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Affine group acting on hyperspaces of compact convex subsets of $\mathbb{R}^n$

by

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Abstract. For every $n \geq 2$, let $cc(\mathbb{R}^n)$ denote the hyperspace of all nonempty compact convex subsets of the Euclidean space $\mathbb{R}^n$ endowed with the Hausdorff metric topology. Let $cb(\mathbb{R}^n)$ be the subset of $cc(\mathbb{R}^n)$ consisting of all compact convex bodies. In this paper we discover several fundamental properties of the natural action of the affine group $\mathrm{Aff}(n)$ on $cb(\mathbb{R}^n)$. We prove that the space $E(n)$ of all n-dimensional ellipsoids is an $\mathrm{Aff}(n)$-equivariant retract of $cb(\mathbb{R}^n)$. This is applied to show that $cb(\mathbb{R}^n)$ is homeomorphic to the product $Q \times \mathbb{R}^{n(n+3)/2}$, where Q stands for the Hilbert cube. Furthermore, we investigate the action of the orthogonal group $O(n)$ on $cc(\mathbb{R}^n)$. In particular, we show that if $K \subset O(n)$ is a closed subgroup that acts nontransitively on the unit sphere $\mathbb{S}^{n-1}$, then the orbit space $cc(\mathbb{R}^n)/K$ is homeomorphic to the Hilbert cube with a point removed, while $cb(\mathbb{R}^n)/K$ is a contractible Q-manifold homeomorphic to the product $(E(n)/K) \times Q$. The orbit space $cb(\mathbb{R}^n)/\mathrm{Aff}(n)$ is homeomorphic to the Banach–Mazur compactum $\mathrm{BM}(n)$, while $cc(\mathbb{R}^n)/O(n)$ is homeomorphic to the open cone over $\mathrm{BM}(n)$.

1. Introduction. Let $cc(\mathbb{R}^n)$ denote the hyperspace of all nonempty compact subsets of the Euclidean space $\mathbb{R}^n$, $n \geq 1$, equipped with the Hausdorff metric:

$$d_{\mathrm{H}}(A, B) = \max\Big\{\sup_{b\in B} d(b, A), \sup_{a\in A} d(a, B)\Big\},$$

where d is the standard Euclidean metric on $\mathbb{R}^n$.

By $cb(\mathbb{R}^n)$ we shall denote the subspace of $cc(\mathbb{R}^n)$ consisting of all compact convex bodies of $\mathbb{R}^n$, i.e.,

$$cb(\mathbb{R}^n) = \{A \in cc(\mathbb{R}^n) \mid \mathrm{Int}\, A \neq \emptyset\}.$$

It is easy to see that $cc(\mathbb{R}^1)$ is homeomorphic to the closed half-plane $\{(x, y) \in \mathbb{R}^2 \mid x \leq y\}$, while $cb(\mathbb{R}^1)$ is homeomorphic to $\mathbb{R}^2$. In [22] it was

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proved that for $n \geq 2$, $cc(\mathbb{R}^n)$ is homeomorphic to the punctured Hilbert cube, i.e., Hilbert cube with a point removed. Furthermore, a simple combination of [6, Corollary 8] and [7, Theorem 1.4] shows that the hyperspace $\mathcal{B}(n)$, consisting of all centrally symmetric (about the origin) convex bodies $A \in cb(\mathbb{R}^n)$, $n \geq 2$, is homeomorphic to $\mathbb{R}^p \times Q$, where Q denotes the Hilbert cube and $p = n(n+1)/2$. However, the topological structure of $cb(\mathbb{R}^n)$ has remained open.

In this paper we study the topological structure of the hyperspace $cb(\mathbb{R}^n)$. Namely, we will show that $cb(\mathbb{R}^n)$ is homeomorphic to the product $Q \times \mathbb{R}^{n(n+3)/2}$. Our argument is based on some fundamental properties of the natural action of the affine group $\mathrm{Aff}(n)$ on $cb(\mathbb{R}^n)$. We prove that $\mathrm{Aff}(n)$ acts properly on $cb(\mathbb{R}^n)$ (Theorem 3.3). Using a well-known result in affine convex geometry about the minimal-volume ellipsoid, we construct a convenient global $O(n)$-slice $L(n)$ for $cb(\mathbb{R}^n)$. Namely, as proved by F. John [17], for each $A \in cb(\mathbb{R}^n)$ there exists a unique minimal-volume ellipsoid $l(A)$ that contains A (see also [15]). It turns out that the map $l : cb(\mathbb{R}^n) \to E(n)$ is an $\mathrm{Aff}(n)$-equivariant retraction onto the subset $E(n)$ of $cb(\mathbb{R}^n)$ consisting of all n-dimensional ellipsoids (Theorem 3.6). Then the convenient global $O(n)$-slice of $cb(\mathbb{R}^n)$ is just the inverse image $L(n) = l^{-1}(\mathbb{B}^n)$ of the n-dimensional closed Euclidean unit ball $\mathbb{B}^n = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$. In other words, $L(n)$ is the subspace of $cb(\mathbb{R}^n)$ consisting of all convex bodies A for which $\mathbb{B}^n$ is the minimal-volume ellipsoid. This fact implies that the two orbit spaces $cb(\mathbb{R}^n)/\mathrm{Aff}(n)$ and $L(n)/O(n)$ are homeomorphic (Corollary 3.7(2)). Taking into account the compactness of $L(n)$ (Proposition 3.4(d)) we recover Macbeath's result [20] from the early fifties to the effect that $cb(\mathbb{R}^n)/\mathrm{Aff}(n)$ is compact (Corollary 3.7(1)).

We show in Corollary 3.9 that $cb(\mathbb{R}^n)$ is homeomorphic (even $O(n)$-equivariantly) to the product $L(n) \times E(n)$. Further, in Section 5 we prove that $L(n)$ is homeomorphic to the Hilbert cube (Corollary 5.9), while $E(n)$ is homeomorphic to $\mathbb{R}^{n(n+3)/2}$ (Corollary 3.10). Thus, we conclude that $cb(\mathbb{R}^n)$ is homeomorphic to $Q \times \mathbb{R}^{n(n+3)/2}$ (Corollary 3.11), one of the main results of the paper.

In Corollary 3.8 we prove that the orbit space $cb(\mathbb{R}^n)/\mathrm{Aff}(n)$ is homeomorphic to the Banach–Mazur compactum $\mathrm{BM}(n)$. Recall that $\mathrm{BM}(n)$ is the set of isometry classes of n-dimensional Banach spaces topologized by the following metric best known in functional analysis as the *Banach–Mazur distance*:

$$d(E,F) = \ln \inf\{\|T\| \cdot \|T^{-1}\| \mid T : E \to F \text{ is a linear isomorphism}\}.$$

These spaces were introduced in 1932 by S. Banach [11] and they continue to be of interest. The original geometric representation of $\mathrm{BM}(n)$ is based on the one-to-one correspondence between norms and odd symmetric convex bodies (see [30, p. 644] and [19, p. 1191]). A. Pełczyński's question of whether the Banach–Mazur compacta $\mathrm{BM}(n)$ are homeomorphic to the Hilbert cube (see [30, Problem 899]) was answered negatively for $n = 2$ by the first author [6]; the case $n \geq 3$ still remains open. The reader can find other results concerning the Banach–Mazur compacta and related spaces in [7].

In Section 4 we study the hyperspace $M(n)$ of all compact convex subsets of the unit ball $\mathbb{B}^n$ which intersect the boundary sphere $\mathbb{S}^{n-1}$. It is established in Corollary 4.13 that for every closed subgroup $K \subset O(n)$ that acts nontransitively on $\mathbb{S}^{n-1}$, the K-orbit space $M(n)/K$ is homeomorphic to the Hilbert cube. In particular, $M(n)$ is homeomorphic to Q. On the other hand, $M_0(n)/K$ is a Hilbert cube manifold for each closed subgroup $K \subset O(n)$, where $M_0(n) = M(n) \setminus \{\mathbb{B}^n\}$. In Theorem 4.16 it is established that $M(n)/O(n)$ is just homeomorphic to the Banach–Mazur compactum $\mathrm{BM}(n)$. The main technique we develop in this section is further applied to Section 5 as well. There we establish analogous properties of the global $O(n)$-slice $L(n)$ of the proper $\mathrm{Aff}(n)$-space $cb(\mathbb{R}^n)$ (Proposition 5.8, Corollary 5.9 and Theorem 5.11).

In Sections 6 and 7 we investigate some orbit spaces of $cc(\mathbb{R}^n)$ and $cb(\mathbb{R}^n)$. We prove in Theorem 7.1 that if K is a closed subgroup of $O(n)$ which acts nontransitively on $\mathbb{S}^{n-1}$, then $cc(\mathbb{R}^n)/K$ is homeomorphic to the punctured Hilbert cube. The orbit space $cc(\mathbb{R}^n)/O(n)$ is homeomorphic to the open cone over the Banach–Mazur compactum $\mathrm{BM}(n)$ (Theorem 7.2). Respectively, $cb(\mathbb{R}^n)/K$ is a contractible Q-manifold homeomorphic to $(E(n)/K) \times Q$ (Theorem 6.1), while the topological structure of $cb(\mathbb{R}^n)/O(n)$ mainly remains unknown.

2. Preliminaries. We refer the reader to the monographs [12] and [23] for basic notions of the theory of G-spaces. However we will recall here some special definitions and results which will be used throughout the paper.

All topological spaces and topological groups are assumed to be Tychonoff.

If G is a topological group and X is a G-space, for any $x \in X$ we denote by G_x the *stabilizer* of x, i.e., $G_x = \{g \in G \mid gx = x\}$. For a subset $S \subset X$ and a subgroup $H \subset G$, $H(S)$ denotes the *H-saturation* of S, i.e., $H(S) = \{hs \mid h \in H,\ s \in S\}$. If $H(S) = S$ then we say that S is an *H-invariant set*. In particular, $G(x)$ denotes the *G-orbit* of x, i.e., $G(x) = \{gx \in X \mid g \in G\}$. The orbit space is denoted by X/G.

For each subgroup $H \subset G$, the *H-fixed point set* X^H is the set $\{x \in X \mid H \subset G_x\}$. Clearly, X^H is a closed subset of X.

The family of all subgroups of G that are conjugate to H is denoted by $[H]$, i.e., $[H] = \{gHg^{-1} \mid g \in G\}$. We will call $[H]$ a *G-orbit type* (or simply an *orbit type*). For two orbit types $[H_1]$ and $[H_2]$, one says that $[H_1] \preceq [H_2]$

iff $H_1 \subset gH_2g^{-1}$ for some $g \in G$. The relation $\preceq$ is a partial ordering on the set of all orbit types. Since $G_{gx} = gG_xg^{-1}$ for any $x \in X$ and $g \in G$, we have $[G_x] = \{G_{gx} \mid g \in G\}$.

A continuous map $f : X \to Y$ between two G-spaces is called *equivariant* or a *G-map* if $f(gx) = g(fx)$ for every $x \in X$ and $g \in G$. If the action of G on Y is trivial and $f : X \to Y$ is an equivariant map, then we will say that f is an *invariant* map.

For any subgroup $H \subset G$, we will denote by G/H the G-space of cosets $\{gH \mid g \in G\}$ equipped with the action induced by left translations.

A G-space X is called *proper* (in the sense of Palais [24]) if it has an open cover consisting of so-called small sets. A set $S \subset X$ is called *small* if any point $x \in X$ has a neighborhood V such that the set $\langle S, V\rangle = \{g \in G \mid gS \cap V \neq \emptyset\}$, called the *transporter* from S to V, has compact closure in G.

Each orbit in a proper G-space is closed, and each stabilizer is compact [24, Proposition 1.1.4]. If G is a locally compact group and Y is a proper G-space, then for every point $y \in Y$ the orbit $G(y)$ is G-homeomorphic to G/G_y [24, Proposition 1.1.5].

For a given topological group G, a metrizable G-space Y is called a *G-equivariant absolute neighborhood retract* (denoted by $Y \in G$-ANR) if for any metrizable G-space M containing Y as an invariant closed subset, there exist an invariant neighborhood U of Y in M and a G-retraction $r : U \to Y$. If we can always take $U = M$, then we say Y is a *G-equivariant absolute retract* (denoted by $Y \in G$-AR).

Let us recall the well known definition of a slice [24, p. 305]:

DEFINITION 2.1. Let X be a G-space and H a closed subgroup of G. An H-invariant subset $S \subset X$ is called an *H-slice* in X if $G(S)$ is open in X and there exists a G-equivariant map $f : G(S) \to G/H$ such that $S = f^{-1}(eH)$. The saturation $G(S)$ is called a *tubular* set. If $G(S) = X$, then we say that S is a *global* H-slice of X.

In the case of a compact group G one has the following intrinsic characterization of H-slices. A subset $S \subset X$ of a G-space X is an H-slice if and only if it satisfies the following four conditions: (1) S is H-invariant, (2) $G(S)$ is open in X, (3) S is closed in $G(S)$, (4) if $g \in G \setminus H$ then $gS \cap S = \emptyset$ (see [12, Ch. II, §4 and §5]).

The following is one of the fundamental results in the theory of topological transformation groups (see, e.g., [12, Ch. II, §4 and §5]):

THEOREM 2.2 (Slice Theorem). *Let G be a compact Lie group, X a Tychonoff G-space and $x \in X$ any point. Then:*

(1) *There exists a G_x-slice $S \subset X$ such that $x \in S$.*
(2) *$[G_y] \preceq [G_x]$ for each point $y \in G(S)$.*

Let G be a compact Lie group and X a G-space. By a *G-normal cover* of X, we mean a family

$$\mathcal{U} = \{gS_\mu \mid g \in G,\ \mu \in M\}$$

where each S_μ is an H_μ-slice for some closed subgroup H_μ of G, the family $\{G(S_\mu)\}_{\mu\in M}$ of saturations is an open cover for X and there exists a locally finite invariant partition of unity $\{p_\mu : X \to [0,1] \mid \mu \in M\}$ subordinated to $\{G(S_\mu)\}_{\mu\in M}$. That is, each p_μ is an invariant function with $\overline{p_\mu^{-1}((0,1])} \subset G(S_\mu)$ and the supports $\{\overline{p_\mu^{-1}((0,1])} \mid \mu \in M\}$ constitute a locally finite family. We refer to [7] for further information on G-normal covers.

Yet another result which plays an important role in the paper is

THEOREM 2.3 (Orbit Space Theorem [4]). *Let G be a compact Lie group and X a G-ANR (resp., a G-AR). Then X/G is an ANR (resp., an AR).*

Let (X,d) be a metric G-space. If $d(gx,gy) = d(x,y)$ for all $x,y \in X$ and $g \in G$, then we say that d is a *G-invariant* (or simply *invariant*) metric.

Suppose that G is a compact group acting on a metric space (X,d). If d is G-invariant, it is well-known [23, Proposition 1.1.12] that the quotient topology of X/G is generated by the metric

$$d^*(G(x),G(y)) = \inf_{g\in G} d(x,gy), \qquad G(x),G(y) \in X/G. \tag{2.1}$$

It is evident that

$$d^*(G(x),G(y)) \le d(x,y), \qquad x,y \in X. \tag{2.2}$$

In the following we will denote by d the Euclidean metric on $\mathbb{R}^n$. For any $A \subset \mathbb{R}^n$ and $\varepsilon > 0$, we denote $N(A,\varepsilon) = \{x \in \mathbb{R}^n \mid d(x,A) < \varepsilon\}$. In particular, for every $x \in \mathbb{R}^n$, $N(x,\varepsilon)$ denotes the open ε-ball around x. On the other hand, if $\mathcal{C} \subset cc(\mathbb{R}^n)$ then for every $A \in \mathcal{C}$ we shall use $O(A,\varepsilon)$ for the ε-open ball in $\mathcal{C}$ centered at A, i.e.,

$$O(A,\varepsilon) = \{B \in \mathcal{C} \mid d_{\mathrm{H}}(A,B) < \varepsilon\},$$

where d_{H} stands for the Hausdorff metric induced by d.

For every subset $A \subset X$ of a topological space X, we write ∂A and $\overline{A}$ for, respectively, the boundary and the closure of A in X.

We will denote by $\mathbb{B}^n$ the n-dimensional Euclidean closed unit ball and by $\mathbb{S}^{n-1}$ the corresponding unit sphere, i.e.,

$$\mathbb{B}^n = \Big\{(x_1,\dots,x_n) \in \mathbb{R}^n \;\Big|\; \sum_{i=1}^n x_i^2 \le 1\Big\},$$

$$\mathbb{S}^{n-1} = \Big\{(x_1,\dots,x_n) \in \mathbb{R}^n \;\Big|\; \sum_{i=1}^n x_i^2 = 1\Big\}.$$

The Hilbert cube $[0,1]^\infty$ will be denoted by Q. By $cc(\mathbb{B}^n)$ we denote the subspace of $cc(\mathbb{R}^n)$ consisting of all $A \in cc(\mathbb{R}^n)$ such that $A \subset \mathbb{B}^n$. It is well known that $cc(\mathbb{B}^n)$ is homeomorphic to Q (see [22, Theorem 2.2]).

A *Hilbert cube manifold* or a *Q-manifold* is a separable, metrizable space that admits an open cover each member of which is homeomorphic to an open subset of Q. We refer to [14] and [21] for the theory of Q-manifolds.

A closed subset A of a metric space (X,d) is called a *Z-set* if the set $\{f \in C(Q,X) \mid f(Q) \cap A = \emptyset\}$ is dense in $C(Q,X)$, where $C(Q,X)$ is the space of all continuous maps from Q to X endowed with the compact-open topology. In particular, if for every $\varepsilon > 0$ there exists a map $f : X \to X \setminus A$ such that $d(x, f(x)) < \varepsilon$, then A is a Z-set.

