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Generating countable sets of surjective functions

by

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Abstract. We prove that any countable set of surjective functions on an infinite set of cardinality $\aleph_n$ with $n \in \mathbb{N}$ can be generated by at most $n^2/2 + 9n/2 + 7$ surjective functions of the same set; and there exist $n^2/2+9n/2+7$ surjective functions that cannot be generated by any smaller number of surjections. We also present several analogous results for other classical infinite transformation semigroups such as the injective functions, the Baer–Levi semigroups, and the Schützenberger monoids.

1. Introduction. If X is a topological space, then we denote by $C(X)$ the semigroup under composition of continuous functions from X to X. If X is a locally compact Hausdorff space, then $C(X)$ with the compact-open topology is a topological semigroup.

In 1934 Schreier and Ulam [38] proved that $C([0,1]^m)$, $m \geq 1$, has a dense subsemigroup generated by 5 elements. In the same issue of Fundamenta Mathematicae that contained Schreier and Ulam's paper, Sierpiński [39] proved that $C([0,1])$ has a dense 4-generated subsemigroup. In fact, Sierpiński proved something stronger: for every countable sequence $f_0, f_1, \ldots \in C([0,1])$ there exists a 4-generated subsemigroup of $C([0,1])$ containing $f_0, f_1, \ldots$. Since $C([0,1])$ is separable, Schreier and Ulam's result, in the case $m = 1$, follows immediately from Sierpiński's. In 1935, Jarník and Knichal [21] proved that Sierpiński's four functions can be generated by two (this result was unwittingly reproduced by Subbiah in [42] and the present authors in [31]). If $f \in C([0,1])$, then the semigroup $\langle f \rangle$ generated by f and the (topological) closure of $\langle f \rangle$ are commutative. However, as $C([0,1])$ is not commutative, 2 is the least number of generators for a dense subsemigroup of $C([0,1])$. The paper immediately following that of Jarník and Knichal in Fundamenta Mathematicae is another work of Sierpiński [40] (see also Banach [2]). In this paper it is shown that if Ω is an infinite

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set and $f_0, f_1, \ldots : \Omega \to \Omega$, then there exist $g_0, g_1 : \Omega \to \Omega$ such that $f_0, f_1, \ldots \in \langle g_0, g_1 \rangle$.

In view of these early papers of Sierpiński, we will say that a semigroup S has *Sierpiński rank* $m \in \mathbb{N}$ if m is the least number such that for all $f_0, f_1, \ldots \in S$, there exist $g_0, \ldots, g_{m-1} \in S$ such that $f_0, f_1, \ldots \in \langle g_0, \ldots, g_{m-1} \rangle$. If no such m exists, then we will say that S has *infinite Sierpiński rank*. Thus $C([0,1])$ and the semigroup Ω^Ω of all functions from Ω to Ω have Sierpiński rank 2.

The following theorem is the main result of this paper.

Main Theorem. *Let Ω be an infinite set and let* $\mathrm{Surj}(\Omega)$ *denote the semigroup of surjective functions from Ω to Ω. Then:*

(i) *if $|\Omega| = \aleph_n$ for $n \in \mathbb{N}$, then* $\mathrm{Surj}(\Omega)$ *has Sierpiński rank $n^2/2 + 9n/2 + 7$;*
(ii) *if $|\Omega| \geq \aleph_\omega$, then* $\mathrm{Surj}(\Omega)$ *has infinite Sierpiński rank.*

In Section 4, we present several analogous results for other classical infinite transformation semigroups. For example, the injective functions on a set of cardinality $\aleph_n$ for $n \in \mathbb{N}$ have Sierpiński rank $n + 4$, the Baer–Levi semigroups on any infinite set have infinite Sierpiński rank, and the Sierpiński rank of the Schützenberger monoid on a set of regular cardinality is 2.

There are several examples of semigroups with finite Sierpiński rank in the literature. It is pointed out in [17] that Banach's proof in [2] can be easily adapted to show that the semigroups of partial mappings, binary relations or partial bijections on an infinite set have finite Sierpiński rank. Examples of semigroups having infinite Sierpiński rank include all non-finitely generated countable semigroups. An uncountable example is that of $\mathbb{R}$ under multiplication, as the natural numbers are not contained in any finitely generated subsemigroup.

Cook and Ingram [8], and independently Subbiah [42], prove that if X is any of: the euclidean m-cell $[0,1]^m$, $m \geq 1$, the Hilbert cube $[0,1]^{\mathbb{N}}$, the Cantor space $2^{\mathbb{N}}$, the rational numbers $\mathbb{Q}$, or the irrational numbers, then $C(X)$ has Sierpiński rank 2. The Sierpiński rank and the minimum number of generators of dense subsemigroups of $C(X)$ and related semigroups for various spaces X were considered in [25, 27, 31, 41, 42]. A survey of some of these results can be found in [33].

Magill [26] proved that the Sierpiński rank of the semigroup of linear mappings of a vector space V over a field F is 2 if and only if V is infinite-dimensional or $\dim V = 1$ and F is finite. Magill's theorem was generalized slightly in [1]. The second author showed in [36, Theorem 3.10.4] that the Sierpiński rank of the endomorphism semigroup of the Rado graph is 2 or 3, although the exact value has not been determined. The Rado graph is the

Fraïssé limit of the class of finite graphs; see [6, Section 5.1] for further details. It would be interesting to know the Sierpiński rank of some further endomorphism semigroups of Fraïssé limits. Galvin [15] proved that any countable subset of the symmetric group $\mathrm{Sym}(\Omega)$ on an infinite set is contained in a 2-generator subgroup. Galvin's proof can be adapted to show that if G is the group of homeomorphisms of the Cantor space, the rationals, or the irrationals, then any countable subset of G is contained in a 2-generator subgroup. It was also shown in [5] that the homeomorphisms of the euclidean m-sphere have finite Sierpiński rank. Mesyan [29] showed that the multiplicative semigroup of the endomorphism ring of the direct sum of infinitely many copies of a non-zero left R-module over a ring R has Sierpiński rank 2.

Perhaps unsurprisingly, having finite Sierpiński rank is a rather strong property that has several consequences, which we now highlight. For instance, as we have seen, if S is a separable topological semigroup with Sierpiński rank $m \in \mathbb{N}$, then S has an m-generated dense subsemigroup. However, the converse does not hold. For example, if $\mathrm{End}(\mathbb{N}, \leq)$ denotes the order-preserving mappings on $\mathbb{N}$, then $\mathrm{End}(\mathbb{N}, \leq)$ has infinite Sierpiński rank ([31, Theorem 4.1]). On the other hand, it is straightforward to show that $\mathrm{End}(\mathbb{N}, \leq)$ contains a finitely generated dense subsemigroup.

A straightforward consequence of Galvin's theorem is an alternative proof to that of Higman, Neumann and Neumann [18] showing that every countable group can be embedded in a 2-generator group. The analogous results for semigroups and rings follow from Sierpiński [40] and Mesyan [29], respectively, reproducing results of Evans [14] and Maltsev (see [34]).

Many of the examples of semigroups with finite Sierpiński rank given above, in fact, have a stronger property. Let Σ be a finite alphabet, let Σ^+ denote the free semigroup over Σ, let $w \in \Sigma^+$, and let S be a semigroup. Then w is *universal* for S if for all $s \in S$ there exists a homomorphism $F : \Sigma^+ \to S$ such that $s = (w)F$. The set of words $\{w_0, w_1, \ldots\}$ over a finite alphabet Σ is *universal* for S if for all $s_0, s_1, \ldots \in S$ there exists a homomorphism $F : \Sigma^+ \to S$ such that $s_i = (w_i)F$ for all $i \in \mathbb{N}$. Replacing 'free semigroup' with 'free group' in the previous sentences, it is clear what is meant by a universal word for a group. Universal words for groups and semigroups have been extensively investigated; see, for example, [10, 12, 13, 24, 32, 35].

If S has an infinite universal set of words over a finite alphabet Σ, then it is clear that S has Sierpiński rank at most $|\Sigma|$. Sierpiński [40] and Banach [2] proved that Ω^Ω has Sierpiński rank 2 by showing that the sets of words $a^2b^3(abab^3)^{i+1}ab^2ab^3$ and $aba^{i+1}b^2$, respectively, with $i \in \mathbb{N}$ are universal for Ω^Ω. Note that *throughout the paper we write functions to the right of* *their arguments and compose from left to right.* Many of the semigroups given above that have finite Sierpiński rank also have an infinite set of universal words over a finite alphabet, and, in fact, it is the stronger property that is shown to hold in the proofs given in the references above. For example: $\mathrm{Sym}(\Omega)$, Ω^Ω, $C([0,1]^m)$ $(m \geq 1)$, $C([0,1]^{\mathbb{N}})$, $C(2^{\mathbb{N}})$, $C(\mathbb{Q})$, $C(\mathbb{R}\setminus\mathbb{Q})$, the linear mappings on an infinite-dimensional vector space, the endomorphisms of the Rado graph. However, we will show that the semigroups of surjective and injective functions have no infinite universal sets of words by showing that they do not have the following weaker property.

A semigroup S is called *strongly distorted* if there exist $m, a_0, a_1, \ldots \in \mathbb{N}$ such that for all $f_0, f_1, \ldots \in S$ there exist $g_0, \ldots, g_m \in S$ such that f_i is a product of $g_0, \ldots, g_m$ with length at most a_i for all $i \in \mathbb{N}$. The notion of strong distortion for groups was introduced by Khelif [23] (under the name Property P). If G is a non-finitely generated group such that G is strongly distorted, then G is not the union of a countable chain of proper subgroups and for all generating sets U for G there exists $m \geq 1$ such that $G = U^m$ (see [5, Remark A.3] or [23]). The latter is referred to as G having *Bergman's property*, after [3]. The analogous properties and results for semigroups were given in [28]. Many groups and semigroups have Bergman's property; see, for example, [11, 22, 28, 37]. As noted above, most of the semigroups known to have finite Sierpiński rank are strongly distorted and hence have Bergman's property.

Clearly, if S has an infinite universal set of words over a finite alphabet, then S is strongly distorted. Also if S is strongly distorted, then S has finite Sierpiński rank. However, none of these notions is equivalent to any of the others. We will prove that $\mathrm{Inj}(\Omega)$ and $\mathrm{Surj}(\Omega)$ do not have Bergman's property and hence are not strongly distorted and have no infinite universal sets of words. Khelif [23] claims that it is possible to construct an example of a group that is strongly distorted but does not have an infinite universal set of words (the latter is referred to as Property P* in [23]). A simple example of a semigroup that is strongly distorted but that does not have an infinite universal set of words is given below.

EXAMPLE 1.1. Let $T = \Omega^\Omega$ for some infinite set Ω and let x, y be two elements not in T. Then define S to be the semigroup with elements $T \cup \{x, y\}$ and multiplication extending that on T such that x and y act as the identity on T and $xy = yx = y^2 = x^2 = y$. Then, since T is strongly distorted and $S \setminus T$ is finite, it follows by Theorem 2.6 below that S is strongly distorted. Seeking a contradiction suppose that there exists an infinite universal set of words $W \subseteq \Sigma^+$ for S over some finite alphabet Σ. Then there exists a homomorphism $F : \Sigma^+ \to S$ such that $(w)F = x$ for all $w \in W$. But no product in S of length greater than 1 equals x and so $|w| = 1$ for all $w \in W$. Thus W is finite, a contradiction.

2. Preliminaries and generalities. In this section we introduce the background material and notation required to prove our Main Theorem. We also give some general results relating to semigroups and their Sierpiński rank.

LEMMA 2.1. *Let S be a semigroup such that every countable subset is contained in a finitely generated subsemigroup. Then S has finite Sierpiński rank.*

Proof. Suppose that for all $m \in \mathbb{N}$ there exists a countable $C_m \subseteq S$ such that C_m is not contained in any m-generated subsemigroup of S. Then $\bigcup_{m\in\mathbb{N}} C_m$ is countable, but not contained in any finitely generated subsemigroup of S, a contradiction. ∎

In the previous section, we mentioned that if S is a semigroup with Sierpiński rank 1, then S is commutative. The following proposition shows that, up to isomorphism, there is only one infinite semigroup with Sierpiński rank 1.

