0,\beta_1}$ is of the form

$$(C(\beta_0) \vee \beta_0 = \min(C)) \wedge (C(\beta_1) \vee \beta_1 = \max(C)) \wedge \beta_0 < y < \beta_1 \wedge \eta^G_{\beta_0,\beta_1},$$

where

$$\eta^G_{\beta_0,\beta_1} \text{ is } \begin{cases} \neg S^{\mathfrak{g}}_{N,C}(\beta_0,\beta_1) \wedge \neg B^{\mathfrak{g}}_{N,C}(\beta_0,\beta_1) & \text{if } G \text{ is } R_C, \\ \neg B^{\mathfrak{g}}_{m,C}(\beta_0,\beta_1) \wedge \neg R_C(\beta_0,\beta_1) & \text{if } G \text{ is } S^{\mathfrak{g}}_{m,C}, \\ \neg S^{\mathfrak{g}}_{m,C}(\beta_0,\beta_1) \wedge \neg R_C(\beta_0,\beta_1) & \text{if } G \text{ is } B^{\mathfrak{g}}_{m,C}. \end{cases}$$

Note that $\vartheta \to \psi^G$ is a minimal y-order formula; thus, the proof of part (1) is finished once we prove the claim.

We prove the claim for R_C; the other cases of G are similar and left to the reader. Suppose that $\mathcal{M} \models \vartheta$ and pick an $a \in M$ such that $\mathcal{M} \models R_C(a,a)$. Then $\mathcal{M} \models \alpha^0_j \le a \le \alpha^1_j$ for some $j \in \{1, \dots, r\}$. If $a = \beta$ for some $\beta \in Y$, we are done, so we assume $a \ne \beta$ for all $\beta \in Y$. Then there are $\beta_0, \beta_1 \in Y$ such that $\mathcal{M} \models \beta_0 < a < \beta_1$ and $\mathcal{M} \models \neg(\beta_0 < \beta < \beta_1)$ for every $\beta \in Y$. Hence by axiom (D4), $\mathcal{M} \models R_C(b,b)$ for every $b \in (\beta_0, \beta_1)$, so $\mathcal{M} \models \neg S^{\mathfrak{g}}_{m,C}(\beta_0,\beta_1) \wedge \neg B^{\mathfrak{g}}_{m,C}(\beta_0,\beta_1)$ as required. The converse of the claim is immediate.

(2) The formula $\vartheta \to G(y,z)$ is in turn equivalent in $T(\Psi)$ to

$$\vartheta \to (G(y,z) \wedge (y = \min(C) \vee y = \max(C) \vee C(y)));$$

since the lemma is immediate for the formulas $\vartheta \to (G(y,z) \wedge y = \min(C))$ and $\vartheta \to (Gyz \wedge y = \max(C))$, we need only consider $\vartheta \to (G(y,z) \wedge C(y))$. We claim that the latter is equivalent to $\vartheta \to \psi^G$, where ψ^G is of the form

$$C(y) \wedge (C(z) \vee z = \max(C)) \wedge y \leq z \wedge ((y = z \wedge G(y,y)) \vee (y < z \wedge \eta^G)),$$

η^G is the formula

$$\bigvee_{\beta \in Y} (y = \beta \wedge G(\beta, z)) \vee \bigvee_{\beta \in Y} (y < \beta < z \wedge G(\beta,\beta)) \vee \bigvee_{\beta_0,\beta_1 \in Y,\ 1 \leq j \leq r} \eta^G_{\beta_0,\beta_1,j},$$

and for each $\beta_0, \beta_1 \in Y$ and $j \in \{1, \ldots, r\}$, the formula $\eta^G_{\beta_0,\beta_1,j}$ is

$$\beta_0 < y \wedge z < \beta_1 \wedge \alpha^0_j \leq \beta_0 \wedge \beta_1 \leq \alpha^1_j \wedge G(\beta_0,\beta_1) \wedge \eta^G_{\beta_0,\beta_1}$$

with $\eta^G_{\beta_0,\beta_1}$ defined as for part (1).

We again prove the claim for R_C, leaving the other cases of G to the reader. Suppose that $\mathcal{M} \models \vartheta$ and $\mathcal{M} \models R_C(a,b) \wedge C(b)$ and work inside $\mathcal{M}$. Suppose that $a \neq \beta$ for all $\beta \in Y$ and that $\mathcal{M} \models \neg(a < \beta < b \wedge R_C(\beta,\beta))$ for every $\beta \in Y$. Then $\mathfrak{f}^N(d) = d$ for some $d \in (a,b)$, and $d \in (\alpha^0_j, \alpha^1_j)$ for some j. Moreover, there are $\beta_0, \beta_1 \in Y$ such that $d \in (\beta_0,\beta_1)$ and $\beta \notin (\beta_0,\beta_1)$ for every $\beta \in Y$. Hence by axiom (D4), we get $\mathcal{M} \models \neg S^{\mathfrak{g}}_{N,C}(\beta_0,\beta_1) \wedge \neg B^{\mathfrak{g}}_{N,C}(\beta_0,\beta_1)$, as required. The converse of the claim is straightforward.

By symmetry, a similar claim holds with $\vartheta \to (G(y,z) \wedge C(z))$ in place of $\vartheta \to (G(y,z) \wedge C(y))$. Combining these two claims with part (1) now yields part (2). ∎

Corollary 9.3. *Every quantifier-free formula $\phi(x,y)$ is equivalent in $T(\Psi)$ to a y-order formula $\psi(x,y)$.*

Proof. It suffices to prove the proposition for all atomic formulas; the relevant atomic formulas are handled in Lemma 9.2. ∎

Our second goal in this section is to show that we only need to consider, for quantifier elimination, y-order formulas in which the complexity of any term involving y is as low as possible. Minimal y-order formulas are examples of such y-order formulas; but we cannot always reduce to minimal y-order formulas.

Definition 9.4. Let t be a term. The *z-height $h_z(t)$ of t* is defined as follows:

(i) if z does not occur in t, then $h_z(t) := 0$;
(ii) $h_z(z) := 1$;
(iii) if t is $\mathfrak{f}t'$ or $\mathfrak{b}t'$ for some term t' and z occurs in t', then $h_z(t) := h_z(t') + 1$.