A map $f : X \to Y$ between topological spaces is called *proper* if $f^{-1}(C)$ is compact for each compact set $C \subset Y$. A proper map $f : X \to Y$ between ANR's is called *cell-like* (abbreviated CE) if it is onto and each point inverse $f^{-1}(y)$ has the *property* UV^∞: for each neighborhood U of $f^{-1}(y)$ there exists a neighborhood $V \subset U$ of $f^{-1}(y)$ such that the inclusion $V \hookrightarrow U$ is homotopic to a constant map of V into U. In particular, if $f^{-1}(y)$ is contractible, then it has the property UV^∞ (see [14, Ch. XIII]).

3. Affine group acting properly on $cb(\mathbb{R}^n)$. Let (X,d) be a metric space and G a topological group acting continuously on X. Consider the hyperspace 2^X consisting of all nonempty compact subsets of X equipped with the Hausdorff metric topology. Define an action of G on 2^X by

$$(g, A) \mapsto gA, \qquad gA = \{ga \mid a \in A\}. \tag{3.1}$$

The reader can easily verify the continuity of this action.

3.1. Properness of the $\mathrm{Aff}(n)$-action on $cb(\mathbb{R}^n)$. Throughout the paper, n will always denote a natural number greater than or equal to 2.

We will denote by $\mathrm{Aff}(n)$ the group of all affine transformations of $\mathbb{R}^n$. Let us recall the definition of $\mathrm{Aff}(n)$. For every $v \in \mathbb{R}^n$ let $T_v : \mathbb{R}^n \to \mathbb{R}^n$ be the translation by v, i.e., $T_v(x) = v + x$ for all $x \in \mathbb{R}^n$. The set of all such translations is a group isomorphic to the additive group of $\mathbb{R}^n$. For every $\sigma \in GL(n)$ and $v \in \mathbb{R}^n$ it is easy to see that $\sigma T_v \sigma^{-1} = T_{\sigma(v)}$. This yields a homomorphism from $GL(n)$ to the group of all linear automorphisms of $\mathbb{R}^n$, and hence we have an (internal) semidirect product

$$\mathbb{R}^n \rtimes GL(n)$$

called the *affine group* of $\mathbb{R}^n$ (see e.g. [2, p. 102]). Each element $g \in \mathrm{Aff}(n)$ is usually represented by $g = T_v + \sigma$, where $\sigma \in GL(n)$ and $v \in \mathbb{R}^n$, i.e.,

$$g(x) = v + \sigma(x) \quad \text{for every } x \in \mathbb{R}^n.$$

As a semidirect product, $\mathrm{Aff}(n)$ is topologized by the product topology of $\mathbb{R}^n \times GL(n)$, thus becoming a Lie group with two connected components. Since the topology of $GL(n)$ is the one inherited from $\mathbb{R}^{n^2}$, we can also consider a natural topological embedding of $\mathrm{Aff}(n)$ into $\mathbb{R}^n \times \mathbb{R}^{n^2} = \mathbb{R}^{n(n+1)}$, which will be helpful in the proof of Theorem 3.3.

Clearly, the natural action of $\mathrm{Aff}(n)$ on $\mathbb{R}^n$ is continuous. This action induces a continuous action on $2^{\mathbb{R}^n}$. Observe that for every $g \in \mathrm{Aff}(n)$ and $A \in cb(\mathbb{R}^n)$, the set $gA = \{ga \mid a \in A\}$ belongs to $cb(\mathbb{R}^n)$, i.e., $cb(\mathbb{R}^n)$ is an invariant subset of $2^{\mathbb{R}^n}$ and thus the restriction of the $\mathrm{Aff}(n)$-action to $cb(\mathbb{R}^n)$ is continuous. We will prove in Theorem 3.3 that this action is proper. First we prove the following two technical lemmas.

Lemma 3.1. *Let $A \in cb(\mathbb{R}^n)$ and let $x_0 \in A$ be such that $\overline{N(x_0, 2\varepsilon)} \subset A$ for a certain $\varepsilon > 0$. If $C \in O(A, \varepsilon)$ then $N(x_0, \varepsilon) \subset C$.*

Proof. Suppose there exists $C \in O(A,\varepsilon)$ such that $N(x_0,\varepsilon) \not\subset C$. Choose $x \in N(x_0,\varepsilon) \setminus C$. Since C is compact, there exists $z \in C$ with $d(x,z) = d(x,C)$. Let H be the hyperplane through z in $\mathbb{R}^n$ orthogonal to the ray $\overrightarrow{xz}$. Since C is convex, it lies in the halfspace determined by H which does not contain x. Let a be the intersection point of $\overrightarrow{zx}$ with $\partial\overline{N(x_0, 2\varepsilon)} \subset A$. Evidently, $d(a, x_0) = 2\varepsilon$ and

$$d(a,z) = d(a,H) \le d(a,C) \le d_{\mathrm{H}}(A,C) < \varepsilon.$$

Since $d(x_0,x) < \varepsilon$ the triangle inequality implies that

$$\varepsilon > d(a,z) > d(a,x) \ge d(a,x_0) - d(x_0,x) > 2\varepsilon - \varepsilon = \varepsilon.$$

This contradiction proves the lemma. ∎

Observe that $cb(\mathbb{R}^n)$ is not closed in $cc(\mathbb{R}^n)$. However, we have the following lemma:

Lemma 3.2. *Let $A \in cb(\mathbb{R}^n)$ and $x_0 \in A$ be such that $\overline{N(x_0, 2\varepsilon)} \subset A$ for a certain $\varepsilon > 0$. Then $\overline{O(A,\varepsilon)}$, the closure of $O(A,\varepsilon)$ in $cb(\mathbb{R}^n)$, is compact.*

Proof. First we observe that $O(A,\varepsilon)$ is contained in $cc(K)$ for some compact convex subset $K \subset \mathbb{R}^n$, where $cc(K)$ stands for the hyperspace of all compact convex subsets of K. By [22], $cc(K)$ is compact, and hence the closure of $O(A,\varepsilon)$ in $cc(K)$, denoted by $[O(A,\varepsilon)]$, is also compact. So, it is enough to prove that $[O(A,\varepsilon)]$ is contained in $cb(\mathbb{R}^n)$.

Let $(D_m)_{m\in\mathbb{N}} \subset O(A,\varepsilon)$ be a sequence of compact convex bodies converging to some $D \in cc(K)$. According to Lemma 3.1, $N(x_0,\varepsilon) \subset D_m$ for every $m \in \mathbb{N}$. Suppose that $N(x_0,\varepsilon) \not\subset D$. Pick $x \in N(x_0,\varepsilon) \setminus D$ and let $\eta = d(x,D) > 0$. Since $x \in D_m$ for each $m \in \mathbb{N}$, it is clear that $d_{\mathrm{H}}(D_m, D) \ge \eta$. This means that $(D_m)_{m\in\mathbb{N}}$ cannot converge to D, a contradiction. This proves that $N(x_0,\varepsilon)$ is contained in D, and therefore D has nonempty interior, so that $D \in cb(\mathbb{R}^n)$. Thus, $[O(A,\varepsilon)]$ is a compact set contained in $cb(\mathbb{R}^n)$, which yields $\overline{O(A,\varepsilon)} = [O(A,\varepsilon)]$, and hence $\overline{O(A,\varepsilon)}$ is compact. ∎

THEOREM 3.3. $\mathrm{Aff}(n)$ *acts properly on* $cb(\mathbb{R}^n)$.

Proof. Let $A \in cb(\mathbb{R}^n)$ and assume that $x_0 \in A$ and $\varepsilon > 0$ are such that $\overline{N(x_0, 2\varepsilon)} \subset A$. We claim that $O(A,\varepsilon)$ is a small neighborhood of A.

Indeed, let $B \in cb(\mathbb{R}^n)$. Since B has nonempty interior, there are $z_0 \in B$ and $\delta > 0$ such that $\overline{N(z_0, 2\delta)} \subset B$. We will prove that the transporter

$$\Gamma = \{g \in \mathrm{Aff}(n) \mid gO(A,\varepsilon) \cap O(B,\delta) \neq \emptyset\}$$

has compact closure in $\mathrm{Aff}(n)$.

It is sufficient to prove that Γ, viewed as a subset of $\mathbb{R}^n \times \mathbb{R}^{n^2}$, is bounded and its closure in $\mathrm{Aff}(n)$ coincides with the one in $\mathbb{R}^n \times \mathbb{R}^{n^2}$.

For every $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, let $\|x\|_\infty = \max_{i=1}^n |x_i|$. There exists $M > 0$ such that, if $C \in O(A,\varepsilon) \cup O(B,\delta)$ then

$$\|c\|_\infty \le M \quad \text{for all } c \in C. \tag{3.2}$$

In particular,

$$\operatorname{diam} C = \sup_{c,c' \in C} \|c - c'\|_\infty \le 2M.$$

Take any $\mu \in \Gamma$. There exist $A' \in O(A,\varepsilon)$ and $B' \in O(B,\delta)$ with $\mu A' = B'$. Since μ is an affine transformation, there are $u \in \mathbb{R}^n$ and $\sigma \in GL(n)$ such that $\mu(x) = u + \sigma(x)$ for all $x \in \mathbb{R}^n$. Let (σ_{ij}) be the matrix representing σ in the canonical basis of $\mathbb{R}^n$, and consider (σ_{ij}) as a point in $\mathbb{R}^{n^2}$.

Since $\mu A' = B' \in O(B,\delta)$, according to inequality (3.2), $\operatorname{diam} \mu A' \le 2M$. Observe that $\mu A' = \sigma A' + u$, and hence $\operatorname{diam} \sigma A' = \operatorname{diam} \mu A' \le 2M$. Let

$$\xi_i = (0, \ldots, 0, \varepsilon/2, 0, \ldots, 0) \in \mathbb{R}^n,$$

where $\varepsilon/2$ is the ith coordinate. Then, by Lemma 3.1, $\xi_i + x_0 \in N(x_0,\varepsilon) \subset A'$ and $-\xi_i + x_0 \in N(x_0,\varepsilon) \subset A'$. Since $\operatorname{diam} \sigma A' \le 2M$, we get

$$\begin{aligned}\|2\sigma(\xi_i)\|_\infty = \|\sigma(2\xi_i)\|_\infty &= \|\sigma((\xi_i + x_0) - (-\xi_i + x_0))\|_\infty \\ &= \|\sigma(\xi_i + x_0) - \sigma(-\xi_i + x_0)\|_\infty \le 2M,\end{aligned}$$

and thus $\|\sigma(\xi_i)\|_\infty \le M$.

However, $\sigma(\xi_i) = (\sigma_{1i}\varepsilon/2, \ldots, \sigma_{ni}\varepsilon/2)$, and therefore $|\sigma_{ji}\varepsilon/2| \le M$ for all $i, j = 1, \ldots, n$. Thus, $|\sigma_{ji}| < 2M/\varepsilon$.

Next, by (3.2), for every $a = (a_1, \ldots, a_n) \in A'$ one has $\|a\|_\infty \le M$. Then

$$\|\sigma(a)\|_\infty = \max_{i=1}^n \Big| \sum_{j=1}^n \sigma_{ij} a_j \Big| \le \sum_{i=1}^n \frac{2M}{\varepsilon} \|a\|_\infty \le \frac{2nM^2}{\varepsilon}.$$

On the other hand, $\mu(a) \in B'$, which yields

$$M \ge \|\mu(a)\|_\infty = \|u + \sigma(a)\|_\infty \ge \|u\|_\infty - \|\sigma(a)\|_\infty \ge \|u\|_\infty - 2nM^2/\varepsilon.$$

This implies that $\|u\|_\infty \le M + 2nM^2/\varepsilon$, and therefore Γ, viewed as a subset of $\mathbb{R}^n \times \mathbb{R}^{n^2}$, is bounded.

To complete the proof, it remains to show that the closure of Γ in $\mathrm{Aff}(n)$ coincides with its closure in $\mathbb{R}^n \times \mathbb{R}^{n^2}$. Observe that here $\mathbb{R}^{n^2}$ represents the space of all real $n \times n$-matrices, i.e., the space of all linear transformations from $\mathbb{R}^n$ into itself. Therefore, an element $\lambda \in \mathbb{R}^n \times \mathbb{R}^{n^2}$ represents a map which is the composition of a linear transformation followed by a translation. In this case, λ is an affine transformation iff it is surjective.

Let $(\lambda_m)_{m\in\mathbb{N}} \subset \Gamma$ be a sequence of affine transformations converging to some $\lambda \in \mathbb{R}^n \times \mathbb{R}^{n^2}$. We need to prove that $\lambda \in \mathrm{Aff}(n)$. Since $\lambda_m \in \Gamma$, there exist $A_m \in O(A,\varepsilon)$ and $B_m \in O(B,\delta)$ such that $\lambda_m A_m = B_m$. By Lemma 3.2, the closures $\overline{O(A,\varepsilon)}$ and $\overline{O(B,\delta)}$ are compact. Hence, we can assume that A_m converges to some $A_0 \in \overline{O(A,\varepsilon)}$ and B_m converges to some $B_0 \in \overline{O(B,\delta)}$. Then the equality $\lambda_m A_m = B_m$ yields $\lambda A_0 = B_0$. Since B_0 has nonempty interior, we infer that $\dim B_0 = n$, and hence the dimension of $\lambda(\mathbb{R}^n)$ also equals n. Thus, $\lambda(\mathbb{R}^n)$ is an n-dimensional hyperplane in $\mathbb{R}^n$, which is possible only if $\lambda(\mathbb{R}^n) = \mathbb{R}^n$. Thus, λ is surjective, as required. ∎

3.2. A convenient global slice for $cb(\mathbb{R}^n)$. A well-known result of F. John [17] (see also [15]) in affine convex geometry states that for each $A \in cb(\mathbb{R}^n)$ there is a unique minimal-volume ellipsoid $l(A)$ containing A (respectively, a maximal-volume ellipsoid $j(A)$ contained in A). Nowadays $j(A)$ is called the *John ellipsoid* of A while $l(A)$ is called its *Löwner ellipsoid.* We denote by $L(n)$ (resp., $J(n)$) the subspace of $cb(\mathbb{R}^n)$ consisting of all convex bodies $A \in cb(\mathbb{R}^n)$ for which the Euclidean unit ball $\mathbb{B}^n$ is the Löwner ellipsoid (resp., the John ellipsoid). By $E(n)$ we denote the subset of $cb(\mathbb{R}^n)$ consisting of all ellipsoids. Below we consider the map $l : cb(\mathbb{R}^n) \to E(n)$ that sends a convex body $A \in cb(\mathbb{R}^n)$ to its minimal-volume ellipsoid $l(A)$. We call l the *Löwner map.*

Proposition 3.4. *$L(n)$ has the following four properties:*

(a) *$L(n)$ is $O(n)$-invariant.*
(b) *The saturation $\mathrm{Aff}(n)(L(n))$ coincides with $cb(\mathbb{R}^n)$.*
(c) *If $gL(n) \cap L(n) \neq \emptyset$ for some $g \in \mathrm{Aff}(n)$, then $g \in O(n)$.*
(d) *$L(n)$ is compact.*

Proof. First we prove the following

Claim. *The Löwner map $l : cb(\mathbb{R}^n) \to E(n)$ is $\mathrm{Aff}(n)$-equivariant, i.e., $l(gA) = gl(A)$ for every $g \in \mathrm{Aff}(n)$ and $A \in cb(\mathbb{R}^n)$.*

Assume that there exist $A \in cb(\mathbb{R}^n)$ and $g \in \mathrm{Aff}(n)$ such that $l(gA) \neq gl(A)$. Clearly, $gl(A)$ is an ellipsoid containing gA. Since the minimal-volume ellipsoid of $g(A)$ is unique, we infer that $\mathrm{vol}(gl(A)) > \mathrm{vol}(l(gA))$.

By the same argument, $\mathrm{vol}(g^{-1}l(gA)) > \mathrm{vol}(l(A))$. Now we apply the well-known fact that each affine transformation preserves the ratio of volumes of any pair of compact convex bodies. Thus

$$\frac{\mathrm{vol}(l(A))}{\mathrm{vol}(A)} = \frac{\mathrm{vol}(gl(A))}{\mathrm{vol}(gA)} > \frac{\mathrm{vol}(l(gA))}{\mathrm{vol}(gA)} = \frac{\mathrm{vol}(g^{-1}l(gA))}{\mathrm{vol}(A)} > \frac{\mathrm{vol}(l(A))}{\mathrm{vol}(A)}.$$

This contradiction proves the claim.

(a) Let $g \in O(n)$ and $A \in L(n)$. The Claim implies that $l(gA) = gl(A) = g\mathbb{B}^n = \mathbb{B}^n$, i.e., $gA \in L(n)$, so $L(n)$ is $O(n)$-invariant.

(b) Let $A \in cb(\mathbb{R}^n)$. There exists $g \in \mathrm{Aff}(n)$ such that $l(A) = g\mathbb{B}^n$. According to the Claim we have

$$\mathbb{B}^n = g^{-1}l(A) = l(g^{-1}A).$$

Then $g^{-1}A \in L(n)$ and $A = g(g^{-1}A)$. This proves $\mathrm{Aff}(n)(L(n)) = cb(\mathbb{R}^n)$.

(c) If there exist $g \in \mathrm{Aff}(n)$ and $A \in L(n)$ such that $gA \in L(n)$, then

$$\mathbb{B}^n = l(gA) = gl(A) = g\mathbb{B}^n.$$

Hence $g \in O(n)$.

