

On almost specification and average shadowing properties

by

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Abstract. We study relations between the almost specification property, the asymptotic average shadowing property and the average shadowing property for dynamical systems on compact metric spaces. We show implications between these properties and relate them to other important notions such as shadowing, transitivity, invariant measures, etc. We provide examples showing that compactness is a necessary condition for these implications to hold. As a consequence, we also obtain a proof that limit shadowing in chain transitive systems implies shadowing.

1. Introduction. Studies on shadowing and specification, originating with the works of Anosov and Bowen, have been developing parallel to the theory of hyperbolic systems. In a crude sense, one may say that these notions are similar. The common goal is to find a true trajectory near an approximate one. The differences are in understanding what constitutes an approximate trajectory. For shadowing one traces a pseudo-orbit, while for specification one follows arbitrarily assembled finite pieces of orbits with a true orbit. A template definition for any generalization of shadowing (or specification) might be: every approximate orbit can be traced by a true one. Moreover, given a quantitative method of measuring to what extent an approximate orbit resembles a true trajectory, and how closely it is traced (followed, reproduced) by the orbit of some point, we may restate our template definition: for every ε there is a δ such that every δ -approximate orbit can be traced with error not greater than ε . This template is a base for the generalizations of all notions we investigate here: the almost specification property, the average shadowing property, and the asymptotic average shadowing property. Pilyugin's book [29] is a good general reference on shad-

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owing and its generalizations, while Palmer's [25] presents their applications. Properties of systems with specification are described in [9].

In the late 1980s Blank introduced the notion of average pseudo-orbits and proved that certain kinds of perturbed hyperbolic systems have the average shadowing property (see [4, 5]). Average pseudo-orbits arise naturally in the realizations of independent Gaussian random perturbations with zero mean and in investigations of most probable orbits of a dynamical system with general Markov perturbations, etc. (see [6, p. 20]). It is proved in [4, Theorem 4] that if Λ is a basic set of a diffeomorphism f satisfying Axiom A, then $f|_{\Lambda}$ has the average shadowing property. The notion gained considerable attention: see [4, 5, 6, 7, 24, 31, 32, 33, 36]. In [31] Sakai analyzed the dynamics of diffeomorphisms with the average shadowing property on a two-dimensional closed manifold. Later, in [32, 33], he compared various shadowing properties of positively expansive maps. The results of [33] were generalized and completed in [18]. In [24] Niu proved that if f has the average shadowing property and the minimal points of f are dense in X , then f is weakly mixing and totally strongly ergodic.

The next property we consider is the *asymptotic average shadowing* introduced by Gu [13]. Gu followed the same scheme as Blank, but with limit shadowing instead of shadowing as the starting point for generalization. The asymptotic average shadowing property was examined, inter alia, in [15, 16]. It was proved that there is a large class of systems with the asymptotic average shadowing property, including all mixing maps of the unit interval and their Denjoy extensions.

More recently, Climenhaga and Thompson [8, 34], inspired by the work of Pfister and Sullivan [28], examined some properties of systems with the almost specification property, which in turn generalizes the notion of specification. As all β -shifts have the almost specification property, their results apply to those important symbolic systems.

We believe that the techniques and notions described above deserve a deeper study and that the results scattered through the literature should be put into a unified framework. Therefore our main goal is to explore the general properties of systems possessing generalized shadowing and/or specification in the abstract setting.

It follows from our work that these generalizations are related much more closely than the original notions. Specification is well known to imply almost specification, and there are examples of systems with almost specification, but without specification. We show that the almost specification property implies the asymptotic average shadowing property (Theorem 3.5), which in turn implies the average shadowing property (Theorem 3.7). We do not know whether the converse implications are true when the space under consideration is compact, which is the usual setting (Question 10.3). Neverthe-

less, we are able to give examples of maps on noncompact metric spaces that either have the average shadowing property, but fail the asymptotic average shadowing property (Example 9.1), or the other way round (Example 9.3).

We also explore recurrence properties of dynamical systems with the average shadowing property. We prove that every dynamical system with the almost specification property and full invariant measure (i.e., positive on each nonempty open set) has a dense set of minimal points (Theorem 4.1) and that every dynamical system with the average shadowing property and full invariant measure is topologically weakly mixing (Theorem 4.3). Moreover, we show that if the supports of invariant measures are not dense, then no recurrence property is implied by the average shadowing property (see Section 8 for several examples). In Section 5 we prove that f has the average shadowing property (almost specification property, asymptotic average shadowing property) provided f restricted to its measure center (the closure of the union of all supports of f -invariant measures) has that property.

As a byproduct, we obtain a somewhat surprising result (Theorem 7.3) that limit shadowing in chain transitive systems implies shadowing (and so transitivity). It was proved recently by Lee and Sakai [21] that expansive systems with shadowing have limit shadowing (in fact they have the so-called strong limit shadowing property, which is stronger than both shadowing and limit shadowing, as shown in [3, Example 3.5]). There are also examples of systems with limit shadowing but without shadowing (see [29, Example 1.21], which may be generalized to a large class of homeomorphisms on $[0, 1]$). By the above evidence it is natural to expect that limit shadowing is a weaker property than shadowing. In fact, Theorems 7.3 and 3.7 are completely opposite to what the authors aimed to prove when they started working on the topics of this paper.

2. Preliminaries. Let $\mathbb{N} = \{0, 1, \dots\}$ denote the set of natural numbers and let $\mathbb{Z}_+, \mathbb{Z}_-$ denote the set of positive and negative integers, respectively. For any $A \subset \mathbb{N}$ and any $n > 0$ we let $\text{card}(A \mid n)$ denote the cardinality of $A \cap \{0, 1, \dots, n - 1\}$.

We define the *lower* and *upper asymptotic density* of a set $J \subset \mathbb{N}$ as

$$d_*(J) = \liminf_{n \rightarrow \infty} \frac{1}{n} \text{card}(J \mid n) \quad \text{and} \quad d^*(J) = \limsup_{n \rightarrow \infty} \frac{1}{n} \text{card}(J \mid n),$$

respectively. The set J is said to have the *asymptotic density* $d(J)$ provided that $d_*(J) = d^*(J) = d(J)$. A set $J \subset \mathbb{N}$ has *positive upper Banach density* if and only if for some sequence $\{k_n\}_{n=1}^\infty$ of integers we have

$$\limsup_{n \rightarrow \infty} \frac{1}{n} (\text{card}(J \mid k_n + n) - \text{card}(J \mid k_n)) > 0.$$

A set $A \subset \mathbb{N}$ is *thick* if it contains arbitrarily long sequences of consecutive natural numbers.

Let $J \subset \mathbb{N}$ be such that $\mathbb{N} \setminus J$ is unbounded. Let $\{a_i\}_{i=0}^\infty$ be a sequence of real numbers. If there is $b \in \mathbb{R}$ such that the sequence obtained from $\{a_i\}_{i=0}^\infty$ by deleting the terms with indices from J has limit b , then we write

$$\lim_{i \notin J} a_i = b.$$

We record the following lemma for further reference.

LEMMA 2.1 ([35, Thm. 1.20]). *Let $\{a_i\}_{i=0}^\infty$ be a bounded sequence of non-negative real numbers. The following conditions are equivalent:*

- (1) $\lim_{n \rightarrow \infty} n^{-1} \sum_{i=0}^{n-1} a_i = 0$,
- (2) *there exists a set $J \subset \mathbb{N}$ such that $d(J) = 0$ and $\lim_{n \notin J} a_n = 0$.*

By a “map” we always understand a continuous map. Given a metric space (X, ρ) and a map $f: X \rightarrow X$ we call the pair (X, f) a *dynamical system*. Throughout this paper, unless stated otherwise, we assume that a dynamical system (X, f) on a compact space X is given.

A dynamical system (Y, g) is a *factor* of a dynamical system (X, f) if there is a continuous surjection $\pi: X \rightarrow Y$ such that $\pi \circ f = g \circ \pi$.

The *orbit* of $x \in X$ is the set $\{f^n(x) : n \in \mathbb{N}\}$. We say that f is *minimal* if the orbit of every $x \in X$ is a dense subset of X . The map f is *transitive* (respectively *mixing*) if for any pair of nonempty open sets $U, V \subset X$ there exists $n > 0$ ($N > 0$) such that $f^n(U) \cap V \neq \emptyset$ (for all $n \geq N$, respectively). We say that f is *totally transitive* if f^n is transitive for all $n \geq 1$. The map f is *weakly mixing* if $f \times f$ is transitive on $X \times X$.

A pair $(x, y) \in X \times X$ is *proximal* if

$$\liminf_{n \rightarrow \infty} \rho(f^n(x), f^n(y)) = 0.$$

A pair $(x, y) \in X \times X$ is *distal* if it is not proximal. We say that a pair $(x, y) \in X \times X$ is *diagonal* if $x = y$. A dynamical system (X, f) is

- *proximal* if all pairs in $X \times X$ are proximal,
- *distal* if all nondiagonal pairs in $X \times X$ are distal.

Compactness of X implies that these properties are independent of the choice of equivalent metric for X .

The map f is *equicontinuous* if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $\rho(f^n(x), f^n(y)) < \varepsilon$ for all $n > 0$ and all $x, y \in X$ such that $\rho(x, y) < \delta$. Every dynamical system (X, f) has a *maximal equicontinuous factor* $(X_{\text{eq}}, f_{\text{eq}})$. That is, $(X_{\text{eq}}, f_{\text{eq}})$ is equicontinuous, and every equicontinuous factor of (X, f) is a factor of $(X_{\text{eq}}, f_{\text{eq}})$ (see [12, pp. 18–19]).

Let $I = \{0, \dots, n\}$ or $I = \mathbb{N}$. A sequence $\{x_i\}_{i \in I}$ is called a δ -*pseudo-orbit* of f (from x_0 to x_n and of *length* n if $I = \{0, \dots, n\}$) if $\rho(f(x_{i-1}), x_i) < \delta$ for

every positive $i \in I$. A sequence $\{x_i\}_{i=0}^\infty \subset X$ is an *asymptotic pseudo-orbit* of f provided that

$$\lim_{n \rightarrow \infty} \varrho(f(x_i), x_{i+1}) = 0.$$

We say that f is *chain transitive* if for any $\delta > 0$ and $x, y \in X$ there is a δ -pseudo-orbit from x to y . A map f is *chain mixing* if for any $\delta > 0$ and $x, y \in X$ there is an integer $N > 0$ such that for every $n \geq N$ there is a finite δ -pseudo-orbit from x to y of length n . This is equivalent to $f \times f$ being chain transitive (see [30]). Given sets $A, B \subset X$ we define the *set of transition times from A to B* by

$$N(A, B) = \{n > 0 : f^n(A) \cap B \neq \emptyset\}.$$

If $x \in X$, then $N(x, B) = N(\{x\}, B) = \{n > 0 : f^n(x) \in B\}$ denotes the *set of visiting times*. Note that there are no universally accepted names for the sets $N(A, B)$ and $N(x, B)$. Some authors (see, e.g., [22]) prefer to call $N(A, B)$ the set of *hitting times of A and B* , and $N(x, B)$ the set of times x enters into B , respectively.

We let $M(X)$ denote the space of all Borel probability measures on X . We say that a measure $\mu \in M(X)$ is *invariant* for $f: X \rightarrow X$ if $\mu(A) = \mu(f^{-1}(A))$ for any Borel set $A \subset X$.

The classical Krylov–Bogolyubov theorem guarantees that every compact dynamical system (X, f) has at least one invariant measure. If the system has exactly one invariant measure then we say that it is *uniquely ergodic*.

We say that a set $Y \subset X$ is *measure saturated* if it is contained in the closure of the union of the topological supports of invariant measures, or equivalently, if for every open set U such that $U \cap Y \neq \emptyset$ there exists an invariant measure μ such that $\mu(U) > 0$. The *measure center* of f is the largest measure saturated set.

We say that an open set $U \subset X$ is *universally null* if $\mu(U) = 0$ for any invariant measure μ . The measure center of f is the complement of the union of all universally null sets.

Given a nonempty Borel set A and $n > 0$ we define

$$\eta(n, A) = \max_{x \in X} \text{card}(N(x, A) \mid n).$$

We note that $\eta(n + m, A) \leq \eta(n, A) + \eta(m, A)$ for all $n, m > 0$, hence $\{\eta(n, A)\}_{n=1}^\infty$ is a subadditive sequence, and we may define the *visit frequency in U* as

$$\xi(A) = \lim_{n \rightarrow \infty} \frac{\eta(n, A)}{n} = \inf_{n > 0} \frac{\eta(n, A)}{n}.$$

The following lemma follows from the ergodic theorem, but here we present a direct topological proof inspired by [14].

LEMMA 2.2. *If (X, f) is a compact dynamical system, then for every nonempty open set $U \subset X$ there exists a point $x \in X$ such that*

$$d(N(x, U)) = \xi(U).$$

Proof. First observe that if there is a point $x \in X$ such that $N(x, U)$ has positive upper Banach density then $\xi(U) > 0$. Therefore $\xi(U) = 0$ implies that $N(x, U)$ has upper Banach density zero for every $x \in X$, hence $d(N(x, U)) = 0$ for every $x \in X$.