LEMMA 2.2. *Let S be an infinite semigroup with Sierpiński rank* 1. *Then S is isomorphic to the natural numbers $\mathbb{N} \setminus \{0\}$ under addition.*

Proof. It suffices to prove that S is 1-generated as every infinite 1-generated semigroup is isomorphic to the natural numbers without zero under addition. Seeking a contradiction assume that S is not 1-generated. Let $s_0 \in S$ be arbitrary. Then $\langle s_0\rangle \neq S$ and so there exists $u \in S$ such that $u \notin \langle s_0\rangle$. But S has Sierpiński rank 1 and so there exists $s_1 \in S$ such that $\langle s_0\rangle \lneq \langle s_0, u\rangle \leq \langle s_1\rangle$. Continuing in this way there exist $s_0, s_1, \ldots \in S$ such that

$$\langle s_0\rangle \lneq \langle s_1\rangle \lneq \cdots .$$

Since S has Sierpiński rank 1, there exists $t \in S$ such that $s_0, s_1, \ldots \in \langle t\rangle$. In particular, for all $i \in \mathbb{N}$ there exists $m_i > 0$ such that $t^{m_i} = s_i$. Hence for all $i \in \mathbb{N}$,

$$\{t^{qm_i} : q \geq 1\} = \langle s_i\rangle \lneq \langle s_{i+1}\rangle = \{t^{qm_{i+1}} : q \geq 1\}.$$

Thus for all $i \in \mathbb{N}$ we see that m_{i+1} divides m_i and $m_{i+1} \neq m_i$. It follows that $m_0 > m_1 > \cdots$, a contradiction. ∎

By a similar argument to that given in the proof of Lemma 2.2 it follows that if G is any group such that every countable subset is contained in a 1-generated subgroup, then G is isomorphic to the integers under addition.

If S is a semigroup and T is a subsemigroup of S, it is natural to ask how the Sierpiński rank of S relates to that of T and vice versa. Of course, the answer is, in general, that they are not related. However, if the subsemigroups are restricted to those that are 'large' in some sense, then more can be said.

If T is a subsemigroup of a semigroup S, then we denote by $\operatorname{rank}(S:T)$ the least cardinality of a subset U of S such that $T\cup U$ generates S. Clearly, if S has Sierpiński rank $m\in\mathbb{N}$ and T is any subsemigroup of S, then $\operatorname{rank}(S:T)\le m$ or $\operatorname{rank}(S:T)>\aleph_0$. The cardinal $\operatorname{rank}(S:T)$ is referred to as the *relative rank of T in S*; see [7, 20, 30]. The following lemma gives an upper bound for the Sierpiński rank of a semigroup in terms of the relative rank and Sierpiński rank of its subsemigroups.

LEMMA 2.3. *Let S be a semigroup and let T be a subsemigroup of S such that* $\operatorname{rank}(S:T)$ *is finite and T has Sierpiński rank $m\in\mathbb{N}$. Then the Sierpiński rank of S is at most* $\operatorname{rank}(S:T)+m$.

Proof. Let $f_0,f_1,\ldots\in S$ be arbitrary and let U be a subset of $S\setminus T$ such that $\langle T,U\rangle=S$ and $|U|=\operatorname{rank}(S:T)$. Then for all $i\in\mathbb{N}$, since f_i is a finite product of elements in T and U, there exists a finite subset V_i of T such that $f_i\in\langle U,V_i\rangle$. Since $V=\bigcup_{i\in\mathbb{N}}V_i$ is countable and T has Sierpiński rank $m\in\mathbb{N}$, there exist $g_0,g_1,\ldots,g_{m-1}\in T$ such that $V\subseteq\langle g_0,g_1,\ldots,g_{m-1}\rangle$. Thus

$$f_0,f_1,\ldots\in\langle U,V\rangle\subseteq\langle U,g_0,g_1,\ldots,g_{m-1}\rangle$$

as required. ∎

If S is a semigroup with finite Sierpiński rank and T is a subsemigroup of S with $\operatorname{rank}(S:T)$ finite, then it is not necessarily true that T has finite Sierpiński rank. For example, $\operatorname{rank}(\mathbb{N}^{\mathbb{N}}:\operatorname{End}(\mathbb{N},\le))=1$ ([17, Proposition 1.8]) but $\operatorname{End}(\mathbb{N},\le)$ has infinite Sierpiński rank ([31, Theorem 4.1]).

If S is a semigroup and T is a strongly distorted subsemigroup of S with $\operatorname{rank}(S:T)$ finite, then S is not always strongly distorted. For example, $\operatorname{rank}(\operatorname{Inj}(\mathbb{N}):\operatorname{Sym}(\mathbb{N}))$ is finite (Proposition 4.2 below) and $\operatorname{Sym}(\mathbb{N})$ is strongly distorted [15] but $\operatorname{Inj}(\mathbb{N})$ is not (see the comments after the proof of Theorem 4.1).

If the subsemigroup T from Lemma 2.3 has the property that $S\setminus T$ is an ideal in S, then the following lemma shows that the upper bound given by Lemma 2.3 is the exact value of the Sierpiński rank of S.

LEMMA 2.4. *Let S be a semigroup and let T be a subsemigroup of S such that* $\operatorname{rank}(S:T)$ *is finite, $S\setminus T$ is an ideal of S, and T has Sierpiński rank $m\in\mathbb{N}$. Then the Sierpiński rank of S is* $\operatorname{rank}(S:T)+m$.

Proof. By Lemma 2.3, it suffices to show that the Sierpiński rank of S is at least $\operatorname{rank}(S:T)+m$. We will prove that there exists a countable subset of S that cannot be generated by fewer than $\operatorname{rank}(S:T)+m$ elements of S.

Let U be a subset of $S \setminus T$ such that $\langle T, U\rangle = S$ and $|U| = \operatorname{rank}(S:T)$. By the definition of Sierpiński rank, there exists a countable $V \subseteq T$ that cannot be generated by fewer than m elements of T.

As $U \cup V$ is a countable subset of S, it follows, by Lemma 2.3, that there exists a finite subset F of S such that $U \cup V \subseteq \langle F\rangle$. Since $U \subseteq \langle F\rangle$, it follows that $\langle T, F\rangle = S$ and hence $|F \setminus T| \geq \operatorname{rank}(S:T)$. On the other hand, $S \setminus T$ is an ideal of T and so V is contained in $\langle F \cap T\rangle$. Since V is not generated by fewer than m elements of T, it follows that $|F \cap T| \geq m$. Thus $|F| \geq \operatorname{rank}(S:T)+m$, as required. ∎

Let G be a group and let H be a subgroup of G with finite index. Clearly, $\operatorname{rank}(G:H)$ is at most the index of H in G. It follows, by Lemma 2.3, that if H has finite Sierpiński rank, then so does G. We define a notion of index for arbitrary semigroups, and prove in Theorem 2.5 that if T is a subsemigroup of finite index in a semigroup S, then T has finite Sierpiński rank if and only if S does. In particular, if a group G has finite Sierpiński rank, then so does any finite index subgroup H.

When considering semigroups in general rather than groups, there are several competing notions of index. The *Rees index* of a subsemigroup T of a semigroup S is just $|S \setminus T|+1$. Although there are some parallels between the usual notion of index in group theory and Rees index, the latter does not, in any sense, generalize the former, since infinite groups have no proper finite Rees index subgroups. Perhaps a more useful notion, that generalizes both Rees index and the group-theoretic index, was defined in [16] using the classical notion of Green's relations from semigroup theory; see [19] for further details relating to Green's relations and semigroup theory in general.

Let S be a semigroup, let T be a subsemigroup of S, and let T^1 denote the subsemigroup T with a new identity 1 adjoined. *Green's relative $\mathcal{R}^T$-relation* and *relative $\mathcal{L}^T$-relation* on S are defined by $s\mathcal{R}^T t$ if $sT^1 = tT^1$ and $s\mathcal{L}^T t$ if $T^1 s = T^1 t$ where $s, t \in S$. *Green's relative $\mathcal{H}^T$-relation* is defined by $\mathcal{H}^T = \mathcal{R}^T \cap \mathcal{L}^T$. These relations are equivalence relations on S and their equivalence classes are referred to as $\mathcal{R}^T$-, $\mathcal{L}^T$-, and $\mathcal{H}^T$-classes, respectively. If C is an $\mathcal{R}^T$-, $\mathcal{L}^T$-, or $\mathcal{H}^T$-class, then it is straightforward to verify that either $C \subseteq T$ or $C \subseteq S\setminus T$; for further details regarding other basic properties of relative Green's relations see [16]. The *Green index* of T in S as defined in [16] is the number of $\mathcal{H}^T$-classes contained in $S \setminus T$ plus 1. Let H be a subgroup of a group G. Then H has finite index in G if and only if H has finite Green index in G; for further details see [16].

The following theorem shows that a semigroup has finite Sierpiński rank if and only if all of its finite Green index subsemigroups have finite Sierpiński rank. Unlike Lemmas 2.3 and 2.4, which are used in the proofs of the Main Theorem and Theorem 4.1, Theorem 2.5 is not used elsewhere in the paper. It seems likely that few of the examples of semigroups with finite Sierpiński rank in this paper have subsemigroups with finite Green index. For example, the Green index of any proper subsemigroup of $\mathbb{N}^{\mathbb{N}}$ is $2^{\aleph_0}$. To see this, let T be a subsemigroup of $\mathbb{N}^{\mathbb{N}}$ with Green index strictly less than $2^{\aleph_0}$. If $f, g \in \mathbb{N}^{\mathbb{N}}$ are such that $f\mathcal{L}^T g$, then $f\mathcal{L}g$ in $\mathbb{N}^{\mathbb{N}}$ and so, by [19, Exercise 2.6.16], $\mathbb{N}f = \mathbb{N}g$. Likewise, if $f\mathcal{R}^T g$, then the kernel $\{(i,j) \in \mathbb{N}\times\mathbb{N} : if = jf\}$ of f equals the kernel of g. It follows from the definitions that there are at most as many $\mathcal{L}^T$-classes and $\mathcal{R}^T$-classes as $\mathcal{H}^T$-classes in $\mathbb{N}^{\mathbb{N}} \setminus T$. Hence T contains $A^{\mathbb{N}}$ for some infinite coinfinite $A \subseteq \mathbb{N}$. If $g : A \to \mathbb{N}$, then there are $2^{\aleph_0}$ many $\mathcal{R}$-classes in $\mathbb{N}^{\mathbb{N}}$ containing extensions of g. Hence T contains an extension of every function in $\mathbb{N}^A$. Clearly, every element of $\mathbb{N}^{\mathbb{N}}$ can be written as a product of an element from $A^{\mathbb{N}}$ and an element from $\mathbb{N}^A$. Thus $T = \mathbb{N}^{\mathbb{N}}$.

The following theorem and a preliminary version of its proof was suggested to us by V. Maltcev. The proof of the direct implication of the theorem is very similar to those of Lemma 3.2 and Theorem 4.3 in [4]. However, Theorem 2.5 does not appear to follow immediately from [4].

Theorem 2.5. *Let S be a semigroup and let T be a subsemigroup of finite Green index in S. Then S has finite Sierpiński rank if and only if T has finite Sierpiński rank.*

Proof. Without loss of generality we may assume that S contains an identity 1 and that $1 \in T$. Let C be the union of $\{1\}$ and a set of representatives of all the $\mathcal{H}^T$-classes contained in $S \setminus T$. Then, by assumption, C is finite.

($\Leftarrow$) Since $\langle T, C\rangle = TC = S$, it follows that $\operatorname{rank}(S : T) \le |C| < \aleph_0$. Hence, by Lemma 2.3, it follows that the Sierpiński rank of S is finite, as required.

($\Rightarrow$) If $s \in S \setminus T$, then there exists a unique $c(s) \in C$ such that $s\mathcal{H}^T c(s)$. Hence there exist $l(s), r(s) \in T$ such that $s = l(s)c(s) = c(s)r(s)$. Throughout the remainder of the proof we fix one such $l(s)$ and one such $r(s)$ for all $s \in S \setminus T$.

Let $t_0, t_1, \ldots \in T$ be arbitrary. It suffices, by Lemma 2.1, to prove that there exists a finite $W \subseteq T$ such that $t_0, t_1, \ldots \in \langle W\rangle$. Since S has finite Sierpiński rank, there exists a finite $U \subseteq S$ such that $t_0, t_1, \ldots \in \langle U\rangle$. Let V be any finite subset of T such that:

- $1 \in V$;

- $\{cu \in T : c \in C,\ u \in U\} = CU \cap T$ is contained in V;
- $\{l(s) : s \in CU \setminus T\}$ is contained in V.

Then, since $1 \in C$, it follows that $U \cap T \subseteq V$ and $l(s) \in V$ for all $s \in U \setminus T$.

We begin by proving that for all $u_0, u_1, \ldots, u_k \in U$ there exist $v_0, v_1, \ldots, v_k \in V$ and $c \in C$ such that

$$u_0 u_1 \cdots u_k = v_0 v_1 \cdots v_k c.$$

We proceed by induction on k. If $k = 0$, then either $u_0 \in U \cap T$ or $u_0 \in U \setminus T \subseteq S \setminus T$. In the first case, $u_0 = u_0 \cdot 1$, and in the second case, $u_0 = l(u_0) \cdot c(u_0)$, as required.

