

Minimal sets of non-resonant torus homeomorphisms

by

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Abstract. As was known to H. Poincaré, an orientation preserving circle homeomorphism without periodic points is either minimal or has no dense orbits, and every orbit accumulates on the unique minimal set. In the first case the minimal set is the circle, in the latter case a Cantor set. In this paper we study a two-dimensional analogue of this classical result: we classify the minimal sets of non-resonant torus homeomorphisms, that is, torus homeomorphisms isotopic to the identity for which the rotation set is a point with rationally independent irrational coordinates.

1. Introduction and statement of results. Let $\mathbb{T}^1 = \mathbb{R} \setminus \mathbb{Z}$ and let $f: \mathbb{T}^1 \rightarrow \mathbb{T}^1$ be an orientation preserving circle homeomorphism. A lift $F: \mathbb{R} \rightarrow \mathbb{R}$ of f satisfies $f \circ p_1 = p_1 \circ F$, with $p_1: \mathbb{R} \rightarrow \mathbb{T}^1$ the canonical projection. The number

$$(1.1) \quad \rho(F, x) := \lim_{n \rightarrow \infty} \frac{F^n(x) - x}{n}$$

exists for all $x \in \mathbb{R}$, is independent of x and well defined up to an integer, that is, if F and \widehat{F} are two lifts of f then $\rho(F) - \rho(\widehat{F}) \in \mathbb{Z}$. The number $\rho(f) := \rho(F, x) \bmod \mathbb{Z}$ is called the *rotation number* of f , and $\rho(f) \in \mathbb{Q}$ if and only if f has periodic points. Denote by $r_\theta: \mathbb{T}^1 \rightarrow \mathbb{T}^1$ the rigid rotation of the circle with rotation number θ . The following classical result classifies the possible topological dynamics of orientation preserving homeomorphisms of the circle without periodic points [P1, P2, P3].

POINCARÉ CLASSIFICATION THEOREM. *Let $f: \mathbb{T}^1 \rightarrow \mathbb{T}^1$ be an orientation preserving homeomorphism such that $\rho(f) \in \mathbb{R} \setminus \mathbb{Q}$. Then*

- (i) *if f is transitive then f is conjugate to the rigid rotation $r_{\rho(f)}$,*
- (ii) *if f is not transitive then f is semi-conjugate to the rotation $r_{\rho(f)}$ via a non-invertible continuous monotone map.*

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Moreover, f has a unique minimal set \mathcal{M} , which is the circle \mathbb{T}^1 in case (i) and a Cantor set in case (ii), and $\mathcal{M} = \Omega(f) = \omega(x) = \alpha(x)$ for all $x \in \mathbb{T}^1$.

Here, $\Omega(f)$ is used to denote the *non-wandering set* of f and $\omega(x), \alpha(x)$ the *omega-limit* and *alpha-limit sets* of f relative to $x \in \mathbb{T}^1$. Every connected component I of the complement of the Cantor minimal set is a wandering interval, i.e. $f^n(I) \cap I = \emptyset$ for all $n \neq 0$. Given a Cantor set in the circle, there exists a circle homeomorphism with any given irrational rotation number that has this Cantor set as its minimal set. This fact was first explicitly mentioned by Denjoy [D], but essentially known already by Bohl [Bo] and Kneser [Kn]. Denjoy [D] proved that an orientation preserving circle diffeomorphism $f \in \text{Diff}^2(\mathbb{T}^1)$ with irrational rotation number is necessarily transitive and hence cannot have a wandering interval; see also [H] where these ideas are further developed.

The key feature of an orientation preserving circle homeomorphism without periodic points is that it has an irrational rotation number which is independent of the basepoint, and the rotation with the corresponding rotation number is minimal. A natural generalization to dimension two is as follows. Let $\mathbb{T}^2 = \mathbb{R}^2 \setminus \mathbb{Z}^2$, where $p: \mathbb{R}^2 \rightarrow \mathbb{T}^2$ is the canonical projection mapping. We denote by $\text{Homeo}(\mathbb{T}^2)$ the class of homeomorphisms of the torus and by $\text{Homeo}_0(\mathbb{T}^2) \subset \text{Homeo}(\mathbb{T}^2)$ the subclass of homeomorphisms isotopic to the identity. Given an element $f \in \text{Homeo}_0(\mathbb{T}^2)$, we denote by $F: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ a lift to the cover. Any two different lifts F, \widehat{F} of f differ by an integer translation, that is, $F(z) = \widehat{F}(z) + (n, m)$ where $(n, m) \in \mathbb{Z}^2$. Given a lift F and $\tilde{z} \in \mathbb{R}^2$, define

$$\rho(F) = \bigcap_{m=1}^{\infty} \text{Cl} \left(\bigcup_{n=m}^{\infty} \left\{ \frac{F^n(\tilde{z}) - \tilde{z}}{n} \right\} \mid \tilde{z} \in \mathbb{R}^2 \right) \subset \mathbb{R}^2.$$

The *rotation set* of f is defined as $\rho(f) = \rho(F) \bmod \mathbb{Z}^2$. In words, the rotation set collects all limit points, modulo \mathbb{Z}^2 , of sequences of the form

$$\frac{F^{n_k}(\tilde{z}) - \tilde{z}}{n_k},$$

where $n_k \rightarrow \infty$ for $k \rightarrow \infty$ and $\tilde{z} \in \mathbb{R}^2$. The rotation set of a homeomorphism $f \in \text{Homeo}_0(\mathbb{T}^2)$ is in general no longer a single point, but a convex connected closed set (see [MZ]). We define

$$(1.2) \quad \text{Homeo}_*(\mathbb{T}^2) \subset \text{Homeo}_0(\mathbb{T}^2)$$

to be the class of homeomorphisms isotopic to the identity for which the rotation set $\rho(f) = (\alpha, \beta) \bmod \mathbb{Z}^2$ is a single point where the numbers $1, \alpha, \beta$ are rationally independent. These homeomorphisms are said to be *non-resonant torus homeomorphisms*.

Generalizations of Poincaré's Theorem have recently seen much progress through the work of F. Béguin, S. Crovisier, T. Jäger, G. Keller, F. Le Roux and J. Stark [Be, JK, JS], where one considers an analogous class of torus homeomorphisms, namely *quasiperiodically forced circle homeomorphisms*, which are torus homeomorphisms of the form $(x, \theta) \mapsto (x + \alpha, g_\theta(x)) \bmod \mathbb{Z}^2$, with $(x, \theta) \in \mathbb{T}^2$, $g_\theta: \mathbb{T}^1 \rightarrow \mathbb{T}^1$ circle homeomorphisms and $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. In [Be, JK, JS], the appropriate analogues of the results of Poincaré and Denjoy in the class of quasiperiodically forced circle homeomorphisms are developed. Analogues of Poincaré's Theorem in the setting of conservative torus homeomorphisms are developed by T. Jäger in [J]. In the setting of non-resonant homeomorphisms of the two-torus, introducing differentiability or other geometrical conditions on the homeomorphisms gives rise to analogues of Denjoy's Theorem; see [No, NV, NS] for results in this direction.

To state our results, we need the following definitions. A connected set $X \subset \mathbb{T}^2$ is said to be *(un)bounded* according to whether a *lift* $\tilde{X} \subset \mathbb{R}^2$ is (un)bounded as a subset of \mathbb{R}^2 , where a *lift* \tilde{X} of X is a connected component of $p^{-1}(X)$. A compact and connected set is called a *continuum*. A continuum in \mathbb{T}^2 is called *non-separating* if its complement in \mathbb{T}^2 is connected. Given a bounded continuum $\mathcal{C} \subset \mathbb{T}^2$, we define $\text{Fill}(\mathcal{C}) \subset \mathbb{T}^2$, the *filled continuum*, to be the smallest (with respect to inclusion) non-separating bounded continuum containing \mathcal{C} . A bounded non-separating continuum in \mathbb{T}^2 is called *acyclic*.

DEFINITION 1 (Extension of a Cantor set). Let \mathcal{M} be a minimal set for $f \in \text{Homeo}_*(\mathbb{T}^2)$, and $\{A_i\}_{i \in I}$ be the collection of connected components of \mathcal{M} . If $\text{Fill}(A_i)$ is acyclic for every $i \in I$, $\text{Fill}(A_i) \cap \text{Fill}(A_j) = \emptyset$ if $i \neq j$ and there exists a continuous $\phi: \mathbb{T}^2 \rightarrow \mathbb{T}^2$, homotopic to the identity, and an $\hat{f} \in \text{Homeo}_*(\mathbb{T}^2)$, such that

- (i) $\phi \circ f = \hat{f} \circ \phi$, i.e. f is semi-conjugate to \hat{f} ,
- (ii) $\hat{\mathcal{M}} := \phi(\hat{\mathcal{Q}}) \subset \mathbb{T}^2$ is a Cantor minimal set for \hat{f} ,

where $\hat{\mathcal{Q}} = \bigcup_{i \in I} \text{Fill}(A_i)$, then we say \mathcal{M} is an *extension of a Cantor set*.

Put in words, \mathcal{M} is an extension of a Cantor set if the semi-conjugacy ϕ between f and \hat{f} sends the collection of filled in components of \mathcal{M} to points in a one-to-one fashion, and the corresponding totally disconnected set $\hat{\mathcal{M}}$ is a Cantor minimal set of the factor \hat{f} . An extension of a Cantor set is called *non-trivial* if there exist components of \mathcal{M} that are not singletons.

Further, we define the following. A *disk* $D \subset \mathbb{T}^2$ in the torus is an injection by a homeomorphism of the open unit disk $\mathbb{D}^2 \subset \mathbb{R}^2$ into the torus. An *annulus* $A \subset \mathbb{T}^2$ is an injection by a homeomorphism of the open annulus $\mathbb{S}^1 \times (0, 1)$ into the torus; an annulus A is said to be *essential* if the inclusion $A \hookrightarrow \mathbb{T}^2$ induces an injection of $\pi_1(A)$ into $\pi_1(\mathbb{T}^2)$.

Let us now state our main results. Our first result gives a classification of the possible minimal sets of homeomorphisms in our class $\text{Homeo}_*(\mathbb{T}^2)$.

THEOREM A (Classification of minimal sets). *Let $f \in \text{Homeo}_*(\mathbb{T}^2)$ and let \mathcal{M} be a minimal set for f . Let $\{\Sigma_k\}_{k \in \mathbb{Z}}$ be the connected components of the complement of \mathcal{M} in \mathbb{T}^2 . If $\mathcal{M} \neq \mathbb{T}^2$, then either*

- (I) $\{\Sigma_k\}$ is a collection of bounded and unbounded disks,
- (II) $\{\Sigma_k\}$ is a collection of essential annuli and bounded disks, or
- (III) \mathcal{M} is an extension of a Cantor set.

We prove this result in Section 2. The proof of Theorem A implies

COROLLARY 1 (Structure of orbits; types I and II). *Let $f \in \text{Homeo}_*(\mathbb{T}^2)$ with a minimal set \mathcal{M} of type I or II. Then*

$$(1.3) \quad \mathcal{M} = \Omega(f) = \omega(z) = \alpha(z)$$

for all $z \in \mathbb{T}^2$. In particular, \mathcal{M} is unique.

In [Be, Thm 1.2], F. Béguin, S. Crovisier, T. Jäger and F. Le Roux construct a counterexample to the conclusion of Corollary 1 in the case where \mathcal{M} is of type III in the setting of quasiperiodically forced circle homeomorphisms. Formulated in our terminology, this result reads

COUNTEREXAMPLE 2 (Structure of orbits; type III [Be]). *There exist homeomorphisms $f \in \text{Homeo}_*(\mathbb{T}^2)$ which have a unique Cantor minimal set \mathcal{M} (and are thus of type III), but are transitive.*

In other words, $\mathcal{M} \neq \mathbb{T}^2$ is the unique Cantor minimal set, but $\Omega(f) = \mathbb{T}^2$. Uniqueness of minimal sets of type III homeomorphisms has not yet been settled: see Question 1 in the final section of this paper. Further, we have

COROLLARY 3 (Connected minimal sets). *Let $f \in \text{Homeo}_*(\mathbb{T}^2)$. If $\mathcal{M} \neq \mathbb{T}^2$, then \mathcal{M} is connected if and only if \mathcal{M} is of type I.*

To state the third corollary, we need the following. Recall that a *null-sequence* is a sequence of positive real numbers for which for every given $\epsilon > 0$, only finitely many elements of the sequence are greater than ϵ .

DEFINITION 2 (Quasi-Sierpiński set). A *quasi-Sierpiński set* is a continuum $S = \mathbb{T}^2 \setminus \bigcup_{k \in \mathbb{Z}} D_k$ with $\{D_k\}_{k \in \mathbb{Z}}$ a family of disks such that $\bigcup_{k \in \mathbb{Z}} D_k$ is dense in \mathbb{T}^2 , and

- (a) D_k is the interior of a closed embedded disk, for every $k \in \mathbb{Z}$,
- (b) $\text{Cl}(D_k) \cap \text{Cl}(D_{k'})$ is at most a single point if $k \neq k'$,
- (c) $\text{diam}(D_k)$, $k \in \mathbb{Z}$, is a null-sequence.

If property (b) above is replaced by the condition that $\text{Cl}(D_k) \cap \text{Cl}(D_{k'}) = \emptyset$ if $k \neq k'$, then we refer to S as a *Sierpiński set*.

A closed subset of a topological space is *locally connected* if every of its points has arbitrarily small connected neighbourhoods. Requiring a minimal set to be locally connected reduces the list of Theorem A to one type of non-trivial minimal set.

COROLLARY 4 (Locally connected minimal sets). *Let $f \in \text{Homeo}_*(\mathbb{T}^2)$ and suppose that the minimal set \mathcal{M} of f is locally connected. Then either $\mathcal{M} = \mathbb{T}^2$ or \mathcal{M} is a quasi-Sierpiński set.*

This result was (essentially) proved by A. Biś, H. Nakayama and P. Walczak in [B1]. We show how this result, for our class of homeomorphisms, can be recovered from Theorem A above, and our line of approach is different. Rather than assuming the minimal set is locally connected from the start as in [B1], here we show that most of the minimal sets of Theorem A are not locally connected, ultimately arriving at the only possible locally connected minimal set, a quasi-Sierpiński set.

Our second result says that the classification of Theorem A is sharp in the following sense.

THEOREM B (Existence of minimal sets). *Every type of minimal set Theorem A allows is realized by homeomorphisms in $\text{Homeo}_*(\mathbb{T}^2)$.*

This result is proved by a number of examples constructed in Section 3 below. Let us briefly discuss these. It is well-known there exist homeomorphisms $f \in \text{Homeo}_*(\mathbb{T}^2)$ for which the minimal set is a Sierpiński set. The first example given, a locally connected quasi-Sierpiński minimal set that is not a Sierpiński set, is known [B1]. The examples constructed below are a minimal set for which the complement is a single unbounded disk (type I), a minimal set for which the complement components are essential annuli and bounded disks (type II) and examples of rather exotic non-trivial extensions of Cantor sets (type III), where the minimal sets constructed are homeomorphic to Cantor dust interspersed with various continua.

2. Classification of minimal sets. In what follows, let $f \in \text{Homeo}_*(\mathbb{T}^2)$ with \mathcal{M} a minimal set of f . This section is devoted to the proof of Theorem A.

2.1. Preliminary results. We first set notation and recollect several basic results to be used in the remainder of this paper.

Let us first recall the following. A vector $(\alpha, \beta) \in \mathbb{R}^2$ is *irrational* if the numbers $1, \alpha, \beta$ are rationally independent, that is, the only solution over the integers of

$$(2.1) \quad N_1 + N_2\alpha + N_3\beta = 0$$

is $N_1 = N_2 = N_3 = 0$. The translation $\tau: \mathbb{T}^2 \rightarrow \mathbb{T}^2$ corresponding to (α, β) ,

$$(2.2) \quad \tau: (x, y) \mapsto (x + \alpha, y + \beta) \bmod \mathbb{Z}^2,$$

is minimal if and only if the vector (α, β) is irrational. The class of homeomorphisms of the torus \mathbb{T}^2 isotopic to the identity with rotation set consisting of a single irrational vector will be denoted by $\text{Homeo}_*(\mathbb{T}^2)$. It is easy to see that a homeomorphism $f \in \text{Homeo}_*(\mathbb{T}^2)$ has no periodic points.

LEMMA 5. *Let $f \in \text{Homeo}_*(\mathbb{T}^2)$. If $X \subset \mathbb{T}^2$ is a bounded connected set, then $f^n(X) \neq X$ for all $n \neq 0$.*

Proof. If $X \subset \mathbb{T}^2$ is bounded and $f^N(X) = X$ for some $N \neq 0$, we can take a lift F of f and a lift \tilde{X} of X such that $F^N(\tilde{X}) = \tilde{X}$. Let $\tilde{z} \in \tilde{X}$. As \tilde{X} is bounded, we must have $\rho(F, \tilde{z}) = (0, 0)$ and thus $\rho(f, z) = (0, 0) \bmod \mathbb{Z}^2$, where $z = p(\tilde{z})$, contrary to our assumption on the rotation set. ■

In other words, if $X \subset \mathbb{T}^2$ is a connected and f -invariant set, then X is necessarily unbounded. In what follows, let

$$(2.3) \quad T_{p,q}: \mathbb{R}^2 \rightarrow \mathbb{R}^2, \quad T_{p,q}(x, y) = (x + p, y + q),$$

where $(p, q) \in \mathbb{Z}^2$.