Next, assume that $\xi(U) > 0$. For every $n > 0$ let $x_n \in X$ be such that

$$\text{card}(N(x_n, U) \mid n) = \max_{x \in X} \text{card}(N(x, U) \mid n) = \eta(n, U).$$

We claim that for each integer $n > 0$ there exists $y_n \in X$ such that

$$(2.1) \quad \xi(U) - 1/n \leq (1/j) \text{card}(N(y_n, U) \mid j) \quad \text{for } 1 \leq j \leq n.$$

Indeed, assume that (2.1) does not hold for some $k > 0$. Then $\xi(U) - 1/k > 0$. Put $m = k^2 + 1$ and let $z = x_m$ be a point defined above so that $\text{card}(N(z, U) \mid m) = \eta(m, U)$. As we assumed that our claim fails we can find a strictly increasing sequence of integers $\{l(s)\}_{s=0}^\infty$ such that $l(0) = 0$, $0 < \lambda_j = l(j) - l(j - 1) \leq k$, and

$$(2.2) \quad \frac{1}{\lambda_j} \text{card}(N(f^{l(j-1)}(z), U) \mid \lambda_j) < \xi(U) - \frac{1}{k}$$

for every $j = 1, 2, \dots$. Let $t > 0$ be such that $l(t) \leq m < l(t + 1)$. Then

$$(2.3) \quad \begin{aligned} &\text{card}(N(z, U) \mid m) \\ &= \left(\sum_{j=0}^t \text{card}(N(f^{l(j-1)}(z), U) \mid \lambda_j) \right) + \text{card}(N(f^{l(t)}(z), U) \mid m - l(t)). \end{aligned}$$

Then by (2.2) and (2.3) we obtain

$$\begin{aligned} m\xi(U) &\leq \eta(m, U) = \text{card}(N(z, U) \mid m) \\ &= \left(\sum_{j=0}^t \text{card}(N(f^{l(j-1)}(z), U \mid \lambda_j) \right) + \text{card}(N(f^{l(t)}(z), U) \mid m - l(t)) \\ &\leq \left[\sum_{j=0}^t \lambda_j \left(\xi(U) - \frac{1}{k} \right) \right] + k < m \left(\xi(U) - \frac{1}{k} \right) + k = m\xi(U) - \frac{1}{k} \\ &< m\xi(U), \end{aligned}$$

which is a contradiction, so our claim holds.

By the claim, for each integer $n > 0$ we can find a point y_n such that (2.1) holds. Since X is compact, we may assume that $\{y_n\}_{n=1}^\infty$ converges to some $x \in X$. Observe that for all $k > 0$ and $n \geq k$ we have $\xi(U) - 1/k \leq$

$(2/k) \text{card}(N(y_n, U) \mid k)$, which implies that

$$\xi(U) - 1/k \leq (1/k) \text{card}(N(x, U) \mid k) \leq (1/k) \cdot \eta(k, U).$$

Letting $k \rightarrow \infty$ we obtain

$$\xi(U) \leq d_*(N(x, U)) \leq d^*(N(x, U)) \leq \xi(U),$$

which finishes the proof. ■

We can now characterize universally null open sets as the sets with visit frequency ξ equal to zero.

THEOREM 2.3. *Let (X, f) be a compact dynamical system. An open set $U \subset X$ is universally null if and only if $\xi(U) = 0$, or equivalently $d(N(x, U)) = 0$ for every $x \in X$.*

Proof. If U is not universally null, then $\mu(U) > 0$ for some f -invariant measure $\mu \in M(X)$. Using ergodic decomposition [35, p. 153] we get an ergodic measure $\mu_e \in M(X)$ such that $\mu_e(U) > 0$. By the Pointwise Ergodic Theorem [35, Thm. 1.14] there is a point $x \in X$ such that $d(N(x, U)) = \mu_e(U) > 0$. Therefore $\xi(U) > 0$.

The proof of the other implication follows the same lines as that of the second part of [11, Lemma 3.17]. We present it for completeness. Assume that $\xi(U) > 0$. By Lemma 2.2 there are $x \in X$ and $0 < \varepsilon < \xi(U)$ such that

$$\text{card}(N(x, U) \mid n) \geq n\varepsilon$$

for all sufficiently large n . Observe that for any continuous function $F: X \rightarrow \mathbb{R}$ there is an increasing function $\sigma: \mathbb{N} \rightarrow \mathbb{N}$ such that the limit

$$L(F) = \lim_{n \rightarrow \infty} \frac{1}{\sigma(n)} \sum_{i=0}^{\sigma(n)-1} F(f^i(x))$$

exists. This implies that for any sequence of continuous functions $F_k: X \rightarrow \mathbb{R}$ we can use a diagonal procedure to find an increasing function $\sigma: \mathbb{N} \rightarrow \mathbb{N}$ such that $L(F_k)$ exists for every k . As the space $C(X)$ of all continuous functions from X to \mathbb{R} with the supremum metric is separable we fix a sequence $\{F_k\}_{k=0}^\infty$ dense in $C(X)$ and choose a sequence σ as above. Then it is elementary to check that with this particular σ the number $L(F)$ is well defined for every continuous function $F: X \rightarrow \mathbb{R}$. Hence we have defined a linear functional L on $C(X)$. By the Riesz Representation Theorem there is a measure $\mu \in M(X)$ such that $L(F) = \int F d\mu$, and since $L(F) = L(F \circ f)$, it must be an invariant measure for f . But

$$\mu(U) = \int \chi_U d\mu = L(\chi_U) = \lim_{n \rightarrow \infty} \frac{1}{\sigma(n)} \sum_{i=0}^{\sigma(n)-1} \text{card}(N(x, U) \mid \sigma(n)) \geq \varepsilon > 0.$$

This concludes the proof. ■

COROLLARY 2.4. *If a set A contains the measure center of a compact dynamical system (X, f) , then for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for every $x \in X$ and $n \geq N$ we have*

$$\frac{1}{n} \#\{0 \leq i < n : \varrho(f^i(x), A) < \varepsilon\} > 1 - \varepsilon.$$

Proof. Fix any $\varepsilon > 0$ and let $U = \{x : \varrho(x, A) > \varepsilon/2\}$. Since A contains the measure center, U is a universally null open set. By Theorem 2.3 we obtain $\xi(U) = 0$ and so for all sufficiently large n we have

$$\begin{aligned} \max_{x \in X} \frac{1}{n} \#\{0 \leq i < n : \varrho(f^i(x), A) \geq \varepsilon\} &\leq \max_{x \in X} \frac{1}{n} \text{card}(N(x, U) \mid n) \\ &= \frac{1}{n} \eta(n, U) \leq \xi(U) + \frac{\varepsilon}{2} < \varepsilon. \blacksquare \end{aligned}$$

The following definitions were introduced by Blank [4, 5]:

DEFINITION 2.5. Given $\delta > 0$ we say that a sequence $\{x_n\}_{n=0}^\infty$ is a δ -average-pseudo-orbit of f if there is an integer $N > 0$ such that for all $n \geq N$ and $k \geq 0$,

$$\frac{1}{n} \sum_{i=0}^{n-1} \varrho(f(x_{i+k}), x_{i+k+1}) < \delta.$$

DEFINITION 2.6. Given $\varepsilon > 0$ and $y \in X$ we say that a sequence $\{x_n\}_{n=0}^\infty$ is ε -shadowed on average by y if

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \varrho(f^i(y), x_i) < \varepsilon.$$

DEFINITION 2.7. We say that f has the *average shadowing property* if for every $\varepsilon > 0$ there is $\delta > 0$ such that every δ -average-pseudo-orbit of f is ε -shadowed on average by some point in X .

The next three definitions were introduced by Gu [13]:

DEFINITION 2.8. A sequence $\{x_i\}_{i=0}^\infty \subset X$ is an *asymptotic average pseudo-orbit* of f provided that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \varrho(f(x_i), x_{i+1}) = 0.$$

DEFINITION 2.9. A sequence $\{x_i\}_{i=0}^\infty \subset X$ is *asymptotically shadowed on average* by a point $y \in X$ provided that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \varrho(f^i(y), x_i) = 0.$$

DEFINITION 2.10. The map f has the *asymptotic average shadowing property* provided that every asymptotic average pseudo-orbit of f is asymptotically shadowed on average by some point in X .

Pfister and Sullivan [28] have introduced a property called the *g -almost product property*. Subsequently Thompson proposed renaming it the *almost specification property*. The only difference between the approach in [28] and the one in [34] is that the mistake function g can depend also on ε (in [28] it depends on n alone). The almost specification property can be verified for every β -shift (see [27]). We follow Thompson, so the version we use here is a priori weaker than that of Pfister and Sullivan.

First we need a few auxiliary definitions.

DEFINITION 2.11. Let $\varepsilon_0 > 0$. A function $g: \mathbb{Z}_+ \times (0, \varepsilon_0] \rightarrow \mathbb{N}$ is called a *mistake function* if for all $\varepsilon \in (0, \varepsilon_0]$ and all $n \in \mathbb{Z}_+$, we have $g(n, \varepsilon) \leq g(n + 1, \varepsilon)$ and

$$\lim_{n \rightarrow \infty} \frac{g(n, \varepsilon)}{n} = 0.$$

Given a mistake function g , we define $g(n, \varepsilon) = g(n, \varepsilon_0)$ for $\varepsilon > \varepsilon_0$.

DEFINITION 2.12. Let g be a mistake function and let $\varepsilon > 0$. For n large enough for the inequality $g(n, \varepsilon) < n$ to hold we define the *set of $(g; n, \varepsilon)$ almost full subsets of $\{0, \dots, n - 1\}$* to be

$$I(g; n, \varepsilon) := \{ \Lambda \subset \{0, 1, \dots, n - 1\} : \#\Lambda \geq n - g(n, \varepsilon) \}.$$

DEFINITION 2.13. For a finite set of indices $\Lambda \subset \{0, 1, \dots, n - 1\}$, we define the *Bowen distance between $x, y \in X$ along Λ* by

$$\varrho_\Lambda(x, y) = \max\{ \varrho(f^j(x), f^j(y)) : j \in \Lambda \}$$

and the *Bowen ball (of radius ε centered at $x \in X$) along Λ* by

$$B_\Lambda(x, \varepsilon) = \{ y \in X : \varrho_\Lambda(x, y) < \varepsilon \}.$$

When g is a mistake function and (n, ε) is such that $g(n, \varepsilon) < n$, we define for $x \in X$ the *$(g; n, \varepsilon)$ -Bowen ball of radius ε , center x , and length n* by

$$\begin{aligned} B_n(g; x, \varepsilon) &:= \{ y \in X : y \in B_\Lambda(x, \varepsilon) \text{ for some } \Lambda \in I(g; n, \varepsilon) \} \\ &= \bigcup_{\Lambda \in I(g; n, \varepsilon)} B_\Lambda(x, \varepsilon). \end{aligned}$$

With the above notation, we are in a position to define the almost specification property.

DEFINITION 2.14. A continuous map $f: X \rightarrow X$ has the *almost specification property* if there exists a mistake function g and a function $k_g: (0, \infty) \rightarrow \mathbb{N}$ such that for any $m \geq 1$, any $\varepsilon_1, \dots, \varepsilon_m > 0$, any points $x_1, \dots, x_m \in X$,

and any integers $n_1 \geq k_g(\varepsilon_1), \dots, n_m \geq k_g(\varepsilon_m)$, setting $n_0 = 0$ and

$$l_j = \sum_{s=0}^{j-1} n_s \quad \text{for } j = 1, \dots, m,$$

we can find a point $z \in X$ such that for every $j = 1, \dots, m$ we have

$$f^{l_j}(z) \in B_{n_j}(g; x_j, \varepsilon_j).$$

In other words, the appropriate part of the orbit of z ε_j -traces the orbit of x_j with at most $g(\varepsilon_j, n_j)$ mistakes.

REMARK 2.15. If the system (X, f) has almost specification with mistake function g we may (and usually will) assume that $k_g(\varepsilon) = n$ implies that $g(m, \varepsilon) < m$ for $m \geq n$.

The function g as above can be interpreted as follows. The integer $g(n, \varepsilon)$ tells us how many mistakes may occur when we use the almost specification property to ε -shadow an orbit of length n .

Note that Pfister and Sullivan use a slightly different notion of a blowup function instead of the mistake function defined above. The blowup function is not allowed to depend on ε . An example of a function which is a mistake function under the definition proposed by Thompson but is not allowed by Pfister and Sullivan is $g(n, \varepsilon) = \lfloor \varepsilon^{-1} \log n \rfloor$. Therefore the almost specification property of Thompson is slightly more general than the g -almost product property of Pfister and Sullivan.

Note that, while in our setting the space X is compact, the definitions of the (almost) specification property and the two generalized shadowing properties remain meaningful without this assumption. But in a noncompact setting none of these properties (specification, almost specification, (asymptotic) average shadowing) is an invariant of topological conjugacy, as can be seen from the example in [20, Section 7] and Theorem 3.8 below. The reader can find several comments on the specification property and its relationship to the (asymptotic) average shadowing property in [15, 16, 18]. Moreover, note the following.

EXAMPLE 2.16. Let $X = \{a, b\}$ be any two points set with the discrete metric ρ and let f be such that $f(a) = b, f(b) = b$. Then f has the (almost) specification property and the (asymptotic) average shadowing property.

3. Connections between almost specification and average shadowing

3.1. Chain mixing. It turns out that if f is surjective, then chain mixing accompanies both the average shadowing and the almost specification

properties, as evidenced by the next two lemmas. Note that surjectivity of f is necessary, in view of

LEMMA 3.1. *Let (X, f) be a compact dynamical system. If f is surjective and has the average shadowing property, then f is chain mixing.*

Proof. It was proved in [24] that if f has the average shadowing property, then so does f^n for every $n > 0$. By [30], if f^n is chain transitive for every $n > 0$, then it is chain mixing. Therefore it is enough to prove that f is chain transitive.

