

Multiple gaps

by

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Abstract. We study a higher-dimensional version of the standard notion of a gap formed by a finite sequence of ideals of the quotient algebra $\mathcal{P}(\omega)/\text{fin}$. We examine different types of such objects found in $\mathcal{P}(\omega)/\text{fin}$ both from the combinatorial and the descriptive set-theoretic side.

1. Introduction. Gaps in the quotient algebra $\mathcal{P}(\omega)/\text{fin}$ are a phenomenon discovered by Hausdorff more than a century ago. Their study has always been a prominent theme in set theory and its applications. For example, Kunen's study of gaps of $\mathcal{P}(\omega)/\text{fin}$ in models of Martin's axiom has led Woodin to a proof that the Proper Forcing Axiom implies Kaplansky's conjecture about automatic continuity of norms in certain Banach algebras (see [5]). Gaps in $\mathcal{P}(\omega)/\text{fin}$ were also the main motivation behind the introduction of the set-theoretic principle about the chromatic number of open graphs on separable metric spaces (see [18]), a principle with many applications (see, for example, [9], [10]). In [19], the second author has initiated the study of gaps in $\mathcal{P}(\omega)/\text{fin}$ from the descriptive set-theoretic side, a study also of independent interest and with important applications (see, for example, [20]).

The purpose of this paper is to extend this theory to higher dimensions, or in other words, to build a theory that involves gaps formed by more than two ideals of $\mathcal{P}(\omega)/\text{fin}$. While this theory is not something that immediately suggests itself when one mentions gaps formed by more than two ideals (see, for example, [8], [17]) it is nevertheless quite natural and it could have been discovered long ago. We came to it while trying to understand the reasons behind the fact that the Banach space ℓ_∞/c_0 is not injective, a result originally due to Amir [2]. While the characterization of 1-injective spaces due to Kelley [13] suggests that gaps of $\mathcal{P}(\omega)/\text{fin}$ should play a role

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in Amir's result, we were surprised to realize that the classical theory of gaps in $\mathcal{P}(\omega)/\text{fin}$ is actually not useful in its proof. What is relevant in this context is a notion of an n -gap formed by a sequence \mathcal{I}_i ($i < n$) of ideals of $\mathcal{P}(\omega)/\text{fin}$ so that the corresponding sequence of open subsets of $\omega^* = \beta\omega \setminus \omega$ has a common boundary point. We give a precise explanation of this in Section 8 where we use Ditor's analysis of lower bounds of norms of averaging operators. This also suggests various refinements of the notion of an n -gap that we study below. As said above we study the new notion both from the combinatorial and descriptive set-theoretic side. For example, we show that there is no Hausdorff phenomenon for n -gaps where $n > 2$, or more precisely, unlike in the case $n = 2$, the usual axioms of set theory are insufficient to construct n -gaps consisting of \aleph_1 -generated ideals for $n > 2$. On the other hand, we show that the descriptive set theory of 2-gaps initiated in [19] extends naturally to all higher dimensions.

2. Notation. For a Boolean algebra \mathcal{B} , we denote by \vee , \wedge and \leq the join operation, meet operation and order in \mathcal{B} , and by 0 and 1 its maximum and minimum. Two subsets I and J of \mathcal{B} are called *orthogonal* if $a \wedge b = 0$ for all $a \in I$ and $b \in J$. Other symbols that we shall use are: $I \vee J = \{a \vee b : a \in I, b \in J\}$ and similarly $I \wedge J$; $I^\perp = \{b \in \mathcal{B} : \forall a \in I \ a \wedge b = 0\}$; $I|_a = \{b \in I : b \leq a\}$; $c \geq I$ if $c \geq a$ for all $a \in I$. Also, $n = \{0, 1, \dots, n-1\}$.

Although we give definitions for arbitrary Boolean algebras, which is the natural context, most of the paper deals only with the algebra $\mathcal{P}(\omega)/\text{fin}$ and the results refer to this algebra unless otherwise explicitly stated. The elements of $\mathcal{P}(\omega)/\text{fin}$ are equivalence classes of the power set $\mathcal{P}(\omega)$ of the natural numbers under the relation of equality modulo finite sets: $a =^* b$ if $(a \setminus b) \cup (b \setminus a)$ is finite. However, manipulating equivalence classes is inconvenient for it makes considerable noise in notation. Instead, we prefer to work in $\mathcal{P}(\omega)$ while referring to properties of $\mathcal{P}(\omega)/\text{fin}$. So we use the following conventions along the text:

Ideals of $\mathcal{P}(\omega)/\text{fin}$ are identified with ideals of $\mathcal{P}(\omega)$ that contain the ideal fin of finite sets. Hence, if I is such an ideal and we write $a \in I$, we understand that $a \in \mathcal{P}(\omega)$ and a belongs to the corresponding ideal of $\mathcal{P}(\omega)$ that contains fin . On the other hand, when we state properties of such ideals like being orthogonal, being a multiple gap, a jigsaw, clover, etc., we always refer to properties of ideals in the Boolean algebra $\mathcal{P}(\omega)/\text{fin}$, never of ideals in $\mathcal{P}(\omega)$. One exception is when we say that an ideal I is analytic or Borel: then we view I as a subset of $\mathcal{P}(\omega) = 2^\omega$.

For $a, b \in \mathcal{P}(\omega)$ when we write $a \subset b$ we mean actual inclusion in $\mathcal{P}(\omega)$ while $a \subset^* b$ means that $b \setminus a$ is finite. The symbols pertaining to the structure of a general Boolean algebra, like \leq and \perp , always refer to $\mathcal{P}(\omega)/\text{fin}$, never to $\mathcal{P}(\omega)$. For instance if $a, b \in \mathcal{P}(\omega)$ and I is an ideal of $\mathcal{P}(\omega)/\text{fin}$, then $a \leq b$

means $a \subset^* b$, $a \perp b$ means $a \cap b =^* \emptyset$, $I \leq a$ means $x \subset^* a$ for all $x \in I$, $a \in I^\perp$ means that $a \cap x =^* \emptyset$ for all $x \in I$, etc. We denote by $\omega^* = \beta\omega \setminus \omega$ the dual compact space of $\mathcal{P}(\omega)/\text{fin}$ under Stone's duality.

Given a set S , we denote by $S^{<\omega}$ the set of all finite sequences of elements of S . Given $s = (s_i)_{i < n}$ and $t = (t_i)_{i < m}$ in $S^{<\omega}$, we write $s \leq t$ when $n \leq m$ and $s_i = t_i$ for all $i < n$. We denote $s \frown t = (s_0, \dots, s_{n-1}, t_0, \dots, t_{m-1})$. The set $S^{<\omega}$ is a tree when endowed with the partial order \leq . A *tree* is a partially ordered set (T, \leq) with a minimum element (its root) such that for every $t \in T$ the set $\{s \in T : s < t\}$ is well ordered. A linearly ordered subset of T is called a *chain*. A *branch* of T is a maximal chain. An *antichain* is a subset of T where no two elements satisfy $s < t$. The set of branches of T is denoted by $[T]$. Given $t \in T$, we write $T_t = \{s \in T : t \leq s\}$ and $[T_t] = \{x \in [T] : t \in x\}$. When $T = S^{<\omega}$ we can identify $[T] = S^\omega$.

Often, instead of working in $\mathcal{P}(\omega)/\text{fin}$ we work in $\mathcal{P}(E)/\text{fin}$ where E is a countable set with some particular structure, like a countable tree or a countable subset of a compact metric space. The same conventions as for $\mathcal{P}(\omega)/\text{fin}$ apply.

3. Basic notions and preliminaries. A *gap* in a Boolean algebra \mathcal{B} is a couple of orthogonal ideals that cannot be separated by two disjoint elements of \mathcal{B} . We introduce a multidimensional generalization:

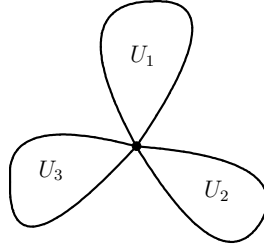
DEFINITION 1. We say that a finite family $\{I_i : i \in n\}$ of mutually orthogonal ideals of a Boolean algebra \mathcal{B} constitutes a *multiple gap*, or an *n-gap*, if for every function $c : n \rightarrow \mathcal{B}$ such that $c(i) \geq I_i$ for all $i \in n$, we have $\bigwedge_{i \in n} c(i) \neq 0$.

There is a simple topological interpretation of this definition. Let $\text{St}(\mathcal{B})$ be the Stone dual compact space of the Boolean algebra \mathcal{B} , that is, the space of all ultrafilters of \mathcal{B} with the topology generated by the sets $\{\mathfrak{U} : a \in \mathfrak{U}\}$ for $a \in \mathcal{B}$. For an ideal I , let $U(I) = \{\mathfrak{U} : \mathfrak{U} \cap I \neq \emptyset\}$ be the associated open subset of $\text{St}(\mathcal{B})$. The ideals $\{I_i : i \in n\}$ form an *n-gap* if and only if $\bigcap_{i \in n} \overline{U(I_i)} \neq \emptyset$. Thus, a multiple gap is nothing other than a finite family of pairwise disjoint open sets whose closures have nonempty intersection.

The multiple gap is called *dense* if $(\bigvee_{i \in n} I_i)^\perp = 0$. Topologically, this means that the union of the open sets considered is dense in $\text{St}(\mathcal{B})$. When $n > 2$, we can distinguish a variety of types of *n-gaps* depending on how subgaps of lower dimension interact. We point out two extreme types that we found of special interest, and called *clovers* and *jigsaws*.

DEFINITION 2. Let $\{I_i : i \in n\}$ be a multiple gap and B a nonempty proper subset of n . We call the multiple gap a *B-clover* if there exists no $b \in \mathcal{B}$ such that $\{I_j|_b : j \in B\}$ is a multiple gap while $b \in I_i^\perp$ for all $i \notin B$.

The multiple gap is called a *clover* if it is a B -clover for all nonempty proper subsets B of n . In topological terms, being a B -clover is equivalent to the equality $\bigcap_{i \in B} \overline{U(I_i)} \setminus \bigcup_{i \notin B} \overline{U(I_i)} = \emptyset$. Hence, for instance the picture of a triple clover would look like this:

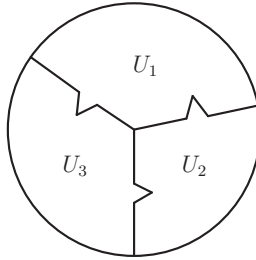


where the intersection of any two closures equals the intersection of all closures (that may consist of more than one point, in spite of the picture).

The second type, opposite to clovers, is that of jigsaws.

DEFINITION 3. Let $\{I_i : i \in n\}$ be an n -gap, and let B be a nonempty proper subset of n . We call the multiple gap a B -jigsaw if for every $A \supset B$ and every $a \in B$, if $\{I_i|_a : i \in A\}$ is a multiple gap, then there exists $b \subset a$ such that $\{I_i|_b : i \in B\}$ is a multiple gap while $b \in I_i^\perp$ for all $i \in A \setminus B$.

The multiple gap is called a *jigsaw* if it is a B -jigsaw for all nonempty proper subsets B of n . In topological terms, being a B -jigsaw is equivalent to $\bigcap_{i \in B} \overline{U(I_i)} \setminus \bigcup_{i \notin B} \overline{U(I_i)}$ being dense in $\bigcap_{i \in B} \overline{U(I_i)}$. The picture of a triple jigsaw is then the following:



where every point in the intersection of the three closures (in the picture, the central point) can be approached by points that lie in the intersection of exactly two closures.

The concept of countable separation plays a central role in our study. Countably separated gaps are introduced in [19]: Two orthogonal ideals I_0 and I_1 are *countably separated* if there exists a sequence $\{c_n : n \in \omega\}$ of elements in the Boolean algebra such that for every $x \in I_0$ and $y \in I_1$ there exists n with $x \leq c_n$ and $y \perp c_n$. Regarding this notion, the following dichotomy is proven in [19]:

THEOREM 4. Let I_0 and I_1 be two orthogonal analytic ideals in $\mathcal{P}(\omega)/\text{fin}$. Then either

- (1) they are countably separated in $\mathcal{P}(\omega)/\text{fin}$, or
- (2) there exist $\{a_s : s \in 2^\omega\} \subset I_0$ and $\{b_s : s \in 2^\omega\} \subset I_1$ such that $a_s \cap b_s = \emptyset$, $(a_s \cap b_t) \cap (a_t \cap b_s) \neq \emptyset$ for $t \neq s$, and the maps $s \mapsto a_s$ and $s \mapsto b_s$ are continuous.