Assume by induction that any product of elements in U with length at most k can be given in the required form. Let $u_0, u_1, \ldots, u_k \in U$. Then, by the inductive hypothesis, there exist $v_0, v_1, \ldots, v_{k-1} \in V$ and $d \in C$ such that $u_0 \cdots u_{k-1} = v_0 \cdots v_{k-1} d$. If $du_k \in T$, then $du_k \in CU \cap T \subseteq V$ and so

$$u_0 \cdots u_k = v_0 \cdots v_{k-1} \cdot du_k \cdot 1,$$

as required. If $du_k \in S \setminus T$, then $du_k \in CU \setminus T$ and so $l(du_k) \in V$. Hence

$$u_0 \cdots u_k = v_0 \cdots v_{k-1} du_k = v_0 \cdots v_{k-1} l(du_k) c(du_k),$$

as required.

Let W be any finite subset of T such that:

- $1 \in W$;
- $\{vc \in T : c \in C,\ v \in V\} = VC \cap T$ is contained in W;
- $\{r(s) : s \in VC \setminus T\}$ is contained in W.

We will prove that $t_0, t_1, \ldots \in \langle W \rangle$. If $i \in \mathbb{N}$ is arbitary, then from the above, there exist $v_0, v_1, \ldots, v_k \in V \subseteq W$ and $d_{k+1} \in C$ such that $t_i = v_0 v_1 \cdots v_k d_{k+1}$. If $d_{k+1} = 1$, then $t_i \in \langle W \rangle$, as required. Otherwise let $N = \max\{j : v_j \cdots v_k d_{k+1} \in T\}$. Of course, N exists since $t_i = v_0 v_1 \cdots v_k d_{k+1} \in T$. It follows that

$$d_{k+1}, v_k d_{k+1}, v_{k-1} v_k d_{k+1}, \ldots, v_{N+1} \cdots v_k d_{k+1} \in S \setminus T.$$

Hence $v_k d_{k+1} = d_k w_k$ where $d_k = c(v_k d_{k+1}) \in C$ and $w_k = r(v_k d_{k+1}) \in W$. This implies that $v_{k-1} v_k d_{k+1} = v_{k-1} d_k w_k \in S \setminus T$ and so $v_{k-1} d_k \in S \setminus T$. It follows that $v_{k-1} d_k = d_{k-1} w_{k-1}$ where $d_{k-1} = c(v_{k-1} d_k) \in C$ and $w_{k-1} = r(v_{k-1} d_k) \in W$.

Continuing in this way we obtain

$$w_{N+1} = r(v_{N+1} d_{N+2}), w_{N+2} = r(v_{N+2} d_{N+3}), \ldots, w_k = r(v_k d_{k+1}) \in W$$

and

$$d_{N+1} = c(v_{N+1} d_{N+2}), d_{N+2} = c(v_{N+2} d_{N+3}), \ldots, d_k = c(v_k d_{k+1}) \in C$$

such that $v_k d_{k+1} = d_k w_k$, $v_{k-1} d_k = d_{k-1} w_{k-1}$, $\dots$, $v_{N+1} d_{N+2} = d_{N+1} w_{N+1}$ and $d_i = c(v_i d_{i+1})\mathcal{H}^T v_i d_{i+1}$ for all i. Hence

$$v_N d_{N+1} \in v_N d_{N+1} T = v_N v_{N+1} d_{N+2} T = \cdots = v_N v_{N+1} \cdots v_k d_{k+1} T.$$

But, by the definition of N, $v_N v_{N+1} \cdots v_k d_{k+1} \in T$ and so $v_N d_{N+1} \in W$. Thus

$$t_i = v_0 v_1 \cdots v_{N-1}(v_N d_{N+1}) w_{N+1} \cdots w_k \in \langle W \rangle$$

and so T has finite Sierpiński rank. ■

The proof of Theorem 2.5 can be modified to show that a semigroup is strongly distorted if and only if all of its finite Green index subsemigroups are.

Theorem 2.6. *Let S be a semigroup and let T be a subsemigroup of finite Green index in S. Then S is strongly distorted if and only if T is strongly distorted.*

Proof. Without loss of generality we may assume that S contains an identity 1 and that $1 \in T$. Let C be as in the proof of Theorem 2.5.

($\Leftarrow$) By assumption, T is strongly distorted with respect to some numbers $m, a_0, a_1, \ldots \in \mathbb{N}$. Let $s_0, s_1, \ldots \in S$. Then for all $i \in \mathbb{N}$ there exist $t_i \in T$ and $c_i \in C$ such that $s_i = t_i c_i$. But there exists $U \subseteq T$ such that $|U| = m$ and v_i is a product of elements of U with length at most a_i for all $i \in \mathbb{N}$. Therefore s_i is a product of length at most $a_i + 1$ of elements in $U \cup C$. Hence S is strongly distorted with respect to $|C| + m, a_0 + 1, a_1 + 1, \ldots \in \mathbb{N}$.

($\Rightarrow$) Again, by assumption, S is strongly distorted with respect to some $m, a_0, a_1, \ldots \in \mathbb{N}$. Let $t_0, t_1, \ldots \in T$ be arbitrary and let $U \subseteq S$ be such that $|U| = m$, and t_i is a product of elements of U with length at most a_i for all $i \in \mathbb{N}$.

Let $i \in \mathbb{N}$ be arbitrary and let $u_0, u_1, \ldots, u_k \in U$ such that $t_i = u_0 u_1 \cdots u_k$ and $k \leq a_i$. If $V \subseteq T$ is defined as in the proof of Theorem 2.5, then $|V| \leq |C| \cdot |U| + 1 = m|C| + 1$ and for all $i \in \mathbb{N}$ there exist $v_0, v_1, \ldots, v_k \in V$ and $d \in C$ such that $t_i = v_0 v_1 \cdots v_k d$. Likewise, if $W \subseteq T$ is defined as in the proof of Theorem 2.5, then $|W| \leq |V| \cdot |C| + 1 \leq m|C|^2 + |C| + 1$ and t_i is a product of length at most a_i over W. Therefore T is strongly distorted with respect to $(m|C|^2 + |C| + 1), a_0, a_1, \ldots \in \mathbb{N}$. ■

The following is an immediate corollary of Theorems 2.5 and 2.6 and of the fact that a subgroup of a group has finite index if and only if it has finite Green index.

Corollary 2.7. *Let G be a group and let H be a subgroup of finite index in G. Then:*

(i) *G has finite Sierpiński rank if and only if H has finite Sierpiński rank;*
(ii) *G is strongly distorted if and only if H is strongly distorted.*

If S is the semigroup from Example 1.1, then the subsemigroup T has finite Rees index and hence finite Green index. But as shown in Example 1.1, S does not have an infinite universal set of words whereas $T = \Omega^\Omega$ does by Sierpiński [40]. Hence the analogue of Theorem 2.5 does not hold in the case where 'finite Sierpiński rank' is replaced with 'an infinite universal set of words'. It is natural to ask: if S is a semigroup with an infinite universal set of words, does every subsemigroup of finite Green index also have an infinite universal set of words? However, we do not know the answer to this question.

We conclude this section by mentioning an application of Corollary 2.7. It was shown in [9, Theorem 1] that if G is a subgroup of $\mathrm{Sym}(\mathbb{N})$ with index less than $2^{\aleph_0}$, then there exists a finite $\Sigma \subseteq \mathbb{N}$ such that G contains the pointwise stabilizer $S_{(\Sigma)}$ of Σ in $\mathrm{Sym}(\mathbb{N})$ and G is contained in the setwise stabilizer $S_{\{\Sigma\}}$ of Σ in $\mathrm{Sym}(\mathbb{N})$. Since the index of $S_{(\Sigma)}$ in $S_{\{\Sigma\}}$ is $|\Sigma|!$, it follows that $S_{(\Sigma)}$ has finite index in G. But $S_{(\Sigma)}$ is isomorphic to $\mathrm{Sym}(\mathbb{N})$ and so $S_{(\Sigma)}$ is strongly distorted, by Galvin [15]. Hence Corollary 2.7(ii) implies that G is strongly distorted, as required.

3. The proof of the Main Theorem. In this section, we prove the main result of the paper:

Main Theorem. *Let Ω be an infinite set and let* $\mathrm{Surj}(\Omega)$ *denote the semigroup of surjective functions from Ω to Ω. Then:*

(i) *if $|\Omega| = \aleph_n$ for $n \in \mathbb{N}$, then* $\mathrm{Surj}(\Omega)$ *has Sierpiński rank $n^2/2 + 9n/2 + 7$;*
(ii) *if $|\Omega| \geq \aleph_\omega$, then* $\mathrm{Surj}(\Omega)$ *has infinite Sierpiński rank.*

It is routine to show that $\mathrm{Surj}(\Omega) \setminus \mathrm{Sym}(\Omega)$ is an ideal in $\mathrm{Surj}(\Omega)$. Thus, by Lemma 2.4, the Sierpiński rank of $\mathrm{Surj}(\Omega)$ is the sum of the Sierpiński rank of $\mathrm{Sym}(\Omega)$ and $\mathrm{rank}(\mathrm{Surj}(\Omega) : \mathrm{Sym}(\Omega))$ if the latter is finite. As mentioned above, the Sierpiński rank of $\mathrm{Sym}(\Omega)$ is 2 for any set Ω. In the case that $|\Omega| = \aleph_n$ for some $n \in \mathbb{N}$, we prove the Main Theorem by calculating the relative rank of $\mathrm{Sym}(\Omega)$ in $\mathrm{Surj}(\Omega)$ and applying Lemma 2.4.

Proposition 3.1. *Let Ω be an infinite set such that $|\Omega| = \aleph_n$ for some $n \in \mathbb{N}$. Then* $\mathrm{rank}(\mathrm{Surj}(\Omega) : \mathrm{Sym}(\Omega)) = n^2/2 + 9n/2 + 5$.

To prove Proposition 3.1 we require the following technical result.

Proposition 3.2. *Let Ω be an infinite set with $|\Omega| = \aleph_n$ for some $n \in \mathbb{N}$. Then there exists a set $\mathcal{K}$ of non-empty subsets of* $\mathrm{Surj}(\Omega) \setminus \mathrm{Sym}(\Omega)$ *such that:*

(i) *if $A \in \mathcal{K}$, then* $\mathrm{Surj}(\Omega) \setminus A$ *is a subsemigroup of* $\mathrm{Surj}(\Omega)$*;*
(ii) *if $A, B \in \mathcal{K}$, then $A \cap B = \emptyset$;*
(iii) $|\mathcal{K}| = n^2/2 + 9n/2 + 5$.

It follows from Proposition 3.1 that $n^2/2 + 9n/2 + 5$ is the largest size of a set $\mathcal{K}$ of non-empty subsets of $\mathrm{Surj}(\Omega) \setminus \mathrm{Sym}(\Omega)$ satisfying the conditions of Proposition 3.2(i)&(ii).

Throughout the remainder of this section we assume, unless stated otherwise, that Ω is an infinite set with $|\Omega| = \aleph_n$ for some $n \in \mathbb{N}$. To prove Propositions 3.1 and 3.2 we require the following parameters of elements of $\mathrm{Surj}(\Omega)$. Let $f \in \mathrm{Surj}(\Omega)$ and let $\lambda \in \mathbb{N} \cup \{\aleph_0, \aleph_1, \ldots, \aleph_{n+1}\}$. Then define

$$\begin{aligned} a(f) &= \min\{\lambda : (\forall \alpha \in \Omega)(|\alpha f^{-1}| < \lambda)\}, \\ b(f, \lambda) &= |\{\alpha \in \Omega : |\alpha f^{-1}| = \lambda\}|, \\ c(f) &= \max\{\lambda : (\forall m \in \mathbb{N})\,(|\{\alpha \in \Omega : |\alpha f^{-1}| \geq m\}| \geq \lambda)\}, \\ d(f) &= |\{\alpha \in \Omega : |(\alpha f) f^{-1}| \geq 2\}|. \end{aligned}$$

It is routine to verify that

$$\begin{aligned} a(f) &\in \{2, 3, \ldots, \aleph_0, \aleph_1, \ldots, \aleph_{n+1}\}, \\ b(f, \lambda), c(f) &\in \mathbb{N} \cup \{\aleph_0, \aleph_1, \ldots, \aleph_n\}, \\ d(f) &\in \{0, 2, 3, \ldots, \aleph_0, \aleph_1, \ldots, \aleph_n\}. \end{aligned}$$

We will make repeated use of the following straightforward observations without reference:

- if $b(f, \aleph_i) > 0$, then $a(f) \geq \aleph_{i+1}$;
- $a(f) < \aleph_0$ if and only if $c(f) = 0$;
- if $a(f) = \aleph_0$, then $c(f) \geq \aleph_0$;
- if $a(f) = \aleph_0$ and $b(f, m) < \aleph_0$ for all but finitely many $m \in \mathbb{N}$, then $c(f) = \aleph_0$;
- if $a(f) = \aleph_0$ and $b(f, m) \geq \aleph_0$ for infinitely many $m \in \mathbb{N}$, then $c(f) = \aleph_i$ where $0 \leq i \leq n$ is the largest number such that $\{m \in \mathbb{N} : b(f, m) = \aleph_i\}$ is infinite;
- $d(f) = 0$ if and only if $f \in \mathrm{Sym}(\Omega)$.