Let $A(t_1, t_2)$ be a binary atomic formula; the *z-height $h_z(A(t_1, t_2))$ of $A(t_1, t_2)$* is defined as the pair $(a, b) \in \mathbb{N}^2$, where

$$a := \begin{cases} 1 & \text{if } z \text{ occurs in both } t_1 \text{ and } t_2, \\ 0 & \text{otherwise,} \end{cases}$$
$$b := \begin{cases} \min\{h_z(t_1), h_z(t_2)\} & \text{if } z \text{ occurs in both } t_1 \text{ and } t_2, \\ \max\{h_z(t_1), h_z(t_2)\} & \text{otherwise.} \end{cases}$$

Let $B(t)$ be a unary atomic formula; the *z-height $h_z(B(t))$ of $B(t)$* is defined by $h_z(B(t)) := (0, h_z(t)) \in \mathbb{N}^2$.

Let ϕ be a quantifier-free formula; the *z-height $h_z(\phi)$ of ϕ* is the maximum of the set $\{h_z(\psi) : \psi \text{ is an atomic subformula of } \phi\}$ with respect to the lexicographic ordering of $\mathbb{N}^2$. We write $h_z(\phi) = (h_z^1(\phi), h_z^2(\phi))$ below.

Finally, a term t is *mixed* if it contains both function symbols $\mathfrak{f}$ and $\mathfrak{b}$; otherwise t is called *unmixed*.

Example 9.5. Let ϕ be a z-order formula. Then $h_z(\phi) \le (0, 1)$ if and only if ϕ is minimal.

Lemma 9.6. *Let $\phi(x, y)$ be a y-order formula. Then there is a y-order formula $\psi(x, y)$ that contains no mixed terms such that ϕ and ψ are equivalent in $T(\Psi)$ and $h_y(\psi) \le h_y(\phi)$.*

Proof. Let ϕ' be the $\mathcal{L}(\Phi)$-formula obtained from ϕ by replacing each constant γ_C^j by a new variable z_C^j, for $C \in \Phi_1$ and $j = 1, \ldots, \nu$. By Lemma 7.1, ϕ' is equivalent in $T(\Phi)$ to a quantifier-free $\mathcal{L}(\Phi)$-formula ψ' that is a disjunction of formulas of the form $\eta \wedge \xi$, where ξ is obtained from ϕ' by replacing each mixed subterm by an unmixed term of lower y-height, and where η is a conjunction of some of the premises of the implications occurring in $\Theta_{(\mathfrak{f},\mathfrak{b})}$ and in $\Theta_{(\mathfrak{b},\mathfrak{f})}$ with x there replaced by various unmixed subterms of ϕ'. Clearly, $h_y(\xi) \le h_y(\phi')$ for every such ξ; since $h_y^1(\eta) = 0$ for every such η, it follows that $h_y(\psi') \le h_y(\phi')$ if $h_y^1(\phi') = 1$. On the other hand, if $h_y^1(\phi') = 0$, then every subterm t of ϕ' satisfies $h_y(t) \le h_y^2(\phi')$; so $h_y(\eta) \le h_y(\phi')$ for every such η. Therefore, we always have $h_y(\psi') \le h_y(\phi') = h_y(\phi)$, and we let ψ be the y-order formula obtained from ψ' by replacing each variable z_C^j again by γ_C^j. ∎

Below we let $\iota(y)$ denote the formula $\bigwedge_{C \in \Phi_{\text{open}}} C(y) \to E_C(y)$ and we put

$$T' := T(\Psi) \cup \{\iota(y)\}.$$

LEMMA 9.7. *Let $\phi(x,y)$ be a y-order formula. Then there is a y-order formula $\psi(x,y)$ such that ϕ is equivalent in T' to ψ and $h_y^2(\psi) \le 1$.*

Proof. By induction on $h_y(\phi)$; the case where $h_y^2(\phi) \le 1$ is trivial, so we assume that $h_y^2(\phi) > 1$ and we prove that

($*$) there exists an order formula $\psi(x,y)$ such that ϕ is equivalent in T' to ψ and $h_y(\psi) < h_y(\phi)$.

To do so, we fix arbitrary $(\mathfrak{g},\mathfrak{h}) \in \{(\mathfrak{f},\mathfrak{b}),(\mathfrak{b},\mathfrak{f})\}$, a unary predicate P, a $C \in \Phi_0$ and terms t_1 and t_2, and we assume that y occurs in t_1, and either y does not occur in t_2 or $h_y(t_1) < h_y(t_2)$. By the definition of $h_y(\phi)$ and axiom (F5), it suffices to prove ($*$) with each of the atomic formulas $P(\mathfrak{g}(t_1))$, $\mathfrak{g}(t_1) = t_2$, $\mathfrak{g}(t_1) <_C t_2$ and $t_2 <_C \mathfrak{g}(t_1)$ in place of ϕ.

CASE 1: ϕ is $P(\mathfrak{g}(t_1))$. By axioms (F7)–(F9), the formula ϕ is equivalent in T' to ψ, where ψ is the formula depending on P defined as follows:

- if $P \in \Phi_{\text{open}}$ or P is E_F for some $F \in \Phi_{\text{open}}$, then ψ is
$$\bigvee_{D\in\Phi_{\text{trans}},\,P=D^{\mathfrak{h}}} D(t_1) \vee \bigvee_{d\in\Phi_{\text{single}},\,P=d^{\mathfrak{h}}} t_1 = d;$$
- if $P \in \Phi_{\tan}$, then ψ is the formula $t_1 = \mathfrak{h}(e_P)$;
- if $P \in \Phi_{\text{trans}}$, then ψ is the formula $E_{P^{\mathfrak{h}}}(t_1)$.

In each case of ψ above, we have $h_y(\psi) < h_y(\phi)$, as required.