Countable separation can be generalized to multiple gaps in several ways:

DEFINITION 5. Let \mathfrak{X} be a family of subsets of n , and let $\{I_i : i \in n\}$ be ideals in a Boolean algebra. We say that a multiple gap $\{I_i : i < n\}$ is \mathfrak{X} -countably separated if there exist elements $\{c_i^k : i \in n, k \in \omega\}$ in the Boolean algebra such that

- (1) $\bigwedge_{i \in A} c_i^k = 0$ for every $k \in \omega$ and every $A \in \mathfrak{X}$,
- (2) for every $x_0 \in I_0, \dots, x_{n-1} \in I_{n-1}$, there exists $k \in \omega$ such that $x_i \leq c_i^k$ for all i .

The two extreme cases of this definition are of special interest for us (we use the notation $[x]^k = \{z \subset x : |x| = k\}$):

- We say that the ideals $\{I_i : i \in n\}$ are *weakly countably separated* if they are \mathfrak{X} -countably separated for $\mathfrak{X} = [n]^n$.
- We say that the ideals $\{I_i : i \in n\}$ are *strongly countably separated* if they are \mathfrak{X} -countably separated for $\mathfrak{X} = [n]^2$.

In Section 4 we prove a generalization of Theorem 4 to multiple gaps. We define for every n a concrete analytic n -gap in $\mathcal{P}(\omega)/\text{fin}$, $\{\mathcal{I}_i : i \in n\}$, which is not weakly countably separated. Roughly speaking, the main result states that an analytic multiple gap $\{I_i : i \in n\}$ in $\mathcal{P}(\omega)/\text{fin}$ is either weakly countably separated or there is a copy of the object $\{\mathcal{I}_i : i \in n\}$ with $\mathcal{I}_i \subset I_i$. The aforementioned n -gap $\{\mathcal{I}_i : i \in n\}$ turns out to be a jigsaw. We also provide an example of an analytic multiple gap which is not a jigsaw. More results on the theory of analytic multiple gaps will appear in [3] and further works.

In Section 5 we deal with multiple gaps which are weakly countably separated. In contrast with the results established in the previous section, now given a weakly countably separated multiple gap $\{I_i : i \in n\}$ in $\mathcal{P}(\omega)/\text{fin}$ we find ideals of a particular form $\{I'_i : i \in n\}$ such that $I_i \subset I'_i$ (that is, above instead of below I_i). The most informative case is when the gap is indeed strongly countably separated, since then these new ideals above constitute a multiple gap. We show that every strongly countably separated multiple gap in $\mathcal{P}(\omega)/\text{fin}$ is below a strongly countably separated multiple gap which is moreover a dense jigsaw. In particular every dense strongly countably separated multiple gap in $\mathcal{P}(\omega)/\text{fin}$ is a jigsaw. We also provide an example

of a strongly countably separated clover. This example is neither analytic nor dense.

In Section 6 we show that there is no analogue of Hausdorff's gap when we deal with n -gaps for $n \geq 3$. Namely, from Hausdorff's gap one finds that there exists a 2-gap $\{I_0, I_1\}$ in $\mathcal{P}(\omega)/\text{fin}$ such that I_0 and I_1 are generated by sets of size \aleph_1 . We prove that if MA_θ holds for a cardinal θ , then for $n \geq 3$ there exists no n -gap made of θ -generated ideals in $\mathcal{P}(\omega)/\text{fin}$. Indeed we show that for every $n \geq 2$ it is consistent that there exists an n -gap of θ -generated ideals in $\mathcal{P}(\omega)/\text{fin}$ but there are no such $n + 1$ -gaps. The technique used allows us to prove some other results on small sets both in $\mathcal{P}(\omega)/\text{fin}$ and in the Banach space ℓ_∞/c_0 .

In Section 7 we construct, for every family \mathfrak{X} of subsets of n , a dense multiple gap which is a B -jigsaw for $B \in \mathfrak{X}$ and a B -clover for $B \notin \mathfrak{X}$. The construction is based on the existence of completely separable almost disjoint families. It is an open problem whether such almost disjoint families exist in ZFC, though they are known to exist under several hypotheses.

In Section 8, we relate jigsaws to injectivity properties of Banach spaces. Let \mathcal{B} be a Boolean algebra and K its dual compact space in Stone's duality. We show that if \mathcal{B} has a dense n -jigsaw for every n , then the Banach space $C(K)$ of continuous functions is uncomplemented in a superspace of the same density. Since we have found dense jigsaws in $\mathcal{P}(\omega)/\text{fin}$ of every size, this provides an alternative proof of a result of Amir [2] that the Banach space ℓ_∞/c_0 is not injective.

We found that many notable examples of multiple gaps in $\mathcal{P}(\omega)/\text{fin}$ happen to be jigsaws, and jigsaws play an important role in the general theory. However we do not know that much about clovers. In particular we do not know whether analytic clovers—or any mixed version of clovers and jigsaws—exist at all, and we do not know if the results of Section 7 hold in ZFC, except for the existence of dense jigsaws.

4. Multiple gaps that are not countably separated

4.1. An example of an analytic jigsaw that is not weakly countably separated. Let $n < \omega$ and $T = n^{<\omega}$ be the n -ary tree. For every branch $x \in [T]$ and $i \in n$, we set

$$a_x^i = \{s \in T : s \hat{\ } i \in x\}.$$

Let \mathcal{J}_i be the ideal of $\mathcal{P}(T)/\text{fin}$ generated by $\{a_x^i : x \in [T]\}$. Notice that each of these ideals is analytic, since its set of generators is an analytic set.

THEOREM 6. *The ideals $\{\mathcal{J}_i : i \in n\}$ constitute an n -jigsaw which is not weakly countably separated.*

Proof. It is obvious that the ideals are mutually orthogonal since $a_x^i \cap a_y^j =^* \emptyset$ whenever $i \neq j$. We prove first that they are not weakly countably separated, hence in particular they are a multiple gap. Denote by $T(m)$ the set of all elements of t of length less than m . Suppose that the ideals are weakly countably separated, so that we have elements $c_i^k \subset T$, $i \in n$, $k < \omega$, such that $\bigcap_{i \in n} c_i^k = \emptyset$ for each k , and whenever we pick $b_i \in \mathcal{J}_i$ there exists k with $b_i \subset^* c_i^k$ for every $i \in n$. In particular, for every $x \in [T]$ and every $i \in n$ there exist $k(x), m(x) \in \omega$ such that $a_x^i \setminus T(m(x)) \subset c_i^{k(x)}$. Let $X(m, k) = \{x \in [T] : \forall i \in n \ a_x^i \setminus T(m) \subset c_i^k\}$. Notice that each $X(m, k)$ is a closed subset of $[T]$, and $\bigcup_{m, k < \omega} X(m, k) = [T]$. By the Baire category theorem, there exist $m_0, k_0 < \omega$ such that $X(m_0, k_0)$ has nonempty interior. This means that we can find $t \in T$ (which we can choose of length greater than m_0) such that $\{x \in [T] : t \in x\} \subset X(m_0, k_0)$. But this actually implies that $\{s \in T : s > t\} \subset \bigcap_{i \in n} c_i^{k_0}$, a contradiction.

We pass now to the proof that the ideals actually constitute a jigsaw. Let $B \subset A \subset n$, and let $a \subset T$ be such that $\{\mathcal{J}_i|_a : i \in A\}$ is a multiple gap. We have to find $b \subset a$ such that $b \perp \mathcal{J}_i$ for $i \in n \setminus B$, while the ideals $\{\mathcal{J}_j|_b : j \in B\}$ are a multiple gap.

A nonempty subset $R \subset T$ is called a *complete A-tree* if for every $r \in R$ and every $i \in A$ there exists $s \in R$ such that $r \widehat{\ } i < s$.

CLAIM. *There exists a complete A-tree $R \subset a$.*

Proof of the claim. Let us say that a set $u \subset T$ is *i-WF* (for *i*-well founded) if it contains no infinite sequence $\{s_k : k < \omega\}$ with $s_k \widehat{\ } i < s_{k+1}$. Notice that an *i-WF* set is orthogonal to the ideal \mathcal{J}_i , therefore its complement is an upper bound of \mathcal{J}_i . It follows that a cannot be a union $\bigcup_{i \in A} u_i$ with each u_i being *i-WF*. Otherwise, we could take $c_i = a \setminus u_i \geq \mathcal{J}_i|_a$ and $\bigcap c_i = \emptyset$, contradicting that $\{\mathcal{J}_i|_a : i \in A\}$ is a multiple gap. We define a derivation procedure on subsets of T . Given a set $v \subset T$ we define $v[i] = \{s \in v : \nexists t \in v \ s \widehat{\ } i < t\}$ for $i \in A$, and then the derived set $v' = v \setminus \bigcup_{i \in A} v[i]$. By induction consider the iterated derivatives $a^{(\alpha)}$ of the set a for ordinals α . Choose an ordinal β with $a^{(\beta)} = a^{(\beta+1)}$. This means that $a^{(\beta)}$ is a complete *A-tree* provided it is nonempty. But notice that $\bigcup_{\alpha < \beta} a^{(\alpha)}[i]$ is an *i-WF* set, so since a cannot be the union of *i-WF* sets for $i \in A$, it follows that $a^{(\beta)} \neq \emptyset$. This finishes the proof of the claim.

Let R be the complete *A-tree* given by the Claim. We can find $b \subset R$ which is a complete *B-tree* and moreover *i-WF* for all $i \in n \setminus B$. We construct b inductively: start with $b[0] \subset R$ a singleton, and then construct $b[n+1]$ by adding one element above $s \widehat{\ } j$ for every $s \in b[n]$ and $j \in B$. At the end, $b = \bigcup_{n < \omega} b[n]$. The set b is orthogonal to \mathcal{J}_i for $i \notin B$ since it is *i-WF*. So it remains to show that the ideals $\{\mathcal{J}_j|_b : j \in B\}$ are a multiple gap. But this

is the same proof as we already used, in the first paragraph of this proof, to show that the $\{\mathcal{J}_i : i \in n\}$ are a multiple gap, but for the integer $|B|$ instead of n . The reason is that we can identify the set b with the $|B|$ -adic tree $B^{<\omega}$ and then the ideals $\mathcal{J}_i|b$ are identified with the originally considered ideals \mathcal{J}_i . ■

4.2. The dichotomy. We show that if an analytic multiple gap is not weakly countably separated, then it contains a copy of the analytic jigsaw $\{\mathcal{J}_i : i \in n\}$ presented above.

THEOREM 7 (Analytic multiple gap dichotomy). *Let $\{I_i : i \in n\}$ be analytic ideals of $\mathcal{P}(\omega)$ which constitute a multiple gap of $\mathcal{P}(\omega)/\text{fin}$. Then either*

- (1) *the ideals are weakly countably separated in $\mathcal{P}(\omega)/\text{fin}$, or*
- (2) *there exists a one-to-one function $u : T \rightarrow \omega$, where $T = n^{<\omega}$ is the n -ary tree, such that $u(\mathcal{J}_i) \subset I_i$ for all $i \in n$.*

Proof. Let $\mathcal{Y} = \omega^{<\omega}$ be the Baire tree. Let $\Sigma = \{(e_i)_{i \in n} \in \prod_{i \in n} I_i : e_i \cap e_j = \emptyset \text{ for } i \neq j\}$. Since Σ is an analytic set, there exists a continuous surjection $f : [\mathcal{Y}] \rightarrow \Sigma$. We write $f(x) = (f_i(x))_{i \in n}$. We distinguish two cases.

In the first case, we suppose that there exists a countable decomposition $[\mathcal{Y}] = \bigcup_{k < \omega} X_k$ such that

$$(\star) \text{ for every } k \text{ and every } \{x_i : i \in n\} \subset X_k \text{ we have } \bigcap_{i \in n} f_i(x_i) = \emptyset.$$

In this case we are in alternative (1) of the dichotomy. The elements $c_i^k = \bigcup_{x \in X_k} f_i(x)$ witness countable separation.

For the second case, we assume that the first case does not hold. Let $\mathcal{Y}' \subset \mathcal{Y}$ be the family of all $s \in \mathcal{Y}$ such that $[\mathcal{Y}_s]$ cannot be decomposed into countably many pieces $X_k, k < \omega$, with the property (\star) specified in the first case. Notice that \mathcal{Y}' is closed under initial segments.

CLAIM 1. *If $t \in \mathcal{Y}'$, then $[\mathcal{Y}'_t]$ cannot be decomposed into countably many pieces with property (\star) .*

Proof of the claim. Consider the set F of all $s > t$ such that $s \notin \mathcal{Y}'$. Then $[\mathcal{Y}_t] = [\mathcal{Y}'_t] \cup \bigcup_{s \in F} [\mathcal{Y}_s]$. All the sets $[\mathcal{Y}_s]$ on the right admit countable decompositions with (\star) , so since $[\mathcal{Y}_t]$ does not have such a decomposition, neither has $[\mathcal{Y}'_t]$.