(d) Clearly, $L(n) \subset cc(\mathbb{B}^n)$. Since $cc(\mathbb{B}^n)$ is compact (in fact, it is homeomorphic to the Hilbert cube [22, Theorem 2.2]), it suffices to show that $L(n)$ is closed in $cc(\mathbb{B}^n)$.

Let $(A_k)_{k\in\mathbb{N}} \subset L(n)$ be a sequence converging to $A \in cc(\mathbb{B}^n)$. We will prove that $A \in L(n)$. To this end, we first prove that A has nonempty interior. If not, there exists an $(n-1)$-dimensional hyperplane $\mathcal{H} \subset \mathbb{R}^n$ such that $A \subset \mathcal{H}$. Let $E' \subset \mathcal{H}$ be an $(n-1)$-dimensional ellipsoid containing A in its interior (with respect to $\mathcal{H}$). For any $r > 0$, consider the line segment T_r of length r which is orthogonal to $\mathcal{H}$ and passes through the center of E'. Let $r > 0$ be small enough that the n-dimensional ellipsoid E generated by E' and T_r has volume less than $\mathrm{vol}(\mathbb{B}^n)$. Since A lies in the interior of E, there exists $\delta > 0$ such that $N(A,\delta) \subset E$. Now, we use the fact that (A_k) converges to A to find $m_0 \in \mathbb{N}$ such that $A_{m_0} \subset N(A,\delta) \subset E$. Thus, E is an ellipsoid containing A_{m_0} and so

$$\mathrm{vol}(\mathbb{B}^n) = \mathrm{vol}(l(A_{m_0})) < \mathrm{vol}(E) < \mathrm{vol}(\mathbb{B}^n).$$

This contradiction proves that A has nonempty interior.

Consequently, $l(A)$ is defined and we have to show that $l(A) = \mathbb{B}^n$. Suppose that $l(A) \neq \mathbb{B}^n$. Since $A_k \subset \mathbb{B}^n$ for every $k \in \mathbb{N}$, it follows that $A \subset \mathbb{B}^n$. Hence, by uniqueness of the minimal-volume ellipsoid, $\mathrm{vol}(l(A)) < \mathrm{vol}(\mathbb{B}^n)$. Let L be an ellipsoid concentric and homothetic with $l(A)$ with ratio > 1 and $\mathrm{vol}(L) < \mathrm{vol}(\mathbb{B}^n)$. As $l(A)$ is contained in the interior of L, the distance $d_{\mathrm{H}}(\partial L, \partial l(A)) = \varepsilon$ is positive. Consider $U = N(\partial l(A), \varepsilon)$, the ε-neighborhood of $\partial l(A)$ in $\mathbb{R}^n$. Since $(A_k)_{k\in\mathbb{N}}$ converges to A and all the sets A_k are convex, $(\partial A_k)_{k\in\mathbb{N}}$ converges to ∂A. Therefore, there exists $k_0 \geq 1$ such that

$\partial A_{k_0} \subset U$. The convexity of A_{k_0} implies that $A_{k_0} \subset L$, and hence

$$\mathrm{vol}(l(A_{k_0})) \leq \mathrm{vol}(L) < \mathrm{vol}(\mathbb{B}^n) = \mathrm{vol}(l(A_{k_0})).$$

This contradiction proves that $A \in L(n)$, and hence $L(n)$ is closed in $cc(\mathbb{B}^n)$. ∎

REMARK 3.5. The first three assertions of Proposition 3.4 are easy modifications of those in [6, proof of Theorem 4], while the fourth one provides a new way of proving Macbeath's result on compactness of $cb(\mathbb{R}^n)/\mathrm{Aff}(n)$ (see Corollary 3.7(1)).

THEOREM 3.6.

(1) *The Löwner map* $l : cb(\mathbb{R}^n) \to E(n)$ *is an* $\mathrm{Aff}(n)$*-equivariant retraction with* $L(n) = l^{-1}(\mathbb{B}^n)$.
(2) $L(n)$ *is a compact global* $O(n)$*-slice for the proper* $\mathrm{Aff}(n)$*-space* $cb(\mathbb{R}^n)$.

Proof. (1) In the proof of Proposition 3.4 we showed that $l : cb(\mathbb{R}^n) \to E(n)$ is $\mathrm{Aff}(n)$-equivariant. Clearly, it is a retraction. Its continuity is a standard consequence of the above four properties in Proposition 3.4, well known in transformation groups (see [12, Ch. II, Theorems 4.2 and 4.4] for compact group actions and [24] for locally compact proper group actions). However, using the compactness of $L(n)$ we shall give here a direct proof of this fact.

Let $(X_m)_{m=1}^{\infty}$ be a sequence in $cb(\mathbb{R}^n)$ that converges to $X \in cb(\mathbb{R}^n)$; we write $X_m \rightsquigarrow X$. We must show that $l(X_m) \rightsquigarrow l(X)$. Assume the contrary is true. Then there exist $\varepsilon > 0$ and a subsequence (A_k) of (X_m) such that $d_{\mathrm{H}}(l(A_k), l(A)) \geq \varepsilon$ for all $k = 1, 2, \ldots$.

By Proposition 3.4(b), there are $g, g_k \in \mathrm{Aff}(n)$, $k = 1, 2, \ldots$, such that $A_k = g_k S_k$ and $A = gP$ for some $P, S_k \in L(n)$. Due to compactness of $L(n)$, without loss of generality, one can assume that $S_k \rightsquigarrow S$ for some $S \in L(n)$. Since $\mathrm{Aff}(n)$ acts properly on $cb(\mathbb{R}^n)$ (see Theorem 3.3), the points S and P have neighborhoods U_S and U_P, respectively, such that the transporter $\langle U_S, U_P \rangle$ has compact closure. Since $S_k \rightsquigarrow S$ and $g^{-1} g_k S_k \rightsquigarrow P$, it then follows that there is a natural number k_0 such that $g^{-1} g_k \in \langle U_S, U_P \rangle$ for all $k \geq k_0$. Consequently, the sequence $(g^{-1} g_k)$ has a convergent subsequence. Again, it is no loss of generality to assume that $g^{-1} g_k \rightsquigarrow h$ for some $h \in \mathrm{Aff}(n)$. This implies that $g^{-1} g_k S_k \rightsquigarrow hS$, which together with $g^{-1} g_k S_k \rightsquigarrow P$ yields $hS = P$. But S and P belong to $L(n)$, and hence Proposition 3.4(c) shows that $h \in O(n)$. Since $g_k \rightsquigarrow gh$, we get

$$l(A_k) = l(g_k S_k) = g_k l(S_k) = g_k \mathbb{B}^n \rightsquigarrow gh\mathbb{B}^n = g\mathbb{B}^n = gl(S) = l(gS) = l(A),$$

which contradicts the inequality $d_{\mathrm{H}}(l(A_k), l(A)) \geq \varepsilon$, $k = 1, 2, \ldots$.

Hence, $l(X_m) \rightsquigarrow l(X)$, as required.

(2) Compactness of $L(n)$ was proved in Proposition 3.4(d). Since $E(n)$ is the $\mathrm{Aff}(n)$-orbit of $\mathbb{B}^n \in cb(\mathbb{R}^n)$ and $O(n)$ is the stabilizer of $\mathbb{B}^n$, one has the $\mathrm{Aff}(n)$-homeomorphism $E(n) \cong \mathrm{Aff}(n)/O(n)$ (see [24, Proposition 1.1.5]). This, together with (1), yields an $\mathrm{Aff}(n)$-equivariant map $f\colon cb(\mathbb{R}^n) \to \mathrm{Aff}(n)/O(n)$ such that $L(n) = f^{-1}(O(n))$. Thus, $L(n)$ is a global $O(n)$-slice for $cb(\mathbb{R}^n)$, as required. ∎

Corollary 3.7.

(1) (Macbeath [20]) *The* $\mathrm{Aff}(n)$*-orbit space* $cb(\mathbb{R}^n)/\mathrm{Aff}(n)$ *is compact.*
(2) *The orbit spaces* $L(n)/O(n)$ *and* $cb(\mathbb{R}^n)/\mathrm{Aff}(n)$ *are homeomorphic.*

Proof. Let $\pi\colon L(n) \to cb(\mathbb{R}^n)/\mathrm{Aff}(n)$ be the restriction of the orbit map $cb(\mathbb{R}^n) \to cb(\mathbb{R}^n)/\mathrm{Aff}(n)$. Then π is continuous and it follows from Proposition 3.4(b) that π is onto. This already implies the first assertion if we remember that $L(n)$ is compact (see Proposition 3.4(d)).

Further, for $A, B \in L(n)$, it follows from Proposition 3.4(c) that $\pi(A) = \pi(B)$ iff A and B have the same $O(n)$-orbit. Hence, π induces a continuous bijective map $p\colon L(n)/O(n) \to cb(\mathbb{R}^n)/\mathrm{Aff}(n)$. Since $L(n)/O(n)$ is compact we conclude that p is a homeomorphism. ∎

In Theorem 5.11 we will prove that $L(n)/O(n)$ is homeomorphic to the Banach–Mazur compactum $\mathrm{BM}(n)$. This, in combination with Corollary 3.7 implies the following:

Corollary 3.8. *The* $\mathrm{Aff}(n)$*-orbit space* $cb(\mathbb{R}^n)/\mathrm{Aff}(n)$ *is homeomorphic to the Banach–Mazur compactum* $\mathrm{BM}(n)$*.*

Corollary 3.9.

(1) *There exists an* $O(n)$*-equivariant retraction* $r\colon cb(\mathbb{R}^n) \to L(n)$ *such that* $r(A)$ *belongs to the* $\mathrm{Aff}(n)$*-orbit of* A*.*
(2) *The diagonal product of the retractions* $r\colon cb(\mathbb{R}^n) \to L(n)$ *and* $l : cb(\mathbb{R}^n) \to E(n)$ *is an* $O(n)$*-equivariant homeomorphism*

$$cb(\mathbb{R}^n) \cong_{O(n)} L(n) \times E(n).$$

Proof. (1) Recall that $O(n)$ is a maximal compact subgroup of $\mathrm{Aff}(n)$. According to the structure theorem (see [16, Ch. XV, Theorem 3.1]), there exists a closed subset $T \subset \mathrm{Aff}(n)$ such that $gTg^{-1} = T$ for every $g \in O(n)$, and the multiplication map

$$(t, g) \mapsto tg\colon T \times O(n) \to \mathrm{Aff}(n) \tag{3.3}$$

is a homeomorphism. In our case it is easy to see that for T one can take the set of all products AS, where A is a translation and S is an invertible symmetric (or self-adjoint) positive operator. This follows easily from two standard facts in linear algebra: (1) each $a \in \mathrm{Aff}(n)$ is uniquely represented as the composition of a translation $t \in \mathbb{R}^n$ and an invertible operator $g \in GL(n)$, (2) by the polar decomposition theorem, every $g \in GL(n)$ can be uniquely represented as the composition of a nondegenerate symmetric positive operator and an orthogonal operator (see, e.g., [18, Sections 2.3 and 2.4]).

Now we define the required $O(n)$-equivariant retraction $r\colon cb(\mathbb{R}^n) \to L(n)$. Let $f\colon \mathrm{Aff}(n) \to E(n)$ be defined by $f(g) = g\mathbb{B}^n$. Then f induces an $\mathrm{Aff}(n)$-equivariant homeomorphism $\widetilde{f}\colon \mathrm{Aff}(n)/O(n) \to E(n)$ [24, Proposition 1.1.5] and f is the composition

$$\mathrm{Aff}(n) \xrightarrow{\pi} \mathrm{Aff}(n)/O(n) \xrightarrow{\widetilde{f}} E(n),$$

where π is the natural quotient map. By compactness of $O(n)$, π is closed, and hence so is f, being the composition of two closed maps.

This implies that the restriction $f|_T\colon T \to E(n)$ is a homeomorphism. Moreover, this homeomorphism is $O(n)$-equivariant if we let $O(n)$ act on T by inner automorphisms and on $E(n)$ by the action induced from $cb(\mathbb{R}^n)$.

Denote by $\xi\colon E(n) \to T$ the inverse map f^{-1}. Then we have the following characteristic property of ξ:

$$[\xi(C)]^{-1}C = \mathbb{B}^n \quad \text{for all } C \in E(n). \tag{3.4}$$

Next, we define

$$r(A) = [\xi(l(A))]^{-1}A \quad \text{for every } A \in cb(\mathbb{R}^n).$$

Clearly, r depends continuously on $A \in cb(\mathbb{R}^n)$.

Since $l(r(A)) = l([\xi(l(A))]^{-1}A) = [\xi(l(A))]^{-1}l(A)$, and since by (3.4), $[\xi(l(A))]^{-1}l(A) = \mathbb{B}^n$, we infer that $r(A) \in L(n)$. If $A \in L(n)$, then $l(A) = \mathbb{B}^n$ and $r(A) = [\xi(l(A))]^{-1}A = [\xi(\mathbb{B}^n)]^{-1}A = 1 \cdot A = A$. Thus, r is a well-defined retraction on $L(n)$.

Let us check that r is $O(n)$-equivariant. Indeed, let $g \in O(n)$ and $A \in cb(\mathbb{R}^n)$. Then $r(gA) = [\xi(l(gA))]^{-1}gA = [\xi(gl(A))]^{-1}gA$. By equivariance of ξ, one has $\xi(gl(A)) = g\xi(l(A))g^{-1}$, and hence $[\xi(gl(A))]^{-1} = g[\xi(l(A))]^{-1}g^{-1}$. Consequently,

$$r(gA) = \big(g[\xi(l(A))]^{-1}g^{-1}\big)gA = g\big([\xi(l(A))]^{-1}A\big) = gr(A),$$

as required. Thus, $r\colon cb(\mathbb{R}^n) \to L(n)$ is an $O(n)$-retraction, and clearly $r(A)$ belongs to the $\mathrm{Aff}(n)$-orbit of A.

(2) Next we define

$$\varphi(A) = (r(A), l(A)) \quad \text{for every } A \in cb(\mathbb{R}^n).$$

Then φ is an $O(n)$-equivariant homeomorphism $cb(\mathbb{R}^n) \to L(n) \times E(n)$ with inverse $\varphi^{-1}((C,E)) = \xi(E)C$ for every $(C,E) \in L(n) \times E(n)$. ■

Corollary 3.10.

(1) *$E(n)$ is an $O(n)$-AR.*
(2) *$E(n)$ is homeomorphic to* $\mathbb{R}^{n(n+3)/2}$.

Proof. (1) follows immediately from Theorem 3.6 and from the fact that $cb(\mathbb{R}^n)$ is an $O(n)$-AR [8, Corollary 4.8].

(2) As observed above, $E(n)$ is homeomorphic to $\mathrm{Aff}(n)/O(n)$ (see [24, Proposition 1.1.5]). Consequently, one should prove that $\mathrm{Aff}(n)/O(n)$ is homeomorphic to $\mathbb{R}^{n(n+3)/2}$.

Since $\mathrm{Aff}(n)$ is the semidirect product of $\mathbb{R}^n$ and $GL(n)$, as a topological space $\mathrm{Aff}(n)/O(n)$ is homeomorphic to $\mathbb{R}^n \times GL(n)/O(n)$. The *RQ*-decomposition theorem in linear algebra states that every invertible matrix can be uniquely represented as the product of an orthogonal matrix and an upper-triangular matrix with positive elements on the diagonal (see, e.g., [13, Fact 4.2.2 and Exercise 4.3.29]). This easily implies that $GL(n)/O(n)$ is homeomorphic to $\mathbb{R}^{(n+1)n/2}$, and hence $\mathrm{Aff}(n)/O(n)$ is homeomorphic to $\mathbb{R}^p$, where $p = n + (n + 1)n/2 = n(n + 3)/2$. ∎

In Section 5 we will prove that $L(n)$ is homeomorphic to the Hilbert cube (Corollary 5.9). This, in combination with Corollaries 3.9 and 3.10, yields the following result, which is one of the main results of the paper:

Corollary 3.11. *$cb(\mathbb{R}^n)$ is homeomorphic to $Q \times \mathbb{R}^{n(n+3)/2}$.*

Remark 3.12. Using maximal-volume ellipsoids instead of minimal-volume ellipsoids, one can prove in a similar way that the subset $J(n)$, defined at the beginning of this subsection, is also a global $O(n)$-slice for $cb(\mathbb{R}^n)$. However, it follows from a result of H. Abels [1, Lemma 2.3] that the two global $O(n)$-slices $J(n)$ and $L(n)$ are equivalent in the sense that there exists an $\mathrm{Aff}(n)$-equivariant homeomorphism $f : cb(\mathbb{R}^n) \to cb(\mathbb{R}^n)$ such that $f(L(n)) = J(n)$. Consequently, all the results stated in terms of $L(n)$ have their analogs in terms of $J(n)$, which can be proven by trivial modification of our proofs.

4. The hyperspace $M(n)$. Let us denote by $M(n)$ the $O(n)$-invariant subspace of $cc(\mathbb{R}^n)$ consisting of all $A \in cc(\mathbb{R}^n)$ such that $\max_{a\in A} \|a\| = 1$. Thus, $M(n)$ consists of all compact convex subsets of $\mathbb{B}^n$ which intersect the boundary sphere $\mathbb{S}^{n-1}$.

It is evident that $M(n)$ is closed in $cc(\mathbb{B}^n) \subset cc(\mathbb{R}^n)$. By compactness of $cc(\mathbb{B}^n)$ (a well-known fact) it follows that $M(n)$ is compact as well. The importance of $M(n)$ lies in the property that $cc(\mathbb{R}^n)$ is the open cone over $M(n)$ (see Section 7). In this section we will prove that $M(n)$ is also homeomorphic to the Hilbert cube (Corollary 4.13) and its orbit space $M(n)/O(n)$ is homeomorphic to the Banach–Mazur compactum $\mathrm{BM}(n)$ (Theorem 4.16).