For simplicity of calculations assume that $\text{diam}(X) \leq 1$. Fix any $\varepsilon > 0$ and $x, y \in X$. Let δ be provided by the average shadowing property for $\varepsilon/2$. Let $n_0 \geq 2$ be such that $2/n_0 < \delta$ and let $n_i = 2^i n_0$ for $i \geq 1$. For $j = 0, 1, 2, \dots$ we use surjectivity to fix a point $y_j \in f^{-n_{2j+1}+1}(y)$ and define a sequence

$$\begin{aligned} \xi = & (x, f(x), \dots, f^{n_0-1}(x), y_0, f(y_0), \dots, f^{n_1-1}(y_0), x, f(x), \dots, f^{n_2-1}(x), \\ & y_1, f(y_1), \dots, f^{n_3-1}(y_1), \dots, x, f(x), \dots, f^{n_{2j-1}-1}(x), \\ & y_j, f(y_j), \dots, f^{n_{2j+1}-1}(y_j), \dots). \end{aligned}$$

Let $l(j) = (2^j - 1)n_0$. By the definition of ξ we have $\xi_{l(2i)} = x$, $\xi_{l(2i+2)-1} = y$ and $\xi_{l(2i+1)} = y_i$ for all $i = 0, 1, 2, \dots$. Therefore $\xi_{l(j)}, \dots, \xi_{l(j+1)-1}$ is the initial segment of length n_j of the orbit of x if j is even, and of y_k , where $k = (j - 1)/2$, if j is odd.

Notice that if we fix any $i \geq 0$ then in the sequence $\xi_i, \dots, \xi_{i+n_0}$ there is at most one position j with $\varrho(f(\xi_j), \xi_{j+1}) > 0$. Therefore for every $k > n_0$ we have

$$\frac{1}{k} \sum_{j=i}^{i+k-1} \varrho(f(\xi_j), \xi_{j+1}) \leq \frac{1}{k} \left(\frac{k}{n_0} + 1 \right) \text{diam}(X) \leq \frac{2}{n_0} < \delta,$$

which shows that ξ is an δ -average-pseudo-orbit. Let $z \in X$ be a point which ε -shadows ξ on average. This implies that there are $p, q \in \mathbb{N}$ such that $\varrho(f^p(x), f^q(z)) < \varepsilon$ and $r, s, t \in \mathbb{N}$ such that $q < l(2t) \leq r$, $s < n_{2t+1}$, and $\varrho(f^r(z), f^s(y_t)) < \varepsilon$, for otherwise we would have

$$\frac{1}{l(j)} \sum_{i=0}^{l(j)-1} \varrho(f^i(z), \xi_i) \geq \frac{2^{j-1}n_0}{(2^j - 1)n_0} \varepsilon \geq \frac{\varepsilon}{2}$$

for every sufficiently large $j \in \mathbb{N}$ of some fixed parity (odd or even). We conclude that for some p, q, r, s, t with $q < r - 1$ and $s < n_{2t+1}$ the sequence

$$x, f(x), \dots, f^{p-1}(x), f^q(z), \dots, f^{r-1}(z), f^s(y_t), \dots, y$$

is a finite ε -pseudo-orbit from x to y , which completes the proof. ■

LEMMA 3.2. *Let (X, f) be a compact dynamical system. If f is surjective and has the almost specification property, then f is chain mixing.*

Proof. Let $x, y \in X$ and fix $\varepsilon > 0$. Use the almost specification property to obtain the constant $N = k_g(\varepsilon)$.

We will show that for each $n \geq 2N + 2$ there is an ε -chain of length n from x to y . To this end we fix $n \geq 2N + 2$. Using surjectivity of f we find $y_0 \in X$ such that $f^{n-N}(y_0) = y$. By the almost specification property there is a point $z \in X$ such that $z \in B_N(f(x); g, \varepsilon)$ and $f^N(z) \in B_N(y_0; g, \varepsilon)$. Equivalently, there is $0 \leq s \leq N-1$ such that $\varrho(f^{s+1}(x), f^s(z)) < \varepsilon$ and there is $N \leq t \leq 2N$ such that $\varrho(f^t(z), f^{t-N}(y_0)) < \varepsilon$. Therefore the sequence

$$x, f(x), \dots, f^s(x), f^s(z), f^{s+1}(z), \dots, f^{t-1}(z), f^{t-N}(y_0), f^{t-N+1}(y_0), \dots, y$$

is an ε -pseudo-orbit of length n from x to y , which completes the proof. ■

LEMMA 3.3. *Let (X, f) be a compact dynamical system. If f is chain mixing and $\{x_j\}_{j=0}^\infty$ is an asymptotic average pseudo-orbit of f , then there exists an asymptotic pseudo-orbit $\{y_j\}_{j=0}^\infty$ of f such that the set $\{j : x_j \neq y_j\}$ has asymptotic density zero.*

Proof. By chain mixing of (X, f) , for every k there exists an integer M_k such that for any $x, y \in X$ there is a $1/2^k$ -chain of length $M_k + 1$ from x to y . Without loss of generality we may assume that M_{k+1} is a multiple of M_k for every k .

Let $\{x_j\}_{j=0}^\infty$ be an asymptotic average pseudo-orbit. By Lemma 2.1 there exists a set J such that $d(J) = 0$ and

$$(3.1) \quad \lim_{j \notin J} \varrho(f(x_j), x_{j+1}) = 0.$$

Since $d(J) = 0$, there exists a strictly increasing sequence $\{n_k\}_{k=1}^\infty$ such that for every k , M_{k+1} divides n_k and

$$M_k \frac{\text{card}(J \mid n)}{n} < \frac{1}{2^k}$$

for every $n > n_k$. By (3.1) we may assume (increasing n_k if necessary) that if $j \notin J$ and $j \geq n_k$ then $\varrho(f(x_j), x_{j+1}) < 1/2^k$. Now we define a set J' and a sequence $\{y_j\}_{j \in J'}$ in the following way: for all k and s such that $[sM_k, (s+1)M_k) \subset [n_k, n_{k+1})$, if $J \cap [sM_k, (s+1)M_k) \neq \emptyset$ then we include the set $[sM_k, (s+1)M_k) \cap \mathbb{N}$ in J' and we define $\{y_j\}_{j=sM_k}^{(s+1)M_k}$ as a $1/2^k$ -chain from x_{sM_k} to $x_{(s+1)M_k}$ of length $M_k + 1$. Note that $y_{sM_k} = x_{sM_k}$ and $y_{(s+1)M_k} = x_{(s+1)M_k}$.

Let J' and $\{y_j\}_{j \in J'}$ be obtained by the above procedure. For $j \notin J'$ we put $y_j = x_j$. First note that for $j \geq n_k$ we have $\varrho(f(y_j), y_{j+1}) < 1/2^k$ and hence $\{y_j\}_{j=0}^\infty$ is an asymptotic pseudo-orbit. Furthermore, if we fix any

$n > n_1$ then there is $k > 0$ such that $n_k < n \leq n_{k+1}$ and then

$$\begin{aligned} \frac{1}{n} \text{card}(J' \mid n) &= \frac{1}{n} \text{card}(J' \mid n_1) + \frac{1}{n} \sum_{s=1}^k \text{card}(J' \cap [n_s, n_{s+1}) \mid n) \\ &\leq \frac{1}{n} \text{card}(J' \mid n_1) + \frac{1}{n} \sum_{s=1}^k M_s \text{card}(J \cap [n_s, n_{s+1}) \mid n) \\ &\leq \frac{1}{n} \text{card}(J' \mid n_1) + M_k \frac{\text{card}(J \mid n)}{n} \\ &< \frac{1}{n} \text{card}(J' \mid n_1) + \frac{1}{2^k}. \end{aligned}$$

This shows that $d(J') = 0$. The proof is completed by noting that $\{j : x_j \neq y_j\} \subset J'$. ■

3.2. Almost specification implies asymptotic average shadowing.

Similarities between almost specification and asymptotic average shadowing are rather vague and it is not obvious whether either one implies the other. In this section we show that almost specification implies asymptotic average shadowing for surjective maps.

LEMMA 3.4. *Let X be compact, let $f: X \rightarrow X$ be a map with the almost specification property, and let g be its mistake function. Assume we are given*

- a sequence $\{\varepsilon_i\}_{i=1}^\infty$ of positive real numbers,
- a sequence $\{x_i\}_{i=1}^\infty \subset X$,
- a sequence $\{n_i\}_{i=0}^\infty$ of integers with $n_0 = 0$ and $n_i \geq k_g(\varepsilon_i)$.

Then, setting

$$l_i = \sum_{s=0}^{i-1} n_s \quad \text{for } i \in \mathbb{Z}_+,$$

we can find a point $z \in X$ such that for every $j \in \mathbb{Z}_+$ we have

$$f^{l_j}(z) \in \overline{B_{n_j}(g; x_j, \varepsilon_j)}.$$

Proof. Since each number $k_g(\varepsilon)$ in the definition of the almost specification property depends only on ε , but not on m , for each $m \in \mathbb{Z}_+$ we can find a point z_m such that for every $j = 1, \dots, m$ we have

$$f^{l_j}(z_m) \in B_{n_j}(g; x_j, \varepsilon_j).$$

We will use a diagonalization procedure to obtain a “good” subsequence of $\{z_i\}_{i=1}^\infty$, which, by an abuse of notation, we will also denote by $\{z_i\}_{i=1}^\infty$. Since each $I(g; n_j, \varepsilon_j)$ is finite, passing to a subsequence of $\{z_i\}_{i=1}^\infty$ we can assume that there is $\Lambda_1 \in I(g; n_1, \varepsilon_1)$ such that $f^{l_1}(z_j) \in B_{\Lambda_1}(x_1, \varepsilon_1)$ for every $j \geq 1$. Similarly, passing again to a subsequence of $\{z_i\}_{i=2}^\infty$ if necessary (i.e. preserving Λ_1 and z_1), we can find $\Lambda_2 \in I(g; n_2, \varepsilon_2)$ such that $f^{l_1}(z_j) \in$

$B_{\Lambda_1}(x_1, \varepsilon_1)$ for every $j \geq 1$ and $f^{l_2}(z_j) \in B_{\Lambda_2}(x_2, \varepsilon_2)$ for every $j \geq 2$. Continuing, for each $i \in \mathbb{Z}_+$ we can construct sets $\Lambda_i \in I(g; n_i, \varepsilon_i)$ such that $f^{l_i}(z_j) \in B_{\Lambda_i}(x_i, \varepsilon_i)$ for every $j \geq i$.

Passing once more to a subsequence we may assume that $z = \lim_{m \rightarrow \infty} z_m$ exists. But then for all $i \in \mathbb{Z}_+, j \geq i$, and $k \in \Lambda_i$ we have $\varrho(f^k(x_i), f^k(f^{l_i}(z_j))) < \varepsilon$ and hence $\varrho(f^k(x_i), f^k(f^{l_i}(z))) \leq \varepsilon$. In other words, for $j \in \mathbb{Z}_+$ we have

$$f^{l_j}(z) \in \overline{B_{\Lambda_j}(x_j, \varepsilon_j)} \subset \overline{B_{n_j}(g; x_j, \varepsilon_j)}. \blacksquare$$

Roughly speaking, the above lemma says that given an infinite sequence of triples consisting of a point, a large enough length and a precision, we can find a point (taken from the orbit of some point z) tracing the orbit of each given point for a given length with the given precision (with possible mistakes, but their number is bounded above by the mistake function g). That point is $f^{l_j}(z)$, and it is tracing the triple $(x_j, m_j, \varepsilon_j)$.

THEOREM 3.5. *If a surjective compact dynamical system (X, f) has the almost specification property, then it has the asymptotic average shadowing property.*

Proof. Let $\{x_j\}_{j=0}^\infty$ be an asymptotic average pseudo-orbit of f . By Lemma 3.2, f is chain mixing, and so we may use Lemma 3.3 to obtain an asymptotic pseudo-orbit of f , denoted $\{y_j\}_{j=0}^\infty$, such that $d(\{j : x_j \neq y_j\}) = 0$. It is enough to show that $\{y_j\}_{j=0}^\infty$ can be asymptotically shadowed on average by some point $z \in X$.

Let g be a mistake function for f . For each $k \geq 1$ we take an integer n_k such that $n_k > k_g(1/2^k)$ and $n_k > 2^k g(n_k, 1/2^k)$. We may assume $n_k < n_{k+1}$. Using compactness of X and continuity of f , for each k we can also find a constant $\delta_k > 0$ such that every δ_k -chain of length n_k is $1/2^k$ -shadowed by its first point. Since $\{y_j\}_{j=0}^\infty$ is an asymptotic pseudo-orbit we can find a sequence $\{m_k\}_{k=1}^\infty$ such that for each $k \in \mathbb{Z}_+$ the sequence $\{y_j\}_{j=m_k}^\infty$ is a δ_k -pseudo-orbit. Clearly we can choose $\{m_k\}_{k=1}^\infty$ so that for each $k \in \mathbb{Z}_+$ we will have in addition:

- $m_{k+1} > 4^k m_k$,
- n_k divides $m_{k+1} - m_k$, $m_{k+1} - m_k = s_k n_k$ for some $s_k > 0$.
- $4^k n_{k+1} < m_{k+1}$.