CLAIM 2. *If $t \in \mathcal{Y}'$, and $F \subset \omega$ is a finite set, then there exist branches $\{x_i : i \in n\} \subset [\mathcal{Y}'_t]$ such that $\bigcap_{i \in n} f_i(x_i) \setminus F \neq \emptyset$.*

Proof of the claim. Suppose that there exists a finite set F such that $\bigcap_{i \in n} f_i(x_i) \subset F$ for all $\{x_i : i \in n\} \subset [\mathcal{Y}'_t]$. Then we can find a countable (even finite) decomposition $[\mathcal{Y}'_t] = \bigcup_{m < \omega} X_m$ and finite sets $F_i(m) \subset F$ for

$i \in n$ and $m \in \omega$ such that $f_i(x) \cap F = F_i(m)$ for every $x \in X_m$. Since $\bigcap_{i \in n} f_i(x) = \emptyset$ for every x , it follows that $\bigcap_{i \in n} F_i(m) = \emptyset$ for every $m < \omega$. Hence for every $\{x_i : i \in n\} \subset X_m$, we have $\bigcap_{i \in n} f_i(x_i) = \emptyset$. This means that every X_m has property (\star) , contrary to Claim 1.

CLAIM 3. *Let $s \in \mathcal{Y}'$ and let $F \subset \omega$ be finite. Then there exist incomparable elements $\{s_i : i \in n\} \subset \mathcal{Y}'$ above s and $k \in \omega \setminus F$ such that $k \in \bigcap_{i \in n} f_i(x_i)$ whenever $s_i \in x_i$ for every i .*

Proof of the claim. By Claim 2, there exist $\{y_i : i \in n\} \subset [\mathcal{Y}'_s]$ such that $\bigcap_{i \in n} f_i(y_i) \setminus F \neq \emptyset$. Fix $k \in \bigcap_{i \in n} f_i(y_i) \setminus F$. Now, using the fact that f is continuous we can find elements $s_i \in \mathcal{Y}'_t$ such that $s < s_i \in y_i$ and which satisfy the requirement of the claim.

We construct inductively two functions $v : T \rightarrow \mathcal{Y}'$ and $u : T \rightarrow \omega$. First, $v(\emptyset)$ is just the root of \mathcal{Y}' . Once $v(t)$ is defined, we apply Claim 3 to $s = v(t)$ and F being the set of all $u(t')$'s that have been previously chosen in the inductive procedure (in order to make u one-to-one). We set $v(t \hat{\ } i) = s_i$ and $u(t) = k$ given by Claim 3. In this way, we have the property that for every $t \in T$, if $x \in [\mathcal{Y}']$ and $v(t \hat{\ } i) \in x$, then $u(t) \in f_i(x)$. To conclude the proof, we have to check that $u(\mathcal{J}_i) \subset I_i$. So let $x \in [T]$ and let $a = a_i^x = \{t \in T : t \hat{\ } i \in x\}$ be one of the generators of \mathcal{J}_i . We want to see that $u(a) \in I_i$. The set $\{v(t) : t \in x\}$ is an infinite chain in \mathcal{Y}' , hence cofinal in some branch $y \in [\mathcal{Y}']$. If $t \in a$, then $v(t \hat{\ } i) \in y$, hence $u(t) \in f_i(y)$. It follows that $u(a) \subset f_i(y) \in I_i$. ■

4.3. Example of an analytic gap that is not a jigsaw. Let $T = 2^{<\omega}$ be the dyadic tree. Again, for $i = 0, 1$, we define elements $a_x^i = \{s \in T : s \hat{\ } i \in x\}$ and the ideal I_i of $\mathcal{P}(T)/\text{fin}$ as generated by all a_x^i for $x \in [T]$. On the other hand, let J be the ideal generated by the antichains of T .

PROPOSITION 8. *The ideals $\{I_0, I_1, J\}$ are analytic ideals which are a multiple gap which is not weakly countably separated and neither a jigsaw nor a clover.*

Proof. The fact that they are analytic is easily checked as they have analytic sets of generators. We see that they are not weakly countably separated (in particular, they are a 3-gap). Let us write $J = I_2$ for convenience. Suppose we have c_i^k with $\bigcap_{i \in 3} c_i^k =^* \emptyset$ for every $k < \omega$, and for every $x_i \in I_i$ ($i \in 3$) there is $k < \omega$ with $x_i \leq c_i^k$. The proof is similar to that of Theorem 6. We denote by $T(m)$ the set of elements of T of length less than m . For every $x \in [T]$ there exist integers $k(x)$ and $m(x)$ such that $a_x^i \subset c_i^{k(x)} \setminus T(m(x))$ for $i = 0, 1$ and

$$\{s \hat{\ } 0 : s \in a_x^1\} \cup \{s \hat{\ } 1 : s \in a_x^0\} \subset c_2^{k(x)} \setminus T(m(x)).$$

By a Baire category argument, there exist $k, m < \omega$ and $t \in T$ of length greater than m such that the properties above hold for all $x \in [T_t]$ with $k(x) = k$ and $m(x) = m$. It follows that $t' \in \bigcap_{i \in \mathbb{3}} c_i^k$ for every $t' > t$. This shows that $\{I_0, I_1, J\}$ are not weakly countably separated.

CLAIM. *If $b \in J^\perp$, then b is contained in a finite number of chains.*

Proof of the claim. Let $A = \{x \in [T] : b \cap x \neq \emptyset\}$. Notice that $b \setminus \bigcup A$ is an antichain, so $b \subset^* \bigcup A$. If we suppose that A is infinite, then it contains either an increasing or a decreasing sequence $\{x_n\}$ in the lexicographical order of $[T] = 2^\omega$. If $t_n \in x_n \cap b$ is chosen high enough so that $t_n > x_n \wedge x_{n+1}$, then $\{t_n : n < \omega\} \subset b$ is an antichain. This finishes the proof of the Claim.

The 3-gap is not a jigsaw, because if $b \in J^\perp$, then by the Claim above, $I_0|_b$ and $I_1|_b$ are finitely generated and therefore separated. Neither is the 3-gap a clover: Consider the set c of all $(k_0, k_1, \dots, k_{m-1}) \in T$ such that m is even and $k_j = 1$ for all k even. Then c is a dyadic subtree of T , $c \in I_0^\perp$ and $\{I_1|_c, J|_c\}$ is a gap (this follows from the Claim, because $I_1|_c$ is the ideal generated by the chains of c , and $I_2|_c$ the ideal generated by the antichains). ■

5. Multiple gaps that are countably separated. Let D be a countable subset of a compact metric space L and let $K = \text{acc}(D)$ be the set of accumulation points of D , those points such that every neighborhood meets infinitely many elements of D . For a subset $G \subset K$, we define

$$\mathcal{I}_G = \{a \subset D : \text{acc}(a) \subset G\}.$$

Note that \mathcal{I}_G is an ideal of $\mathcal{P}(D)/\text{fin}$. Note also that \mathcal{I}_G is orthogonal to $\mathcal{I}_{G'}$ if and only if $G \cap G' = \emptyset$. Theorem 10 below shows that this kind of ideals characterize countable separation.

LEMMA 9. *Let L be a compact space, \mathfrak{X} a family of subsets of n , \mathfrak{B} a basis for the topology of L which is closed under finite unions and intersections, and $\{Z_i : i \in n\}$ closed subsets of L such that $\bigcap_{i \in A} Z_i = \emptyset$ whenever $A \in \mathfrak{X}$. Then there exist sets $Z_i \subset V_i \in \mathfrak{B}$ with $\bigcap_{i \in A} \overline{V_i} = \emptyset$ whenever $A \in \mathfrak{X}$.*

Proof. First we prove the case in which \mathfrak{X} consists only of the full subset of n . This is done by induction on m , the case $m = 2$ being just the standard compactness argument for separation. For $m > 2$, the sets $Z = \bigcap_{i \in m-1} Z_i$ and Z_{m-1} are disjoint closed subsets of L , so by the case $m = 2$, there are $V, W \in \mathfrak{B}$ with $Z_{m-1} \subset V$, $Z \subset W$ and $\overline{V} \cap \overline{W} = \emptyset$. The inductive hypothesis provides $\{U_i : i \in m-1\} \subset \mathfrak{B}$ with $\bigcap_{i \in m-1} \overline{U_i} = \emptyset$ and $Z_i \setminus W \subset U_i$ for $i \in m-1$, and then we define $V_{m-1} = V$ and $V_i = U_i \cup W$ for $i \in m-1$. Now, for a general family \mathfrak{X} of subsets of n , the previous case allows us to find, for every A , $Z_i \subset V_i^A \in \mathfrak{B}$ such that $\bigcap_{i \in A} \overline{V_i^A} = \emptyset$, and take $V_i = \bigcap_{A \in \mathfrak{X}} V_i^A$. ■

THEOREM 10. Let $\{I_i : i \in n\}$ be ideals of $\mathcal{P}(\omega)/\text{fin}$ and let \mathfrak{X} be a family of subsets of n . The following are equivalent:

- (1) The ideals are \mathfrak{X} -countably separated.
- (2) There exists a continuous function ⁽¹⁾ $\phi : \omega^* \rightarrow 2^\omega$ with $\bigcap_{i \in A} \phi(U(I_i)) = \emptyset$ for every $A \in \mathfrak{X}$.
- (3) There exists a bijection $f : \omega \rightarrow D$, where D is a countable subset of a compact metric space L , and sets $G_i \subset K = \text{acc}(D)$, $i \in n$, such that
 - (a) $\bigcap_{i \in A} G_i = \emptyset$ for every $A \in \mathfrak{X}$,
 - (b) $f(I_i) = \{f(a) : a \in I_i\} \subset \mathcal{I}_{G_i}$ for every $i \in n$.
- (4) Condition (3) holds for $D = \{\xi \in 2^\omega : \exists m_0 \forall m > m_0 \xi_m = 0\}$.

Proof. We prove (4) \Rightarrow (3) \Rightarrow (1) \Rightarrow (2) \Rightarrow (4). The first implication is obvious, so we start by showing that (3) \Rightarrow (1). We prove that the ideals $\{\mathcal{I}_{G_i} : i \in n\}$ are \mathfrak{X} -countably separated. Let \mathfrak{B} be a countable basis for the topology of L which is closed under finite unions and intersections. Consider the family of all n -tuples of subsets of D of the form $(V_i \cap D)_{i \in n}$ such that $V_i \in \mathfrak{B}$ and $\bigcap_{i \in A} \overline{V_i} = \emptyset$ for all $A \in \mathfrak{X}$. This is a countable family of n -tuples of elements of $\mathcal{P}(D)$. We see that it witnesses \mathfrak{X} -countable separation. The first condition to check is obvious: $\bigcap_{i \in A} V_i \cap D =^* \emptyset$ for $A \in \mathfrak{X}$ whenever $(V_i \cap D)_{i \in n}$ is in the family. For the second condition, we pick $a_i \in \mathcal{I}_{G_i}$. Since $\text{acc}(a_i) \subset G_i$ for each i , we have $\bigcap_{i \in A} \text{acc}(a_i) = \emptyset$ for $A \in \mathfrak{X}$. By Lemma 9, we can find $V_i \in \mathfrak{B}$ such that $\text{acc}(a_i) \subset V_i$ and $\bigcap_{i \in A} \overline{V_i} = \emptyset$ for all $A \in \mathfrak{X}$. It follows that $a_i \subset^* V_i \cap D$ and the tuple $(V_i \cap D)_{i \in n}$ belongs to our family.

We pass now to implication (1) \Rightarrow (2). Using Stone's duality, \mathfrak{X} -countable separation can be restated as the existence of clopen sets $\{C_m^i : i \in n, m \in \omega\}$ of ω^* such that

- $\bigcap_{i \in A} C_m^i = \emptyset$ for every $m \in \omega$ and every $A \in \mathfrak{X}$,
- for any clopen sets $x_i \subset U(I_i)$, $i \in n$, there exists m such that $x_i \subset C_m^i$ for every $i \in n$.

For every $(m, i) \in \omega \times n$ define $\phi_{(m,i)} : \omega^* \rightarrow 2$ to be the characteristic function of the clopen set C_m^i . All these functions together provide a continuous function $\phi : \omega^* \rightarrow 2^{\omega \times n}$. Suppose now for contradiction that there exists $\xi \in \bigcap_{i \in A} \phi(U(I_i))$ with $A \in \mathfrak{X}$. Then for every $i \in A$, $\xi = \phi(t_i)$ with $t_i \in U(I_i)$. There must exist $m \in \omega$ such that $t_i \in C_m^i$ for every $i \in A$. For every $i, j \in A$ we have

$$1 = \phi_{(m,i)}(t_i) = \xi_{(m,i)} = \phi_{(m,i)}(t_j).$$

It follows that $t_j \in \bigcap_{i \in A} C_m^i = \emptyset$, a contradiction.

