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Coherent adequate sets and forcing square

by

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Abstract. We introduce the idea of a coherent adequate set of models, which can be used as side conditions in forcing. As an application we define a forcing poset which adds a square sequence on ω_2 using finite conditions.

In previous work [3] we introduced the idea of an adequate set of models and showed how to use adequate sets as side conditions in forcing with finite conditions. We gave several examples of forcing with adequate sets, including forcing posets for adding a generic function on ω_2, adding a nonreflecting stationary subset of ω_2, adding a Kurepa tree on ω_1, and in [4] adding a club to a fat stationary subset of ω_2. The main result of the present paper is to define a forcing poset using adequate sets which adds a $\square_{\omega_1}$-sequence.

The idea of using models as side conditions in forcing goes back to Todorčević [6], where the method was applied to add generic objects of size ω_1 with finite approximations. In the original context of applications of PFA, the preservation of ω_2 was not necessary. To preserve ω_2, Todorčević introduced the requirement of a system of isomorphisms on the models in a condition.

In the present paper we introduce the idea of a coherent adequate set of models. A coherent adequate set is essentially an adequate set in the sense of [3] which also satisfies the existence of a system of isomorphisms in the sense of Todorčević. Combining these two ideas turns out to provide a powerful method for forcing with side conditions. As an application we define a forcing poset which adds a square sequence on ω_2 using finite conditions.

We assume that the reader is familiar with the basic theory of adequate sets as described in Sections 1–3 of [3]. Our treatment of coherent adequate sets owes a lot to the presentation of nicely arranged families given by Abra-

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ham and Cummings [1]. Forcing a square sequence with finite conditions was first achieved by Dolinar and Džamonja [2] using the Mitchell style of models as side conditions [5]. An important difference is that the clubs which appear in the square sequence added by their forcing poset belong to the ground model, whereas for us the clubs are themselves generically approximated by finite fragments.

1. Adequate sets. In this section we review the material on adequate sets which we will use. Throughout the paper we assume that $2^\omega = \omega_1$ and $2^{\omega_1} = \omega_2$.

Let π be a bijection of ω_2 onto $H(\omega_2)$. Fix a set of definable Skolem functions for the structure $(H(\omega_2), \in, \pi)$. For any set $a \subseteq \omega_2$, let $\mathrm{Sk}(a)$ denote the closure of a under these Skolem functions. Let C^* be the club set of $\beta < \omega_2$ such that $\mathrm{Sk}(\beta) \cap \omega_2 = \beta$. Let $\Lambda := C^* \cap \mathrm{cof}(\omega_1)$. Note that any ordinal in Λ is also a limit point of C^*.

Let $\mathcal{X}$ be the set of countable $M \subseteq \omega_2$ such that $\mathrm{Sk}(M) \cap \omega_2 = M$ and for all $\gamma \in M$, $\sup(C^* \cap \gamma) \in M$. Note that $\mathcal{X}$ is a club subset of $P_{\omega_1}(\omega_2)$. If $M \in \mathcal{X}$ then $\mathrm{Sk}(M) = \pi[M]$. It follows that if M and N are in $\mathcal{X}$ and $N \in \mathrm{Sk}(M)$, then $\mathrm{Sk}(N) \in \mathrm{Sk}(M)$. If a and b are in $\mathcal{X} \cup \Lambda$, then $\mathrm{Sk}(a) \cap \mathrm{Sk}(b) = \mathrm{Sk}(a \cap b)$ (see [3, Lemma 1.4]). This implies that if $M \in \mathcal{X}$ and $\beta \in \Lambda$, then $M \cap \beta \in \mathcal{X}$.