We call any point $y_{m_k + s n_k}$, where $k \in \mathbb{Z}_+$ and $0 \leq s < s_k$, an *initial point of order k* . Note that by our choice of parameters, for every $k \in \mathbb{Z}_+$ and every $0 \leq s < s_k$ the sequence $\{y_j\}_{j=m_k + s n_k}^{m_k + (s+1)n_k - 1}$ is $1/2^k$ -shadowed by its first (initial) point $y_{m_k + s n_k}$. Set $\varepsilon_0 = 1$, $k_0 = m_1$ and let $\{\varepsilon_j\}_{j=1}^\infty$ ($\{k_j\}_{j=1}^\infty$, respectively) be a nonincreasing (nondecreasing, respectively) sequence in which $1/2^k$ (n_k , respectively) repeats s_k times. We may plug this data (the sequence of all initial points and the sequences $\varepsilon = \{\varepsilon_j\}_{j=0}^\infty$, $k = \{k_j\}_{j=0}^\infty$) into Lemma 3.4 to obtain a point $z \in X$ such that for all $k \in \mathbb{Z}_+$ and $0 \leq s < s_k$ we

$$\begin{aligned}
 \varepsilon &= (1, \underbrace{1/2, \dots, 1/2}_{s_1 \text{ times}}, \underbrace{1/2^2, \dots, 1/2^2}_{s_2 \text{ times}}, \dots, \underbrace{1/2^k, \dots, 1/2^k}_{s_k \text{ times}}, \dots), \\
 x &= (y_0, \underbrace{y_{m_1}, y_{m_1+n_1}, y_{m_1+2n_1}, \dots, y_{m_2-n_1}, \dots}_{s_1 \text{ initial points of order 1}}, \\
 &\quad \underbrace{y_{m_k}, y_{m_k+n_k}, y_{m_k+2n_k}, \dots, y_{m_k-n_k}, \dots}_{s_k \text{ initial points of order } k}), \\
 k &= (m_1, \underbrace{n_1, \dots, n_1}_{s_1 \text{ times}}, \underbrace{n_2, \dots, n_2}_{s_2 \text{ times}}, \dots, \underbrace{n_k, \dots, n_k}_{s_k \text{ times}}, \dots)
 \end{aligned}$$

Fig. 1. Sequences ε, x, k to which Lemma 3.4 is applied in the proof of Theorem 3.5

have $f^{m_k+sn_k}(z) \in \overline{B_{n_k}(g; y_{m_k+sn_k}, 1/2^k)}$. The corresponding set of mistakes is defined as $\{m_k + sn_k \leq j < m_k + (s + 1)n_k : \varrho(f^j(z), f^{j-m_k-sn_k}) \geq 1/2^k\}$.

Let $J \subset \mathbb{N}$ be the complement of the union of all sets of mistakes. Note that if $j \in J \cap [m_k, m_{k+1})$ then $\varrho(f^j(z), y_j) < 1/2^k + 1/2^k$, because y_j is $1/2^k$ -traced by the orbit of the closest preceding initial point, and $j \in J$ means that the orbit of z traces the orbit of that preceding initial point without mistake at j . Therefore we have $\lim_{j \in J} d(f^j(z), y_j) = 0$. But by the choice of z we also have

$$\begin{aligned}
 \frac{1}{m_{k+1}} \#(J \cap [m_k, m_{k+1})) &\geq \frac{1}{m_{k+1}} \cdot \frac{m_{k+1} - m_k}{n_k} (n_k - g(n_k, 1/2^k)) \\
 &= (1 - m_k/m_{k+1})(1 - g(n_k, 1/2^k)/n_k) \\
 &\geq (1 - 1/4^k)(1 - 1/2^k).
 \end{aligned}$$

Next, if we fix $n \in [m_{k+1}, m_{k+2})$, where $n = m_{k+1} + sn_{k+1} + r, 0 \leq r < n_{k+1}$, then we also have

$$\begin{aligned}
 \frac{1}{n} \text{card}(J \mid n) &\geq \frac{1}{m_{k+1}} \cdot \frac{m_{k+1}}{n} \#(J \cap [m_k, m_{k+1})) + \frac{1}{n} \#(J \cap [m_{k+1}, n)) \\
 &\geq (1 - 1/4^k)(1 - 1/2^k) \frac{m_{k+1}}{n} + \frac{s(n_{k+1} - g(n_{k+1}, 1/2^{k+1}))}{n} \\
 &\geq (1 - 1/4^k)(1 - 1/2^k) \frac{m_{k+1}}{n} + \frac{(1 - 1/2^{k+1})sn_{k+1}}{n} \\
 &\geq (1 - 1/4^k)(1 - 1/2^k) \frac{m_{k+1}}{n} + \frac{(1 - 1/4^k)(1 - 1/2^k)sn_{k+1}}{n} \\
 &\geq (1 - 1/4^k)(1 - 1/2^k)(1 - n_{k+1}/m_{k+1}) \\
 &\geq (1 - 1/4^k)(1 - 1/2^k)(1 - 1/4^k) \xrightarrow[k \rightarrow \infty]{} 1.
 \end{aligned}$$

We conclude that $d(J) = 1$, which by Lemma 2.1 ends the proof. ■

3.3. Asymptotic average shadowing implies average shadowing.

In this section we show that asymptotic average shadowing implies average shadowing. The next theorem shows that, under the assumption of chain mixing, the average shadowing property is all about average shadowing of pseudo-orbits.

THEOREM 3.6. *Assume that a compact dynamical system (X, f) is chain mixing. Then the following conditions are equivalent:*

- (1) *f has the average shadowing property,*
- (2) *for every $\varepsilon > 0$ there is $\delta > 0$ such that every δ -pseudo-orbit $\{x_i\}_{i=0}^\infty$ is ε -shadowed on average by some point $x \in X$.*

Proof. Since every δ -pseudo-orbit is also a δ -average-pseudo-orbit we only need to prove (2) \Rightarrow (1).

Fix any $\varepsilon > 0$ and let γ be such that every γ -pseudo-orbit is $\varepsilon/5$ -shadowed on average by some point in X . By chain mixing and compactness of X there is M such that for every $x, y \in X$ there is a γ -pseudo-orbit of length M from x to y .

Put $\delta = \varepsilon\gamma/(3M \text{diam}(X))$, fix any δ -average-pseudo-orbit $\{x_i\}_{i=0}^\infty$, and let N be the corresponding constant from the definition of a δ -average-pseudo-orbit. Without loss of generality we may assume that $M < N$.

We will now construct a sequence $\{y_i\}_{i=0}^\infty$ which is a genuine pseudo-orbit and shadows $\{x_i\}_{i=0}^\infty$ on average. Starting with $j = 0$ and going up through all the natural numbers, we initially put (some of these blocks will be modified later)

$$y_{jN}, y_{jN+1}, \dots, y_{jN+N} = x_{jN}, x_{jN+1}, \dots, x_{jN+N}.$$

Next, if there is $jN \leq i < jN + N$ such that $\varrho(f(y_i), y_{i+1}) \geq \gamma$, we choose k so that $jN \leq k \leq i < k + M - 1 \leq jN + N$ and replace $y_k, y_{k+1}, \dots, y_{k+M-1}$ with any γ -pseudo-orbit of length M from y_k to y_{k+M-1} . We repeat these replacements until $y_{jN}, y_{jN+1}, \dots, y_{jN+N}$ becomes a γ -pseudo-orbit. Note that no replacement changes y_{jN} or y_{jN+N} and therefore the resulting infinite sequence $\{y_i\}_{i=0}^\infty$ is a γ -pseudo-orbit.

Observe that for all $j \in \mathbb{N}$ we have

$$M\#\{i \in [jN, jN + N) : \varrho(f(x_i), x_{i+1}) \geq \gamma\} \geq \#\{i \in [jN, jN + N) : x_i \neq y_i\},$$

and therefore

$$\begin{aligned} \delta M &> \frac{M}{N} \sum_{i=0}^{N-1} \varrho(f(x_{jN+i}), x_{jN+i+1}) \\ &\geq \frac{M}{N} \gamma \#\{i \in [jN, jN + N) : \varrho(f(x_i), x_{i+1}) \geq \gamma\}. \end{aligned}$$

Hence

$$\frac{\text{diam}(X)}{N} \#\{i \in [jN, jN + N) : x_i \neq y_i\} \leq \frac{\delta M \text{diam}(X)}{\gamma}.$$

But by the definition of δ we have

$$\frac{\delta M \text{diam}(X)}{\gamma} = \frac{M \text{diam}(X)}{\gamma} \cdot \frac{\varepsilon \gamma}{3M \text{diam}(X)} \leq \frac{\varepsilon}{3}.$$

We are now ready for the final calculation. Let z be a point that $\varepsilon/5$ -shadows on average the γ -pseudo-orbit $\{y_i\}_{i=0}^\infty$. There is $K \in \mathbb{N}$ such that for every $n > K$ we have

$$\frac{1}{n} \sum_{i=0}^n d(f^i(z), y_i) < \varepsilon/4.$$

We may also assume that $N/K < \varepsilon/4$. Fix any $n > K$ and let $s, l \geq 0$ be such that $n = sN + l$, where $l < N$. Let $A = \{i < n : x_i = y_i\}$ and $B = \{i < n : x_i \neq y_i\}$. Then

$$\begin{aligned} \frac{1}{n} \sum_{i=0}^n d(f^i(z), x_i) &= \frac{1}{n} \sum_{i \in A} d(f^i(z), y_i) + \frac{1}{n} \sum_{i \in B} d(f^i(z), x_i) \\ &\leq \frac{1}{n} \sum_{i=0}^n d(f^i(z), y_i) + \frac{l}{n} + \frac{1}{n} \sum_{j=0}^{s-1} \sum_{i \in B \cap [Nj, Nj+N)} \text{diam}(X) \\ &\leq \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{N}{n} \sum_{j=0}^{s-1} \frac{\text{diam } X}{N} \#\{B \cap [Nj, Nj + N)\} \\ &\leq \frac{\varepsilon}{2} + \frac{sN}{n} \frac{\varepsilon}{3} \leq \frac{5\varepsilon}{6}. \end{aligned}$$

This implies that $\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^n d(f^i(z), x_i) \leq 5\varepsilon/6 < \varepsilon$, and the proof is complete. ■

THEOREM 3.7. *If $f : X \rightarrow X$ is a surjection and the compact dynamical system (X, f) has the asymptotic average shadowing property, then it also has the average shadowing property.*

Proof. By [16, Theorem 3.1] every surjective dynamical system with the asymptotic average shadowing property is chain mixing. Assume that (X, f) does not have the average shadowing property. Then by Theorem 3.6, there is an $\varepsilon > 0$ such that for every $n \in \mathbb{Z}_+$ there is a $1/n$ -pseudo-orbit $\{\alpha_j^{(n)}\}_{j=0}^\infty$ which is not ε -shadowed on average by any point in X . Then by compactness of X for every $n \in \mathbb{Z}_+$ there exist $k(n) \in \mathbb{Z}_+$ and an initial block $\gamma^{(n)} = \{\gamma_i^{(n)}\}_{i=0}^{k(n)-1} = \{\alpha_0^{(n)}, \alpha_1^{(n)}, \dots, \alpha_{k(n)-1}^{(n)}\}$ of $\alpha^{(n)}$ such that for every $z \in X$ we have $\frac{1}{k(n)} \sum_{i=0}^{k(n)-1} \varrho(f^i(z), \alpha_i^{(n)}) \geq \varepsilon$. Note that $\lim_{n \rightarrow \infty} k(n) = \infty$, as

otherwise by the continuity of f for a sufficiently large n the pseudo-orbit $\gamma^{(n)}$ would be ε -shadowed by its initial point $\alpha_0^{(n)}$.

By compactness of X and chain mixing of f , for every $n \in \mathbb{Z}_+$ there is $m(n) \in \mathbb{N}$ such that any two points can be connected by a $1/n$ -pseudo-orbit of length $m(n)$. For $a < b$ any $1/b$ -pseudo-orbit is also a $1/a$ -pseudo-orbit and the sequence $m(n)$ is already fixed, so, without loss of generality, passing to a subsequence of $\{k(n)\}_{n=1}^\infty$ if necessary, we may assume that for every $n \in \mathbb{Z}_+$ we have $k(n+1) > 2 \sum_{i=1}^n (k(i) + m(i))$. For every $n \in \mathbb{Z}_+$ denote by $\eta^{(n)}$ a $1/n$ -pseudo-orbit of length $m(n) - 2$ such that $\gamma_{k(n)-1}^{(n)}, \eta^{(n)}, \gamma_1^{(n+1)}$ is also a $1/n$ -pseudo-orbit (such an $\eta^{(n)}$ exists by the choice of $m(n)$).

Denote by ζ the following concatenation of pseudo-orbits:

$$\zeta = \{\zeta_i\}_{i=0}^\infty = \gamma^{(1)}\eta^{(1)}\gamma^{(2)}\eta^{(2)} \dots \gamma^{(n)}\eta^{(n)} \dots$$

and observe that ζ is an asymptotic pseudo-orbit. Let $z \in X$ be a point provided by the asymptotic average shadowing property that asymptotically shadows ζ on average. Denote by $r(n)$ the number of elements in the pseudo-orbit $\gamma^{(1)}\eta^{(1)} \dots \eta^{(n-1)}\gamma^{(n)}$. There exists $N \in \mathbb{Z}_+$ such that

$$\frac{1}{r(N)} \sum_{i=0}^{r(N)-1} \varrho(f^i(z), \zeta_i) < \varepsilon/2.$$

Note that

$$r(N) \leq k(N) + \sum_{i=1}^{N-1} (k(i) + m(i)) \leq (1 + 1/2)k(N) \leq 2k(N),$$

and so

$$\begin{aligned} \frac{1}{k(N)} \sum_{i=0}^{k(N)-1} \varrho(f^{r(N)-k(N)+i}(z), \gamma_i^{(N)}) &= \frac{1}{k(N)} \sum_{i=r(N)-k(N)}^{r(N)-1} \varrho(f^i(z), \zeta_i) \\ &\leq 2 \frac{1}{r(N)} \sum_{i=r(N)-k(N)}^{r(N)-1} \varrho(f^i(z), \zeta_i) < \varepsilon, \end{aligned}$$

which contradicts the choice of $\gamma^{(N)}$, and completes the proof. ■

3.4. Under shadowing. Combining our new results with previous research we obtain the following:

THEOREM 3.8. *Let X be a compact metric space. If $f: X \rightarrow X$ is a continuous map with the shadowing property, then the following conditions are equivalent:*

- (1) f is totally transitive,
- (2) f is topologically weakly mixing,

- (3) f is topologically mixing,
- (4) f is surjective and has the specification property,
- (5) f is surjective and has the almost specification property,
- (6) f is surjective and has the asymptotic average shadowing property.
- (7) f is surjective and has the average shadowing property,

Moreover, if f is c -expansive, then any of the above conditions is equivalent to the periodic specification property of f .

