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Open mapping theorems for capacities

by

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Abstract. For the functor of upper semicontinuous capacities in the category of compact Hausdorff spaces and two of its subfunctors, we prove open mapping theorems. These are counterparts of the open mapping theorem for the probability measure functor proved by Ditor and Eifler.

1. Introduction. The notion of capacity in its general form was introduced by Choquet [4]. Interest in topological properties of capacities stems, in particular, from their important applications in mathematical economics (see, e.g., [6–8]).

The space of upper semicontinuous capacities (see the definition below) is introduced in [20] and it is proved therein that this construction determines a functor in the category of compact Hausdorff spaces. The functor is systematically investigated in [19]. In particular, it is proved in [19] that the functor of upper semicontinuous capacities has all the properties from the definition of the normal functor in the sense of Shchepin [14] except the preimage-preserving property.

The notion of upper semicontinuous capacity is a generalization of that of probability measure and one can expect that some results known for measures can be carried over to capacities.

The paper is devoted to the open mapping theorem for upper semicontinuous capacities. The corresponding result for probability measures was first proved by Ditor and Eifler [5] and found numerous applications. It asserts that, for any open onto map $f\colon X \to Y$ of compact Hausdorff spaces, the map $Pf\colon PX \to PY$ is also open, where P denotes the functor of probability measures. (Recall that a map of topological spaces is *open* if the image of every open set is also open.) Our proof is based on ideas different from

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those of Ditor and Eifler and exploits the properties of so-called Milyutin maps; see [18] for an analogous technique. The results of the present paper (they also include open mapping theorems for ∪-capacities and ∩-capacities) demonstrate the universality of the method and we expect that it can also be applied to other functors in the category of compact Hausdorff spaces.

Note that an alternative approach to the open mapping theorem for capacities can be found in [19].

2. Upper semicontinuous capacities. We denote by **Comp** the category of compact Hausdorff spaces and their continuous maps. Unless explicitly stated otherwise, all spaces and maps under consideration are from this category. The *identity functor* in **Comp** is denoted by $\mathbf{1}_{\mathbf{Comp}}$.

We write $A \subset_{\mathrm{cl}} X$ if A is a closed set in a space X. We denote by I the *unit segment* $[0,1]$ with the natural topology, and $\mathbf{1}_A\colon A \to A$ is the *identity map* from a set A onto itself. Let $\exp X$ denote the set of all nonempty closed subsets of a space X. If X is a compact Hausdorff space, the set $\exp X$ is endowed with the *Vietoris topology*, whose base consists of the sets of the form

$$\langle U_1,\dots,U_n\rangle = \Big\{A \in \exp X \,\Big|\, A \subset \bigcup_{i=1}^n U_i,\ A\cap U_i \neq \emptyset,\ i=1,\dots,n\Big\},$$

where $U_1,\dots,U_n$ are open subsets in X.

A *capacity* on a space X is a function $c\colon \exp X \cup \{\emptyset\} \to I$ with the properties:

(1) $c(\emptyset)=0$, $c(X)=1$;
(2) if F, G are closed in X and $F \subset G$, then $c(F) \le c(G)$ (monotonicity);
(3) if $F \subset X$ is closed and $c(F) < a$, then there exists an open set $U \supset F$ such that $c(G) < a$ for every $G \subset U$ (upper semicontinuity).

Remark. A capacity on a Hausdorff space X is often defined (e.g. in [12]) as a function c on the set of *all* subsets of X, but the property of *inner regularity* is demanded, which states that for any set $A \subset X$ the value $c(A)$ is equal to the supremum of $c(K)$ for all compact sets $K \subset X$ such that $K \subset A$. The property (3) is required only for *compact* sets $F \subset X$ and is called *outer regularity*. It is obvious that the restriction of such a capacity to the set of all *compact* subsets of X satisfies (1)–(3), and any function that satisfies (1)–(3) extends to a unique capacity in the latter sense. Thus we will regard a capacity as a function defined for compact (= closed) subsets of X only, but inner regularity is considered as a *useful convention* that extends a capacity to all subsets.

Any probability measure on a compactum X is a capacity, but the converse is false. Choose upper semicontinuous functions $f_1,\dots,f_n : X \to I$

such that $\max f_i = 1$ for all $i = 1, \dots, n$. Then the formulae

$$c(F) = \min\{\max f_1|_F, \dots, \max f_n|_F\}$$

and

$$c'(F) = \max\{\inf(1 - f_1)|_{X\setminus F}, \dots, \inf(1 - f_n)|_{X\setminus F}\}$$

define capacities that are not probability measures in general.

The set of all capacities on X is denoted by MX. We endow the set MX with the topology whose subbase consists of the sets of the form

$$O_-(F, a) = \{c \in MX \mid c(F) < a\},$$

where F is a closed subset in X, $a \in \mathbb{R}$, and

$$\begin{aligned} O_+(U, a) &= \{c \in MX \mid c(U) > a\} \\ &= \{c \in MX \mid c(F) > a \text{ for some closed subset } F \subset U\}, \end{aligned}$$

where U is an open subset in X and $a \in \mathbb{R}$. Then MX is a compact Hausdorff space.