Let us recall that a G-space X is called *strictly G-contractible* if there exists a G-homotopy $F : X \times [0,1] \to X$ and a G-fixed point $a \in X$ such that $F(x,0) = x$ for all $x \in X$ and $F(x,t) = a$ if and only if $t = 1$ or $x = a$.

Lemma 4.1. *$M(n)$ is strictly $O(n)$-contractible to its only $O(n)$-fixed point $\mathbb{B}^n$.*

Proof. The map $F : M(n) \times [0,1] \to M(n)$ defined by

$$F(A,t) = (1-t)A + t\mathbb{B}^n$$

is the desired $O(n)$-contraction. ∎

Consider the map $\nu : cc(\mathbb{R}^n) \to [0,\infty)$ defined by

$$\nu(A) = \max_{a\in A} \|a\|, \qquad A \in cc(\mathbb{R}^n). \tag{4.1}$$

Lemma 4.2. *ν is a uniformly continuous $O(n)$-invariant map.*

Proof. Let $\varepsilon > 0$ and $A, B \in cc(\mathbb{R}^n)$, and suppose that $d_{\mathrm{H}}(A,B) < \varepsilon$. Let $a \in A$ be such that $\nu(A) = \|a\|$. Then there exists $b \in B$ with $\|a-b\| < \varepsilon$. Since $\|b\| \le \nu(B)$ we have

$$\varepsilon > \|a-b\| \ge \|a\| - \|b\| \ge \nu(A) - \nu(B).$$

Similarly, we can prove that $\nu(B) - \nu(A) < \varepsilon$, and hence ν is uniformly continuous.

Now, if $g \in O(n)$ then $\|gx\| = \|x\|$ for every $x \in \mathbb{R}^n$. Thus,

$$\nu(gA) = \max_{a'\in gA} \|a'\| = \max_{a\in A} \|ga\| = \max_{a\in A} \|a\| = \nu(A).$$

This proves that ν is $O(n)$-invariant, as required. ∎

Lemma 4.3. *$M(n)$ is an $O(n)$-AR with a unique $O(n)$-fixed point, $\mathbb{B}^n$.*

Proof. By [8, Corollary 4.8], $cc(\mathbb{R}^n)$ is an $O(n)$-AR. Hence, $cc(\mathbb{R}^n) \setminus \{0\}$ is an $O(n)$-ANR. The map $r : cc(\mathbb{R}^n) \setminus \{0\} \to M(n)$ defined by

$$r(A) = \frac{1}{\nu(A)} A \tag{4.2}$$

is an $O(n)$-retraction, where ν is defined in (4.1). Thus $M(n)$, being an $O(n)$-retract of an $O(n)$-ANR, is itself an $O(n)$-ANR. On the other hand, it was shown in Lemma 4.1 that $M(n)$ is $O(n)$-contractible to its point $\mathbb{B}^n$. Since every $O(n)$-contractible $O(n)$-ANR space is an $O(n)$-AR (see [3]) we conclude that $M(n)$ is an $O(n)$-AR. ∎

The following lemma will be used several times:

Lemma 4.4. *Let $p_1, \ldots, p_k \in \mathbb{R}^n$ be a finite number of points. Let $K \subset O(n)$ be a closed subgroup which acts nontransitively on $\mathbb{S}^{n-1}$. Then the boundary of the convex hull*

$$D = \operatorname{conv}(K(p_1) \cup \cdots \cup K(p_k))$$

does not contain an $(n-1)$*-dimensional elliptic domain, i.e.,* ∂D *contains no open subset* V *which is at the same time an open connected subset of some* $(n-1)$*-dimensional ellipsoid surface lying in* $\mathbb{R}^n$*.*

Proof. Assume that there exists an open subset $V \subset \partial D$ which is an $(n-1)$-dimensional elliptic domain. Recall that a convex body $A \subset \mathbb{R}^n$ is called *strictly convex* if every boundary point $a \in \partial A$ is an extreme point, that is, $A \setminus \{a\}$ is convex. Since every ellipsoid in $\mathbb{R}^n$ is strictly convex, it will follow that every $v \in V$ is an extreme point of D too, as we now show.

Indeed, suppose that there are distinct points $b, c \in D$ such that v belongs to the relative interior of the line segment $[b, c] = \{\lambda b + (1-\lambda)c \mid \lambda \in [0, 1]\}$. Since v is a boundary point of D, the whole segment $[b, c]$ lies in ∂D. Next, since V is open in ∂D, we infer that for b and c sufficiently close to v, the segment $[b, c]$ is contained in V. However, this is impossible because V is an elliptic domain.

Thus, we have proved that every $v \in V$ is an extreme point of D. Next, since D is the convex hull of $\bigcup_{i=1}^{k} K(p_i)$, each extreme point of D lies in $\bigcup_{i=1}^{k} K(p_i)$ (see, e.g., [29, Corollary 2.6.4]). This implies that $V \subset \bigcup_{i=1}^{k} K(p_i)$. Further, by connectedness, V is contained in only one $K(p_i)$. However, we now show this is impossible.

Indeed, since $K(p_i)$ lies on the $(n-1)$-sphere $\partial N(0, \|p_i\|)$ centered at the origin and having radius $\|p_i\|$, V should be a domain in this sphere. As $K(p_i)$ is a homogeneous compact space, there exists a finite cover $\{V_1, \ldots, V_m\}$ of $K(p_i)$, where each V_j is homeomorphic to V. Then, by the Domain Invariance Theorem (see, e.g., [26, Ch. 4, Section 7, Theorem 16]), each V_j is open in $\partial N(0, \|p_i\|)$. Hence, $V_1 \cup \cdots \cup V_m = K(p_i)$ is open in $\partial N(0, \|p_i\|)$. But $K(p_i)$ is also compact, and therefore closed in $\partial N(0, \|p_i\|)$. Thus $K(p_i)$ is an open and closed subset of the connected space $\partial N(0, \|p_i\|)$, and consequently $K(p_i) = \partial N(0, \|p_i\|)$. This implies that K acts transitively on $\mathbb{S}^{n-1}$, which is a contradiction. ∎

The *Fell topology* in $cc(\mathbb{R}^n)$ is generated by all sets of the form

$$U^- = \{A \in cc(\mathbb{R}^n) \mid A \cap U \neq \emptyset\} \quad \text{and}$$
$$(\mathbb{R}^n \setminus K)^+ = \{A \in cc(\mathbb{R}^n) \mid A \subset \mathbb{R}^n \setminus K\},$$

where $U \subset \mathbb{R}^n$ is open and $K \subset \mathbb{R}^n$ is compact.

It is well known that the Fell topology and the Hausdorff metric topology coincide in $cc(\mathbb{R}^n)$ (see, e.g., [25, Remark 2]). In particular, they coincide in $cb(\mathbb{R}^n)$. This will be used in the proof of the following lemma:

Lemma 4.5. *Let* $T \in cb(\mathbb{R}^n)$ *be a convex body and* $\mathcal{H} \subset cb(\mathbb{R}^n)$ *a subset such that for every* $A \in \mathcal{H}$*, the intersection* $A \cap T$ *has nonempty interior.*

Then the map $\upsilon : \mathcal{H} \to cb(\mathbb{R}^n)$ *defined by*

$$\upsilon(A) = A \cap T, \qquad A \in \mathcal{H},$$

is continuous.

Proof. It is enough to show that $\upsilon^{-1}(U^-)$ and $\upsilon^{-1}((\mathbb{R}^n \setminus K)^+)$ are open in $\mathcal{H}$ for every open $U \subset \mathbb{R}^n$ and compact $K \subset \mathbb{R}^n$.

First, suppose that $U \subset \mathbb{R}^n$ is open and $A \in \upsilon^{-1}(U^-)$. Then $U \cap (A \cap T) \neq \emptyset$. Since U is open and $A \cap T$ is a convex body, there exists a point x_0 in the interior of $A \cap T$ such that $x_0 \in U$. So, one can find $\delta > 0$ satisfying

$$\overline{N(x_0, 2\delta)} \subset U \cap (A \cap T).$$

In view of Lemma 3.1, if $C \in O(A, \delta) \cap \mathcal{H}$ then $N(x_0, \delta) \subset C$. Since $x_0 \in U \cap T$, we conclude that $U \cap \upsilon(C) = U \cap (C \cap T) \neq \emptyset$. This proves that $O(A, \delta) \cap \mathcal{H} \subset \upsilon^{-1}(U^-)$, and hence $\upsilon^{-1}(U^-)$ is open in $\mathcal{H}$.

Consider now a compact subset $K \subset \mathbb{R}^n$ and suppose $A \in \mathcal{H}$ is such that $\upsilon(A) \cap K = \emptyset$. If $K \cap T = \emptyset$ then $\mathcal{H} = \upsilon^{-1}((\mathbb{R}^n \setminus K)^+)$, which is open in $\mathcal{H}$. If $K \cap T \neq \emptyset$ then we define

$$\eta = \inf\{d(a, x) \mid a \in A,\ x \in K \cap T\}.$$

Since $(A \cap T) \cap K = \emptyset$, we have $\eta > 0$. Let $C \in O(A, \eta) \cap \mathcal{H}$ and suppose that $\upsilon(C)$ meets K. Then there exists $x_0 \in C \cap T \cap K$. Since C belongs to the η-neighborhood of A, we can find $a \in A$ such that $d(a, x_0) < \eta$, contradicting the choice of η. Thus we conclude that

$$O(A, \eta) \cap \mathcal{H} \subset \upsilon^{-1}((\mathbb{R}^n \setminus K)^+),$$

and hence $\upsilon^{-1}((\mathbb{R}^n \setminus K)^+)$ is open in $\mathcal{H}$. ∎

Denote by $M_0(n)$ the complement $M(n) \setminus \{\mathbb{B}^n\}$.

PROPOSITION 4.6. *For each closed subgroup* $K \subset O(n)$ *that acts non-transitively on* $\mathbb{S}^{n-1}$ *and each* $\varepsilon > 0$, *there exists a* K*-equivariant map* $\chi_\varepsilon : M(n) \to M_0(n)$ *which is* ε*-close to the identity map of* $M(n)$. *In particular,* $\chi_\varepsilon(M(n)^K) \subset M_0(n)^K$.

Proof. Let $r : cc(\mathbb{R}^n) \setminus \{0\} \to M(n)$ be the $O(n)$-equivariant retraction defined in (4.2). Since $M(n)$ is compact, one can find $0 < \delta < \varepsilon/2$ such that $d_{\mathrm{H}}(r(A), A) < \varepsilon/2$ for all A in the δ-neighborhood of $M(n)$ in $cc(\mathbb{R}^n) \setminus \{0\}$.

Choose a convex polyhedron $P \subset \mathbb{B}^n$ with nonempty interior, $\delta/4$-close to $\mathbb{B}^n$, such that all the vertices $p_1, \ldots, p_k$ of P lie on $\mathbb{S}^{n-1} = \partial\mathbb{B}^n$. Then

$$T = \mathrm{conv}(K(p_1) \cup \cdots \cup K(p_k))$$

is a compact convex K-invariant subset of $\mathbb{R}^n$. By Lemma 4.4, ∂T contains no $(n-1)$-dimensional elliptic domain. Furthermore,

$$d_{\mathrm{H}}(\mathbb{B}^n, T) \le d_{\mathrm{H}}(\mathbb{B}^n, P) < \delta/4. \tag{4.3}$$

Let $h : M(n) \to M(n)$ be defined as follows:

$$h(A) = \{x \in \mathbb{B}^n \mid d(x, A) \leq \delta/2\} \quad \text{for every } A \in M(n).$$

Clearly, $h(A) \cap T$ is a nonempty set with nonempty interior.

Then setting

$$\chi'(A) = h(A) \cap T$$

we obtain a map $\chi' : M(n) \to cc(\mathbb{R}^n)$. Since T is a K-fixed point of $cc(\mathbb{R}^n)$, we see that χ' is K-equivariant.

Continuity of χ' follows from the one of h and Lemma 4.5.

We claim that for any $A \in M(n)$, $\chi'(A)$ is not a closed Euclidean ball centered at the origin.

Indeed, if $h(A) \subset T$ then $h(A) \neq \mathbb{B}^n$ since T is strictly contained in $\mathbb{B}^n$. In this case $\chi'(A) = h(A) \cap T = h(A)$, and hence $\chi'(A) \in M(n)$. However, the only Euclidean ball centered at the origin that belongs to $M(n)$ is $\mathbb{B}^n$. But $\chi'(A) = h(A) \neq \mathbb{B}^n$.

If $h(A) \not\subset T$, then the boundary of $\chi'(A)$ contains a domain lying in ∂T. Since ∂T contains no $(n-1)$-dimensional elliptic domain (as shown in Lemma 4.4), $\chi'(A)$ is not an ellipsoid. In particular, it is not a Euclidean ball centered at the origin, and the claim is proved.

Now we assert that $\chi = r \circ \chi'$ is the desired map. Indeed, $r(A) = \mathbb{B}^n$ if and only if A is a Euclidean ball centered at the origin. Since $\chi'(A)$ is not such a ball, we infer that $\chi(A) = r(\chi'(A)) \neq \mathbb{B}^n$ for every $A \in M(n)$. Thus $\chi : M(n) \to M_0(n)$ is a well-defined map. It is continuous and K-equivariant because χ' and r are.

Now, if $x \in \chi'(A)$ then $x \in h(A)$. Hence, $d(x, A) \leq \delta/2 < \delta$ and $\chi'(A) \subset N(A, \delta)$. On the other hand, if $a \in A \subset \mathbb{B}^n$, then by (4.3) there exists $x \in T$ such that $d(x, a) < \delta/4 < \delta/2$. Therefore, $x \in h(A) \cap T = \chi'(A)$, and hence $A \subset N(\chi'(A), \delta/2)$. This proves that $d_{\mathrm{H}}(A, \chi'(A)) < \delta$.

By the choice of δ the last inequality implies $d_{\mathrm{H}}(r(\chi'(A)), \chi'(A)) \leq \varepsilon/2$. Then for all $A \in M(n)$ we have

$$\begin{aligned} d_{\mathrm{H}}(\chi(A), A) &\leq d_{\mathrm{H}}(\chi(A), \chi'(A)) + d_{\mathrm{H}}(\chi'(A), A) \\ &= d_{\mathrm{H}}\big(r(\chi'(A)), \chi'(A)\big) + d_{\mathrm{H}}(\chi'(A), A) \\ &< \varepsilon/2 + \delta < \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

This proves that χ is ε-close to the identity map of $M(n)$, and the proof is complete. ■

Observe that the induced action of $O(n)$ on $cc(\mathbb{R}^n)$ is isometric with respect to the Hausdorff metric. In particular, for every closed subgroup $K \subset O(n)$, the Hausdorff metric on $cc(\mathbb{R}^n)$ is K-invariant.

Let d_{H}^* be the metric on $M(n)/K$ induced by the Hausdorff metric on $M(n)$ as defined in (2.1):

$$d_{\mathrm{H}}^*(K(A), K(B)) = \inf_{k\in K} d_{\mathrm{H}}(A, kB), \qquad A, B \in M(n).$$

COROLLARY 4.7. *Let $K \subset O(n)$ be a closed subgroup that acts nontransitively on $\mathbb{S}^{n-1}$. Then*

(1) *the singleton $\{\mathbb{B}^n\}$ is a Z-set in $M(n)^K$,*
(2) *the class of $\{\mathbb{B}^n\}$ is a Z-set in $M(n)/K$.*

Proof. The first statement follows directly from Proposition 4.6. For the second statement, take $\varepsilon > 0$. By Proposition 4.6, there exists a K-map $\chi_\varepsilon : M(n) \to M_0(n)$ such that $d_{\mathrm{H}}(A, \chi(A)) < \varepsilon$ for every $A \in M(n)$. This induces a continuous map $\widetilde{\chi}_\varepsilon : M(n)/K \to M_0/K$ as follows:

$$\widetilde{\chi}_\varepsilon(K(A)) = \pi(\chi_\varepsilon(A)) = K(\xi_\varepsilon(A)), \qquad A \in M(n),$$

where $\pi : M(n) \to M(n)/K$ is the K-orbit map. By (2.2) we have

$$d_{\mathrm{H}}^*\big(K(\chi_\varepsilon(A)), K(A)\big) \le d_{\mathrm{H}}(\chi_\varepsilon(A), A) < \varepsilon,$$

and thus $\widetilde{\chi}_\varepsilon$ is ε-close to the identity map of $M(n)/K$.

On the other hand, since $\{\chi_\varepsilon(A)\} \neq \{\mathbb{B}^n\} = K(\mathbb{B}^n)$ for every $A \in M(n)$, we conclude that

$$\widetilde{\chi}_\varepsilon(M(n)/K) \cap \{\mathbb{B}^n\} = \emptyset,$$

which proves that the class of $\{\mathbb{B}^n\}$ is a Z-set on $M(n)/K$. ∎

Now, we shall give a sequence of lemmas and propositions culminating in Corollary 4.15.

Denote by $\mathcal{R}(n)$ the subspace of $M(n)$ consisting of all $A \in M(n)$ such that the contact set $A \cap \mathbb{S}^{n-1}$ has empty interior in $\mathbb{S}^{n-1}$.

For every $A \in M(n)$, $A \cap \mathbb{S}^{n-1}$ is nonempty, and therefore there exists $a \in A \cap \mathbb{S}^{n-1}$. If $O(n)_A$ is the $O(n)$-stabilizer of A then $O(n)_A(a) \subset A \cap \mathbb{S}^{n-1}$. Therefore, if $A \neq \mathbb{B}^n$, the subset $O(n)_A(a)$ should be different from $\mathbb{S}^{n-1}$, and thus $O(n)_A$ acts nontransitively on $\mathbb{S}^{n-1}$.

