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Covering spaces and irreducible partitions

by

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Abstract. An irreducible partition of a space is a partition of that space into solid sets with a certain minimality property. Previously, these partitions were studied using the cup product in cohomology. This paper obtains similar results using the fundamental group instead. This allows the use of covering spaces to obtain information about irreducible partitions. This is then used to generalize Knudsen's construction of topological measures on the torus. We give examples of such measures that are invariant under Hamiltonian flows on certain symplectic manifolds.

1. Introduction. Topological measures were first constructed by Johan Aarnes in [1] under the name of quasi-measures. They are generalizations of regular Borel measures which represent functionals on $C(X)$ spaces that are linear on singly generated subalgebras. In a later paper [2], Aarnes gave a general construction theorem for topological measures. However, except in very nice spaces (those with $g(X) = 0$, see below) there are topological difficulties in carrying out the specifics of this construction. In [4], a new construction was discovered for the case where X is the torus. In [3], the current author analyzed the "irreducible partitions" in the construction theorem and related them to aspects of the cohomology ring of the underlying space. It is the goal of this paper to show that the fundamental group of the space can be used instead and to generalize the recent construction given by Knudsen in [4] to this more abstract setting.

We begin with a simple result from algebraic topology.

PROPOSITION 1. *Let X be a connected, locally path connected space. Suppose that $\{U_i\}_{i=1}^n$ are disjoint connected open sets as are $\{V_j\}_{j=1}^m$ such that $X = (\bigcup_{i=1}^n U_i) \cup (\bigcup_{j=1}^m V_j)$. Suppose that each U_i intersects each V_j and that $(\bigcup_{i=1}^n U_i) \cap (\bigcup_{j=1}^m V_j)$ has p components. Then there is a connected covering space $\tilde{X}$ of X so that the group of deck transformations of $\tilde{X}$ is the*

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free group on $q = p - m - n + 1$ generators, FG(q) *and so that each U_i and each V_j is evenly covered.*

Proof. This proof is a generalization of one in [5] for the case $n = m = 1$, $p = 2$. We will construct $\tilde{X}$ by stitching together the sets U_i and V_j. Write $U_i \cap V_j = \bigcup_{k=1}^{p_{ij}} W_{ijk}$ where $p = \sum_{ij} p_{ij}$ and each W_{ijk} is connected. Make a group with generators $\{\gamma_{ijk}\}$ and require relations $\gamma_{i11} = \gamma_{1j1} = 1$ for each i and j. This leaves q generators for the free group FG(q) where $q = p - m - n + 1$.

Let

$$Y = \Big(\bigcup_{i=1}^{n} U_i \times \{i\} \times \{0\} \times \mathrm{FG}(q)\Big) \cup \Big(\bigcup_{j=1}^{m} V_j \times \{j\} \times \{1\} \times \mathrm{FG}(q)\Big).$$

We make identifications $(x, i, 0, w) \equiv (x, j, 1, w\gamma_{ijk})$ when $x \in W_{ijk}$ and $w \in \mathrm{FG}(q)$. Let $\tilde{X}$ denote the resulting quotient space. With the obvious projection, both U_i and V_j are evenly covered.

Clearly, for each $w \in \mathrm{FG}(q)$, the image of

$$\bigcup_i (U_i \times \{i\} \times \{0\} \times \{w\}) \cup \bigcup_j (V_j \times \{j\} \times \{1\} \times \{w\})$$

is connected since U_1 connects to V_j via W_{1j1} and V_1 connects to U_i via W_{i11}. But now, the image of $U_i \times \{i\} \times \{0\} \times \{w\}$ connects to the image of $V_j \times \{j\} \times \{1\} \times \{w\gamma_{ijk}\}$ through W_{ijk}. Hence $\tilde{X}$ is connected. ■

One consequence of this result is that there is a surjective map of $\pi_1(X)$ onto FG(q). If $x_0 \in U_1$, then a loop at x_0 with image γ_{ijk} can be obtained by starting at x_0, moving through U_1 into W_{111}, through V_1 to W_{i11}, through U_i into W_{ijk}, through V_j into W_{1j1}, and through U_1 to x_0. If $i = 1$, then the loop just goes from x_0 to W_{1jk}, through V_j to W_{1j1} and through U_1 to x_0.