Given a continuous map $f\colon X \to Y$ of compact Hausdorff spaces, we denote by $Mf\colon MX \to MY$ the map acting as follows: $Mf(c)(F) = c(f^{-1}(F))$ for any closed subset F in Y. This map is continuous, and we obtain a functor M in the category **Comp**. We let $M^2 = MM$.

The functor M is part of the *capacity monad* $\mathbb{M} = (M, \eta, \mu)$ that was investigated in detail in [19].

If X is a compact Hausdorff space, then the mappings $\eta X\colon X \to MX$ and $\mu X\colon M^2X \to MX$ are defined by the formulae

$$\eta X(x)(F) = \begin{cases} 1, & x \in F, \\ 0, & x \notin F, \end{cases}$$

$$\mu X(\mathcal{C})(F) = \sup\{a \in I \mid \mathcal{C}(\{c \in MX \mid c(F) \geq a\}) \geq a\}$$

for $\mathcal{C} \in M^2X$ and $F \in \exp X \cup \{\emptyset\}$.

The collections of ηX and μX for all compacta X are *natural transformations* [11] $\eta\colon \mathbf{1}_{\mathbf{Comp}} \to M$ and $\mu X\colon M^2 \to M$. This means that for any continuous map $f : X \to Y$ of compacta we have $Mf \circ \eta X = \eta Y \circ f$ and $Mf \circ \mu X = \mu Y \circ M^2 f$. Moreover, the triple $\mathbb{M} = (M, \eta, \mu)$ is a *monad* [19], i.e. $\mu X \circ M\eta X = \mu X \circ \eta MX = \mathbf{1}_{MX}$ and $\mu X \circ M\mu X = \mu X \circ \mu MX$ for each compactum X. For general questions concerning monads (also called *triples*) see [11].

A capacity c on a compactum X is called a $\cup$-*capacity* if $c(F \cup G) = \max\{c(F), c(G)\}$ for any closed sets F, G in X. The set $M_\cup X$ of all $\cup$-capacities on X is closed in MX, and for any continuous map of compacta $f\colon X \to Y$ we have $Mf(M_\cup X) \subset M_\cup Y$. Therefore we obtain a subfunctor $M_\cup$ of M. Moreover, the inclusions $\eta X(X) \subset M_\cup X$ and $\mu X(M_\cup(M_\cup X)) \subset$

$M_\cup X$ allow us to define a submonad $\mathbb{M}_\cup = (M_\cup, \eta_\cup, \mu_\cup)$ of the capacity monad $\mathbb{M} = (M, \eta, \mu)$ [10].

A capacity c on a compactum X is called a $\cap$-*capacity* if $c(F \cap G) = \min\{c(F), c(G)\}$ for any closed sets F, G in X. The set $M_\cap X$ of all $\cap$-capacities on X is closed in MX, and for any continuous map of compacta $f\colon X \to Y$ we have $Mf(M_\cap X) \subset M_\cap Y$. Therefore we obtain a subfunctor $M_\cap$ of M. The inclusions $\eta X(X) \subset M_\cap X$ and $\mu X(M_\cap(M_\cap X)) \subset M_\cap X$ allow us to define a submonad $\mathbb{M}_\cap = (M_\cap, \eta_\cap, \mu_\cap)$ of the capacity monad $\mathbb{M} = (M, \eta, \mu)$ [10].

The functors $M_\cup$ and $M_\cap$ have all the properties of normal functor (see [14] for the definition). Similarity between $M_\cup$ and $M_\cap$ is not accidental. For a capacity $c \in MX$, the function $\tilde{c}\colon \exp X \cup \{\emptyset\} \to I$ defined by the formula

$$\tilde{c}(F) = 1 - \sup\{c(G) \mid G \subset_{\mathrm{cl}} X,\, G \cap F = \emptyset\}$$

is a capacity on X as well. It is called the *dual capacity* to c. The map $\varkappa X\colon MX \to MX$ that sends each capacity to its dual is a homeomorphism. We use the facts from [19] that $\varkappa X \circ \varkappa X = \mathbf{1}_X$ and $\varkappa X(M_\cup X) = M_\cap X$ for any compactum X. In fact, the collection of maps $(\varkappa X)$ for all compacta X is a natural isomorphism between the functors $M_\cup$ and $M_\cap$.

$\cup$-capacities are called *sup-measures* in [12], but we prefer a non-standard terminology to emphasize the duality between $\cup$- and $\cap$-capacities.

Let $C_+(X)$ denote the set of all nonnegative continuous functions on X. The *Choquet integral* of $\varphi \in C_+(X)$ with respect to $c \in MX$ is defined as follows:

$$I_c(\varphi) = \int \varphi(x)\, dc(x) = \int_0^\infty c(\{x \in X \mid \varphi(x) \ge a\})\, da.$$

One can identify every capacity c with the corresponding Choquet integral I_c. If $\varphi \in C_+(X)$, we write $c(\varphi)$ instead of $I_c(\varphi)$. The diagonal map $c \mapsto (c(\varphi))_{\varphi \in C_+(X)}$ embeds the set MX into the product $\mathbb{R}^{C_+(X)}$. We endow MX with the topology induced by this embedding (weak* topology). A subbase of this topology is formed by the sets of the form

$$O_-(\varphi, a) = \{c \in MX \mid c(\varphi) < a\} \quad \text{and} \quad O_+(\varphi, a) = \{c \in MX \mid c(\varphi) > a\},$$

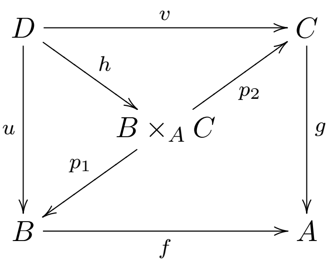
where $\varphi \in C_+(X)$, $a \in \mathbb{R}$. It is known [19] that the weak* topology on MX coincides with the previously defined topology. The space PX of probability measures on X endowed with the weak* topology is clearly a closed subspace of MX.