If $M \in \mathcal{X}$ and $\beta \in \Lambda \cap \sup(M)$, then $\min(M \setminus \beta)$ is in Λ. Clearly $\min(M \setminus \beta)$ has cofinality ω_1. If this ordinal is not in Λ, then it is not a limit point of C^*. Also $\beta \neq \min(M \setminus \beta)$, so $\sup(M \cap \beta) < \beta < \min(M \setminus \beta)$. Hence $\sup(C^* \cap \min(M \setminus \beta))$ is below $\min(M \setminus \beta)$ and is in M by the definition of $\mathcal{X}$. In particular this supremum is below β. This is a contradiction since β is in C^*.

Let M be in $\mathcal{X}$. A set K is an *initial segment* of M if either $K = M$ or there exists $\beta \in M \cap \Lambda$ such that $K = M \cap \beta$. So any initial segment of M is also in $\mathcal{X}$. If M and N are in $\mathcal{X}$ and $N \in \mathrm{Sk}(M)$, then since N has only countably many initial segments, they are all members of $\mathrm{Sk}(M)$.

Since $2^\omega = \omega_1$, for all $\beta \in \Lambda$, $\mathcal{X} \cap P(\beta) \subseteq \mathrm{Sk}(\beta)$. For since $\mathrm{cf}(\beta) = \omega_1$, every member of $\mathcal{X} \cap P(\beta)$ belongs to $P_{\omega_1}(\gamma)$ for some $\gamma < \beta$. And since $\omega_1 \subseteq \mathrm{Sk}(\beta)$, we have $P_{\omega_1}(\gamma) \subseteq \mathrm{Sk}(\beta)$. In particular, if $M \in \mathcal{X}$ and $\beta \in \Lambda$, then $M \cap \beta \in \mathrm{Sk}(\beta)$.

For a set $M \in \mathcal{X}$, let Λ_M denote the set of $\beta \in \Lambda$ such that

$$\beta = \min(\Lambda \setminus \sup(M \cap \beta)).$$

In other words, $\beta \in \Lambda_M$ if $\beta \in \Lambda$ and there are no elements of Λ strictly between $\sup(M \cap \beta)$ and β. For M and N in $\mathcal{X}$, $\Lambda_M \cap \Lambda_N$ has a largest element (see [3, Lemma 2.4]). We denote this largest element by $\beta_{M,N}$, which is called the *comparison point* of M and N. An important property

of the comparison point is the following:

$$(M \cup \lim(M)) \cap (N \cup \lim(N)) \subseteq \beta_{M,N}$$

(see [3, Proposition 2.6]).

DEFINITION 1.1. A set $A \subseteq \mathcal{X}$ is *adequate* if for all M and N in A, either $M \cap \beta_{M,N} \in \mathrm{Sk}(N)$, $N \cap \beta_{M,N} \in \mathrm{Sk}(M)$, or $M \cap \beta_{M,N} = N \cap \beta_{M,N}$.

Note that a set $A \subseteq \mathcal{X}$ is adequate iff for all M and N in A, $\{M, N\}$ is adequate. If $\{M, N\}$ is adequate, then $M \cap \beta_{M,N} \in \mathrm{Sk}(N)$ iff $M \cap \omega_1 < N \cap \omega_1$, and $M \cap \beta_{M,N} = N \cap \beta_{M,N}$ iff $M \cap \omega_1 = N \cap \omega_1$.

Suppose that $\{M, N\}$ is adequate. If $M \cap \beta_{M,N} = N \cap \beta_{M,N}$, then $M \cap N = M \cap \beta_{M,N}$. And if $M \cap \beta_{M,N} \in \mathrm{Sk}(N)$, then $M \cap N = M \cap \beta_{M,N}$.

The next lemma records some important technical facts about comparison points which are used frequently. The proofs can be found in Section 3 of [3].

LEMMA 1.2.