CASE 2: ϕ is $\mathfrak{g}(t_1) = t_2$. Then by axioms (F5), (F7)–(F9) and (F13) the formula ϕ is equivalent in T' to ψ, where ψ is the conjuction of the formulas

(i) $t_2 = s \ \vee \bigvee_{C\in\Phi_1} C(t_2) \ \vee \bigvee_{c\in\Phi_{\text{single}}} t_2 = c \ \vee \bigvee_{C\in\Phi_{\tan}} t_2 = e_C$,

(ii) $t_2 = c \to t_1 = \mathfrak{h}(c)$ for each constant c different from s,

(iii) $$t_2 = s \to \Big((t_1 = s) \vee \bigvee_{C\in\Phi_{\text{open}}} \Big(E_C(t_1) \wedge \bigwedge_{D\in S_C} \neg(r_D^{\mathfrak{h}} <_C t_1 <_C s_D^{\mathfrak{h}}) \wedge \bigwedge_{c\in\Phi_{\text{single}}} (\neg t_1 = \mathfrak{h}(c))\Big) \vee \bigvee_{C\in\Phi_{\tan}} ((\mathfrak{g}(e_C) <_C t_1 \le_C e_C \vee e_C \le_C t_1 <_C \mathfrak{g}(e_C)) \wedge \mathfrak{g}(e_C) = s)\Big)$$
with $S_C := \{D \in \Phi_{\text{trans}} : D^{\mathfrak{h}} = C\}$,

(iv) $C(t_2) \to t_1 = \mathfrak{h}(t_2)$ for $C \in \Phi_1$.

If y does not occur in t_2, then $h_y(\psi) < h_y(\phi)$; so we assume that y occurs in t_2. In this case, the only atomic subformula ξ of ψ with $h_y^1(\xi) = 1$ is $t_1 = \mathfrak{h}(t_2)$, and $h_y(t_1 = \mathfrak{h}(t_2)) = (1, h_y(t_1)) < (1, h_y(\mathfrak{g}(t_1))) = h_y(\phi)$ by hypothesis, so $h_y(\psi) < h_y(\phi)$ as well.

CASE 3: ϕ is $\mathfrak{g}(t_1) <_C t_2$. There are various subcases depending on C.

- If $C \in \Phi_{\text{trans}}$, we write $D := C^{\mathfrak{h}}$; then by axioms (F8) and (F13) the formula ϕ is equivalent in T' to ψ, where ψ is the conjunction of the formulas
$$(C(t_2) \vee t_2 = \max(C)) \wedge ((E_D(t_1) \wedge r_C^{\mathfrak{h}} <_D t_1 <_D r_C^{\mathfrak{h}}) \vee t_1 = \mathfrak{h}(\min(C)))$$
and
$$(E_D(t_1) \wedge r_C^{\mathfrak{h}} <_D t_1 <_D r_C^{\mathfrak{h}}) \to (t_1 <_D \mathfrak{h}(t_2) \vee t_2 = \max(C)).$$
- If $C \in \Phi_{\text{open}}$, then by axioms (F2), (F9), (F10), (F12) and (F13) the formula ϕ is equivalent in T' to ψ, where ψ is the conjunction of the formulas
  - (i) $\displaystyle\bigvee_{D \in \Phi_{\text{trans}},\, D^{\mathfrak{g}} = C} D(t_1) \vee \bigvee_{d \in \Phi_{\text{single}},\, P = d^{\mathfrak{h}}} t_1 = d$,
  - (ii) $(C(t_2) \wedge \neg E_C(t_2) \wedge E_C(\mathfrak{g}(t_2))) \vee \big(C(t_2) \wedge \neg E_C(t_2) \wedge E_C(\mathfrak{h}(t_2))\big) \vee E_C(t_2) \vee \big(t_2 = \max(C)\big)$,
  - (iii) $(D(t_1) \wedge E_C(t_2)) \to ((r_D^{\mathfrak{g}} <_C t_2 <_C s_D^{\mathfrak{g}} \wedge t_1 <_D \mathfrak{h}(t_2)) \vee (s_D^{\mathfrak{g}} \leq_C t_2))$ for each $D \in \Phi_{\text{trans}}$ with $D^{\mathfrak{g}} = C$,
  - (iv) $(D(t_1) \wedge \neg E_C(t_2) \wedge E_C(\mathfrak{g}(t_2))) \to ((r_D^{\mathfrak{g}} <_C \mathfrak{g}(t_2) <_C s_D^{\mathfrak{g}} \wedge t_1 <_D \mathfrak{h}(t_2)) \vee (s_D^{\mathfrak{g}} \leq_C \mathfrak{g}(t_2))$ for each $D \in \Phi_{\text{trans}}$ with $D^{\mathfrak{g}} = C$,
  - (v) $(D(t_1) \wedge \neg E_C(t_2) \wedge E_C(\mathfrak{h}(t_2))) \to ((r_D^{\mathfrak{g}} <_C \mathfrak{h}(t_2) <_C s_D^{\mathfrak{g}} \wedge t_1 \leq_D \mathfrak{h}(\mathfrak{h}(t_2))) \vee (s_D^{\mathfrak{g}} \leq_C \mathfrak{h}(t_2)))$ for each $D \in \Phi_{\text{trans}}$ with $D^{\mathfrak{g}} = C$,
  - (vi) $t_1 = d \to \mathfrak{g}d <_C t_2$ for $d \in \Phi_{\text{single}}$ with $P = d^{\mathfrak{h}}$.
- If $C \in \Phi_{\tan}$, then by axioms (F2) and (F7) the formula ϕ is equivalent in T' to ψ', where ψ' is
$$(C(t_2) \vee t_2 = \max(C)) \wedge \big((t_1 = \mathfrak{h}(e_C) \wedge e_C <_C t_2) \vee \mathfrak{g}(t_1) = \min(C)\big).$$
In this case we let ψ be the formula obtained from ψ' by replacing the subformula $\mathfrak{g}(t_1) = \min(C)$ by the corresponding formula obtained in Case 2.

We leave it to the reader to verify that $h_y(\psi) < h_y(\phi)$ in each of these subcases.

CASE 4: ϕ is $t_2 <_C \mathfrak{g}(t_1)$. This case is similar to Case 3; we leave the details to the reader. ∎

PROPOSITION 9.8. *Let $\phi(x, y)$ be a quantifier-free formula. Then there is a minimal y-order formula $\psi(x, y)$ such that ϕ is equivalent in T' to ψ.*

Proof. By Corollary 9.3 and Lemma 9.7, we may assume that ϕ is a y-order formula such that $h_y^2(\phi) \leq 1$. By Lemma 9.6, there is a y-order formula $\psi'(x, y)$ such that ϕ is equivalent in T' to ψ', ψ' contains no mixed terms and $h_y(\psi) \leq h_y(\phi)$.