LEMMA 6. *Let $f \in \text{Homeo}_*(\mathbb{T}^2)$ and F a lift of f . Let $D \subset \mathbb{R}^2$ be a closed topological disk. Then there exist no $N \neq 0$ and $(p, q) \in \mathbb{Z}^2$ such that*

$$F^N(D) \subseteq T_{p,q}(D) \quad \text{or} \quad T_{p,q}(D) \subseteq F^N(D).$$

Proof. Suppose that there exist $N \neq 0$ and $(p, q) \in \mathbb{Z}^2$ such that $F^N(D) \subseteq T_{p,q}(D)$. Choosing a different lift \hat{F} if necessary, we may assume that $\hat{F}^N(D) \subseteq D$. By the Brouwer Fixed Point Theorem, \hat{F}^N has a fixed point on D , and thus f has a periodic point, contrary to our assumptions. The case where $T_{p,q}(D) \subseteq F^N(D)$ follows by considering the inverse F^{-1} . ■

Next, we turn to the topology of domains in the torus. In the subsequent proof, the various topological types of domains on the torus play an important role. In what follows, a *domain* is an open connected set. Let $\gamma \subset \mathbb{T}^2$ be an essential simple closed curve. We say the curve γ has *homotopy type* (p, q) if γ lifts to a curve $\tilde{\gamma} \subset \mathbb{R}^2$ such that, up to a suitable translation, $\tilde{\gamma}$ connects the lattice points $(0, 0) \in \mathbb{Z}^2$ and $(p, q) \in \mathbb{Z}^2$ with p and q coprime. Then $\tilde{\gamma}$ is periodic in the sense that

$$(2.4) \quad \tilde{\gamma} = \bigcup_{n \in \mathbb{Z}} T_{p,q}^n(\eta),$$

where $\eta \subset \tilde{\gamma}$ is the arc connecting $(0, 0)$ and (p, q) . Let $D \subset \mathbb{T}^2$ be a domain. The inclusion $D \hookrightarrow \mathbb{T}^2$ naturally induces an injection of $\pi_1(D)$ into $\pi_1(\mathbb{T}^2)$. This gives rise to the following.

DEFINITION 3 (Types of domains). A domain $D \subset \mathbb{T}^2$ is said to be *trivial*, *essential* or *doubly essential* according to whether the inclusion of $\pi_1(D)$ into $\pi_1(\mathbb{T}^2)$ is isomorphic to 0 , \mathbb{Z} or \mathbb{Z}^2 respectively.

DEFINITION 4. An essential domain $D \subset \mathbb{T}^2$ has *characteristic* (p, q) if an essential closed curve $\gamma \subset D$ has homotopy type (p, q) .

Note that Definition 4 is correct, in the sense that any other essential simple closed curve in D must have the same homotopy type (as otherwise the domain D would be doubly essential). The following lemma relates the notion of a trivial and essential domain to that of a disk and essential annulus in the torus respectively.

LEMMA 7. A domain $D \subset \mathbb{T}^2$ such that \tilde{D} is simply connected is trivial (resp. essential) if and only if it is a disk (resp. essential annulus) in the torus.

Proof. As the “if” part is evident, we need only prove the “only if” part. First suppose that D is trivial and let \tilde{D} be a lift of D . By the Riemann mapping theorem, there exists a biholomorphism $\phi: \mathbb{D}^2 \rightarrow \tilde{D}$, where $\mathbb{D}^2 \subset \mathbb{R}^2$ is the unit (Poincaré) disk. As D is trivial, no two points in \tilde{D} are identified under the action of the translation group $\mathbb{Z}^2 \subset \mathbb{R}^2$ and thus $p|_{\tilde{D}}$ is injective, where $p: \mathbb{R}^2 \rightarrow \mathbb{T}^2$ is the projection mapping. Therefore, the map $p \circ \phi: \mathbb{D}^2 \rightarrow \mathbb{T}^2$ satisfies $p \circ \phi(\mathbb{D}^2) = D$, and thus $D \subset \mathbb{T}^2$ is a disk.

Next, suppose that D is essential. Then there exists a unique pair $(p, q) \in \mathbb{Z}^2$ with $\gcd(p, q) = 1$ such that the translation $T_{p,q}$ leaves \tilde{D} invariant, i.e. $T_{p,q}(\tilde{D}) = \tilde{D}$. Further, as \tilde{D} is simply connected, again by the Riemann mapping theorem, there exists a biholomorphism $\phi: \mathbb{D}^2 \rightarrow \tilde{D}$. As $T_{p,q}: \tilde{D} \rightarrow \tilde{D}$ is a biholomorphism, the map

$$\mu: \mathbb{D}^2 \rightarrow \mathbb{D}^2, \quad \mu = \phi^{-1} \circ T_{p,q} \circ \phi,$$

is itself a biholomorphism and thus a Möbius transformation. Moreover, as $T_{p,q}$ does not fix any point in \tilde{D} , μ does not fix any point in \mathbb{D}^2 and thus μ is either a hyperbolic or a parabolic Möbius transformation.

As is well known, $\mathbb{D}^2 \setminus \langle \mu \rangle$ is conformally equivalent to an annulus if μ is hyperbolic, and conformally equivalent to the once punctured disk $\mathbb{D}^2 \setminus \{0\}$ if μ is parabolic (see e.g. [Ab]). Thus both are topologically equivalent to an annulus $\mathbb{S}^1 \times (0, 1)$, therefore so is $\tilde{D} \setminus \langle T_{p,q} \rangle$. As \tilde{D} admits no translations other than (multiples of) $T_{p,q}$ that leave \tilde{D} invariant, the continuous projection p restricted to $\tilde{D} \setminus \langle T_{p,q} \rangle$ into the torus \mathbb{T}^2 is an injection and thus D is indeed topologically equivalent to the annulus $\mathbb{S}^1 \times (0, 1)$. ■

We recall some standard results from decomposition theory, to be used in the proof of Theorem A. In the following statements, let M be a closed surface.

DEFINITION 5. A collection $\mathcal{U} = \{\mathcal{U}_i\}_{i \in I}$ of continua in a surface M is said to be *upper semicontinuous* if the following holds:

- (1) If $\mathcal{U}_i, \mathcal{U}_j \in \mathcal{U}$, then $\mathcal{U}_i \cap \mathcal{U}_j = \emptyset$.
- (2) Each \mathcal{U}_i is non-separating.
- (3) $M = \bigcup_{i \in I} \mathcal{U}_i$.
- (4) If \mathcal{U}_{i_k} with $k \in \mathbb{N}$ is a sequence that has the Hausdorff limit \mathcal{C} , then there exists $\mathcal{U}_j \in \mathcal{U}$ such that $\mathcal{C} \subset \mathcal{U}_j$.

In a compact metric space, every Hausdorff limit of continua is again a continuum. We have the following classical result (see for example [W] or [Da]).

MOORE'S THEOREM. *Let \mathcal{U} be an upper semicontinuous decomposition of M so that every element of \mathcal{U} is acyclic. Then there is a continuous map $\phi: M \rightarrow M$ that is homotopic to the identity and such that for every $z \in M$, $\phi^{-1}(z) = \mathcal{U}_i$ for some $\mathcal{U}_i \in \mathcal{U}$.*

The following is a corollary of Moore's Theorem (see also [Br]). We give a proof here for the convenience of the reader.

LEMMA 8. *Given an upper semicontinuous decomposition \mathcal{U} of \mathbb{T}^2 into acyclic elements and an $f \in \text{Homeo}_0(\mathbb{T}^2)$ that sends elements of \mathcal{U} onto elements of \mathcal{U} . Then the map $\hat{f}: \mathbb{T}^2 \rightarrow \mathbb{T}^2$ defined by $\phi \circ f(z) = \hat{f} \circ \phi(z)$, for every $z \in \mathbb{T}^2$, is an element of $\text{Homeo}_0(\mathbb{T}^2)$. In other words, f is semiconjugate to \hat{f} through ϕ .*

Proof. The map $\phi: \mathbb{T}^2 \rightarrow \mathbb{T}^2$ given by Moore's Theorem, sending elements of \mathcal{U} to points, is continuous. The map \hat{f} , where $\phi \circ f(z) = \hat{f} \circ \phi(z)$, is one-to-one as f sends elements of \mathcal{U} onto elements of \mathcal{U} . To prove that \hat{f} is continuous, we observe that, whenever $\mathcal{C} \subset \mathbb{T}^2$ is closed, so are $\mathcal{C}' = (\phi \circ f)^{-1}(\mathcal{C})$ and $\mathcal{C}'' = \phi(\mathcal{C}')$, since ϕ and f are continuous and \mathbb{T}^2 is compact. Since $\mathcal{C}'' = \hat{f}^{-1}(\mathcal{C})$, this shows that \hat{f} is continuous and hence a homeomorphism, again by compactness of \mathbb{T}^2 . ■

2.2. Topology of the domains Σ_k . In what follows, let $\Sigma = \Sigma_k$ be any element of $\{\Sigma_k\}$, the collection of connected components of the complement of \mathcal{M} , a minimal set of an element $f \in \text{Homeo}_*(\mathbb{T}^2)$. Further, let $\tilde{d}(\cdot, \cdot)$ be the standard Euclidean metric on \mathbb{R}^2 , and $d(\cdot, \cdot)$ the (induced) metric on \mathbb{T}^2 .

LEMMA 9. *If $f^n(\Sigma) \cap \Sigma = \emptyset$ for all $n \neq 0$, then Σ is a disk or an essential annulus.*

Proof. First, suppose that Σ is doubly essential and let $\gamma, \gamma' \subset \Sigma$ be two non-homotopic essential simple closed curves. As f is homotopic to the identity, the homotopy classes of $f(\gamma)$ and γ are equal. As any two non-homotopic simple closed curves on the torus intersect, it follows that

$f(\gamma) \cap \gamma' \neq \emptyset$. Therefore $f(\Sigma) \cap \Sigma \neq \emptyset$ and thus Σ has to be either trivial or essential.

Thus let Σ be a trivial or essential domain and let $\tilde{\Sigma}$ be a lift of Σ . In order to show that Σ is a disk or an essential annulus respectively, by Lemma 7 it suffices to show that $\tilde{\Sigma}$ is simply connected. Suppose it is not. Then there exists a simple closed curve $\gamma \subset \tilde{\Sigma}$ such that the open disk D_γ with boundary curve γ has the property that

$$(2.5) \quad D_\gamma \cap p^{-1}(\mathcal{M}) \neq \emptyset.$$

Let F be a lift of f . As every point in \mathcal{M} is recurrent, there exists a subsequence n_k such that $f^{n_k}(z) \rightarrow z$ for $k \rightarrow \infty$. Therefore, by passing to a subsequence if necessary, we may assume that for all $k \geq 1$,

$$(2.6) \quad F^{n_k}(D_\gamma) \cap T_{p_k, q_k}(D_\gamma) \neq \emptyset$$

for certain $(p_k, q_k) \in \mathbb{Z}^2$. Given (2.6), there are two possibilities. For a given $k \geq 1$, we have either

- (a) $F^{n_k}(D_\gamma) \subset T_{p_k, q_k}(D_\gamma)$ or $T_{p_k, q_k}(D_\gamma) \subset F^{n_k}(D_\gamma)$, or
- (b) $F^{n_k}(\gamma) \cap T_{p_k, q_k}(\gamma) \neq \emptyset$.

Case (a) can be excluded as, by Lemma 6, this yields periodic points for f . Furthermore, case (b) is ruled out as this implies that

$$(2.7) \quad F^{n_k}(\tilde{\Sigma}) \cap T_{p_k, q_k}(\tilde{\Sigma}) \neq \emptyset,$$

implying that $f^{n_k}(\Sigma) \cap \Sigma \neq \emptyset$, contrary to our assumption. Therefore, $\tilde{\Sigma}$ must be simply connected indeed. ■

In what follows, a fundamental domain of \mathbb{T}^2 is defined as the standard square $[0, 1] \times [0, 1] \subset \mathbb{R}^2$ and the integer translates thereof.

LEMMA 10. *If Σ is trivial, then Σ is a disk. Moreover, if Σ is bounded, then $f^n(\Sigma) \cap \Sigma = \emptyset$ for all $n \neq 0$.*

Proof. Because \mathcal{M} is invariant, we have either (a) $f^n(\Sigma) \cap \Sigma = \emptyset$ for all $n \neq 0$, or (b) $f^N(\Sigma) = \Sigma$ for some $N \neq 0$. In case (a), Σ is a disk by Lemma 9. In case (b), Lemma 5 implies that Σ is necessarily unbounded.

Thus we need to show that for unbounded Σ , the set $\tilde{\Sigma}$ is simply connected if $f^N(\Sigma) = \Sigma$ for some $N \neq 0$. We may as well assume that $N = 1$. Take a lift F of f such that $F(\tilde{\Sigma}) = \tilde{\Sigma}$. If $\tilde{\Sigma}$ is not simply connected, then there exists a simple closed curve $\gamma \subset \tilde{\Sigma}$ such that the disk $D_\gamma \subset \mathbb{R}^2$ with boundary curve γ has the property that $D_\gamma \cap p^{-1}(\mathcal{M}) \neq \emptyset$. Similarly to Lemma 9, there exists a subsequence n_k such that, for $k \geq 1$,

$$(2.8) \quad F^{n_k}(D_\gamma) \cap T_{p_k, q_k}(D_\gamma) \neq \emptyset$$

for certain $(p_k, q_k) \in \mathbb{Z}^2$. As

$$\rho(f) = \rho(f, z) = (\alpha, \beta) \bmod \mathbb{Z}^2 \neq (0, 0) \bmod \mathbb{Z}^2$$

for every $z \in \mathbb{T}^2$, it follows that $\tilde{d}(F^{n_k}(\tilde{z}), \tilde{z}) \rightarrow \infty$ for $k \rightarrow \infty$. In particular, passing to a subsequence once again, we may assume that $F^{n_k}(\tilde{z})$ is contained in a fundamental domain different from that of \tilde{z} , for all $k \geq 1$. Condition (2.8) gives again the two possibilities (a) and (b) of Lemma 9, and we can exclude case (a) as this would yield periodic points for f . Therefore, for all $k \geq 1$, (2.8) reduces to the condition that

$$(2.9) \quad F^{n_k}(\gamma) \cap T_{p_k, q_k}(\gamma) \neq \emptyset$$

for some $(p_k, q_k) \in \mathbb{Z}^2$. Thus for every $k \geq 1$,

- (i) $F^{n_k}(\gamma) \cap T_{p_k, q_k}(\gamma) \neq \emptyset$,
- (ii) $F^{n_k}(\gamma)$ lies in a fundamental domain different from that of γ ,
- (iii) $F^{n_k}(\gamma) \subset \tilde{\Sigma}$.

Condition (iii) follows simply from the fact that $\tilde{\Sigma}$ is F -invariant and $\gamma \subset \tilde{\Sigma}$. Fix any $k \geq 1$, choose $\tilde{w} \in F^{n_k}(\gamma) \cap T_{p_k, q_k}(\gamma)$ and let $\tilde{w}' = T_{p_k, q_k}^{-1}(\tilde{w}) \in \gamma$. As $\tilde{\Sigma}$ is a domain, it is path-connected and thus there exists an arc $\eta \subset \tilde{\Sigma}$ connecting \tilde{w} and \tilde{w}' . As these endpoints lie in different fundamental domains of \mathbb{T}^2 , η projects under p to an essential closed curve, as its endpoints differ by an integer translate. However, this contradicts our assumption that Σ is trivial (and thus does not contain any essential simple closed curves). This contradiction shows that $\tilde{\Sigma}$ must be simply connected, and this completes the proof. ■

Using the irrationality of the rotation vector (α, β) , we now deduce the following.

LEMMA 11. *If Σ is essential, then Σ is an essential annulus and $f^n(\Sigma) \cap \Sigma = \emptyset$ for all $n \neq 0$.*

Proof. It suffices to show that if Σ is essential, then $f^n(\Sigma) \cap \Sigma \neq \emptyset$ for all $n \neq 0$. It then follows from Lemma 9 that Σ is an essential annulus. Assume that Σ has characteristic (p, q) . We will show that, by our choice of translation number, f^N cannot fix an essential domain, for any $N \neq 0$. To derive a contradiction, suppose there exists an $N \neq 0$ such that $f^N(\Sigma) = \Sigma$. We may assume that $N = 1$, i.e. $f(\Sigma) = \Sigma$. Let $\gamma \subset \Sigma$ be an essential simple closed curve and let $\tilde{\gamma}$ be a lift of γ . We may assume that $\tilde{\gamma}$ passes through $(0, 0) \in \mathbb{R}^2$, and by definition it also passes through $(p, q) \in \mathbb{R}^2$, where $p, q \in \mathbb{Z}$ and $\gcd(p, q) = 1$. The arc $\eta \subset \tilde{\gamma}$ connecting $(0, 0)$ and (p, q) is compact and therefore bounded. Therefore, the curve $\tilde{\gamma}$ divides \mathbb{R}^2 into two unbounded connected components \mathcal{H}_l and \mathcal{H}_r , homeomorphic to half-planes, so that $\mathbb{R}^2 \setminus \tilde{\gamma} = \mathcal{H}_l \cup \mathcal{H}_r$ and $\mathcal{H}_l \cap \mathcal{H}_r = \emptyset$. Further, as γ is a simple closed curve, any integer translate $\tilde{\gamma}' = T_{p', q'}(\tilde{\gamma})$, where (p', q') is not an integer multiple of (p, q) , has the property that $\tilde{\gamma}' \cap \tilde{\gamma} = \emptyset$. This follows from the fact that Σ is essential, but not doubly essential; if $\tilde{\gamma}' \neq \tilde{\gamma}$, then there exists an arc

$\zeta \subset \tilde{\Sigma}$ connecting $(0, 0)$ to a point $(p', q') = T_{p', q'}(0, 0)$ which is not a multiple of (p, q) ; the projection of ζ under p would then lie in a homotopy class other than that of γ , implying that Σ would be doubly essential, contrary to our assumption. Therefore, we can choose integer translates $\tilde{\gamma}_l$ and $\tilde{\gamma}_r$ of $\tilde{\gamma}$ contained in \mathcal{H}_l and \mathcal{H}_r respectively and we can define $\Gamma \subset \mathbb{R}^2$ to be the infinite strip bounded by $\tilde{\gamma}_l \cup \tilde{\gamma}_r$.