LEMMA 4.8. *Let $\varepsilon > 0$. For each $D \in M_0(n)$ there exist $A \in \mathcal{R}(n)$ such that $d_{\mathrm{H}}(D, A) < \varepsilon$ and the $O(n)$-stabilizer $O(n)_A$ coincides with $O(n)_D$.*

Proof. According to Theorem 2.2, there is $0 < \eta < \varepsilon$ such that if $d_{\mathrm{H}}(C, D) < \eta$ then the stabilizer $O(n)_C$ is conjugate to a subgroup of $O(n)_D$. Let $p_1, \ldots, p_k \in D$ be such that $P = \operatorname{conv}(\{p_1, \ldots, p_k\}) \in M(n)$ (it is enough to choose one of the p_i's lying in $\partial D \cap \mathbb{S}^{n-1}$) and $d_{\mathrm{H}}(D, P) < \eta$. Next, we define

$$A = \operatorname{conv}\big(O(n)_D(p_1) \cup \cdots \cup O(n)_D(p_k)\big).$$

Clearly, $A \in M(n)$ and

$$d_{\mathrm{H}}(D, A) \le d_{\mathrm{H}}(D, P) < \eta < \varepsilon.$$

Since $O(n)_D$ acts nontransitively on $\mathbb{S}^{n-1}$, Lemma 4.4 show that ∂A contains no $(n-1)$-elliptic domain. In particular, $\partial A \cap \mathbb{S}^{n-1}$ has empty interior in $\mathbb{S}^{n-1}$, i.e., $A \in \mathcal{R}(n)$.

By the choice of η the stabilizer $O(n)_A$ is conjugate to a subgroup of $O(n)_D$. On the other hand, A is an $O(n)_D$-invariant subset, so that $O(n)_D \subset O(n)_A$. This implies that $O(n)_A = O(n)_D$, as required. ∎

The following lemma is just a special case of [8, Theorem 4.5].

LEMMA 4.9. *Let $X \in cc(\mathbb{R}^n)$. For every $\varepsilon > 0$, the open ball in $cc(\mathbb{R}^n)$ with radius ε centered at X is convex, i.e., if $\{A_1, \ldots, A_k\} \subset cc(\mathbb{R}^n)$ is a finite family such that for every $i = 1, \ldots, k$, $d_{\mathrm{H}}(A_i, X) < \varepsilon$, then the set*

$$\sum_{i=1}^{k} t_i A_i = \Big\{ \sum_{i=1}^{k} t_i a_i \;\Big|\; a_i \in A_i,\ i = 1, \ldots, k \Big\}$$

is ε-close to X, where $t_1, \ldots, t_k \in [0,1]$ with $\sum_{i=1}^{k} t_i = 1$.

The following is perhaps the key result of this section:

PROPOSITION 4.10. *For every $\varepsilon > 0$, there exists an $O(n)$-map $f_\varepsilon : M_0(n) \to \mathcal{R}(n)$, ε-close to the identity map of $M_0(n)$.*

Proof. Let $\mathcal{V} = \{O(X, \varepsilon/4)\}_{X \in M_0(n)}$ be the open cover of $M_0(n)$ consisting of all open balls of radius $\varepsilon/4$. By [7, Lemma 4.1], there exists an $O(n)$-normal cover of $M_0(n)$ (see Section 2 for the definition),

$$\mathcal{W} = \{gS_\mu \mid g \in O(n),\ \mu \in \mathcal{M}\}$$

satisfying the following two conditions:

(a) $\mathcal{W}$ is a star-refinement of $\mathcal{V}$, that is, for each $gS_\mu \in \mathcal{W}$, there exists $V \in \mathcal{V}$ that contains the star of gS_μ with respect to $\mathcal{W}$, i.e.,

$$\mathrm{St}(gS_\mu, \mathcal{W}) = \bigcup \{hS_\lambda \in \mathcal{W} \mid hS_\lambda \cap gS_\mu \neq \emptyset\} \subset V.$$

(b) For each $\mu \in \mathcal{M}$, the set S_μ is an H_μ-slice, where H_μ coincides with the stabilizer $O(n)_{X_\mu}$ of a certain point $X_\mu \in S_\mu$.

Since $X_\mu \in M_0(n)$, we see that H_μ acts nontransitively on $\mathbb{S}^{n-1}$. Thus, by Lemma 4.8, there exists $A_\mu \in \mathcal{R}(n)$ which is $\varepsilon/4$-close to X_μ and $O(n)_{A_\mu} = H_\mu$.

For every $\mu \in \mathcal{M}$, set $O_\mu = O(n)(S_\mu)$. Define $F_\mu : O_\mu \to O(n)(A_\mu)$ by

$$F_\mu(gZ) = gA_\mu, \qquad Z \in S_\mu,\ g \in O(n).$$

Clearly F_μ is a well-defined continuous $O(n)$-map.

Fix an invariant locally finite partition of unity $\{p_\mu\}_{\mu \in \mathcal{M}}$ subordinated to the open cover $\mathcal{U} = \{O_\mu\}_{\mu \in \mathcal{M}}$, i.e.,

$$\overline{p_\mu^{-1}((0,1])} \subset O_\mu \quad \text{for every } \mu \in \mathcal{M}.$$

Let $\mathcal{N}(\mathcal{U})$ be the nerve of the cover $\mathcal{U}$ and suppose that $\mathcal{M}$ is its vertex set. Denote by $|\mathcal{N}(\mathcal{U})|$ the geometric realization of $\mathcal{N}(\mathcal{U})$. Recall that every point $\alpha \in |\mathcal{N}(\mathcal{U})|$ can be expressed as a sum $\alpha = \sum_{\mu\in\mathcal{M}} \alpha_\mu v_\mu$, where v_μ is the geometric vertex corresponding to $\mu \in \mathcal{M}$ and α_μ, $\mu \in \mathcal{M}$, are the barycentric coordinates of α.

For a simplex σ of $\mathcal{N}(\mathcal{U})$ with vertices $\mu_0, \dots, \mu_k$, we will use the notation $\sigma = \langle\mu_0, \dots, \mu_k\rangle$. By $|\langle\mu_0, \dots, \mu_k\rangle|$ we denote the corresponding geometric simplex with geometric vertices $v_{\mu_0}, \dots, v_{\mu_k}$.

For every geometric simplex $|\sigma| = |\langle\mu_0, \dots, \mu_k\rangle| \subset |\mathcal{N}(\mathcal{U})|$, denote by $\beta(\sigma) \in |\mathcal{N}(\mathcal{U})|$ the geometric barycenter of $|\sigma|$, i.e., $\beta(\sigma) = \sum_{\mu\in\mathcal{M}} \beta(\sigma)_\mu v_\mu$ where

$$\beta(\sigma)_\mu = \begin{cases} 1/(k+1) & \text{if } \mu \in \{\mu_0, \dots, \mu_k\}, \\ 0 & \text{if } \mu \notin \{\mu_0, \dots, \mu_k\}. \end{cases}$$

Consider the map $\Psi : |\mathcal{N}(\mathcal{U})| \to |\mathcal{N}(\mathcal{U})|$ defined in each $\alpha = \sum_{\mu\in\mathcal{M}} \alpha_\mu v_\mu \in |\mathcal{N}(\mathcal{U})|$ as follows: if $|\langle\mu_0, \dots, \mu_k\rangle|$ is the carrier of α and $\alpha_{\mu_0} \ge \alpha_{\mu_1} \ge \dots \ge \alpha_{\mu_k}$, then

$$\Psi(\alpha) = \sum_{\sigma\in\mathcal{N}(\mathcal{U})} \Psi(\alpha)_\sigma \beta(\sigma)$$

where

$$\Psi(\alpha)_\sigma = \begin{cases} (i+1)(\alpha_{\mu_i} - \alpha_{\mu_{i+1}}) & \text{if } \sigma = \langle\mu_0, \dots, \mu_i\rangle,\ i = 0, \dots, k-1, \\ (k+1)\alpha_{\mu_k} & \text{if } \sigma = \langle\mu_0, \dots, \mu_k\rangle, \\ 0 & \text{if } \sigma \neq \langle\mu_0, \dots, \mu_i\rangle,\ i = 0, \dots, k. \end{cases} \tag{4.4}$$

It is not difficult to see that Ψ is the identity map of $|\mathcal{N}(\mathcal{U})|$ written in the barycentric coordinates with respect to the first barycentric subdivision of $|\mathcal{N}(\mathcal{U})|$; we shall need this representation in what follows.

Let $p : M_0(n) \to |\mathcal{N}(\mathcal{U})|$ be the canonical map defined by

$$p(X) = \sum_{\mu\in\mathcal{M}} p_\mu(X) v_\mu, \qquad X \in M_0(n).$$

Since each p_μ is $O(n)$-invariant, the map p is also $O(n)$-invariant.

For every simplex $\sigma = \langle\mu_0, \dots, \mu_k\rangle \in \mathcal{N}(\mathcal{U})$ the set $V_\sigma = O_{\mu_0} \cap \dots \cap O_{\mu_k}$ is a nonempty open subset of $M_0(n)$. Continuity of the union operator and the convex hull operator (see, e.g., [21, Corollary 5.3.7] and [29, Theorem 2.7.4(iv)]) imply that the map $\Omega'_\sigma : V_\sigma \to M_0(n)$ given by

$$\Omega'_\sigma(X) = \operatorname{conv}\Big(\bigcup_{\mu\in\sigma} F_\mu(X)\Big), \qquad X \in V_\sigma,$$

is a continuous $O(n)$-map.

Observe that $\Omega'_\sigma(X) \in M_0(n)$ and

$$\Omega'_\sigma(X) \cap \mathbb{S}^{n-1} \subset \Big(\bigcup_{\mu\in\sigma} F_\mu(X)\Big) \cap \mathbb{S}^{n-1} = \bigcup_{\mu\in\sigma} (F_\mu(X) \cap \mathbb{S}^{n-1}),$$

and hence

$$\Omega'_\sigma(X)\cap\mathbb{S}^{n-1} \quad \text{has empty interior in}\,\mathbb{S}^{n-1}. \tag{4.5}$$

Fix $B\in M_0(n)$. For each simplex σ of $\mathcal{N}(\mathcal{U})$, we extend the map Ω'_σ to a function $\Omega_\sigma : M_0(n)\to M_0(n)$ as follows:

$$\Omega_\sigma(X)=\begin{cases}\Omega'_\sigma(X) & \text{if } X\in V_\sigma,\\ B & \text{if } X\notin V_\sigma.\end{cases}$$

The desired map $f_\varepsilon : M_0(n)\to M_0(n)$ can now be defined by

$$f_\varepsilon(X)=\sum_{\sigma\in\mathcal{N}(\mathcal{U})}\Psi(p(X))_\sigma\Omega_\sigma(X),\qquad X\in M_0(n).$$

For every $X\in M_0(n)$, let $Q(X)$ be the subset of $\mathcal{M}$ consisting of all $\mu\in\mathcal{M}$ such that $X\in p_\mu^{-1}((0,1])$. Similarly, denote by $Q'(X)$ the subset of $\mathcal{M}$ consisting of all $\mu\in\mathcal{M}$ such that $X\in\overline{p_\mu^{-1}((0,1])}$.

It is clear that $Q(X)\subset Q'(X)$ and, by local finiteness of the cover $\{\overline{p_\mu^{-1}((0,1])}\}_{\mu\in\mathcal{M}}$, both sets are finite. Moreover, it follows from (4.4) that $\Psi(p(X))_\sigma=0$ whenever $\sigma\not\subset Q'(X)$.

Then, for every $X\in M_0(n)$ we have

$$f_\varepsilon(X)=\sum_{\substack{\sigma\in\mathcal{N}(\mathcal{U})\\ \sigma\subset Q(X)}}\Psi(p(X))_\sigma\Omega_\sigma(X)=\sum_{\substack{\sigma\mathcal{N}(\mathcal{U})\\ \sigma\subset Q'(X)}}\Psi(p(X))_\sigma\Omega_\sigma(X). \tag{4.6}$$

To see the continuity of f_ε, fix $C\in M_0(n)$ and define

$$V=\Big(\bigcap_{\mu\in Q'(C)}O_\mu\Big)\setminus\bigcup_{\mu\notin Q'(C)}\overline{p_\mu^{-1}((0,1])}.$$

Since the family $\{p_\mu^{-1}((0,1])\}_{\mu\in\mathcal{M}}$ is locally finite, $\bigcup_{\mu\notin Q'(C)}\overline{p_\mu^{-1}((0,1])}$ is closed, and therefore V is a neighborhood of C. It is evident that $Q(X)\subset Q'(C)$ for every $X\in V$. Using (4.6), we infer that

$$f_\varepsilon(X)=\sum_{\substack{\sigma\in\mathcal{N}(\mathcal{U})\\ \sigma\subset Q'(C)}}\Psi(p(X))_\sigma\Omega_\sigma(X)\quad\text{for every } X\in V.$$

Observe that $V\subset V_\sigma$ for every simplex $\sigma\in\mathcal{N}(\mathcal{U})$ such that $\sigma\subset Q'(C)$, and hence $\Omega_\sigma|_V=\Omega'_\sigma|_V$ is continuous.

On the other hand, $\Psi(p(X))_\sigma$ is just the $\beta(\sigma)$th barycentric coordinate of $\Psi(p(X))$. Thus, for every $\sigma\in\mathcal{N}(\mathcal{U})$, the map $X\mapsto\Psi(p(X))_\sigma$ depends continuously on X. So, $f_\varepsilon|_V$ is a finite sum of continuous functions and so it is also continuous in V. Consequently, f_ε is continuous at $C\in M_0(n)$, as required.

If $g \in O(n)$ and $X \in M_0(n)$, then

$$\begin{aligned} f_\varepsilon(gX) &= \sum_{\substack{\sigma\in\mathcal{N}(\mathcal{U})\\ \sigma\subset Q(X)}} \Psi(p(gX))_\sigma \Omega_\sigma(gX) = \sum_{\substack{\sigma\in\mathcal{N}(\mathcal{U})\\ \sigma\subset Q(X)}} \Psi(p(X))_\sigma \Omega'_\sigma(gX) \\ &= \sum_{\sigma\in\mathcal{N}(\mathcal{U})} \Psi(p(X))_\sigma (g\Omega'_\sigma(X)) = g\Big(\sum_{\substack{\sigma\in\mathcal{N}(\mathcal{U})\\ \sigma\subset Q(X)}} \Psi(p(X))_\sigma \Omega'_\sigma(X)\Big) \\ &= g\Big(\sum_{\substack{\sigma\in\mathcal{N}(\mathcal{U})\\ \sigma\subset Q(X)}} \Psi(p(X))_\sigma \Omega_\sigma(X)\Big) = gf_\varepsilon(X), \end{aligned}$$

which shows that f_ε is $O(n)$-equivariant.

To see that $f_\varepsilon(X) \in M_0(n)$, suppose that

$$Q(X) = \{\mu_0, \dots, \mu_k\} \quad \text{and} \quad p_{\mu_0}(X) \ge p_{\mu_1}(X) \ge \dots \ge p_{\mu_k}(X).$$

Then, by (4.4) and (4.6), the set $f_\varepsilon(X)$ can be seen as a convex sum:

$$\begin{aligned} f_\varepsilon(X) &= (k+1)p_{\mu_k}(X)\Omega_{\langle\mu_0,\dots,\mu_k\rangle}(X) \\ &\quad + \sum_{i=0}^{k-1}(i+1)(p_{\mu_i}(X) - p_{\mu_{i+1}}(X))\Omega_{\langle\mu_0,\dots,\mu_i\rangle}(X) \\ &= (k+1)p_{\mu_k}(X)\Omega'_{\langle\mu_0,\dots,\mu_k\rangle}(X) \\ &\quad + \sum_{i=0}^{k-1}(i+1)(p_{\mu_i}(X) - p_{\mu_{i+1}}(X))\Omega'_{\langle\mu_0,\dots,\mu_i\rangle}(X). \end{aligned}$$

Thus, $f_\varepsilon(X)$ is a convex subset contained in $\mathbb{B}^n$. Furthermore, observe that $F_{\mu_0}(X) \subset \Omega'_{\langle\mu_0,\dots,\mu_i\rangle}(X)$ for every $i = 0, \dots, k$. This implies that

$$\begin{aligned} F_{\mu_0}(X) &= (k+1)p_{\mu_k}(X)F_{\mu_0}(X) + \sum_{i=0}^{k-1}(i+1)(p_{\mu_i}(X) - p_{\mu_{i+1}}(X)g)F_{\mu_0}(X) \\ &\subset (k+1)p_{\mu_k}(X)\Omega'_{\langle\mu_0,\dots,\mu_k\rangle}(X) \\ &\quad + \sum_{i=0}^{k-1}(i+1)(p_{\mu_i}(X) - p_{\mu_{i+1}}(X))\Omega'_{\langle\mu_0,\dots,\mu_i\rangle}(X) \\ &= f_\varepsilon(X). \end{aligned}$$

Since $F_{\mu_0}(X) \in M_0(n)$, the inclusion $F_{\mu_0}(X) \subset f_\varepsilon(X)$ yields $f_\varepsilon(X) \in M_0(n)$.