For a closed subset $A \subset X$ we identify a capacity $c \in MA$ with its image $Mi(c) \in MX$, where $i\colon A \hookrightarrow X$ is the inclusion map. Then MA is a *closed subspace* of MX. The same identification is applicable to probability measures.

An onto map of compacta is *open* if and only if the preimage of each point depends on this point continuously with respect to the Vietoris topology [17]. For onto maps of compact metric spaces, the property of openness of an onto map is equivalent to that of 0-softness [15]. A map $f : X \to Y$ is 0-*soft* if for any zero-dimensional paracompact space Z, a closed subset Z_0 of Z and maps $g_0 : Z_0 \to X$ and $g : Z \to Y$ such that $f \circ g_0 = g|_{Z_0}$, there exists a map $\bar{g} : Z \to X$ such that $\bar{g}|_{Z_0} = g_0$ and $f \circ \bar{g} = g$.

Given continuous maps of compacta $f\colon B \to A$ and $g\colon C \to A$, we denote by $B \times_A C$ the subspace $\{(b, c) \mid b \in B, c \in C, f(b) = g(c)\} \subset B \times C$. Below, the notation $B \times_A C$ is used when there is no ambiguity about the maps $f\colon B \to A$ and $g\colon C \to A$.

Let $p_1\colon B \times_A C \to B$, $p_2\colon B \times_A C \to C$ be the restrictions of $\mathrm{pr}_1^{B\times C}$ and $\mathrm{pr}_2^{B\times C}$. Then $B \times_A C$ along with p_1, p_2 is the *pullback* [11] of f, g in the category of compacta. If $u : D \to B$ and $v : D \to C$ are such that $f \circ u = g \circ v$, i.e. the outer square in the diagram



commutes, then the map $h : D \to B \times_A C$, $h(t) = (u(t), v(t))$ for $t \in D$, is called the *characteristic map* of this square. It is the unique map such that $p_1 \circ h = u$ and $p_2 \circ h = v$. If it is a surjection, then the square that consists of u, v, f, g is called *bicommutative* [14, 15].

Recall that an *inverse σ-system* $\mathcal{S} = \{X_\alpha, p_{\alpha\beta}\}_{\alpha,\beta\in\mathcal{A}}$ is an inverse system satisfying the conditions:

(1) the directed set $\mathcal{A}$ is σ-complete (i.e., each of its countable subsets has the least upper bound);
(2) all X_α are metrizable compacta;
(3) $\mathcal{S}$ is continuous, i.e., for any $\gamma \in \mathcal{A}$, $\varprojlim\{X_\alpha, p_{\alpha\beta}\}_{\alpha,\beta<\gamma} = X_\gamma$

(see [14]). A morphism of a σ-system $\mathcal{S} = \{X_\alpha, p_{\alpha\beta}\}_{\alpha,\beta\in\mathcal{A}}$ into a σ-system $\mathcal{S}' = \{X'_\alpha, p'_{\alpha\beta}\}_{\alpha,\beta\in\mathcal{A}}$ is a collection $(f_\alpha)_{\alpha\in\mathcal{A}}$ of maps such that $p'_{\alpha\beta} f_\alpha = f_\beta p_{\alpha\beta}$ for all $\alpha, \beta \in \mathcal{A}$, $\beta \le \alpha$.

3. Milyutin maps. Recall that a map $f\colon X \to Y$ is called a *Milyutin map* if there exists a map $s\colon Y \to PX$ such that, for any $y \in Y$, $s(y) \in P(f^{-1}(y)) \subset PX$ (see [13]; the term “Milyutin map” was first used by Shchepin [16]). It is known [13] that for any compact metrizable space

X there exists a Milyutin map $f\colon Z \to X$, where Z is a zero-dimensional compact metrizable space.

We call a map $f\colon X \to Y$ a *Milyutin map for capacities* if there exists a map $s\colon Y \to MX$ such that, for any $y \in Y$, $s(y) \in M(f^{-1}(y)) \subset M(X)$. As PX is a subset of MX, and $s(y) \in P(f^{-1}(y))$ implies $s(y) \in M(f^{-1}(y))$, we immediately obtain

PROPOSITION 3.1. *Let X be a compact metrizable space. There exists a Milyutin map for capacities $f\colon Z \to X$, where Z is a zero-dimensional compact metrizable space.*

A map $f\colon X \to Y$ is called a *Milyutin map for $\cup$-capacities* if there exists a map $s\colon Y \to M_\cup X$ such that, for any $y \in Y$, $s(y) \in M_\cup(f^{-1}(y)) \subset M_\cup X$.