In particular, for every binary atomic subformula η of ψ' in which both terms contain y, one of the terms is y itself and the other is either $\mathfrak{f}^m(y)$ or $\mathfrak{b}^m(y)$ for some $m = m(\eta) \in \mathbb{N}$. We now replace each such binary atomic subformula η of ψ' with $m(\eta) > 1$ by the formula η' defined as follows:

- if η is $y = \mathfrak{g}^m(y)$ with $\mathfrak{g} \in \{\mathfrak{f}, \mathfrak{b}\}$, then η' is the disjunction of the formulas $y = c \wedge \mathfrak{g}^m(c) = c$, for each constant symbol c, and $C(\mathfrak{g}^m(y)) \wedge R_C(y, y)$, for each $C \in \Phi_1$;
- if η is $y <_C \mathfrak{g}^m(y)$ with $\mathfrak{g} \in \{\mathfrak{f}, \mathfrak{b}\}$, then η' is $B^{\mathfrak{g}}_{m,C}(y, y)$;
- if η is $\mathfrak{g}^m(y) <_C y$ with $\mathfrak{g} \in \{\mathfrak{f}, \mathfrak{b}\}$, then η' is $S^{\mathfrak{g}}_{m,C}(y, y)$.

We also replace each occurrence of $y = y$ by $s = s$ and each occurrence of $y <_C y$ by $s \neq s$, and we denote by ψ'' the resulting formula. Clearly, $h_y(\psi'') \leq h_y(\psi')$, and every binary atomic subformula of ψ'' in which both terms contain y is of the form $G(y, y)$ for some $G \in \mathcal{L}(\Psi) \setminus \mathcal{L}(\Phi)$. Moreover, by axioms (D1)–(D4), $(\mathrm{D5})_\nu$ and $(\mathrm{D6})_\nu$, the formula ψ' is equivalent in T' to ψ''.

Next, we replace each subformula of ψ'' of the form $G(y, y)$, where $G \in \mathcal{L}(\Psi) \setminus \mathcal{L}(\Phi)$, by the corresponding minimal y-order formula $\psi(y)$ obtained in Lemma 9.2(1). If ψ''' is the resulting y-order formula, then ψ'' is equivalent in $T(\Psi)$ to ψ''' and $h^1_y(\psi''') = 0$.

Finally, by Lemmas 9.7 and 9.6, there is a y-order formula ψ such that $h_y(\psi) \leq (0, 1)$, ψ contains no mixed terms and ψ is equivalent in T' to ψ'''. ■

Finally, note that

$$T(\Phi) \cup \{C(y)\} \models \neg E_C(y) \leftrightarrow (C(\mathfrak{f}(y)) \vee C(\mathfrak{b}(y)))$$

for each $C \in \Phi_{\mathrm{open}}$, by axioms (F5), (F10) and (F12). Hence, for each $C \in \Phi_{\mathrm{open}}$ and each $\mathfrak{g} \in \{\mathfrak{f}, \mathfrak{b}\}$, we put $T_{C,\mathfrak{g}} := T(\Psi) \cup \{C(y) \wedge C(\mathfrak{g}(y))\}$; by the previous proposition, it remains to reduce quantifier-free formulas in each $T_{C,\mathfrak{g}}$. It turns out, however, that we cannot entirely reduce to minimal y-order formulas in these situations.

Instead, given $\mathfrak{g} \in \{\mathfrak{f}, \mathfrak{b}\}$, we call a formula ϕ $\mathfrak{g}$-*almost minimal* if ϕ is quantifier-free, the only subterms of ϕ containing z are z and $\mathfrak{g}(z)$, and every binary atomic subformula $A(t_1, t_2)$ of ϕ is such that at most one of t_1 and t_2 contains z.

PROPOSITION 9.9. *Let $\phi(x, y)$ be a quantifier-free formula, $C \in \Phi_{\mathrm{open}}$ and $\mathfrak{g} \in \{\mathfrak{f}, \mathfrak{b}\}$. Then there is a $\mathfrak{g}$-almost minimal y-order formula $\psi_{C,\mathfrak{g}}(x, y)$ such that ϕ is equivalent in $T_{C,\mathfrak{g}}$ to $\psi_{C,\mathfrak{g}}$.*

Proof. By Corollary 9.3 and Lemma 9.6, we may assume that ϕ is a y-order formula containing no mixed terms. On the other hand, we have $T \models \iota(\mathfrak{f}(y))$ and $T \models \iota(\mathfrak{b}(y))$ by axiom (F5). Let $\eta(x, y)$ be an atomic subformula of ϕ; it suffices to show that there is a $\mathfrak{g}$-almost minimal y-order formula $\xi_\eta(x, y)$ such that η and ξ_η are equivalent in $T_{C,\mathfrak{g}}$. If $h^2_y(\eta) = 0$, there is nothing to do, so we assume $h^2_y(\eta) > 0$, and we distinguish two cases to define ξ_η.

CASE 1: $h^2_y(\eta) > 1$. We first replace each occurrence of $\mathfrak{g}(y)$ in η by a new variable z and each occurrence of $\mathfrak{h}(y)$ in η by $\mathfrak{h}(z)$. Denote the resulting atomic formula by $\eta'(x,z)$; by axiom (F12), $\eta'(x,\mathfrak{g}(y))$ is equivalent in $T_{C,\mathfrak{g}}$ to $\eta(x,y)$. By Proposition 9.8, the formula $\eta'(x,z)$ is equivalent in T' to a minimal z-order formula $\eta''(x,z)$. Since $T(\Psi) \models \iota(\mathfrak{g}(y))$, it follows that η is equivalent in $T_{C,\mathfrak{g}}$ to the $\mathfrak{g}$-almost minimal y-order formula ξ_η given by $\eta''(x,\mathfrak{g}(y))$.