Proof. The implications (5) \Rightarrow (6) \Rightarrow (7) were proved in Theorems 3.5 and 3.7, respectively. To see (7) \Rightarrow (1), first observe that f is chain transitive by [26, Theorem 3.4]. It is also known that if f has the average shadowing property then so does f^n for every $n \geq 1$ (see e.g. [24, Lemma 3.3]). In particular, by shadowing, f is totally transitive.

Equivalence of conditions (1)–(4) as well as the “moreover” part are [18, Theorem 1]. The proof that specification implies almost specification can be found in [28, Proposition 2.1] (see also comments in [34]). ■

4. Weak mixing. In general, no recurrence property is implied by almost specification or average shadowing. We show this in Section 8 below. Nevertheless, under some additional assumptions one can get a positive result. For example, Niu [24, Corollary 3.8] proved that a system with the average shadowing property and a dense set of minimal points must be weakly mixing. Here we prove that a compact system with almost specification and an invariant measure with full support has a dense set of minimal points. We do not know whether the same holds if we assume only (asymptotic) average shadowing (Question 10.7). However, we can generalize Niu’s result and show that his conclusion holds under the (a priori weaker) assumption of the existence of an invariant measure with full support. But if the answer to Question 10.7 or Question 10.3 is positive, then Theorem 4.3 is merely a reproof of Niu’s result. Note that there are examples of nontrivial proximal systems with an invariant measure of full support. For example one may consider the restriction of a shift constructed in [17, Theorem 5.6] to the support of any of its invariant measures of positive entropy (there is at least one such measure).

THEOREM 4.1. *Assume that a compact dynamical system (X, f) has an invariant measure with full support. If f has the almost specification property, then it has a dense set of minimal points.*

Proof. Fix any nonempty open set U . We can find $\varepsilon > 0$ and nonempty open sets $W \subset V \subset U$ such that the ε -neighborhood of \overline{W} is contained in V and $\overline{V} \subset U$. Since U is not universally null we conclude from Theorem 2.3 and Lemma 2.2 that we can find $\gamma > 0$ and a point $x \in W$

such that $d(N(x, W)) = \gamma$. Let $N \in \mathbb{N}$ be such that for every $n \geq N$ we have $\text{card}(N(x, W) \mid n) \geq n\gamma/2$. Using the almost specification property we can find an $M > 0$ such that for all $m \geq M$ we have $g(\varepsilon, m) < m\gamma/2$. Let $n = \max\{N, M, k_g(\varepsilon)\}$. Let $\{x_j\}_{j=0}^\infty, \{\varepsilon_j\}_{j=0}^\infty, \{n_j\}_{j=0}^\infty$ be constant sequences, where $x_j = x, \varepsilon_j = \varepsilon, n_j = n$ for every $j \in \mathbb{N}$. By Lemma 3.4 there is a point y such that $f^{jn}(y) \in B_n(g; x, \varepsilon)$ for every $j \in \mathbb{N}$.

We claim that $N(y, V)$ is syndetic (has gaps bounded by $2n$). Assume conversely that $f^l(y) \notin V$ for $2n$ consecutive indices l . In particular, for some $j \geq 0$ and every $0 \leq i < n$ such that $f^i(x) \in W$ we necessarily have $\varrho(f^{jn+i}(z), f^i(x)) \geq \varepsilon$. But this leads to a contradiction: $n\gamma/2 \leq g(\varepsilon, n) < n\gamma/2$. Therefore $N(y, V)$ is syndetic with gaps bounded by $2n$ as claimed.

By the Auslander–Ellis Theorem [11, Theorem 8.7] there is a minimal point z proximal to y . It is easy to see that $f^l(z) \in U$ for some $l \geq 0$. Since $f^l(z)$ is also a minimal point, the proof is finished. ■

LEMMA 4.2. *Let $p, q \in X$. Let $A, B \subset \mathbb{N}$ be such that $d^*(A) > 0$ and $d^*(B) > 0$. If X is compact and $f: X \rightarrow X$ has the average shadowing property, then for every $\varepsilon > 0$ there exist a point $z \in X$ and integers $a_0 < b_0 < a_1 < b_1 < \dots$ such that for every $i \in \mathbb{N}$ we have $a_i \in A, b_i \in B, \varrho(f^{a_i}(z), f^{a_i}(p)) < \varepsilon$, and $\varrho(f^{b_i}(z), f^{b_i}(q)) < \varepsilon$.*

Proof. We may assume without loss of generality that $\text{diam}(X) = 1$. Let $0 < \gamma < \min(d^*(A), d^*(B))$. Fix any $\varepsilon > 0$ and let $\alpha > 0$ correspond to $\gamma\varepsilon$ by the average shadowing property. Using the definition of the upper asymptotic density we can easily construct an increasing sequence of integers $0 = n_0 < n_1 < \dots$ such that $2 < \alpha n_1$ and for any $i \in \mathbb{Z}_+$ we have:

- $2n_i < n_{i+1}$,
- $\frac{1}{n_{2i+1}} \text{card}(A \cap [n_{2i}, n_{2i+1}] \mid n_{2i+1}) > \gamma$,
- $\frac{1}{n_{2i}} \text{card}(B \cap [n_{2i-1}, n_{2i}] \mid n_{2i}) > \gamma$.

Next we define a sequence of points $y_j \in X$ by

$$y_j = \begin{cases} f^j(p) & \text{if } j \in [n_{2i}, n_{2i+1}) \text{ for some } i, \\ f^j(q) & \text{otherwise.} \end{cases}$$

We claim that $\{y_j\}_{j=0}^\infty$ is an α -average-pseudo-orbit of f . To prove this observe that $\varrho(f(y_{j-1}), y_j)$ can be positive only if $j \in \{n_i\}_{i=0}^\infty$. But if we put $N = n_1$ then, by the definition of $\{n_j\}_{i=0}^\infty$, for every $j \in \mathbb{N}$ the interval $[j, j + N]$ can contain at most one element of this sequence. If we now fix any $n \geq N$ we may find $s > 0$ such that $Ns \leq n < (s + 1)N$. Then for every $k \geq 0$ we have

$$\frac{1}{n} \sum_{i=0}^{n-1} \varrho(f(y_{i+k}), y_{i+k+1}) < \frac{s + 1}{sN} \leq \frac{2}{n_1} < \alpha,$$

proving that $\{y_j\}_{j=0}^\infty$ is an α -average-pseudo-orbit. It is, therefore, $\gamma\varepsilon$ -shadowed on average by some $z \in X$.

If the choice of sequences $\{a_i\}_{i=0}^\infty$ and $\{b_i\}_{i=0}^\infty$ is not possible, then, for all i large enough, at least one of the following two conditions must be satisfied:

- for all $j \in A \cap [n_{2i}, n_{2i+1})$ we have $\varrho(f^j(z), f^j(p)) \geq \varepsilon$,
- for all $j \in B \cap [n_{2i-1}, n_{2i})$ we have $\varrho(f^j(z), f^j(q)) \geq \varepsilon$.

We may assume without loss of generality that the first condition holds. Then for all i large enough we have

$$\frac{1}{n_{2i+1}} \sum_{s=0}^{n_{2i+1}-1} \varrho(f^s(z), f^s(p)) \geq \varepsilon \frac{1}{n_{2i+1}} \text{card}(A \cap [n_{2i}, n_{2i+1}) \mid n_{2i+1}) > \gamma\varepsilon,$$

contradicting that z $\gamma\varepsilon$ -shadows on average the α -average-pseudo-orbit $\{y_j\}_{j=0}^\infty$. ■

The following theorem generalizes [24, Theorem 3.7].

THEOREM 4.3. *Assume that a compact dynamical system (X, f) has an invariant measure with full support. If f has the average shadowing property, then it is weakly mixing.*

Proof. Fix any nonempty open sets U, V . We can find $\varepsilon > 0$ and nonempty open sets U', V' such that the ε -neighborhood of $\overline{U'}$ is contained in U and the ε -neighborhood of $\overline{V'}$ is contained in V . Since neither U' nor V' is universally null, we conclude from Theorem 2.3 that we can find $\gamma > 0$, points $p \in U', q \in V'$, and sets $A, B \subset \mathbb{N}$ such that $d^*(A) > \gamma, d^*(B) > \gamma$, and $f^i(p) \in U', f^j(q) \in V'$ for every $i \in A, j \in B$. Then by Lemma 4.2 there are $z \in X$ and $i < j$ such that $d(f^i(z), f^i(p)) < \varepsilon$ and $d(f^j(z), f^j(q)) < \varepsilon$. In particular, $f^i(z) \in U$ and $f^j(z) \in V$. This shows that f is transitive.

It was shown in [24, Proposition 3.5] that if f has the average shadowing property, then so does $f \times f$. Obviously $\mu \times \mu$ is a fully supported invariant measure for $f \times f$, so by the above arguments $f \times f$ is transitive, which ends the proof. ■

5. Measure center. In this section we prove that the generalized shadowing and specification are connected with the behavior of the dynamical system on its measure center.

5.1. Almost specification. In what follows, we will use the notation introduced in Section 2, in particular the definition of the function ξ .

THEOREM 5.1. *If a closed invariant set A contains the measure center of a compact dynamical system (X, f) and $f|_A$ has the almost specification property on A , then so does f on X .*

Proof. Let $g_A: \mathbb{Z}_+ \times (0, \varepsilon_0] \rightarrow \mathbb{N}$ be a mistake function for $f|_A$ and let k_{g_A} be provided by the almost specification property. We claim that it is sufficient to prove that there is a mistake function $\gamma: \mathbb{Z}_+ \times (0, \varepsilon_0] \rightarrow \mathbb{N}$ satisfying

- (\star) for every $\varepsilon \in (0, \varepsilon_0]$ there is $N = N_\gamma(\varepsilon)$ such that for every $x \in X$ and every $n \geq N$ there is a point $z \in A \cap B_n(\gamma; x, \varepsilon)$.

First we claim that if there is γ satisfying (\star), then f has the almost specification property on X with the mistake function

$$g(n, \varepsilon) = g_A(n, \varepsilon/2) + \gamma(n, \varepsilon/2), \quad \text{where } (n, \varepsilon) \in \mathbb{Z}_+ \times (0, \varepsilon_0].$$

and with $k_g(\varepsilon) = \max\{k_{g_A}(\varepsilon), N_\gamma(\varepsilon)\}$. To prove the claim, fix any $m \geq 1$, any $\varepsilon_1, \dots, \varepsilon_m > 0$, any points $x_1, \dots, x_m \in X$, and any integers $n_1 \geq k_g(\varepsilon_1), \dots, n_m \geq k_g(\varepsilon_m)$. Put $n_0 = 0$ and denote

$$l_j = \sum_{s=0}^{j-1} n_s \quad \text{for } j = 1, \dots, m.$$

By the choice of γ , for each $j = 1, \dots, m$ there is a set $\Gamma_j \in I(\gamma; n_j, \varepsilon_j/2)$ and a point $z_j \in A \cap B_{\Gamma_j}(x_j, \varepsilon_j/2)$. Let z be a point obtained by the almost specification property for the aforementioned points z_1, \dots, z_m and constants n_j and $\varepsilon_j/2$. Strictly speaking, $z \in X$ is such that for every $j = 1, \dots, m$ we have

$$f^{l_j}(z) \in B_{n_j}(g_A; z_j, \varepsilon_j/2).$$

In particular, for every j there is a set $\Lambda_j \in I(g_A; n_j, \varepsilon_j/2)$ such that $f^{l_j}(z) \in B_{\Lambda_j}(g_A; y_j, \varepsilon_j/2)$. Put $K_j = \Gamma_j \cap \Lambda_j$ and observe that

$$n_j - \#K_j \leq (n_j - \#\Gamma_j) + (n_j - \#\Lambda_j) \leq \gamma(n_j, \varepsilon_j/2) + g_A(n_j, \varepsilon_j/2) = g(n_j, \varepsilon_j),$$

so $K_j \in I(g; n_j, \varepsilon_j)$. Additionally, if $i \in K_j$, then $\varrho(f^{l_j+i}(z), f^i(z_j)) < \varepsilon_j/2$ and $\varrho(f^i(x_j), f^i(z_j)) < \varepsilon_j/2$, hence $\varrho(f^{l_j+i}(z), f^i(x_j)) < \varepsilon_j$. This proves that

$$f^{l_j}(z) \in B_{K_j}(x_j, \varepsilon_j) \subset B_{n_j}(g; x_j, \varepsilon_j),$$

and therefore it only remains to prove that γ can be constructed in such a way that (\star) is satisfied.