Lemma 3.3. *Let $f, g \in \mathrm{Surj}(\Omega)$ and let $i \in \mathbb{N}$ be such that $0 \leq i \leq n$. Then:*

(i) $\max\{a(f), a(g)\} \leq a(fg) \leq a(f)\, a(g)$*;*
(ii) *if* $\max\{a(f), a(g)\} = \aleph_{i+1}$*, then*

$$b(g, \aleph_i) \leq b(fg, \aleph_i) \leq b(f, \aleph_i) + b(g, \aleph_i);$$

(iii) *if* $\max\{a(f), a(g)\} = \aleph_{i+1}$ *and* $b(f, \aleph_i) \geq \max\{a(g), \aleph_0\}$*, then*

$$b(f, \aleph_i) \leq b(fg, \aleph_i);$$

(iv) *if* $\max\{a(f), a(g)\} = \aleph_0$*, then*

$$c(fg) = \max\{c(f), c(g)\};$$

(v) $\max\{d(f), d(g)\} \leq d(fg) \leq d(f) + d(g)$.

Proof. (i) Let $\alpha \in \Omega$ be arbitrary. Then clearly $|\alpha g^{-1}| \leq |\alpha(fg)^{-1}|$ and so $a(g) \leq a(fg)$. Also $|\alpha f^{-1}| \leq |(\alpha)g(fg)^{-1}|$ and so $a(f) \leq a(fg)$.

On the other hand, $\alpha(fg)^{-1} = \bigcup_{\beta\in\alpha g^{-1}} \beta f^{-1}$ and

$$\Big|\bigcup_{\beta\in\alpha g^{-1}} \beta f^{-1}\Big| < a(f)\, |\alpha g^{-1}| \leq a(f)\, a(g).$$

(ii) It follows from (i) that $a(fg) = \aleph_{i+1}$. If $\alpha \in \Omega$ with $|\alpha g^{-1}| = \aleph_i$, then $\aleph_i \geq |\alpha(fg)^{-1}| = |\alpha g^{-1} f^{-1}| \geq \aleph_i$, giving equality throughout. Thus $b(g, \aleph_i) \leq b(fg, \aleph_i)$.

Let $\alpha \in \Omega$ with $|\alpha(fg)^{-1}| = \aleph_i$. Then either $|\alpha g^{-1}| = \aleph_i$ or there exists $\beta \in \alpha g^{-1}$ with $|\beta f^{-1}| = \aleph_i$. Hence

$$\{\alpha \in \Omega : |\alpha(fg)^{-1}| = \aleph_i\} \subseteq \{\alpha \in \Omega : |\alpha g^{-1}| = \aleph_i\} \cup \{\beta \in \Omega : |\beta f^{-1}| = \aleph_i\}g.$$

Therefore $b(fg, \aleph_i) \leq b(f, \aleph_i) + b(g, \aleph_i)$.

(iii) As $0 < a(g) \leq b(f, \aleph_i)$, it follows that $a(f) = \aleph_{i+1}$ and $|\{\alpha \in \Omega : |\alpha f^{-1}| = \aleph_i\}g| = b(f, \aleph_i)$. Let $\alpha \in \Omega$ with $|\alpha f^{-1}| = \aleph_i$. Then, since $a(fg) = \aleph_{i+1}$ by (i), $|\alpha g(fg)^{-1}| = \aleph_i$. Thus $b(f, \aleph_i) \leq b(fg, \aleph_i)$, as required.

(iv) Let $k \in \mathbb{N}$ and let $\alpha \in \Omega$ with $|\alpha g^{-1}| \geq k$. Then $|\alpha(fg)^{-1}| \geq k$. Thus $\{\beta \in \Omega : |\beta g^{-1}| \geq k\} \subseteq \{\beta \in \Omega : |\beta(fg)^{-1}| \geq k\}$ and so $c(g) \leq c(fg)$.

On the other hand, if $\alpha \in \Omega$ with $|\alpha f^{-1}| \geq k$, then $|(\alpha g)g^{-1}f^{-1}| \geq k$. Hence $\{\beta \in \Omega : |\beta f^{-1}| \geq k\}g \subseteq \{\beta \in \Omega : |\beta(fg)^{-1}| \geq k\}$. If $\{\beta \in \Omega : |\beta f^{-1}| \geq k\}$ is infinite, then since $a(g) \leq \aleph_0$, we have

$$\begin{aligned}|\{\beta \in \Omega : |\beta f^{-1}| \geq k\}| &= |\{\beta \in \Omega : |\beta f^{-1}| \geq k\}g| \\ &\leq |\{\beta \in \Omega : |\beta(fg)^{-1}| \geq k\}|.\end{aligned}$$

It follows that $c(f) \leq c(fg)$. If $\{\beta \in \Omega : |\beta f^{-1}| \geq k\}$ is finite, then $a(f) < \aleph_0$ and so $c(f) = 0 \leq c(fg)$. Therefore $\max\{c(f), c(g)\} \leq c(fg)$.

From the comments immediately before the lemma, if $\aleph_j > c(f)$, then $\{m \in \mathbb{N} : b(f, m) = \aleph_j\}$ is finite. Hence there exists $k \in \mathbb{N}$ such that $b(f, m) \leq c(f)$ for all $m \geq k$. It follows that $|\{\alpha \in \Omega : |\alpha f^{-1}| \geq k\}| = c(f)$ and $|\{\alpha \in \Omega : |\alpha g^{-1}| \geq k\}| = c(g)$. Let $\alpha \in \Omega$ with $|\alpha(fg)^{-1}| \geq k^2$. Then either $|\alpha g^{-1}| \geq k$ or there exists $\beta \in \alpha g^{-1}$ with $|\beta f^{-1}| \geq k$. Thus $|\{\alpha \in \Omega : |\alpha(fg)^{-1}| \geq k^2\}| \leq c(f) + c(g) = \max\{c(f), c(g)\}$, since either $c(f)$ or $c(g)$ is infinite. Thus $c(fg) \leq \max\{c(f), c(g)\}$, as required.

(v) Let

$$\begin{aligned}A &= \{\alpha \in \Omega : |(\alpha f)f^{-1}| \geq 2\},\\ B &= \{\alpha \in \Omega : |(\alpha g)g^{-1}| \geq 2\},\\ C &= \{\alpha \in \Omega : |(\alpha fg)(fg)^{-1}| \geq 2\}.\end{aligned}$$

Of course, for any $\alpha \in \Omega$ we see that $(\alpha fg)(fg)^{-1}$ is the union of βf^{-1} for all $\beta \in (\alpha fg)g^{-1}$. Hence $|(\alpha fg)(fg)^{-1}| \geq 2$ if and only if $|(\alpha fg)g^{-1}| \geq 2$ or $\alpha(fg)g^{-1} = \{\alpha f\}$ and $|(\alpha f)f^{-1}| \geq 2$. Thus $|(\alpha fg)(fg)^{-1}| \geq 2$ if and only if $\alpha \in A \cup Bf^{-1}$. It follows that $C = A \cup Bf^{-1}$. Also, $|Bf^{-1}| \geq |B|$ since $f \in \mathrm{Surj}(\Omega)$. Thus

$$\begin{aligned}\max\{d(f), d(g)\} = \max\{|A|, |B|\} &\leq \max\{|A|, |Bf^{-1}|\}\\ &\leq |A \cup Bf^{-1}| = |C| = d(fg).\end{aligned}$$

On the other hand, C is the disjoint union of A and $Bf^{-1} \setminus A$. Since f is injective on $\Omega \setminus A$, it is in particular injective on $Bf^{-1} \setminus A$. Hence $|(Bf^{-1} \setminus A)f| = |Bf^{-1} \setminus A|$. So,

$$\begin{aligned}d(fg) = |C| = |A| + |Bf^{-1} \setminus A| = |A| + |(Bf^{-1} \setminus A)f| = |A| + |B \setminus Af|\\ \leq |A| + |B| = d(f) + d(g),\end{aligned}$$

as required. ∎

Proof of Proposition 3.2. Let $\mathcal{K}$ be the set consisting of the following subsets of $\mathrm{Surj}(\Omega) \setminus \mathrm{Sym}(\Omega)$:

$$\begin{aligned}U_{i,j} &= \{f \in \mathrm{Surj}(\Omega) : a(f) = \aleph_{i+1} \text{ and } b(f, \aleph_i) = \aleph_j\},\\ V_i &= \{f \in \mathrm{Surj}(\Omega) : a(f) = \aleph_{i+1} \text{ and } 1 \leq b(f, \aleph_i) < \aleph_i\},\\ W_i &= \{f \in \mathrm{Surj}(\Omega) : a(f) = \aleph_0 \text{ and } c(f) = \aleph_i\},\\ X_i &= \{f \in \mathrm{Surj}(\Omega) : a(f) \in \mathbb{N} \text{ and } d(f) = \aleph_i\},\\ Y &= \{f \in \mathrm{Surj}(\Omega) : a(f) \in \mathbb{N} \text{ and } 1 < d(f) < \aleph_0\},\end{aligned} \tag{3.1}$$

for all $0 \leq i \leq j \leq n$. Then $|\mathcal{K}| = n^2/2 + 9n/2 + 5$, and if $A, B \in \mathcal{K}$, then $A \cap B = \emptyset$.

To conclude the proof, it suffices to prove that if $A \in \mathcal{K}$, then $\mathrm{Surj}(\Omega) \setminus A$ is a subsemigroup of $\mathrm{Surj}(\Omega)$.

Let $f, g \in \mathrm{Surj}(\Omega) \setminus U_{i,j}$. We will prove that $fg \in \mathrm{Surj}(\Omega) \setminus U_{i,j}$ by showing that either $a(fg) \neq \aleph_{i+1}$ or $b(fg, \aleph_i) \neq \aleph_j$. If $\max\{a(f), a(g)\} \neq \aleph_{i+1}$, then, by Lemma 3.3(i), $a(fg) \neq \aleph_{i+1}$. Assume that $\max\{a(f), a(g)\} = \aleph_{i+1}$. If $b(f, \aleph_i) = \aleph_j$, then $a(f) = \aleph_{i+1}$ and so $f \in U_{i,j}$. Hence $b(f, \aleph_i) \neq \aleph_j$ and likewise $b(g, \aleph_i) \neq \aleph_j$. If $b(g, \aleph_i) > \aleph_j$, then, by Lemma 3.3(ii), $b(fg, \aleph_i) \geq b(g, \aleph_i) > \aleph_j$. If $b(g, \aleph_i) < \aleph_j$ and $b(f, \aleph_i) > \aleph_j$, then, in particular, $b(f, \aleph_i) \geq a(g)$. Hence, by Lemma 3.3(iii), $\aleph_j < b(f, \aleph_i) \leq b(fg, \aleph_i)$. Finally, if $b(f, \aleph_i)$

$< \aleph_j$ and $b(g,\aleph_i) < \aleph_j$, then, by Lemma 3.3(ii), $b(fg,\aleph_j) \le b(f,\aleph_i) + b(g,\aleph_i) < \aleph_j$. Hence, in any case, $fg \in \mathrm{Surj}(\Omega) \setminus U_{i,j}$.

Let $f,g \in \mathrm{Surj}(\Omega) \setminus V_i$. We prove that $fg \in \mathrm{Surj}(\Omega) \setminus V_i$ by showing that either $a(fg) \neq \aleph_{i+1}$ or $b(fg,\aleph_i) \ge \aleph_i$. As above, if $\max\{a(f),a(g)\} \neq \aleph_{i+1}$, then, by Lemma 3.3(i), $a(fg) \neq \aleph_{i+1}$. Hence we may assume that $\max\{a(f),a(g)\} = \aleph_{i+1}$. So, either $b(f,\aleph_i) \ge \aleph_i$ or $b(g,\aleph_i) \ge \aleph_i$. In the latter case, it follows by Lemma 3.3(ii) that $b(fg,\aleph_i) \ge b(g,\aleph_i) \ge \aleph_i$. On the other hand, if $b(f,\aleph_i) \ge \aleph_i$ and $b(g,\aleph_i) < \aleph_i$, then $b(g,\aleph_i) = 0$ since $g \notin V_i$. Therefore $a(g) \le \aleph_i = b(f,\aleph_i)$. By Lemma 3.3(iii) it follows that $b(fg,\aleph_i) \ge b(f,\aleph_i) \ge \aleph_i$. Hence, in any case, $fg \in \mathrm{Surj}(\Omega) \setminus V_i$.

