FUNDAMENTA
MATHEMATICAE
224 (2014)

Reflecting Lindelöf and converging ω_1-sequences

by

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Abstract. We deal with a conjectured dichotomy for compact Hausdorff spaces: each such space contains a non-trivial converging ω-sequence or a non-trivial converging ω_1-sequence. We establish that this dichotomy holds in a variety of models; these include the Cohen models, the random real models and any model obtained from a model of CH by an iteration of property K posets. In fact in these models every compact Hausdorff space without non-trivial converging ω_1-sequences is first-countable and, in addition, has many $\aleph_1$-sized Lindelöf subspaces. As a corollary we find that in these models all compact Hausdorff spaces with a small diagonal are metrizable.

Introduction. This paper deals with converging sequences of type ω and ω_1. If γ is a limit ordinal then a sequence $\langle x_\alpha : \alpha < \gamma\rangle$ in a topological space is said to converge to a point x if for every neighbourhood U of x there is an $\alpha < \gamma$ such that $x_\beta \in U$ for $\beta \geq \alpha$. To avoid non-relevant cases we shall always assume that our sequences are injective.

In [9] Juhász and Szentmiklóssy showed that if a compact space has a free sequence of length ω_1 then it has a converging free sequence of that length—a sequence $\langle x_\alpha : \alpha \in \omega_1\rangle$ is *free* if for all α the sets $\{x_\beta : \beta < \alpha\}$ and $\{x_\beta : \beta \geq \alpha\}$ have disjoint closures. One may rephrase this as: a compact space without converging ω_1-sequences must have countable tightness.

The authors of [9] also recall two questions of Hušek and Juhász regarding converging ω_1-sequences:

Hušek: Does every compact Hausdorff space contain a non-trivial converging ω-sequence or a non-trivial converging ω_1-sequence?

Juhász: Does every non-first-countable compact Hausdorff space contain a non-trivial converging ω_1-sequence?

In [1] it was shown that the space $\beta\mathbb{N}$ does contain a converging ω_1-sequence, which shows that Hušek's question is a weakening of Efimov's

2010 *Mathematics Subject Classification*: Primary 54D30; Secondary 03E35, 54A20, 54A25, 54A35, 54D20.

Key words and phrases: compact space, first-countable space, Lindelöfness, converging sequence, L-reflection, small diagonal, forcing, property K.

DOI: 10.4064/fm224-3-1

well-known question in [4] whether every compact Hausdorff space contains a converging ω-sequence or a copy of $\beta\mathbb{N}$.

For the remainder of the paper we refer to a space without converging ω_1-sequences as an *ω_1-free space.* Our main result shows that the answer to Juhász' question (and hence to Hušek's) is positive in a large class of models. The precise definition will be given later but examples are those obtained by adding Cohen and random reals and models obtained by iterations of Hechler forcing.

An important class of ω_1-free spaces consists of those having a small diagonal—introduced by Hušek in [6]. We say that a space, X, *has a small diagonal* if there is no ω_1-sequence in X^2 that converges to the diagonal $\Delta(X)$; a sequence $\langle\langle x_\alpha, y_\alpha\rangle : \alpha < \omega_1\rangle$ in X^2 *converges to the diagonal* $\Delta(X)$ if every neighbourhood of the diagonal contains a tail of the sequence. Note that if $\langle x_\alpha : \alpha \in \omega_1\rangle$ converges to x then $\langle\langle x, x_\alpha\rangle : \alpha \in \omega\rangle$ converges to $\langle x, x\rangle$ and hence to the diagonal. A well-studied problem is whether a compact Hausdorff space with a small diagonal (which we abbreviate by *csD space*) is metrizable.

The second part of our main result is that, in the same models, all ω_1-free spaces have many $\aleph_1$-sized Lindelöf subspaces. This will then imply that, in these models, all csD spaces are metrizable.

We benefit from the results in [11] concerning the notion of L-reflecting and the preservation of the Lindelöf property by forcing.

Finally, we should mention that in [8] it was shown that in the Cohen model all csD spaces are metrizable and Juhász' question has a positive answer.

Status of the problems. There are consistent examples of compact ω_1-free spaces that are not first-countable; we review some of these at the end of the paper, where we will also show that Hušek's question is strictly weaker than that of Efimov: there are many consistent counterexamples to Efimov's question but none to Hušek's question (yet).

As to the consistency of the existence of non-metrizable csD spaces—that is still an open problem.

1. Preliminaries

1.1. Elementary sequences and L-reflection. For a cardinal θ we let $H(\theta)$ denote the collection of all sets whose transitive closure has cardinality less than θ (see [13, Chapter IV]). An ω_1-sequence $\langle M_\alpha : \alpha \in \omega_1\rangle$ of countable elementary substructures of $H(\theta)$ that satisfies $\langle M_\beta : \beta \le \alpha\rangle \in M_{\alpha+1}$ for all α and $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$ for all limit α will simply be called an *elementary sequence.*

Let X be a compact Hausdorff space; an *elementary sequence for X* is an elementary sequence $\langle M_\alpha : \alpha \in \omega_1\rangle$ such that $X \in M_0$; of course in this case θ should be large enough in order that $X \in H(\theta)$; in most cases $\theta = (2^\kappa)^+$, where $\kappa = w(X)$, suffices.