On the other hand, the contact set $f_\varepsilon(X) \cap \mathbb{S}^{n-1}$ is contained in

$$\Big(\bigcup_{i=0}^{k} \Omega'_{\langle\mu_0,\dots,\mu_i\rangle}(X)\Big) \cap \mathbb{S}^{n-1} = \bigcup_{i=0}^{k}(\Omega'_{\langle\mu_0,\dots,\mu_i\rangle}(X) \cap \mathbb{S}^{n-1}).$$

Further, since by (4.5), each $\Omega'_{\langle\mu_0,\dots,\mu_i\rangle}(X) \cap \mathbb{S}^{n-1}$ has empty interior in $\mathbb{S}^{n-1}$, we infer that the finite union $\bigcup_{i=0}^{k}(\Omega'_{\langle\mu_0,\ldots,\mu_i\rangle}(X)\cap\mathbb{S}^{n-1})$ also has empty interior in $\mathbb{S}^{n-1}$. This shows that $f_\varepsilon(X)\cap\mathbb{S}^{n-1}$ has empty interior in $\mathbb{S}^{n-1}$, as required.

It remains only to prove that $d_{\mathrm{H}}(X, f_\varepsilon(X)) < \varepsilon$ for every $X \in M_0(n)$.

Since $f_\varepsilon(X)$ is a convex sum of the sets $\Omega_{\langle\mu_0,\ldots,\mu_i\rangle}(X)$ for $i = 0, \ldots, k$, according to Lemma 4.9 it is enough to prove that $\Omega_{\langle\mu_0,\ldots,\mu_i\rangle}(X)$ is ε-close to X for every $i = 0, \ldots, k$.

Recall that $\Omega_{\langle\mu_0,\ldots,\mu_i\rangle}(X) = \operatorname{conv}(\bigcup_{j=0}^{i} F_{\mu_j}(X))$, and hence we have only to prove that $d_{\mathrm{H}}(X, F_{\mu_j}(X)) < \varepsilon$ for each j.

For this purpose, suppose that $g_j \in O(n)$ is such that $F_{\mu_j}(X) = g_j A_{\mu_j}$. Then $X \in g_j S_{\mu_j}$ and $g_j X_{\mu_j} \in g_j S_{\mu_j}$.

Since $\mathcal{W}$ is a star-refinement of $\mathcal{V}$, there exists $Z \in M_0(n)$ such that $\mathrm{St}(X, \mathcal{W}) = \bigcup\{gS_\mu \in \mathcal{W} \mid X \in gS_\mu\} \subset O(Z, \varepsilon/4)$. In particular,

$$d_{\mathrm{H}}(X, Z) < \varepsilon/4 \quad \text{and} \quad d_{\mathrm{H}}(g_j X_{\mu_j}, Z) < \varepsilon/4. \tag{4.7}$$

This implies that $d_{\mathrm{H}}(g_j X_{\mu_j}, X) < \varepsilon/2$. By the choice of A_{μ_j}, we see that $d_{\mathrm{H}}(A_{\mu_j}, X_{\mu_j}) < \varepsilon/4$. Since the Hausdorff metric is $O(n)$-invariant we get

$$d_{\mathrm{H}}(g_j A_{\mu_j}, g_j X_{\mu_j}) = d_{\mathrm{H}}(A_{\mu_j}, X_{\mu_j}) < \varepsilon/4,$$

and hence

$$\begin{aligned} d_{\mathrm{H}}(X, F_{\mu_j}(X)) = d_{\mathrm{H}}(X, g_j A_{\mu_j}) &\le d_{\mathrm{H}}(X, g_j X_{\mu_j}) + d_{\mathrm{H}}(g_j X_{\mu_j}, g_j A_{\mu_j}) \\ &< \varepsilon/2 + \varepsilon/4 < \varepsilon, \end{aligned}$$

as required. ∎

Proposition 4.11. *For every $\varepsilon > 0$, there is an $O(n)$-map $h_\varepsilon : M_0(n) \to M_0(n) \setminus \mathcal{R}(n)$, ε-close to the identity map of $M_0(n)$.*

Proof. Define a continuous map $\gamma : M_0(n) \to \mathbb{R}$ by

$$\gamma(A) = \tfrac{1}{2}\min\{\varepsilon, d_{\mathrm{H}}(\mathbb{B}^n, A)\} \quad \text{for every } A \in M_0(n).$$

Let $h_\varepsilon(A)$ be the closed $\gamma(A)$-neighborhood of A in $\mathbb{B}^n$, i.e.,

$$h_\varepsilon(A) = A_{\gamma(A)} = \{x \in \mathbb{B}^n \mid d(x, A) \le \gamma(A)\}, \quad A \in M_0(n).$$

By the choice of $\gamma(A)$, the set $h_\varepsilon(A)$ is different from $\mathbb{B}^n$, and since $A \subset h_\varepsilon(A)$, we see that $h_\varepsilon(A) \in M_0(n)$. Even more, $h_\varepsilon(A) \cap \mathbb{S}^{n-1}$ has nonempty interior in $\mathbb{S}^{n-1}$. Thus, $h_\varepsilon(A) \in M_0(n) \setminus \mathcal{R}(n)$.

By [7, Lemma 5.3], $d_{\mathrm{H}}(A, A_{\gamma(A)}) < \gamma_A < \varepsilon$, which implies that h_ε is ε-close to the identity map of $M_0(n)$.

Let us check the continuity of h_ε. For any $A, C \in M_0(n)$,

$$d_{\mathrm{H}}(h_\varepsilon(A), h_\varepsilon(C)) = d_{\mathrm{H}}(A_{\gamma(A)}, C_{\gamma(C)}) \le d_{\mathrm{H}}(A_{\gamma(A)}, A_{\gamma(C)}) + d_{\mathrm{H}}(A_{\gamma(C)}, C_{\gamma(C)}).$$

But

$$d_{\mathrm{H}}(A_{\gamma(A)}, A_{\gamma(C)}) \le |\gamma(A) - \gamma(C)| \quad \text{and} \quad d_{\mathrm{H}}(A_{\gamma(C)}, C_{\gamma(C)}) \le d_{\mathrm{H}}(A, C)$$

(see, e.g., [7, Lemma 5.3]). Consequently,

$$d_{\rm H}(h_\varepsilon(A), h_\varepsilon(C)) \le |\gamma(A) - \gamma(C)| + d_{\rm H}(A, C).$$

Now the continuity of γ implies the one of h_ε. ∎

As a consequence of Propositions 4.10 and 4.11 we have the following corollaries.

COROLLARY 4.12. *For any closed subgroup $K \subset O(n)$, the K-orbit space $M_0(n)/K$ is a Q-manifold.*

Proof. Consider the metric on $M_0(n)/K$ induced by $d_{\rm H}$ according to (2.1).

Clearly, $M_0(n)$ is a locally compact space, and thus $M_0(n)/K$ is also locally compact. Since $M(n)$ is an $O(n)$-AR, and $M_0(n)$ is an open $O(n)$-invariant set in $M(n)$, we infer that $M_0(n)$ is an $O(n)$-ANR. This in turn implies that $M_0(n)$ is a K-ANR (see, e.g., [28]). Then, by Theorem 2.3, $M_0(n)/K$ is an ANR.

According to Toruńczyk's Characterization Theorem [27, Theorem 1], it remains to check that for every $\varepsilon > 0$, there exist continuous maps $\widetilde{f}_\varepsilon, \widetilde{h}_\varepsilon : M_0(n)/K \to M_0(n)/K$, ε-close to the identity map of $M_0(n)/K$ such that the images $\operatorname{Im} \widetilde{f}_\varepsilon$ and $\operatorname{Im} \widetilde{h}_\varepsilon$ are disjoint.

Let f_ε and h_ε be the $O(n)$-maps from Propositions 4.10 and 4.11, respectively. They induce continuous maps $\widetilde{f}_\varepsilon : M_0(n)/K \to M_0(n)/K$ and $\widetilde{h}_\varepsilon : M_0(n)/K \to M_0(n)/K$. Since $\operatorname{Im} \widetilde{f}_\varepsilon = (\operatorname{Im} f_\varepsilon)/K$, $\operatorname{Im} \widetilde{h}_\varepsilon = (\operatorname{Im} h_\varepsilon)/K$ and $\operatorname{Im} f_\varepsilon \cap \operatorname{Im} h_\varepsilon = \emptyset$, we infer that $\operatorname{Im} \widetilde{f}_\varepsilon \cap \operatorname{Im} \widetilde{h}_\varepsilon = \emptyset$.

On the other hand, since f_ε and h_ε are ε-close to the identity map of $M_0(n)$, using inequality (2.2), we see that $\widetilde{f}_\varepsilon$ and $\widetilde{h}_\varepsilon$ are ε-close to the identity map of $M_0(n)/K$. ∎

COROLLARY 4.13. *For any closed subgroup $K \subset O(n)$ that acts nontransitively on $\mathbb{S}^{n-1}$, the K-orbit space $M(n)/K$ is a Hilbert cube. In particular, $M(n)$ is homeomorphic to Q.*

Proof. We have already seen in Corollary 4.7 that $\{\mathbb{B}^n\}$ is a Z-set in $M(n)/K$. Observe that the Q-manifold $M_0(n)/K$ can be seen as the complement $(M(n)/K) \setminus \{\mathbb{B}^n\}$. It then follows from [27, §3] that $M(n)/K$ is also a Q-manifold. Furthermore, $M(n)/K$ is compact and contractible. But since the only compact contractible Q-manifold is the Hilbert cube (see [21, Theorem 7.5.8]), we conclude that $M(n)/K$ is homeomorphic to Q. ∎

COROLLARY 4.14. *For any closed subgroup $K \subset O(n)$ that acts nontransitively on $\mathbb{S}^{n-1}$, the K-fixed point set $M(n)^K$ is homeomorphic to the Hilbert cube.*

Proof. Since $M(n)$ is compact and $M(n)^K$ is closed in $M(n)$, we see that $M(n)^K$ is also compact. By Theorem 4.3, $M(n)$ is an $O(n)$-AR. This, in combination with [9, Theorem 3.7], implies that $M(n)^K$ is an AR. In particular, $M(n)^K$ is contractible.

Let f_ε and h_ε be the $O(n)$-maps from Propositions 4.10 and 4.11, respectively. By equivariance, we have

$$f_\varepsilon(M_0(n)^K) \subset M_0(n)^K \quad \text{and} \quad h_\varepsilon(M_0(n)^K) \subset M_0(n)^K.$$

By Toruńczyk's Characterization Theorem [27, Theorem 1], $M_0(n)^K$ is a Q-manifold. But $M_0(n)^K = M(n)^K \setminus \{\mathbb{B}^n\}$ and Corollary 4.7 implies that $\{\mathbb{B}^n\}$ is a Z-set in $M(n)^K$. This shows that $M(n)^K$ is also a Q-manifold (see [27, §3]). Furthermore, $M(n)^K$ is compact and contractible. Since the only compact contractible Q-manifold is the Hilbert cube (see [21, Theorem 7.5.8]), we conclude that $M(n)^K$ is homeomorphic to Q. ∎

We summarize all the above results about the $O(n)$-space $M(n)$ in the following corollary:

Corollary 4.15. *$M(n)$ is a Hilbert cube endowed with an $O(n)$-action satisfying the following properties:*

1. *$M(n)$ is an $O(n)$-AR with a unique $O(n)$-fixed point, $\mathbb{B}^n$,*
2. *$M(n)$ is strictly $O(n)$-contractible to $\mathbb{B}^n$,*
3. *for a closed subgroup $K \subset O(n)$, the set $M(n)^K$ equals the singleton $\{\mathbb{B}^n\}$ if and only if K acts transitively on $\mathbb{S}^{n-1}$, and $M(n)^K$ is homeomorphic to the Hilbert cube whenever $M(n)^K \neq \{\mathbb{B}^n\}$,*
4. *for any closed subgroup $K \subset O(n)$, the K-orbit space $M_0(n)/K$ is a Q-manifold.*

This corollary in combination with [10, Theorem 3.3] yields

Theorem 4.16. *The orbit space $M(n)/O(n)$ is homeomorphic to the Banach–Mazur compactum* BM(n).

5. Some properties of $L(n)$. Recall that $L(n)$ is the hyperspace of all compact convex bodies for which the Euclidean unit ball is the minimum-volume ellipsoid of Löwner.

In [7] the subset $L'(n)$ of $L(n)$ consisting of all $A \in L(n)$ with $A = -A$ was studied. It turns out that $L(n)$ enjoys all the properties of $L'(n)$ established in [7], and an easy modification of the method developed in [7, Section 5] allows one to establish similar properties of $L(n)$. However, for completeness, we shall provide in this section some more specific details and appropriate new references.

Proposition 5.1. *$L(n)$ is an $O(n)$-AR.*

Proof. It was proved in [8, Corollary 4.8] that $cb(\mathbb{R}^n)$ is an $O(n)$-AR. Since $L(n)$ is a global $O(n)$-slice in $cb(\mathbb{R}^n)$, according to Corollary 3.9(2), there exists an $O(n)$-equivariant retraction $r : cb(\mathbb{R}^n) \to L(n)$. This implies that $L(n)$ is also an $O(n)$-AR. ∎

PROPOSITION 5.2. *The map $F : L(n) \times [0,1] \to L(n)$ defined by*

$$F(A,t) = (1-t)A + t\mathbb{B}^n$$

is an $O(n)$-strict contraction such that $F(A,1) = \mathbb{B}^n$. In particular, for every closed subgroup $K \subset O(n)$, the orbit space $L(n)/K$ is contractible to its point $\mathbb{B}^n$.

Proof. It is evident that F satisfies the first assertion of the proposition. Letting $\widetilde{F}(K(A),t) = K(F(A,t))$ we obtain a deformation of $L(n)/K$ to the point $\mathbb{B}^n \in L(n)/K$, thus proving that $L(n)/K$ is contractible. ∎

By $\mathcal{P}(n)$ we will denote the subset of $L(n)$ consisting of all compact convex bodies $A \in L(n)$ such that $A \cap \partial\mathbb{B}^n$ has empty interior in $\partial\mathbb{B}^n = \mathbb{S}^{n-1}$.

Denote by $L_0(n)$ the complement $L(n) \setminus \{\mathbb{B}^n\}$.

LEMMA 5.3. *Let $\varepsilon > 0$. For each convex body $X \in L_0(n)$, there exists a convex body $A \in \mathcal{P}(n)$ such that $d_{\mathrm{H}}(X,A) < \varepsilon$ and the $O(n)$-stabilizer $O(n)_A$ coincides with $O(n)_X$.*

Although the proof of Lemma 5.3 is similar to the one of Lemma 4.8, there is a significant difference, and for this reason we present a complete proof here.

Proof. Let $r : cb(\mathbb{R}^n) \to L(n)$ be the $O(n)$-equivariant retraction used in the proof of Proposition 5.1 (cf. Corollary 3.9(2)). By Theorem 2.2, there is a $O(n)_X$-slice S such that $X \in S$ and $[O(n)_C] \preceq [O(n)_X]$ whenever $C \in O(n)(S)$. Since $O(n)(S)$ is open, there exists $0 < \eta < \varepsilon$ such that $O(X,\eta) \subset O(n)(S)$. In particular, if $C \in O(X,\eta)$ then $[O(n)_C] \preceq [O(n)_X]$.

Since $L(n)$ is compact, there exists $0 < \delta < \eta/2$ such that $d_{\mathrm{H}}(r(C),C) < \eta/2$ for every C in the δ-neighborhood of $L(n)$.

Let $p_1,\dots,p_k \in \partial X$ be such that $P = \operatorname{conv}(\{p_1,\dots,p_k\})$ has nonempty interior in $\mathbb{R}^n$ and $d_{\mathrm{H}}(P,X) < \delta$. Set

$$D = \operatorname{conv}\big(O(n)_X(p_1) \cup \dots \cup O(n)_X(p_k)\big).$$

Since $P \subset D$, we see that D has nonempty interior, and hence $D \in cb(\mathbb{R}^n)$. Since $O(n)_X$ acts nontransitively on $\mathbb{S}^{n-1}$, Lemma 4.4 states that ∂D contains no $(n-1)$-elliptic domain. In particular, $D \cap \partial l(D)$ contains no elliptic domain (recall that here $l(D)$ denotes the minimal-volume ellipsoid containing D).

Let $A = r(D)$. Since $A \in L(n)$ and A lies in the Aff(n)-orbit of D (see Corollary 3.9(1)), there exists an affine transformation g such that $A = gD$. The contact set $A \cap \mathbb{S}^{n-1}$ is the image under g of $D \cap \partial l(D)$, and thus it has empty interior in $\mathbb{S}^{n-1}$. Hence, $A \in \mathcal{P}(n)$. The construction of P guarantees that $P \subset D \subset X$, and therefore

$$d_{\mathrm{H}}(D, X) \le d_{\mathrm{H}}(P, X) < \delta < \eta/2.$$

By the choice of δ one has $d_{\mathrm{H}}(r(D), D) < \eta/2$, and hence

$$\begin{aligned} d_{\mathrm{H}}(A, X) &\le d_{\mathrm{H}}(A, D) + d_{\mathrm{H}}(D, X) \\ &= d_{\mathrm{H}}(r(D), D) + d_{\mathrm{H}}(D, X) < \eta/2 + \eta/2 = \eta. \end{aligned}$$

Thus, $d_{\mathrm{H}}(A, X) < \eta < \varepsilon$, as required.

Furthermore, by the choice of η, $O(n)_A$ is conjugate to a subgroup of $O(n)_X$. It remains to prove that $O(n)_X = O(n)_A$. Since D is an $O(n)_X$-invariant subset, one has $O(n)_X \subset O(n)_D$. Also, as r is an $O(n)$-map, we have

$$O(n)_D \subset O(n)_{r(D)} = O(n)_A.$$

Thus, $O(n)_X \subset O(n)_A$, which implies, together with $[O(n)_A] \preceq [O(n)_X]$, that $O(n)_A = O(n)_X$, as required. ∎

Proposition 5.4. *For every $\varepsilon > 0$, there is an $O(n)$-map $f_\varepsilon : L_0(n) \to \mathcal{P}(n)$, ε-close to the identity map of $L_0(n)$.*

Proof. Repeat the proof of Proposition 4.10, replacing $M_0(n)$ by $L_0(n)$, until the construction of the family $\{X_\mu\}_{\mu \in \mathcal{M}}$. Next, use Lemma 5.3 to find, for every index μ, a compact set A_μ, $\varepsilon/4$-close to X_μ, such that $O(n)_{A_\mu} = H_\mu$.