We say a subset $A \subseteq X$ is *solid* if both A and $X \setminus A$ are connected. If A is closed and solid, we say that A *has* $\leq n$ *sides* if there are arbitrarily small neighborhoods V of A so that $V \setminus A$ has $\leq n$ components. We let $g(X) + 1$ denote the largest number of sides for a closed solid subset of X. In [3], basic facts about the number $g(X)$ are given when X is compact, connected and locally connected.

Definition 2. Let $n(X)$ denote the largest integer n so that there is a surjective map from $\pi_1(X)$ onto FG(n).

Proposition 3. *For X compact, we have $g(X) \leq n(X)$. In fact, if $\{C_i\}_{i=1}^m$ is a disjoint collection of closed solid sets such that $X \setminus \bigcup_i C_i$ is connected, and if C_i has s_i sides, then $\sum_{i=1}^m s_i \leq m + n(X)$.*

Proof. Choose mutually disjoint solid open sets $U_1, \ldots, U_m$ with $C_i \subseteq U_i$ and so that $U_i \setminus C_i$ have s_i components. Let $V_1 = X \setminus \bigcup_{i=1}^m C_i$. The result then follows from Proposition 1. ■

This result should be compared to Corollary 18 and Proposition 20 of [3]. There, similar results are obtained with $d(X)$ in place of $n(X)$ where $d(X)$ is the maximal rank of an isotropic subgroup of the cohomology group $H^1(X)$. In fact, $n(X) \leq d(X)$ so the proposition above is stronger. To see this, note that a surjection of $\pi_1(X)$ to F_n is induced by a map of X to the wedge of n circles, W_n. The corresponding map of $H^1(W_n)$ to $H^1(X)$ is injective and has isotropic image of rank n.

Corollary 4. *If $\pi_1(X)$ is finite, then $g(X) = 0$, and if $\pi_1(X)$ is abelian, $g(X) \leq 1$. Also, if $g(X) = n(X) = 1$, then any two disjoint closed two-sided sets disconnect X.*

In particular, if X is any topological group, $g(X) \leq 1$.

Now suppose that disjoint closed solid sets C_i are given so that $X \setminus \bigcup_{i=1}^m C_i$ is connected. The U_i in the proof can be chosen so that whenever $C_i \subseteq V \subseteq U_i$ with V open solid, $V \setminus C$ has s_i components. Under this restriction, the covering space obtained does not depend on the specific sets U_i chosen. In fact, shrinking U_i does not change $\tilde{X}$ and any two choices of U_i have a common shrinking. In this way we obtain a well defined covering space $\tilde{X}_{\{C_i\}}$. In the case of just one set C, we denote this space by $\tilde{X}_C$ and the corresponding covering map by p_C.

Recall that a collection of closed solid sets $\{C_i\}_{i=1}^n$ is said to *generate an irreducible partition* if $X \setminus \bigcup_{i=1}^n C_i$ is disconnected while $X \setminus \bigcup_{i\in I} C_i$ is connected for every $I \subsetneq \{1,\ldots,n\}$. In that case, each component of $X \setminus \bigcup_{i=1}^n C_i$ is a solid open set and the collection $\mathcal{P}$ consisting of $\{C_i\}$ together with all such components $\{U_j\}$ is said to be an irreducible partiton of X. The *trace* of this partition, $\operatorname{tr}(\mathcal{P})$, is the sum $\sum s_i$ where s_i is the number of sides of C_i. We have the following proposition concerning the trace.

Proposition 5. *Let $\mathcal{P} = \{C_i\}_{i=1}^n \cup \{U_j\}_{j=1}^m$ be an irreducible partition of X. Then $\operatorname{tr}(\mathcal{P}) \leq n(X) + m + n - 1$.*

Proof. By a result from [3], there are open solid sets $\{V_i\}_{i=1}^n$ so that $C_i \subseteq V_i$ and $V_i \setminus C_i$ has exactly s_i components. Furthermore, each U_j intersects each V_i. The result now follows from Proposition 1. ∎

In fact, the proposition gives a covering space $\tilde{X}$ which evenly covers each element of the partition $\mathcal{P}$. Once again, this partition does not depend on the sets V_i as long as they are sufficiently small. Thus we define a covering space $\tilde{X}_{\mathcal{P}}$ for each irreducible partition $\mathcal{P}$.