Definition 1.1. We say that a space is *weakly L-reflecting* if every $\aleph_1$-sized subspace is contained in a Lindelöf subspace of cardinality $\aleph_1$. We say that X is *L-reflecting* if, for some regular θ with $X \in H(\theta)$ and any countable M_0 with $X \in M_0 \prec H(\theta)$, there is an elementary sequence $\langle M_\alpha : \alpha \in \omega_1\rangle$ for X such that $X \cap \bigcup_{\alpha\in\omega_1} M_\alpha$ is Lindelöf.

1.2. csD spaces. We will need a technical improvement of Gruenhage's result [5] that hereditarily Lindelöf csD spaces are metrizable. It is clear that compact hereditarily Lindelöf spaces are both first-countable and L-reflecting; we shall show that the last two properties suffice to make csD spaces metrizable. This will allow us to conclude that in the models from Section 2 all csD spaces are metrizable, because they will be seen to be first-countable and L-reflecting.

We use a convenient characterization of csD spaces obtained by Gruenhage [5]: for every sequence $\langle\langle x_\alpha, y_\alpha\rangle : \alpha \in \omega_1\rangle$ of pairs of points there is an uncountable subset A of ω_1 such that $\{x_\alpha : \alpha \in A\}$ and $\{y_\alpha : \alpha \in A\}$ have disjoint closures.

Theorem 1.2. *If a compact first-countable space is L-reflecting and has a small diagonal, then it is metrizable.*

Proof. Let X be a csD space that is first-countable and L-reflecting.

Let $\langle M_\alpha : \alpha \in \omega_1\rangle$ be an elementary sequence for X such that $X \cap \bigcup_{\alpha<\omega_1} M_\alpha$ is Lindelöf; we denote this subspace by Y. For each α let $\mathcal{B}_\alpha$ be the family of those open subsets of X that belong to M_α.

If we assume that X is not metrizable then there is no α for which $\mathcal{B}_\alpha$ is a base for the topology of X, or even, by compactness, a T_2-separating open cover of X. Therefore we can find, by elementarity, points $x_\alpha, y_\alpha \in X \cap M_{\alpha+1}$ that do not have disjoint neighbourhoods that belong to $\mathcal{B}_\alpha$.

We apply Gruenhage's criterion to find an uncountable subset A of ω_1 such that the closed sets $F = \mathrm{cl}\{x_\alpha : \alpha \in A\}$ and $G = \mathrm{cl}\{y_\alpha : \alpha \in A\}$ are disjoint. We take disjoint open sets U and V around F and G respectively.

Since X is first-countable we know that M_α contains a local base at each point of $X \cap M_\alpha$. Thus we may choose for each $x \in Y$ a neighbourhood B_x such that

- $B_x \subseteq U$ if $x \in F$,
- $B_x \subseteq V$ if $x \in G$,
- $B_x \cap (F \cup G) = \emptyset$ otherwise,

and in addition $B_x \in \mathcal{B}_\alpha$ whenever $x \in M_\alpha$.

Since Y is Lindelöf there is an α such that $Y \subseteq \bigcup\{B_x : x \in X \cap M_\alpha\}$. But now take $\beta \in A$ above α and take x and y in M_α such that $x_\beta \in B_x$ and $y_\beta \in B_y$. It follows readily that $B_x \subseteq U$ and $B_y \subseteq V$, so that x_β and y_β do have disjoint neighbourhoods that belong to $\mathcal{B}_\alpha$.

This contradiction concludes the proof. ∎

1.3. ω_1-free spaces. Here we include two technical results on ω_1-free spaces that will be useful in Section 2. The first follows from [10, Lemma 2.2] by setting $\rho = \mu = \aleph_1$; we give a proof for completeness and to illustrate the use of elementary sequences.

THEOREM 1.3. *If every separable subspace of a compact ω_1-free space is first-countable, then the space is first-countable.*

Proof. Assume that X is compact and ω_1-free. Working contrapositively we assume that X is not first-countable and produce a separable subspace that is not first-countable.

To this end we let $\langle M_\alpha : \alpha \in \omega_1\rangle$ be an elementary sequence for X. By elementarity there will be a point x in $X \cap M_0$ at which X is not first-countable; by compactness this means that $\{x\}$ is not a G_δ-set in X. This implies that for each $\alpha \in \omega_1$ there is a point $x_\alpha \in M_{\alpha+1}$ that is in $\bigcap\{U : U \in M_\alpha$ and U is open in $X\}$ and distinct from x.

Since X is ω_1-free, there is a complete accumulation point, z, of $\{x_\alpha : \alpha \in \omega_1\}$ that is distinct from x. Since X has countable tightness there is a $\delta \in \omega_1$ such that z is in the closure of $X \cap M_\delta$.

We show that $\mathrm{cl}(X \cap M_\delta)$ is not first-countable at x. Indeed, if W is an open neighbourhood of x that belongs to $M_{\delta+1}$ then $x_\alpha \in W$ for all $\alpha > \delta$ and in particular $z \in \mathrm{cl}\, W$. This more than shows that $M_{\delta+1}$ does not contain a countable family of neighbourhoods of x that would determine a local base at x in $\mathrm{cl}(X \cap M_\delta)$; by elementarity there is no such family at all. ∎

For the next result we need a piece of notation and the notion of a local π-net.