(1) *Let $M \in \mathcal{X}$, $\beta \in \Lambda$, and suppose $M \subseteq \beta$. Then $\beta_{M,N} \leq \beta$ for all $N \in \mathcal{X}$.*
(2) *Let $K, M, N \in \mathcal{X}$, and suppose $M \subseteq N$. Then $\beta_{K,M} \leq \beta_{K,N}$.*
(3) *Let M and N be in $\mathcal{X}$, and $\beta \in \Lambda$. If $\beta_{M,N} \leq \beta$, then $\beta_{M,N} = \beta_{M \cap \beta, N}$.*
(4) *Let M and N be in $\mathcal{X}$, and $\beta \in \Lambda$. If $N \subseteq \beta$, then $\beta_{M,N} = \beta_{M \cap \beta, N}$.*

Another important fact is that if $\{M, N\}$ is adequate and $\beta \in \Lambda$, then $\{M \cap \beta, N \cap \beta\}$ is adequate (see [3, Lemma 3.3]).

LEMMA 1.3. *If $\{M \cap \beta_{M,N}, N \cap \beta_{M,N}\}$ is adequate, then so is $\{M, N\}$.*

Proof. Let $\beta := \beta_{M,N}$. Since $\beta \leq \beta$, Lemma 1.2(3) implies that $\beta = \beta_{M \cap \beta, N}$. And as $M \cap \beta \subseteq \beta$, Lemma 1.2(4) implies that $\beta_{M \cap \beta, N} = \beta_{M \cap \beta, N \cap \beta}$. Hence $\beta = \beta_{M \cap \beta, N \cap \beta}$. In particular, $(M \cap \beta) \cap \beta_{M \cap \beta, N \cap \beta} = M \cap \beta$ and $(N \cap \beta) \cap \beta_{M \cap \beta, N \cap \beta} = N \cap \beta$. So if $(M \cap \beta) \cap \beta_{M \cap \beta, N \cap \beta} \in \mathrm{Sk}(N \cap \beta)$ then $M \cap \beta \in \mathrm{Sk}(N)$, and similarly if $(N \cap \beta) \cap \beta_{M \cap \beta, N \cap \beta} \in \mathrm{Sk}(M \cap \beta)$ then $N \cap \beta \in \mathrm{Sk}(M)$. Also the equality $(M \cap \beta) \cap \beta_{M \cap \beta, N \cap \beta} = (N \cap \beta) \cap \beta_{M \cap \beta, N \cap \beta}$ is equivalent to the equality $M \cap \beta = N \cap \beta$. ∎

2. Coherent adequate sets. In the basic theory of adequate sets, we identify a set M in $\mathcal{X}$ with $\mathrm{Sk}(M)$, and oftentimes with the structure $(\mathrm{Sk}(M), \in, \pi \cap \mathrm{Sk}(M))$, which is an elementary substructure of $(H(\omega_2), \in, \pi)$. For any set $P \subseteq H(\omega_2)$ and $M \in \mathcal{X}$, let $P_M := P \cap \mathrm{Sk}(M)$. In the context of coherent adequate sets we are interested in the expanded structure

$$\mathfrak{M} = (\mathrm{Sk}(M), \in, \pi_M, \mathcal{X}_M, \Lambda_M).$$

Note that $\mathfrak{M}$ is not necessarily an elementary substructure of $(H(\omega_2), \in, \pi, \mathcal{X}, \Lambda)$. In general if a set in $\mathcal{X}$ is denoted with a particular letter, we use

the Fraktur version of the letter to denote the above structure on its Skolem hull.

Let M and N be in $\mathcal{X}$. We say that M and N are *isomorphic* if the structures $\mathfrak{M}$ and $\mathfrak{N}$ are isomorphic. In other words, M and N are isomorphic if there exists a bijection $\sigma : \mathrm{Sk}(M) \to \mathrm{Sk}(N)$ such that for all x and y in $\mathrm{Sk}(M)$:

(1) $x \in y$ iff $\sigma(x) \in \sigma(y)$;
(2) $\pi(x) = y$ iff $\pi(\sigma(x)) = \sigma(y)$;
(3) $x \in \mathcal{X}$ iff $\sigma(x) \in \mathcal{X}$;
(4) $x \in \Lambda$ iff $\sigma(x) \in \Lambda$.