⁽¹⁾ We recall that we denote by ω^* the Stone space of $\mathcal{P}(\omega)/\text{fin}$.

We prove now that (2) \Rightarrow (4). Given ϕ , we construct inductively a tree of subsets of ω , $\{a_s : s \in 2^{<\omega}\}$, with

- $\phi^{-1}(\{\xi \in 2^\omega : \xi|_{\text{length}(s)} = s\}) = \text{clopen}(a_s)$, where $\text{clopen}(a)$ denotes the clopen subset of ω^* associated to the set $a \in \mathcal{P}(\omega)$,
- $a_\emptyset = \omega$,
- $a_s = a_{s \smallfrown 0} \cup a_{s \smallfrown 1}$,
- $\emptyset = a_{s \smallfrown 0} \cap a_{s \smallfrown 1}$,
- if we set $m_s = \min(a_s \setminus \{m_t : t < s\})$, then $m_s \in a_r$ whenever $s < r$ and $r_k = 0$ for all $k \geq \text{length}(s)$.

Notice that the function $s \mapsto m_s$ is a bijection from $2^{<\omega}$ onto ω . Let $g : \omega \rightarrow 2^{<\omega}$ be its inverse, so that $m_{g(m)} = m$ for all m . We define $f : \omega \rightarrow D$ by $f(m) = g(m) \smallfrown 1 \smallfrown (0, 0, 0, \dots)$. Let $G_i = \phi(U(I_i))$. We have to prove that if $x \in I_i$, then $\text{acc}\{f(m) : m \in x\} \subset G_i$. So fix $x \in I_i$ and suppose that $\xi \in \text{acc}\{f(m) : m \in x\}$. There exists $\{m_1, m_2, \dots\} \subset x$ such that the sequence $\{f(m_k)\}_{k < \omega}$ converges to ξ in the space 2^ω . This means that for every $k \in \omega$ all but finitely many $p \in \omega$ satisfy $(\xi_0, \dots, \xi_k) < g(m_p)$. This implies that, for every k ,

$$\{m_0, m_1, \dots\} \subset^* a_{(\xi_0, \dots, \xi_k)}.$$

This is because if $t < g(m)$ then $m \in a_t$ (notice that $m_s \in a_s$, which implies $m = m_{g(m)} \in a_{g(m)} \subset a_t$). Now, by Stone's duality

$$\begin{aligned} \text{clopen}\{m_0, m_1, \dots\} &\subset \text{clopen}(a_{(\xi_0, \dots, \xi_k)}) \\ &= \phi^{-1}(\{\zeta \in 2^\omega : \zeta_i = \xi_i, i = 0, \dots, k\}). \end{aligned}$$

It follows that $\text{clopen}\{m_0, m_1, \dots\} \subset \phi^{-1}(\xi)$. But

$$\text{clopen}\{m_0, m_1, \dots\} \subset \text{clopen}(x),$$

and $\text{clopen}(x) \subset U(I_i)$ since $x \in I_i$. Hence $\xi \in \phi(U(I_i)) = G_i$ as desired. ■

In Theorem 10, notice that if the ideals $\{I_i : i \in n\}$ are strongly countably separated, then the sets G_i are pairwise disjoint, and then the ideals $\{\mathcal{I}_{G_i} : i \in n\}$ are mutually orthogonal. Hence, in that case, if $\{I_i : i \in n\}$ are a multiple gap, then so are the larger ideals $\{\mathcal{I}_{G_i} : i \in n\}$.

PROPOSITION 11. *In Theorem 10, if the ideals I_i are analytic, then the sets G_i can be taken to be Borel.*

Proof. Let us compute the complexity of the sets G_i obtained in the proof of (2) \Rightarrow (4) of Theorem 10. The third equality below follows from the

fact that nonempty G_δ subsets of ω^* have nonempty interior. We have

$$\begin{aligned} G_i &= \phi(U(I_i)) = \{\xi \in 2^\omega : \exists t \in U(I_i) \phi(t) = \xi\} \\ &= \{\xi \in 2^\omega : \exists c \subset U(I_i), c \text{ clopen}, \phi|_c = \xi\} \\ &= \{\xi \in 2^\omega : \exists a \in I_i \phi|_{\text{clopen}(a)} = \xi\} \\ &= \{\xi \in 2^\omega : \exists a \in I_i \forall k \exists m a \setminus m \subset a_{(\xi_0, \dots, \xi_k)}\}. \end{aligned}$$

Hence, if the ideals I_i are analytic, then the sets G_i are analytic. Using a generalized version of Lusin's separation theorem (which is proved in the same manner as Lemma 9), we can find Borel sets G'_i with $G_i \subset G'_i$ and still $\bigcap_{i \in A} G'_i = \emptyset$ for all $A \in \mathfrak{X}$. ■

Notice that the fact that G is Borel does not mean that the ideal \mathcal{I}_G is Borel. Indeed one can check that $\mathcal{I}_\mathbb{Q}$ is true coanalytic. However, when G is G_δ we have:

PROPOSITION 12. *If G is a G_δ set, then \mathcal{I}_G is Borel, indeed an $F_{\sigma\delta}$ set.*

Proof. Let $G = \bigcap_{m < \omega} U_m$ with U_m open. For every m , let $U_m = \bigcup_{k < \omega} V_m^k$ where each V_m^k is open, $V_m^1 \subset V_m^2 \subset \dots$ and $\overline{V_m^k} \subset U_m$. Then

$$\mathcal{I}_G = \bigcap_{m < \omega} \bigcup_{k < \omega} \bigcup_{F \subset \omega \text{ finite}} \{a \subset D : a \setminus F \subset V_m^k\}$$

and the sets on the right are closed. Let us check each inclusion. For $[\subset]$, assume that $a \in \mathcal{I}_G$. Then $\text{acc}(a) \subset U_m$ for every m . Fix such an m . By compactness of $\text{acc}(a)$, there exists k such that $\text{acc}(a) \subset V_m^k$. Since V_m^k is open and L is compact, there exists a finite $F \subset \omega$ such that $a \setminus F \subset V_m^k$. For $[\supset]$, assume that for every m there exist k and a finite $F \subset \omega$ such that $a \setminus F \subset V_m^k$. Then $\text{acc}(a) \subset \overline{V_m^k} \subset U_m$ for every m , so $\text{acc}(a) \subset G$. ■

For the rest of this section, L is a compact metric space, $D \subset L$ is countable and $K = \text{acc}(D)$.

LEMMA 13. *Let $Z \subset L$ and let $\{V_i : i \in n\}$ be relatively open subsets of Z with $\bigcap_{i \in n} V_i = \emptyset$. Then there exist open subsets $\{U_i : i \in n\}$ of L such that $U_i \cap Z = V_i$ and $\bigcap_{i \in n} U_i = \emptyset$.*

Proof. It is enough to prove the case when Z is closed in L (if it is not, we can deal with relatively open subsets of \overline{Z}). Let $f_i : Z \rightarrow [0, 1]$ be continuous functions such that $f_i^{-1}(0) = Z \setminus V_i$. Let $f : Z \rightarrow [0, 1]^n$ be given by putting together the functions f_i , $i < n$. Then $f(Z) \subset H = \{(t_i)_{i \in n} : \exists i t_i = 0\} \subset [0, 1]^n$. Notice that H is a retract of $[0, 1]^n$, the retraction being $r((t_i)_{i \in n}) = (t_i - \min[t_j : j \in n])_{i \in n}$. Hence, there exists a continuous function $\hat{f} : L \rightarrow H$ such that $\hat{f}|_Z = f$. The sets $U_j = \hat{f}^{-1}(\{(t_i)_{i \in n} \in H : t_j > 0\})$ are the ones we are looking for. ■

Given a countable compact set S , we denote by S' its Cantor–Bendixson derivative, that is, the set of all points of S which are not isolated in S . Inductively, we define $S^{(n)} = [S^{(n-1)}]'$, assuming that $S^{(0)} = S$. The space S has *height* n if $S^{(n)} = \emptyset$ but $S^{(n-1)} \neq \emptyset$.

THEOREM 14. *Let $\{G_i : i \in n\}$ be pairwise disjoint subsets of K . The following are equivalent:*

- (1) $\{\mathcal{I}_{G_i} : i \in n\}$ is an n -gap.
- (2) For any open subsets $\{U_i : i \in n\}$ of L such that $G_i \subset U_i$ for every $i \in n$, we have $\bigcap_{i \in n} U_i \neq \emptyset$.
- (3) There exists a countable compact $S \subset K$ of height n and a bijection $\sigma : n \rightarrow n$ such that
 - $S^{(j)} \setminus S^{(j+1)} \subset G_{\sigma(j)}$ for $j \in n$.

Proof. For (1) \Rightarrow (2), suppose that there exist open sets $U_i \subset L$ with $G_i \subset U_i$ and $\bigcap_{i \in n} U_i = \emptyset$. Let $a_i = D \cap U_i$. Then $\bigcap_{i \in n} a_i = \emptyset$ and if $x \in \mathcal{I}_{G_i}$, then $x \subset^* a_i$. Hence the ideals $\{\mathcal{I}_{G_i} : i \in n\}$ are separated by the a_i 's and they are not an n -gap. For (2) \Rightarrow (1), suppose that the ideals are not a multiple gap, so there exist $a_i \subset D$ such that $\bigcap_{i \in n} a_i = \emptyset$ and $x \subset^* \mathcal{I}_{G_i}$ whenever $x \in \mathcal{I}_{G_i}$. Let $V_i = K \setminus \text{acc}(D \setminus a_i)$. These are relatively open subsets of K . On the one hand,

$$\begin{aligned} \bigcap_{i \in n} V_i &= K \setminus \bigcup_{i \in n} \text{acc}(D \setminus a_i) = K \setminus \text{acc}\left(\bigcup_{i \in n} D \setminus a_i\right) \\ &= K \setminus \text{acc}\left(D \setminus \bigcap_{i \in n} a_i\right) = K \setminus \text{acc}(D) = \emptyset. \end{aligned}$$

On the other hand, $G_i \subset V_i$ because if $\xi \notin V_i$, then there is a convergent sequence (d_n) contained in $D \setminus a_i$ that converges to ξ . If, in addition $\xi \in G_i$, then $x = \{d_n : n < \omega\} \in \mathcal{I}_{G_i}$, hence $x \subset^* a_i$, a contradiction. We use Lemma 13 to get the open sets U_i from the relatively open sets V_i .

For (3) \Rightarrow (2), we proceed by induction on n . Suppose that $G_i \subset U_i$ with U_i open, $i \in n$. Then $G_{\sigma(n-1)} \subset U_{\sigma(n-1)}$, hence $U_{\sigma(n-1)}$ contains a point ξ_1 of the level $S^{(n-1)}$ of the scattered space S . Being open, $U_{\sigma(n-1)}$ contains a copy of a countable compact space R of height $n-1$ contained in $S \setminus S^{(n-1)}$, which is a clopen subset of S . Notice that $R^{(j)} \setminus R^{(j+1)} = (S^{(j)} \setminus S^{(j+1)}) \cap R$ for every $j \in n-1$. Hence $R^{(k)} \setminus R^{(k+1)} \subset U_{\sigma(k)} \cap U_{\sigma(n-1)}$ for $k \in n-1$. We can then apply the inductive hypothesis to conclude that $\bigcap_{k \in n-1} U_{\sigma(k)} \cap U_{\sigma(n-1)} \neq \emptyset$.