If $\mathcal{F}$ is filter on a set X then $\mathcal{F}^+$ denotes the family of sets that are positive with respect to $\mathcal{F}$, i.e., $G \in \mathcal{F}^+$ iff G intersects every member of $\mathcal{F}$.

A *local π-net* at a point, x, of a topological space is a family, $\mathcal{A}$, of non-empty subsets of the space such that every neighbourhood of x contains a member of $\mathcal{A}$. Clearly $\{x\}$ is a local π-net at x but it may not always be a very useful one; the next result produces a local π-net that consists of somewhat larger sets.

THEOREM 1.4. *Let X be a compact space of countable tightness and let $\mathcal{F}$ be a countable filter base in X. Then there is a point x in $\bigcap\{\mathrm{cl}\, F : F \in \mathcal{F}\}$ that has a countable π-net that is contained in $\mathcal{F}^+$.*

Proof. Without loss of generality we assume that $\mathcal{F}$ is enumerated as $\{F_n : n \in \omega\}$ such that $F_{n+1} \subseteq F_n$ for all n. Take a sequence $\langle a_n : n \in \omega\rangle$ in X such that $a_n \in F_n$ for all n and let K be the set of cluster points of this sequence.

If K has an isolated point, x, then some subsequence of $\langle a_n : n \in \omega\rangle$ converges to x; the tails of that sequence form the desired π-net at x.

In the other case there is a point x such that K has a countable local π-base $\{U_n : n \in \omega\}$ at x. Shrink each member U_n of this local π-base to a compact relative G_δ-set G_n.

Write $G_n = K \cap \bigcap_{m\in\omega} O_{n,m}$, where each $O_{n,m}$ is open in X and $\operatorname{cl} O_{n,m+1} \subseteq O_{n,m}$ for all m. Choose an infinite subset A_n of $\{a_k : k \in \omega\}$ such that $A_n \setminus O_{n,m}$ is finite for all m. Observe that all accumulation points of A_n belong to G_n.

Now, if O is an open set that contains x then $G_n \subseteq O$ for some n and hence $A_n \setminus O$ is finite. This shows that the family $\{A_n \setminus F : n \in \omega,\ F \text{ is finite}\}$ is the desired π-net at x. ∎

1.4. A strengthening of the Fréchet–Urysohn property. We shall need a version of the Fréchet–Urysohn property where the convergent sequences are guided by ultrafilters.

Definition 1.5. A space X will be said to be *ultra-Fréchet* if it has countable tightness and for each countable subset D of X and free ultrafilter $\mathcal{U}$ on D there is a countable subfamily $\mathcal{U}'$ of $\mathcal{U}$ with the property that every infinite pseudointersection of $\mathcal{U}'$ converges.

A set P is an *infinite pseudointersection* of a family $\mathcal{F}$ of subsets of ω if $P \setminus F$ is finite for all $F \in \mathcal{F}$.

We shall use this property in the proof of Theorem 2.6, where we will have to distinguish between a subspace being ultra-Fréchet or not. To see how the property will be used we prove the following lemma.

Lemma 1.6. *Let D be a countable subset of an ultra-Fréchet space X and let $x \in \operatorname{cl} D$. Then whenever $\langle A_n : n \in \omega\rangle$ is a decreasing sequence of infinite subsets of D such that $x \in \bigcap_{n\in\omega} \operatorname{cl} A_n$ there is an infinite pseudointersection of the A_n that converges to x.*

Proof. Let $\langle A_n : n \in \omega\rangle$ be given and let $\mathcal{U}$ be an ultrafilter on D that converges to x and contains all A_n. Let $\mathcal{U}'$ be a countable subset of $\mathcal{U}$ as in the definition of ultra-Fréchet.

We first claim that *every* infinite pseudointersection of $\mathcal{U}'$ converges to x. Indeed, since the union of two pseudointersections is again a pseudointersection all pseudointersections converge to the same point, y say. Next, every neighbourhood of x contains such a pseudointersection, which implies that y belongs to every neighbourhood of x and therefore $y = x$.

To finish the proof take any infinite pseudointersection of the countable family $\mathcal{U}' \cup \{A_n : n \in \omega\}$. ∎

1.5. A preservation result. We finish this section by quoting a preservation result on the Lindelöf property. The Tychonoff cube $[0,1]^{\omega_1}$ is compact but when we pass to a forcing extension the ground model cube is at best a (proper) dense subset of the cube in the extension and thus no longer a compact space. Under certain circumstances, however, it will still have the Lindelöf property.

PROPOSITION 1.7 ([11]). *If $\mathbb{P}$ is a poset each of whose finite powers satisfies the countable chain condition then in any forcing extension by $\mathbb{P}$ the set of points in $[0,1]^{\omega_1}$ from the ground model is still Lindelöf.* ∎

Note that this result applies to closed subsets of $[0,1]^{\omega_1}$ as well.