We claim that $\tilde{\Sigma} \subset \Gamma$. Indeed, if $\tilde{\Sigma} \cap \Gamma^c \neq \emptyset$, where $\Gamma^c := \mathbb{R}^2 \setminus \Gamma$, then $\tilde{\Sigma} \cap (\tilde{\gamma}_l \cup \tilde{\gamma}_r) \neq \emptyset$. Suppose that $\tilde{\Sigma} \cap \tilde{\gamma}_l \neq \emptyset$. The case where $\tilde{\Sigma} \cap \tilde{\gamma}_r \neq \emptyset$ (or both) is similar. Let $\tilde{z}' \in \tilde{\Sigma} \cap \tilde{\gamma}_l$. Because $\tilde{\gamma} \subset \tilde{\Sigma}$ and $\tilde{\Sigma}$ is path-connected, there exists an arc $\zeta \subset \tilde{\Sigma}$ connecting \tilde{z}' to a point $\tilde{z} \in \tilde{\gamma}$ such that $z = p(\tilde{z}') = p(\tilde{z})$. As the arc ζ connects two lattice points whose projection lies in a homotopy class different from γ , this would imply that Σ is doubly essential, contrary to our assumption. Thus $\tilde{\Sigma} \subset \Gamma$.

To finish the proof, choose a lift F of f such that $F(\tilde{\Sigma}) = \tilde{\Sigma}$. As Γ is invariant under the translation $T_{p, q}$, and $\tilde{\Sigma} \subset \Gamma$, we must have

$$(2.10) \quad \rho(F, \tilde{z}) = \lim_{n \rightarrow \infty} \frac{F^n(\tilde{z}) - \tilde{z}}{n} = (a, b), \quad \text{where} \quad \frac{b}{a} = \frac{q}{p},$$

for every $\tilde{z} \in \tilde{\Sigma}$. As $(a, b) = (\alpha + s, \beta + t)$ for certain $s, t \in \mathbb{Z}$ and $\alpha, \beta \notin \mathbb{Q}$, we see that $a = \alpha + s \neq 0$ and $b = \beta + t \neq 0$, and we obtain

$$(2.11) \quad \frac{\alpha + s}{\beta + t} = \frac{a}{b} = \frac{p}{q}.$$

Rewriting (2.11) gives

$$q\alpha - p\beta - (qs - pt) = 0.$$

As $p, q, qs - pt \in \mathbb{Z}$, with $(p, q) \neq (0, 0)$, this gives a non-trivial solution of (2.1), which contradicts the irrationality of (α, β) . ■

The following lemma shows that not all combinations of types of domains can occur.

LEMMA 12. *The collection $\{\Sigma_k\}$ cannot contain both an essential annulus and an unbounded disk.*

Proof. Suppose, to derive a contradiction, that $\{\Sigma_k\}$ contains both an essential annulus and an unbounded disk. By Lemma 11, $\{\Sigma_k\}$ contains infinitely many essential annuli; denote these by $\{\Sigma_k^a\}$. Note further that, as all these annuli are disjoint, they all have the same characteristic, which we assume to be $(0, 1)$; the proof in the case of any other characteristic is entirely similar. Denote by Σ an element of $\{\Sigma_k\}$ homeomorphic to an unbounded disk.

Let $\tilde{\Sigma} \subset \mathbb{R}^2$ and $\tilde{\Sigma}_k^a \subset \mathbb{R}^2$ be lifts of Σ and Σ_k^a respectively. Take $\tilde{z} \in \tilde{\Sigma}$ and let $\ell_{\tilde{z}}$ be the horizontal (Euclidean) line through \tilde{z} . Let $I \subset \ell_{\tilde{z}} \cap \tilde{\Sigma}$ be

the connected component containing \tilde{z} . As the line $\ell_{\tilde{z}}$ is horizontal and the characteristic of the essential annuli Σ_k^a is $(0, 1)$, the length of the interval I is finite. Indeed, as the characteristic of the annuli is $(0, 1)$, curves in these annuli lift to curves in \mathbb{R}^2 which are homotopic, and therefore within uniformly bounded distance of genuinely vertical Euclidean lines. As the length of a horizontal line segment disjoint from the genuinely Euclidean vertical curves is at most 1 by periodicity, the length of I has to be finite, as a homotopy moves points a uniformly bounded distance. Let $\tilde{z}^-, \tilde{z}^+ \in \partial I$ be the left and right endpoints of the interval I respectively. As $\tilde{z}^-, \tilde{z}^+ \in \partial \tilde{\Sigma}$,

$$(2.12) \quad z^\pm := p(\tilde{z}^\pm) \in \partial \Sigma \subset \mathcal{M}.$$

Define $I^{\pm 1} = T_{0,1}^{\pm 1}(I)$. Let $\gamma_k \subset \Sigma_k^a$ be a simple closed curve and $\tilde{\gamma}_k$ a lift of γ_k . Certainly, $\tilde{\gamma}_k \cap T_{0,1}^n(I)$ for all $n \in \mathbb{Z}$. As every orbit in \mathcal{M} is dense, we can take a point $z' \in \partial \Sigma_k^a$, for some $k \in \mathbb{Z}$, and find subsequences k_t and k'_t such that $f^{k_t}(z') \rightarrow z^+$ and $f^{k'_t}(z') \rightarrow z^-$ for $t \rightarrow \infty$. After appropriately labelling the annuli if necessary, we find points $z_{k_t} \in \gamma_{k_t} \subset \Sigma_{k_t}^a$ and $z_{k'_t} \in \gamma_{k'_t} \subset \Sigma_{k'_t}^a$ such that $z_{k_t} \rightarrow z^+$ and $z_{k'_t} \rightarrow z^-$ for $t \rightarrow \infty$. Thus we can find lifts $\tilde{\gamma}_{k_t}$ and $\tilde{\gamma}_{k'_t}$ and points $\tilde{z}_{k_t} \in \tilde{\gamma}_{k_t}$ and $\tilde{z}_{k'_t} \in \tilde{\gamma}_{k'_t}$ such that $\tilde{z}_{k_t} \rightarrow \tilde{z}^+$ and $\tilde{z}_{k'_t} \rightarrow \tilde{z}^-$ for $t \rightarrow \infty$. As the curves $\tilde{\gamma}_{k_t}, \tilde{\gamma}_{k'_t}$ are periodic (in the sense of (2.4)), they define an infinite strip Γ_t that contains the line segments $T_{0,1}^n(I)$ for all $n \in \mathbb{Z}$. Further, after a relabelling if necessary, we may assume that $\Gamma_{t'} \subset \Gamma_t$ if $t' > t$. We now see that $\tilde{\Sigma} \subset \Gamma_t$ for every $t \geq 1$ and $I \subset \tilde{\Sigma}$.

By periodicity, $T_{0,1}^{\pm 1}(\tilde{z}_{k_t})$ and $T_{0,1}^{\pm 1}(\tilde{z}_{k'_t})$ limit to $T^{\pm 1}(\tilde{z}^+)$ and $T^{\pm 1}(\tilde{z}^-)$ respectively. Therefore, as $\tilde{\Sigma}$ is unbounded, we must have either $\tilde{\Sigma} \cap T_{0,1}(I) \neq \emptyset$ or $\tilde{\Sigma} \cap T_{0,1}^{-1}(I) \neq \emptyset$, or both. As $\tilde{\Sigma}$ is path-connected, we can find an arc $\eta \subset \tilde{\Sigma}$ that projects under p to an essential closed curve $\gamma \subset \Sigma$, which is the desired contradiction: Σ is a disk and thus does not contain any essential closed curves. ■

We recall that a Cantor set can be characterized topologically as being compact, perfect and totally disconnected.

LEMMA 13. *If Σ is doubly essential, then \mathcal{M} is an extension of a Cantor set.*

Proof. Denote by $\{A_i\}_{i \in I}$ the collection of connected components of \mathcal{M} . Because Σ is doubly essential, $f(\Sigma) \cap \Sigma \neq \emptyset$, hence $f(\Sigma) = \Sigma$. As $f(\Sigma) = \Sigma$, we have $f(\partial \Sigma) = \partial \Sigma$. Further, as $\partial \Sigma \subset \mathcal{M}$ and $\partial \Sigma$ is closed, $\partial \Sigma = \mathcal{M}$ by minimality of \mathcal{M} . Let $A := A_i$ for some $i \in I$. As A is closed in \mathcal{M} and \mathcal{M} is closed in \mathbb{T}^2 , A is closed, and thus compact, in \mathbb{T}^2 . Therefore, A is a continuum. Further, as \mathcal{M} is nowhere dense (as $\mathcal{M} \neq \mathbb{T}^2$), it follows that A is nowhere dense.

We need to show that a lift $\tilde{\Lambda}$ of Λ is bounded. As Σ is doubly essential, there exist two non-homotopic essential simple closed curves $\gamma, \gamma' \subset \Sigma$. As these curves are non-homotopic, their respective lifts $\tilde{\gamma}, \tilde{\gamma}' \subset \tilde{\Sigma}$ and the integer translates of these lifts tile \mathbb{R}^2 into bounded disks. As $\tilde{\Lambda} \cap \tilde{\Sigma} = \emptyset$, it follows that $\tilde{\Lambda}$ has to be contained in one of these bounded disks, implying $\tilde{\Lambda}$ itself is bounded.

As Λ is a connected component of \mathcal{M} , we must have either $f^n(\Lambda) \cap \Lambda = \emptyset$ for all $n \neq 0$, or $f^N(\Lambda) = \Lambda$ for some finite $N \neq 0$. However, the latter is excluded by Lemma 5 as Λ is bounded, and thus $f^n(\Lambda) \cap \Lambda = \emptyset$ for all $n \neq 0$. As Λ is a bounded continuum, $\mathbb{T}^2 \setminus \Lambda$ has a unique unbounded component and all the other components are bounded disks. Let D be one such disk. Then Σ is contained in the unbounded component: indeed, otherwise we can take a point $z \in D \cap \Sigma$ and an essential simple closed curve $\gamma \subset \Sigma$ passing through z ; as $\mathcal{M} \cap \Sigma = \emptyset$ and $z \in D$, this implies that $\gamma \subset D$, contradicting that D is a disk. We thus conclude that $D \cap \Sigma = \emptyset$. Further, if $D \cap \mathcal{M} \neq \emptyset$, then $\Sigma \cap D \neq \emptyset$ as $\partial\Sigma = \mathcal{M}$, which contradicts our earlier conclusion. In other words, to a component Λ , we can uniquely adjoin the open disks which, apart from the unique doubly essential component containing Σ , form the connected components of $\mathbb{T}^2 \setminus \Lambda$. This proves that $\text{Fill}(\Lambda_i) \cap \text{Fill}(\Lambda_j) = \emptyset$ if $i \neq j$, with $\text{Fill}(\Lambda_i)$ a bounded non-separating continuum, for every $i \in I$.

Let again Λ_i be any component of \mathcal{M} and define $\hat{\mathcal{Q}} = \bigcup_{i \in I} \text{Fill}(\Lambda_i)$. Define the decomposition \mathcal{U} of \mathbb{T}^2 into the continua $\{\text{Fill}(\Lambda_i)\}_{i \in I}$ and singletons in the complement of these continua. In order to show that \mathcal{M} is an extension of a Cantor set, we first show that the decomposition \mathcal{U} is upper semicontinuous. By Moore's Theorem, this implies there exists a continuous $\phi: \mathbb{T}^2 \rightarrow \mathbb{T}^2$ such that $\phi^{-1}(z) = \mathcal{U}_j$ for every $z \in \mathbb{T}^2$. We have already shown that the decomposition \mathcal{U} satisfies conditions (1)–(3) of Definition 5. To prove it satisfies condition (4), we need to show that if a sequence of continua \mathcal{U}_{j_k} , with $k \in \mathbb{Z}$, has Hausdorff limit \mathcal{C} , then $\mathcal{C} \subset \mathcal{U}_j$ for some $j \in J$. This being obvious if \mathcal{C} is a singleton, assume \mathcal{C} is a non-trivial continuum. Note that every non-degenerate element $\mathcal{U}_{j_k} \in \mathcal{U}$ has $\partial\mathcal{U}_{j_k} \subset \partial\Sigma = \mathcal{M}$. Further, without loss of generality, we may assume that no $\partial\mathcal{U}_{j_k}$ is a singleton and that they are mutually disjoint.

We first claim that the interior of \mathcal{C} has to be empty. Indeed, if not, there would exist a subsequence of elements for which the largest open disk contained in the interior of \mathcal{U}_{j_k} would be bounded from below, contradicting that the torus is compact and the elements mutually disjoint. Therefore, as $\partial\mathcal{U}_{j_k} \subset \mathcal{M}$, every point of the Hausdorff limit \mathcal{C} is the limit point of a sequence of points of \mathcal{M} . As \mathcal{M} is closed, this implies \mathcal{C} is itself contained in \mathcal{M} . In particular, as \mathcal{C} is connected, \mathcal{C} is contained in a connected component of \mathcal{M} , i.e. $\mathcal{C} \subset \Lambda_i \subset \mathcal{U}_j$ for some $j \in J$. So \mathcal{U} is upper semicontinuous indeed.

We have already shown that all non-trivial elements of \mathcal{U} are non-separating and bounded, and thus acyclic.

Thus, by Moore's Theorem, there exists a continuous $\phi: \mathbb{T}^2 \rightarrow \mathbb{T}^2$, homotopic to the identity, such that for every $z \in \mathbb{T}^2$, $\phi^{-1}(z)$ is a unique element of \mathcal{U} . By Lemma 8, as \mathcal{U} is upper semicontinuous and f sends elements of \mathcal{U} into elements of \mathcal{U} , the mapping \widehat{f} defined by $\phi \circ f = \widehat{f} \circ \phi$ is an element of $\text{Homeo}_0(\mathbb{T}^2)$. By a standard argument, $\rho(f) = \rho(\widehat{f}) \bmod \mathbb{Z}^2$, thus $\widehat{f} \in \text{Homeo}_*(\mathbb{T}^2)$. This proves condition (i) of Definition 1.

To prove condition (ii) of Definition 1, we need to show that $\widehat{\mathcal{M}} := \phi(\widehat{\mathcal{Q}}) \subset \mathbb{T}^2$ is a Cantor minimal set for \widehat{f} . As \mathcal{M} is a minimal set for f , $\widehat{\mathcal{M}}$ is a minimal set for \widehat{f} . Further, as $\widehat{\mathcal{M}}$ is totally disconnected by construction, it suffices to show that it is compact and perfect. First, $\widehat{\mathcal{Q}}$ is compact as the complement Σ is open. Because ϕ is continuous, $\widehat{\mathcal{M}}$ is compact. To show $\widehat{\mathcal{M}}$ is perfect, we observe that, because $\text{Fill}(A_i) \cap \text{Fill}(A_j) = \emptyset$ if $i \neq j$, no element $\text{Fill}(A_i)$ is isolated, as this would imply that a component A_i is isolated. Therefore, by continuity of ϕ , no point of $\widehat{\mathcal{M}}$ is isolated, and thus $\widehat{\mathcal{M}}$ is perfect. ■

2.3. Proof of Theorem A and its corollaries

Proof of Theorem A. To show that the minimal set \mathcal{M} of f is either of type I, II or III as given above, assume that $\mathcal{M} \neq \mathbb{T}^2$ and let $\{\Sigma_k\}$ be the collection of connected components of the complement of \mathcal{M} . If no element of $\{\Sigma_k\}$ is doubly essential, then Σ_k is either trivial or essential, for all $k \in \mathbb{Z}$. By Lemmas 10 and 11, $\{\Sigma_k\}$ are all disks and/or essential annuli; however, by Lemma 12, $\{\Sigma_k\}$ cannot both contain an essential annulus and an unbounded disk. In case no element Σ_k is essential, we have a type I minimal set. In case at least one, and therefore infinitely many, connected components are essential, we have a type II minimal set. If for some k , Σ_k is doubly essential, then \mathcal{M} is an extension of a Cantor set by Lemma 13 and these correspond to type III minimal sets. This concludes the proof. ■

Proof of Corollary 1. Let \mathcal{M} be a minimal set of f of type I. It suffices to show that $\mathcal{M} = \Omega(f)$. Indeed, if this is shown, then by minimality of \mathcal{M} and the inclusions $\alpha(z), \omega(z) \subseteq \Omega(f)$, with $\omega(z), \alpha(z)$ closed and f -invariant sets for every $z \in \mathbb{T}^2$, we obtain (1.3). Uniqueness then also follows, as any other minimal set \mathcal{M}' of f has to be contained in the complement of $\Omega(f)$, which is clearly impossible: if $z \in \mathcal{M}'$ then z would be both recurrent and wandering, which are incompatible conditions to hold simultaneously.