Lemma 6. *Suppose that C and D are closed, solid, two-sided subsets of X such that some neighborhood of D is evenly covered and such that some lift $\tilde{D}$ of D disconnects $\tilde{X}_C$. Then $\tilde{X}_C$ and $\tilde{X}_D$ are equivalent as covering spaces of X.*

Proof. The group of deck transformations of $\tilde{X}_C$ is singly generated. Let $\sigma : \tilde{X}_C \to \tilde{X}_C$ be a generator. Write $\tilde{X}_C \setminus \tilde{D} = U \cup V$ where U and V are disjoint open sets. Since $\tilde{D} \cap \sigma(\tilde{D}) = \emptyset$, we may assume that $\sigma(\tilde{D}) \subseteq U$. Let A be a neighborhood of D which is evenly covered by $\tilde{X}_C$ and let $\tilde{A}$ be the component of $p_C^{-1}(A)$ which contains $\tilde{D}$. We may assume that A is solid, that $A \setminus D$ has exactly two components and that $\sigma(\tilde{A}) \subseteq U$.

Now, $\tilde{X} = \tilde{A} \cup U \cup V$ is connected with U and V disjoint, so $\tilde{A} \cap U \neq \emptyset \neq \tilde{A} \cap V$. Hence, $\tilde{A} \cap U$ and $\tilde{A} \cap V$ must be the two components of $\tilde{A} \setminus \tilde{D}$. Thus, both of these sets are connected. Furthermore, the connectedness of $\tilde{X}$ again shows that both U and V must be connected.

This allows us to write $U = [U \cap \sigma(V)] \cup \sigma(\tilde{D}) \cup \sigma(U)$, a disjoint union. Let $W = U \cap \sigma(V)$. Since $\sigma(A \cap V) \subseteq W$, $W \neq \emptyset$. Furthermore $\sigma^n(W) \cap \sigma^m(W) = \emptyset$ if $n \neq m$, and $\tilde{X}_C = \bigcup_{n=-\infty}^{\infty} \sigma^n(\tilde{D}) \cup \sigma^n(W)$. This shows that p_C takes W homeomorphically to $X \setminus D$ and so p_C evenly covers $X \setminus D$. It follows that $\tilde{X}_C$ and $\tilde{X}_D$ are equivalent as covering spaces. ■

Another worthwhile observation is the following:

LEMMA 7. *Let $p : \tilde{X} \to X$ be a covering map and $U \subseteq X$ an open solid set. Pick $x_0 \in U$. Then U is evenly covered by p if and only if $\pi_1(U, x_0) \subseteq \operatorname{Im} p_*$ where p_* is the induced map between the fundamental groups of $\tilde{X}$ and X.*

This follows since U is evenly covered if and only if p is one to one on each component of $p^{-1}U$. Clearly, if p is one-to-one on some component of U, then $\pi_1(U, x_0) \subseteq \operatorname{Im} p_*$. Conversely, a path between inverse images of x_0 will project to an element of $\pi_1(U, x_0)$ which is not in $\operatorname{Im} p_*$.

2. Construction in the case $g = 1$. Let $\mathcal{A}_s$ denote the collection of solid subsets of X that are either open or closed. A *solid set function* is a map $\mu : \mathcal{A}_s \to [0, 1]$ with the following properties:

(i) If $C_1, \ldots, C_n$ are disjoint closed solid sets, then $\sum \mu(C_i) \leq 1$.
(ii) If U is open and solid, then $\mu(U) = \sup\{\mu(C) : C \subseteq U,\ C \text{ closed, solid}\}$.
(iii) If $\mathcal{P} = \{C_i\}_{i=1}^n \cup \{U_j\}_{j=1}^m$ is an irreducible partition of X, then $\sum \mu(C_i) + \sum \mu(U_j) = \mu(X) = 1$.