2. Forcing extensions. In this section we prove the main result of this paper; it establishes first-countability of ω_1-free spaces in a variety of models. The better known of these are the Cohen model, the random real model and Hechler's models with various cofinal subsets in ${}^{\omega}\omega$ with the order $<^*$.

We begin by defining the particular type of poset our result will apply to.

2.1. ω_1-finally property K. We recall that a subset A of a poset $\mathbb{P}$ is *linked* if any two elements are compatible, i.e., whenever p and q are in A there is an $r \in \mathbb{P}$ such that $r \le p$ and $r \le q$. A poset has *property K*, or *satisfies the Knaster condition*, if every uncountable subset has an uncountable linked subset. Any measure algebra has property K and any finite support iteration of property K posets again has property K.

Our result uses a modification of this notion.

DEFINITION 2.1. We say that a poset $\mathbb{P}$ is *ω_1-finally property K* if for each completely embedded poset $\mathbb{Q}$ of cardinality at most ω_1 the quotient $\mathbb{P}/\dot{G}$ is forced, by $\mathbb{Q}$, to have property K.

A poset $\mathbb{Q}$ is *completely embedded* in $\mathbb{P}$ if for every generic filter H on $\mathbb{P}$ the intersection $H \cap \mathbb{Q}$ is generic on $\mathbb{Q}$. As explained in [13, Section VII.7] the factor (or quotient) $\mathbb{P}/\dot{G}$ is a name for the poset obtained from a generic filter G on $\mathbb{Q}$ in the following way: it is the subset of those elements of $\mathbb{P}$ that are compatible with all elements of G. We shall use the important fact that $\mathbb{P}$ is forcing equivalent to the two-step iteration $\mathbb{Q} * (\mathbb{P}/\dot{G})$ ([13, Chapter VII, Exercises D3–5] or [14, Lemma V.4.45]).

2.2. Forcing and elementarity. Throughout our proofs we will be working with elementary sequences and we shall frequently be using the following fact, a proof of which can be found in [15, Theorem III, 2.11].

PROPOSITION 2.2. *Let $M \prec H(\theta)$ and let $\mathbb{P} \in M$ be a poset. Then $M[G]$ is an elementary substructure of $H(\theta)[G]$ (which is the $H(\theta)$ of $V[G]$).* ∎

Here $M[G] = \{\mathrm{val}_G(\tau) : \tau \in M \text{ and } \tau \text{ is a } \mathbb{P}\text{-name}\}$.

The general situation that we will consider is one where we have an elementary sequence $\langle M_\alpha : \alpha \in \omega_1\rangle$ and a poset $\mathbb{P}$ that belongs to M_0. The union $M = \bigcup_{\alpha\in\omega_1} M_\alpha$ is also an elementary substructure of the $H(\theta)$ under consideration.

Since we will be assuming CH it follows that ${}^\omega M \subseteq M$; using this, one readily proves the following proposition.

PROPOSITION 2.3. *If $\mathbb{P} \in M$ is a partial order that satisfies the countable chain condition then $\mathbb{P} \cap M$ is a complete suborder of $\mathbb{P}$.* ∎

Thus the intersection G_M of a generic filter G on $\mathbb{P}$ with M will be generic on $\mathbb{P}_M$.

Furthermore, if $\dot{X} \in M_0$ is a $\mathbb{P}$-name for a compact space then the above implies that in $V[G]$ the sequence $\langle M_\alpha[G] : \alpha \in \omega_1\rangle$ is an elementary sequence for X.

Finally, a $\mathbb{P}$-name $\dot{A}$ for a subset of ω can be represented by the subset $\{\langle p, n\rangle : p \Vdash n \in \dot{A}\}$ of $\mathbb{P}\times\omega$ and even by a countable subset of this product: simply choose, for each n, a maximal antichain A_n in $\{p : p \Vdash n \in \dot{A}\}$ and let $A' = \{\langle p, n\rangle : p \in A_n\}$. Now if $\dot{A} \in M$ then, by elementarity, there is such a countable A' in M, and then $A' \in M_\alpha$ for some α. But then $A' \subseteq M_\alpha$ and therefore $\mathrm{val}_G(\dot{A}) \in M_\alpha[G_M] \subseteq M[G_M]$.

The facts above are well known but we reviewed them because they are crucial to some of our arguments.

2.3. Pre-Luzin gaps. In our proof we shall construct converging ω_1-sequences in an intermediate model and use the following combinatorial structure to lift these to the full generic extension.

DEFINITION 2.4. For a countable set D, a family $\{\langle a_\alpha, b_\alpha\rangle : \alpha \in \omega_1\}$ of ordered pairs of disjoint subsets of D will be called a *pre-Luzin gap* if for all $E \subset D$ the set of α such that $E \cap (a_\alpha \cup b_\alpha) =^* a_\alpha$ is countable.

The lifting of the sequence to the full extension is accomplished using the following lemma.