Take any $\varepsilon \in (0, \varepsilon_0]$ and $s \geq k_{g_A}(\varepsilon/2)$. Fix a constant $\delta > 0$ such that for every $x, y \in X$ with $\varrho(x, y) < \delta$ we have $\varrho(f^j(x), f^j(y)) < \varepsilon/2$ for $j = 0, \dots, s - 1$. Denote $P(n) = \sup_{x \in X} \#\{0 \leq i < n : \varrho(f^i(x), A) \geq \delta\}$ and observe that, by Corollary 2.4, we have $\lim_{n \rightarrow \infty} P(n)/n = 0$. Pick $N_\gamma^s(\varepsilon)$ large enough for the inequality

$$s(P(n) + 1) \leq \frac{n \cdot g_A(\varepsilon/2, s)}{s}$$

to hold for every $n \geq N_\gamma^s(\varepsilon)$. Let $n \geq N_\gamma^s(\varepsilon)$ be given. Assume that $n = ps+r$, where $p, r \in \mathbb{N}$ and $r < s$. For $j = 1, \dots, p$ define $n_j = s$, $\varepsilon_j = \delta$, and let $z_j \in A$ be such that $\varrho(z_j, f^{(j-1)s}(x))$ realizes the distance between the closed set A and the point $f^{(j-1)s}(x)$. Let $z \in A$ be obtained by the almost specification property of $f|_A$ for the points z_1, \dots, z_p and constants n_j and ε_j . Observe that

$$\begin{aligned} \#\{0 \leq i < n : \varrho(f^i(x), f^i(z)) \geq \varepsilon\} &\leq p \cdot g_A(\varepsilon/2, s) + r + sP(n) \\ &\leq \frac{n \cdot g_A(\varepsilon/2, s)}{s} + s(P(n) + 1) \\ &\leq \frac{2n \cdot g_A(\varepsilon/2, s)}{s}. \end{aligned}$$

To sum up, if $s \geq k_{g_A}(\varepsilon/2)$ and $n \geq N_\gamma^s(\varepsilon)$, then the orbit of length n of every point $x \in X$ can be ε -shadowed by the orbit of length n of some point $z \in A$ with at most $2n \cdot g_A(\varepsilon/2, s)/s$ errors.

Fix a sequence $\{s_i\}_{i=1}^\infty$ such that $k_{g_A}(\varepsilon/2) = s_1 < s_2 < \dots$ and put $N_\gamma(\varepsilon) = k_{g_A}(\varepsilon/2)$. Finally observe that the function

$$\alpha(n, \varepsilon) = \min_{N_\gamma^s(\varepsilon) \leq n} \frac{2n \cdot g_A(\varepsilon/2, s_i)}{s_i}$$

satisfies (\star) except that it may not be increasing with respect to n . Therefore $\gamma(n, \varepsilon) = \max_{N_\gamma(\varepsilon) \leq k \leq n} \alpha(k, \varepsilon)$ is a mistake function satisfying (\star) . ■

5.2. Average and asymptotic average shadowing properties. In [16] the authors considered properties that are sufficient to extend the asymptotic average shadowing property from a closed invariant set A to the whole space (Theorem 5.1 is, in fact, motivated by these studies). Recall that by Corollary 2.4 if a set A contains the measure center of f then for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for every $x \in X$ and $n \geq N$ we have

$$\frac{1}{n} \#\{0 \leq i < n : \varrho(f^i(x), A) < \varepsilon\} > 1 - \varepsilon.$$

Combining this observation with results of [16] we obtain the following.

THEOREM 5.2. *If a closed invariant set $A \subset X$ contains the measure center of a compact dynamical system (X, f) and $f|_A$ has the asymptotic average shadowing property on A , then so does f on X .*

Proof. This follows from Corollary 2.4 and [16, Theorem 3.3]. ■

To complete our considerations we will also prove a similar theorem for the average shadowing property. In the proof we will make use of Corollary 2.4 together with [16, Lemma 3.5], which combine to yield the following lemma.

LEMMA 5.3. *If $f: X \rightarrow X$ is a continuous map of a compact metric space and $A \subset X$ is a closed invariant set containing the measure center of f , then for every $\eta > 0$ there exist $K \in \mathbb{N}$ and $\beta > 0$ such that for every $k \geq K$ and for every $\{x_i\}_{i=0}^{k-1}$ which is a β -pseudo-orbit of f we have*

$$(5.1) \quad \frac{1}{k} \#\{0 \leq i < k : \varrho(x_i, A) < \eta\} > 1 - \eta.$$

This lemma can be applied to obtain the following property.

LEMMA 5.4. *Let $f: X \rightarrow X$ be a continuous map of a compact metric space and let $A \subset X$ be a closed invariant set containing the measure center of f . Then for every $\varepsilon > 0$ there is $0 < \delta < \varepsilon$ such that for every δ -average-pseudo-orbit $\{x_n\}_{n=0}^\infty$ there is an ε -average-pseudo-orbit $\{y_n\}_{n=0}^\infty \subset A$ such that*

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \#\{0 \leq i < n : \varrho(x_i, y_i) \geq \varepsilon\} < \varepsilon.$$

Proof. Fix any $\varepsilon > 0$ and choose $N \in \mathbb{Z}_+$ with $3 \operatorname{diam}(X)/N < \varepsilon$. Use continuity of f to obtain $\eta > 0$ such that $\eta < \varepsilon/2$ and every η -pseudo-orbit z_0, \dots, z_N is ε -shadowed by any point $x \in X$ with $\varrho(x, z_0) < \eta$. Let K and β correspond to η via Lemma 5.3. Without loss of generality we may assume that N divides K . Fix $\delta < \min\{\varepsilon\eta/2K, \beta, \eta\}$ and any δ -average-pseudo-orbit $\{x_i\}_{i=0}^\infty$. Note that by the choice of β the set $\{j : \varrho(x_j, A) < \eta\}$ is infinite.

We are now ready to define $\{y_i\}_{i=0}^\infty$. We choose $y_0 \in A$ arbitrarily, start with $i = 0$, and perform indefinitely the following procedure:

- (1) find the first $j > i$ such that $\varrho(x_j, a) < \eta$ for some $a \in A$,
- (2) for all $i < p < j$ put $y_p = f^{p-i}(y_i)$,
- (3) for all $j \leq p < j + N$ put $y_p = f^{p-j}(a)$,
- (4) increase i to $j + N - 1$,
- (5) go back to step (1).

Observe that the sequence $\{y_i\}_{i=0}^\infty$ obtained in this way is a concatenation of fragments of orbits of points from A , and the fragments are (with the possible exception of the first one) of length at least N . It follows that for every $n \geq N$, where $n = sN + r$ for some $0 \leq r < N$, and for every $k \geq 0$,

$$\frac{1}{n} \sum_{i=0}^{n-1} \varrho(f(y_{i+k}), y_{i+k+1}) \leq \frac{(s+2) \operatorname{diam}(X)}{sN} \leq \frac{3 \operatorname{diam}(X)}{N} < \varepsilon,$$

and therefore $\{y_i\}_{i=0}^\infty$ is an ε -average-pseudo-orbit in A .

Additionally, by the definition of η , the first N points of every fragment (again except possibly the first one) ε -shadow the N points in $\{x_i\}_{i=0}^\infty$ with corresponding indices. Let l denote the length of the first fragment of $\{y_i\}_{i=0}^\infty$.

It follows that for n large enough we have

$$\begin{aligned} \frac{1}{n} \#\{0 \leq i < n : \varrho(x_i, y_i) \geq \varepsilon\} &\leq \frac{1}{n} (l + n - \#\{0 \leq i < n : \varrho(x_i, A) < \eta\}) \\ &\leq \frac{l}{n} + 1 - (1 - \eta) < \frac{\varepsilon}{4} + \frac{\varepsilon}{2} = \frac{3}{4}\varepsilon, \end{aligned}$$

and the assertion follows. ■

THEOREM 5.5. *If a closed invariant set $A \subset X$ contains the measure center of a compact dynamical system (X, f) and $f|_A$ has the average shadowing property on A , then so does f on X .*

Proof. Without loss of generality we may assume that $\text{diam}(X) = 1$. Fix any $\varepsilon > 0$. Let $\kappa < \varepsilon/4$ be such that every κ -average-pseudo-orbit in A is $\varepsilon/4$ -shadowed on average by some $x \in A$. Let δ be provided by Lemma 5.4 for κ . Take any δ -average-pseudo-orbit $\{x_n\}_{n=0}^\infty$ and let $\{y_n\}_{n=0}^\infty$ be a κ -average-pseudo-orbit in A obtained by application of Lemma 5.4. Pick $z \in A$ which $\varepsilon/4$ -shadows it on average. Note that there is $M > 0$ such that for every $n \geq M$ we have the inequalities

$$\begin{aligned} \frac{1}{n} \#\{0 \leq i < n : \varrho(x_i, y_i) \geq \varepsilon/4\} &< \frac{\varepsilon}{4}, \\ \frac{1}{n} \sum_{i=0}^{n-1} \varrho(f^i(z), y_i) &< \frac{\varepsilon}{4}. \end{aligned}$$

Thus if we denote $\Gamma_n = \{i < n : \varrho(x_i, y_i) < \varepsilon/4\}$ we have

$$\begin{aligned} \frac{1}{n} \sum_{i=0}^{n-1} \varrho(f^i(z), x_i) &\leq \frac{n - \#\Gamma_n}{n} + \frac{1}{n} \sum_{i \in \Gamma_n} (\varrho(f^i(z), y_i) + \varrho(y_i, x_i)) \\ &\leq \frac{\varepsilon}{4} + \frac{1}{n} \sum_{i=0}^{n-1} \varrho(f^i(z), y_i) + \frac{\varepsilon}{4} \leq \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4}. \end{aligned}$$

This proves that $\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \varrho(f^i(z), x_i) \leq 3\varepsilon/4 < \varepsilon$. Consequently, $\{x_n\}_{n=0}^\infty$ is ε -shadowed on average by z , and the proof is complete. ■

6. Factors and equicontinuity. We do not know (see Question 10.6) if a factor of a system with the (asymptotic) average shadowing property shares this property. If the answer is yes, then the following theorem follows from [24, Corollary 3.8].

THEOREM 6.1. *Suppose that a compact dynamical system (X, f) has the average shadowing property. If (Y, ϱ') is a metric space and (Y, g) is a maximal equicontinuous factor of (X, f) , then Y is a singleton.*

Proof. Let $\pi : X \rightarrow Y$ be a factor map onto a maximal equicontinuous factor. If Y has at least two elements, then there are $x, y \in X$ such that

$\pi(x) \neq \pi(y)$. Since (Y, g) is equicontinuous, it is distal, and there is a $\lambda > 0$ such that $\liminf_{n \rightarrow \infty} \varrho'(g^n(\pi(x)), g^n(\pi(y))) = 2\lambda > 0$. By equicontinuity there is $\varepsilon > 0$ such that if $\varrho(p, q) < \varepsilon$, then $\varrho(f^n(p), f^n(q)) < \lambda/3$ for every n . Using compactness we can find a $\delta > 0$ such that if $\varrho(p, q) < \delta$ then $\varrho'(\pi(p), \pi(q)) < \varepsilon$. If we fix any $\eta > 0$, then we can find a sufficiently large N such that the sequence

$$\xi = x, f(x), \dots, f^{N-1}(x), f^N(y), f^{N+1}(y), \dots, f^{2N-1}(y), \\ f^{3N}(x), f^{3N+1}(x), \dots, f^{4N-1}(x), f^{4N}(y), \dots$$

is an η -average-pseudo-orbit. In particular, for sufficiently small η (and large N) there is a point z which δ -shadows it on average. Note that the set of all indices j such that $\xi_j = f^l(x)$ for some $l \geq 0$ has upper density at least $1/2$, and an analogous statement holds with x replaced by y . Therefore there are $0 \leq i < j$ such that $\varrho(f^i(z), f^i(x)) < \delta$ and $\varrho(f^j(z), f^j(y)) < \delta$. This implies that $\varrho'(g^i(\pi(z)), g^i(\pi(x))) < \varepsilon$ and $\varrho'(g^j(\pi(z)), g^j(\pi(y))) < \varepsilon$. In particular, for all sufficiently large n , we have

$$\lambda < \varrho'(g^n(\pi(x)), g^n(\pi(y))) \\ \leq \varrho'(g^n(\pi(x)), g^n(\pi(z))) + \varrho'(g^n(\pi(z)), g^n(\pi(y))) \leq 2\lambda/3 < \lambda,$$

which is a contradiction. ■

Using the above theorem or the well known fact that every distal system has fully supported invariant measure, and Theorem 4.3, one obtains another proof of [24, Corollary 3.8] which says that the only compact, distal dynamical system with the average shadowing property is a trivial one-point system.

7. Consequences of asymptotic shadowing. The limit shadowing property was introduced by Eirola, Nevanlinna, and Pilyugin [10]. It is known that limit shadowing is always present in a neighborhood of a hyperbolic set for a diffeomorphism f of \mathbb{R}^n (see [29, Theorem 1.4.1]). Limit shadowing is an important notion with many possible applications (see [29]). It is also known that f has the shadowing property in a neighborhood of its hyperbolic set. Our Theorem 7.3 below shows that such a situation is more general than it seems at first sight.

DEFINITION 7.1. A compact dynamical system (X, f) has *limit shadowing* if every asymptotic pseudo-orbit $\{x_i\}_{i=0}^\infty$ is *asymptotically shadowed* by some point $z \in X$, that is,

$$\lim_{n \rightarrow \infty} \varrho(f^n(z), x_n) = 0.$$

We say that the compact system (X, f) has *s-limit shadowing* if it has shadowing and for every $\varepsilon > 0$ there is $\delta > 0$ such that every δ -pseudo-orbit

that is also an asymptotic pseudo-orbit is both ε -shadowed and asymptotically shadowed by some $z \in X$.

REMARK 7.2. If f is surjective, then s-limit shadowing clearly implies limit shadowing. There are, however, examples of dynamical systems with shadowing and limit shadowing but without s-limit shadowing (see for instance [3, Example 3.5]).