PROPOSITION 3.2. *Let X be a compact metrizable space. There exists a Milyutin map $f\colon Z \to X$ for $\cup$-capacities, where Z is a zero-dimensional compact metrizable space.*

Proof. Actually, the proof is a modification of that of [1, Lemma 8] (see also [2] and [3]). One can easily construct a sequence $(\mathcal{W}_n)_{n\in\omega}$, where each $\mathcal{W}_n$ is a finite set of pairs (A, B) such that:

(1) $\mathcal{W}_0 = \{(X, X)\}$;
(2) $A, B \subset_{\mathrm{cl}} X$, $A \subset \operatorname{Int} B$ for any $(A, B) \in \mathcal{W}_n$, $n \in \omega$;
(3) $\operatorname{diam} B < 1/n$ for any $(A, B) \in \mathcal{W}_n$, $n \in \mathbb{N}$;
(4) $\{A \mid (A, B) \in \mathcal{W}_n\}$ is a cover of X for each $n \in \omega$.

We denote by $\coprod$ the topological sum of disjoint copies of topological spaces. Let $X_n = \coprod\{B \mid (A, B) \in \mathcal{W}_n\}$, $n \in \omega$, in particular, $X_0 = X$. Define a map $f_n\colon X_n \to X$ so that its restriction to each B for $(A, B) \in \mathcal{W}_n$ is the inclusion map $B \hookrightarrow X$. Denote by Z the subspace of $\prod_{n\in\omega} X_n$ that consists of all (x_n) such that there exists $x_0 \in X$ with $f_n(x_n) = x_0$ for each $n \in \mathbb{N}$.

Let $p_k\colon Z \to X_k$ be the kth projection: $p_k((x_n)) = x_k$, and denote $p_0\colon Z \to X$ by f. It is obvious that Z is a zero-dimensional metrizable compactum, and f is a continuous surjective map.

We also choose a sequence of continuous maps $\varphi_n\colon X_n \to I$, $n \in \mathbb{N}$, such that for any $(A, B) \in \mathcal{W}_n$ we have $\varphi_n(A) \subset \{1\}$, $\varphi_n(B \setminus \operatorname{Int} B) \subset \{0\}$ ($\operatorname{Int} B$ is taken in X, but A and $B \setminus \operatorname{Int} B$ are regarded as subsets of X_n).

For any $x \in X$ define a function $c_x\colon \exp Z \cup \{\emptyset\} \to I$ by the formula

$$c_x(F) = \sup\{\inf\{\varphi_k(x_k) \mid k = 1, 2, \dots\} \mid (x_n) \in F,\ f((x_n)) = x\}$$

(here we assume $\sup \emptyset = 0$).

All sets $O_{k,a} = \varphi_k^{-1}((-\infty, a))$ for $a \in \mathbb{R}$, $k \in \mathbb{N}$ are open in the corresponding X_k. If $c_x(F) < a$, then for any $z = (x_n) \in f^{-1}(x) \cap F$ we can fix $k_z \in \{1, 2, \dots\}$ such that $\varphi_{k_z}(x_{k_z}) < a$, thus $(x_{k_z}) \in O_{k_z,a}$. Then

$\{p_{k_z}^{-1}(O_{k_z,a}) \mid z \in f^{-1}(x) \cap F\}$ is an open cover of the compactum $f^{-1}(x) \cap F$, therefore we can choose a finite subcover $p_{k_1}^{-1}(O_{k_1,a}), \dots, p_{k_m}^{-1}(O_{k_m,a})$, with all k_i being pairwise distinct. Let

$$W = p_{k_1}^{-1}(O_{k_1,a}) \cup \dots \cup p_{k_m}^{-1}(O_{k_m,a}) \cup (Z \setminus f^{-1}(x)).$$

Then W is an open neighborhood of F.

If $G \subset_{\mathrm{cl}} Z$ and $G \subset W$, choose any $z' = (x'_n) \in G \cap f^{-1}(x)$. Then $x'_{k_i} \in O_{k_i,a}$ for some $i \in \{1, \dots, m\}$, and

$$\inf\{\varphi_k(x'_k) \mid k \in \mathbb{N}\} \le \varphi_{k_i}(x'_{k_i}) < a.$$

This implies $c_x(G) < a$, and c_x is a capacity on X. It is straightforward to show that c_x is a $\cup$-capacity and $c_x \in M(f^{-1}(x))$.

Define a map $s\colon X \to M_\cup Z$ by the formula $s(x) = c_x$. To prove the continuity of s, consider the preimages under s of subbase elements $O_-(F,a) = \{c \in MX \mid c(F) < a\}$, where F is a closed subset in X, $a \in \mathbb{R}$, $a \le 1$, and of

$$\begin{aligned} O_+(U,a) &= \{c \in MX \mid c(U) > a\} \\ &= \{c \in MX \mid c(F) > a \text{ for some closed subset } F \subset U\}, \end{aligned}$$

where U is an open subset in X, $a \in \mathbb{R}$, $a \ge 0$.