LEMMA 2.5. *If $\mathbb{P}$ has property K and $\{\langle a_\alpha, b_\alpha\rangle : \alpha \in \omega_1\}$ is a pre-Luzin gap on ω then it remains a pre-Luzin gap in every forcing extension by $\mathbb{P}$.*

Proof. Let $\dot{E}$ be a $\mathbb{P}$-name of a subset of ω. Arguing contrapositively we assume there is an uncountable subset A of ω_1 so that for each $\alpha \in A$ there are a $p_\alpha \in \mathbb{P}$ and an integer n_α such that $p_\alpha \Vdash_{\mathbb{P}} \dot{E} \cap (a_\alpha \cup b_\alpha) \setminus n_\alpha = a_\alpha \setminus n_\alpha$. We shall build a subset E of ω such that $E \cap (a_\alpha \cup b_\alpha) =^* a_\alpha$ for uncountably many $\alpha \in A$.

We apply property K and assume, without loss of generality, that $\{p_\alpha : \alpha \in A\}$ is linked and that there is a single integer n such that $n_\alpha = n$ for all $\alpha \in A$.

We let $E = \bigcup\{a_\alpha \setminus n : \alpha \in A\}$. To verify that E is as required we let $\alpha \in A$ and $j \in E \setminus n$. Fix $\beta \in A$ such that $j \in a_\beta$. Then $p_\beta \Vdash j \in \dot{E}$ and $p_\alpha \Vdash \dot{E} \cap b_\alpha \subseteq n$; as p_α and p_β are compatible this implies that $j \notin b_\alpha$. ∎

2.4. The main theorem

Theorem 2.6. (CH) *Let $\mathbb{P}$ be a poset that is ω_1-finally property K. Then in every generic extension by $\mathbb{P}$ every compact ω_1-free space is first-countable and L-reflecting.*

We prove this theorem in three steps.

Let $\dot{X}$ be a $\mathbb{P}$-name for a compact Hausdorff space that is ω_1-free. We take an elementary sequence $\langle M_\alpha : \alpha \in \omega_1\rangle$ such that $\mathbb{P}$ and $\dot{X}$ are in M_0.

We continue to write $M = \bigcup_{\alpha\in\omega_1} M_\alpha$, $\mathbb{P}_M = \mathbb{P} \cap M$ and $G_M = G \cap M$. To save on writing we put $N_\alpha = M_\alpha[G]$ and $N = M[G]$.

Proposition 2.7. *In $V[G]$ the space X is ultra-Fréchet.*

Proof. We assume X is not ultra-Fréchet and take a countable set D witnessing this—we know already that X has countable tightness. Let $e : \omega \to D$ be a bijection and let $\mathcal{U}$ be an ultrafilter on ω such that every countable subfamily of it has an infinite pseudointersection whose image under e does not converge. Let z be the limit of $e(\mathcal{U})$ and observe that $\{z\} \neq \bigcap\{\operatorname{cl} e[U] : U \in \mathcal{U}'\}$ whenever $\mathcal{U}'$ is a countable subfamily of $\mathcal{U}$; indeed, if equality were to hold then the image of every infinite pseudointersection of $\mathcal{U}'$ would converge to z.

By elementarity we can assume that D, $e, \mathcal{U}$ and (hence) z belong to N_0.

For each α we find a point $x_\alpha \in X$ and a countable family $\mathcal{A}_\alpha$ of subsets of ω as follows. We apply the property above to $\mathcal{U} \cap N_\alpha$ and fix $y \in N_{\alpha+1}$ such that $y \neq z$ and $y \in \bigcap\{\operatorname{cl} e[U] : U \in \mathcal{U} \cap N_\alpha\}$. Next we take a neighbourhood W of y with $z \notin \operatorname{cl} W$.

We apply Theorem 1.4 to the filter base generated by $\{W \cap D\} \cup (e(\mathcal{U}) \cap N_\alpha)$ to find a point x_α and a countable local π-net at x_α that can be written as $\{e[A] : A \in \mathcal{A}_\alpha\}$, where $\mathcal{A}_\alpha$ is a countable subfamily of $(\{e^{\leftarrow}[W]\} \cup (\mathcal{U} \cap N_\alpha))^+$, which itself is a subfamily of $(\mathcal{U} \cap N_\alpha)^+$. Note that $x_\alpha \neq z$ because $x_\alpha \in \operatorname{cl} W$.

By elementarity the choices above—W, x_α and $\mathcal{A}_\alpha$—can all be made in $N_{\alpha+1}$.

As noted above in Subsection 2.2, since $\mathcal{A}_\alpha$ is a *countable* family of subsets of ω we can find a name for it that is actually a subset of $M_{\alpha+1}$. Therefore we know that each $\mathcal{A}_\alpha$ and its members belong to $V[G_M]$.

Since we assume that X is ω_1-free the sequence $\langle x_\alpha : \alpha \in \omega_1 \rangle$ will have (at least) two distinct complete cluster points, so that there are two open sets O_1 and O_2 with disjoint closures that each contain x_α for uncountably many α.