THEOREM 7.3. *Assume that a compact dynamical system (X, f) is chain transitive. If (X, f) has limit shadowing then it also has shadowing.*

Proof. Suppose on the contrary that (X, f) does not have shadowing. Then there is $\varepsilon > 0$ such that for any $n > 0$ there is a finite $1/n$ -pseudo-orbit α_n which cannot be ε -shadowed by any point in X . By chain transitivity, for every n there exists a $1/n$ -pseudo-orbit β_n such that the sequence $\alpha_n\beta_n\alpha_{n+1}$ forms a $1/n$ -pseudo-orbit. Then the infinite concatenation

$$\alpha_1\beta_1\alpha_2\beta_2\alpha_3\beta_3\dots$$

is an asymptotic pseudo-orbit, and therefore, by limit shadowing, it is asymptotically shadowed by some $z \in X$. Asymptotic shadowing is also, starting at some point, ε -shadowing, so this would mean that almost all α_n are ε -shadowed by some point of the form $f^{i_n}(z)$, which is a contradiction. ■

REMARK 7.4. It is easy to verify that if (X, f) is chain transitive and has shadowing or limit shadowing then it is transitive.

By the above facts, we can extend results of [21] to a complete characterization of shadowing in expansive systems.

COROLLARY 7.5. *Let (X, f) be a compact, c -expansive, and transitive dynamical system. Then the following conditions are equivalent:*

- (1) (X, f) has shadowing,
- (2) (X, f) has limit shadowing,
- (3) (X, f) has s-limit shadowing.

8. Examples. In this section we present some examples of systems with the almost specification property (average and asymptotic average shadowing properties). They show that there is no clear relation between these properties and standard notions of recurrence such as transitivity, mixing, etc.

EXAMPLE 8.1. Consider the map $f : \mathbb{S}^1 \ni e^{2\pi ix} \mapsto e^{2\pi ix^2} \in \mathbb{S}^1$ (where $x \in [0, 1]$ and \mathbb{S}^1 denotes the unit circle). Then there is a unique recurrent point of f in \mathbb{S}^1 , in particular the measure center of f is a singleton.

It was proved in [16] that the map in Example 8.1 is a nontransitive system with the asymptotic average shadowing property. By Theorem 5.1 this map actually has the almost specification property. By Theorem 3.7 this implies that there also exist nontransitive systems with the average shadowing property, so all three properties can exist in nontransitive dynamical systems.

Motivated by Example 8.1 and with the help of Lemma 8.2 below we are able to provide numerous examples of proximal systems with different degrees of transitivity. All these systems are uniquely ergodic. Moreover, they all have the almost specification property (hence the average and asymptotic average shadowing properties). This shows, in contrast to Theorem 3.8, that none of these properties is strong enough to induce any form of transitivity when only weaker kind of transitivity is assumed and the measure center is a proper subsystem (e.g., weak mixing and the average shadowing property are not sufficient to induce mixing, etc.).

LEMMA 8.2. *Every uniquely ergodic and proximal compact dynamical system has the almost specification property.*

Proof. Unique ergodicity means that there exists only one invariant measure. Proximality implies that the system has a fixed point. The atomic measure concentrated on the fixed point is an invariant measure. Therefore the measure center of a uniquely ergodic and proximal compact dynamical system is a singleton. It contains only one point—the unique fixed point of the system. Since the trivial one-point system has the almost specification property, we may apply Theorem 5.1 to complete the proof. ■

The next few examples will be *subshifts*, that is, subsystems of the full shift $(\{0, 1\}^{\mathbb{N}}, \sigma)$. The set $\{0, 1\}^{\mathbb{N}}$ is endowed with a metric compatible with the Tikhonov topology and the map σ is defined as $\sigma(x)_i = x_{i+1}$ for every $i \in \mathbb{N}$ and $x \in \{0, 1\}^{\mathbb{N}}$. If X is a subshift then we will simply write (X, σ) instead of $(X, \sigma|_X)$.

Fix $P \subset \mathbb{N}$ and observe that the set

$$\Omega_P = \{x \in \{0, 1\}^{\mathbb{N}} : x_i = x_j = 1 \Rightarrow |i - j| \in P \cup \{0\}\}$$

is a subshift called the *spacing shift generated by P*.

Specific properties of P induce some properties of (Ω_P, σ) (see e.g. [2] or [17]).

- (i) P is thick if and only if (Ω_P, σ) is weakly mixing,
- (ii) if P has a thick complement then it is proximal, uniquely ergodic, and has entropy zero (the unique invariant measure is supported on the fixed point).

EXAMPLE 8.3. Let (Σ, σ) be the subshift of the full shift on $\{0, 1\}$ such that $x \in \Sigma$ if and only if $\#\{j : x_j = 1\} \cap [i, i + 2^n] \leq n$ for every $i, n > 0$. Then (Σ, σ) is uniquely ergodic, proximal, and has entropy zero. In particular, it does not have the specification property.

REMARK 8.4. There is also a large class of dynamical systems with almost specification property, but without specification. In particular it contains all β -shifts (see e.g. [34]).

EXAMPLE 8.5. Let $P \subset \mathbb{N}$ be any thick set with a thick complement. Then (Ω_P, σ) is a weakly mixing, uniquely ergodic, and proximal dynamical system which is not mixing.

EXAMPLE 8.6. In [2] and [19] the authors constructed P such that (Ω_P, σ) is totally transitive, but not weakly mixing (i.e. P is not thick). In their example P has a thick complement, therefore the system is proximal and uniquely ergodic.

EXAMPLE 8.7. Let $P \subset \mathbb{N}$ be any thick set with a thick complement. Define $Y = \Omega_P \times \{-1, 1\}$. Let $g: Y \rightarrow Y$ be defined by $g(x, i) = (\sigma_P(x), -i)$. Let X be obtained from Y by collapsing $\{(0^\infty, -1), (0^\infty, 1)\}$ to a single point p and let (X, f) be obtained from (Y, g) by the natural projection. Then f is transitive, but not totally transitive. It is also easy to see that the only invariant ergodic measure for (X, f) must be concentrated on p , and therefore (X, f) is uniquely ergodic and proximal.

9. The noncompact case. The next two examples illustrate that without the assumption of compactness of X there is no implication in either direction between average shadowing and asymptotic average shadowing.

EXAMPLE 9.1. Let p, a_0 and b_0 be vertices of an equilateral triangle with side 1. For $n \in \{-1, -2, \dots\}$ define inductively $\overline{a_n}$ to be the middle of the line segment $\overline{pa_{n+1}}$ and $\overline{b_n}$ to be the middle of $\overline{pb_{n+1}}$. For every $n \in \mathbb{Z}_+$ choose a_n and b_n so that $\{a_n\}_{n \in \mathbb{Z}}$ and $\{b_n\}_{n \in \mathbb{Z}}$ are disjoint sequences and p is not a term in either of them. Define $X_1 = \{p\} \cup \bigcup_{n \in \mathbb{Z}} \{a_n, b_n\}$. We define a metric ϱ_1 on X_1 in the following way:

- (i) if $x, y \in \{p\} \cup \bigcup_{n \in \{0, -1, \dots\}} \{a_n, b_n\}$ then $\varrho_1(x, y)$ is the Euclidean distance between x and y in the triangle,
- (ii) for every $n \in \mathbb{N}$ we put $\varrho_1(a_n, a_{n+1}) = \varrho_1(b_n, b_{n+1}) = 2^n$,
- (iii) for every $n \in \mathbb{Z}_+$ we put $\varrho_1(a_n, b_n) = \sum_{i=1}^n 1/i$,
- (iv) for every pair of points $x, y \in X_1$ for which $\varrho_1(x, y)$ has not been defined in steps (i)–(iii) we put

$$\varrho_1(x, y) = \inf \left\{ \sum_{i=0}^n \varrho_1(c_i, c_{i+1}) \mid n \in \mathbb{N}, c_0, \dots, c_{n+1} \in X_1, c_0 = x, c_{n+1} = y, \right. \\ \left. \varrho_1(c_0, c_1), \dots, \varrho_1(c_n, c_{n+1}) \text{ have been defined in steps (i)–(iii)} \right\}.$$

It is elementary to verify that ϱ_1 is a metric. Finally, define $f_1: X_1 \rightarrow X_1$ by setting $f_1(p) = p$ and by putting $f_1(a_n) = a_{n+1}$ and $f_1(b_n) = b_{n+1}$ for every $n \in \mathbb{Z}$. It can be easily verified that f_1 is continuous.

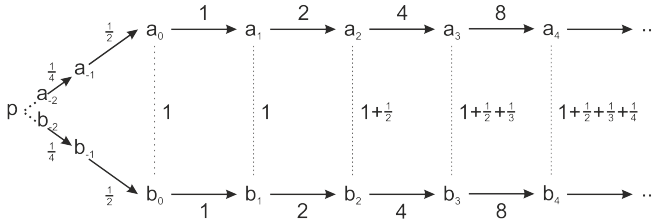


Fig. 2. The space X_1 and the map f_1

THEOREM 9.2. *The map f_1 has the average shadowing property, but not the asymptotic average shadowing property.*

Proof. Average shadowing property. Given $\varepsilon > 0$ set $\delta < \varepsilon/3$. For every δ -average-pseudo-orbit $\{z_i\}_{i=0}^\infty$ with a constant N as in Definition 2.5, there are two possibilities:

CASE 1: *The sequence $\{z_i\}_{i=0}^\infty$ is bounded.* Consider all the indices r_i where $\varrho_1(p, z_{r_i+1}) \leq \varrho_1(p, z_{r_i})$. As the sequence $\{\varrho_1(p, z_i)\}_{i=0}^\infty$ is bounded and takes values from the set $\{2^i: i \in \mathbb{Z}\}$, these indices can be written as an infinite increasing sequence $\{r_i\}_{i=0}^\infty$. Put $r_{-1} = -1$. Observe that the structure of X_1 guarantees that for every $i \in \mathbb{N}$ we have

$$\sum_{k=r_{i-1}+1}^{r_i} \varrho_1(p, z_k) \leq \varrho_1(p, f_1(z_{r_i})) \leq 2\varrho_1(f_1(z_{r_i}), z_{r_i}) \leq 2\varrho_1(f_1(z_{r_i}), z_{r_{i+1}}),$$

which implies that for every $n \in \mathbb{N}$ we have

$$\frac{1}{r_n + 1} \sum_{i=0}^{r_n} \varrho_1(p, z_i) \leq \frac{2}{r_n + 1} \sum_{i=0}^{r_n} \varrho_1(f_1(z_i), z_{i+1}).$$

Note that since $\{z_i\}_{i=0}^\infty$ is bounded, the sums $\sum_{k=r_{i-1}+1}^{r_i} \varrho_1(p, z_k)$ are also bounded by some constant $R > 0$. Take $n \geq N$ large enough for the inequality $R/(n + 1) \leq \delta$ to hold and such that $n > r_0$. Let r_j be the largest term of the sequence $\{r_i\}_{i=0}^\infty$ smaller than n . Finally, observe that

$$\begin{aligned} \frac{1}{n+1} \sum_{i=0}^n \varrho_1(p, z_i) &= \frac{1}{n+1} \sum_{i=0}^{r_j} \varrho_1(p, z_i) + \frac{1}{n+1} \sum_{i=r_j+1}^n \varrho_1(p, z_i) \\ &\leq \frac{2}{n+1} \sum_{i=0}^n \varrho_1(f_1(z_i), z_{i+1}) + \frac{R}{n+1} \leq 3\delta < \varepsilon, \end{aligned}$$

which proves ε -shadowing on average by p .

CASE 2: *The pseudo-orbit $\{z_i\}_{i=0}^\infty$ is not bounded.* Notice that it follows straight from the definition of the δ -average-pseudo-orbit that the maximum possible error $\varrho_1(f(z_i), z_{i+1})$ is bounded by δN . Therefore, once the pseudo-orbit strays far enough from p , it will be unable to deviate from the trajectory of the point it landed on and will be ε -shadowed on average by the orbit of that point.

Asymptotic average shadowing property. Define the sequence $\{z_i\}_{i=0}^\infty$ by

$$z_i = \begin{cases} b_i & \text{when } i \in \{0, 1\}, \\ a_i & \text{when } i \in (2^{2k}, 2^{2k+1}] \text{ for some } k \in \mathbb{N}, \\ b_i & \text{when } i \in (2^{2k+1}, 2^{2k+2}] \text{ for some } k \in \mathbb{N}. \end{cases}$$

Observe that

$$\begin{aligned} \frac{1}{n} \sum_{i=0}^n \varrho_1(f_1(z_i), z_{i+1}) &= \frac{1}{n} \sum_{i \in \{1,2,4,\dots\} \cap [0,n]} \varrho_1(f_1(z_i), z_{i+1}) \\ &\leq \frac{1}{n} (1 + \log_2 n) \sum_{i=1}^{n+1} \frac{1}{i} \leq \frac{1 + \log_2 n}{\sqrt{n}} \sum_{i=1}^{n+1} \frac{1}{i\sqrt{i}}. \end{aligned}$$

Observe that this last expression tends to 0 as $n \rightarrow \infty$, proving that $\{z_i\}_{i=0}^\infty$ is an asymptotic average pseudo-orbit of f_1 .

This pseudo-orbit may not be asymptotically shadowed on average by any point other than a_0 or b_0 , as it maintains a distance bounded from below by $1/2$ from the iterations of any point but these two. To exclude the possibility of shadowing by a_0 note that for n large enough we have

$$\frac{1}{2^{2n}} \sum_{i=0}^{2^{2n}} \varrho_1(f_1^i(a_i), z_i) \geq \frac{1}{2^{2n}} 2^{2n-1} = 1/2.$$

The shadowing by b_0 is handled in an analogous way, proving that f_1 does not have the asymptotic average shadowing property. ■

EXAMPLE 9.3. First we define a family of pairwise disjoint metric spaces $\{(A_{i,j}, \varrho_{i,j})\}_{i \in \mathbb{Z}_-, j \in \mathbb{N}}$ in the following way:

- for every $i \in \mathbb{Z}_-$ and $j \in \mathbb{N}$ we put $A_{i,j} = \{a_{i,j}\}$,
- for every $j \in \mathbb{N}$ we put $A_{0,j} = \{a_{0,j}, b_{0,j}\}$ and $\varrho_{0,j}(a_{0,j}, b_{0,j}) = 1/2^j$,

- for every $i \in \mathbb{Z}_+$ and $j \in \mathbb{N}$ we put $A_{i,j} = \{a_{i,j}, b_{i,j}, c_{i,j}\}$, $\varrho_{i,j}(a_{i,j}, b_{i,j}) = 1/2^j$, $\varrho_{i,j}(b_{i,j}, c_{i,j}) = 1 - 1/2^j$, and $\varrho_{i,j}(a_{i,j}, c_{i,j}) = 1$.