Let $x \in s^{-1}(O_-(F,a))$, i.e. $c_x(F) < a$, and let $O_{k_1,a}, \dots, O_{k_m,a}$ be the open sets defined above. The set

$$V = X \setminus f(F \setminus (p_{k_1}^{-1}(O_{k_1,a}) \cup \dots \cup p_{k_m}^{-1}(O_{k_m,a})))$$

is open and contains x. If $x' \in V$, $z' = (x'_n) \in F$, $f(z') = x'$, then $x'_{k_i} \in O_{k_i,a}$ for some $i \in \{1, \dots, m\}$. Thus

$$\inf\{\varphi_k(x'_k) \mid k = 1, 2, \dots\} \le \varphi_{k_i}(x'_{k_i}) < a$$

again, and $c_{x'}(F) < a$, i.e. $x' \in s^{-1}(O_-(F,a))$. This implies that the preimage $s^{-1}(O_-(F,a))$ is open.

Let $x \in s^{-1}(O_+(U,a))$ for $a \ge 0$, i.e. there exist $b > a$, $z = (x_n) \in f^{-1}(x) \cap U$ such that $\varphi_k(x_k) > b$ for all $k \in \mathbb{N}$. We can choose an open neighborhood of the form $V = p_{k_1}^{-1}(V_1) \cap \dots \cap p_{k_m}^{-1}(V_m)$ such that $z \in V \subset U$ and all V_i are open in the corresponding X_{k_i}. If $x_{k_i} \in B \subset X_{k_i}$ for $(A,B) \in \mathcal{W}_{k_i}$, then $\varphi_{k_i}(x_{k_i}) > b > 0$ implies $x_{k_i} \in \operatorname{Int} B$. Let

$$V'_i = V_i \cap \operatorname{Int} B \cap \varphi_{k_i}^{-1}((b,\infty)), \qquad V' = f_{k_1}(V'_1) \cap \dots \cap f_{k_m}(V'_m).$$

Then V' is an open neighborhood of x in X, and for any $x' \in V'$ we choose $z' = (x'_n)$ such that

$$x'_k = \begin{cases} t \in V'_i \text{ such that } f_k(t) = x' & \text{if } k \in \{k_1, \dots, k_m\}, \\ \text{any } t \in X_k \text{ such that } f_k(t) = x',\ \varphi_k(t) = 1 & \text{if } k \notin \{k_1, \dots, k_m\}. \end{cases}$$

In the latter case t exists due to (4) and properties of φ_k. Then $z' \in V \subset U$ and $\varphi_k(x'_k) > b$ for all $k = 1, 2, \dots$, thus $c_{x'}(U) \ge b > a$, and $x' \in s^{-1}(O_+(U,a))$. This implies that the preimage $s^{-1}(O_+(U,a))$ is open as well, and the map $s\colon X \to M_\cup Z$ is continuous. Thus s is the required Milyutin map for $\cup$-capacities. ■

Recall that a capacity $c \in MX$ is a $\cap$-capacity if and only if $\varkappa X(c)$ is a $\cup$-capacity. It is also obvious that $c \in MA$ for $A \subset_{\mathrm{cl}} X$ if and only if $\varkappa X(c) \in MA$.

This allows us to easily obtain a statement dual to the previous one. A map $f\colon X \to Y$ is called a *Milyutin map* for *$\cap$-capacities* if there exists a map $s\colon Y \to M_\cap X$ such that for any $y \in Y$, $s(y) \in M_\cap(f^{-1}(y)) \subset M_\cap X$.

PROPOSITION 3.3. *Let X be a compact metrizable space. There exists a Milyutin map $f\colon Z \to X$ for $\cap$-capacities, where Z is a zero-dimensional compact metrizable space.*

Proof. Let $f\colon Z \to X$ be the Milyutin map for $\cup$-capacities, that was constructed in the previous theorem, and suppose $s\colon X \to M_\cup Z$ satisfies the condition $s(x) \in M_\cup(f^{-1}(x)) \subset M_\cup Z$ for all $x \in X$. Then the map $s' = \varkappa X \circ s\colon X \to M_\cap Z$ satisfies the condition $s'(x) \in M_\cap(f^{-1}(x)) \subset M_\cap Z$ for all $x \in X$. Thus f is also a Milyutin map for $\cap$-capacities. ■

REMARK. It is obvious that if $\tilde{Z}$ is a zero-dimensional metrizable compactum, $r : \tilde{Z} \to Z$ is a retraction and $f : Z \to X$ is a Milyutin map for either of the above mentioned functors, then $f \circ r : \tilde{Z} \to X$ is a Milyutin map as well. Any metrizable zero-dimensional compactum Z is a retract of $Z \times C \cong C$, therefore we can assume in the last three propositions that $Z = C$.

4. Open mapping theorems. The following is the main result of the paper.

THEOREM 4.1. *Let $f\colon X \to Y$ be an open continuous map of compact Hausdorff spaces. Then the map $Mf\colon MX \to MY$ is also open.*

Proof. For any cartesian product $X_1 \times \dots \times X_n$ we denote by $\mathrm{pr}_i^{X_1\times\dots\times X_n}$ its projection onto the ith factor and by $\mathrm{pr}_{ij}^{X_1\times\dots\times X_n}$ its projection onto the product of the ith and jth factors.

By Shchepin's spectral theorem [15, 17] an open surjective map of compacta is induced by a morphism of σ-systems with all components being open surjective maps of metrizable compacta. Thus it is sufficient to consider the case of metrizable X and Y. We first assume that X and Y are finite, and $f\colon X \to Y$ is surjective. Then the map $Mf\colon MX \to MY$ is an affine map of convex subsets of linear spaces. These sets, being the convex hulls of finite sets, are convex polyhedra, which implies the openness of Mf.