Now we consider $V[G]$ as an extension of $V[G_M]$ by the poset $\mathbb{P}/G_M$ and choose a sequence $\langle p_\alpha : \alpha \in \omega_1 \rangle$ of conditions in the latter, two strictly increasing sequences of ordinals $\langle \beta_\alpha : \alpha \in \omega_1 \rangle$ and $\langle \gamma_\alpha : \alpha \in \omega_1 \rangle$, and two sequences $\langle a_\alpha : \alpha \in \omega_1 \rangle$ and $\langle b_\alpha : \alpha \in \omega_1 \rangle$ of subsets of ω, such that

$$p_\alpha \Vdash \dot{x}_{\beta_\alpha} \in \dot{O}_1 \quad \text{and} \quad p_\alpha \Vdash \dot{x}_{\gamma_\alpha} \in \dot{O}_2$$

and $a_\alpha \in \mathcal{A}_{\beta_\alpha}$ and $b_\alpha \in \mathcal{A}_{\gamma_\alpha}$, and

$$p_\alpha \Vdash e[a_\alpha] \subseteq \dot{O}_1 \quad \text{and} \quad p_\alpha \Vdash e[b_\alpha] \subseteq \dot{O}_2.$$

We apply the ccc to find $q \in \mathbb{P}/G_M$ that forces $G \cap \{p_\alpha : \alpha \in \omega_1\}$ to be uncountable.

In $V[G]$ we form $A = \{\alpha : p_\alpha \in G\}$ and $E = e^{\leftarrow}[O_1]$; then $a_\alpha \subseteq E$ and $b_\alpha \cap E = \emptyset$ for $\alpha \in A$, so that $\{\langle a_\alpha, b_\alpha \rangle : \alpha \in \omega_1\}$ is not a pre-Luzin gap in $V[G]$.

However, in $V[G_M]$ the set $\{\langle a_\alpha, b_\alpha \rangle : \alpha \in \omega_1\}$ *is* a pre-Luzin gap. For if $E \subseteq \omega$ belongs to $V[G_M]$ then it belongs to N_α for some α and either it or its complement belongs to $\mathcal{U}$, say $E \in \mathcal{U}$. But then a_β and b_β both meet E in an infinite set whenever $\beta > \alpha$. ∎

The next step is to prove that X is first-countable.

Proposition 2.8. *In $V[G]$ the space X is first-countable.*

Proof. We assume it is not and apply Theorem 1.3 to find a countable subset D of X and a point z in $\operatorname{cl} D$ that does not have a countable local base in $\operatorname{cl} D$. By elementarity, z and D can be found in N_0; as above we take a bijection e from ω to D, also in N_0.

Using the fact that X is ultra-Fréchet we shall construct an ω_1-sequence that converges to z. To this end we observe that for every α one can build a family $\{A(\alpha, s) : s \in {}^{<\omega}2\}$ of subsets of ω that has the following properties:

(1) $A(\alpha, \emptyset) = \omega$,
(2) $A(\alpha, s) = A(\alpha, s * 0) \cup A(\alpha, s * 1)$ (the $*$ denotes concatenation),
(3) $A(\alpha, s * 0) \cap A(\alpha, s * 1) = \emptyset$,
(4) for every subset A of ω that is in N_α there is an n such that the family $\{A(\alpha, s) : s \in {}^n2\}$ refines $\{A, \omega \setminus A\}$.

This can be done in a simple recursion, using the fact that N_α is countable; by elementarity we can assume that the family $\{A(\alpha, s) : s \in {}^{<\omega}2\}$ belongs to $N_{\alpha+1}$. Each $A(\alpha, s)$ determines, via e, a subset $D(\alpha, s)$ of D.

Fix an α and form

$$C_\alpha = \{s \in {}^{<\omega}2 : z \notin \operatorname{cl} D(\alpha, s)\};$$

the set $\bigcap\{\operatorname{cl} D \setminus \operatorname{cl} D(\alpha, s)\}$ is a G_δ-set in $\operatorname{cl} D$ and hence it is not equal to $\{z\}$. Thus we may pick $y_\alpha \neq z$ in this intersection and a function f_α in ${}^\omega 2$ such that $y_\alpha \in \operatorname{cl} D(\alpha, f_\alpha{\restriction}m)$ for all m; note that then also $z \in \operatorname{cl} D(\alpha, f_\alpha{\restriction}m)$ for all m.

Now apply Lemma 1.6 and choose infinite pseudointersections a_α and b_α of the $A(\alpha, f_\alpha{\restriction}m)$ such that $e[a_\alpha]$ and $e[b_\alpha]$ converge to z and y_α respectively. By elementarity, C_α, y_α, f_α, a_α and b_α can all be chosen in $N_{\alpha+1}$, and as in the previous proof, the countable objects—in particular a_α and b_α—actually belong to $M_{\alpha+1}[G_M]$.

Again as in the previous proof: any subset of ω that belongs to $V[G_M]$ has a name that is in M and hence belongs to N_α for some α, which implies that it either contains or is disjoint from both a_β and b_β whenever $\beta \geq \alpha$. Thus, $\{\langle a_\alpha, b_\alpha\rangle : \alpha \in \omega_1\}$ is a pre-Luzin gap in $V[G_M]$ and hence, because $\mathbb{P}/G_M$ has property K, it is also a pre-Luzin gap in $V[G]$.