The implication (2) \Rightarrow (3) requires more work. We suppose that (3) does not hold and we will find open sets $V_i \supset G_i$ with $\bigcap_{i \in n} V_i = \emptyset$. We proceed by induction on n . In case $n = 2$, the failure of (3) simply means that there is no sequence of elements of G_0 converging to a point of G_1 or vice versa. Then $G_0 \subset V_0 = \overline{G_0} \cup \overline{G_1} \setminus \overline{G_1}$ and $G_1 \subset V_1 = \overline{G_0} \cup \overline{G_1} \setminus \overline{G_0}$. The sets V_0

and V_1 are relatively open in $\overline{G_0 \cup G_1}$ and $V_0 \cap V_1 = \emptyset$, hence by Lemma 13, G_0 and G_1 are separated by open sets. Now we prove the implication for a fixed integer n assuming it holds for all integers less than n . We start with a claim showing that separation by open sets holds locally:

CLAIM. *If $\xi \in \bigcup_{i \in n} G_i$ then there exists an open neighborhood W of ξ and open sets $V_j \subset W$ for $j \in n$ such that $G_j \cap W \subset V_j$ and $\bigcap_{j \neq n} V_j = \emptyset$.*

Proof of the claim. Fix $i \in n$ such that $\xi \in G_i$. We will prove that the claim holds with the additional fact that $V_i = W$. If, reasoning towards a contradiction, we were unable to find $\{V_j : j \neq i\}$ as required for any neighborhood W of ξ , then the sets $\{G_j \cap W : j \neq i\}$ cannot be separated by open sets for any neighborhood W of ξ . By inductive hypothesis, this means that we can find a bijection $\sigma : n - 1 \rightarrow n \setminus \{i\}$ and a countable compact S of height $n - 1$ such that $S^{(k)} \setminus S^{(k+1)} \subset G_{\sigma(k)} \cap W$ for every $k \in n - 1$. Since we can do this for every neighborhood of ξ , we can find a sequence $\{S_m : m < \omega\}$ of pairwise disjoint ⁽²⁾ countable compact spaces of height $n - 1$ such that

- $S_m^{(k)} \setminus S_m^{(k-1)} \subset G_{\sigma(k)}$ for every $k \in n - 1$ and every $m < \omega$,
- $\overline{\bigcup_{m < \omega} S_m} = \{\xi\} \cup \bigcup_{m < \omega} S_m$.

Now, the space $S_\infty = \{\xi\} \cup \bigcup_{m < \omega} S_m$ is a countable compact space such that $S_\infty^{(k)} \setminus S_\infty^{(k+1)} \subset G_{\sigma(k)}$ and $S_\infty^{(n-1)} \setminus S_\infty^{(n)} = \{\xi\} \subset G_i$. This means that condition (3) holds, and we are supposing that it does not. This finishes the proof of the claim.

Now let \mathcal{F}_0 be the family of all open subsets W of K such that there exist open sets $V_i(W) \subset W$ with $\bigcap_{i \in n} V_i(W) = \emptyset$ and $G_i \cap W \subset V_i(W)$. Let \mathcal{F}_1 be the family of all open subsets W_1 of K such that there exists $W_0 \in \mathcal{F}_0$ such that $\overline{W_1} \subset W_0$. Let $Z = \bigcup \mathcal{F}_0 = \bigcup \mathcal{F}_1$. By the Claim above, $\bigcup_{i \in n} G_i \subset Z$. Since Z is a metric space, it is paracompact [16], so there exists a locally finite open refinement \mathcal{F}_2 of \mathcal{F}_1 which covers Z . For every $W \in \mathcal{F}_2$, we fix $W_0 \in \mathcal{F}_0$ such that $\overline{W} \subset W_0$. For every $i \in n$, let

$$V_i = \bigcap \{V_i(W_0) \cup (Z \setminus \overline{W}) : W \in \mathcal{F}_2\}.$$

We have the following properties:

- Each V_i is an open set because it is locally a finite intersection of open sets. Indeed, since \mathcal{F}_2 is locally finite, every point ξ has a neighborhood $U \subset Z$ that meets only finitely many sets $W \in \mathcal{F}_2$. Let U_0 be an open neighborhood of ξ with $\overline{U_0} \subset U$. Then $U_0 \subset Z \setminus \overline{W}$ whenever $W \cap U = \emptyset$, hence

⁽²⁾ Inductively, we find S_m inside $W_m \setminus \bigcup_{m' < m} S_{m'}$, where $\{W_m : m < \omega\}$ is a neighborhood basis of ξ .

$$U_0 \cap V_i = U_0 \cap \bigcap \{V_i(W_0) \cup (Z \setminus \overline{W}) : W \cap U \neq \emptyset\}$$

is open because it is a finite intersection of open sets.

- $G_i \subset V_i$ for every $i \in n$. Indeed, if $\xi \in G_i$ and $W \in \mathcal{F}_2$ then we have two options. First, if $\xi \in W_0$, then $\xi \in G_i \cap W_0 \subset V_i(W_0)$. Second, if $\xi \notin W_0$, then $\xi \in Z \setminus \overline{W}$ since $\overline{W} \subset W_0$.
- $\bigcap_{i \in n} V_i = \emptyset$. Indeed, suppose $\xi \in \bigcap_{i \in n} V_i$. Since $\xi \in Z$, consider $W \in \mathcal{F}_2$ such that $\xi \in W$. Then $\xi \notin Z \setminus \overline{W}$, so since $\xi \in \bigcap_{i \in n} V_i$, it follows that $\xi \in \bigcap_{i \in n} V_i(W_0) = \emptyset$, a contradiction.

This together with Lemma 13 finishes the proof. ■

THEOREM 15. *Let $\{G_i : i \in n\}$ be disjoint subsets of K such that $\{\mathcal{I}_{G_i} : i \in n\}$ are an n -gap. Then the ideals $\{\mathcal{I}_{G_i} : i \in n\}$ are a jigsaw. Moreover, in the definition of jigsaw, the element b can always be chosen so that the ideals $\mathcal{I}_{G_i}|_b$ are Borel $F_{\sigma\delta}$.*

Proof. Let $a \subset D$ and $A \subset n$ be such that $\{\mathcal{I}_{G_i}|_a : i \in A\}$ form a multiple gap. Let $B \subset A$. We have to find $b \subset a$ such that $\{\mathcal{I}_{G_i}|_b : i \in B\}$ is a multiple gap and $b \in \mathcal{I}_{G_i}^\perp$ for $i \in A \setminus B$. We can suppose, without loss of generality, that $D = a$, $\text{acc}(a) = K$ and $A = n$, because $\{\mathcal{I}_{G_i}|_a : i \in A\}$ is just $\{\mathcal{I}_{G_i \cap \text{acc}(a)} : i \in A\}$ (taking in the latter case a instead of D as the ground countable set). By Theorem 14 there exists a countable compact $S \subset K$ of height n and a bijection $\sigma : n \rightarrow n$ such that $S^{(j)} \setminus S^{(j+1)} \subset G_{\sigma(j)}$ for all $j \in n$. Let us enumerate the set B as $B = \{\sigma(i_j) : j < r\}$ with $i_0 < i_1 < \dots$. Elementary manipulation of countable compact spaces shows that we can find a compact subset $R \subset S$ of height r such that

$$R^{(j)} \setminus R^{(j+1)} \subset S^{(i_j)} \setminus S^{(i_j+1)} \subset G_{\sigma(i_j)}$$

for all $j \in r$. Also, it is possible to find $b \subset D$ such that $\text{acc}(b) = R$ (for example consider a distance ρ metrizing L , enumerate $R = \{\xi_p : p < \omega\}$, let $\xi_p^m \in D$ be such that $\rho(\xi_p^m, \xi_p) < 1/m$ and set $b = \{\xi_p^m : m > p\}$). The set b is the one we are looking for. On the one hand,

$$\text{acc}(b) \subset R \subset \bigcup_{j < r} G_{\sigma(i_j)} = \bigcup_{i \in B} G_i,$$

hence $b \in \mathcal{I}_{G_i}^\perp$ for $i \notin B$. On the other hand $\{\mathcal{I}_{G_i}|_b : i \in B\}$ is the same as $\{\mathcal{I}_{G_i \cap R} : i \in B\}$, and this is a multiple gap by Theorem 14, since $R^{(j)} \setminus R^{(j+1)} \subset G_{\sigma(i_j)}$. Moreover, the ideals $\mathcal{I}_{G_i \cap R}$ are Borel $F_{\sigma\delta}$ sets by Proposition 12: since R is countable, all its subsets are G_δ . ■

COROLLARY 16. *If $\{I_i : i \in n\}$ is a strongly countably separated multiple gap in $\mathcal{P}(\omega)/\text{fin}$, then there exists a strongly countably separated dense jigsaw $\{J_i : i \in n\}$ such that $I_i \subset J_i$ for every $i \in n$.*

Proof. By Theorem 10 we have $I_i \subset \mathcal{I}_{G_i}$ where the G_i 's are pairwise disjoint. We can suppose that $\bigcup G_i = K$, so that $\{\mathcal{I}_{G_i} : i \in n\}$ is dense. This is obviously a multiple gap since the smaller ideals I_i were already a multiple gap. By Theorem 15 it is indeed a jigsaw. ■

COROLLARY 17. *If $\{I_i : i \in n\}$ is a strongly countably separated dense n -gap in $\mathcal{P}(\omega)/\text{fin}$, then it is a jigsaw.*

Proof. By Corollary 16, we have $I_i \subset J_i$ with $\{J_i : i \in n\}$ being a jigsaw. But this together with density implies that $\{I_i : i \in n\}$ is also a jigsaw. Indeed, suppose $\{I_i|_a : i \in A\}$ is a multiple gap, and consider $B \subset A$. Then $\{J_i|_a : i \in A\}$ is also a multiple gap, hence there exists $b \subset a$ such that $\{J_i|_b : i \in B\}$ is a multiple gap and $b \in J_i^\perp$ for $i \notin B$. Then $b \in I_i^\perp$ for $i \notin B$ since $I_i \subset J_i$. Also $\{I_i|_b : i \in B\}$ is a multiple gap. Indeed, if $a_i \geq I_i|_b$, then $a_i \geq J_i|_b$. For suppose $x \not\geq a_i$ with $x \in J_i$, $x \leq b$. Then by density, there exist j and $z \in I_j$ with $z \subset x \setminus a_i$. Then $z \in J_i$ so we must have $j = i$. Hence $z \in I_i$ and $z \not\geq a_i$, a contradiction. ■

COROLLARY 18. *If $\{I_i : i \in n\}$ is a strongly countably separated dense n -gap in $\mathcal{P}(\omega)/\text{fin}$, then there exists $a \subset \omega$ and a strongly countably separated jigsaw $\{J_i : i \in n\}$ made of Borel $F_{\sigma\delta}$ ideals such that $I_i|_a \subset J_i$ for every $i \in n$.*

5.1. Example of a strongly countably separated clover. Let $T = 2^{<\omega}$ be the dyadic tree. We work in $\mathcal{P}(T)/\text{fin}$. Every element $x \in 2^\omega$ is viewed as a branch of T , in particular it is an infinite subset of T . Given $S \subset 2^\omega$, we denote by I_S the ideal of $\mathcal{P}(T)/\text{fin}$ generated by S . A Bernstein set is a subset $S \subset 2^\omega$ that uncountably meets every uncountable Borel subset of 2^ω . Bernstein sets can be constructed by transfinite induction by enumerating 2^ω and its Borel subsets by the ordinal \mathfrak{c} . Bernstein's classical result states that 2^ω can be divided into two Bernstein sets, but the same argument can provide a decomposition into any finite number of Bernstein sets.

PROPOSITION 19. *Let $\{S_i : i \in n\}$ be pairwise disjoint Bernstein subsets of 2^ω . Then the ideals $\{I_{S_i} : i \in n\}$ constitute a clover n -gap which is strongly countably separated.*

Proof. It is clear that the ideals are mutually orthogonal since $x \cap y =^* \emptyset$ for any branches $x \neq y$. If the ideals were not a multiple gap, then we would have elements $c_i \geq I_i$ with $\bigcap_{i \in n} c_i =^* \emptyset$. Consider the sets $Z_i = \{x \in [T] : x \subset^* c_i\} \subset 2^\omega$ (remember that $[T]$ is naturally identified with 2^ω). The sets Z_i are Borel and $S_i \subset Z_i$. Hence each Z_i is co-countable and in particular $\bigcap_{i \in n} Z_i \neq \emptyset$. On the other hand $\bigcap_{i \in n} Z_i = \emptyset$ because $\bigcap_{i \in n} c_i =^* \emptyset$, a contradiction. The ideals are strongly countably separated because if we put on $[T] = 2^\omega$ the Cantor set topology and we identify

each $t = (t_0, \dots, t_k) \in T$ with the branch $(t_0, t_1, \dots, t_k, 0, 0, \dots)$, then for any $S \subset [T]$ we have $I_S \subset \mathcal{I}_S$, so we can use Theorem 10. To show that the ideals are a clover, suppose that $d \subset T$ with $d \in I_{S_j}^\perp$ for some j . Then $X = \{x \in [T] : x \cap d \neq \emptyset\}$ is a Borel set disjoint from S_j . Since S_j is a Bernstein set, it follows that X is countable. Therefore each ideal $I_{S_i}|_d$ is countably generated for $i \neq j$. Countably generated orthogonal ideals in $\mathcal{P}(\omega)/\text{fin}$ can always be separated. ■

6. Multiple gaps made of small ideals. Given a cardinal θ , we say that an ideal is θ -generated if it is generated by a set of cardinality θ . Given $k \in \omega$, a partially ordered set (poset, for short) \mathbb{P} is σ - k -linked if $\mathbb{P} = \bigcup_{m < \omega} P_m$ where for every m , any k elements of P_m are compatible (that is, for every $\{p_i : i < k\} \subset P_m$ there exists $p \in \mathbb{P}$ such that $p \leq p_i$ for every $i \in k$). We write $\text{MA}_\theta(\sigma$ - k -linked) for Martin's axiom for the cardinal θ and σ - k -linked posets: In a σ - k -linked poset, every family of θ many dense subsets has a generic filter. This is weaker than the usual MA_θ , the same statement but for ccc partial orders. Let \mathfrak{X} be a family of subsets of n which is *upwards closed*, that is: if $A \in \mathfrak{X}$ and $B \supset A$, then $B \in \mathfrak{X}$. We denote by $\mathfrak{X}[k]$ the family of all $A \subset n$ such that whenever $A = \bigcup_{j \in k} B_j$ then there exists $j \in k$ such that $B_j \in \mathfrak{X}$. One example is given by $\mathfrak{X} = \{A \subset n : |A| \geq 2\}$ and $\mathfrak{X}[k] = \{A \subset n : |A| > k\}$. All along this section, θ is a cardinal and $k \geq 1$ an integer. We notice that the results of this section hold in ZFC when we take $k = 1$ and $\theta = \aleph_0$.