Let $f,g \in \mathrm{Surj}(\Omega)\setminus W_i$. We prove that $fg \in \mathrm{Surj}(\Omega)\setminus W_i$ by showing that either $a(fg) \neq \aleph_0$ or $c(fg) \neq \aleph_i$. If $\max\{a(f),a(g)\} \neq \aleph_0$, then, by Lemma 3.3(i), $a(fg) \neq \aleph_0$. Hence we may assume that $\max\{a(f),a(g)\} = \aleph_0$. If $c(f) = \aleph_i$, then $a(f) = \aleph_0$, and so $f \in W_i$. Hence $c(f) \neq \aleph_i$ and likewise $c(g) \neq \aleph_i$. If $c(f) > \aleph_i$ or $c(g) > \aleph_i$, then Lemma 3.3(iv) implies that $c(fg) > \aleph_i$. If $c(f) < \aleph_i$ and $c(g) < \aleph_i$, then, again by Lemma 3.3(iv), $c(fg) < \aleph_i$. Hence, in any case, $fg \in \mathrm{Surj}(\Omega) \setminus W_i$.

Let $f,g \in \mathrm{Surj}(\Omega) \setminus X_i$. We prove that $fg \in \mathrm{Surj}(\Omega) \setminus X_i$ by showing that either $a(fg) \notin \mathbb{N}$ or $d(fg) \neq \aleph_i$. If $a(f) \notin \mathbb{N}$ or $a(g) \notin \mathbb{N}$, then, by Lemma 3.3(i), $a(fg) \notin \mathbb{N}$. Hence we may assume that $a(f), a(g) \in \mathbb{N}$ and so $d(f) \neq \aleph_i$ and $d(g) \neq \aleph_i$. Then, by Lemma 3.3(v), $d(fg) \neq \aleph_i$. Hence, in any case, $fg \in \mathrm{Surj}(\Omega) \setminus X_i$.

It follows that $\mathrm{Surj}(\Omega) \setminus Y$ is a semigroup by the same argument as in the previous paragraph. ∎

The following three lemmas are used to reduce the problem of generating $\mathrm{Surj}(\Omega)$ to the problem of generating a particular subset of $\mathrm{Surj}(\Omega)$. The first lemma is straightforward and its proof omitted.

LEMMA 3.4. *Let Ω be any infinite set and let $f,g \in \mathrm{Surj}(\Omega)$. Then there exist $h,k \in \mathrm{Sym}(\Omega)$ such that $hfk = g$ if and only if $b(f,\lambda) = b(g,\lambda)$ for all $\lambda \le |\Omega|$.* ∎

LEMMA 3.5. *Let $f \in \mathrm{Surj}(\Omega)$. Then there exist $g,h \in \mathrm{Surj}(\Omega)$ with $b(g,1) = b(h,1) = \aleph_n$ such that $f = gh$.*

Proof. It suffices, by Lemma 3.4, to prove that there exist $g,h \in \mathrm{Surj}(\Omega)$ with $b(g,1) = b(h,1) = \aleph_n$ such that $b(gh,\lambda) = b(f,\lambda)$ for all $\lambda \in \mathbb{N} \cup \{\aleph_0,\aleph_1,\ldots,\aleph_n\}$.

Since f is surjective, Ω can be partitioned into sets A and B such that $Af \cap Bf = \emptyset$ and $|A| = |B| = |Af| = |Bf| = |\Omega|$. Let $g' : Af \to A$ and $h' : Bf \to B$ be arbitrary bijections. Then define $g = f|_A g' \cup 1_B$ and $h = f|_B h' \cup 1_A$. Clearly, $g,h \in \mathrm{Surj}(\Omega)$ with $b(g,1) = b(h,1) = |A| =$

$|B| = \aleph_n$. Moreover, $gh = f|_A g' \cup f|_B h'$ satisfies $b(gh, \lambda) = b(f, \lambda)$ for all $\lambda \in \mathbb{N} \cup \{\aleph_0, \aleph_1, \dots, \aleph_n\}$. ∎

LEMMA 3.6. *Let $f, g \in \mathrm{Surj}(\Omega)$ with $b(f,1) = b(g,1) = \aleph_n$. Then there exists $h \in \mathrm{Sym}(\Omega)$ such that $b(fhg, 1) = \aleph_n$ and*

$$b(fhg, \lambda) = b(f, \lambda) + b(g, \lambda)$$

for all $\lambda \in \{2, 3, \dots, \aleph_0, \aleph_1, \dots, \aleph_n\}$.

Proof. Let $F_0 = \{\alpha \in \Omega : |\alpha f^{-1}| \geq 2\}$ and $G_0 = \{\alpha \in \Omega : |\alpha g^{-1}| \geq 2\}$. Then $|\Omega \setminus F_0| = b(f, 1) = \aleph_n$ and so we may partition $\Omega \setminus F_0$ into sets F_1 and F_2 such that $|F_1| = \aleph_n$ and $|F_2| = |G_0 g^{-1}|$. Likewise, we can partition $\Omega \setminus G_0$ into G_1 and G_2 where $|G_1| = \aleph_n$ and $|G_2| = |F_0| = |G_2 g^{-1}|$. The required $h \in \mathrm{Sym}(\Omega)$ is any element such that $F_0 h = G_2 g^{-1}$, $F_1 h = G_1 g^{-1}$, and $F_2 h = G_0 g^{-1}$.

If $\alpha \in G_1$, then $|\alpha g^{-1}| = 1$ and $|\alpha g^{-1} h^{-1}| = 1$ since h is a bijection. Also $\alpha g^{-1} h^{-1} \in F_1$ and so $|\alpha g^{-1} h^{-1} f^{-1}| = |\alpha (fhg)^{-1}| = 1$. Thus $\aleph_n \geq b(fhg, 1) \geq |G_1| = \aleph_n$, giving equality throughout.

If $\alpha \in G_0$, then $|\alpha (fhg)^{-1}| = |\alpha g^{-1}|$ since $(fh)^{-1}$ maps $G_0 g^{-1}$ bijectively to $F_2 f^{-1}$. Let $\alpha \in G_2$. Then, since $(hg)^{-1}$ maps G_2 bijectively to F_0, $|\alpha (fhg)^{-1}| = |\beta f^{-1}|$ where $\beta = \alpha (hg)^{-1} \in F_0$.

It follows that if $\lambda \in \{2, 3, \dots, \aleph_0, \aleph_1, \dots, \aleph_n\}$, then

$$\begin{aligned}
b(fhg, \lambda) &= |\{\alpha \in \Omega : |\alpha (fhg)^{-1}| = \lambda\}| = |\{\alpha \in G_0 \cup G_2 : |\alpha (fhg)^{-1}| = \lambda\}| \\
&= |\{\alpha \in G_0 : |\alpha (fhg)^{-1}| = \lambda\}| + |\{\alpha \in G_2 : |\alpha (fhg)^{-1}| = \lambda\}| \\
&= |\{\alpha \in G_0 : |\alpha g^{-1}| = \lambda\}| + |\{\beta \in F_0 : |\beta f^{-1}| = \lambda\}| \\
&= b(g, \lambda) + b(f, \lambda),
\end{aligned}$$

as required. ∎

We next specify a subset of $\mathrm{Surj}(\Omega)$ with $n^2/2 + 9n/2 + 5$ elements that together with $\mathrm{Sym}(\Omega)$ generates $\mathrm{Surj}(\Omega)$. Let $u_{i,j}, v_i, w_i, x_i, y \in \mathrm{Surj}(\Omega)$, where $0 \leq i \leq j \leq n$, be any functions satisfying:

- $b(u_{i,j}, 1) = b(v_i, 1) = b(w_i, 1) = b(x_i, 1) = b(y, 1) = \aleph_n$;
- $a(u_{i,j}) = \aleph_{i+1}$, $b(u_{i,j}, \aleph_i) = \aleph_j$ and $b(u_{i,j}, \lambda) = 0$ for all $\lambda \notin \{1, \aleph_i\}$;
- $a(v_i) = \aleph_{i+1}$, $b(v_i, \aleph_i) = 1$ and $b(v_i, \lambda) = 0$ for all $\lambda \notin \{1, \aleph_i\}$;
- $a(w_i) = \aleph_0$ and $b(w_i, m) = \aleph_i$ for all $m > 1$;
- $a(x_i) = 3$ and $b(x_i, 2) = \aleph_i$;
- $a(y) = 3$ and $b(y, 2) = 1$.

Then $u_{i,j} \in U_{i,j}$, $v_i \in V_i$, $w_i \in W_i$, $x_i \in X_i$, and $y \in Y$, where $U_{i,j}, V_i, W_i, X_i$ and Y denote the sets comprising $\mathcal{K}$ defined in (3.1) in the proof of Proposition 3.2.

We proceed by a sequence of lemmas that will finally be combined to prove Proposition 3.1.

LEMMA 3.7. *If* $m \in \mathbb{N}$, *then there exists* $g \in \langle \mathrm{Sym}(\Omega), v_i\rangle$ *such that* $b(g,1) = \aleph_n$, $b(g,\aleph_i) = m$, *and* $b(g,\lambda) = 0$ *for all* $\lambda \notin \{1, \aleph_i\}$.

Proof. If $m = 0$, then any element g of $\mathrm{Sym}(\Omega)$ has the required properties. Assume that $m > 0$. By definition, $b(v_i, \aleph_i) = 1$ and so, by applying Lemma 3.6 ($m-1$ times), there exists $g \in \langle \mathrm{Sym}(\Omega), v_i\rangle$ with the required properties. ∎

LEMMA 3.8. *If* $0 \le j < i$, *then there exists* $g \in \langle \mathrm{Sym}(\Omega), u_{i,i}, v_i\rangle$ *such that* $b(g,1) = \aleph_n$, $b(g,\aleph_i) = \aleph_j$, *and* $b(g,\lambda) = 0$ *for all* $\lambda \notin \{1, \aleph_i\}$.

Proof. Let $\beta \in \Omega$ be the unique element such that $|\beta v_i^{-1}| = \aleph_i$ and let

$$A = \{\alpha \in \Omega : |\alpha u_{i,i}^{-1}| = \aleph_i\}.$$

Then $|A| = \aleph_i$ and so there exists $B \subseteq A$ with $|B| = \aleph_j$. If $p \in \mathrm{Sym}(\Omega)$ is any element such that $(A \setminus B)p = \beta v_i^{-1}$, then $|\alpha(u_{i,i}pv_i)^{-1}| = \aleph_i$ for all $\alpha \in Bpv_i \cup \{\beta\}$. Moreover, $|\alpha(u_{i,i}pv_i)^{-1}| = 1$ for all $\alpha \notin Bpv_i \cup \{\beta\}$ and $|Bpv_i \cup \{\beta\}| = |B| = \aleph_j$. Thus $g = u_{i,i}pv_i \in \langle \mathrm{Sym}(\Omega), u_{i,i}, v_i\rangle$ satisfies $b(g,1) = \aleph_n$, $b(g,\aleph_i) = \aleph_j$, and $b(g,\lambda) = 0$ for all $\lambda \notin \{1, \aleph_i\}$, as required. ∎

LEMMA 3.9. *If* $m \in \mathbb{N}$ *and* $m \ge 2$, *then there exists* $g \in \langle \mathrm{Sym}(\Omega), x_i\rangle$ *such that* $b(g,1) = \aleph_n$, $b(g,m) = \aleph_i$, *and* $b(g,\lambda) = 0$ *for all* $\lambda \notin \{1, m\}$.

Proof. Let Σ_λ be countable subsets of Ω for all $\lambda < \aleph_i$ such that $\Sigma_\lambda \cap \Sigma_\mu = \emptyset$ if $\lambda \ne \mu$; write $\Sigma_\lambda = \{\sigma_{\lambda,1}, \sigma_{\lambda,2}, \ldots\}$. Then define $h \in \mathrm{Surj}(\Omega)$ by

$$\alpha h = \begin{cases} \sigma_{\lambda,j-1} & \text{if } \alpha = \sigma_{\lambda,j},\ j > 1, \\ \alpha & \text{otherwise.} \end{cases} \tag{3.2}$$

As $b(h,\lambda) = b(x_i,\lambda)$ for all λ, it follows by Lemma 3.4 that $h \in \langle \mathrm{Sym}(\Omega), x_i\rangle$. Hence $g = h^{m-1} \in \langle \mathrm{Sym}(\Omega), x_i\rangle$ and

$$\alpha g^{-1} = \begin{cases} \{\sigma_{\lambda,1}, \ldots, \sigma_{\lambda,m}\} & \text{if } \alpha = \sigma_{\lambda,1},\ \lambda < \aleph_i, \\ \{\sigma_{\lambda,j+m-1}\} & \text{if } \alpha = \sigma_{\lambda,j},\ j \ne 1,\ \lambda < \aleph_i, \\ \{\alpha\} & \text{otherwise.} \end{cases}$$

Therefore $b(g,1) = \aleph_n$, $b(g,m) = \aleph_i$, and $b(g,\lambda) = 0$ if $\lambda \notin \{1, m\}$, as required. ∎

LEMMA 3.10. *If* $m, r \in \mathbb{N}$, $m \ge 2$, *and* $r \ge 1$, *then there exists* $g \in \langle \mathrm{Sym}(\Omega), y\rangle$ *such that* $b(g,1) = \aleph_n$, $b(g,m) = r$, *and* $b(g,\lambda) = 0$ *for all* $\lambda \notin \{1, m\}$.