Next we put $X_2 = \bigcup_{i \in \mathbb{Z}, j \in \mathbb{N}} A_{i,j}$ and define a metric ϱ_2 on X_2 in the following way:

- (i) for every $i \in \mathbb{Z}$, $j \in \mathbb{N}$, $x, y \in A_{i,j}$ we put $\varrho_2(x, y) = \varrho_{i,j}(x, y)$,
- (ii) for every $i, j \in \mathbb{N}$, $x \in A_{i,j}$ and $y \in A_{i+1,j}$ we put $\varrho_2(x, y) = 2^i$,
- (iii) for every $i \in \mathbb{Z}_-$, $j \in \mathbb{N}$, $x \in A_{i,j}$ and $y \in A_{i+1,j}$ we put $\varrho_2(x, y) = 2^{-i-1}$,
- (iv) for every $i \in \mathbb{Z}$, $j, k \in \mathbb{N}$, $x \in A_{i,j}$ and $y \in A_{i,k}$ we put $\varrho_2(x, y) = 2^{|i|}$,
- (v) for any $x, y \in X_2$ for which $\varrho_2(x, y)$ has not been defined in steps (i)–(iv) we put

$$\varrho_2(x, y) = \inf \left\{ \sum_{i=0}^n \varrho_2(c_i, c_{i+1}) \mid n \in \mathbb{N}, c_0, \dots, c_{n+1} \in X_2, c_0 = x, c_{n+1} = y, \right. \\ \left. \varrho_2(c_0, c_1), \dots, \varrho_2(c_n, c_{n+1}) \text{ have been defined in steps (i)–(iv)} \right\}.$$

Once again, it is elementary to check that ϱ_2 is a metric. Finally, define $f_2: X_2 \rightarrow X_2$ as follows:

- for every $i \in \mathbb{Z}$ and $j \in \mathbb{N}$ put $f_2(a_{i,j}) = a_{i+1,j}$,
- for every $i, j \in \mathbb{N}$ put $f_2(b_{i,j}) = c_{i+1,j}$,
- for every $i \in \mathbb{Z}_+$ and $j \in \mathbb{N}$ put $f_2(c_{i,j}) = a_{i+1,j}$.

Since the topology on X_2 is discrete, the map f_2 is continuous.

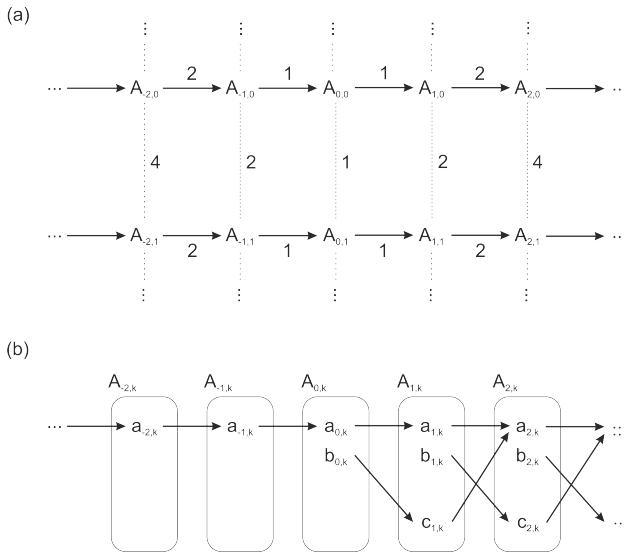


Fig. 3. (a) The space X_2 and the map f_2 ; (b) a detailed view of one of the rows

THEOREM 9.4. *The map f_2 does not have the average shadowing property, but has the asymptotic average shadowing property.*

Proof. Average shadowing property. Let $k \in \mathbb{N}$ be fixed. Consider the sequence $\{z_i\}_{i=0}^\infty = \{b_{0,k}, c_{1,k}, b_{2,k}, c_{3,k}, \dots\}$. Since for every $k \in \mathbb{N}$ we have

$$\frac{1}{n} \sum_{i=0}^{n-1} \varrho_2(f_2(x_{i+k}), x_{i+k+1}) \leq \frac{n}{n2^k} = \frac{1}{2^k},$$

this sequence is a $1/2^{k-1}$ -average-pseudo-orbit. Notice that $\{z_i\}_{i=0}^\infty$ may not be shadowed on average by any point except for $a_{0,k}$ or $b_{0,k}$, as $\{z_i\}_{i=0}^\infty$ maintains a positive distance from the orbit of any other point. Note also that

$$\frac{1}{n} \sum_{i=0}^{n-1} \varrho_2(f_2(a_{0,k}), z_i) \geq \left\lfloor \frac{n}{2} \right\rfloor / n \geq \frac{1}{4}.$$

The possibility of shadowing by $b_{0,k}$ is excluded in an analogous way. Therefore, for $\varepsilon = 1/5$ there is no $\delta > 0$ for which every δ -average-pseudo-orbit would be ε -shadowed on average by some point in X_2 .

Asymptotic average shadowing property. Let $\{z_i\}_{i=0}^\infty$ be an asymptotic average pseudo-orbit of f_2 . Consider all the indices r_i for which the points $f_2(z_{r_i})$ and $z_{r_{i+1}}$ belong to different spaces $A_{i,j}$.

Let us first assume that there are infinitely many such indices; write them as an increasing sequence $\{r_i\}_{i=0}^\infty$. Put $r_{-1} = -1$. Observe that the structure of X_2 guarantees that for every $i \in \mathbb{Z}_+$ we have $\varrho_2(f_2(z_{r_{i-1}}), z_{r_{i-1}+1}) + \varrho_2(f_2(z_{r_i}), z_{r_i+1}) \geq 2^{\lfloor (r_i - r_{i-1})/2 \rfloor}$. Consequently for every $k \in \mathbb{N}$ we have

$$\begin{aligned} (9.1) \quad & \limsup_{k \rightarrow \infty} \frac{1}{r_k + 1} \sum_{i=0}^{r_k} \varrho_2(f_2(z_k), z_{k+1}) \\ & \geq \frac{1}{2} \limsup_{k \rightarrow \infty} \frac{1}{r_k + 1} \left(\sum_{i=0}^{r_k} \varrho_2(f_2(z_k), z_{k+1}) + \sum_{i=1}^{r_k-1} \varrho_2(f_2(z_k), z_{k+1}) \right) \\ & = \frac{1}{2} \limsup_{k \rightarrow \infty} \frac{1}{r_k + 1} \sum_{i=1}^{r_k} (\varrho_2(f_2(z_{k-1}), z_k) + \varrho_2(f_2(z_k), z_{k+1})) \\ & = \frac{1}{2} \limsup_{k \rightarrow \infty} \frac{1}{r_k + 1} \sum_{i=0}^{r_k} (\varrho_2(f_2(z_{k-1}), z_k) + \varrho_2(f_2(z_k), z_{k+1})) \\ & \geq \frac{1}{2} \limsup_{k \rightarrow \infty} \frac{1}{r_k + 1} \sum_{i=0}^k 2^{\lfloor (r_i - r_{i-1})/2 \rfloor}. \end{aligned}$$

Since $2^{\lfloor n/2 \rfloor} \geq n/2$ for every $n \in \mathbb{N}$, and the $k+1$ terms $r_0 - r_{-1}, \dots, r_k - r_{k-1}$

add up to $r_k + 1$, we deduce from (9.1) that

$$\limsup_{k \rightarrow \infty} \frac{1}{r_k + 1} \sum_{i=0}^{r_k} \varrho_2(f_2(z_k), z_{k+1}) \geq \frac{1}{2} \limsup_{k \rightarrow \infty} \frac{1}{r_k + 1} \cdot \frac{r_k + 1}{2} = \frac{1}{4}.$$

This contradicts the fact that $\{z_i\}_{i=0}^\infty$ is an asymptotic average pseudo-orbit, so there are only finitely many indices r_i , and therefore there exist $p \in \mathbb{Z}$ and $q, k \in \mathbb{N}$ such that for every $n \geq k$ the points $f_2^n(a_{p,q})$ and z_n belong to the same space $A_{p+n,q}$.

To prove that $a_{p,q}$ asymptotically shadows $\{z_i\}_{i=0}^\infty$ on average, observe that every positive term in the sum $\sum_{i=k}^n \varrho_2(f_2(z_i), z_{i+1})$ adds at most $2^q + 1$ times its value to the sum $\sum_{i=k}^n \varrho_2(f_2^i(a_{p,q}), z_i)$. Strictly speaking, if $\varrho_2(f_2(z_j), z_{j+1}) > 0$ for some $j \geq k$, then, assuming $f_2(z_{j+1}) = z_{j+2}$, the following situations are possible:

$f_2(z_j)$	z_{j+1}	$\varrho_2(f_2(z_j), z_{j+1})$	$\varrho_2(f_2^{j+1}(a_{p,q}), z_{j+1})$	$\varrho_2(f_2^{j+2}(a_{p,q}), z_{j+2})$
$a_{p+j+1,q}$	$b_{p+j+1,q}$	$1/2^q$	$1/2^q$	1
$a_{p+j+1,q}$	$c_{p+j+1,q}$	1	1	0
$b_{p+j+1,q}$	$a_{p+j+1,q}$	$1/2^q$	0	0
$b_{p+j+1,q}$	$c_{p+j+1,q}$	$1 - 1/2^q$	1	0
$c_{p+j+1,q}$	$a_{p+j+1,q}$	1	0	0
$c_{p+j+1,q}$	$b_{p+j+1,q}$	$1 - 1/2^q$	$1/2^q$	1

Starting from z_{j+3} the sequence $\{z_i\}_{i=0}^\infty$ will follow the trajectory of $a_{p,q}$ until the next deviation.

If $f_2(z_{j+1}) \neq z_{j+2}$ then the deviation at $j + 1$ contributes only the fourth column of the above table (instead of the fourth and fifth) to the total error $\sum_{i=k}^n \varrho_2(f_2^i(a_{p,q}), z_i)$.

Therefore

$$\begin{aligned} \lim_{s \rightarrow \infty} \frac{1}{s} \sum_{i=0}^{s-1} \varrho_2(f_2^i(a_{p,q}), z_i) &= \lim_{s \rightarrow \infty} \frac{1}{s} \sum_{i=k}^{s-1} \varrho_2(f_2^i(a_{p,q}), z_i) \\ &\leq (2^q + 1) \lim_{s \rightarrow \infty} \frac{1}{s} \sum_{i=k}^{s-1} \varrho_2(f_2(z_i), z_{i+1}) = 0. \end{aligned}$$

This proves that f_2 has the asymptotic average shadowing property. ■

10. Open problems. In this section we collect a few open questions for further research.

If $f: X \rightarrow X$ is a continuous map of a compact metric space with the shadowing property, then so does f restricted to its nonwandering set $\Omega(f)$ (see e.g. [1, Theorem 3.4.2] extended by [23, Lemma 1]). Motivated by this result and Theorem 5.1 above, we raise the following question.

QUESTION 10.1. *Assume that (X, f) has the almost specification property (resp. asymptotic average shadowing property, average shadowing property) and let A be any compact invariant set containing the measure center of f . Does the dynamical system $(A, f|_A)$ has the almost specification property (resp. asymptotic average shadowing property, average shadowing property)?*

Note that if the answer to the above question is positive, then we can get rid of the surjectivity assumption in Theorems 3.5 and 3.7.

By Theorem 6.1, any minimal system with the average shadowing property has to be weakly mixing. We are unable, however, to provide any example of a nonsingleton minimal system with almost specification or generalized shadowing.

QUESTION 10.2. *Is there a nontrivial minimal system (X, f) with the almost specification property or the (asymptotic) average shadowing property?*

We proved that almost specification implies asymptotic average shadowing, which in turn implies average shadowing. It is natural to ask if the converse is true. We conjecture it is not.

QUESTION 10.3. *Does the (asymptotic) average shadowing property imply the almost specification property? Does the average shadowing property imply the asymptotic average shadowing property?*

On the other hand, we provided examples that when the space is not compact, there are no implications between average shadowing and asymptotic average shadowing; their construction relied heavily on the fact that the space was unbounded. In this context it is natural to ask the following.

QUESTION 10.4. *Can examples analogous to Examples 9.1 and 9.3 be presented in a bounded metric space?*

In view of Theorem 4.3 and the well-known properties of maps with the specification property we conjecture that the answer to the following question is positive.

QUESTION 10.5. *Does the (asymptotic) average shadowing property or the almost specification property imply topological mixing or positive topological entropy provided that the measure center is the whole space?*

It is easy to see that a factor of a system with the (almost) specification property also has (almost) specification. Any sofic shift which is not of finite type shows that a factor of a system with shadowing does not necessarily have shadowing. We do not know if every factor of a system with (asymptotic) average shadowing has (almost) specification.

otic) average shadowing also has this property. Any counterexample would show that (asymptotic) average shadowing does not imply almost specification.

QUESTION 10.6. *Is the (asymptotic) average shadowing property or the almost specification property inherited by factors?*

We are also unable to answer the following question related to Theorem 4.1.

QUESTION 10.7. *Is it true that if a compact system has the (asymptotic) average shadowing property and an invariant measure of full support, then the minimal points are dense?*

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