First, suppose that \mathcal{M} is of type I. Fix a component $\Sigma := \Sigma_k$. Then Σ is a disk. If Σ is bounded, then by Lemma 10 we have $f^n(\Sigma) \cap \Sigma = \emptyset$ for all $n \neq 0$, and thus $\Sigma \cap \Omega(f) = \emptyset$. So it remains to consider the case where Σ is an unbounded disk. We may assume that there exists an $N \neq 0$

such that $f^N(\Sigma) = \Sigma$, as otherwise we are done by the previous argument. A straightforward modification of the argument of Lemma 10 shows that $\Omega(f) \cap \Sigma = \emptyset$ in this case as well. Finally, if \mathcal{M} is of type II, then all elements of $\{\Sigma_k\}$ are essential annuli or bounded disks. We need only show that if Σ_k is an essential annulus, then $\Omega(f) \cap \Sigma_k = \emptyset$. This follows from Lemma 11 stating that $f^n(\Sigma_k) \cap \Sigma_k = \emptyset$ for all $n \neq 0$, and this finishes the proof. ■

Proof of Corollary 3. First, it is readily verified that no minimal set of type II or III is connected. Conversely, let \mathcal{M} be of type I, so that Σ_k is an open topological disk for every $k \in \mathbb{Z}$. In each disk Σ_k , we can find a sequence D_k^t of nested disks, i.e. $D_k^t \subset D_k^{t+1}$, embedded in Σ_k , such that $\text{Cl}(D_k^t)$ is a closed disk and $\bigcup_{t \geq 1} D_k^t = \Sigma_k$. We can accomplish this by uniformizing each disk Σ_k to the unit disk \mathbb{D}^2 , taking nested such disks centred at the origin in \mathbb{D}^2 , and pulling these back to Σ_k . Define $\Gamma_t = \mathbb{T}^2 \setminus \bigcup_{k \in \mathbb{Z}} D_k^t$. We claim that Γ_t is connected. Indeed, define the compact sets $\Gamma_t^s = \mathbb{T}^2 \setminus \bigcup_{k=-s}^s D_k^t$. Clearly, Γ_t^s , as the torus with finitely many disjoint disks whose closures are disjoint deleted, is connected. As $\Gamma_t^{s+1} \subset \Gamma_t^s$, we see that $\Gamma_t = \bigcap_{s \geq 1} \Gamma_t^s$ is connected as well. By the same token, as $\Gamma_{t+1} \subset \Gamma_t$ with Γ_t compact and connected for every $t \geq 1$, we conclude that $\mathcal{M} = \bigcap_{t \geq 1} \Gamma_t$ is connected. ■

Sketch of the proof of Corollary 4. First, it is not difficult to show that a locally connected minimal set \mathcal{M} has to be of type I and that the collection $\{\Sigma_k\}$ cannot contain an unbounded disk. By the same arguments as in [B1, Lemma 7], one can now show that $\text{diam}(\Sigma_k)$ is a null-sequence. Further, as no minimal set of a homeomorphism of a compact metric space can have cut points by [B1, Lemma 2], it follows from e.g. [Ku, Thm. 61-4] that Σ_k is the interior of a closed embedded disk.

To show that $\text{Cl}(\Sigma_k) \cap \text{Cl}(\Sigma_{k'})$ consists of at most one point if $k \neq k'$, we reason as follows. Denote $\Sigma = \Sigma_k$ and $\Sigma' = \Sigma_{k'}$ and let $\gamma = \partial\Sigma$ and $\gamma' = \partial\Sigma'$, both simple closed (trivial) curves. Assume, to the contrary, that $\gamma \cap \gamma'$ contains at least two points z_1, z_2 . There exists an arc $\eta \subset \text{Cl}(\Sigma)$ starting at z_1 and ending at z_2 such that $\eta \cap \partial\Sigma = \{z_1, z_2\}$. Similarly, there exists an arc $\eta' \subset \text{Cl}(\Sigma')$ starting at z_1 and ending at z_2 such that $\eta' \cap \partial\Sigma' = \{z_1, z_2\}$. Then $\eta \cup \eta'$ forms a simple closed curve that bounds a disk D . As the diameters of Σ and Σ' tend to zero, $\text{diam}(f^n(D)) \rightarrow 0$ for $|n| \rightarrow \infty$. Furthermore, as $z_1 \neq z_2$, and D contains arcs contained in $\partial\Sigma$ and $\partial\Sigma'$ joining z_1 and z_2 in its interior, we have $D \cap \mathcal{M} \neq \emptyset$. As $\text{diam}(f^n(D))$ is a null-sequence, for sufficiently large N we have $f^N(D) \subset D$, which by Lemma 6 implies f has periodic points. Therefore, $\text{Cl}(\Sigma) \cap \text{Cl}(\Sigma')$ can consist of at most a single point and thus \mathcal{M} is indeed a quasi-Sierpiński set. ■

3. Existence of minimal sets. Having given a classification of the possible minimal sets of homeomorphisms $f \in \text{Homeo}_*(\mathbb{T}^2)$, in this section we aim at constructing such homeomorphisms admitting a minimal set of every type Theorem A allows. More precisely, we show there exist $f \in \text{Homeo}_*(\mathbb{T}^2)$ whose minimal set \mathcal{M}

- (1) is a quasi-Sierpiński set, but not a Sierpiński set,
- (2) has complement consisting of a single unbounded disk,
- (3) has complement consisting of essential annuli and disks,
- (4) is a non-trivial extension of a Cantor set.

It is well-known there exist $f \in \text{Homeo}_*(\mathbb{T}^2)$ for which the minimal set is a Sierpiński set. Example 1, constructed in [B1], is derived from the Sierpiński set; see also Section 3.3 below for a discussion of this example. The remainder of this section is devoted to the construction of Example 2 (type I), Example 3 (type II) and Example 4 (type III). Combined, these examples prove Theorem B.

3.1. Homeomorphisms semiconjugate to an irrational translation. There is a natural subclass of $\text{Homeo}_*(\mathbb{T}^2)$, namely those homeomorphisms that are semiconjugate to an irrational translation of the torus. Indeed, a standard argument shows that an element $f \in \text{Homeo}_0(\mathbb{T}^2)$ semiconjugate to a translation τ , through a continuous map homotopic to the identity, has the property that $\rho(f) = \rho(\tau) \bmod \mathbb{Z}^2$. Given a continuous map $\pi: \mathbb{T}^2 \rightarrow \mathbb{T}^2$, we call the set of points

$$(3.1) \quad \mathcal{R}_\pi = \{z \in \mathbb{T}^2 \mid \#(\pi^{-1}(\pi(z))) = 1\} \subset \mathbb{T}^2$$

the *regular set* of π .

DEFINITION 6. We define $\text{Homeo}_\#(\mathbb{T}^2) \subset \text{Homeo}_*(\mathbb{T}^2)$ to be the class of homeomorphisms which satisfy the following:

- (i) f is isotopic to the identity, i.e. $f \in \text{Homeo}_0(\mathbb{T}^2)$,
- (ii) there exists a monotone ⁽¹⁾ and continuous $\pi: \mathbb{T}^2 \rightarrow \mathbb{T}^2$, homotopic to the identity, and an irrational translation τ such that $\pi \circ f = \tau \circ \pi$ (cf. (2.2)),
- (iii) the regular set \mathcal{R}_π is uncountable.

The following simple but important observation plays a crucial role in the constructions below. In what follows, we denote by $\mathcal{O}_f(z)$ the full orbit of a point z under f .

LEMMA 14. *Let $f \in \text{Homeo}_\#(\mathbb{T}^2)$, with π the corresponding semiconjugacy. Then f has a unique minimal set \mathcal{M} and*

$$(3.2) \quad \mathcal{M} = \text{Cl}(\mathcal{R}_\pi) = \text{Cl}(\mathcal{O}_f(z)) \quad \text{for any } z \in \mathcal{R}_\pi.$$

⁽¹⁾ A map is said to be *monotone* if every point-inverse is connected.

Proof. Let us first prove the uniqueness of the minimal set. Let \mathcal{M} and \mathcal{M}' be two minimal sets for f . Because \mathcal{M} is closed and f -invariant, $\pi(\mathcal{M})$ is closed and τ -invariant. In particular, $\pi(\mathcal{M})$ contains the complete τ -orbit of every point $\pi(z) \in \pi(\mathcal{M})$, where $z \in \mathcal{M}$. Since every orbit of τ is dense, we have $\pi(\mathcal{M}) = \mathbb{T}^2$. Similarly, $\pi(\mathcal{M}') = \mathbb{T}^2$. Take $z \in \mathcal{R}_\pi \neq \emptyset$. Then $\{z\} = \pi^{-1}(\pi(z))$ is contained in both \mathcal{M} and \mathcal{M}' . As two minimal sets are either identical or disjoint, this implies that $\mathcal{M} = \mathcal{M}'$.

Next we prove that $\mathcal{M} = \text{Cl}(\mathcal{O}_f(z))$ for every $z \in \mathcal{R}_\pi$. For any $z \in \mathcal{R}_\pi$ the set $\text{Cl}(\mathcal{O}_f(z))$ is closed and invariant, hence it contains the (unique) minimal set \mathcal{M} of f , i.e. $\mathcal{M} \subseteq \text{Cl}(\mathcal{O}_f(z))$. We need to show that $\text{Cl}(\mathcal{O}_f(z)) \subseteq \mathcal{M}$. If we had $\mathcal{M} \cap \mathcal{O}_f(z) = \emptyset$, then as $\pi(\mathcal{M}) = \mathbb{T}^2$ by the above, there would exist a point $z' \in \mathcal{M}$ for which $\pi(z') = \pi(z)$. However, this contradicts the assumption that $z \in \mathcal{R}_\pi$ and thus $\pi^{-1}(\pi(z)) = \{z\}$. It follows that $\mathcal{M} \cap \mathcal{O}_f(z) \neq \emptyset$. Let $z' \in \mathcal{M} \cap \mathcal{O}_f(z)$; then $\mathcal{O}_f(z') = \mathcal{O}_f(z) \subseteq \mathcal{M}$, since \mathcal{M} is invariant. But since \mathcal{M} is also closed, $\text{Cl}(\mathcal{O}_f(z)) \subseteq \mathcal{M}$. Hence $\mathcal{M} = \text{Cl}(\mathcal{O}_f(z))$ for any $z \in \mathcal{R}_\pi$, and consequently $\mathcal{M} = \text{Cl}(\mathcal{R}_\pi)$. ■

Let us further introduce the following notation, to be used in the proofs of Examples 2–4 below. A non-transitive orientation preserving circle homeomorphism with irrational rotation number will be referred to as a *Denjoy counterexample*. Moreover, given a Cantor set in the circle $\mathcal{Q} = \mathbb{T}^1 \setminus \bigcup_{k \in \mathbb{Z}} I_k$, we denote by $\mathcal{Q}_{\text{rat}} \subset \mathcal{Q}$ and $\mathcal{Q}_{\text{irr}} = \mathcal{Q} \setminus \mathcal{Q}_{\text{rat}}$ the *rational* and *irrational* parts of \mathcal{Q} , which are the set of all endpoints of the deleted intervals and its complement in \mathcal{Q} . It is readily verified, using Poincaré's Theorem, that

- (1) a product of a Denjoy counterexample and an irrational rotation, and
- (2) a product of two Denjoy counterexamples,

are elements of $\text{Homeo}_\#(\mathbb{T}^2)$, provided the rotation numbers of the factors are rationally independent.

3.2. A topological blow-up procedure. In order to construct the examples, we devise a tool that enables us to *blow up* an orbit of a point under a homeomorphism to a collection of disks. A. Biś, H. Nakayama and P. Walczak in [B2] define such a blow-up procedure that works for (groups of) diffeomorphisms. J. Aarts and L. Oversteegen in [A] defined a similar blow-up construction for a homeomorphism of the annulus that preserves radial lines. Both constructions allow for the mapping to be extended to the disks glued to the surface by the infinitesimal behaviour of the mapping. As this would not work for a general homeomorphism, we circumvent this by inductively blowing up punctures to disks, by pulling back points near a puncture along leaves of a dynamically defined foliation emanating from

the puncture. We use the continuity of the foliation to define an extension of the mapping to the disks.

Let $f \in \text{Homeo}_\#(\mathbb{T}^2)$, with $\pi: \mathbb{T}^2 \rightarrow \mathbb{T}^2$ the semiconjugacy between f and an irrational translation τ . Take a point $z_0 \in \mathcal{R}_\pi$; we do not require f to be minimal, so $\mathcal{O}_f(z_0)$ may or may not be dense. Define $\Gamma = \mathbb{T}^2 \setminus \mathcal{O}_f(z_0)$. Clearly, $f(\Gamma) = \Gamma$ and $f|_\Gamma$ is a homeomorphism. Let $B_\delta := B(z_0, \delta) \subset \mathbb{T}^2$ be the embedded closed Euclidean disk of radius $0 < \delta \leq 1/4$ centred at z_0 . Choose $0 < \delta_0 \leq 1/4$ and let \mathcal{F}_0 be the foliation of B_{δ_0} by straight rays emanating from z_0 (see Figure 1). The leaves $\rho_\theta \in \mathcal{F}_0$ are parametrized by $\theta \in [0, 2\pi)$. Fix $0 < \epsilon_0 < 1$. To blow up the punctures to disks, we define the following auxiliary planar map, in polar coordinates:

$$(3.3) \quad g_\epsilon: \tilde{B}_1 \setminus \{0\} \rightarrow \tilde{B}_1 \setminus \tilde{B}_\epsilon, \quad g_\epsilon(r, \theta) = \left(\frac{r + \epsilon}{1 + r\epsilon}, \theta \right),$$

where $\tilde{B}_\rho \subset \mathbb{R}^2$ is the closed Euclidean disk centred at $0 \in \mathbb{R}^2$ of radius $0 < \rho \leq 1$. Conjugating g_{ϵ_0} with a linear injection (into the torus) $\lambda_0: \tilde{B}_1 \hookrightarrow B_{\delta_0}$ yields a homeomorphism

$$(3.4) \quad h_0 = \lambda_0 \circ g_{\epsilon_0} \circ \lambda_0^{-1}: A_0 \rightarrow h_0(A_0),$$

where $A_0 = B_{\delta_0} \setminus \{z_0\}$ and $h_0(A_0) = B_{\delta_0} \setminus B_{\epsilon_0 \delta_0}$ is the corresponding annulus. We can extend h_0 to $\mathbb{T}^2 \setminus \{z_0\}$ by declaring it to be the identity off A_0 ; the extension is a homeomorphism we denote again by h_0 , and it naturally acts on $\Gamma \subset \mathbb{T}^2 \setminus \{z_0\}$ by restriction. Note that h_0 acts on $B_{\delta_0} \setminus \{z_0\}$ along the leaves of the foliation \mathcal{F}_0 .

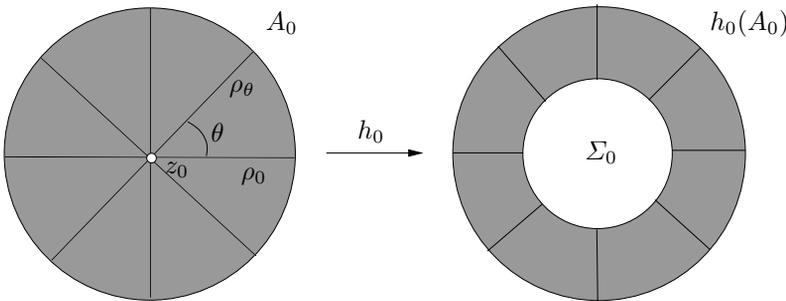


Fig. 1. Radial blow up of a puncture to a disk

Define $\Gamma_0 = h_0(\Gamma)$, define the homeomorphism

$$(3.5) \quad f_0: \Gamma_0 \rightarrow \Gamma_0, \quad f_0 = h_0 \circ f|_\Gamma \circ h_0^{-1},$$

and consider the continuous map $\phi_0: \Gamma_0 \rightarrow \Gamma$ given by $\phi_0 = h_0^{-1}$. Note that, by construction, $f_0 = \phi_0^{-1} \circ f|_\Gamma \circ \phi_0$. Define $\Sigma_0 := \text{Int}(B_{\epsilon_0 \delta_0})$ and $\gamma_0 := \partial \Sigma_0$

(see again Figure 1). Consider the points $z_{\pm 1} := \phi_0^{-1}(f^{\pm 1}(z_0))$ and define

$$(3.6) \quad d_1 = \frac{1}{4} \min\{(1/4)^2, d(z_{-1}, z_1), d(z_{-1}, \Sigma_0), d(z_1, \Sigma_0)\} > 0.$$

Given $0 < \epsilon_0 < \epsilon < 1$, set $\epsilon' = (\epsilon + \epsilon_0)/2$ and define the second auxiliary planar map

$$(3.7) \quad q_\epsilon: \tilde{B}_\epsilon \setminus \tilde{B}_{\epsilon_0} \rightarrow \tilde{B}_\epsilon \setminus \tilde{B}_{\epsilon'}, \quad q_\epsilon = \hat{g}_{\epsilon'/\epsilon} \circ \hat{g}_{\epsilon_0/\epsilon}^{-1},$$

where

$$(3.8) \quad \hat{g}_{\delta/\epsilon} := r_\epsilon \circ g_{\delta/\epsilon} \circ r_\epsilon^{-1}$$

and $r_\epsilon: \tilde{B}_1 \rightarrow \tilde{B}_\epsilon$ is a linear (planar) rescaling for $\epsilon_0 \leq \delta < \epsilon$. Let $\lambda_\epsilon: \tilde{B}_\epsilon \hookrightarrow B_{\epsilon\delta_0}$ be the linear injection of the disk $\tilde{B}_\epsilon \subset \mathbb{R}^2$ onto the disk $B_{\epsilon\delta_0} \subset \mathbb{T}^2$. Define $A_{\epsilon, \epsilon'} = B_{\epsilon'\delta_0} \setminus B_{\epsilon\delta_0}$ and

$$(3.9) \quad \hat{q}_\epsilon: A_{\epsilon, \epsilon_0} \rightarrow A_{\epsilon, \epsilon'}, \quad \hat{q}_\epsilon = \lambda_\epsilon \circ q_\epsilon \circ \lambda_\epsilon^{-1}.$$

In words, \hat{q}_ϵ has the effect of mapping the annulus A_{ϵ, ϵ_0} radially, i.e. along (part of) the foliation \mathcal{F}_0 , to the annulus $A_{\epsilon, \epsilon'}$ with the same outer boundary curve, but larger inner boundary curve, so as to halve the modulus of the annulus. There exist $0 < \epsilon_0 < \epsilon_{\pm 1} < 1$ such that if we denote $A_{\pm 1} := f_0^{\pm 1}(A_{\epsilon_{\pm 1}, \epsilon_0})$, then $\text{diam}(A_{\pm 1}) \leq d_1$. Define

$$(3.10) \quad h_{\pm 1}: A_{\pm 1} \rightarrow h_{\pm 1}(A_{\pm 1}), \quad h_{\pm 1} = f_0^{\mp 1} \circ \hat{q}_{\epsilon_{\pm 1}} \circ f_0^{\pm 1};$$

we extend $h_{\pm 1}$ to \mathbb{T}^2 by declaring it to be the identity off $A_{\pm 1}$. The annuli $A_{\pm 1}$ are foliated by $\mathcal{F}_{\pm 1} = f_0|_{A_{\epsilon_{\pm 1}, \epsilon_0}}(\mathcal{F}_0)$. The maps $h_{\pm 1}$ have the effect of blowing up the puncture $z_{\pm 1}$ along the foliation $\mathcal{F}_{\pm 1}$ to a disk $\Sigma_{\pm 1}$ (see Figure 2). Denote by $\Sigma_{\pm 1}$ the open disks obtained by blowing up the corresponding puncture $z_{\pm 1}$. As $\partial\Sigma_{\pm 1} = f_0(C_{\epsilon'_{\pm 1}})$, where $C_{\epsilon'_{\pm 1}}$ is the Euclidean circle centred at z_0 of radius $\epsilon_0 < \epsilon'_{\pm 1} < \epsilon_{\pm 1}$, $\gamma_{\pm 1} := \partial\Sigma_{\pm 1}$ is a simple closed curve, as $f_0^{\pm 1}|_{A_{\epsilon_{\pm 1}, \epsilon_0}}$ is a homeomorphism.