CASE 2: $h^2_y(\eta) = 1$. In this case, we take ξ_η equal to η if η contains a unary predicate symbol; so we assume that η is a binary atomic formula $A(t_1,t_2)$. If η is $y = y$, we take ξ_η to be $s = s$, and if η is $y <_D y$ for some $D \in \Phi_0$, we take ξ_η to be $s \neq s$; so we also assume from now on that $\max\{h^2_y(t_1), h^2_y(t_2)\} > 1$. By axiom (F5), the formulas $y = \mathfrak{g}^m(y)$, $y = \mathfrak{h}^m(y)$, $y <_D \mathfrak{g}^m(y)$, $y <_D \mathfrak{h}^m(y)$, $\mathfrak{g}^m(y) <_D y$ and $\mathfrak{h}^m(y) <_D y$, for $m > 0$ and $D \in \Phi_0 \setminus \{C\}$, are all equivalent in $T_{C,\mathfrak{g}}$ to $s \neq s$, so we are left with four subcases:

(i) if η is $y <_C \mathfrak{g}^m(y)$ for some $m > 0$, then we let η' be the formula $(y <_C \mathfrak{g}(y) \wedge C(\mathfrak{g}^m(y)) \wedge R_C(\mathfrak{g}(y))\mathfrak{g}y) \vee B^{\mathfrak{g}}_{m-1,C}(\mathfrak{g}(y),\mathfrak{g}(y))$;
(ii) if η is $y <_C \mathfrak{h}^m(y)$ for some $m > 0$, then we let η' be the formula $(y <_C \mathfrak{g}(y) \wedge C(\mathfrak{h}^m(y)) \wedge R_C(\mathfrak{g}(y),\mathfrak{g}(y))) \vee B^{\mathfrak{h}}_{m,C}(\mathfrak{g}(y),\mathfrak{g}(y))$;
(iii) if η is $\mathfrak{g}^m(y) <_C y$ for some $m > 0$, then we let η' be the formula $(\mathfrak{g}(y) <_C y \wedge C(\mathfrak{g}^m(y)) \wedge R_C(\mathfrak{g}(y),\mathfrak{g}(y))) \vee S^{\mathfrak{g}}_{m-1,C}(\mathfrak{g}(y),\mathfrak{g}(y))$;
(iv) if η is $\mathfrak{h}^m(y) <_C y$ for some $m > 0$, then we let η' be the formula $(\mathfrak{g}(y) <_C y \wedge C(\mathfrak{h}^m(y)) \wedge R_C(\mathfrak{g}(y),\mathfrak{g}(y))) \vee S^{\mathfrak{h}}_{m,C}(\mathfrak{g}(y),\mathfrak{g}(y))$.

We claim that η and η' are equivalent in $T_{C,\mathfrak{g}}$. We prove this for subcase (i); the other cases are similar and left to the reader. Let $b \in M$ be such that $\mathcal{M} \models C(b) \wedge C(\mathfrak{g}(b))$. Assume that $\mathcal{M} \models b <_C \mathfrak{g}^m(b) \wedge \neg B^{\mathfrak{g}}_{m-1,C}(\mathfrak{g}(b),\mathfrak{g}(b))$. Then $\mathfrak{g}^m(b) \in E_C$ and $\mathfrak{g}^m(b) \leq_C \mathfrak{g}(b)$ by axioms (F2) and (F5). Hence $b <_C \mathfrak{g}(b)$, so $\mathcal{M} \models \phi^{\mathfrak{f}}(b,\mathfrak{g}(b))$ by axioms (F10) and (F12), which implies $\mathfrak{g}^m(b) = \mathfrak{g}(b)$ as required. Conversely, assume first that $\mathcal{M} \models b <_C \mathfrak{g}(b) \wedge C(\mathfrak{g}^m(b)) \wedge R_C(\mathfrak{g}(b),\mathfrak{g}(b))$; then $b <_C \mathfrak{g}^m(b)$ by axioms (D2) and (F14). Now assume that $\mathcal{M} \models B^{\mathfrak{g}}_{m-1,C}(\mathfrak{g}(b),\mathfrak{g}(b))$; then $\mathfrak{g}(b) <_C \mathfrak{g}^m(b)$ by axiom (D3), and hence $b <_C \mathfrak{g}^m(b)$ by axioms (F10) and (F12).

Finally, by Proposition 9.8, the formulas $B^{\mathfrak{g}}_{k,C}(z,z)$, $S^{\mathfrak{g}}_{k,C}(z,z)$, $C(\mathfrak{g}^k(z)) \wedge R_C(z,z)$ and $C(\mathfrak{h}^k(z)) \wedge R_C(z,z)$ are each equivalent in T' to minimal z-order formulas. It follows from the above claim that we are left with subcases (i)–(iv) for $m = 1$. But by axioms (F5), (F10) and (F12) we have $T_{C,\mathfrak{g}} \models \neg C(\mathfrak{h}(y))$. Hence $T_{C,\mathfrak{g}} \models \neg\phi^{\mathfrak{h}}_C(y,\mathfrak{h}(y))$, so from axioms (F10) and (F12) we get $T_{C,\mathfrak{g}} \models \phi_C^{\mathfrak{g}}(y, \mathfrak{g}y)$. Therefore, $y <_C \mathfrak{g}(y)$ is equivalent in $T_{C,\mathfrak{g}}$ to $s = s$ if $\mathfrak{g}$ is $\mathfrak{f}$, and to $\neg s = s$ if $\mathfrak{g}$ is $\mathfrak{b}$; the other subcases follow similarly. ■

The previous two propositions allow us to reduce the problem of eliminating quantifiers in $T(\Psi)$ to that of eliminating quantifiers in two simpler theories: for $C \in \Phi_1 \cup \Phi_{\tan}$ we let $\mathcal{L}_C$ be the language $\{<_C, \min(C), \max(C)\}$ and T_C be the $\mathcal{L}_C$-theory consisting of the universal closures of

(A1) the sentences stating that $<_C$ is a dense linear ordering on C, together with the formula $x = \min(C) \vee x = \max(C) \vee \min(C) <_C x <_C \max(C)$.

For $C \in \Phi_{\text{open}}$ we let $\mathcal{L}_C$ be the language $\{<_C, \pi_C, E_C, \min(C), \max(C)\}$, where π_C a unary function symbol, and we let T_C be the $\mathcal{L}_C$-theory consisting of the universal closures of (A1) as well as

(B1) the formula $E_C(\pi_C(x)) \wedge (E_C(x) \to \pi_C(x) = x)$;
(B2) the formula $\pi_C(x) <_C x \to \neg\exists y(E_C(y) \wedge \pi_C(x) <_C y <_C x)$;
(B3) the formula $x <_C \pi_C(x) \to \neg\exists y(E_C(y) \wedge x <_C y <_C \pi_C(x))$;
(B4) the sentences stating that for every $x \in E_C$, the restriction of $<_C$ to the set $\{y : \pi_C(y) = x\}$ is a dense linear ordering without endpoints.