It is a fundamental result of Aarnes that every solid set function extends to a topological measure on X. That is, μ can be defined on all open or closed sets to be additive on disjoint sets, regular, and monotone. In practice it is the third condition on solid set functions that is the hardest to deal with. For spaces with $g(X) = 0$, however, this condition reduces to $\mu(C) + \mu(X \setminus C) = 1$ for all closed solid sets. It is this simplification that has allowed a wide variety of topological measures to be constructed in that case. It is the goal of this paper to provide constructions for spaces with $g(X) = 1$. To do this, we quickly review the main construction result from [3]. For convenience, all sets are assumed to be solid unless otherwise mentioned.

Let X be a compact, connected, locally path connected space with $g(X) = 1$ and with the property that whenever C and D are disjoint two-sided sets in X, then $X \setminus (C \cup D)$ is disconnected. For example, if $n(X) = 1$, this is the case. It follows from [3] that there is an equivalence relation on the class of closed two-sided sets, $\mathcal{C}_{s2}$, defined as follows: $C \sim D$ whenever there exist $C = C_0, C_1, \ldots, C_n = D$ in $\mathcal{C}_{s2}$ with $C_k \cap C_{k+1} = \emptyset$ for all $k \le n-1$. We will denote by $\mathcal{E}$ equivalence classes under this relation. Notice also that $C \sim D$ implies that $\tilde{X}_C$ and $\tilde{X}_D$ are equivalent as covering spaces by Lemma 6.

We classify the closed one-sided sets into three classes. The class $\mathcal{B}$ will consist of all one-sided sets that are disjoint from some closed two-sided set; the class $\mathcal{F}$ will consist of all one-sided sets which intersect every closed two-sided set; and $\mathcal{T}$ will denote the class of those one-sided sets all of whose neighborhoods contain some closed two-sided set.

The main result from [3] now says the following:

THEOREM 8. *With notation and assumptions as above, suppose that for each equivalence class $\mathcal{E}$ we have a set function $\tau_{\mathcal{E}}$ on $\mathcal{E}$ with the following properties:*

- (i) *If $C_1, \ldots, C_n \in \mathcal{E}$ are disjoint, then $\sum \tau_{\mathcal{E}}(C_i) \le 1$,*
- (ii) *If $C \in \mathcal{E}$, then $1 - \tau_{\mathcal{E}}(C) = \sup\{\tau_{\mathcal{E}}(D) : C \cap D = \emptyset\}$. We define $\tau_{\mathcal{E}}(X \setminus C) = 1 - \tau_{\mathcal{E}}(C)$.*
- (iii) *If $X = C \cup D \cup U \cup V$ is an irreducible partition of X with $C \in \mathcal{E}$, then $\tau_{\mathcal{E}}(C) + \tau_{\mathcal{E}}(D) + \tau_{\mathcal{E}}(U) + \tau_{\mathcal{E}}(V) = 1$. By the results of* [3], *this sum is defined.*

Suppose also that we have a set function $\tau_{\mathcal{F}}$ on $\mathcal{F}$ which satisfies (i) *and* (ii) *above. If we then define τ to be* 0 *on $\mathcal{B}$,* 1 *on $\mathcal{T}$, $\tau_{\mathcal{F}}$ on $\mathcal{F}$, and $\tau_{\mathcal{E}}$ on each $\mathcal{E}$, then τ extends uniquely to a topological measure on X.*

Our main task will be the construction of the set functions $\tau_{\mathcal{E}}$. With this in mind, fix $\mathcal{E}$ and let $K \in \mathcal{E}$. Let $\tilde{X} = \tilde{X}_K$. By construction, there is a generator for the group of deck transformations of $\tilde{X}$ over X. Pick such a generator and denote it by $\sigma = \sigma_K$. Notice that if C' is a lift of C in $\tilde{X}$, then by construction, $\tilde{X} \setminus C'$ has exactly two components, one of which contains $\sigma(C')$ and the other contains $\sigma^{-1}(C')$. If $A \subseteq \tilde{X}$ is disjoint from C', we say that A is *upstream from C'* with respect to σ if it is contained in the same component of $\tilde{X} \setminus C'$ as $\sigma(C')$, and *downstream from C'* if it is contained in the same component as $\sigma^{-1}(C')$. Every connected set A disjoint from C' is either upstream or downstream from C'.