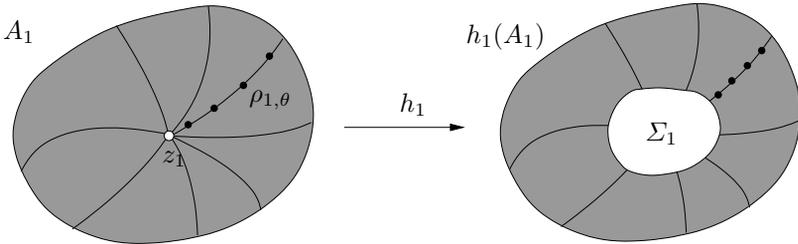


Fig. 2. Blowing up the puncture z_1 to a disk; $\rho_{1, \theta}$ is a leaf of the foliation \mathcal{F}_1 emanating from z_1 , and h_1 has the effect of pulling back points on $\rho_{1, \theta} \in \mathcal{F}_1$ along this leaf, for every $\theta \in [0, 2\pi)$.

Define $\hat{h}_1 := h_{-1} \circ h_1$ on $A_{-1} \cup A_1$, set $\Gamma_1 = \hat{h}_1(\Gamma_0)$ and define the homeomorphism

$$(3.11) \quad f_1: \Gamma_1 \rightarrow \Gamma_1, \quad f_1 := \hat{h}_1 \circ f_0 \circ \hat{h}_1^{-1}.$$

Further, consider the continuous map $\phi_1: \Gamma_1 \rightarrow \Gamma$ given by $\phi_1 = \phi_0 \circ \hat{h}_1^{-1}$.

We proceed by induction. Assume we have blown up the punctures z_k to disks Σ_k for $-n+1 \leq k \leq n-1$, and consider the points $z_{\pm n} := \phi_{n-1}^{-1}(f^{\pm n}(z_0))$. Define $\Delta_{n-1} = \bigcup_{k=-n+1}^{n-1} \Sigma_k$ and

$$(3.12) \quad d_n = \frac{1}{4} \min\{(1/4)^{n+1}, d(z_{-n}, z_n), d(z_{-n}, \Delta_{n-1}), d(z_n, \Delta_{n-1})\} > 0.$$

There exist $0 < \epsilon_0 < \epsilon_{\pm n} < 1$ such that $\text{diam}(A_{\pm n}) \leq d_n$. Define

$$(3.13) \quad h_{\pm n}: A_{\pm n} \rightarrow h_{\pm n}(A_{\pm n}), \quad h_{\pm n} = f_{n-1}^{\mp n} \circ \hat{g}_{\epsilon_{\pm n}} \circ f_{n-1}^{\pm n},$$

which we extend to \mathbb{T}^2 by declaring them to be the identity off $A_{\pm n}$. The annuli $A_{\pm 1}$ are foliated by $\mathcal{F}_{\pm n} = f_{n-1}^n|_{A_{\epsilon_{\pm n}, \epsilon_0}}(\mathcal{F}_0)$. The maps $h_{\pm n}$ blow up the puncture $z_{\pm n}$ along the foliation $\mathcal{F}_{\pm n}$ to a disk $\Sigma_{\pm n}$. The boundaries $\gamma_{\pm n}$ are again simple closed curves, as $\gamma_{\pm n} = f_{n-1}^{\pm n}(C_{\epsilon'_{\pm n}})$, where $C_{\epsilon'_{\pm n}}$ is the Euclidean circle centred at z_0 of radius $\epsilon_0 < \epsilon'_{\pm n} < \epsilon_{\pm n}$ and $f_{n-1}^{\pm n}|_{A_{\epsilon_{\pm n}, \epsilon_0}}$ is a homeomorphism. Define $\hat{h}_n := h_{-n} \circ h_n$ on $A_{-n} \cup A_n$, set $\Gamma_n = \hat{h}_n(\Gamma_{n-1})$ and define the homeomorphism

$$(3.14) \quad f_n: \Gamma_n \rightarrow \Gamma_n, \quad f_n := \hat{h}_n \circ f_{n-1} \circ \hat{h}_n^{-1}.$$

Further, consider the continuous map $\phi_n: \Gamma_n \rightarrow \Gamma$ given by $\phi_n = \phi_{n-1} \circ \hat{h}_n^{-1}$.

Next, we show that the above sequences of maps and homeomorphisms converge and have the desired properties. First, from (3.12) combined with (3.13), it follows that $\Gamma_n \subset \Gamma_{n-1}$ and $\Gamma_\infty = \lim_{n \rightarrow \infty} \Gamma_n$ exists in the Hausdorff sense, as $\sum_{n \geq 0} 1/4^{n+1} < 1 < \infty$. Denote $\mathcal{N} = \text{Cl}(\Gamma_\infty)$. Notice that $\mathcal{N} = \Gamma_\infty \cup \bigcup_{k \in \mathbb{Z}} \gamma_k$, since no point in Σ_k can be the limit point of points in Γ_∞ as $\Sigma_k \cap \Gamma_\infty = \emptyset$. Furthermore, as the boundary curves γ_k are simple closed curves, the extension $\bar{\phi}_n: \text{Cl}(\Gamma_n) \rightarrow \mathbb{T}^2$ of ϕ_n is continuous.

LEMMA 15. *The homeomorphisms $f_n: \Gamma_n \rightarrow \Gamma_n$ converge to a homeomorphism $f_\infty: \Gamma_\infty \rightarrow \Gamma_\infty$ that extends to a homeomorphism $f' \in \text{Homeo}_*(\mathbb{T}^2)$ with $f'(\mathcal{N}) = \mathcal{N}$. Further, the disks $\{\Sigma_k\}$ in the complement of \mathcal{N} are interiors of closed topological disks. Similarly, the continuous maps ϕ_n converge to a continuous map $\phi_\infty: \Gamma_\infty \rightarrow \Gamma$ that extends to a continuous map $\phi: \mathbb{T}^2 \rightarrow \mathbb{T}^2$ for which $\phi(\mathcal{N}) = \mathbb{T}^2$. Furthermore, f' is semiconjugate to f through ϕ .*

Proof. First, we show that the f_n converge to a homeomorphism f_∞ of Γ_∞ . Indeed, for every $n \geq 0$, $f_n: \Gamma_n \rightarrow \Gamma_n$ is a homeomorphism and we observed above that $\Gamma_n \rightarrow \Gamma_\infty$. As \hat{h}_n moves points by no more than $d_n \leq 1/4^{n+1}$, and $\sum_{n \geq 0} 1/4^{n+1} < 1 < \infty$, f_n converges uniformly and

thus the limit f_∞ is a homeomorphism. Further, we observed that γ_k is a simple closed curve for every $k \in \mathbb{Z}$ and thus Σ_k is the interior of the closed topological disk $\text{Cl}(\Sigma_k) = \Sigma_k \cup \gamma_k$.

Next, we show that f_∞ extends to a homeomorphism f' of \mathcal{N} . To this end, we first show that f_∞ induces a homeomorphism from γ_k to γ_{k+1} for every $k \in \mathbb{Z}$. To prove this, we note that the disks Σ_k , for $-n \leq k \leq n$, which have been constructed after n steps, are left unmoved by future perturbations by virtue of our choice of d_n . Moreover, again by our choice of d_n , we have $f_n|_{\gamma_k} = f_\infty|_{\gamma_k} : \gamma_k \rightarrow \gamma_{k+1}$ for $-n \leq k \leq n-1$, where $f_n|_{\gamma_k}$ and $f_\infty|_{\gamma_k}$ are the extensions of f_n and f_∞ to γ_k . To prove that $f_n|_{\gamma_k}$ is a homeomorphism, it suffices to show that $f_n|_{\gamma_k}$ is one-to-one and continuous. We prove this by induction for $0 \leq k \leq n$, the case of negative k being handled by considering the inverse.

Assume that after step $n-1$, we have shown that $f_{n-1}|_{\gamma_k} : \gamma_k \rightarrow \gamma_{k+1}$ for $0 \leq k \leq n-2$ are homeomorphisms, and consider step n , where we have to show that $f_n|_{\gamma_{n-1}} : \gamma_{n-1} \rightarrow \gamma_n$ is a homeomorphism. By choice of ϵ_n , A_n is disjoint from the previously constructed disks and disjoint from A_{-n} . Restricting to a smaller neighbourhood of A_{n-1} if necessary, we may as well assume that $A_{n-1} \cap A_n = \emptyset$. As $h_n = \hat{h}_n|_{A_n}$ (as defined by (3.10)) acts along the foliation \mathcal{F}_n , f_n sends leaves of \mathcal{F}_{n-1} to leaves of \mathcal{F}_n which foliate A_{n-1} and A_n respectively. To each $\theta \in [0, 2\pi)$ corresponds a unique point $z(\theta) \in \gamma_{n-1}$ lying on $\rho_{n-1, \theta} \in \mathcal{F}_{n-1}$, which is mapped by f_n to a unique point $z'(\theta) = f_n(z(\theta)) \in \gamma_n$ lying on $\rho_{n, \theta} = f_n(\rho_{n-1, \theta})$. As these foliations are continuous, and as γ_{n-1} and γ_n are (continuous) simple closed curves, the points $z'(\theta)$ vary continuously as θ varies, and thus continuously as $z(\theta)$ varies, which is what we needed to show.

By induction, f_∞ extends homeomorphically to every boundary curve γ_k , $k \in \mathbb{Z}$. It thus follows that the extension f' to \mathcal{N} is one-to-one, as $\mathcal{N} = \Gamma_\infty \cup \bigcup_{k \in \mathbb{Z}} \gamma_k$. To show f' is continuous, we distinguish between two cases.

First, let $z \in \Gamma_\infty$. As f_∞ is a homeomorphism, given a neighbourhood $V \subset \mathcal{N}$ containing z' , we can find a small neighbourhood $U \subset \mathcal{N}$, containing the point z for which $f'(z) = z'$, such that $\text{Cl}(f_\infty(W)) \subset V$, where $W = U \cap \Gamma_\infty$. As $\text{Cl}(\Gamma_\infty) = \mathcal{N}$ and f' extends homeomorphically to \mathcal{N} , we have $f'(U) = f'(\text{Cl}(W)) = \text{Cl}(f_\infty(W)) \subset V$.

Secondly, suppose that $z \in \gamma_k$ for some $k \in \mathbb{Z}$. For $N \geq k+1$, $f_N|_{\gamma_k} = f_\infty|_{\gamma_k} : \gamma_k \rightarrow \gamma_{k+1}$ is a homeomorphism. Therefore, given a neighbourhood $V \subset \mathcal{N}$ containing $z' = f_N(z) = f'(z)$, there exists a small neighbourhood $U \ni z$ such that $f_N(U) \subset V$. Choosing N larger, and a smaller neighbourhood $U' \subset U$ containing z if necessary, as $\sum_{n \geq N} d_n \rightarrow 0$ for $N \rightarrow \infty$, it follows that $f'(U') \subset V$ as well. Thus f' is continuous, and therefore a homeomorphism, being one-to-one as well.

We can extend the homeomorphism $f': \mathcal{N} \rightarrow \mathcal{N}$ to a homeomorphism of \mathbb{T}^2 by extending, e.g. by Alexander's trick, the induced homeomorphisms of the boundary curves γ_k to homeomorphisms of the corresponding closed disks $\text{Cl}(\Sigma_k) = \Sigma_k \cup \gamma_k$. As the disks Σ_k are disjoint, and $\text{diam}(\Sigma_k)$ forms a null-sequence, the extension of $f': \mathcal{N} \rightarrow \mathcal{N}$ to \mathbb{T}^2 is a homeomorphism, which we denote again by f' .

To show that $\phi: \mathcal{N} \rightarrow \mathbb{T}^2$ is continuous, we recall that $\text{Cl}(\Gamma_n) = \mathbb{T}^2 \setminus \Delta_n$ for $n \geq 0$. As we observed, for every $n \geq 0$, $\phi_n: \Gamma_n \rightarrow \Gamma$ is continuous and it extends to a continuous $\bar{\phi}_n: \text{Cl}(\Gamma_n) \rightarrow \mathbb{T}^2$. As $\phi_n = \phi_{n-1} \circ \hat{h}_n^{-1}$ with \hat{h}_n as in (3.13), whose norm is bounded by d_n , $\phi = \lim_{n \rightarrow \infty} \bar{\phi}_n: \mathcal{N} \rightarrow \mathbb{T}^2$ is continuous as the limit of uniformly converging continuous maps $\bar{\phi}_n$. By declaring $\phi(\Sigma_k) = f^k(z_0)$, ϕ extends to a continuous map defined on \mathbb{T}^2 .

Finally, we show that $f' \in \text{Homeo}_{\#}(\mathbb{T}^2)$. First, we observe that, as \mathcal{R}_π is uncountable, and a countable number of points of \mathcal{R}_π is blown up to disks, $\mathcal{R}_{\pi'}$ is uncountable as well. Further, the ϕ thus constructed is homotopic to the identity. Thus it suffices to show that $\phi \circ f' = f \circ \phi$. For this, we note that for every $n \geq 0$ we have $\phi_n \circ f_n = f|_\Gamma \circ \phi_n$, where $f_n: \Gamma_n \rightarrow \Gamma_n$ and $\phi_n: \Gamma_n \rightarrow \Gamma$. As both f_n and ϕ_n converge uniformly and extend continuously to \mathcal{N} , it follows that $\phi \circ f' = f \circ \phi$ for some $f': \mathcal{N} \rightarrow \mathcal{N}$. Further, as Σ_k , along with γ_k , is mapped to a single point by ϕ , we also have $\phi \circ f' = f \circ \phi$ when f' is extended to \mathbb{T}^2 . ■

The following lemma, which combines Lemmas 14 and 15, is the key ingredient in the construction of Examples 3 and 4. Let $B_{\delta_0} \setminus \{z_0\} \subset \mathbb{T}^2$ be an embedded punctured disk centred at $z_0 \in \mathcal{R}_\pi$ with $\delta_0 \leq 1/4$, and \mathcal{F}_0 the corresponding foliation of $B_{\delta_0} \setminus \{z_0\}$ by straight rays emanating from z_0 , in the notation of the construction above. A wedge $\mathcal{W}(r, \theta_1, \theta_2) \subset B_{r\delta_0} \setminus \{z_0\}$ is the region bounded by two leaves $\rho_{\theta_1}, \rho_{\theta_2} \in \mathcal{F}_0$, where $0 < |\theta_1 - \theta_2| < \pi$ and $0 < r \leq 1$. Define $\mathcal{R}_\pi^n := \phi_n^{-1}(\mathcal{R}_\pi \setminus \mathcal{O}_f(z_0))$ for $n \geq 0$.

LEMMA 16. *In the construction above, let f' be semiconjugate to f through ϕ by blowing up the orbit $\mathcal{O}_f(z_0)$, with $z_0 \in \mathcal{R}_\pi$, to disks whose interiors are Σ_k , and let $\gamma_k = \partial\Sigma_k$, where $k \in \mathbb{Z}$. Let \mathcal{M}' be the minimal set of f' and define $\pi' = \pi \circ \phi$. Then*

- (1) $\mathcal{M}' = \text{Cl}(\mathcal{R}_{\pi'}) = \text{Cl}(\phi^{-1}(\mathcal{R}_\pi \setminus \mathcal{O}_f(z_0)))$,
- (2) $\gamma_k \subset \mathcal{M}'$, for all $k \in \mathbb{Z}$, if for every $0 < r \leq 1$ and every $\theta_1, \theta_2 \in [0, 2\pi)$ with $0 < |\theta_1 - \theta_2| < \pi$, we have $\mathcal{W}(r, \theta_1, \theta_2) \cap (\mathcal{R}_\pi \setminus \mathcal{O}_f(z_0)) \neq \emptyset$.

Proof. To prove (1), as \mathcal{R}_π is uncountable, $\mathcal{R}_{\pi'}^0 \neq \emptyset$. As the points $f^k(z_0)$ are blown up to disks, i.e. $\phi^{-1}(f^k(z_0)) = \text{Cl}(\Sigma_k)$, where $\text{Cl}(\Sigma_k)$ is a closed topological disk, we have $\mathcal{R}_{\pi'} = \phi^{-1}(\mathcal{R}_\pi \setminus \mathcal{O}_f(z_0))$. By Lemma 14, $\mathcal{M}' = \text{Cl}(\mathcal{R}_{\pi'})$, and this proves (1).