THEOREM 20 ($\text{MA}_\theta(\sigma$ - k -linked)). *Let $\{I_i : i \in n\}$ be θ -generated ideals of $\mathcal{P}(\omega)/\text{fin}$ and \mathfrak{X} an upwards closed family of subsets of n such that $\bigcap_{i \in A} x_i =^* \emptyset$ whenever $A \in \mathfrak{X}$ and $x_i \in I_i$ for every i . Then there exist $c_i \subset \omega$, $i \in n$, such that $I_i \leq c_i$ for $i \in n$, and $\bigcap_{i \in A} c_i = \emptyset$ for $A \in \mathfrak{X}[k]$.*

Proof. To apply $\text{MA}_\theta(\sigma$ - k -linked) we consider a poset \mathbb{P} , whose elements are of the form $p = (x_p^i, i \in n)$ where $x_p^i \in I_i$ for every $i \in n$ and $\bigcap_{i \in A} x_p^i = \emptyset$ for every $A \in \mathfrak{X}[k]$. The order relation is given by $p \leq q$ if and only if $x_p^i \supset x_q^i$ for all $i \in n$.

CLAIM. \mathbb{P} is σ - k -linked.

Proof of the claim. Given a family $\{u^B : B \in \mathfrak{X}\}$ of finite subsets of ω , let Q be the set of all $p \in \mathbb{P}$ such that $u^B = \bigcap_{i \in B} x_p^i$ for $B \in \mathfrak{X}$. We prove that $p[0], \dots, p[k-1] \in Q$ are compatible for each k . For this we show that if we define $r_i = \bigcup_{j \in k} x_{p[j]}^i$, then $r = (r_i)_{i \in n} \in \mathbb{P}$ and $r \leq p[j]$ for every $j \in k$. The second part is obvious, so we only have to check that $r \in \mathbb{P}$. So we fix $A \in \mathfrak{X}[k]$ and we prove that $\bigcap_{i \in A} r_i = \emptyset$. Suppose that $m \in \bigcap_{i \in A} r_i$. Set $B_j = \{i \in A : m \in x_{p[j]}^i\}$. Then $A = \bigcup_{j \in k} B_j$, so since $A \in \mathfrak{X}[k]$, there exists j_0 such that $B_{j_0} \in \mathfrak{X}$. For simplicity we suppose $j_0 = 0$. Keep in mind that,

since \mathfrak{X} is upwards closed, this implies that $B_0 \cup \dots \cup B_l \in \mathfrak{X}$ for all l . Then

$$\begin{aligned}
m &\in \bigcap_{i \in B_0} x_{p[0]}^i \cap \bigcap_{i \in B_1} x_{p[1]}^i \cap \dots \cap \bigcap_{i \in B_{k-1}} x_{p[k-1]}^i \\
&= u^{B_0} \cap \bigcap_{i \in B_1} x_{p[1]}^i \cap \dots \cap \bigcap_{i \in B_{k-1}} x_{p[k-1]}^i \\
&= \bigcap_{i \in B_0} x_{p[1]}^i \cap \bigcap_{i \in B_1} x_{p[1]}^i \cap \bigcap_{i \in B_2} x_{p[2]}^i \cap \dots \cap \bigcap_{i \in B_{k-1}} x_{p[k-1]}^i \\
&= u^{B_0 \cup B_1} \cap \bigcap_{i \in B_2} x_{p[2]}^i \cap \bigcap_{i \in B_3} x_{p[3]}^i \cap \dots \cap \bigcap_{i \in B_{k-1}} x_{p[k-1]}^i \\
&= \dots = u^{B_0 \cup \dots \cup B_{k-1}} = u^A = \emptyset.
\end{aligned}$$

This contradiction finishes the proof of the claim.

Now, for every $x = (x^i, i \in n) \in \prod_{i \in n} I_i$, the set

$$D_x = \{p \in \mathbb{P} : x^i \subset^* x_p^i, i \in n\}$$

is a dense subset of \mathbb{P} . Since the ideals are θ -generated, we have cofinal subsets $S_i \subset I_i$ with $|S_i| = \theta$. By $\text{MA}_\theta(\sigma$ - k -linked), there is a filter G in \mathbb{P} which is generic for all the dense sets D_x , $x \in \prod_{i \in n} S_i$. This implies that G is generic for all the dense sets D_x , $x \in \prod_{i \in n} I_i$. The sets $c_i = \bigcup \{x_p^i : p \in G\}$ for $i \in n$ are as desired. ■

COROLLARY 21 ($\text{MA}_\theta(\sigma$ - k -linked)). *For $n > k$, there exist no n -gaps of θ -generated ideals in $\mathcal{P}(\omega)/\text{fin}$.*

Proof. Let $\{I_i : i \in n\}$ be mutually orthogonal ideals. Then the hypotheses of Theorem 20 hold for $\mathfrak{X} = \{A \subset n : |A| \geq 2\}$, hence there exist elements $c_i \geq I_i$ with $\bigcap_{i \in A} c_i = \emptyset$ for all $A \in \mathfrak{X}[k] = \{A \subset n : |A| > k\}$. ■

The next corollary is just a topological interpretation of Theorem 20. An open subset of a topological space X is called F_θ if it is the union of θ many closed sets. We point out that, by Lemma 9, the conclusion of Corollary 22 is equivalent to saying that $\bigcap_{i \in A} \overline{U_i} = \emptyset$ for all $A \in \mathfrak{X}[k]$.

COROLLARY 22 ($\text{MA}_\theta(\sigma$ - k -linked)). *Let $\{U_i : i \in n\}$ be open F_θ subsets of ω^* , and \mathfrak{X} an upwards closed family of subsets of n such that $\bigcap_{i \in A} U_i = \emptyset$ for all $A \in \mathfrak{X}$. Then there exist clopen sets $C_i \supset U_i$ such that $\bigcap_{i \in A} C_i = \emptyset$ for all $A \in \mathfrak{X}[k]$.*

We present a result for the Banach lattice $C(\omega^*)$ of real-valued continuous functions on ω^* (that is, the Banach lattice ℓ_∞/c_0), which is a consequence of Theorem 20. We use the following notation: Given $\mathcal{F} \subset C(\omega^*)$ and $g \in C(\omega^*)$, $g \leq \mathcal{F}$ means that $g \leq f$ for all $f \in \mathcal{F}$.

THEOREM 23 (MA $_{\theta}(\sigma$ - k -linked)). *Let $\{\mathcal{F}_i \geq 0 : i < n\}$ be n subsets of $C(\omega^*)$ of cardinality θ such that $\sum_{i \in n} f_i \leq 1$ whenever $f_i \in \mathcal{F}_i$. Then there exist functions $g_i \in C(\omega^*)$ such that $g_i \geq \mathcal{F}_i$ and $\sum_{i \in n} g_i \leq k$.*

Proof. As a first step, we prove the following claim:

CLAIM. *Given $\varepsilon > 0$ there exist functions $g_i \geq \mathcal{F}_i$ such that $\sum_{i \in n} g_i \leq k + \varepsilon$.*

Proof of the claim. Consider m such that $n/2^m < \varepsilon$. Let $\Delta = \{j/2^m : j = 0, \dots, 2^m - 1\}$. For every $\delta \in \Delta$ and $i \in n$, let

$$U_{\delta}^i = \{x \in \omega^* : \exists f \in \mathcal{F}_i \ f(x) > \delta\}.$$

These are F_{θ} open subsets of ω^* . Given $\delta_* = (\delta_0, \dots, \delta_{n-1}) \in \Delta^n$, we define $\mathfrak{X}_{\delta_*} = \{A \subset n : \sum_{i \in A} \delta_i \geq 1\}$. Notice that $\mathfrak{X}_{\delta_*}[k] \supset \{A \subset n : \sum_{i \in A} \delta_i \geq k\}$. By Corollary 22 we obtain clopen sets $C_{\delta_*}^i \supset U_{\delta_i}^i$ such that $\bigcap_{i \in A} C_{\delta_*}^i = \emptyset$ whenever $A \in \mathfrak{X}_{\delta_*}[k]$. For $\delta \in \Delta$ and $i \in n$, set $D_{\delta}^i = \bigcap \{C_{\delta_*}^i : \delta_i = \delta\}$. Observe that $D_{\delta}^i \supset U_{\delta}^i$. Define then

$$g_i(x) = 2^{-m} + \max\{\delta \in \Delta : x \in D_{\delta}^i\}.$$

We define that maximum to be 0 if $x \notin \bigcup_{\delta \in \Delta} D_{\delta}^i$. We check the desired properties:

- The functions g_i are continuous because the D_{δ}^i are finitely many clopen sets.
- $g_i \geq \mathcal{F}_i$. Let $f \in \mathcal{F}_i$ and $x \in \omega^*$ with $f(x) > 0$. Pick $\delta \in \Delta$ such that $\delta < f(x) \leq \delta + 2^{-m}$. Then $x \in U_{\delta}^i \subset D_{\delta}^i$, hence $g_i(x) \geq 2^{-m} + \delta \geq f(x)$.
- $\sum_{i \in n} g_i \leq k + \varepsilon$. Pick $x \in \omega^*$. Write $g_i(x) = 2^{-m} + \delta_i$, so that $x \in D_{\delta_i}^i$ for $i \in A = \{j : \delta_j \neq 0\}$. Consider $\delta_* = (\delta_0, \dots, \delta_{n-1})$. We have

$$x \in \bigcap_{i \in A} D_{\delta_i}^i \subset \bigcap_{i \in A} C_{\delta_*}^i,$$

so in particular the intersection is nonempty, so $A \notin \mathfrak{X}_{\delta_*}[k]$, hence $\sum_{i \in n} \delta_i = \sum_{i \in A} \delta_i < k$. Therefore $\sum_{i \in n} g_i(x) < n2^{-m} + k < \varepsilon + k$.

This finishes the proof of the Claim.

The Claim allows us to find functions g_i^m for $i \in n$ and $m \in \omega$ such that $g_i^m \geq \mathcal{F}_i$ and $\sum_{i \in n} g_i^m \leq k + 2^{-m}$. We can take these functions so that $g_i^0 \geq g_i^1 \geq g_i^2 \geq \dots$. Moreover, the hypotheses of the theorem imply that $0 \leq \mathcal{F}_i \leq 1$ for each i , so we can suppose that $g_i^m : \omega^* \rightarrow [0, 1]$ (we could replace g_i^m by $\min(g_i^m, 1)$). To finish the proof, we find functions $g_i \in C(\omega^*)$ such that $g_i \geq \mathcal{F}_i$ and $g_i \leq g_i^m$ for all m . This is a consequence of the fact that under MA $_{\theta}(\sigma$ -centered) there are no (\aleph_0, θ) -gaps in $\mathcal{P}(\omega)/\text{fin}$, since gaps in $\mathcal{P}(\omega)/\text{fin}$ are related to gaps in $C(\omega^*)$. Anyway, we provide a direct proof. We show that if $\mathcal{F} \geq 0$ has cardinality θ and $g^1 \geq g^2 \geq \dots \geq \mathcal{F}$, then there exists g such that $g^1 \geq g^2 \geq \dots \geq g \geq \mathcal{F}$. We consider continuous functions

$\hat{g}_m : \beta\omega \rightarrow [0, 1]$ such that $\hat{g}_m|_{\omega^*} = g^m$. Notice that for $\phi, \psi \in C(\beta\omega)$ we have $\phi|_{\omega^*} \leq \psi|_{\omega^*}$ if and only if $\phi(k) \leq \psi(k)$ for all but finitely many $k \in \omega$. Consider a poset \mathbb{P} whose elements are pairs of the form $p = (h^p, s^p)$ where $h^p : \beta\omega \rightarrow [0, 1]$ is continuous and $s^p \in \omega^{<\omega}$ with the following properties:

- $h^p|_{\omega^*} \leq g^m$ for every $m < \omega$,
- $h^p(k) \leq \hat{g}_m(k)$ for every $m < \text{length}(s^p)$ and every $k > s^p_m$.