Lemma 9. *Let $D \in \mathcal{E}$ be another two-sided set equivalent to C. Then there is a lift D' of D and a component V of $\tilde{X} \setminus (C' \cup D')$ such that V is upstream from C' and downstream from D' and $C' \cup V \cup D'$ is connected.*

Notice, in particular, that C' and D' are disjoint.

Proof. First assume that C and D are disjoint. Then $X \setminus (C \cup D)$ has exactly two components, say U and V. Furthermore, all of C, D, U, V are evenly covered by $\tilde{X}$. Also, the closures of U and V both intersect both C and D. Hence, there are lifts U' and V' of U and V whose closures intersect C'. We may assume that V' is upstream from C'. Then there is a lift, D', of D that intersects the closure of V'. Then D' and V' satisfy the conditions of the lemma.

Now suppose that $C \sim D$ in $\mathcal{E}$. If $C = C_0, C_1, \ldots, C_n = D$ is a sequence of two-sided sets with $C_k \cap C_{k+1} = \emptyset$, choose V_1' and C_1' with C_1' a lift of C_1, and V_1' a component of $\tilde{X} \setminus (C' \cup C_1')$ which is upstream from C' and downstream from C_1'. Then pick V_2' and C_2' a lift of C_2 with V_2' upstream from C_1' and downstream from C_2'. Continue in this way to obtain $C_1', \ldots, C_n'$ and $V_1', \ldots, V_n'$. Then set $D' = C_n'$ and $V = V_1' \cup C_2' \cup V_2' \cup \cdots \cup V_n'$. ■

Definition 10. A positive Borel probability measure μ is *$\mathcal{E}$-adapted* if whenever $C \in \mathcal{E}$, we have $\mu(C) < 1$.

We may lift $\mu = \mu_{\mathcal{E}}$ to a measure $\tilde{\mu}$ on $\tilde{X}$ so that whenever p takes C' to C homeomorphically, we have $\tilde{\mu}(C') = \mu(C)$. Notice that if $C \in \mathcal{E}$ and C' is a lift of C, then the union of C' and the bounded component of $\tilde{X} \setminus (C' \cup \sigma(C'))$ has measure 1.

Now let $\nu = \nu_{\mathcal{E}}$, a probability measure on the circle S^1, and $K \in \mathcal{E}$ be fixed. We regard S^1 as the unit interval with endpoints identified. Define $\tau_{\mathcal{E}}(C)$ for $C \in \mathcal{E}$ as follows: Pick lifts K' and C' of K and C so that $\tilde{X} \setminus (K' \cup C')$ has only one bounded component, V, which is upstream from K' and downstream from C'. Then the closed interval $[\tilde{\mu}(V), \tilde{\mu}(V \cup C')]$ has length less than 1 and we regard it as a subset of S^1. Set $\tau_{\mathcal{E}}(C) = \nu([\tilde{\mu}(V), \tilde{\mu}(V \cup C')])$.

Notice that if C'' is another lift of C upstream from K' and if V' is the bounded component of $\tilde{X} \setminus (K' \cup C'')$, then $\tilde{\mu}(V) - \tilde{\mu}(V')$ is an integer (since $\mu(X) = 1$) and $\tilde{\mu}(C') = \tilde{\mu}(C'') = \mu(C)$, so the intervals $[\tilde{\mu}(V), \tilde{\mu}(V \cup C')]$ and $[\tilde{\mu}(V'), \tilde{\mu}(V' \cup C'')]$ are the same in S^1. Similar considerations involving the lift K' show that $\tau_{\mathcal{E}}(C)$ does not depend on K'.

Now suppose that $U = X \setminus C$ where $C \in \mathcal{E}$. Then we define $\tau_{\mathcal{E}}(U) = 1 - \tau_{\mathcal{E}}(C)$. If V and C' are as above, then $\tau_{\mathcal{E}}(U) = \nu([\tilde{\mu}(V \cup C'), \tilde{\mu}(V \cup C' \cup U'))$ where U' is the lift of U immediately upstream from C', i.e. so that $C' \cup U'$ is connected. To see this, notice that $\tilde{\mu}(C' \cup U') = \tilde{\mu}(C') + \tilde{\mu}(U') = \mu(C) + \mu(U) = \mu(X) = 1$.