Let C denote the Cantor set. Let us prove that the map $M(\mathrm{pr}_1^{C\times C})$ is open. To this end, represent C as $\varprojlim\{C_i, f_{ij}\}$, where C_i are finite sets and

$f_{ij}\colon C_i \to C_j$ are surjections for $i \geq j$. According to [14], in order to prove that $M(\mathrm{pr}_1^{C\times C})$ is open, it is sufficient to prove that the diagram

$$\begin{array}{ccc} M(C_i \times C_i) & \xrightarrow{M(f_{ij}\times f_{ij})} & M(C_j \times C_j) \\ {\scriptstyle M\mathrm{pr}_1^{C_i\times C_i}}\downarrow & & \downarrow{\scriptstyle M\mathrm{pr}_1^{C_j\times C_j}} \\ MC_i & \xrightarrow{Mf_{ij})} & MC_j \end{array}$$

is bicommutative.

Let $(c', c'') \in M(C_j\times C_j)\times_{M(C_j)} M(C_i)$ and $c = Mf_{ij}(c') = M\mathrm{pr}_1^{C_j\times C_j}(c'')$. Given $A \subset C_j$, find $B \subset C_i$ and $D \subset C_j \times C_j$ such that $f_{ij}(B) = \mathrm{pr}_1^{C_j\times C_j}(D) = A$ and $c'(B) = c''(D) = c(A)$. Let

$$\alpha(A, B, D) = (\mathrm{pr}_1^{C_i\times C_i}, f_{ij} \times f_{ij})^{-1}(B \times_A D) \subset C_i \times C_i.$$

Define $\hat{c} \in M(C_i \times C_i)$ as follows: given $X \subset C_i \times C_i$, let $\hat{c}(X)$ be the maximal value of $c(A)$, where $A \subset C_j$ is such that there exist B, D as above, for which $X \supset \alpha(A, B, D)$. It is easy to see that $\hat{c}$ is well-defined and

$$M\mathrm{pr}_1^{C_i\times C_i}(\hat{c}) = c', \qquad M(f_{ij} \times f_{ij})(\hat{c}) = c''.$$

Therefore, the map $M\mathrm{pr}_1^{C\times C}\colon M(C \times C) \to MC$ is open.

Let X be a metrizable compactum and $K \subset C \times X$ a closed subset such that the restriction $\pi\colon K \to C$ of $\mathrm{pr}_1^{C\times X}$ is an open onto map. We choose a continuous surjection $\varphi : C \to X$ and by the 0-openness of π we can add a dotted arrow to the following commutative diagram (i is the inclusion):

$$\begin{array}{ccc} (\mathbf{1}_C \times \varphi)^{-1}(K) & \xrightarrow{\mathbf{1}_C\times\varphi} & K \\ {\scriptstyle i}\downarrow & \overset{h}{\nearrow} & \downarrow{\scriptstyle \pi} \\ C \times C & \xrightarrow{\mathrm{pr}_1^{C\times C}} & C \end{array}$$

Then $\mathrm{pr}_1^{C\times C} = \pi \circ h$ implies $M\mathrm{pr}_1^{C\times C} = M\pi \circ Mh$. The functor M preserves the property of h being surjective, thus $M\pi$ is open as well (see also [9, Lemma 1]).

Now, consider an open surjective map $f\colon X \to Y$ of compact metrizable spaces. Let $p\colon Z \to Y$ be a Milyutin map, where Z is a compact metrizable zero-dimensional space. We may assume that Z is homeomorphic to the Cantor set and $s\colon Y \to MZ$ is a map such that $s(y) \in M(p^{-1}(y))$, thus $Mp \circ s(y) = \eta Y(y)$, for every $y \in Y$. Let

$$K = Z \times_Y X = \{(z, x) \in Z \times X \mid p(z) = f(x)\}.$$

Denote by $\pi_1\colon K \to Z$ and $\pi_2\colon K \to X$ the restrictions to K of $\mathrm{pr}_1^{Z\times X}$ and $\mathrm{pr}_2^{Z\times X}$. Since f is open, the preimage $\pi_1^{-1}(z) = \{z\} \times f^{-1}(p(z))$ is nonempty and depends on $z \in Z$ continuously, which implies that $\pi_1 \colon K \to Z$ is open and surjective. By the above $M\pi_1 \colon MK \to MZ$ is also open and surjective.

For every $x \in X$ let $i_x \colon Z \to Z \times X$ be defined by $i_x(z) = (z, x)$. It is obvious that all values of the map $g \colon X \to M(Z \times X)$ defined by $g(x) = Mi_x(s(f(x)))$ are contained in MK.

Consider a sequence (c_i) in MY converging to c_0 and $\tilde{c}_0 \in MX$ such that $Mf(\tilde{c}_0) = c_0$. Let $\tilde{c}'_0 = \mu K \circ Mg(\tilde{c}_0)$.