Proof. By applying Lemma 3.6 to y ($r-1$ times), we find $k \in \langle \mathrm{Sym}(\Omega), y\rangle$ such that $b(k,1) = \aleph_n$, $b(k,2) = r$, and $b(k,\lambda) = 0$ if $\lambda \notin \{1, 2\}$.

Replacing $\aleph_i$ with r in the proof of Lemma 3.9, we obtain sets $\Sigma_0, \Sigma_1, \dots, \Sigma_{r-1}$ and a function h defined analogously to that in (3.2) such that $b(h,\lambda) = b(k,\lambda)$ for all λ. It follows by Lemma 3.4 that $h \in \langle \mathrm{Sym}(\Omega), y\rangle$. As in the proof of Lemma 3.9, if $g = h^{m-1}$, then g satisfies $b(g,1) = \aleph_n$, $b(g,m) = r$, and $b(g,\lambda) = 0$ for all $\lambda \notin \{1,m\}$, as required. ∎

Lemma 3.11. *If $0 \le i \le n$ and $h \in \mathrm{Surj}(\Omega)$ with $a(h) = \aleph_0$, $b(h,1) = \aleph_n$, $c(h) = \aleph_i$, and $b(h,m) \le \aleph_i$ for all $m > 1$, then $h \in \langle \mathrm{Sym}(\Omega), w_i, x_i\rangle$.*

Proof. Let $\Sigma = \{\alpha \in \Omega : |\alpha h^{-1}| > 1\}$. Then, since $c(h) = \aleph_i$ and $b(h,m) \le \aleph_i$ for all $m > 1$, it follows that $|\Sigma| = \aleph_i$ and

$$|\{\alpha \in \Sigma : |\alpha h^{-1}| > m\}| = \aleph_i \tag{3.3}$$

for all $m \in \mathbb{N}$. Also since $b(h,1) = \aleph_n$ we have $|\Omega \setminus \Sigma| = \aleph_n$. Let $w, x \in \mathrm{Surj}(\Omega)$ be any elements such that Σ is fixed setwise, $\Omega\setminus\Sigma$ is fixed pointwise, $a(w) = \aleph_0$, $a(x) = 3$,

$$|\{\beta \in \Sigma : |\beta w^{-1}| = m\}| = \aleph_i \tag{3.4}$$

for all $1 \le m < \aleph_0$, and $|\alpha x^{-1}| = 2$ for all $\alpha \in \Sigma$. Then $b(w,\lambda) = b(w_i,\lambda)$ and $b(x,\lambda) = b(x_i,\lambda)$ for all λ. Hence, by Lemma 3.4, $w, x \in \langle \mathrm{Sym}(\Omega), w_i, x_i\rangle$ and so it suffices to prove that $h \in \langle \mathrm{Sym}(\Omega), w, x\rangle$.

We construct $p \in \mathrm{Sym}(\Omega)$ such that $b(wpx,\lambda) = b(h,\lambda)$ for all λ. We identify Σ with the least ordinal of cardinality $\aleph_i$ and proceed by transfinite induction over Σ. Let $\alpha \in \Sigma$ be arbitrary. Assume that for all $\beta < \alpha$ we have defined bijections p_β from subsets of Σ into $\bigcup_{\gamma \le \beta} \gamma x^{-1}$ such that $p_{\beta_1} \subseteq p_{\beta_2}$ for all $\beta_1 \le \beta_2 < \alpha$. Let $q_\alpha = \bigcup_{\beta<\alpha} p_\beta$.

Since $|\mathrm{dom}(q_\alpha)| < \aleph_i$ and $|\alpha h^{-1}| > 1$, it follows from (3.4) that

$$\{\beta \in \Sigma : |\beta w^{-1}| < |\alpha h^{-1}|\} \setminus \mathrm{dom}(q_\alpha) \ne \emptyset.$$

Let β_α denote the least element of this set. Likewise, by (3.4) there exists $\gamma_\alpha \in \Sigma \setminus \mathrm{dom}(q_\alpha) \cup \{\beta_\alpha\}$ such that

$$|\gamma_\alpha w^{-1}| = |\alpha h^{-1}| - |\beta_\alpha w^{-1}|. \tag{3.5}$$

Let p_α be an extension of q_α that maps $\{\beta_\alpha, \gamma_\alpha\}$ bijectively to αx^{-1}.

Let $q = \bigcup_{\alpha\in\Sigma} p_\alpha$. Then q is a bijection from a subset of Σ to $\Sigma x^{-1} = \Sigma$. We will prove that $q \in \mathrm{Sym}(\Sigma)$. It suffices to show that $\gamma \in \mathrm{dom}(q)$ for all $\gamma \in \Sigma$. If

$$A = \{\alpha \in \Sigma : |\alpha h^{-1}| > |\gamma w^{-1}|\},$$

then $|A| = \aleph_i$ by (3.3). If $\alpha \in A$, then $|\gamma w^{-1}| < |\alpha h^{-1}|$ and so $\gamma \in \{\beta \in \Sigma : |\beta w^{-1}| < |\alpha h^{-1}|\}$. Hence if $\alpha \in A$ and $\gamma \notin \mathrm{dom}(p_\alpha)$, then, in particular, $\gamma \in \{\beta \in \Sigma : |\beta w^{-1}| < |\alpha h^{-1}|\} \setminus \mathrm{dom}(q_\alpha)$ and so $\gamma > \beta_\alpha$. Thus

$$\{\alpha \in A : \gamma \notin \mathrm{dom}(p_\alpha)\} \subseteq \{\alpha \in A : \beta_\alpha < \gamma\}.$$

But $\alpha \mapsto \beta_\alpha$ is an injective function, and so

$$|\{\alpha \in A : \beta_\alpha < \gamma\}| \leq |\{\delta \in \Sigma : \delta < \gamma\}| < \aleph_i.$$

Therefore $|\{\alpha \in A : \gamma \in \mathrm{dom}(p_\alpha)\}| = |A \setminus \{\alpha \in A : \gamma \notin \mathrm{dom}(p_\alpha)\}| = \aleph_i$. In particular, $\gamma \in \mathrm{dom}(q)$, as required.

Let $p \in \mathrm{Sym}(\Omega)$ be the identity on $\Omega \setminus \Sigma$ and equal to q on Σ. If $\alpha \in \Omega \setminus \Sigma$, then $\alpha(wpx)^{-1} = \{\alpha\}$ and so $b(wpx, 1) = \aleph_n = b(h, 1)$. If $\alpha \in \Sigma$, then $\alpha(wpx)^{-1} = (\alpha x^{-1})p^{-1}w^{-1} = \{\beta_\alpha, \gamma_\alpha\}w^{-1}$ since p maps $\{\beta_\alpha, \gamma_\alpha\}$ to αx^{-1}. Hence $|\alpha(wpx)^{-1}| = |\beta_\alpha| + |\gamma_\alpha| = |\alpha h^{-1}|$ by (3.5). In particular, if $m > 1$, then $b(wpx, m) = b(h, m)$, as required. ∎

Proof of Proposition 3.1. We start by showing that $\mathrm{rank}(\mathrm{Surj}(\Omega) : \mathrm{Sym}(\Omega)) \geq n^2/2 + 9n/2 + 5$. Seeking a contradiction assume that there exists $F \subseteq \mathrm{Surj}(\Omega)$ such that $\langle \mathrm{Sym}(\Omega), F\rangle = \mathrm{Surj}(\Omega)$ and $|F| < n^2/2 + 9n/2 + 5 = |\mathcal{K}|$. Then there exists $A \in \mathcal{K}$ such that $A \cap F = \emptyset$. Hence $\langle \mathrm{Sym}(\Omega), F\rangle$ is contained in the subsemigroup $\mathrm{Surj}(\Omega) \setminus A$ of $\mathrm{Surj}(\Omega)$. In particular, $\langle \mathrm{Sym}(\Omega), F\rangle \neq \mathrm{Surj}(\Omega)$, a contradiction.

To show that $\mathrm{rank}(\mathrm{Surj}(\Omega) : \mathrm{Sym}(\Omega)) \leq n^2/2 + 9n/2 + 5$, let F be the set consisting of $u_{i,j}, v_i, w_i, x_i, y$ for all $0 \leq i \leq j \leq n$. Then $|F| = n^2/2 + 9n/2 + 5$. We will show that $\langle \mathrm{Sym}(\Omega), F\rangle$ generates $\mathrm{Surj}(\Omega)$. By Lemmas 3.4 and 3.5, it suffices to prove that for all $f \in \mathrm{Surj}(\Omega)$ satisfying $b(f, 1) = \aleph_n$ there exists $f' \in \langle \mathrm{Sym}(\Omega), F\rangle$ such that $b(f, \lambda) = b(f', \lambda)$ for all $\lambda \in \mathbb{N} \cup \{\aleph_0, \aleph_1, \dots, \aleph_n\}$.

Let $f \in \mathrm{Surj}(\Omega)$ with $b(f, 1) = \aleph_n$. By the definition of $u_{i,j}$ and Lemmas 3.7 and 3.8, for all $0 \leq i \leq n$ there exists $g_i \in \langle \mathrm{Sym}(\Omega), F\rangle$ such that $b(g_i, 1) = \aleph_n$, $b(g_i, \aleph_i) = b(f, \aleph_i)$, and $b(g_i, \lambda) = 0$ for all $\lambda \notin \{1, \aleph_i\}$. Hence, by Lemma 3.6, there exists $g \in \langle \mathrm{Sym}(\Omega), F\rangle$ such that $b(g, 1) = \aleph_n$, $b(g, \aleph_i) = b(f, \aleph_i)$ for all $i \in \{0, 1, \dots, n\}$, and $b(g, m) = 0$ for all $m > 1$.

Hence, again by Lemma 3.6, it suffices to prove that there exists $h \in \langle \mathrm{Sym}(\Omega), F\rangle$ such that $b(h, m) = b(f, m)$ for all $m \in \mathbb{N}$ and $b(h, \lambda) = 0$ for all $\lambda \in \{\aleph_0, \dots, \aleph_n\}$.

Let $h \in \mathrm{Surj}(\Omega)$ be such that $b(h, m) = b(f, m)$ for all $m \geq 1$ and $b(h, \lambda) = 0$ for all $\lambda \in \{\aleph_0, \dots, \aleph_n\}$. From the definition of $c(h)$, there exists $M \in \mathbb{N}$ such that $b(h, m) \leq c(h)$ for all $m > M$. Let $h_2, h_3, \dots, h_M, t \in \mathrm{Surj}(\Omega)$ be such that $b(h_2, 1) = b(h_3, 1) = \dots = b(h_M, 1) = b(t, 1) = \aleph_n$ and let:

- $b(h_i, i) = b(h, i)$, and $b(h_i, \lambda) = 0$ if $\lambda \notin \{1, i\}$,
- $b(t, \lambda) = b(h, \lambda)$ if $M < \lambda < \aleph_0$, and $b(t, \lambda) = 0$ if $1 < \lambda \leq M$ or $\lambda \in \{\aleph_0, \dots, \aleph_n\}$,

for all $i \in \{2, 3, \dots, M\}$. It follows from Lemmas 3.4, 3.9, and 3.10 that $h_2, \dots, h_M \in \langle \mathrm{Sym}(\Omega), F\rangle$. Likewise, it follows, by Lemmas 3.4 and 3.11

where $\aleph_i = c(h)$, that $t \in \langle \mathrm{Sym}(\Omega), F\rangle$. We conclude, by Lemmas 3.4 and 3.6, that $h \in \langle \mathrm{Sym}(\Omega), F\rangle$, as required. ∎

Proof of the Main Theorem. (i) The set $\mathrm{Surj}(\Omega) \setminus \mathrm{Sym}(\Omega)$ is an ideal in $\mathrm{Surj}(\Omega)$. Thus, by Lemma 2.4, the Sierpiński rank of $\mathrm{Surj}(\Omega)$ is the sum of the Sierpiński rank of $\mathrm{Sym}(\Omega)$ and $\mathrm{rank}(\mathrm{Surj}(\Omega) : \mathrm{Sym}(\Omega))$ when the latter is finite. In particular, if $|\Omega| = \aleph_n$, then the Sierpiński rank of $\mathrm{Surj}(\Omega)$ is $n^2/2 + 9n/2 + 7$.