Now suppose that $C_1, \ldots, C_n$ are disjoint elements of $\mathcal{E}$. Then $X \setminus (C_1 \cup \cdots \cup C_n)$ has exactly n components, $U_1, \ldots, U_n$, and we may assume that the sets are numbered so that $C_k \cup U_k \cup C_{k+1}$ is always connected. Let C_1' be a lift of C_1 upstream from K'. Then there are lifts C_k' and U_k' upstream from C_1' and downstream from $\sigma(C_1')$. If V is the bounded component of $\tilde{X} \setminus (K' \cup C_1')$, then $V \cup C_1' \cup U_1'$ is the bounded component of $\tilde{X} \setminus (K' \cup C_2')$, with similar expressions for other C_k'. By construction, the intervals $[\tilde{\mu}(V), \tilde{\mu}(V \cup C_1')]$, $[\tilde{\mu}(V \cup C_1' \cup U_1'), \tilde{\mu}(V \cup C_1' \cup U_1' \cup C_2')], \ldots$ are disjoint in S^1. Thus, since ν is a probability measure on S^1, $\sum \tau_{\mathcal{E}}(C_k) \le \nu(S^1) = 1$. In fact, $\sum(\tau_{\mathcal{E}}(C_k) + \tau_{\mathcal{E}}(U_k)) = 1$. Thus, $\tau_{\mathcal{E}}$ is additive on irreducible partitions of X from $\mathcal{E}$.

Now suppose that $C \subseteq X$ and $\varepsilon > 0$ are given. Using regularity of ν, pick an interval $[\alpha, \beta]$ of S^1 disjoint from $[\tilde{\mu}(V), \tilde{\mu}(V \cup C')]$ so that $\nu[\alpha, \beta] + \tau_{\mathcal{E}}(C) > 1 - \varepsilon$. Next, use regularity of μ on X to find $D \in \mathcal{E}$ disjoint from C with $\mu(C) + \mu(D) > 1 - \delta$ where 2δ is the distance between $[\alpha, \beta]$ and $[\tilde{\mu}(V), \tilde{\mu}(V \cup C')]$ in S^1. Then by an analysis as above we see that $\tau_{\mathcal{E}}(D) + \tau_{\mathcal{E}}(C) > 1 - \varepsilon$.

This completes the construction of the set functions $\tau_{\mathcal{E}}$.

Finally, let $\tau_{\mathcal{F}} : \mathcal{F} \to [0, 1]$ be any function satisfying (i) and (ii) in Theorem 8 with $\mathcal{E}$ replaced by $\mathcal{F}$. For example, if $\{x_1, x_2, x_3\}$ are in X, we may define $\tau_{\mathcal{F}}(F)$ to be 1 if F contains at least two x_i and 0 otherwise. More generally, analogs of the finitely determined topological measures on spaces with $g(X) = 0$ can be used for $\tau_{\mathcal{F}}$. Using the result from [3] mentioned above, we then get a topological measure τ on X.

THEOREM 11. *Let X be a compact, connected, locally path connected space with $g(X) = 1$ and such that whenever C and D are disjoint two-sided sets, $X \setminus (C \cup D)$ is disconnected. Suppose that for each equivalence class of two-sided sets, $\mathcal{E}$, we are given an $\mathcal{E}$-adapted measure $\mu_{\mathcal{E}}$ on X. Then the above procedure gives a topological measure on X.*

Now suppose that X is a closed oriented smooth n-manifold and suppose that $\Omega_{\mathcal{E}}$ is an n-form with full support for each equivalence class $\mathcal{E}$. If $K \in \mathcal{E}$ is an $(n-1)$-dimensional submanifold, the covering space X_K is a non-compact n-manifold, so $H^n(X_K) = 0$. If $p_K : X_K \to X$ is the covering map, then $p_K^* \Omega_{\mathcal{E}}$ is an exact form on X_K, so there is an $(n-1)$-form $\lambda_{\mathcal{E}}$ with $d\lambda_{\mathcal{E}} = p_K^* \Omega_{\mathcal{E}}$. Let K' be a lift of K in X_K.