A routine application of a quantifier elimination test such as Theorem 3.1.4 in [8] gives the following result; we leave the details to the reader.

PROPOSITION 9.10. *For each unary predicate symbol C of $\mathcal{L}(\Phi)$, the theory T_C admits quantifier elimination in the language $\mathcal{L}_C$.* ■

THEOREM 9.11. *The theory $T(\Psi)$ admits quantifier elimination.*

Proof. Let $\phi(x, y)$ be a quantifier-free formula; we show that $\exists y\phi(x, y)$ is equivalent in $T(\Psi)$ to a quantifier-free formula. First, note that $\exists y\phi(x, y)$ is equivalent in $T(\Psi)$ to the disjunction of the formulas

(1) $\phi(x, c)$ for each constant c;
(2) $\exists y(C(y) \wedge \phi(x, y))$ for each $C \in \Phi_1 \cup \Phi_{\tan}$;
(3) $\exists y(C(y) \wedge C\mathfrak{g}(y) \wedge \phi(x, y))$ for each $C \in \Phi_{\text{open}}$ and each $\mathfrak{g} \in \{\mathfrak{f}, \mathfrak{b}\}$.

We deal with each disjunct separately; since formulas of type (1) are trivial to handle, we deal with types (2) and (3).

Type (2). Let $C \in \Phi_1 \cup \Phi_{\tan}$. Since $T(\Psi) \models C(y) \to \iota(y)$, we may assume by Proposition 9.8 that ϕ is a minimal y-order formula. Without loss of generality, we may also assume that ϕ is a conjunction of atomic formulas, that y occurs in each of the atomic subformulas of ϕ and, by axiom (F1), that ϕ contains only the relation symbols $=$ and $<_C$. Let $t_1, \dots, t_k$ be all maximal subterms of ϕ that do not contain y, and let $\phi'(z_1, \dots, z_k, y)$ be the formula obtained from ϕ by replacing each t_i by a new variable z_i. Then ϕ' is a $<_C$-formula without parameters; by Proposition 9.10, there is a quantifier-free $\mathcal{L}_C$-formula $\psi'(z_1,\dots,z_k)$ such that $\exists y\phi'$ and ψ' are equivalent in T_C. Let $\psi(x)$ be the $\mathcal{L}(\Psi)$-formula obtained from ψ' by replacing each z_i by t_i; then $\exists y\phi$ and ψ are equivalent in $T(\Psi)$, as required.

Type (3). Let $C \in \Phi_{\text{open}}$ and $\mathfrak{g} \in \{\mathfrak{f},\mathfrak{b}\}$; by Proposition 9.9, we may assume that ϕ is a $\mathfrak{g}$-almost minimal y-order formula. Without loss of generality, we may also assume that ϕ is a conjunction of atomic formulas, that y occurs in each of the atomic subformulas of ϕ and, by axiom (F1), that ϕ contains only the relation symbols $=$, $<_C$ and E_C. Let $t_1,\dots,t_k$ be all maximal subterms of ϕ that do not contain y, and let $\phi'(z_1,\dots,z_k,y)$ be the formula obtained from ϕ by replacing each t_i by a new variable z_i. Note that ϕ' contains no parameters. Arguing as for Type (2), it now suffices to find a quantifier-free formula $\psi'(z_1,\dots,z_k)$ equivalent in $T(\Psi)$ to $\exists y\phi'(z_1,\dots,z_k,y)$.

To do so, we let π_C be a new unary function symbol and let $T(\Psi)_C$ be the theory $T(\Psi)$ together with the universal closure of the formula

$$\begin{aligned} y=\pi_C(x) \leftrightarrow ((&E_C(x)\wedge y=x)\\ &\vee(C(x)\wedge C(\mathfrak{f}(x))\wedge y=\mathfrak{f}(x))\vee(C(x)\wedge C(\mathfrak{b}(x))\wedge y=\mathfrak{b}(x))).\end{aligned}$$

Since $T(\Psi)_C$ is an extension by definitions of $T(\Psi)$ in the sense of [13, Section 4.6], it suffices to find a quantifier-free $\mathcal{L}(\Psi)$-formula $\psi'(z_1,\dots,z_k)$ equivalent in $T(\Psi)_C$ to $\exists y\phi'(z_1,\dots,z_k,y)$.

Let ϕ'' be the $\mathcal{L}_C$-formula obtained from ϕ' by replacing each occurrence of $\mathfrak{g}(y)$ by $\pi(y)$; then ϕ' and ϕ'' are equivalent in $T(\Psi)_C$. Since $T(\Psi)_C \models T_C$, there is by Proposition 9.10 a quantifier-free $\mathcal{L}_C$-formula $\psi''(z_1,\dots,z_k)$ that is equivalent in $T(\Psi)_C$ to $\exists y\phi''(z_1,\dots,z_k,y)$; without loss of generality, we may assume that the only subterms of ψ'' are z_i and πz_i for $i=1,\dots,k$.

Finally, we let ψ' be the $\mathcal{L}(\Psi)$-formula obtained from ψ'' by replacing each atomic subformula η of ψ'' by an $\mathcal{L}(\Psi)$-formula η' determined as follows:

(i) if η is $E_C(\pi_C(z_i))$, we let η' be $C(z_i)\wedge(E_C(z_i)\vee C(\mathfrak{f}(z_i))\vee C(\mathfrak{b}(z_i)))$;

(ii) if η is $\pi_C(z_i) * z_j$ with $* \in \{=,<_C,>_C\}$, we let η' be

$$C(z_i)\wedge C(z_j)\wedge\Big(\bigvee_{\mathfrak{g}\in\{\mathfrak{f}^0,\mathfrak{f},\mathfrak{b}\}} E_C(\mathfrak{g}(z_i))\wedge\mathfrak{g}(z_i)*z_j\Big);$$

(iii) if η is $\pi_C(z_i) <_C \pi_C(z_j)$ and $* \in \{=,<_C\}$, we let η' be

$$C(z_i)\wedge C(z_j)\wedge\Big(\bigvee_{\mathfrak{g},\mathfrak{h}\in\{\mathfrak{f}^0,\mathfrak{f},\mathfrak{b}\}} E_C(\mathfrak{g}(z_i))\wedge E_C(\mathfrak{h}(z_j))\wedge\mathfrak{g}(z_i)*\mathfrak{h}(z_j)\Big);$$

and if η is not of one of the forms (i)–(iii) above, we let η' be η. This ψ' is equivalent in $T(\Psi)_C$ to ψ'' and is of the required form. ■

10. Consequences for the model theory of $T(\Psi)$. The quantifier elimination result established in the previous section allows us to show that the theory $T(\Psi)$ is very well-behaved: it is a theory of finite rank in the sense developed by Onshuus [10].