(ii) Lemma 3.3(i) was only stated for sets Ω of cardinality $\aleph_n$ for $n \in \mathbb{N}$. However, the proof actually shows that:

(a) if Ω is any infinite set and $f, g \in \mathrm{Surj}(\Omega)$, then
$$a(fg) \geq \max\{a(f), a(g)\};$$

(b) if Ω is any infinite set and $f, g \in \mathrm{Surj}(\Omega)$ such that $a(f)$ and $a(g)$ are regular cardinals, then $a(fg) \leq a(f)\, a(g)$,

If $|\Omega| \geq \aleph_\omega$, then there exist $f_0, f_1, \ldots \in \mathrm{Surj}(\Omega)$ such that $a(f_i) = \aleph_i$ for all $i \in \mathbb{N}$. Let U be any subset of $\mathrm{Surj}(\Omega)$ such that $f_0, f_1, \ldots \in \langle U\rangle$. If there exists $u \in U$ such that $a(u) \geq \aleph_\omega$, then $f_0, f_1, \ldots \in \langle U \setminus \{u\}\rangle$ by (a). Hence we may assume without loss of generality that $a(u) < \aleph_\omega$ for all $u \in U$. Since $\aleph_i$ is regular for all $i \in \mathbb{N}$, it follows from (b) that for all $i \in \mathbb{N}$ there exists $g \in U$ such that $a(g) = \aleph_i$. Thus U is infinite. ∎

We conclude by showing that if Ω is any infinite set, then there exists a generating set U for $\mathrm{Surj}(\Omega)$ such that $\mathrm{Surj}(\Omega) \neq U \cup U^2 \cup \cdots \cup U^m$ for any $m \geq 1$. In other words, we prove that $\mathrm{Surj}(\Omega)$ does not have Bergman's property, and so, by [28, Lemma 2.4], $\mathrm{Surj}(\Omega)$ is not strongly distorted and has no infinite universal set of words.

Let Ω be any infinite set. If $f \in \mathrm{Surj}(\Omega)$, then $d(f)$ was defined to be
$$|\{\alpha \in \Omega : |(\alpha f)f^{-1}| \geq 2\}|.$$
In (3.1) we defined
$$Y = \{f \in \mathrm{Surj}(\Omega) : a(f) \in \mathbb{N} \text{ and } 1 < d(f) < \aleph_0\}.$$
In the original definitions of d and Y we assumed that the set Ω had cardinality $\aleph_n$ for some $n \in \mathbb{N}$. However, both d and Y are well-defined for any infinite Ω. Likewise, the conclusion
$$\max\{d(f), d(g)\} \leq d(fg) \leq d(f) + d(g)$$
of Lemma 3.3(v) holds when Ω is any infinite set, and the proof is identical to that given above.

It is straightforward to verify that $Y \subseteq \langle \mathrm{Sym}(\Omega), y\rangle$. In particular, if $U = (\mathrm{Surj}(\Omega) \setminus Y) \cup \{y\}$, then $\langle U\rangle = \mathrm{Surj}(\Omega)$. Since $\max\{d(f), d(g)\} \leq d(fg)$ for all $f, g \in \mathrm{Surj}(\Omega)$, it follows that $\mathrm{Surj}(\Omega) \setminus (\mathrm{Sym}(\Omega) \cup Y)$ is an ideal in

$\mathrm{Surj}(\Omega)$. Hence if $f \in Y$ and $g_0, g_1, \ldots, g_r \in U$ are such that $f = g_0 g_1 \cdots g_r$, then $g_0, g_1, \ldots, g_r \in \mathrm{Sym}(\Omega) \cup \{y\}$. Therefore

$$d(f) \leq d(g_0) + d(g_1) + \cdots + d(g_r) \leq 2r + 2.$$

In particular, if $m \in \mathbb{N}$ is such that $d(f) > 2m$, then $f \notin U \cup U^2 \cup \cdots \cup U^m$. Since Y contains elements f with arbitrarily large $d(f) \in \mathbb{N}$, it follows that $\mathrm{Surj}(\Omega)$ does not have Bergman's property, as required.

4. Further classical transformation semigroups. In this section, we determine the Sierpiński rank of several further classical transformation semigroups including the injective functions, Baer–Levi semigroups, and Schützenberger monoids.

THEOREM 4.1. *Let Ω be an infinite set and let* $\mathrm{Inj}(\Omega)$ *be the semigroup of injective functions from Ω to Ω. Then:*

(i) *if $|\Omega| = \aleph_n$ for some $n \in \mathbb{N}$, then* $\mathrm{Inj}(\Omega)$ *has Sierpiński rank $n+4$;*
(ii) *if $|\Omega| \geq \aleph_\omega$, then* $\mathrm{Inj}(\Omega)$ *has infinite Sierpiński rank.*

Let Ω be any set and let $f, g \in \mathrm{Inj}(\Omega)$. Then it is straightforward to verify that

$$|\Omega \setminus \Omega fg| = |\Omega \setminus \Omega g| + |\Omega \setminus \Omega f|. \tag{4.1}$$

Consequently, $\mathrm{Inj}(\Omega) \setminus \mathrm{Sym}(\Omega)$ is an ideal in $\mathrm{Inj}(\Omega)$. Thus, by Lemma 2.4, the Sierpiński rank of $\mathrm{Inj}(\Omega)$ is the sum of the Sierpiński rank of $\mathrm{Sym}(\Omega)$ and $\mathrm{rank}(\mathrm{Inj}(\Omega) : \mathrm{Sym}(\Omega))$ when the latter is finite.

PROPOSITION 4.2. *Let Ω be an infinite set such that $|\Omega| = \aleph_n$ for some $n \in \mathbb{N}$. Then* $\mathrm{rank}(\mathrm{Inj}(\Omega) : \mathrm{Sym}(\Omega)) = n + 2$.

Proof. We start by showing that $\mathrm{rank}(\mathrm{Inj}(\Omega) : \mathrm{Sym}(\Omega)) \geq n+2$. Let U be any subset of $\mathrm{Inj}(\Omega)$ such that $\langle \mathrm{Sym}(\Omega), U\rangle = \mathrm{Inj}(\Omega)$. It follows by (4.1) that for all $0 \leq m \leq n$ there exists $f \in U$ such that $|\Omega \setminus \Omega f| = \aleph_m$. Also by (4.1) there exists $f \in U$ such that $|\Omega \setminus \Omega f| = 1$. Hence $|U| \geq n+2$.

To prove that $\mathrm{rank}(\mathrm{Inj}(\Omega) : \mathrm{Sym}(\Omega)) \leq n+2$, let $f \in \mathrm{Inj}(\Omega) \setminus \mathrm{Sym}(\Omega)$ be arbitrary, let $g_m \in \mathrm{Inj}(\Omega)$ be any element with $|\Omega \setminus \Omega g_m| = \aleph_m$ for all $0 \leq m \leq n$, and let $h \in \mathrm{Inj}(\Omega)$ be any element with $|\Omega \setminus \Omega h| = 1$. If $|\Omega \setminus \Omega f| = r$ for some $r \in \mathbb{N}$, then let $k = h^r$. If $|\Omega \setminus \Omega f| = \aleph_m$ for some $0 \leq m \leq n$, then let $k = g_m$ where $|\Omega \setminus \Omega f| = \aleph_m$. In either case, $|\Omega \setminus \Omega k| = |\Omega \setminus \Omega f|$. Hence there exists a bijection $t : \Omega \setminus \Omega k \to \Omega \setminus \Omega f$. Let $l \in \mathrm{Sym}(\Omega)$ be defined by

$$\alpha l = \begin{cases} \alpha k^{-1} f & \text{if } \alpha \in \Omega k, \\ \alpha t & \text{if } \alpha \in \Omega \setminus \Omega k. \end{cases}$$

Then $f = hl$ and so f belongs to the semigroup generated by $\mathrm{Sym}(\Omega)$ and $\{h\} \cup \{g_m : 0 \le m \le n\}$. Thus $\mathrm{rank}(\mathrm{Inj}(\Omega) : \mathrm{Sym}(\Omega)) \le |\{h\} \cup \{g_m : 0 \le m \le n\}| = n + 2$. ∎

Proof of Theorem 4.1. (i) If $|\Omega| = \aleph_n$ for some $n \in \mathbb{N}$, then, by Proposition 4.2, $\mathrm{rank}(\mathrm{Inj}(\Omega) : \mathrm{Sym}(\Omega)) = n + 2$. As the Sierpiński rank of $\mathrm{Sym}(\Omega)$ is 2 by Galvin [15], it follows by Lemma 2.4 that the Sierpiński rank of $\mathrm{Inj}(\Omega)$ is $n + 4$.

(ii) If $|\Omega| \ge \aleph_\omega$, then there exist $f_0, f_1, \ldots \in \mathrm{Inj}(\Omega)$ such that $|\Omega \setminus \Omega f_i| = \aleph_i$ for all $i \in \mathbb{N}$. So, if U is any subset of $\mathrm{Inj}(\Omega)$ such that $f_0, f_1, \ldots \in \langle U \rangle$, then, by (4.1), for all $i \in \mathbb{N}$ there exists $g \in U$ such that $|\Omega \setminus \Omega g| = \aleph_i$. Thus U is infinite. ∎

Let Ω be any infinite set, let $f \in \mathrm{Inj}(\Omega)$ be such that $|\Omega \setminus \Omega f| = 1$, and let

$$I = \{g \in \mathrm{Inj}(\Omega) : |\Omega \setminus \Omega g| \ge \aleph_0\}.$$

It is straightforward to prove that $\mathrm{Inj}(\Omega)$ is generated by $U = \mathrm{Sym}(\Omega) \cup I \cup \{f\}$. Also, by (4.1), I is an ideal of $\mathrm{Inj}(\Omega)$. Hence if $g \in \mathrm{Inj}(\Omega)$ is such that $|\Omega \setminus \Omega g| < \aleph_0$ and $g_0, g_1, \ldots, g_r \in U$ are such that $g = g_0 g_1 \cdots g_r$, then $g_0, g_1, \ldots, g_r \in \mathrm{Sym}(\Omega) \cup \{f\}$. Therefore

$$|\Omega \setminus \Omega g| \le |\Omega \setminus \Omega g_0| + |\Omega \setminus \Omega g_1| + \cdots + |\Omega \setminus \Omega g_r| \le r + 1.$$

In particular, if $m \in \mathbb{N}$ and $|\Omega \setminus \Omega g| > m$, then $g \notin U \cup U^2 \cup \cdots \cup U^m$. Since $\mathrm{Inj}(\Omega) \setminus I$ contains elements g with $|\Omega \setminus \Omega g|$ arbitrarily large, it follows that $\mathrm{Inj}(\Omega)$ does not have Bergman's property, is not strongly distorted, and does not have an infinite universal set of words.

Let Ω be an infinite set and let λ be any infinite cardinal less than $|\Omega|$. Then the *Baer–Levi semigroup* $\mathrm{BL}(\Omega, \lambda)$ is defined by

$$\mathrm{BL}(\Omega, \lambda) = \{f \in \mathrm{Inj}(\Omega) : |\Omega \setminus \Omega f| = \lambda\}.$$

THEOREM 4.3. *Let Ω be an infinite set and let λ be any infinite cardinal less than $|\Omega|$. Then $\mathrm{BL}(\Omega, \lambda)$ has infinite Sierpiński rank.*

Proof. Let $\Omega_0, \Omega_1, \ldots$ be disjoint subsets of Ω with $|\Omega_i| = \lambda$ for all $i \in \mathbb{N}$ and for all $i \in \mathbb{N}$ let $f_i \in \mathrm{BL}(\Omega, \lambda)$ be any element with $\Omega f_i = \Omega \setminus \Omega_i$. Seeking a contradiction, assume that there exists a finite subset $F \subseteq \mathrm{BL}(\Omega, \lambda)$ such that $f_0, f_1, \ldots \in \langle F \rangle$.

If $g \in F$, then let

$$A_g = \{i \in \mathbb{N} : (\exists w \in \langle F \rangle)(f_i = wg)\}.$$

By the pigeonhole principle, there exists $g \in F$ such that A_g is infinite. In particular, there exist $i, j \in \mathbb{N}$ such that $i \ne j$ and $f_i, f_j \in A_g$. Hence $\Omega = \Omega f_i \cup \Omega f_j \subseteq \Omega g$, a contradiction. ∎

The injectivity of elements of $\mathrm{BL}(\Omega,\lambda)$ was not used to prove Theorem 4.3. A similar argument can be used to prove that the Sierpiński rank of $\{f\in\Omega^\Omega : |\Omega\setminus\Omega f|=\lambda\}$ is infinite. If S is a semigroup such that $S\setminus S^2$ is infinite, then S has infinite Sierpiński rank. For example, if $S=\mathrm{Inj}(\Omega)\setminus\mathrm{Sym}(\Omega)$, $S=\mathrm{Surj}(\Omega)\setminus\mathrm{Sym}(\Omega)$, or $S=\{f\in\mathrm{Inj}(\Omega):\Omega f\subseteq\Omega\setminus\Sigma\}$ for any fixed $\Sigma\subseteq\Omega$, then $S\setminus S^2$ is infinite and so these semigroups have infinite Sierpiński rank.