The order relation is that $p \leq q$ if $h_q \leq h_p$ (as functions on $\beta\omega$) and $s^q \leq s^p$ (as elements of the tree $\omega^{<\omega}$). This partial order is σ - k -linked (even σ -centered) since a finite number of conditions with a fixed s^p are compatible. We have a filter G that is generic for each of the dense sets $D_m = \{p : \text{length}(s_p) > m\}$ for $m \in \omega$ and $D_f = \{p : h_p|_{\omega^*} \geq f\}$ for $f \in \mathcal{F}$. Let $g(k) = \sup_{p \in G} h_p(k)$ for $k \in \omega$, and let $g : \beta\omega \rightarrow [0, 1]$ be a continuous extension. The function $g|_{\omega^*}$ is as desired. ■

We notice that if there exists a k -gap of θ -generated ideals in $\mathcal{P}(\omega)/\text{fin}$, then the constant k of Theorem 23 cannot be improved. Namely, consider a multiple gap $\{I_i : i \in k\}$, and $\mathcal{F}_i \subset C(\omega^*)$ the set of characteristic functions of θ many clopens whose union is $U(I_i)$. Suppose that $g_i \geq \mathcal{F}_i$. Pick $x \in \bigcap_{i \in k} \overline{U(I_i)}$. Then $g_i(x) \geq 1$ for every i , hence $\sum_{i \in k} g_i(x) \geq k$.

THEOREM 24. *Let M be a model of ZFC. Let $\aleph_0 < \theta_1 < \theta_2 < \dots$ be cardinals in M and $1 < n_1 \leq n_2 \leq \dots$ be integers. Then there exists a ccc generic extension $M(G)$ of M such that for every $j \in \omega$ the following hold in $M(G)$:*

- (1) $\text{MA}_{\theta_j}(\sigma\text{-}n_j\text{-linked})$, so there are no θ_j -generated n -gaps for $n > n_j$.
- (2) There exists an n_j -gap of θ_j -generated ideals in $\mathcal{P}(\omega)/\text{fin}$.

Proof. Let M_1 be the generic extension of M obtained by adding θ_ω many Cohen reals, where θ_ω is any cardinal larger than all cardinals θ_j . For every j , we can consider a set $X_j \subset n_j^\omega$ of Cohen reals of cardinality θ_j . We identify the set n_j^ω with the set of branches of the n_j -adic tree $n_j^{<\omega}$. A set X of Cohen reals is always a *Lusin set*, that is, every nowhere dense subset of X is countable. This implies in particular that each of the sets X_j has property $A(\aleph_1, n_j)$ described as follows:

DEFINITION. Let θ be a cardinal, $n \in \omega$, and $X \subset n^\omega$. We say that X has *property $A(\theta, n)$* if for every $Y \subset X$ with $|Y| = \theta$, there exists $t \in n^{<\omega}$ and elements $\{y^i : i \in n\} \subset Y$ such that $t \frown i \in y^i$ for every $i \in n$.

In M_1 pick a cardinal $\kappa \geq \theta_\omega$ such that $\kappa^{\theta_\omega} = \kappa$. Let M_2 be the generic extension of M_1 obtained by a natural finite support iteration $(\mathbb{P}_\alpha, \mathbb{Q}_\alpha : \alpha < \kappa)$ of posets whose limit \mathbb{P}_κ forces $\text{MA}_{\theta_j}(\sigma\text{-}n_j\text{-linked})$ for every j . Thus, for each of the iterands \mathbb{Q}_α such there is j so that \mathbb{P}_α forces both that $|\mathbb{Q}_\alpha| = \theta_j$ and that \mathbb{Q}_α is σ - n_j -linked. The model M_2 is the one we are looking for.

It remains to find an n_j -gap of θ_j -generated ideals in the model M_2 . For every j we fix a regular cardinal λ_j such that $\theta_{j-1} < \lambda_j \leq \theta_j$ (assume $\theta_0 = \aleph_0$).

DEFINITION. Given a regular cardinal λ and $n \in \omega$, we say that a poset \mathbb{P} has *property* $K(\lambda, n)$ if every subset of \mathbb{P} of cardinality λ has a further subset of cardinality λ which is n -linked.

It is well-known and easily seen that these Knaster-type chain conditions are preserved under finite support iterations. If a poset \mathbb{Q} satisfies either $|\mathbb{Q}| < \lambda$ or it is σ - n' -linked for $n' \geq n$, then \mathbb{Q} has property $K(\lambda, n)$. It follows that the poset that forces the generic extension from M_1 to M_2 has property $K(\lambda_j, n_j)$ for every j .

CLAIM 1. *Let $n \in \omega$ and λ be a regular cardinal. Let $X \subset n^\omega$ have property $A(\lambda, n)$. Let \mathbb{P} be a poset that satisfies $K(\lambda, n)$. Then the set X still has property $A(\lambda, n)$ in the generic extension forced by \mathbb{P} .*

Proof of the claim. Suppose not. Then

$$M(G) \models \exists Y \subset X \ |Y| = \lambda \text{ and } \forall t \in n^{<\omega} \ \exists \iota(t) \in n \ \forall y \in Y \ t \hat{\cap} \iota(t) \notin y.$$

Let \dot{Y} and i be names for Y and the function $\iota : n^{<\omega} \rightarrow n$ respectively. Let $\tilde{Y} = \{x \in X : \exists p \in \mathbb{P} \ p \Vdash x \in \dot{Y}\}$. Then \tilde{Y} is a set in the ground model with $Y \subset \tilde{Y} \subset X$. Since the forcing is $K(\lambda, n)$, in particular λ -cc, the cardinal λ is preserved, hence $|\tilde{Y}| \geq \lambda$. For every $x \in \tilde{Y}$ there exists $p(x) \in \mathbb{P}$ such that

$$\begin{aligned} p(x) \Vdash \dot{Y} \subset X, \ |\dot{Y}| = \lambda \text{ and } x \in \dot{Y}, \\ p(x) \Vdash \forall t \in T \ i(t) \in n, \\ p(x) \Vdash \forall t \in T \ t \hat{\cap} i(t) \notin x. \end{aligned}$$

Since \mathbb{P} has property $K(\lambda, n)$ and $|\tilde{Y}| \geq \lambda$ is a regular cardinal, there exists $Z \subset \tilde{Y}$ with $|Z| = \lambda$ such that $\{p(x) : x \in Z\}$ is n -linked. Since X has property $A(\lambda, n)$ there exist $t \in n^{<\omega}$ and $\{x^j : j \in n\} \subset X$ such that $t \hat{\cap} j \in x^j$ for all $j \in n$. Since Z is n -linked, we can pick a condition $p \leq p(x_j)$ for $j \in n$. Then $p \Vdash t \hat{\cap} i(t) \notin x^j$, hence $p \Vdash i(t) \neq j$ for every $j \in n$, which contradicts the fact that $p \Vdash i(t) \in n$. This finishes the proof of Claim 1.

From Claim 1 and the remarks before it, we see that $M_2 \models X_j$ has property $A(\lambda_j, n_j)$ for every j . It remains to show the following:

CLAIM 2. *Let $n \in \omega$ and λ a regular cardinal. Let $X \subset n^\omega$ with property $A(\lambda, n)$. As in Section 4.1, for every $x \in X$ and $i \in n$ set $a_x^i = \{s \in T : s \hat{\cap} i \in x\}$. Let J_i be the ideal generated by $\{a_x^i : x \in X\}$. Then the ideals $\{J_i : i \in n\}$ form an n -gap that is not weakly countably separated (indeed not even weakly $(<\lambda)$ -separated).*

Proof of the claim. We just follow the same argument as for Theorem 6. Let us denote by $T(m)$ the set of all elements of $t \in T = n^\omega$ of length

less than m . Suppose that the ideals are weakly countably separated, so that we have elements $c_i^k \subset \omega$, $i \in n$, $k \in \omega$, such that $\bigcap_{i \in n} c_i^k = \emptyset$ for each k , and whenever we pick $b_i \in \mathcal{J}_i$ there exists k with $b_i \subset^* c_i^k$. In particular, for every $x \in X$ and every $i \in n$ there exist $k(x), m(x) \in \omega$ such that $a_x^i \setminus T(m(x)) \subset c_i^{k(x)}$ for every $i \in n$. For every $m, k \in \omega$ and every $s \in n^{m+2}$, let

$$X(m, k, s) = \{x \in X : x|_{m+2} = s \text{ and } \forall i \in n \ a_x^i \setminus T(m) \subset c_i^k\}.$$

There exist m_0, k_0, s_0 such that $|X(m_0, k_0, s_0)| \geq \lambda$, and by property $A(\lambda, n)$ we can find $t \in T$ (necessarily of length greater than m_0) and $\{x^i : i \in n\} \subset X(m_0, k_0, s_0)$ with $t \frown i \in x^i$. This implies that $t \in \bigcap_{i \in n} a_{x^i}^i \setminus T(m_0) \subset \bigcap_{i \in n} c_i^{k_0} = \emptyset$, a contradiction. ■

7. Dense multiple gaps from completely separable almost disjoint families. An *almost disjoint family* is a family \mathcal{A} of infinite subsets of ω such that $a \cap b =^* \emptyset$ for all different $a, b \in \mathcal{A}$.

DEFINITION 25. An almost disjoint family \mathcal{A} is called *completely separable* if for every subset $x \subset \omega$ that is not in the ideal generated by \mathcal{A} we have $|\{a \in \mathcal{A} : a \subset^* x\}| = \mathfrak{c}$.

The concept traces back to [11]. Completely separable almost disjoint families exist under the assumption that $\mathfrak{a} = \mathfrak{c}$ [14] or $\mathfrak{s} = \omega_1$ [4, Corollary 2.8]. A recent work of Shelah [15] shows their existence under other assumptions, in particular if $\mathfrak{s} < \mathfrak{a}$ or $\mathfrak{c} < \aleph_\omega$. The consistency of the nonexistence of such families is unknown.

THEOREM 26. *Assume there exists a completely separable almost disjoint family. Let $n \in \omega$ and \mathfrak{X} be a family of proper nonempty subsets of n . Then there exists a dense n -gap $\{I_i : i \in n\}$ in $\mathcal{P}(\omega)/\text{fin}$ that is a B -jigsaw for $B \in \mathfrak{X}$ but a B -clover for $B \notin \mathfrak{X}$ ($|B| \geq 2$).*

Proof. We proceed by induction on n , so we fix n and we suppose the theorem is proven for all $m < n$. For convenience we will assume that \mathfrak{X} includes all singletons. For $n = 1$ we need an ideal and for $n = 2$ we need a 2-gap. Let \mathcal{A} be a completely separable almost disjoint family. We decompose it into disjoint subfamilies \mathcal{A}_B , $B \in \mathfrak{X}$, in such a way that if $x \subset \omega$ is not in the ideal generated by \mathcal{A} , then $|\{a \in \mathcal{A}_B : a \subset^* x\}| = \mathfrak{c}$ for all $B \in \mathfrak{X}$. The decomposition can be made as follows: Let $\{x_\alpha : \alpha < \mathfrak{c}\}$ be an enumeration of all infinite subsets of ω that are not in the ideal generated by \mathcal{A} in which each set is repeated \mathfrak{c} many times. Inductively on α , we will choose elements $\{a_B^\alpha : \alpha < \mathfrak{c}, B \in \mathfrak{X}\} \subset \mathcal{A}$. At step α , since $|\{a \in \mathcal{A} : a \subset^* x\}| = \mathfrak{c}$ and we have chosen only less than \mathfrak{c} elements in the previous steps, we can pick $a_B^\alpha \subset^* x_\alpha$ and different from all previous choices. At the end, it is

enough to declare $a_B^\alpha \in \mathcal{A}_B$ for all $\alpha < \mathfrak{c}$ and all $B \in \mathfrak{X}$ and we will have a decomposition as required.