We first rephrase the results from the previous section. For a flow configuration Φ, $C \in \Phi_{\mathrm{open}}$, $\mathcal{M} \models T(\Psi)$ and $x \in E_C^{\mathcal{M}}$, we put

$$C_x^{\mathcal{M}} := \{y \in C^{\mathcal{M}} : y = x \vee \mathfrak{f}(y) = x \vee \mathfrak{b}(y) = x\}$$

and $\overline{C}_x^{\mathcal{M}} := C_x^{\mathcal{M}} \cup \{\mathfrak{f}(x), \mathfrak{g}(x)\}$. The following corollary implies Theorem C:

COROLLARY 10.1. *Let Ψ be a Dulac flow configuration and $\mathcal{M} \models T(\Psi)$.*

(1) *For $C \in \Phi_1 \cup \Phi_{\mathrm{tan}}$, every definable subset of $C^{\mathcal{M}}$ is a finite union of points and open $<_C$-intervals with endpoints in $\overline{C}$.*
(2) *For $C \in \Phi_{\mathrm{open}}$ and $x \in E_C^{\mathcal{M}}$, every definable subset of $C_x^{\mathcal{M}}$ is a finite union of points and open $<_C$-intervals with endpoints in $\overline{C}_x^{\mathcal{M}}$.*

Proof. This follows immediately from Theorem 9.11, Propositions 9.8 and 9.9 and axioms (F2) and (F11). ■

Below we use the terminology of rosy theories.

THEOREM 10.2. *Let Ψ be a Dulac flow configuration and T be any completion of $T(\Psi)$. Then T is rosy with $\mathrm{U}^{\text{þ}}(T) \leq 2$.*

Proof. Let $p(x)$ be a complete 1-type in T, $\mathcal{M} \models T$ and $a \in M$ be such that $\mathcal{M} \models p(a)$. If $C(x) \in p$ for some $C \in \Phi_{\mathrm{tan}} \cup \Phi_1$, then by Corollary 10.1(1) the type p is determined by the $<_C$-order type of x over the constants; hence $\mathrm{U}^{\text{þ}}(p) \leq 1$. If $C(x) \wedge \neg E_C(x) \in p$ for some $C \in \Phi_{\mathrm{open}}$, then by Corollary 10.1(2) the type p is determined by the $<_C$-order type $o(x)$ of a over the constants and $\pi_C(a)$, where $\pi_C : C \to E_C$ is given by

$$\pi_C(z) := \begin{cases} z & \text{if } z \in E_C^{\mathcal{M}}, \\ \mathfrak{f}(z) & \text{if } \mathfrak{f}(z) \in E_C^{\mathcal{M}}, \\ \mathfrak{b}(z) & \text{if } \mathfrak{b}(z) \in E_C^{\mathcal{M}}. \end{cases}$$

Again by Corollary 10.1(1), the type of $\pi_C(a)$ over the constants is determined by the $<_C$-order type of $\pi_C(a)$ over the constants.

Since p contains either one of the above formulas or a formula $x = c$ for some constant symbol c, it follows from Fact 2 in the introduction that $\mathrm{U}^{\text{þ}}(T) \leq 2$. ■

In fact, the $\mathrm{U}^{\text{þ}}$-rank in the previous theorem is actually equal to 2:

PROPOSITION 10.3. *Let Φ be a flow configuration and $\mathcal{M} \models T(\Phi)$, and assume that $\Phi_{\mathrm{open}} \neq \emptyset$. Then $\mathrm{U}^{\text{þ}}(\mathcal{M}) \geq 2$.*

Proof. Let $C \in \Phi_{\mathrm{open}}$. Then by the example in the introduction, the theory of $(C, <_C, E_C)$ has $\mathrm{U}^{\text{þ}}$-rank at least two. Hence $\mathrm{U}^{\text{þ}}(\mathcal{M}) \geq 2$. ■

There is a certain converse to Theorem 10.2 based on Remark 8.8: We let Φ be a flow configuration and consider the theory $T(\Phi)^+$ obtained by adding the universal closures of the following formulas to $T(\Phi)'$ for each $C \in \Phi_{\mathrm{trans}}$:

$$\begin{aligned} &C(x) \to \exists y(\overline{C}(y) \wedge y = \inf\{z : x <_C z \wedge \mathrm{Bd}_C(z)\}), \\ &C(x) \to \exists y(\overline{C}(y) \wedge y = \sup\{z : z <_C x \wedge \mathrm{Bd}_C(z)\}). \end{aligned} \tag{10.1}$$

EXAMPLES 10.4.

(1) Let Ψ be a Dulac flow configuration. Then any model $\mathcal{M}$ of $T(\Psi)$ satisfies (10.1).
(2) Let ξ be a definable vector field on $\mathbb{R}^2$, and let $\mathcal{M}_\xi$ be an $\mathcal{L}(\Phi_\xi)$-structure associated to ξ as in Example 6.4. Then $\mathcal{M}_\xi$ satisfies (10.1) by Corollary 7.5, and by Remark 8.8 the structure $\mathcal{M}_\xi$ can be expanded to a model $\mathcal{M}_\xi^+$ of $T(\Phi_\xi)^+$.

Below for each $\nu \in \mathbb{N}$ we abbreviate the formula stating that $\mathrm{Bd}_C(x)$ defines a set with at most ν elements by "$|\mathrm{Bd}_C(x)| \le \nu$".