The *Schützenberger monoid* on an infinite set Ω of regular cardinality is defined as

$$\mathrm{Sch}(\Omega)=\{f\in\Omega^\Omega : |\alpha f^{-1}|<|\Omega|\ \forall\alpha\in\Omega\}.$$

THEOREM 4.4. *Let Ω be an infinite set where $|\Omega|$ is a regular cardinal. Then* $\mathrm{Sch}(\Omega)$ *has Sierpiński rank* 2.

Proof. The proof has two steps. First, we show that any countable set of elements of $\mathrm{Sch}(\Omega)$ can be generated by five elements of $\mathrm{Sch}(\Omega)$, and then that any finite number of elements can be generated by two.

STEP 1. Let $f_0,f_1,\ldots\in\mathrm{Sch}(\Omega)$ be arbitrary and let $\Omega_0,\Omega_1,\ldots$ be sets partitioning Ω of cardinality $|\Omega|$ each. We define five functions in $\mathrm{Sch}(\Omega)$ that generate $f_0,f_1,\ldots$. Let $g_0\in\Omega^\Omega$ be a bijection from Ω to Ω_0, let g_1 be any extension of g_0^{-1} to an element of $\mathrm{Sch}(\Omega)$, let g_2 be any function that maps Ω_i bijectively to Ω_{i+1} for all $i\in\mathbb{N}$, and let g_3 be any extension of g_2^{-1} to an element of $\mathrm{Sch}(\Omega)$.

Then $g_0g_2^i$ is a bijection from Ω to Ω_i and so $(g_0g_2^i)^{-1}f_ig_0g_2^i$ is a function from Ω_i to Ω_i for all $i\in\mathbb{N}$. The fifth required function $g_4\in\Omega^\Omega$ is defined by

$$g_4=\bigcup_{i=0}^{\infty}(g_0g_2^i)^{-1}f_ig_0g_2^i.$$

As g_0 is a bijection and $f_0\in\mathrm{Sch}(\Omega)$, it follows that

$$|\Omega g_4|\geq|\Omega_0g_4|=|\Omega_0g_0^{-1}f_0g_0|=|\Omega f_0|=|\Omega|.$$

Furthermore, if $\alpha\in\Omega$, then there exists $i\in\mathbb{N}$ such that $\alpha\in\Omega_i$ and so

$$|\alpha g_4^{-1}|=|\alpha((g_0g_2^i)^{-1}f_ig_0g_2^i)^{-1}|=|\alpha(g_0g_2^i)^{-1}f_i^{-1}g_0g_2^i|=|\beta f_i^{-1}|<|\Omega|$$

where $\beta=\alpha(g_0g_2^i)^{-1}$, since $g_0g_2^i$ is a bijection and $f_i\in\mathrm{Sch}(\Omega)$. Hence $g_4\in\mathrm{Sch}(\Omega)$, as required.

To conclude, let $i\in\mathbb{N}$ and $\alpha\in\Omega$. Then, as $g_3^ig_1$ is an extension of $(g_0g_2^i)^{-1}$, we have

$$\alpha g_0g_2^ig_4(g_3)^ig_1=\alpha g_0g_2^ig_4(g_0g_2^i)^{-1}=\alpha g_0g_2^i(g_0g_2^i)^{-1}f_ig_0g_2^i(g_0g_2^i)^{-1}=\alpha f_i.$$

Thus $f_i=g_0g_2^ig_4(g_3)^ig_1$ and so $f_0,f_1,\ldots\in\langle g_0,g_1,g_2,g_3,g_4\rangle$.

STEP 2. Let $f_0,f_1,\ldots,f_m\in\mathrm{Sch}(\Omega)$ be arbitrary and let $\Omega_0,\Omega_1,\ldots,\Omega_{m+1}$ be sets partitioning Ω of cardinality $|\Omega|$ each. We will prove that there exist two functions in $\mathrm{Sch}(\Omega)$ that generate $f_0, f_1, \dots, f_m$. Let $g_0 \in \mathrm{Sch}(\Omega)$ be any function that maps Ω_i bijectively to Ω_{i+1} for all $0 \le i \le m-1$ and that maps $\Omega_m \cup \Omega_{m+1}$ bijectively to Ω_{m+1}. If h is any bijection from Ω_{m+1} to Ω_0, then $g_0^{m+1}hg_0^i$ is a bijection from Ω to Ω_i for all $0 \le i \le m$. The second required function is the extension g_1 of h defined by

$$\alpha g_1 = \begin{cases} \alpha h & \text{if } \alpha \in \Omega_{m+1}, \\ \alpha(g_0^{m+1}hg_0^i)^{-1} f_i & \text{if } \alpha \in \Omega_i \text{ and } 0 \le i \le m. \end{cases}$$

To show that $g_1 \in \mathrm{Sch}(\Omega)$, note that

$$|\Omega g_1| \ge |\Omega_{m+1} g_1| = |\Omega_{m+1} h| = |\Omega_0| = |\Omega|.$$

Moreover, if $\alpha \in \Omega g_1$ is arbitrary, then αg_1^{-1} is the union of αh^{-1} and the sets $\alpha((g_0^{m+1}hg_0^i)^{-1}f_i)^{-1}$ for all $0 \le i \le m$. Note that any, but not all, of the sets αh^{-1} and $\alpha((g_0^{m+1}hg_0^i)^{-1}f_i)^{-1}$ can be empty. Since h and $(g_0^{m+1}hg_0^i)^{-1}f_i$ are bijections and $f_i \in \mathrm{Sch}(\Omega)$, it follows that $|\alpha h^{-1}| = 1$ and $|\alpha((g_0^{m+1}hg_0^i)^{-1}f_i)^{-1}| = |\alpha f_i^{-1}| < |\Omega|$ for all $0 \le i \le m$. Thus αg_1^{-1} is a finite union of sets of cardinality strictly less than $|\Omega|$ and so $|\alpha g_1^{-1}| < |\Omega|$.

To finish the proof, let $0 \le i \le m$ be arbitrary and let $\alpha \in \Omega$. Then

$$\alpha g_0^{m+1}hg_0^i g_1 = \alpha(g_0^{m+1}hg_0^i)(g_0^{m+1}hg_0^i)^{-1}f_i = \alpha f_i.$$

Thus $f_i = g_0^{m+1}hg_0^i g_1$ and so $f_0, f_1, \dots, f_m \in \langle g, h\rangle$, as required. ■

The set of *bounded functions* on the rationals is defined by

$$\mathrm{BSelf}(\mathbb{Q}) = \{f \in \mathbb{Q}^{\mathbb{Q}} : (\exists k \in \mathbb{Q})(\forall q \in \mathbb{Q})(|(q)f - q| \le k)\}.$$

THEOREM 4.5. $\mathrm{BSelf}(\mathbb{Q})$ *has Sierpiński rank* 2.

Proof. The proof has two steps. First, we show that any countable set of elements of $\mathrm{BSelf}(\mathbb{Q})$ can be generated by three elements of $\mathrm{BSelf}(\mathbb{Q})$, and then that any finite number of elements can be generated by two. Throughout the proof we denote $\{q \in \mathbb{Q} : a \le q < b\}$ by $[a, b)$.

STEP 1. Let $f_0, f_1, \ldots \in \mathrm{BSelf}(\mathbb{Q})$ be arbitrary and let $\Omega_{i,0}, \Omega_{i,1}, \ldots$ be infinite sets partitioning the interval $[i, i+1)$ for all $i \in \mathbb{Z}$. It is straightforward to verify that $\mathrm{BSelf}(\mathbb{Q})$ is generated by those of its elements f satisfying $|(q)f - q| \le 1$ for all $q \in \mathbb{Q}$. Hence we may assume without loss of generality that $|(q)f_j - q| \le 1$ for all $q \in \mathbb{Q}$ and for all $j \in \mathbb{N}$. We define three functions in $\mathrm{BSelf}(\mathbb{Q})$ that generate $f_0, f_1, \ldots$. Let $g_0 : \Omega \to \Omega$ be any function that maps $[i, i+1)$ bijectively to $\Omega_{i,0}$ for all $i \in \mathbb{Z}$ and let $g_1 : \Omega \to \Omega$ be any function that maps $\Omega_{i,j}$ bijectively to $\Omega_{i,j+1}$ for all $i \in \mathbb{Z}$ and for all $j \in \mathbb{N}$. Then $g_1^{-j}g_0^{-1}$ is a bijection from $\Omega_{i,j}$ to $[i, i+1)$ for all $i \in \mathbb{Z}$ and for all $j \in \mathbb{N}$. The third function g_2 is defined to be the union of the functions $(g_0 g_1^j)^{-1} f_j|_{[i,i+1)}$ for all $i \in \mathbb{Z}$ and $j \in \mathbb{N}$. Since $[i, i+1)g_0 = \Omega_{i,0} \subseteq [i, i+1)$,

$[i,i+1)g_1 = [i,i+1) \setminus \Omega_{i,0}$, and $[i,i+1)g_2 \subseteq [i-1,i+2)$, it follows that $g_0, g_1, g_2 \in \mathrm{BSelf}(\mathbb{Q})$.

Finally, for any $q \in \mathbb{Q}$, there exists $i \in \mathbb{Z}$ such that $q \in [i,i+1)$. Hence

$$qg_0g_1^jg_2 = g_0g_1^j(g_0g_1^j)^{-1}f_j|_{[i,i+1)} = qf_j$$

and so $g_0g_1^jg_2 = f_j$ for all $j \in \mathbb{N}$.

Step 2. Let $f_0, f_1, \ldots, f_m \in \mathrm{BSelf}(\mathbb{Q})$ be arbitrary and let $\Omega_{i,0}, \Omega_{i,1}, \ldots$ $\ldots, \Omega_{i,m+1}$ be infinite sets partitioning $[i,i+1)$ for all $i \in \mathbb{Z}$. We will prove that there exist two elements of $\mathrm{BSelf}(\mathbb{Q})$ that generate $f_0, f_1, \ldots, f_m$. Let $g_0 : \Omega \to \Omega$ be any function that maps $\Omega_{i,j}$ bijectively to $\Omega_{i,j+1}$ for all $0 \leq j \leq m-1$ and that maps $\Omega_{i,m} \cup \Omega_{i,m+1}$ bijectively to $\Omega_{i,m+1}$ for all $i \in \mathbb{Z}$.

If $h : \bigcup_{i\in\mathbb{Z}} \Omega_{i,m+1} \to \Omega$ maps $\Omega_{i,m+1}$ bijectively to $\Omega_{i,0}$ for all $i \in \mathbb{Z}$, then $g_0^{m+1}hg_0^j$ is a bijection from $[i,i+1)$ to $\Omega_{i,j}$ for all $i \in \mathbb{Z}$ and for all $j \in \mathbb{N}$. The second required function g_1 is the extension of h defined by

$$qg_1 = \begin{cases} qh & \text{if } q \in \Omega_{i,m+1},\ i \in \mathbb{Z}, \\ q(g_0^{m+1}hg_0^j)^{-1}f_j & \text{if } q \in \Omega_{i,j},\ i \in \mathbb{Z},\ 0 \leq j \leq m. \end{cases}$$

We will show that $g_1 \in \mathrm{BSelf}(\mathbb{Q})$. Let $q \in \mathbb{Q}$ be arbitrary. If $q \in \Omega_{i,m+1} \subseteq [i,i+1)$ for some $i \in \mathbb{Z}$, then $qg_1 \in \Omega_{i,0} \subseteq [i,i+1)$ and so $|qg_1 - q| \leq 1$. Since there are only finitely many functions f_j, there exists $k \in \mathbb{N}$ such that $|qf_j - q| \leq k$ for all $q \in \mathbb{Q}$ and for all $0 \leq j \leq m$. If $q \in \Omega_{i,j}$ for some $i \in \mathbb{Z}$ and $0 \leq j \leq m$, then $qg_1 \in [i,i+1)f_j \subseteq [i-k,i+k+1)$ and so $g_1 \in \mathrm{BSelf}(\mathbb{Q})$.

Finally, if $q \in \mathbb{Q}$, then

$$qg_0^{m+1}g_1g_0^jg_1 = qg_0^{m+1}hg_0^jg_1 = qg_0^{m+1}hg_0^j(g_0^{m+1}hg_0^j)^{-1}f_j = qf_j$$

for all $0 \leq j \leq m$, as required. ∎

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