We show that this implies that $\langle y_\alpha : \alpha \in \omega_1\rangle$ converges to z. Indeed, let U be a neighbourhood of z and consider $e^{\leftarrow}[U \cap D]$; this set contains a cofinite part of every set a_α and hence an infinite part of all but countably many b_α. This implies that $y_\alpha \in \operatorname{cl} U$ for all but countably many α. ∎

Proof that X is L-reflecting. Since $\langle M_\alpha : \alpha \in \omega_1\rangle$ is an arbitrary elementary sequence it suffices to show that $X \cap N$ is Lindelöf. This will require some more notation.

To begin we fix, in V, a cardinal κ and we assume that $\dot{X}$ is forced to be a subset of $[0,1]^\kappa$; we can take κ in M_0. We also write $\Gamma = \kappa \cap M$ and let π_Γ denote the projection of $[0,1]^\kappa$ onto $[0,1]^\Gamma$. We shall show three things:

(1) π_Γ is a homeomorphism between $X \cap N$ and $\pi_\Gamma[X \cap N]$ in $V[G]$,
(2) $\pi_\Gamma[X \cap N]$ is in $V[G_M]$, and
(3) $\pi_\Gamma[X \cap N]$ is a closed subset of $[0,1]^\Gamma$ in $V[G_M]$.

Proposition 1.7 then implies that $\pi_\Gamma[X \cap N]$ is Lindelöf in $V[G]$ and hence that $X \cap N$ is Lindelöf too.

The first item is a consequence of the first-countability of X.

Lemma 2.9. *The map π_Γ is a homeomorphism between $X \cap N$ and $\pi_\Gamma[X \cap N]$.*

Proof. Because X is first-countable there is a countable local base at each point of $X \cap N$ that consists of basic open sets and that, by elementarity, may be taken to be a member of N. The latter means that all members of such a local base have their supports in Γ. This is enough to establish the lemma. ∎

In the proof of the other two statements we abbreviate $M_\alpha[G_M]$ by M_α^+ and $M[G_M]$ by M^+. We also need a way to code members of $[0,1]^\kappa$ that makes it easy to calculate (names for) projections of members of X. A point of $[0,1]^\Gamma$ is determined by a function $x : \kappa \times \omega \to 2$: its γth coordinate is given by $\sum_{n\in\omega} x(\gamma, n) \cdot 2^{-n-1}$.

If $\dot{x} \in M$ is such a name and $x = \mathrm{val}_G(\dot{x})$ then one readily checks that $\pi_\Gamma(x) = \mathrm{val}_{G_M}(\dot{x})$.

We let X^+ denote the set of $\mathbb{P}$-names of such functions that are forced by $\mathbb{P}$ to determine members of X. Note that $X^+ \in M_0$ by elementarity.

Using these names it is easy to prove the second item in our list.

Lemma 2.10. *The set $\pi_\Gamma[X \cap N]$ belongs to $V[G_M]$.*

Proof. Using the coding described above we deduce that $\pi_\Gamma[X \cap N] = \{\mathrm{val}_{G_M}(\dot{x}) : x \in X^+\}$; the latter set belongs to $V[G_M]$. ∎

In preparation for the proof of the third item in our list we prove:

Lemma 2.11. *For every α we have $\pi_\Gamma[X \cap N_\alpha] = \pi_\Gamma[X] \cap M_\alpha^+$.*

Proof. The equality $\mathrm{val}_{G_M}(\dot{x}) = \pi_\Gamma(\mathrm{val}_G(\dot{x}))$ for $\dot{x} \in M_\alpha$ establishes the inclusion $\pi_\Gamma[X \cap N_\alpha] \subseteq \pi_\Gamma[X] \cap M_\alpha^+$.

For the converse let $\dot{x} \in M_\alpha$ be such that $x_M = \mathrm{val}_{G_M}(\dot{x}) \in \pi_\Gamma[X]$; then $x = \mathrm{val}_G(\dot{x})$ belongs to N_α and, by elementarity, $y = x{\restriction}(\kappa \times \omega)$ is a function that also belongs to N_α and whose domain contains $\Gamma \times \omega$. By elementarity $\mathrm{dom}\, y = \kappa \times \omega$ and so $x_M = \pi_\Gamma(y)$. ∎

The proof of our third statement will almost be a copy of that of Proposition 2.7.

Proposition 2.12. *In $V[G_M]$ the set $\pi_\Gamma[X \cap N]$ is closed in $[0,1]^\Gamma$.*

Proof. In $V[G_M]$ let $z \in \mathrm{cl}\, \pi_\Gamma[X \cap N]$. Of course z is a point of $\mathrm{cl}\, \pi_\Gamma[X \cap N]$ as computed in $V[G]$ as well and hence $z \in \pi_\Gamma[X]$ as the latter set is compact; the task is to find $x \in X \cap N$ such that $z = \pi_\Gamma(x)$.

In $V[G]$ the set $\pi_\Gamma[X]$ is of countable tightness. Hence z is in the closure of $\pi_\Gamma[X \cap N_\delta]$ for some $\delta < \omega_1$.

We assume, to reduce indexing, that $\delta = 0$ and we write D for $\pi_\Gamma[X \cap N_0]$; we also take an enumeration $e : \omega \to D$ that belongs to M_1^+; we shall use e also, implicitly, to enumerate $X \cap N_0$.