Now we define the ideals $\{I_i : i \in n\}$. For $B \in \mathfrak{X}$ with $2 \leq |B| < n$ and $x \in \mathcal{A}_B$, we will apply the inductive hypothesis to find a dense multiple gap $\{I_i^x : i \in B\}$ in the Boolean algebra $\mathcal{P}(x)/\text{fin}$ that is a C -jigsaw for all $C \subset B$ with $C \in \mathfrak{X}$ but a C -clover for all $C \subset B$ with $C \notin \mathfrak{X}$. For every $i \in n$, I_i is defined to be the ideal generated by

$$\mathcal{A}_{\{i\}} \cup \bigcup \left\{ I_i^x : x \in \bigcup \{ \mathcal{A}_B : B \in \mathfrak{X}, i \in B, 2 \leq |B| < n \} \right\}.$$

Let us prove that this is a dense multiple gap. Density is clear. The ideals are mutually orthogonal, because if u and v belong to the set of generators of I_i and I_j respectively ($i \neq j$), then: either $u \subset x$ and $v \subset y$ for different $x, y \in \mathcal{A}$ (and \mathcal{A} is almost disjoint); or there exists $x \in \mathcal{A}$ such that $u \in I_i^x$ and $v \in I_j^x$. Finally, suppose that we have elements c_i with $c_i \geq I_i$. We prove inductively on $k < n$ that $\bigcap_{i \in k} c_i$ is not in the ideal generated by \mathcal{A} , in particular it is infinite. So suppose $x = \bigcap_{i \in k} c_i$ is not in the ideal generated by \mathcal{A} , hence there are \mathfrak{c} many $a \in \mathcal{A}_{\{k\}}$ such that $a \subset^* x$. On the other hand, $a \subset^* c_k$ for all $a \in \mathcal{A}_{\{k\}}$, so there are \mathfrak{c} many $a \in \mathcal{A}_{\{k\}}$ with $a \subset^* c_k \cap x = \bigcap_{i \in k+1} c_i$. This proves that the ideals form a multiple gap.

Let us check now that the ideals form a B -jigsaw for all $B \in \mathfrak{X}$. So assume we have $a \subset \omega$ and $A \supset B$ such that $\{I_i|_a : i \in A\}$ form a multiple gap. There are two possibilities: either a is in the ideal generated by \mathcal{A} or it is not. If it is not, then a contains an element $b \in \mathcal{A}_B$. Then $b \in I_i^\perp$ for $i \notin B$ while $\{I_i|_b : i \in B\} = \{I_i^b : i \in B\}$ is a multiple gap, so we are done. The other case is that a is in the ideal generated by \mathcal{A} . Let $F = \{u \in \mathcal{A} : u \cap a \neq \emptyset\}$. This set is finite and $a = \bigcup_{u \in F} a \cap u$. Since we assume that $\{I_i|_a : i \in A\}$ is a multiple gap, there must exist $u \in F$ such that $\{I_i|_{a \cap u} : i \in A\} = \{I_i^u|_{a \cap u} : i \in A\}$ is a multiple gap. For that u , we must have $u \in \mathcal{A}_C$ with $C \supset A$. Since we assumed that the ideals $\{I_i^u : i \in A\}$ are a B -jigsaw, we find $b \subset a \cap u$ such that $b \in I_i^\perp$ for $i \in A \setminus B$ but $\{I_i|_b : i \in B\}$ is a multiple gap. This finishes the proof that the ideals are a B -jigsaw for $B \in \mathfrak{X}$.

We now prove that the ideals are a B -clover for all $B \notin \mathfrak{X}$. Suppose for contradiction that there exists an infinite set $a \subset \omega$ such that $\{I_i|_a : i \in B\}$ is a multiple gap but $a \in I_i^\perp$ for $i \notin B$. Again we distinguish two cases, according to whether a is in the ideal generated by \mathcal{A} or not. If a is not in the ideal, then it contains \mathfrak{c} many elements from $\mathcal{A}_{\{i\}}$ for every $i \in n$, which contradicts $a \in I_i^\perp$ for $i \notin B$. So suppose that a is in the ideal generated by \mathcal{A} and again consider the finite set $F = \{u \in \mathcal{A} : u \cap a \neq \emptyset\}$, so that $a = \bigcup_{u \in F} a \cap u$. Since $\{I_i|_a : i \in B\}$ is a multiple gap, again it follows that for some $u \in F$, $\{I_i|_{a \cap u} : i \in B\}$ is a multiple gap, so $u \in \mathcal{A}_C$ with $C \supset B$. Notice that $C \neq B$ since $B \notin \mathfrak{X}$. We find that $\{I_i|_{a \cap u} : i \in B\} = \{I_i^u|_{a \cap u} : i \in B\}$

is a multiple gap but $a \cap u \in (I_i^u)^\perp$ for $i \in C \setminus B$. This contradicts the hypothesis that the ideals $\{I_i^u : i \in C\}$ are a B -clover. ■

8. Dense jigsaws and injective Banach spaces. In this section we consider an application to the theory of Banach spaces. We refer to [1, 7] for the basic facts and terminology on the subject.

The existence of a gap in a Boolean algebra is equivalent to the failure of completeness. Namely, in a complete Boolean algebra orthogonal ideals can always be separated by their suprema, and conversely if a set S lacks a supremum, the ideal generated by S forms a gap together with its orthogonal ideal. At the same time, completeness is equivalent to injectivity (cf. for instance [21]), meaning that \mathcal{B} is complete if and only if for every superalgebra $\mathcal{B}' \supset \mathcal{B}$ there is a Boolean projection from \mathcal{B}' onto \mathcal{B} . If we look at the Banach space $C(\text{St}(\mathcal{B}))$ of continuous functions we find in a similar spirit that \mathcal{B} is complete if and only if $C(\text{St}(\mathcal{B}))$ is 1-*injective*, meaning that for every superspace $X \supset C(\text{St}(\mathcal{B}))$ there exists a projection $T : X \rightarrow C(\text{St}(\mathcal{B}))$ of norm 1 (cf. for instance [1]). The situation becomes more complicated if we deal with injectivity instead of 1-injectivity of Banach spaces. The definition of injectivity is the same as 1-injectivity but we require the projection T to be just a bounded operator, not necessarily of norm 1. A characterization of injective Banach spaces is an important open problem. In this section we prove that while the existence of a gap is necessary and sufficient for the failure of 1-injectivity, the existence of arbitrarily large dense jigsaws is sufficient for the failure of injectivity.

THEOREM 27. *If \mathcal{B} contains a dense n -jigsaw for every n , then $C(\text{St}(\mathcal{B}))$ is uncomplemented in some superspace X of density character $|\mathcal{B}|$.*

This will be obtained as an application of a result by Ditor [6]. Theorem 27 is a corollary of Theorem 29, whose proof is given at the end of this section, after some lemmas. For $\mathcal{B} = \mathcal{P}(\omega)/\text{fin}$, the assumption of Theorem 27 above holds by Theorem 15 (when $\bigcup_{i \in n} G_i = K$). In this way, we have a proof of a result of Amir [2] that $C(\omega^*)$ is uncomplemented in some superspace of density character \mathfrak{c} . In this connection, we mention the open problem whether $C(\omega^*)$ is uncomplemented in a superspace X with $\text{dens}(X/C(\omega^*)) = \aleph_1$. Theorem 27 can be stated more generally, removing the zero-dimensionality assumption. For this we define:

DEFINITION 28. Let K be a compact space and $\{F_i : i \in n\}$ closed subspaces. We say that they form a *jigsaw of closed sets* if for any nonempty proper subsets $A \supset B$ of n , the set $\bigcap_{i \in B} F_i \setminus \bigcap_{i \in A} F_i$ is dense in $\bigcap_{i \in B} F_i$. Such a jigsaw is called *dense* if $\bigcup_{i \in n} F_i = K$.

A finite family $\{I_i : i \in n\}$ of ideals of \mathcal{B} is a dense jigsaw if and only if the sets $\{\overline{U(I_i)}\}$ form a dense jigsaw of closed sets in $\text{St}(\mathcal{B})$. The more general statement is:

THEOREM 29. *Let K be a compact space. Suppose that for every n there exists a dense jigsaw of n closed sets in K . Then $C(K)$ is uncomplemented in some superspace of the same density character as $C(K)$.*

Let $\{F_i : i \in n\}$ be a dense jigsaw of closed sets. We consider the compact space $L = \bigcup_{i \in n} F_i \times \{i\}$, the disjoint sum of the compact spaces F_i , and $\phi : L \rightarrow K$ the continuous surjection given by $\phi(x, i) = x$. The composition operator $\phi^0 : C(K) \rightarrow C(L)$ given by $\phi^0(f) = f \circ \phi$ is an isometric embedding of Banach spaces. Let $H(L)$ be the hyperspace of L consisting of all nonempty closed subsets of L endowed with the Vietoris topology. We define a derivation process on subsets of K , the derived set $D(X)$ of $X \subset K$ being defined to be the set of all $x \in X$ for which there exist two disjoint closed nonempty sets $R, S \subset \phi^{-1}(x)$ that both belong to the closure in $H(L)$ of the set $\{\phi^{-1}(y) : y \in X\}$. As usual, the iterated derived sets are $D^{(n+1)}(X) = D(D^{(n)}(X))$. This derivation procedure is defined by Ditor [6, Definition 5.3]; in his notation our $D^{(n)}(K)$ would be $\Delta_\phi^{(n)}(2, \dots, 2)$. He proves the following [6, Corollary 5.4]:

LEMMA 30. *If $D^{(k)}(K) \neq \emptyset$ and $T : C(L) \rightarrow C(K)$ is an operator with $T\phi^0 = 1_{C(K)}$, then $\|T\| \geq 1 + k$.*

Here $1_{C(K)}$ denotes the identity map on $C(K)$. In our particular situation, we have:

LEMMA 31. $D^{(k)}(K) = \{x \in K : |\{i \in n : x \in F_i\}| \geq 2^k\}$.

Proof. We prove the lemma by induction. It is obvious for $k=0$ ($D^{(0)}(K) = K$), so we suppose it holds for k and prove it for $k+1$. Suppose $|\{i \in n : x \in F_i\}| \geq 2^{k+1}$. Write $\{i \in n : x \in F_i\}$ as the union $A \cup B$ of two disjoint sets with $|A|, |B| \geq 2^k$. Notice that $\phi^{-1}(x) = \{(x, i) : i \in A \cup B\}$. Because the sets F_i form a jigsaw, x is in the closure of the sets $V = \bigcap_{i \in A} F_i \setminus \bigcup_{i \notin A} F_i$ and $W = \bigcap_{i \in B} F_i \setminus \bigcup_{i \notin B} F_i$, which are both contained in $D^{(k)}(K)$ by the inductive hypothesis. Consider nets $x_\alpha \subset V$ and $y_\alpha \subset W$ that converge to x , with $\phi^{-1}(x_\alpha)$ and $\phi^{-1}(y_\alpha)$ being convergent nets in the Vietoris topology to certain sets R and S respectively. Notice that $R \subset \{(x, i) : i \in A\}$ and $S \subset \{(x, i) : i \in B\}$. This shows that $x \in D^{(k+1)}(K)$. For the converse, observe that for every $y \in D^{(k)}(K)$, $|\phi^{-1}(y)| = |\{(y, i) : y \in F_i\}| \geq 2^k$, and for any R in the closure of $\{\phi^{-1}(y) : y \in D^{(k)}(K)\}$ in the Vietoris topology we have $|R| \geq 2^k$. Therefore, if $|\phi^{-1}(x)| < 2^{k+1}$, then $x \notin D^{(k+1)}(K)$. ■

Putting together the previous two lemmas, we get:

LEMMA 32. *If we have a jigsaw of 2^m closed sets in K , and L and ϕ are as above, then for any $T : C(L) \rightarrow C(K)$ with $T\phi^0 = 1_{C(K)}$ we have $\|T\| \geq 1 + m$.*

Proof of Theorem 29. By Lemma 32, for every m , we have a continuous surjection $\phi_m : L_m \rightarrow K$ such that L_m has the same weight as K , and for any $T : C(L_m) \rightarrow C(K)$ with $T\phi_m^0 = 1_{C(K)}$ we have $\|T\| \geq 1 + m$. Consider

$$L = \left\{ (x_m)_{m < \omega} \in \prod_{m < \omega} L_m : \forall n, m < \omega \ \phi_m(x_m) = \phi_n(x_n) \right\}.$$

The weight of L equals the weight of K , and we have a continuous surjection $\phi : L \rightarrow K$ given by $\phi((x_m)_{m < \omega}) = \phi_n(x_n)$ for any fixed n . Then $Y = C(L)$ will be the superspace that we look for, and we consider $C(K) \subset C(L)$ through the embedding $\phi^0 : C(K) \rightarrow C(L)$. For every m we can express $\phi = \phi_m \pi_m$ as the composition of the m th coordinate projection $\pi_m : L \rightarrow L_m$ with $\phi_m, \phi : L \rightarrow L_m \rightarrow K$. Hence we can write $C(K) \subset C(L_m) \subset C(L)$. Since any projection $T : C(L_m) \rightarrow C(K)$ must have norm at least $1 + m$, no projection $T : C(L) \rightarrow C(K)$ exists. ■

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