Recall that (M, η, μ) is a triple, thus we have

$$\begin{aligned} M\pi_2(\tilde{c}'_0) &= M\mathrm{pr}_2^{Z\times X} \circ \mu(Z \times X) \circ Mg(\tilde{c}_0)) \\ &= \mu X \circ M^2\mathrm{pr}_2^{Z\times X} \circ Mg(\tilde{c}_0) = \mu X \circ M\eta X(\tilde{c}_0) = \tilde{c}_0. \end{aligned}$$

For every $i = 0, 1, \dots,$ define $c'_i = \mu Z \circ Ms(c_i)$. Then

$$\begin{aligned} Mp(c'_i) &= Mp \circ \mu Z \circ Ms(c_i) = \mu Y \circ M^2 p \circ Ms(c_i) \\ &= \mu Y \circ M(Mp \circ s)(c_i) = \mu Y \circ M\eta Y(c_i) = c_i. \end{aligned}$$

We also have

$$\begin{aligned} M\pi_1(\tilde{c}'_0) &= M\mathrm{pr}_1^{Z\times X} \circ \mu(Z \times X) \circ Mg(\tilde{c}_0) \\ &= \mu Z \circ M^2\mathrm{pr}_1^{Z\times X} \circ Mg(\tilde{c}_0) = \mu Z \circ M(M\mathrm{pr}_1^{Z\times X} \circ g)(\tilde{c}_0) \\ &= \mu Z \circ M(s \circ f)(\tilde{c}_0) = \mu Z \circ Ms \circ Mf(\tilde{c}_0) \\ &= \mu Z \circ Ms(c_0) = c'_0. \end{aligned}$$

By the openness of the map $M\pi_1 \colon MK \to MZ$, there exists a sequence $(\tilde{c}'_i)$ in MK such that $\lim_{i\to\infty} \tilde{c}'_i = \tilde{c}'_0$ and $M\pi_1(\tilde{c}'_i) = c'_i$. Denote $\tilde{c}_i = M\pi_2(\tilde{c}'_i)$, $i = 1, 2, \dots$. Then

$$\lim_{i\to\infty} \tilde{c}_i = \lim_{i\to\infty} M\pi_2(\tilde{c}'_i) = M\pi_2(\lim_{i\to\infty} \tilde{c}'_i) = M\pi_2(\tilde{c}'_0) = \tilde{c}_0.$$

Note also that

$$Mf(\tilde{c}_i) = Mf \circ M\pi_2(\tilde{c}'_i) = M(f \circ \pi_2)(\tilde{c}'_i) = M(p \circ \pi_1)(\tilde{c}'_i) = Mp(c'_i) = c_i.$$

This proves that Mf is an open map. ∎

THEOREM 4.2. *Let $f \colon X \to Y$ be an open continuous map of compact Hausdorff spaces. Then the map $M_\cup(f) \colon M_\cup(X) \to M_\cup(Y)$ is also open.*

Proof. This theorem is a counterpart of the previous one, and the method is the same, so we only point out the parts of the proof that cannot be obtained *mutatis mutandis.*

First, for a finite compactum X the set $M_\cup X$ is not convex in the usual sense: for $c_1, c_2 \in M_\cup X$ and $0 < \lambda < 1$ the function c on $\exp X \cup \{\emptyset\}$, defined by $c(F) = \lambda c_1(F) + (1 - \lambda)c_2(F)$, does not necessarily belong to $M_\cup X$. Nevertheless, a capacity $c \in M_\cup X$ allows a simple representation. If $X = \{x_1, \dots, x_n\}$, denote $\alpha_X(c) = (a_1, \dots, a_n)$, where $a_i = c(\{x_i\})$. Then $c(F) = \max\{a_i \mid x_i \in F,\ 1 \le i \le n\}$ for $F \in \exp X \cup \{\emptyset\}$ (here we assume $\max \emptyset = 0$).

All vectors $\alpha_X(c)$ that correspond to capacities c from $M_\cup X$ form a closed subset I^n_* in I^n that is determined by the equality $\max\{a_1,\dots,a_n\}=1$. It is easy to check that $\alpha_X\colon M_\cup X\to I^n_*$ is a homeomorphism.

To prove that $M_\cup$ preserves openness of maps of finite compacta, it suffices to consider only the case $f\colon X\to Y$, $X=\{x_1,\dots,x_n\}$, $Y=\{y_1,\dots,y_{n-1}\}$, $f(x_1)=y_1$, $f(x_2)=y_2$, $\dots$, $f(x_{n-2})=y_{n-2}$, $f(x_{n-1})=f(x_n)=y_{n-1}$. This is equivalent to proving the openness of $F=\alpha_Y\circ M_\cup f\circ\alpha_X^{-1}\colon I^n_*\to I^{n-1}_*$. Obviously,

$$F(a_1,\dots,a_n)=(a_1,\dots,a_{n-2},\max\{a_{n-1},a_n\}).$$

Then for $b=(b_1,\dots,b_{n-1})\in I^{n-1}_*$ we have

$$F^{-1}(b)=\{b_1\}\times\dots\times\{b_{n-2}\}\times(\{b_{n-1}\}\times[0;b_{n-1}]\cup[0;b_{n-1}]\times\{b_{n-1}\})\}.$$

Then $F^{-1}\colon I^{n-1}_*\to\exp I^n_*$ is continuous, thus F and therefore $M_\cup F$ are open.