We take an ultrafilter $\mathcal{U}$ on ω such that $e(\mathcal{U})$ converges to z. Note that, in contrast with the proof of Proposition 2.7, neither z nor $\mathcal{U}$ need belong to M^+. However, because ${}^\omega M \subseteq M$ and because $\mathbb{P}_M$ is a ccc poset of cardinality $\aleph_1$ we also have ${}^\omega M^+ \subseteq M^+$. Therefore we know that for every α there is $\beta_\alpha > \alpha$ such that $\mathcal{U}_\alpha = \mathcal{U} \cap M_\alpha^+ \in M_{\beta_\alpha}^+$.

Thus, if there is some α such that $\{z\} = \bigcap\{\mathrm{cl}\, e[U] : U \in \mathcal{U}_\alpha\}$ then $z \in M_{\beta_\alpha}^+$ and Lemma 2.11 applies to show that $z \in \pi_\Gamma[X \cap N_{\beta_\alpha}]$. From now on we assume $\{z\} \neq \bigcap\{\mathrm{cl}\, e[U] : U \in \mathcal{U}_\alpha\}$ for all α and follow the proof Proposition 2.7 to reach a contradiction.

The only modification that needs to be made is when choosing the point x_α and the family $\mathcal{A}_\alpha$. Our assumption now implies that $\bigcap\{\mathrm{cl}\, e[U] : U \in \mathcal{U}_\alpha\}$ has more than one point, hence there are two basic open sets W_1 and W_2 in M_{β_α} with disjoint closures that both meet this intersection. We let W be the one of the two that does not have z in its closure.

We now use first-countability of X to find $x_\alpha \in X \cap N_{\beta_\alpha}$ and an infinite pseudointersection c_α of $\mathcal{U}_\alpha \cup \{e^{\leftarrow}[W]\}$, also in N_{β_α}, such that $e[c_\alpha]$ converges to x_α (remember that e also enumerates $X \cap N_0$); $\mathcal{A}_\alpha$ consists of the cofinite subsets of c_α.

From here on the proof is the same as that of Proposition 2.7. ∎

3. Examples

Juhász' question. Since our main result establishes a consistent positive answer to Juhász' question we should begin by recording a consistent negative answer as well.

Example 3.1 ([7]). It is consistent to have a compact ω_1-free space that is not first-countable.

The space is the one-point compactification of a locally compact, first-countable and initially ω_1-compact space that is locally of cardinality $\aleph_1$. The space is not L-reflecting either but this is not easily shown so we omit the proof.

Question 1. If X is locally compact, not Lindelöf, and initially ω_1-compact, does it fail to be L-reflecting?

Hušek versus Efimov. We can also use our main result to show that Hušek's question is strictly weaker than Efimov's: in [3] it is shown that $\mathfrak{b} = \mathfrak{c}$ implies there is an Efimov space, that is, a compact Hausdorff space that contains neither a converging ω-sequence nor a copy of $\beta\mathbb{N}$. Since we can use Hechler forcing to create models for $\mathfrak{b} = \mathfrak{c}$, where $\mathfrak{c}$ can have any regular value we please, we get a slew of models where Hušek's question has a positive answer and Efimov's a negative one.

The need for property K. To demonstrate the need for property K in the proof of Theorem 2.6 we quote the following example.

Example 3.2 ([12]). There is a poset $\mathbb{P}$, with all finite powers ccc, that forces the existence of a compact first-countable space that is not weakly L-reflecting.

This space is constructed in Theorems 7.5 and 7.6 of [12]. Theorem 7.5 produces a compact space K with ${}^{\omega_1}2$ as its underlying set and the property that whenever a point f and a sequence $\langle f_\alpha : \alpha \in \omega_1\rangle$ in ${}^{\omega_1}2$ are given such that $f_\alpha \cap f \in 2^\alpha$ for all α then in K the point f is the limit of the converging ω_1-sequence $\langle f_\alpha : \alpha \in \omega_1\rangle$. Theorem 7.6 then produces a compact first-countable space X that maps onto K.

The poset is constructed in a ground model V that satisfies CH. We let $T = ({}^{<\omega_1}2)^V$ and we choose for each $t \in T$ an $f_t \in {}^{\omega_1}2$ such that $t \subset f_t$. The closure of the set $Y = \{f_t : t \in T\}$ will contain all the cofinal branches of T, and so will contain $\aleph_2$ many converging ω_1-sequences with distinct limits of K. It then follows that any subset of X that maps onto Y will not be contained in a Lindelöf subset of cardinality $\aleph_1$.

Question 2. Are compact spaces with small diagonal metrizable in the model described in Example 3.2?

While we do not know the answer to this question, let us remark that the space constructed in Example 3.2 does not have a small diagonal. In fact it was the space X of Example 3.2 that was the motivation for Proposition 2.4 of [2]. The space X has copies of Cantor sets and ω_1-sequences that co-countably converge to these. By the aforementioned proposition this implies that X does not have a small diagonal.

Acknowledgments. We thank the referee for suggesting that the proof of the L-reflecting property of X could be simplified and made more direct.

Research of the first author was supported by NSF grant No. NSF-DMS-0901168.

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Received 15 November 2012;
in revised form 18 July 2013