Now repeat the rest of the proof of Proposition 4.10, replacing $M_0(n)$ by $L_0(n)$, and $\mathcal{R}(n)$ by $\mathcal{P}(n)$. ∎

Proposition 5.5. *For every $\varepsilon > 0$, there is an $O(n)$-map $h_\varepsilon : L_0(n) \to L_0(n) \setminus \mathcal{P}(n)$, ε-close to the identity map of $L(n)$, such that $h_\varepsilon(A) \neq \mathbb{B}^n$ for every $A \in L(n)$.*

Proof. Repeat the proof of Proposition 4.11, replacing $M_0(n)$ by $L_0(n)$, and $M_0(n) \setminus \mathcal{R}(n)$ by $L_0(n) \setminus \mathcal{P}(n)$. ∎

Proposition 5.6. *Let $K \subset O(n)$ be a a closed subgroup that acts non-transitively on $\mathbb{S}^{n-1}$. Then, for every $\varepsilon > 0$, there exists a K-equivariant map $\chi_\varepsilon : L(n) \to L_0(n)$, ε-close to the identity map of $L(n)$.*

Proof. The proof goes as the one of Proposition 4.6 if we replace $M(n)$ by $L(n)$, $M_0(n)$ by $L_0(n)$, $cc(\mathbb{R}^n)$ by $cb(\mathbb{R}^n)$, and the retraction r of (4.2) by the retraction $r : cb(\mathbb{R}^n) \to L(n)$ given in Corollary 3.9(2). We omit the details. ∎

In the same manner that Proposition 4.6 implies Corollary 4.7, we deduce from Proposition 5.6 the following corollary:

Corollary 5.7. *For every closed subgroup $K \subset O(n)$ that acts non-transitively on $\mathbb{S}^{n-1}$,*

(1) *$\{\mathbb{B}^n\}$ is a Z-set in $L(n)^K$,*
(2) *the class of $\{\mathbb{B}^n\}$ is a Z-set in $L(n)/K$.*

Proposition 5.8. *For every closed subgroup $K \subset O(n)$, $L_0(n)/K$ is a Q-manifold.*

Proof. By Proposition 5.1, $L(n)$ is an $O(n)$-AR, hence a K-AR (see, e.g., [28]). Then Theorem 2.3 implies that $L(n)/K$ is an AR. Since $L_0(n)/K$ is open in $L(n)/K$ we conclude that $L_0(n)/K$ is a locally compact ANR.

According to Toruńczyk's Characterization Theorem [27, Theorem 1], it is enough to check that for every $\varepsilon > 0$, there exist continuous maps $\widetilde{f}_\varepsilon, \widetilde{h}_\varepsilon : L_0(n)/K \to L_0(n)/K$ ε-close to the identity map of $L_0(n)/K$ and with disjoint images.

Let f_ε and h_ε be the $O(n)$-maps constructed in Propositions 5.4 and 5.5, respectively. They induce continuous maps $\widetilde{f}_\varepsilon : L_0(n)K \to L_0(n)/K$ and $\widetilde{h}_\varepsilon : L_0(n)/K \to L_0(n)/K$. Since $\operatorname{Im} \widetilde{f}_\varepsilon = (\operatorname{Im} f_\varepsilon)/K$, $\operatorname{Im} \widetilde{h}_\varepsilon = (\operatorname{Im} h_\varepsilon)/K$ and $\operatorname{Im} f_\varepsilon \cap \operatorname{Im} h_\varepsilon = \emptyset$, we infer that $\operatorname{Im} \widetilde{f}_\varepsilon \cap \operatorname{Im} \widetilde{h}_\varepsilon = \emptyset$. Since f_ε and h_ε are ε-close to the identity map of $L_0(n)$, using inequality (2.2), we conclude that $\widetilde{f}_\varepsilon$ and $\widetilde{h}_\varepsilon$ are ε-close to the identity map of $L_0(n)/K$, as required. ∎

Now, Proposition 5.8, Corollary 5.7 and [27, §3] imply that $L(n)/K$ is a Q-manifold if $K \subset O(n)$ is a closed subgroup that acts nontransitively on $\mathbb{S}^{n-1}$. Since $L(n)/K$ is compact and contractible, we deduce from [21, Theorem 7.5.8] the following corollary:

Corollary 5.9. *For every closed subgroup $K \subset O(n)$ that acts nontransitively on $\mathbb{S}^{n-1}$, the K-orbit space $L(n)/K$ is a Hilbert cube. In particular, $L(n)$ is a Hilbert cube.*

Repeating the same steps used in the proof of Corollary 4.14, we can infer from Corollary 5.7 and Propositions 5.4 and 5.5 the following result:

Corollary 5.10. *For any closed subgroup $K \subset O(n)$ that acts nontransitively on $\mathbb{S}^{n-1}$, the K-fixed point set $L(n)^K$ is homeomorphic to the Hilbert cube.*

Finally, similarly to the case of $M(n)$, we can infer from all previous results of this section that $L(n)$ is a Hilbert cube endowed with an $O(n)$-action that satisfies the following conditions:

(1) $L(n)$ is an $O(n)$-AR with a unique $O(n)$-fixed point, $\mathbb{B}^n$,
(2) $L(n)$ is strictly $O(n)$-contractible to $\mathbb{B}^n$,
(3) for a closed subgroup $K \subset O(n)$, the set $L(n)^K$ equals $\{\mathbb{B}^n\}$ if and only if K acts transitively on $\mathbb{S}^{n-1}$, and $L(n)^K$ is homeomorphic to the Hilbert cube whenever $L(n)^K \neq \{\mathbb{B}^n\}$,
(4) for any closed subgroup $K \subset O(n)$, the K-orbit space $L_0(n)/K$ is a Q-manifold.

These properties in combination with [10, Theorem 3.3] yield

THEOREM 5.11. *The orbit space $L(n)/O(n)$ is homeomorphic to the Banach–Mazur compactum* $\mathrm{BM}(n)$.

6. Orbit spaces of $cb(\mathbb{R}^n)$. In what follows we will denote by $cb_0(\mathbb{R}^n)$ the complement

$$cb_0(\mathbb{R}^n) = cb(\mathbb{R}^n) \setminus E(n).$$

In this section we shall prove the following main result:

THEOREM 6.1. *Let $K \subset O(n)$ be a closed subgroup that acts nontransitively on $\mathbb{S}^{n-1}$. Then:*

(1) *$cb_0(\mathbb{R}^n)/K$ is a Q-manifold.*
(2) *$cb(\mathbb{R}^n)/K$ is a Q-manifold homeomorphic to $(E(n)/K) \times Q$.*

By Corollary 3.9(2) we have an $O(n)$-equivariant homeomorphism

$$cb(\mathbb{R}^n) \cong_{O(n)} L(n) \times E(n).$$

Under this homeomorphism, $cb_0(\mathbb{R}^n)$ corresponds to $E(n) \times L_0(n)$, thus we have the $O(n)$-equivariant homeomorphism

$$cb(\mathbb{R}^n) \cong_{O(n)} L(n) \times E(n). \tag{6.1}$$

We will consider the following $O(n)$-invariant metric on the product $E(n) \times L(n)$:

$$D((A_1, E_1), (A_2, E_2)) = d_{\mathrm{H}}(A_1, A_2) + d_{\mathrm{H}}(E_1, E_2).$$

PROPOSITION 6.2. *For each $\varepsilon > 0$ and every closed subgroup $K \subset O(n)$ that acts nontransitively on $\mathbb{S}^{n-1}$, there exists a K-equivariant map $\eta : cb(\mathbb{R}^n) \to cb_0(\mathbb{R}^n)$ which is ε-close to the identity map of $cb(\mathbb{R}^n)$.*

Proof. Let $\varepsilon > 0$. By Proposition 5.6, there exists a K-map $\chi_\varepsilon : L(n) \to L_0(n)$ such that $d_{\mathrm{H}}(A, \xi(A)) < \varepsilon$ for every $A \in L(n)$. Then the map

$$\eta = \chi_\varepsilon \times \mathrm{Id} : L(n) \times E(n) \to L_0(n) \times E(n)$$

is a K-map such that

$$D(\eta(A, E), (A, E)) = d_{\mathrm{H}}(\xi(A), A) < \varepsilon. \ \blacksquare$$

The map η of Proposition 6.2 induces a map

$$\widetilde{\eta} : \frac{L(n) \times E(n)}{K} \longrightarrow \frac{L_0(n) \times E(n)}{K}$$

which, by (2.2), is ε-close to the identity map of $\frac{L(n)\times E(n)}{K}$. This yields the following corollary:

COROLLARY 6.3. *For every closed subgroup $K \subset O(n)$ that acts nontransitively on $\mathbb{S}^{n-1}$, $E(n)/K$ is a Z-set in $cb(\mathbb{R}^n)/K$. In particular, $E(n)$ is a Z-set in $cb(\mathbb{R}^n)$.*

Proposition 6.4. *Let $K \subset O(n)$ be a closed subgroup that acts non-transitively on $\mathbb{S}^{n-1}$ and $\pi : L(n) \times E(n) \to E(n)$ be the second projection. Then the induced map $\widetilde{\pi} : (L(n) \times E(n))/K \to E(n)/K$ is proper and has contractible fibers.*

Proof. Consider the following commutative diagram:

$$\begin{array}{ccc} L(n) \times E(n) & \xrightarrow{\pi} & E(n) \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_2} \\ \frac{L(n)\times E(n)}{K} & \xrightarrow{\widetilde{\pi}} & \frac{E(n)}{K} \end{array}$$

where p_1 and p_2 are the respective K-orbit maps.

Properness of $\widetilde{\pi}$ easily follows from compactness of $L(n)$ and K. That the fibers of $\widetilde{\pi}$ are contractible follows immediately from the fact that $L(n)$ is $O(n)$-equivariantly contractible (see Proposition 5.2). ■

Theorem 6.5 (R. D. Edwards). *Let M be a Q-manifold and Y a locally compact ANR. If there exists a CE-map $f : M \to Y$, then M is homeomorphic to $Y \times Q$.*

Proof. Since f is a CE-map, by a theorem of R. D. Edwards [14, Theorem 43.1] the product map

$$f \times \mathrm{Id} : M \times Q \to Y \times Q$$

is a near homeomorphism. According to the Stability Theorem [14, Theorem 15.1], M is homeomorphic to $M \times Q$. Thus, we have the homeomorphisms

$$M \cong M \times Q \cong Y \times Q. \ ■$$

Proof of Theorem 6.1. (1) By (6.1), $cb_0(\mathbb{R}^n)$ is $O(n)$-homeomorphic to $L_0(n) \times E(n)$. This implies that the orbit spaces $cb_0(\mathbb{R}^n)/K$ and $\frac{L_0(n)\times E(n)}{K}$ are homeomorphic. Hence, it is enough to prove that the latter is a Q-manifold.

Suppose that $\frac{L_0(n)\times E(n)}{K}$ is equipped with the metric D^* induced by D as defined in (2.1).

By Proposition 5.1, $L(n) \in O(n)$-AR, and by Corollary 3.9(2), $E(n) \in O(n)$-AR. Consequently, $L_0(n)\times E(n)$ is a locally compact $O(n)$-ANR, which in turn implies that $L_0(n) \times E(n) \in K$-AR (see, e.g., [28]). Then, by Theorem 2.3, $\frac{L_0(n)\times E(n)}{K}$ is a locally compact ANR.

Let f_ε and h_ε be the maps from Propositions 5.4 and 5.5, respectively. Consider the maps

$$f = f_\varepsilon \times \mathrm{Id} : L_0(n) \times E(n) \to L_0(n) \times E(n),$$
$$h = h_\varepsilon \times \mathrm{Id} : L_0(n) \times E(n) \to L_0(n) \times E(n),$$

where Id denotes the identity map of $E(n)$. Since f_ε and h_ε are $O(n)$-maps with disjoint images, so are f and h. Hence they induce continuous maps

$$\widetilde{f}, \widetilde{h} : \frac{L_0(n) \times E(n)}{K} \to \frac{L_0(n) \times E(n)}{K}$$

which make the following diagrams commutative:

$$\begin{array}{ccc} L_0(n) \times E(n) & \xrightarrow{f} & L_0(n) \times E(n) \\ \Big\downarrow p & & \Big\downarrow p \\ \frac{L_0(n)\times E(n)}{K} & \overset{\widetilde{f}}{\dashrightarrow} & \frac{L_0(n)\times E(n)}{K} \end{array} \qquad \begin{array}{ccc} L_0(n) \times E(n) & \xrightarrow{h} & L_0(n) \times E(n) \\ \Big\downarrow p & & \Big\downarrow p \\ \frac{L_0(n)\times E(n)}{K} & \overset{\widetilde{h}}{\dashrightarrow} & \frac{L_0(n)\times E(n)}{K} \end{array}$$

Since, $d_{\mathrm{H}}(f_\varepsilon(A), A) < \varepsilon$, we infer that

$$D(f(A,E),(A,E)) = D((f_\varepsilon(A),E),(A,E)) = d_{\mathrm{H}}(f_\varepsilon(A), A) < \varepsilon$$

Similarly, we can prove that $D(h(A,E),(A,E)) < \varepsilon$. Thus, f and h are ε-close to the identity map of $L_0(n) \times E$. Next, using (2.2) we find that $\widetilde{f}$ and $\widetilde{h}$ are ε-close to the identity map of $\frac{L_0(n)\times E(n)}{K}$.

Finally, since $\operatorname{Im} \widetilde{f} = (\operatorname{Im} f)/K$, $\operatorname{Im} \widetilde{h} = (\operatorname{Im} h)/K$ and $\operatorname{Im} f \cap \operatorname{Im} h = \emptyset$, we infer that $\operatorname{Im} \widetilde{f} \cap \operatorname{Im} \widetilde{h} = \emptyset$. Consequently, by Toruńczyk's Characterization Theorem ([27, Theorem 1]), $\frac{L_0(n)\times E}{K}$ is a Q-manifold, as required.

(2) Since, by Corollary 3.9(2), $cb(\mathbb{R}^n)$ and $L(n) \times E(n)$ are $O(n)$-homeomorphic, so are the K-orbit spaces $cb(\mathbb{R}^n)/K$ and $\frac{L(n)\times E(n)}{K}$. On the other hand, $cb(\mathbb{R}^n)$ is an $O(n)$-AR ([8, Corollary 4.8]), and hence a K-AR (see, e.g., [28]). Then Theorem 2.3 shows that $cb(\mathbb{R}^n)/K \cong \frac{L(n)\times E(n)}{K}$ is an AR. By the previous case (1), $cb_0(\mathbb{R}^n)/K$ is a Q-manifold while its complement in $cb(\mathbb{R}^n)/K$ is a Z-set (see Corollary 6.3). Now a result of Toruńczyk [27, §3] implies that $cb(\mathbb{R}^n)/K$ is a Q-manifold too.

Furthermore, by Corollary 3.10, $E(n)$ is an $O(n)$-AR, and hence a K-AR (see, e.g., [28]). Then, according to Theorem 2.3, $E(n)/K$ is an AR.

Since, by Proposition 6.4, the map

$$\widetilde{\pi} : \frac{L(n) \times E(n)}{K} \to E(n)/K$$

is proper and has contractible fibers, it is a CE-map (see [14, Ch. XIII]) between AR's. Since $\frac{cb(\mathbb{R}^n)}{K} \cong \frac{L(n)\times E(n)}{K}$ is a Q-manifold, Edwards' Theorem 6.5 shows that $cb(\mathbb{R}^n)/K$ is homeomorphic to $(E(n)/K) \times Q$, as required. ∎

7. Orbit spaces of $cc(\mathbb{R}^n)$. In this section we shall prove the following two main results:

Theorem 7.1. *For every closed subgroup $K \subset O(n)$ that acts nontransitively on $\mathbb{S}^{n-1}$, the orbit space $cc(\mathbb{R}^n)/K$ is homeomorphic to the punctured Hilbert cube.*

Theorem 7.2. *The orbit space $cc(\mathbb{R}^n)/O(n)$ is homeomorphic to the open cone over* $\mathrm{BM}(n)$.

The proofs require some preparation.

Lemma 7.3. *The map ν defined in* (4.1) *is proper and has contractible fibers.*

Proof. Clearly, ν is onto. Take a compact subset $C \subset [0,\infty)$. Let b be the supremum of C and denote by N_b the closed ball of radius b centered at the origin of $\mathbb{R}^n$. Clearly, $\nu^{-1}(C)$ is a closed subset of $cc(N_b)$. According to [22, Theorem 2.2], $cc(N_b)$ is compact, and thus $\nu^{-1}(C)$ is also compact. This shows that ν is a proper map.