To prove (2), first assume that $\gamma_0 \subset \text{Cl}(\mathcal{R}_\pi^0)$. As the size of the perturbations \hat{h}_n , by virtue of our choice of d_n , converges to zero as the perturbations approach γ_0 , it follows that $\gamma_0 \subset \text{Cl}(\mathcal{R}_\pi^n)$ for every $n \geq 0$. As the maps ϕ_n converge, we thus have $\gamma_0 \subset \text{Cl}(\mathcal{R}_{\pi'}) = \mathcal{M}'$, by (1). As \mathcal{M}' is f' -invariant, and $f'(\gamma_k) = \gamma_{k+1}$, we find that $\gamma_k \subset \mathcal{M}'$, for every $k \in \mathbb{Z}$. To finish the proof, suppose that $\mathcal{W}(r, \theta_1, \theta_2) \cap (\mathcal{R}_\pi \setminus \mathcal{O}_f(z_0)) \neq \emptyset$ for every $0 < r \leq 1$ and every $\theta_1, \theta_2 \in [0, 2\pi)$ for which $0 < |\theta_1 - \theta_2| < \pi$. We have to show that $\gamma_0 \subset \text{Cl}(\mathcal{R}_\pi^0)$.

Suppose, to derive a contradiction, that $\gamma_0 \cap \text{Cl}(\mathcal{R}_\pi^0) \neq \gamma_0$. As $\gamma_0 \cap \text{Cl}(\mathcal{R}_\pi^0)$ is closed, this implies there exists an open subarc $\eta \subset \gamma_0$ such that $\eta \cap \text{Cl}(\mathcal{R}_\pi^0) = \emptyset$. Let $z \in \eta$ be the midpoint of η and let $\eta' \subset \eta$ be a closed subsegment properly contained in η , and containing $z \in \eta$, with endpoints $\{z^-, z^+\} = \partial\eta'$. Let ρ_{θ_1} and ρ_{θ_2} be the two rays passing through z^- and z^+ , and $\mathcal{W}(1, \theta_1, \theta_2)$ the corresponding wedge. As $\text{Cl}(\mathcal{R}_\pi^0)$ is closed, there exists an open neighbourhood $U \supset \eta'$ such that $U \cap \text{Cl}(\mathcal{R}_\pi^0) = \emptyset$. However, this implies that $\mathcal{W}(r, \theta_1, \theta_2) \cap (\mathcal{R}_\pi \setminus \mathcal{O}_f(z_0)) = \emptyset$ for $r > 0$ sufficiently small, contrary to our assumption. ■

3.3. Minimal sets of type I. It is well-known that, given any Sierpiński set $S \subset \mathbb{T}^2$, there exists a homeomorphism $f \in \text{Homeo}_*(\mathbb{T}^2)$ for which the minimal set \mathcal{M} is S . The following example can be found in [B1, Thm. 3]. We will only sketch the proof.

EXAMPLE 1 (Type I: a quasi-Sierpiński set [B1]). There exist homeomorphisms $f \in \text{Homeo}_*(\mathbb{T}^2)$ for which the minimal set \mathcal{M} is a quasi-Sierpiński set, but not a Sierpiński set.

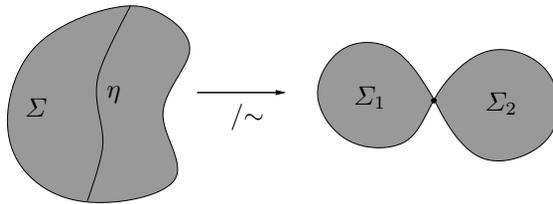


Fig. 3. Construction of a quasi-Sierpiński set: collapsing arcs to points

Sketch of the proof. Let \mathcal{M} be a Sierpiński minimal set of an $f \in \text{Homeo}_*(\mathbb{T}^2)$ and let Σ be a component of the complement of \mathcal{M} . Denote $\Sigma_n = f^n(\Sigma)$ and $\gamma_n = \partial\Sigma_n$. Take an arc $\eta \subset \text{Cl}(\Sigma)$ such that only the endpoints of η intersect $\gamma = \gamma_0$ (see Figure 3). Let $\eta_n := f^n(\eta) \subset \text{Cl}(\Sigma_n)$ the corresponding arcs in the image disks. Using techniques from decomposition theory, it can be shown that \mathbb{T}^2/\sim , where $z \sim z'$ if and only if $z, z' \in \eta_n$ (i.e. collapsing the arcs η_n to points), yields a well-defined quotient space homeomorphic to \mathbb{T}^2 and that \mathcal{M} quotients to a quasi-Sierpiński set $\mathcal{M}' = \mathcal{M}/\sim$,

which is not a Sierpiński set. The corresponding quotient homeomorphism $f' \in \text{Homeo}_*(\mathbb{T}^2)$ has \mathcal{M}' as its minimal set and this minimal set is locally connected. ■

Next, we give an example of a minimal set which is of type I, but not locally connected. It shows the existence of homeomorphisms $f \in \text{Homeo}_*(\mathbb{T}^2)$ for which the complement of the minimal set \mathcal{M} consists of a single unbounded disk Σ , which is f -invariant.

EXAMPLE 2 (Type I: unbounded disks). There exist minimal sets \mathcal{M} of homeomorphisms $f \in \text{Homeo}_*(\mathbb{T}^2)$ of type I such that the complement of \mathcal{M} in \mathbb{T}^2 is a single unbounded disk.

The example uses a *derived-from-Anosov* type argument ⁽²⁾, constructed initially by P. McSwiggen [McS1] on the 3-torus and subsequently generalized to the general $(k + 1)$ -torus for $k \geq 1$ in [McS2]. For completeness, we recall this discussion for the case $k = 1$ and collect the necessary results needed for our construction. Almost all of this material, up to the proof of Example 2, is taken *verbatim* from [McS2].

Start with a hyperbolic $A \in \text{SL}(2, \mathbb{Z})$ and let $g_0: \mathbb{T}^2 \rightarrow \mathbb{T}^2$ be the induced linear toral automorphism. Let $z_0 \in \mathbb{T}^2$, with $z_0 = p(0)$ where $0 \in \mathbb{R}^2$ is the origin and $p: \mathbb{R}^2 \rightarrow \mathbb{T}^2$ is the canonical projection, be the fixed point of g_0 . Let $v^s, v^u \in \mathbb{R}^2$ be the stable and unstable eigenvector of A respectively and λ_s, λ_u the corresponding eigenvalues. Let $\mathcal{F}_{\text{lin}}^s, \mathcal{F}_{\text{lin}}^u$ be the stable and unstable foliations respectively relative to g_0 of \mathbb{T}^2 by parallel lines and let $\ell_0 \subset \mathbb{T}^2$ be the unstable leaf of $\mathcal{F}_{\text{lin}}^u$ passing through the saddle fixed point z_0 . Relative to the standard basis, the eigenvectors v^s, v^u have irrational slope, and consequently every leaf of $\mathcal{F}_{\text{lin}}^s$ and $\mathcal{F}_{\text{lin}}^u$ is an isometric immersion into \mathbb{T}^2 of a copy of \mathbb{R} . For future reference, define \mathcal{F}_{hor} to be the foliation of \mathbb{T}^2 by horizontal (relative to the standard basis) simple closed curves, parametrized by $y \in \mathbb{T}^1$, i.e. $\mathcal{F}_{\text{hor}} = \{C_y\}_{y \in \mathbb{T}^1}$ with $C_y \subset \mathbb{T}^2$ the curve of height $y \in \mathbb{T}^1$.

Next we perturb g_0 on a small neighbourhood $U \ni z_0$ of the original saddle fixed point of g_0 to turn it into a repeller and create two additional saddle fixed points z_{-1}, z_1 close to z_0 through a pitch-fork bifurcation, resulting in a DA diffeomorphism. We first describe the perturbation and subsequently we scale it into the neighbourhood U of z_0 . For this, we use the following (see [McS2, Lemmas 1.3 and 1.4]).

LEMMA 17. *Fix $\rho > 0$. Given any $v > \rho$ and any arbitrarily small $\delta > 0$, for each κ with $\rho > \kappa > 0$, there exists a C^∞ function $\lambda_\kappa: \mathbb{R} \rightarrow \mathbb{R}$ satisfying the following conditions.*

⁽²⁾ See for example [PdM, Chapter 4] for these and related constructions.

- (i) $\lambda_\kappa(t) \equiv v$ for $t \leq 0$ and $\lambda_\kappa(t) \equiv \kappa$ for $t \geq 1$.
- (ii) $-\delta \leq t\lambda'_\kappa(t) \leq 0$ for all t and $\lambda'_\kappa(t) < 0$ if $v > \lambda_\kappa(t) > \kappa$. In addition, if $\kappa \geq \kappa'$, then $0 \leq \lambda_\kappa(t) - \lambda_{\kappa'}(t) \leq \kappa - \kappa'$ for all t .
- (iii) There exists $t_0 > 0$, depending only on ρ and δ , where $\lambda_\kappa(t_0) = \rho$ and such that $\lambda_\kappa(t)$ is independent of κ for $t \leq t_0$ and independent of v for $t \geq t_0$.

Further, if we define $\eta(t) = t\lambda(t)$, then $0 \leq (\lambda - \eta')(t) \leq \delta$ and $0 \leq \eta(t) - \kappa t \leq \delta$ for $t \geq 0$.

Fix v with $\lambda_u > v > 1$. Choose ρ with $1 > \rho > \lambda_s$ and $\delta > 0$ small enough so that $1 - \delta > \rho$. Choose \bar{v} satisfying $1 - \delta \geq \bar{v} > \rho$. Let $\lambda, \bar{\lambda}$ be the functions of Lemma 17 relative to the choices of $\kappa = \lambda_s$ and these v and \bar{v} respectively. Denote by $E_1 \oplus E_2$ the splitting of the tangent bundle $T\mathbb{R}^2$ into the unstable and stable direction relative to $A \in \text{SL}(2, \mathbb{Z})$. By a linear change of coordinates B taking $E_1 \oplus E_2$ to \mathbb{R}^2 , the action reads $BAB^{-1}(x, y) = (\lambda_u x, \lambda_s y)$. Let $\chi_0, \chi_1, \chi_2: \mathbb{R} \rightarrow [0, 1]$ be even C^∞ functions satisfying $\chi_0 + \chi_1 + \chi_2 \equiv 1$, $\chi_0(0) = 1$; $\chi_0(t) \equiv 0$ for $|t| \geq 1$; $\chi_2(t) \equiv 0$ for $|t| \leq 1$ and $\chi_2(t) \equiv 1$ for $|t| \geq 2$. We may further assume that $|\chi'_i| \leq 2$ for all i .

Define $G(x, y) = (\lambda_u x, L(x, y)y)$, where $L(x, y) = \chi_0 \lambda(y) + \chi_1(x) \bar{\lambda}(y) + \chi_2 \lambda_s$. The derivative of G has the following form:

$$(3.15) \quad DG_{(x,y)} = \begin{pmatrix} \lambda_u & 0 \\ c(x, y) & N(x, y) \end{pmatrix},$$

where

$$(3.16) \quad c(x, y) = \begin{cases} \chi'_0(x)(\lambda(y) - \bar{\lambda}(y))y, & |x| \leq 1, \\ \chi'_1(x)(\lambda(y) - \lambda_s)y, & 1 \leq |x| \leq 2, \end{cases}$$

and $N(x, y) = L(x, y) + (\chi_0(x)\lambda'(y) + \chi_1(x)\bar{\lambda}'(y))y$. The following estimates hold (cf. [McS2, Lemma 2.2]).

LEMMA 18. *We have $|c(x, y)| \leq 2\delta$ and $\lambda_s \leq |N(x, y)| \leq v$. Moreover, if $|y| \geq t_0$ or $|x| \geq 1$, then $N(x, y) < 1$.*

Proof. Set $t = |y|$ and $\hat{t} = y/|y|$. For $|x| \leq 1$, we have $c(x, y) = \chi'_0(x)(\lambda(t)t - \bar{\lambda}(t)t)\hat{t}$. As $\lambda(t)t - \bar{\lambda}(t)t = \eta(t) - \bar{\eta}(t) \leq \eta(t) - \lambda_s t \leq \delta$ by Lemma 17, it follows that $|c(x, y)| \leq \delta|\chi'_0(x)| \leq 2\delta$ by choice of χ_0 . For $1 \leq |x| \leq 2$, $\lambda(t) - \bar{\lambda}(t)$ is replaced by $\bar{\lambda}(t)t - \lambda_s t = \bar{\eta}(t) - \lambda_s$, which has the same estimate. For $|x| \geq 2$, we find that $c(x, y) = 0$.

To estimate $N(x, y)$, we first estimate $(\chi_0(x)\lambda'(y) + \chi_1(x)\bar{\lambda}'(y))y$. As $|y\lambda'(y)| \leq \delta$ and $|y\bar{\lambda}'(y)| \leq \delta$ by Lemma 17, it follows that $|(\chi_0(x)\lambda'(y) + \chi_1(x)\bar{\lambda}'(y))y| \leq \delta$. As $\lambda(t), \bar{\lambda}(t) \geq \lambda_s$, we have $|N(x, y)| \geq \lambda_s$. For $t \geq t_0$, $\lambda(t) = \bar{\lambda}(t) \leq \rho$, again by Lemma 17. Therefore, $|N(x, y)| \leq \rho + \delta < 1$. If

$|x| \geq 1$, we have $|N(x, y)| \leq \bar{\lambda}(t) + \delta \leq \bar{v} + \delta < 1$. Therefore, $|N(x, y)| < 1$ for $|y| \geq t_0$ or $|x| \geq 1$. Finally, if $t \leq t_0$ and $|x| \leq 1$, then $\lambda(t) \geq \bar{\lambda}(t)$. We have $|N(x, y)| = |\chi_0(\lambda(t) + t\lambda'(t)) + \chi_1(\bar{\lambda}(t) + t\bar{\lambda}'(t))|$. As $\eta'(t) \leq \lambda(t)$ and $\bar{\eta}'(t) \leq \bar{\lambda}(t) \leq \lambda(t)$ by Lemma 17, and as $\eta'(t) = \lambda(t) + t\lambda'(t)$ and $\bar{\eta}'(t) = \bar{\lambda}(t) + t\bar{\lambda}'(t)$ by definition, it follows that $|N(x, y)| = |\chi_0\eta'(t) + \chi_1\bar{\eta}'(t)| \leq \lambda(t) \leq v$ for all t . This finishes the proof. ■

In particular, the map $G: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a C^∞ diffeomorphism agreeing with BAB^{-1} outside the square $\{(x, y) \mid |x| \leq 2, |y| \leq 1\}$ and, as $N(0, 0) = v > 1$, the origin is turned into a repeller. We can now rescale the perturbation so that the support is contained in $U \ni z_0$, the fixed point of the original linear g_0 , and we obtain a perturbed system $g: \mathbb{T}^2 \rightarrow \mathbb{T}^2$. Since rescaling does not affect estimates on first derivatives, relative to the splitting $E_1 \oplus E_2$, the derivative of g has the form

$$(3.17) \quad Dg_z = \begin{pmatrix} \lambda_u & 0 \\ \tilde{c}(z) & \tilde{N}(z) \end{pmatrix},$$

where $|\tilde{c}(z)| \leq 2\delta$ and $\lambda_u \leq |\tilde{N}(z)| \leq v$ for all $z \in \mathbb{T}^2$. With g defined as such, Dg leaves the line bundle E_2 invariant and, although it is not a contraction on E_2 , it expands in this direction by no more than $v < \lambda_u$. Set $E^{\text{ps}} := E_2$. Even though E_1 is no longer invariant, there is a new invariant line bundle E^u everywhere close to E_1 which Dg expands uniformly by λ_u in a suitable metric and thus E^u is a strong unstable direction. To see this, let $s: \mathcal{L}(E_1, E_2) \rightarrow \mathbb{T}^2$ be the standard vector bundle over \mathbb{T}^2 with fibres $s^{-1}(z) \cong \text{Lin}(E_1, E_2)$, where $\text{Lin}(E_1, E_2)$ is the space of linear maps from E_1 to E_2 . A continuous section of $\mathcal{L}(E_1, E_2)$ corresponds to a continuous line field on \mathbb{T}^2 . If $S_z \in \text{Lin}(E_1(z), E_2(z))$, then the induced action of Dg_z is given by $S_z \mapsto \tilde{c}(z)\lambda_u^{-1} + \tilde{N}(z)S_z\lambda_u^{-1} \in \text{Lin}(E_1(g(z)), E_2(g(z)))$. As $|\tilde{N}(z)\lambda_u^{-1}| \leq v\lambda_u^{-1} < 1$, Dg contracts the section, hence there is a unique invariant section $\mathcal{S} \in \mathcal{L}(E_1, E_2)$. If we define E^u to be the corresponding invariant line field for Dg , then $E^u \oplus E^{\text{ps}}$ is the continuous invariant splitting under Dg of \mathbb{T}^2 . As the image of the zero section is bounded in norm by

$$\sup_{z \in \mathbb{T}^2} \tilde{c}(z)\lambda_u^{-1} \leq 2\delta\lambda_u^{-1},$$

Dg takes the disk of radius $r := 2\delta(\lambda_u - v)^{-1}$ into itself, which implies $E^u(z)$ is contained in the cone $\{(u, v) \in T_z\mathbb{T}^2 \mid |v| \leq r|u|\}$ for each $z \in \mathbb{T}^2$. Choose $\delta > 0$ so that $r \leq 1$. Defining an equivalent metric $\|(u, v)\|$ on \mathbb{T}^2 by declaring that $\|(u, v)\| = \max\{|u|, |v|\}$ for $(u, v) \in E_1 \oplus E_2$, we have $\|(u, v)\| = |u|$ for $(u, v) \in E^u$. As E^u is invariant, this implies that $Dg_z(u, v) \in E^u$ and thus $\|Dg_z(u, v)\| = |\lambda_u u| = \lambda_u|u| = \lambda_u\|(u, v)\|$. Therefore, under this metric, Dg expands E^u by λ_u . Thus, all of \mathbb{T}^2 is a pseudo-hyperbolic set for g with strong unstable direction E^u .