Let the measure $\mu_{\mathcal{E}}$ correspond to $\Omega_{\mathcal{E}}$ and let $\nu_{\mathcal{E}}$ be the point mass at the real number $\int_{K'} \lambda_{\mathcal{E}}$. Then, if $C \in \mathcal{E}$ is another $(n-1)$-submanifold C and if C' is a lift of C disjoint from K', we let U be the bounded component of $X_K \setminus (K' \cup C')$ and find

$$\tilde{\mu}_{\mathcal{E}}(U \cup C') = \tilde{\mu}_{\mathcal{E}}(U) = \tilde{\mu}_{\mathcal{E}}(\overline{U}) = \int_{\overline{U}} \Omega_{\mathcal{E}} = \int_{C'} \lambda_{\mathcal{E}} - \int_{K'} \lambda_{\mathcal{E}}.$$

Hence, $\tau_{\mathcal{E}}(C) = 1$ if $\int_{K'} \lambda_{\mathcal{E}} = \int_{C'} \lambda_{\mathcal{E}}$ (mod 1) and $\tau_{\mathcal{E}}(C) = 0$ otherwise.

This construction was inspired by the results of Knudsen in [4]. The topological measures in that paper correspond to taking $\Omega_{\mathcal{E}} = f(x_\epsilon)dx_\epsilon dy_\epsilon$ with the lift K' corresponding to the set $y_\epsilon = 0$.

Now suppose that (X, ω) is a $2n$-dimensional symplectic manifold and we choose $\Omega_{\mathcal{E}} = \omega^n$, the corresponding volume form, for each $\mathcal{E}$. If $\{H_t\}_{0\le t\le 1}$ is a Hamiltonian with corresponding vector field $\{X_t\}_{0\le t\le 1}$ where $i(X_t)\omega = dH$, and flow $\{\phi_t\}_{0\le t\le 1}$, we may lift to any cover X_K to get the symplectic form $p_K^*\omega$ on X_K, Hamiltonian $H\circ p_K$, and vector field $\{\tilde{X}_t\}$ with $p_{K*}(\tilde{X}_t) = X_t$ and flow $\{\tilde{\phi}_t\}$ with $p_K \circ \tilde{\phi}_t = \phi_t \circ p_K$. Then

$$L_{\tilde{X}_t}\lambda_{\mathcal{E}} = i(\tilde{X}_t)d\lambda_{\mathcal{E}} + d(i(\tilde{X}_t)\lambda_{\mathcal{E}}) = d(n(H\circ p_K)(p_K^*\omega)^{n-1} + i(\tilde{X}_t\lambda_{\mathcal{E}})).$$

Hence, if $C \in \mathcal{E}$ is a closed $(2n-1)$-dimensional submanifold, the integral $\int_{\tilde{\phi}_t(C')} \lambda_{\mathcal{E}}$ is independent of t. Thus, $\tau_{\mathcal{E}}(C) = \tau_{\mathcal{E}}(\phi_t(C))$. In particular, since $\tau_{\mathcal{E}}(K) = 1$, we have $\phi_t(K) \cap K \neq \emptyset$ for each t. This proves the following result.

THEOREM 12. *Suppose that (X, ω) is a compact symplectic manifold with $g(X) = 1$ such that every pair $\{C, D\}$ of closed, two-sided sets disconnects X. Then every two-sided, closed, $(2n-1)$-dimensional submanifold is non-displaceable via Hamiltonian flows.*

Every symplectic manifold of the form $M \times \mathbb{T}^n$ where M is simply connected and $\mathbb{T}$ is the torus satisfies the conditions of this theorem.

References

[1] J. F. Aarnes, *Quasi-states and quasi-measures*, Adv. Math. 86 (1991), 41–67.

[2] —, *Construction of non-subadditive measures and discretization of Borel measures*, Fund. Math. 147 (1995), 213–237.

[3] D. J. Grubb, *Irreducible partitions and the construction of quasi-measures*, Trans. Amer. Math. Soc. 353 (2001), 2059–2072.

[4] F. F. Knudsen, *New topological measures on the torus*, Fund. Math. 185 (2005), 287–293.

[5] J. R. Munkres, *Topology*, 2nd ed., Prentice Hall, 2000.

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