We show that for every $t \in [0,\infty)$ the inverse image $\nu^{-1}(t)$ is contractible. Consider the homotopy $H : \nu^{-1}(t) \times [0,1] \to \nu^{-1}(t)$ defined by

$$H(A,s) = sN_t + (1-s)A, \qquad A \in \nu^{-1}(t),\ s \in [0,1]. \tag{7.1}$$

It is easy to see that $H(A,s) \in \nu^{-1}(t)$, and hence H defines a (strict) homotopy of $\nu^{-1}(t)$ to its point $N_t \in \nu^{-1}(t)$. Thus, $\nu^{-1}(t)$ is contractible, as required. ∎

Since ν is $O(n)$-invariant, it induces, for every closed subgroup $K \subset O(n)$, a continuous map

$$\widetilde{\nu} : cc(\mathbb{R}^n)/K \to [0,\infty)$$

given by

$$\widetilde{\nu}(K(A)) = \nu(A), \qquad K(A) \in cc(\mathbb{R}^n)/K.$$

Proposition 7.4. *$\widetilde{\nu}$ is proper and has contractible fibers.*

Proof. Clearly, $\widetilde{\nu}$ is an onto map. Let $p : cc(\mathbb{R}^n) \to cc(\mathbb{R}^n)/K$ be the K-orbit map. Then we have the following commutative diagram:

$$\begin{array}{ccc} cc(\mathbb{R}^n) & \xrightarrow{\ \nu\ } & [0,\infty) \\ {\scriptstyle p}\big\downarrow & \nearrow_{\widetilde{\nu}} & \\ \frac{cc(\mathbb{R}^n)}{K} & & \end{array}$$

If $C \subset [0,\infty)$ is a compact set, then

$$\widetilde{\nu}^{-1}(C) = \{K(A) \mid \nu(A) \in C\} = p(\nu^{-1}(C)),$$

which is compact because ν is proper and p is continuous. Thus $\widetilde{\nu}$ is a proper map.

To finish the proof, let us show that $\widetilde{\nu}^{-1}(t)$ is contractible for every $t \in [0,\infty)$. Consider the homotopy H defined in (7.1). Observe that H is equivariant. Indeed, for every $g \in O(n)$ one has

$$\begin{aligned} H(gA,s) &= sN_t + (1-s)gA = sgN_t + (1-s)gA \\ &= g(sN_t + (1-s)A) = gH(A,s). \end{aligned} \tag{7.2}$$

Hence, H induces a homotopy $\widetilde{H} : \widetilde{\nu}^{-1}(t) \times [0,1] \to \widetilde{\nu}^{-1}(t)$ defined as follows:

$$\widetilde{H}(K(A),s) = K(H(A,s)).$$

Clearly, $\widetilde{H}$ is a contraction to the point $K(N_t)$, which proves that $\widetilde{\nu}^{-1}(t)$ is contractible, as required. ∎

Proposition 7.5. *The complement*

$$\frac{cc(\mathbb{R}^n)}{K} \setminus \frac{cb(\mathbb{R}^n)}{K}$$

is a Z-set in $cc(\mathbb{R}^n)/K$.

Proof. For every positive ε, the map $\zeta_\varepsilon : cc(\mathbb{R}^n) \to cb(\mathbb{R}^n)$ defined by

$$\zeta_\varepsilon(A) = A_\varepsilon = \{x \in \mathbb{R}^n \mid d(x,A) \le \varepsilon\}$$

is an $O(n)$-equivariant map which is ε-close to the identity map of $cc(\mathbb{R}^n)$. Hence, for every closed subgroup $K \subset O(n)$ it induces a continuous map

$$\widetilde{\zeta}_\varepsilon : cc(\mathbb{R}^n)/K \to cb(\mathbb{R}^n)/K.$$

Since the Hausdorff metric d_{H} is $O(n)$-invariant, it induces a metric in $cc(\mathbb{R}^n)/K$ as in (2.1). Then, by (2.2), the map $\widetilde{\zeta}_\varepsilon$ is ε-close to the identity map of $cc(\mathbb{R}^n)/K$. This proves that

$$\frac{cc(\mathbb{R}^n) \setminus cb(\mathbb{R}^n)}{K} = \frac{cc(\mathbb{R}^n)}{K} \setminus \frac{cb(\mathbb{R}^n)}{K}$$

is a Z-set in $cc(\mathbb{R}^n)/K$. ∎

Proof of Theorem 7.1. Since by Theorem 6.1, $cb(\mathbb{R}^n)/K$ is a Q-manifold and the complement $\frac{cc(\mathbb{R}^n)}{K} \setminus \frac{cb(\mathbb{R}^n)}{K}$ is a Z-set, it follows from [27, §3] that $cc(\mathbb{R}^n)/K$ is also a Q-manifold.

Next, since by Proposition 7.4, the map $\widetilde{\nu} : cc(\mathbb{R}^n)/K \to [0,\infty)$ is proper and has contractible fibers, it is a CE-map (see [14, Ch. XIII]). Then we can use Edwards' Theorem 6.5 to conclude that $cc(\mathbb{R}^n)/K$ is homeomorphic to $[0,\infty) \times Q$. As shown in the proof of [14, Theorem 12.2], the product $[0,\infty) \times Q$ is homeomorphic to the punctured Hilbert cube, which completes the proof. ∎

Now we turn to the proof of Theorem 7.2.

The open cone over a topological space X is defined to be the quotient space

$$\mathrm{OC}(X) = X \times [0,\infty)/X \times \{0\}.$$

We will denote by $[A,t]$ the equivalence class of the pair $(A,t)\in X\times[0,\infty)$ in this quotient space. It is evident that $[A,t]=[A',t']$ iff $t=0=t'$ or $A=A'$ and $t=t'$. For convenience, the class $[A,0]$ will be denoted by θ.

Denote the open cone over $M(n)$ by $\widetilde{M}(n)$. The orthogonal group $O(n)$ acts continuously on $\widetilde{M}(n)$ by the rule

$$g*[A,t]=[gA,t].$$

PROPOSITION 7.6. *The hyperspace $cc(\mathbb{R}^n)$ is $O(n)$-homeomorphic to $\widetilde{M}(n)$.*

Proof. Define $\Phi: cc(\mathbb{R}^n)\to\widetilde{M}(n)$ by

$$\Phi(A)=\begin{cases}\theta & \text{if } A=\{0\},\\ [r(A),\nu(A)] & \text{if } A\neq\{0\},\end{cases}$$

where ν and r are the maps defined in (4.1) and (4.2), respectively.

Since r is $O(n)$-equivariant and ν is $O(n)$-invariant, we infer that Φ is $O(n)$-equivariant.

Clearly, Φ is a bijection with $\Phi^{-1}:\widetilde{M}(n)\to cc(\mathbb{R}^n)$ given by

$$\Phi^{-1}([A,t])=tA.$$

Continuity of $\Phi|_{cc(\mathbb{R}^n)\setminus\{0\}}$ and $\Phi^{-1}|_{\widetilde{M}(n)\setminus\{\theta\}}$ is evident. Let us prove simultaneously the continuity of Φ at $\{0\}$ and the continuity of Φ^{-1} at θ.

Let $\varepsilon>0$ and let O_ε be the open ε-ball in $cc(\mathbb{R}^n)$ centered at $\{0\}$. Denote $U_\varepsilon=\{[A,t]\in\widetilde{M}(n)\mid t<\varepsilon\}$. Since U_ε is an open neighborhood of θ in $\widetilde{M}(n)$, it is enough to prove that $\Phi(O_\varepsilon)=U_\varepsilon$.

If $B\in O_\varepsilon$ then $B\subset N(\{0\},\varepsilon)$, and hence $\nu(B)<\varepsilon$. This proves that $\Phi(B)=[r(B),\nu(B)]\in U_\varepsilon$, implying that

$$\Phi(O_\varepsilon)\subset U_\varepsilon. \tag{7.3}$$

On the other hand, if $[A,t]\in U_\varepsilon$ then $t<\varepsilon$, implying that $tA\subset N(\{0\},\varepsilon)$. This shows that for every $a\in A$, $d(ta,0)<\varepsilon$. In particular, $0\in N(tA,\varepsilon)$, and hence $d_{\mathrm{H}}(\{0\},tA)<\varepsilon$. Thus, $\Phi^{-1}(U_\varepsilon)\subset O_\varepsilon$ and

$$U_\varepsilon=\Phi(\Phi^{-1}(U_\varepsilon))\subset\Phi(O_\varepsilon). \tag{7.4}$$

Combining (7.3) and (7.4) we get $\Phi(O(\{0\},\varepsilon))=U_\varepsilon$. ■

Since Φ is an $O(n)$-homeomorphism, it induces a homeomorphism between $cc(\mathbb{R}^n)/O(n)$ and $\widetilde{M}(n)/O(n)$. Thus, we have

COROLLARY 7.7. *The orbit spaces $cc(\mathbb{R}^n)/O(n)$ and $\widetilde{M}(n)/O(n)$ are homeomorphic.*

LEMMA 7.8. *For every closed subgroup $K\subset O(n)$, the orbit space $\widetilde{M}(n)/K$ is homeomorphic to the open cone over $M(n)/K$.*

Proof. The map $\Psi : \widetilde{M}(n)/K \to \mathrm{OC}(M(n)/K)$ defined by

$$\Psi(K[A,t]) = [K(A),t]$$

is a homeomorphism. ∎

Proof of Theorem 7.2. According to Corollary 7.7 and Lemma 7.8, the orbit space $cc(\mathbb{R}^n)/O(n)$ is homeomorphic to the open cone $\mathrm{OC}(M(n)/O(n))$. By Corollary 4.16, $M(n)/O(n)$ is homeomorphic to the Banach–Mazur compactum $\mathrm{BM}(n)$, and hence $cc(\mathbb{R}^n)/O(n)$ is homeomorphic to $\mathrm{OC}(\mathrm{BM}(n))$, as required. ∎

7.1. Conic structure of $cc(\mathbb{R}^n)$ and related spaces. It is easy to see that $\mathbb{R}^n$ is $O(n)$-homeomorphic to the open cone over $\mathbb{S}^{n-1}$. This conic structure induces a conic structure in $cc(\mathbb{R}^n)$, as shown in Proposition 7.6.

Furthermore, the $O(n)$-homeomorphism between $cc(\mathbb{R}^n)$ and $\widetilde{M}(n)$, in combination with Lemma 7.8, yields the following:

Theorem 7.9. *For every closed subgroup $K \subset O(n)$, the K-orbit space $cc(\mathbb{R}^n)/K$ is homeomorphic to the open cone $OC(M(n)/K)$.*

On the other hand, if we restrict the $O(n)$-homeomorphism from Proposition 7.6 to $cc(\mathbb{B}^n)$, we get an $O(n)$-homeomorphism between $cc(\mathbb{B}^n)$ and the cone over $M(n)$.

As in Lemma 7.8, we can prove that the K-orbit space of the cone over $M(n)$ is homeomorphic to the cone over $M(n)/K$ for every closed subgroup K of $O(n)$. This implies the following result:

Proposition 7.10. *For every closed subgroup $K \subset O(n)$, the K-orbit space $cc(\mathbb{B}^n)/K$ is homeomorphic to the cone over $M(n)/K$.*

Corollary 7.11. *For every closed subgroup $K \subset O(n)$ that acts nontransitively on $\mathbb{S}^{n-1}$, the K-orbit space $cc(\mathbb{B}^n)/K$ is homeomorphic to the Hilbert cube.*

Proof. By Proposition 7.10, $cc(\mathbb{B}^n)/K$ is homeomorphic to the cone over $M(n)/K$. Since K acts nontransitively on $\mathbb{S}^{n-1}$, we infer from Corollary 4.13 that $M(n)/K$ is homeomorphic to the Hilbert cube. Thus, $cc(\mathbb{B}^n)/K$ is homeomorphic to the cone over Q, which according to [14, Theorem 12.2] is homeomorphic to Q itself. ∎

On the other hand, Theorem 4.16 and Proposition 7.10 imply our final result:

Corollary 7.12. *The orbit space $cc(\mathbb{B}^n)/O(n)$ is homeomorphic to the cone over the Banach–Mazur compactum* $\mathrm{BM}(n)$.

It is well known that $\mathrm{BM}(n)$ is an absolute retract for all $n \geq 2$ (see [5]) and the only compact absolute retract that is homeomorphic to its own cone is the Hilbert cube (see, e.g., [21, Theorem 8.3.2]). Therefore, it follows from Corollary 7.12 and Theorem 4.16 that Pełczyński's question of whether BM(n) is homeomorphic to Q is equivalent to the following one:

QUESTION 7.13. *Are $cc(\mathbb{B}^n)/O(n)$ and $M(n)/O(n)$ homeomorphic?*

In conclusion we would like to formulate two more questions suggested by the referee of this paper.

QUESTION 7.14. *What is the topological type of the pair $(cc(\mathbb{R}^n), cb(\mathbb{R}^n))$?*

For any $0 \leq k \leq n$, define

$$cc_{\geq k}(\mathbb{R}^n) = \{A \in cc(\mathbb{R}^n) \mid \dim A \geq k\}$$

and observe that $cb(\mathbb{R}^n) = cc_{\geq n}(\mathbb{R}^n)$ and $cc(\mathbb{R}^n) = cc_{\geq 0}(\mathbb{R}^n)$.

QUESTION 7.15. *What is the topological structure of the spaces $cc_{\geq k}(\mathbb{R}^n)$ and of the complements $cc_k(\mathbb{R}^n) = cc_{\geq k}(\mathbb{R}^n) \setminus cc_{\geq k+1}(\mathbb{R}^n)$ for $0 \leq k < n$?*

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References

[1] H. Abels, *Parallelizability of proper actions, global K-slices and maximal compact subgroups*, Math. Anal. 212 (1974), 1–19.

[2] J. L. Alperin and R. B. Bell, *Groups and Representations*, Grad. Texts in Math. 162, Springer, New York, 1995.

[3] S. A. Antonyan, *Retracts in the categories of G-spaces*, Izv. Akad. Nauk Arm. SSR. Ser. Mat. 15 (1980), 365–378 (in Russian); English transl.: Soviet J. Contemp. Math. Anal. 15 (1980), 30–43.

[4] S. A. Antonyan, *Retraction properties of an orbit space*, Mat. Sb. 137 (1988), 300–318 (in Russian); English transl.: Math. USSR-Sb. 65 (1990), 305–321.

[5] S. A. Antonyan, *The Banach–Mazur compacta are absolute retracts*, Bull. Polish Acad. Sci. Math. 46 (1998), 113–119.

[6] S. A. Antonyan, *The topology of the Banach–Mazur compactum*, Fund. Math. 166 (2000), 209–232.

[7] S. A. Antonyan, *West's problem on equivariant hyperspaces and Banach–Mazur compacta*, Trans. Amer. Math. Soc. 355 (2003), 3379–3404.

[8] S. A. Antonyan, *Extending equivariant maps into spaces with convex structures*, Topology Appl. 153 (2005), 261–275.

[9] S. A. Antonyan, *A characterization of equivariant absolute extensors and the equivariant Dugundji theorem*, Houston J. Math. 31 (2005), 451–462.

[10] S. A. Antonyan, *New topological models for Banach–Mazur compacta*, J. Math. Sci. 146 (2007), 5465–5473.

[11] S. Banach, *Théorie des Opérations Linéaires*, Monografje Mat., Warszawa, 1932.

[12] G. E. Bredon, *Introduction to Compact Transformation Groups*, Academic Press, New York, 1972.

[13] O. Bretscher, *Linear Algebra with Applications*, Prentice-Hall, Upper Saddle River, NJ, 1997.
[14] T. A. Chapman, *Lectures on Hilbert Cube Manifolds*, CBMS Reg. Conf. Ser. Math. 28, Amer. Math. Soc., Providence, RI, 1975.
[15] L. Danzer, D. Laugwitz und H. Lenz, *Über das Löwnersche Ellipsoid und sein Analogon unter den einem Eikörper einbeschriebenen Ellipsoiden*, Arch. Math. (Basel) 8 (1957), 214–219.
[16] G. Hochschild, *The Structure of Lie Groups*, Holden-Day, San Francisco, 1965.
[17] F. John, *Extremum problems with inequalities as subsidiary conditions*, in: F. John, Collected Papers, Vol. 2, J. Moser (ed.), Birkhäuser, 1985, 543–560.
[18] I. Kaplansky, *Linear Algebra and Geometry. A second course*, Dover Publ., New York, 1974.
[19] J. Lindenstrauss and V. D. Millman, *The local theory of normed spaces and its applications to convexity*, in: Handbook of Convex Geometry, P. M. Gruber and J. M. Wills (eds.), Elsevier, Amsterdam, 1993, 1149–1220.
[20] A. M. Macbeath, *A compactness theorem for affine equivalence-classes of convex regions*, Canad. J. Math. 3 (1951), 54–61.
[21] J. van Mill, *Infinite-Dimensional Topology: Prerequisites and Introduction*, North-Holland Math. Library 43, North-Holland, Amsterdam, 1989.
[22] S. B. Nadler, Jr., J. E. Quinn and N. M. Stavrakas, *Hyperspaces of compact convex sets*, Pacific J. Math. 83 (1979), 441–462.
[23] R. Palais, *The classification of G-spaces*, Mem. Amer. Math. Soc. 36 (1960).
[24] R. Palais, *On the existence of slices for actions of non-compact Lie groups*, Ann. of Math. 73 (1961), 295–323.
[25] K. Sakai and Z. Yang, *The space of closed convex sets in Euclidean spaces with the Fell topology*, Bull. Polish Acad. Sci. 55 (2007), 139–143.
[26] E. H. Spanier, *Algebraic Topology*, McGraw-Hill, New York, 1966.
[27] H. Toruńczyk, *On* CE*-images of the Hilbert cube and characterization of Q-manifolds*, Fund. Math. 106 (1980), 31–40.
[28] J. de Vries, *Topics in the theory of topological groups*, in: Topological Structures II, Math. Centre Tracts 116, Math. Centrum, Amsterdam, 1979, 291–304.
[29] R. Webster, *Convexity*, Oxford Univ. Press, Oxford, 1994.
[30] J. E. West, *Open problems in infinite-dimensional topology*, in: Open Problems in Topology, J. van Mill and G. Reed (eds.), North-Holland, Amsterdam, 1990, 524–586.

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