The classical theory of (un)stable manifolds now gives that, through each point $z \in \mathbb{T}^2$, there is a unique strong unstable manifold, which is a C^∞ immersed copy of \mathbb{R} . As two such manifolds are either disjoint or equal, the union of these leaves foliates the torus \mathbb{T}^2 and this foliation, denoted $\mathcal{F}_{\text{DA}}^u$, is continuous. If r is sufficiently small, implying that δ is sufficiently small, $\mathcal{F}_{\text{DA}}^u$ will be everywhere transverse to the leaves of the foliation \mathcal{F}_{hor} . Let us summarize this.

LEMMA 19. *The diffeomorphism $g: \mathbb{T}^2 \rightarrow \mathbb{T}^2$ admits a g -invariant strong unstable lamination (i.e. a C^0 foliation by smooth leaves) $\mathcal{F}_{\text{DA}}^u$ whose leaves are everywhere nearly parallel to the linear unstable foliation $\mathcal{F}_{\text{lin}}^u$ and hence everywhere transversal to the horizontal foliation \mathcal{F}_{hor} of \mathbb{T}^2 .*

Recall that $z_0 \in \mathbb{T}^2$ is the repelling fixed point of g . Let K be the set of points at which g does not contract E_2 . By Lemma 18, K is contained in the rectangle described in local coordinates by $|x| \leq 1$ and $|y| \leq t_0$. Let $\{z_{-1}, z_1\}$ be the two intersection points of ∂K with the local pseudo-stable leaf of $\mathcal{F}_{\text{DA}}^{\text{ps}}$ passing through z_0 , which in local coordinates corresponds to $\{(0, y) \mid \lambda(y) = 1\}$. Since λ is strictly decreasing, there is a unique r_0 such that $\lambda(r_0) = 1$. Let $J \subset \mathbb{T}^2$ be the open arc contained in the pseudo-stable leaf of $\mathcal{F}_{\text{DA}}^{\text{ps}}$ passing through z_0 whose endpoints coincide with z_{-1} and z_1 . As $\lambda(r_0) = 1$, the points z_{-1}, z_1 are hyperbolic fixed points for g . Let $\Sigma \subset \mathbb{T}^2$ be the basin of repulsion of z_0 . Clearly, Σ is the domain obtained by taking the union of all unstable leaves of $\mathcal{F}_{\text{DA}}^u$ passing through points $z \in J$, and Σ is bounded by the two unstable leaves $W_{\pm 1}^u \in \mathcal{F}_{\text{DA}}^u$ passing through z_{-1} , and z_1 respectively. Further note that Σ is g -invariant.

In what follows, let $\{I_y^i\}_{i \in \mathbb{Z}}$ be the collection of connected components of $C_y \cap \Sigma$. The following holds (see [McS2, Thm 2.4] for a detailed proof).

LEMMA 20. *If we denote by $h_y, h'_y: C_y \rightarrow C_y$ the holonomy homeomorphisms, i.e. the return maps to C_y along the unstable foliation, of the linear and perturbed systems respectively, then for every $y \in \mathbb{T}^1$, there exists a semiconjugacy $\pi_y: C_y \rightarrow C_y$ between the perturbed and the linear systems, i.e. a continuous π_y such that $\pi_y \circ h'_y = h_y \circ \pi_y$. Further, for every $y \in \mathbb{T}^1$, $\Sigma \cap C_y$ is dense in C_y and $h'_y(\bar{I}_y^i) \cap \bar{I}_y^i = \emptyset$ for every $i \neq 0$, where \bar{I} denotes the closure $\text{Cl}(I)$ of the interval I in C_y .*

Let us further observe the following.

LEMMA 21. *The domain Σ is an unbounded disk which is dense in \mathbb{T}^2 .*

Proof. By Lemma 20, $\Sigma \cap C_y$ is dense in C_y for every $y \in \mathbb{T}^2$, thus $\Sigma \subset \mathbb{T}^2$ is dense in \mathbb{T}^2 as $\bigcup_{y \in \mathbb{T}^1} C_y = \mathbb{T}^2$. To show that Σ is an unbounded disk, we first recall the following. By [McS2, Thm. 2.4], g is semiconjugate to g_0 , through a semiconjugacy which is homotopic to the identity, taking unstable leaves of $\mathcal{F}_{\text{DA}}^u$ injectively onto unstable leaves of $\mathcal{F}_{\text{lin}}^u$. Consequently, the lifts

of the leaves of $\mathcal{F}_{\text{DA}}^u$ are homotopic (and thus uniformly close), through the semiconjugacy between g and g_0 , to the corresponding lifts of the leaves of $\mathcal{F}_{\text{lin}}^u$. As the latter leaves are clearly unbounded in the lift, so are the lifts of the leaves of $\mathcal{F}_{\text{DA}}^u$ and, in particular, $\tilde{\Sigma}$ is unbounded, where $\tilde{\Sigma}$ is a lift of Σ to the cover \mathbb{R}^2 . Further, as $\tilde{\Sigma}$ is foliated by leaves homeomorphic to \mathbb{R} passing through the cross-section $\tilde{J} \subset \tilde{\Sigma}$, where \tilde{J} is a lift of the open interval $J \subset \Sigma$ as defined above, $\tilde{\Sigma}$ is homeomorphic to $(0, 1) \times \mathbb{R}$ and thus simply connected.

To show that Σ is a disk, we need to show that Σ contains no essential simple closed curves. To derive a contradiction, suppose there exists an essential simple closed curve $\gamma \subset \Sigma$. As γ lifts to a curve $\tilde{\gamma}$ which is homotopic to a curve of type (p, q) , where $p, q \in \mathbb{Z}$, $\tilde{\gamma}$ is homotopic, and thus uniformly close, to a straight line with rational slope. However, as the unstable leaves $W_{\pm 1}^u$ have lifts that are homotopic to a leaf of $\mathcal{F}_{\text{lin}}^u$ whose slope is irrational, this implies that $\tilde{\gamma}$ has to intersect a lift of W_1^u , implying that $\gamma \cap W_1^u \neq \emptyset$, a contradiction. Thus Σ is an unbounded disk indeed. ■

Let us now proceed to the construction of our example.

Proof of Example 2. As $C_y \cap \Sigma$ is a dense union of disjoint intervals, we find that $\mathcal{Q}_y = C_y \setminus (C_y \cap \Sigma)$, being the circle minus a dense union of disjoint intervals, is a Cantor set. Note that the endpoints $\mathcal{Q}_{y,\text{rat}}$ of \mathcal{Q}_y are exactly the set of points $C_y \cap \partial\Sigma$ and that $\partial\Sigma = \bigcup_{y \in \mathbb{T}^1} \mathcal{Q}_{y,\text{rat}}$. The complement in \mathcal{Q}_y of these endpoints is the irrational points, i.e. $\mathcal{Q}_{y,\text{irr}} = \mathcal{Q}_y \setminus \mathcal{Q}_{y,\text{rat}}$. Further, as the slope of v^u is irrational, there exists a suitable $\nu \in \mathbb{R}$ such that $\nu v^u = (\alpha, \beta)$ with $1, \alpha, \beta$ rationally independent. For example, if we take $g_0: \mathbb{T}^2 \rightarrow \mathbb{T}^2$ to be Arnold's cat map, where $v^u = (1, \frac{1+\sqrt{5}}{2})$, we may choose $\nu = e$ so that (α, β) is an irrational vector, as $1, \sqrt{5}, e$ are rationally independent. Let $\tau := \tau_{\alpha, \beta}$ be the corresponding irrational translation of the torus where, by construction, $\tau(\ell_0) = \ell_0$.

Choose compatible orientations on the foliations $\mathcal{F}_{\text{DA}}^u$ and $\mathcal{F}_{\text{lin}}^u$. Given $y \in \mathbb{T}^1$, let $h_y, h'_y: C_y \rightarrow C_y$ be the holonomy homeomorphisms, i.e. the return maps to C_y of the unstable foliations of the linear and the perturbed systems respectively. For every $y \in \mathbb{T}^1$, there exists a semiconjugacy $\pi_y: C_y \rightarrow C_y$ such that $\pi_y \circ h'_y = h_y \circ \pi_y$, by Lemma 20. As the continuous foliations $\mathcal{F}_{\text{DA}}^u$ and $\mathcal{F}_{\text{lin}}^u$ are everywhere transversal to the horizontal foliation, the holonomy homeomorphisms h'_y , and consequently the maps π_y , depend continuously on $y \in \mathbb{T}^1$. In other words, as $\mathbb{T}^2 = \bigcup_{y \in \mathbb{T}^1} C_y$, this defines a continuous map

$$(3.18) \quad \pi: \mathbb{T}^2 \rightarrow \mathbb{T}^2, \quad \pi(x, y) = (\pi_y(x), y),$$

that has the property that $\pi(\Sigma \cup W_{-1}^u \cup W_1^u) = \ell_0$, as $\pi_y(\bar{I}_y^i) \in C_y \cap \ell_0$ by Lemma 18 for every $i \in \mathbb{Z}$ and $y \in \mathbb{T}^1$. For every $y \in \mathbb{T}^1$, there exists a

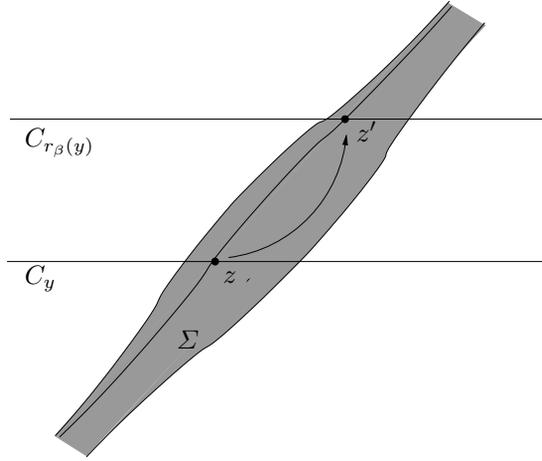


Fig. 4. f_y maps the point $z \in C_y$ along a leaf of the foliation $\mathcal{F}_{\text{DA}}^u$ to a point $z' = f_y(z) \in C_{r_\beta(y)}$.

homeomorphism $f_y: C_y \rightarrow C_{r_\beta(y)}$, defined by mapping $z \in C_y$ to the first intersection point of the unique leaf of $\mathcal{F}_{\text{DA}}^u$ through z with $C_{r_\beta(y)}$ (along the positive direction of the leaf), where $r_\beta: \mathbb{T}^1 \rightarrow \mathbb{T}^1$ is the irrational rotation with rotation number $\rho(r) = \beta \bmod \mathbb{Z}$ (see Figure 4).

As $\mathcal{F}_{\text{DA}}^u$ is a foliation which is transversal to $\mathcal{F}_{\text{hor}}^u$, f_y is one-to-one. Further, as $\mathcal{F}_{\text{DA}}^u$ is continuous, f_y is continuous as well and thus a homeomorphism, for every $y \in \mathbb{T}^1$. Further, it follows from the definitions that $\pi_{r_\beta(y)} \circ f_y = \tau \circ \pi_y$. Define

$$(3.19) \quad f: \mathbb{T}^2 \rightarrow \mathbb{T}^2, \quad f(x, y) = (f_y(x), r_\beta(y)).$$

As the maps f_y are homeomorphisms for every $y \in \mathbb{T}^1$ and depend continuously on $y \in \mathbb{T}^1$ (by virtue again of the foliation $\mathcal{F}_{\text{DA}}^u$ being transversal and continuous), it follows that f as defined by (3.19) is a homeomorphism. It is clear that this f is isotopic to the identity, π is homotopic to the identity and, by construction, $\pi \circ f = \tau \circ \pi$ and $f(\Sigma) = \Sigma$ with Σ the unbounded disk (by Lemma 21) bounded by the leaves $W_{\pm 1}^u \in \mathcal{F}_{\text{DA}}^u$, as $\tau(\ell_0) = \ell_0$. Let \mathcal{M} be the minimal set of f . As π is one-to-one except on $\Sigma \cup W_{-1}^u \cup W_1^u$ (again by Lemma 18), it follows that

$$(3.20) \quad \mathcal{R}_\pi = \pi^{-1}(\mathbb{T}^2 \setminus \ell_0) = \mathbb{T}^2 \setminus (\Sigma \cup W_{-1}^u \cup W_1^u) = \bigcup_{y \in \mathbb{T}^1} \mathcal{Q}_{y, \text{irr}}.$$

As $\mathcal{Q}_y = \text{Cl}(\mathcal{Q}_{y, \text{irr}})$, combining (3.20) with Lemma 14, we conclude that

$$(3.21) \quad \mathcal{M} = \text{Cl} \left(\bigcup_{y \in \mathbb{T}^1} \mathcal{Q}_{y, \text{irr}} \right) = \bigcup_{y \in \mathbb{T}^1} \text{Cl}(\mathcal{Q}_{y, \text{irr}}) = \bigcup_{y \in \mathbb{T}^1} \mathcal{Q}_y = \mathbb{T}^2 \setminus \Sigma,$$

where the second equality holds as the Cantor sets \mathcal{Q}_y (and therefore their irrational parts $\mathcal{Q}_{y,\text{irr}}$) depend continuously on $y \in \mathbb{T}^1$. Thus $\mathcal{M} = \mathbb{T}^2 \setminus \Sigma$, with Σ an unbounded and f -invariant disk, and $f \in \text{Homeo}_{\#}(\mathbb{T}^2)$, as required. This finishes the proof. ■

3.4. Examples of type II. Let us next give examples of homeomorphisms for which the connected components $\{\Sigma_k\}$ of the complement of \mathcal{M} are essential annuli and disks.

EXAMPLE 3 (Type II: essential annuli and disks). There exist $f \in \text{Homeo}_*(\mathbb{T}^2)$ with minimal sets of the form $\mathcal{M} = \mathbb{T}^2 \setminus \bigcup_{n \in \mathbb{Z}} \Sigma_k$ with $\{\Sigma_k\}_{k \in \mathbb{Z}}$ a collection of essential annuli and disks. Furthermore, the essential annuli can be constructed to have any characteristic (p, q) , where $\gcd(p, q) = 1$.

The proof of Example 3 uses the following.

LEMMA 22. *Let $f \in \text{Homeo}_*(\mathbb{T}^2)$ and $f' = L_A^{-1} \circ f \circ L_A$ with $L_A: \mathbb{T}^2 \rightarrow \mathbb{T}^2$ induced by some $A \in \text{SL}(2, \mathbb{Z})$. Then $f' \in \text{Homeo}_*(\mathbb{T}^2)$.*

Proof. Let L_A be the linear conjugation induced by $A \in \text{SL}(2, \mathbb{Z})$. As $f \in \text{Homeo}_0(\mathbb{T}^2)$, $f' \in \text{Homeo}_0(\mathbb{T}^2)$ as well. Let F, F' be lifts of f, f' respectively. By [KK, Lemma 2.4], we have

$$\rho(L_A^{-1} \circ F \circ L_A) = L_A^{-1} \rho(F) \bmod \mathbb{Z}^2.$$

Therefore, $\rho(f') = (\alpha', \beta') \bmod \mathbb{Z}^2$, where

$$\alpha' = a\alpha + b\beta \quad \text{and} \quad \beta' = c\alpha + d\beta$$

with $a, b, c, d \in \mathbb{Z}$. The condition $N_1 + N_2\alpha' + N_3\beta' = 0$ implies that $N_1 = N_2a + N_3c = N_2b + N_3d = 0$, as $1, \alpha, \beta$ are rationally independent. Multiplying $N_2a + N_3c$ by b and $N_2b + N_3d$ by a and subtracting yields $N_2(ad - bc) = 0$, which yields $N_2 = 0$, as $A \in \text{SL}(2, \mathbb{Z})$ and thus $ad - bc = 1$. Similarly, $N_3 = 0$ and it thus follows that $1, \alpha', \beta'$ are rationally independent as well. Therefore, $f' \in \text{Homeo}_*(\mathbb{T}^2)$. ■

Proof of Example 3. First, we construct a homeomorphism $f \in \text{Homeo}_{\#}(\mathbb{T}^2)$ for which the complement of \mathcal{M} is a collection of essential annuli of a given characteristic. Let (p, q) with $\gcd(p, q) = 1$ be given. Let $f \in \text{Homeo}_{\#}(\mathbb{T}^2)$ be the product of a Denjoy counterexample $\varphi \in \text{Homeo}(\mathbb{T}^1)$ with rotation number $\alpha \notin \mathbb{Q}$ and an irrational rotation r_β with $\beta \notin \mathbb{Q}$ chosen so that $1, \alpha, \beta$ are rationally independent. The corresponding semi-conjugacy π is of the form $\pi = (\pi_1, \text{Id})$, where π_1 is the semiconjugacy of φ to the irrational rotation r_α . As $\mathcal{R}_\pi = \mathcal{Q}_{\text{irr}} \times \mathbb{T}^1$, the minimal set \mathcal{M} of f is $\mathcal{M} = \text{Cl}(\mathcal{Q}_{\text{irr}} \times \mathbb{T}^1) = \mathcal{Q} \times \mathbb{T}^1$, where $\mathcal{Q} \subset \mathbb{T}^1$ is the Cantor minimal set of φ . The characteristic of the corresponding essential annuli is $(0, 1)$. For later reference, denote by $\{\Sigma_t^a\}$ the collection of essential annuli in the complement of \mathcal{M} . Given any pair $(p, q) \in \mathbb{Z}^2$ such that $\gcd(p, q) = 1$, there exists

an element $A \in \mathrm{SL}(2, \mathbb{Z})$ such that the (linear) $L_A \in \mathrm{Homeo}(\mathbb{T}^2)$ induced by A has the property that an essential simple closed curve of characteristic $(0, 1)$ is mapped to an essential simple closed curve of characteristic (p, q) . By Lemma 22, conjugating f with L_A gives a homeomorphism $f' \in \mathrm{Homeo}_*(\mathbb{T}^2)$ and the components of the complement of the minimal set $\mathcal{M}' = L_A^{-1}(\mathcal{M})$ are now essential annuli of characteristic (p, q) .