PROPOSITION 10.5. *Let Φ be a flow configuration and T be a completion of $T(\Phi)^+$, and assume that* $\mathrm{U}^\flat(T) \le 2$. *Then there is a $\nu \in \mathbb{N}$ such that*

(1) $T \models |\mathrm{Bd}_C(x)| \le \nu$;
(2) *every model $\mathcal{M}$ of T can be expanded to a model of $T(\Phi, \nu)$.*

Proof. (1) Assume that $T \not\models |\mathrm{Bd}_C(x)| \le \nu$ for any $\nu \in \mathbb{N}$. Then by model-theoretic compactness, there are an $\mathcal{M} \models T$ and a $C \in \Phi_1$ such that the set $\mathrm{Bd}_C(M)$ is infinite; we may assume that $\mathcal{M}$ is $\aleph_1$-saturated. Moreover, by axiom (F8), we may assume that $C \in \Phi_{\mathrm{trans}}$. Also, by axiom (F8) and an argument as in the proof of Proposition 10.3, it suffices to find a $d \in C^{\mathcal{M}}$ such that $\mathrm{U}^\flat(d) \ge 2$.

Since $\mathcal{M}$ is $\aleph_1$-saturated, there is an interval $I \subseteq C^{\mathcal{M}}$ such that $I \cap \mathrm{acl}(\emptyset) = \emptyset$ and $I \cap \mathrm{Bd}_C(M)$ is infinite. By (10.1) and since $\mathrm{Bd}_C(M)$ is nowhere dense, there is a $c \in I \setminus \mathrm{Bd}_C(M)$ such that the elements $a := \sup\{x \in I : x <_C c \wedge \mathrm{Bd}_C(x)\}$ and $b := \inf\{x \in C : a <_C x \wedge \mathrm{Bd}_C(x)\}$ exist in I. Then $a <_C b$, $a, b \notin \mathrm{acl}(\emptyset)$, $b \in \mathrm{dcl}(a)$ and

$$\mathcal{M} \models a <_C b \wedge \mathrm{Bd}_C(a) \wedge \neg\exists x(C(x) \wedge a <_C x <_C b \wedge \mathrm{Bd}_C(x)).$$

It follows that the formula $\phi(x) := a <_C x <_C b$ strongly divides over $\emptyset$; hence $\mathrm{U}^\flat(d) \ge 2$ for some $d \in C^{\mathcal{M}}$, as required.

Part (2) follows from Proposition 8.7 and part (1). ∎

We can now prove our restatement of Dulac's Problem:

Proof of Theorem B. (1) If ξ has finitely many boundary cycles, then by Proposition 8.7 the structure $\mathcal{M}_\xi$ can be expanded into a model $\mathcal{M}_\xi^D$ of $T(\Phi_\xi, \nu)$ for some $\nu \in \mathbb{N}$. Since $(\Phi_\xi)_{\mathrm{open}} \ne \emptyset$, it follows that $2 \le \mathrm{U}^\flat(\mathcal{M}_\xi) \le \mathrm{U}^\flat(\mathcal{M}_\xi^D) \le 2$ by Proposition 10.3 and Theorem 10.2.

Conversely, if $\mathrm{U}^{\flat}(\mathcal{M}_\xi) = 2$ then by Proposition 10.5, the structure $\mathcal{M}_\xi$ can be expanded into a model of $T(\Phi_\xi, \nu)$ for some $\nu \in \mathbb{N}$, so by Example 6.8 the vector field ξ has finitely many boundary cycles.

(2) follows from (1) and Poincaré's Theorem [12] (see also [11, p. 217]). The "moreover" clause follows from (1) and Theorem 10.2. ∎

11. Final questions and remarks

(1) In the situation of Theorem B, is it possible for $\mathcal{M}_\xi$ to be rosy of $\mathrm{U}^{\flat}$-rank strictly greater than 2?

(2) Can a restatement of Hilbert's 16th Problem be obtained in the spirit of Theorem B?

A naïve approach to this question is as follows: Let $\{\xi_a : a \in A\}$ be a family of vector fields on $\mathbb{R}^2$ definable in $\mathcal{R}$. Since the arguments in Sections 1 through 5 are uniform in parameters, we may assume that there is a flow configuration Φ such that $\Phi_{\xi_a} = \Phi$ for all $a \in A$. In this situation, one can readily reformulate the theory $T(\Phi)$ for the parametric situation; and if one also assumes the existence of a uniform bound $\nu \in \mathbb{N}$ on the number of boundary cycles of each ξ_a, such a reformulation extends to $T(\Phi, \nu)$. We suspect that under the latter assumption, the corresponding theory is rosy of $\mathrm{U}^{\flat}$-rank 3; however, this does not appear to us to be a completely trivial generalization of the results in Section 10, and we plan to pursue it in a future project.

(3) The structure $\mathcal{M}_\xi^D$ in Example 8.5 does not define any algebraic operations (by Theorem 9.11). Assume here that $S(\xi) = \emptyset$; is it possible to expand $\mathcal{M}_\xi^D$ by some (or all) of the sets definable in the original o-minimal structure $\mathcal{R}$ without increasing the $\mathrm{U}^{\flat}$-rank? We know very little about this question. However, if (a) the x-axis, the projection from $\mathbb{R}^2$ onto the x-axis, and both addition and multiplication are definable in an expansion $\mathcal{M}'$ of $\mathcal{M}_\xi^D$, and if (b) the expansion $\mathcal{M}'$ still has $\mathrm{U}^{\flat}$-rank 2, then $\mathcal{M}'$ (and hence $\mathcal{M}_\xi^D$) would be definable in an o-minimal structure. (The assumption that $\mathcal{M}'$ has $\mathrm{U}^{\flat}$-rank two is necessary here.) Thus, question (3) is related to the following question:

(4) Is the structure $\mathcal{M}_\xi^D$ of Example 8.5 definable in some o-minimal expansion of the real field?

(5) Consider a Dulac flow configuration Ψ and $\mathcal{M} \models T(\Psi)$. Corollary 10.1, Theorem 10.2 and their respective proofs may be loosely interpreted as indicating that $\mathcal{M}$ is built-up from sets $D \subseteq M$ on which the induced structure is o-minimal. Is there a theory of structures built-up from sets with induced o-minimal structure, say in the spirit of Zilber's results on the fine structure of uncountably categorical theories [16]?

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