We also need to reprove that the characteristic map

$$\begin{aligned}(M_\cup\mathrm{pr}_1^{C_i\times C_i},M_\cup(f_{ij}\times f_{ij}))\colon M_\cup(C_i\times C_i)&\to M_\cup C_i\times_{M_\cup C_j}M_\cup(C_j\times C_j)\\ &=\{(c',c'')\in M_\cup C_i\times M_\cup(C_j\times C_j)\mid M_\cup f_{ij}(c')=M_\cup\mathrm{pr}_1^{C_j\times C_j}(c'')\}\end{aligned}$$

is an onto map.

Consider a slightly more general case. Let

$$\begin{array}{ccc} D & \xrightarrow{\ g\ } & C\\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle k}\\ B & \xrightarrow{\ h\ } & A\end{array}$$

be a bicommutative diagram that consists of finite compacta and subjections. If $c_1\in M_\cup B$, $c_2\in M_\cup C$, $M(h)(c_1)=M(h)(c_2)$, then define a function $\hat c:\exp D\cup\{\emptyset\}\to I$ by

$$\hat c(F)=\max\{\min\{c_1(\{f(d)\}),c_2(\{g(d)\})\}\mid d\in F\}.$$

It is straightforward to verify that $\hat c\in M_\cup D$, $M_\cup f(\hat c)=c_1$, $M_\cup g(\hat c)=c_2$, i.e., the characteristic map $(M_\cup f,M_\cup g)\colon M_\cup D\to M_\cup B\times_{M_\cup A}M_\cup C$, $(M_\cup f,M_\cup g)(c)=(M_\cup f(c),M_\cup g(c))$ for $c\in M_\cup D$, is surjective. ∎

By duality it is trivial to obtain also

THEOREM 4.3. *Let $f\colon X\to Y$ be an open continuous map of compact Hausdorff spaces. Then the map $M_\cap f\colon M_\cap X\to M_\cap Y$ is also open.*

5. Final remarks. A capacity $c\in MX$ is called *convex* (resp. *concave*) if $c(F\cup G)+c(F\cap G)\ge c(F)+c(G)$ (resp. $c(F\cup G)+c(F\cap G)\le c(F)+c(G)$) for any closed sets F, G in X.

We denote by M^cX (resp. M_cX) the set of convex (resp. concave) capacities on a compactum X. It is easily seen that M^c and M_c are subfunctors of the functor M. We leave it as an open question whether the maps M^cf and M_cf are open for an open map f.

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References

[1] S. A. Argyros and A. D. Arvanitakis, *A characterization of regular averaging operators and its consequences*, Studia Math. 151 (2002), 207–226.
[2] —, —, *Regular averaging and regular extension operators in weakly compact subsets of Hilbert spaces*, Serdica Math. J. 30 (2004), 527–548.
[3] S. Ageev and E. D. Tymchatyn, *On exact atomless Milutin maps*, Topology Appl. 153 (2005), 227–238.
[4] G. Choquet, *Theory of capacities*, Ann. Inst. Fourier (Grenoble) 5 (1955), 131–295.
[5] S. Z. Ditor and L. Eifler, *Some open mapping theorems for measures*, Trans. Amer. Math. Soc. 164 (1972), 287–293.
[6] J. Eichberger and D. Kelsey, *Non-additive beliefs and strategic equilibria*, Games Econom. Behavior 30 (2000), 183–215.
[7] L. Epstein and T. Wang, *"Beliefs about beliefs" without probabilities*, Econometrica 64 (1996), 1343–1373.
[8] I. Gilboa and D. Schmeidler, *Updating ambiguous beliefs*, J. Econom. Theory 59 (1993), 33–49.
[9] R. Haydon, *On a problem of Pełczyński: Milutin spaces, Dugundji spaces and* AE(0-dim), Studia Math. 52 (1974), 23–31.
[10] I. D. Hlushak and O. R. Nykyforchyn, *Submonads of the capacity monad*, Carpathian J. Math. 24 (2008), 56–67.
[11] S. MacLane, *Categories for the Working Mathematician*, 2nd ed., Springer, New York, 1998.
[12] G. L. O'Brien and W. Verwaat, *How subadditive are subadditive capacities?*, Comment. Math. Univ. Carolin. 35 (1994), 311–324.
[13] A. Pełczyński, *Linear extensions, linear averagings, and their applications to linear topological classification of spaces of continuous functions*, Dissertationes Math. (Rozprawy Mat.) 58 (1968), 92 pp.
[14] E. V. Shchepin, *Functors and uncountable powers of compacta*, Uspekhi Mat. Nauk 36 (1981), no. 3, 3–62 (in Russian).
[15] —, *Method of inverse spectra in the topology of bicompacta*, Math. Notes 31 (1982), 154–162.
[16] —, *Topology of limit spaces of uncountable inverse spectra*, Russian Math. Surveys 31 (1976), 155–191.
[17] A. Teleiko and M. Zarichnyi, *Categorical Topology of Compact Hausdorff Spaces*, VNTL Publ., Lviv, 1999.
[18] M. M. Zarichnyi, *Idempotent probability measures I*, arXiv:math.GN/0608754v1, 2006.
[19] M. M. Zarichnyi and O. R. Nykyforchyn, *Capacity functor in the category of compacta*, Mat. Sb. 199 (2008), 3–26.

[20] L. Zhou, *Integral representation of continuous comonotonically additive functionals*, Trans. Amer. Math. Soc. 350 (1998), 1811–1822.

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