To obtain an example of a homeomorphism $f' \in \mathrm{Homeo}_*(\mathbb{T}^2)$ with a minimal set \mathcal{M}' for which the complementary components include both essential annuli and disks, we modify the above example as follows. Let f again be the product homeomorphism given above. Choose $z_0 \in \mathcal{R}_\pi$ and blow up the orbit $\mathcal{O}_f(z_0)$ to disks by the procedure of Section 3.2. This gives a homeomorphism $f' \in \mathrm{Homeo}_\#(\mathbb{T}^2)$ and a continuous map $\phi: \mathbb{T}^2 \rightarrow \mathbb{T}^2$ such that $\phi \circ f' = f \circ \phi$, and we define $\pi' = \pi \circ \phi$. We have $\phi^{-1}(f^k(z_0)) = \mathrm{Cl}(\Sigma_k)$, where Σ_k is the interior of the closed disk $\mathrm{Cl}(\Sigma_k)$ and $\gamma_k = \partial\Sigma_k$ is a simple closed curve. Denote by \mathcal{M}' the corresponding minimal set of f' .

In order to show that the complement of \mathcal{M}' consists of essential annuli and disks, it suffices to show that $\gamma_k \subset \mathcal{M}'$. Indeed, as ϕ is one-to-one on $\mathbb{T}^2 \setminus \bigcup_{k \in \mathbb{Z}} \mathrm{Cl}(\Sigma_k)$, $\phi^{-1}(\Sigma_t^a)$ is again an essential annulus for every $t \in \mathbb{Z}$, where $\phi^{-1}(\Sigma_t^a) \cap \mathcal{M}' = \emptyset$. Thus to show that $\gamma_k \subset \mathcal{M}'$ for every $k \in \mathbb{Z}$, by Lemma 16(2) it suffices to show that for every $0 < r \leq 1$ and every $\theta_1, \theta_2 \in [0, 2\pi)$ with $0 < |\theta_1 - \theta_2| < \pi$, we have $\mathcal{W}(r, \theta_1, \theta_2) \cap (\mathcal{R}_\pi \setminus \mathcal{O}_f(z_0)) \neq \emptyset$, where we recall that the wedge $\mathcal{W}(r, \theta_1, \theta_2) \subset B_{r\delta_0} \setminus \{z_0\}$ is the region bounded by two leaves $\rho_{\theta_1}, \rho_{\theta_2} \in \mathcal{F}_0$, where $0 < |\theta_1 - \theta_2| < \pi$ and $0 < r \leq 1$ (see again Section 3.2).

This is proved as follows: as $z_0 \in \mathcal{R}_\pi = \mathcal{Q}_{\mathrm{irr}} \times \mathbb{T}^1$, through every wedge $\mathcal{W}(r, \theta_1, \theta_2)$ pass infinitely many vertical simple closed curves (i.e. the connected components of $\mathcal{Q}_{\mathrm{irr}} \times \mathbb{T}^1$), arbitrarily close to z_0 . As only countably many of these points are deleted from these curves, every wedge $\mathcal{W}(r, \theta_1, \theta_2)$ contains points of $\mathcal{R}_\pi \setminus \mathcal{O}_f(z_0)$, for any $r > 0$. Therefore, $\gamma_k \subset \mathcal{M}'$ for every $k \in \mathbb{Z}$ indeed, where $\mathcal{M}' = \mathrm{Cl}(\mathcal{R}_{\pi'})$ by Lemma 16(1). This finishes the proof. ■

3.5. Examples of type III. The simplest example of an extension of a Cantor set is of course a Cantor set itself. A homeomorphism $f \in \mathrm{Homeo}_*(\mathbb{T}^2)$ admitting such a Cantor minimal set is obtained by taking the product of two Denjoy counterexamples with rationally independent rotation numbers. Its minimal set is the product of the Cantor minimal sets of its factors, and thus itself a Cantor set. Recall that an extension of a Cantor set \mathcal{M} is said to be *non-trivial* if not all connected components of \mathcal{M} are singletons. Below we give examples of non-trivial extensions of a Cantor set. Recently, F. Béguin, S. Crovisier, T. Jäger and F. Le Roux in [Be, Thm. 1.2] also constructed an example of a homeomorphism $f \in \mathrm{Homeo}_*(\mathbb{T}^2)$ for which

the minimal set \mathcal{M} is a non-trivial extension of a Cantor set (in our terminology). This homeomorphism is constructed by adapting a quasiperiodically forced circle homeomorphism (see [Be, Thm. 1.2]; cf. Counterexample 2) with a Cantor minimal set, and blowing up an orbit of points to arcs, using a construction due to M. Rees [R1, R2]. The minimal set thus constructed has a countable number of arcs among its connected components. The examples below show that in our class $\text{Homeo}_*(\mathbb{T}^2)$ there exist minimal sets which are non-trivial extensions of Cantor sets and which have separating connected components among its connected components.

EXAMPLE 4 (Type III: extensions of Cantor sets). There exist $f \in \text{Homeo}_*(\mathbb{T}^2)$ for which \mathcal{M} is a non-trivial extension of a Cantor set whose non-degenerate components are simple closed curves.

REMARK 1. In a way similar to [B1, Thm. 3] (cf. Example 1), by defining a suitable family of arcs in the disks enclosed by the non-degenerate components of the above minimal set, if we pass to the quotient by collapsing these arcs to points, this gives new quotient homeomorphisms of type III for which the corresponding minimal set is again a non-trivial extension of a Cantor set, possible connected components of which include flowers and dendrites.

The proof of the above example needs two further lemmas. Let $\mathcal{Q} \subset \mathbb{T}^1$ be a Cantor set and denote by $\{I_k\}_{k \in \mathbb{Z}}$ the collection of connected components of $\mathbb{T}^1 \setminus \mathcal{Q}$. In what follows, we denote by $|I|$ the length (relative to the Haar measure on the circle) of an interval $I \subset \mathbb{T}^1$, and by \tilde{d}_1 the Euclidean metric on \mathbb{R} .

LEMMA 23. *There exist Cantor sets $\mathcal{Q} \subset \mathbb{T}^1$ with the following property: there exists a point $x_0 \in \mathcal{Q}_{\text{irr}}$, an interval $J \subset \mathbb{T}^1$ with $\partial J \subset \mathcal{Q}_{\text{irr}}$ and x_0 as midpoint of J , and a constant $C > 0$, such that for every interval $I_k \subset J \setminus \mathcal{Q}$, $k \geq 1$, we have $|I_k| \leq C(d(x_0, x_k))^2$, where x_k is the midpoint of the interval I_k .*

Proof. First, let $[-1, 1] \subset \mathbb{R}$ with midpoint $0 \in [-1, 1]$. Inductively delete intervals from $[-1, 1]$: at each step $t \geq 1$, choose a point $x_t \in (-1, 1) \setminus \bigcup_{s=0}^{t-1} I'_s$ with $x_t \neq 0$, and delete an interval $I'_t \subset (-1, 1) \setminus \bigcup_{s=0}^{t-1} I'_s$, centred at x_t and not overlapping 0, of length at most $(\tilde{d}_1(0, x_t))^2$. Repeating this ad infinitum produces a Cantor set $\mathcal{Q}' \subset [-1, 1]$. Given a Cantor set $\mathcal{Q} \subset \mathbb{T}^1$, take a small (closed) interval $J \subset \mathbb{T}^1$ for which $\partial J \subset \mathcal{Q}_{\text{irr}}$. Replacing $J \cap \mathcal{Q} \subset \mathbb{T}^1$ with a rescaled copy of \mathcal{Q}' into J yields the desired Cantor set in \mathbb{T}^1 , with $C = |J|/2$. ■

LEMMA 24. *Let $f \in \text{Homeo}_{\#}(\mathbb{T}^2)$ be the product of two Denjoy counterexamples $\varphi, \psi \in \text{Homeo}(\mathbb{T}^1)$, semiconjugate to an irrational translation τ*

through π . Let $\mathcal{Q}_1, \mathcal{Q}_2 \subset \mathbb{T}^1$ be two Cantor minimal sets of φ and ψ respectively, with points $x_0 \in \mathcal{Q}_{1,\text{irr}}$ and $y_0 \in \mathcal{Q}_{2,\text{irr}}$ and corresponding intervals J_1 and J_2 satisfying the conditions of Lemma 23. Set $z_0 := (x_0, y_0) \in \mathcal{R}_\pi$. For every $0 < r \leq 1$ and $\theta_1, \theta_2 \in [0, 2\pi)$ with $0 < |\theta_1 - \theta_2| < \pi$, we have $\mathcal{W}(r, \theta_1, \theta_2) \cap (\mathcal{R}_\pi \setminus \mathcal{O}_f(z_0)) \neq \emptyset$.

Proof. First, we observe that $\mathcal{R}_\pi = \mathcal{Q}_{1,\text{irr}} \times \mathcal{Q}_{2,\text{irr}}$, so the minimal set of f is $\mathcal{M} = \text{Cl}(\mathcal{Q}_{1,\text{irr}} \times \mathcal{Q}_{2,\text{irr}}) = \mathcal{Q}_1 \times \mathcal{Q}_2$, the product of the Cantor sets of the factors φ and ψ . Let $B_{\delta_0} \subset \mathbb{T}^2$ be the closed embedded disk centred at z_0 , where δ_0 is small enough so that the disk is contained in the rectangle of width $|J_1|$ and height $|J_2|$ centred at z_0 .

Relative to the disk B_{δ_0} , let $\mathcal{W}(r, \theta_1, \theta_2)$ be any wedge, denoted \mathcal{W} for brevity from now on. Let $0 < \nu < \pi$, where $\nu = |\theta_1 - \theta_2|$ is the angle between the two rays $\rho_{\theta_1}, \rho_{\theta_2}$ that bound the wedge, and define $\bar{\rho} := \rho_{(\theta_1 + \theta_2)/2}$, the bisector of the two rays. Further, let $\nu' \in [0, 2\pi)$ be the angle between $\bar{\rho}$ and the positive horizontal line through z_0 . As the vertical and horizontal lines through z_0 contain points of $\mathcal{R}_\pi \setminus \mathcal{O}_f(z_0)$ arbitrarily close to z_0 , by symmetry we may assume that $0 < \nu' < \pi/2$. Given a point $w = (x, y) \in \bar{\rho} \cap \mathcal{W}$, let $\ell_h(w), \ell_v(w)$ be the horizontal and vertical straight lines through w respectively and define the intercepts $\ell'_h(w) = \ell_h(w) \cap \mathcal{W}$ and $\ell'_v(w) = \ell_v(w) \cap \mathcal{W}$, which for w sufficiently close to z_0 only pass through ρ_{θ_1} and ρ_{θ_2} (and not through the circular arc that cuts off the wedge).

There exist constants $K_h, K_v > 0$, depending only on θ_1 and θ_2 , such that $\ell'_h(w) = K_h d(y, y_0)$ and $\ell'_v(w) = K_v d(x, x_0)$. Given any $0 < r \leq 1$, choose $w \in \bar{\rho}$ such that $d(w, z_0) < \delta_0 r$. As the lengths $\ell'_h(w), \ell'_v(w)$ behave linearly, and the lengths $|I_{1,k}| \leq C_1 (d(x_0, x_k))^2$ and $|I_{2,t}| \leq C_2 (d(y_0, y_t))^2$ behave quadratically, with $C_1, C_2 > 0$ uniform constants, if we choose w sufficiently close to z_0 , then $w \in R := I_{1,k} \times I_{2,t}$ for some $k, t \in \mathbb{Z}$, with $\text{Cl}(R)$ properly contained in \mathcal{W} . The cornerpoints of the rectangle R are limit points of \mathcal{R}_π , and thus also of $\mathcal{R}_\pi \setminus \mathcal{O}_f(z_0)$, as a neighbourhood of any point $z \in \mathcal{R}_\pi$ contains uncountably many points and only a countable orbit is deleted. Therefore, as $\text{Cl}(R) \subset \mathcal{W}$, we see that $\mathcal{W} \cap (\mathcal{R}_\pi \setminus \mathcal{O}_f(z_0)) \neq \emptyset$, as required. ■

Proof of Example 4. We start with $f \in \text{Homeo}_\#(\mathbb{T}^2)$ which is the product of two Denjoy counterexamples $\varphi, \psi \in \text{Homeo}(\mathbb{T}^1)$, with Cantor minimal sets $\mathcal{M}_1 = \mathcal{Q}_1$ and $\mathcal{M}_2 = \mathcal{Q}_2$ respectively, semiconjugate to an irrational translation τ through π . As every Cantor set in \mathbb{T}^1 can be realized as a minimal set of a Denjoy counterexample, we can choose φ and ψ such that the Cantor sets \mathcal{Q}_i with $i = 1, 2$, with $x_0 \in \mathcal{Q}_{1,\text{irr}}$ and $y_0 \in \mathcal{Q}_{2,\text{irr}}$ and corresponding intervals J_1 and J_2 , satisfy the conditions of Lemma 23. Let $z_0 = (x_0, y_0) \in \mathcal{R}_\pi$ and let $B_{\delta_0} \subset \mathbb{T}^2$ be the closed embedded Euclidean disk with radius $\delta_0 > 0$ small enough so that B_{δ_0} is contained in the rectangle of width $|J_1|$ and height $|J_2|$ centred at z_0 . Through the procedure in Sec-

tion 3.2, we blow up the orbit $\mathcal{O}_f(z_0)$ to a collection of disks to obtain a homeomorphism $f' \in \text{Homeo}_*(\mathbb{T}^2)$ and a continuous map $\phi: \mathbb{T}^2 \rightarrow \mathbb{T}^2$ such that $\phi \circ f' = f \circ \phi$ and $\phi^{-1}(f^k(z_0)) = \text{Cl}(\Sigma_k)$ is a closed topological disk, where $\gamma_k = \partial\Sigma_k$ is a simple closed curve, for every $k \in \mathbb{Z}$.

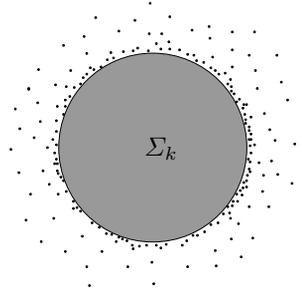


Fig. 5. A non-trivial extension of a Cantor set \mathcal{M}' ; Cantor dust accumulates on the boundary $\gamma_k \subset \mathcal{M}'$ of each disk Σ_k and, conversely, every point of the Cantor dust is a limit point of increasingly small disks Σ_k .

If we denote by Σ the doubly essential component of the complement of the minimal set \mathcal{M} of f , then $\Sigma' := \phi^{-1}(\Sigma) \subset \mathbb{T}^2$ is open as ϕ is continuous. As $\phi: \Sigma' \rightarrow \Sigma$ is one-to-one and continuous, it is a homeomorphism. Therefore, as Σ is doubly essential, take for example the vertical and horizontal essential simple closed curves in Σ ; then these are taken by $\phi^{-1}|_{\Sigma}$ to essential simple closed curves in Σ' with the same homotopy type, as ϕ is homotopic to the identity. Therefore, Σ' is doubly essential indeed and thus \mathcal{M}' is of type III. The other connected components of the complement of \mathcal{M}' are, by construction, the disks Σ_k for $k \in \mathbb{Z}$.

Now, $\phi^{-1}(\mathcal{R}_\pi \setminus \mathcal{O}_f(z_0)) \subset \mathcal{M}'$ are all singletons, which by Lemma 16(2) combined with Lemma 24 accumulate on the boundaries of the disks Σ_k to form the non-trivial connected components γ_k (see Figure 5). Thus, as $\mathcal{M}' = \text{Cl}(\phi^{-1}(\mathcal{R}_\pi \setminus \mathcal{O}_f(z_0)))$ by Lemma 16(1), $\gamma_k \subset \mathcal{M}'$ for every $k \in \mathbb{Z}$, which are the desired non-degenerate components of \mathcal{M}' . ■

REMARK 2. In the proof of Example 4, we explicitly constructed a semi-conjugacy between the extension of the Cantor set \mathcal{M}' and the original Cantor set \mathcal{M} . Theorem A in essence says that all extensions of Cantor sets are of this form.

4. Concluding remarks. Let us conclude this paper with the following remarks. Corollary 1 states that homeomorphisms that admit a type I or II minimal set have in fact a unique minimal set. This poses the natural

QUESTION 1 (Uniqueness of type III minimal sets). Let $f \in \text{Homeo}_*(\mathbb{T}^2)$ with a minimal set \mathcal{M} of type III. Is the minimal set \mathcal{M} unique?

Note that, by Lemma 14, a (possible) counterexample to the above question could not be an element of $\text{Homeo}_{\#}(\mathbb{T}^2)$. Further, it would be interesting to get a completer description of the possible topology of extensions of Cantor sets.

QUESTION 2 (Topology of type III minimal sets). Let $f \in \text{Homeo}_*(\mathbb{T}^2)$ with a minimal set \mathcal{M} of type III.

- (i) Exactly what continua can arise as a connected component of \mathcal{M} ?
- (ii) Can \mathcal{M} have uncountably many non-degenerate connected components? Can all components be non-degenerate?

For example, a classical result by R. Moore [M] implies that not all components of \mathcal{M} can be triodic continua, as the number of connected components of \mathcal{M} is uncountable⁽³⁾. However, it is not entirely clear whether uncountably many components could be for example an arc.

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⁽³⁾ A planar continuum \mathcal{C} is said to be *triodic* if it contains a subcontinuum $\mathcal{C}_0 \subset \mathcal{C}$ such that $\mathcal{C} \setminus \mathcal{C}_0$ has at least three connected components.

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