In particular, such a map σ is an isomorphism from $(\mathrm{Sk}(M), \in)$ to $(\mathrm{Sk}(N), \in)$. Since these structures model the axiom of extensionality, such an isomorphism is unique if it exists. In that case, let $\sigma_{M,N}$ denote the unique isomorphism from $\mathfrak{M}$ to $\mathfrak{N}$. Note that if M, N, and K are isomorphic, then $\sigma_{M,N} = \sigma_{K,N} \circ \sigma_{M,K}$.

For $M \in \mathcal{X}$, let $\overline{\mathfrak{M}}$ denote the transitive collapse of the structure $\mathfrak{M}$, and let $\sigma_M : \mathfrak{M} \to \overline{\mathfrak{M}}$ be the collapsing map. Note that M and N are isomorphic iff $\overline{\mathfrak{M}} = \overline{\mathfrak{N}}$. In that case, by the uniqueness of isomorphisms we have

$$\sigma_{M,N} = \sigma_N^{-1} \circ \sigma_M.$$

Suppose that M and N are isomorphic and $a \in \mathrm{Sk}(M)$ is countable. We claim that $\sigma_{M,N}(a) = \sigma_{M,N}[a]$. Since a and $\sigma_{M,N}(a)$ are countable, we have $a \subseteq \mathrm{Sk}(M)$ and $\sigma_{M,N}(a) \subseteq \mathrm{Sk}(N)$. Hence $x \in a$ implies $\sigma_{M,N}(x) \in \sigma_{M,N}(a)$, so that $\sigma_{M,N}[a] \subseteq \sigma_{M,N}(a)$. On the other hand, if $z \in \sigma_{M,N}(a)$, then for some $x \in \mathrm{Sk}(M)$, $\sigma_{M,N}(x) = z$, which implies that $x \in a$. So $z \in \sigma_{M,N}[a]$.

LEMMA 2.1. *Let M and N be isomorphic, and $K \in \mathrm{Sk}(M) \cap \mathcal{X}$. Let $K^* = \sigma_{M,N}(K)$. Then $\sigma_{M,N}(\mathrm{Sk}(K)) = \mathrm{Sk}(K^*)$, K and K^* are isomorphic, and $\sigma_{M,N} \restriction \mathrm{Sk}(K) = \sigma_{K,K^*}$.*

Proof. Since K is countable, $K^* = \sigma_{M,N}[K]$. For all $\alpha \in K$, we have $\sigma_{M,N}(\pi(\alpha)) = \pi(\sigma_{M,N}(\alpha))$. It follows that

$$\begin{aligned}\sigma_{M,N}(\mathrm{Sk}(K)) &= \sigma_{M,N}[\mathrm{Sk}(K)] = \sigma_{M,N}[\pi[K]] \\ &= \pi[\sigma_{M,N}[K]] = \pi[K^*] = \mathrm{Sk}(K^*).\end{aligned}$$

So $\sigma_{M,N} \restriction \mathrm{Sk}(K)$ is a bijection from $\mathrm{Sk}(K)$ to $\mathrm{Sk}(K^*)$, and it clearly preserves the predicates $\in$, π, $\mathcal{X}$, and Λ. Hence $\sigma_{M,N} \restriction \mathrm{Sk}(K)$ is an isomorphism of $\mathfrak{K}$ to $\mathfrak{K}^*$. So K and K^* are isomorphic and $\sigma_{K,K^*} = \sigma_{M,N} \restriction \mathrm{Sk}(K)$. ∎

LEMMA 2.2. *Let M and N be isomorphic, and let K be an initial segment of M. Let $K^* := \sigma_{M,N}[K]$. Then K^* is an initial segment of N, $\sigma_{M,N}[\mathrm{Sk}(K)] = \mathrm{Sk}(K^*)$, K and K^* are isomorphic, and $\sigma_{M,N} \restriction \mathrm{Sk}(K) = \sigma_{K,K^*}$.*

Proof. This is clear if $M = K$. Otherwise there is $\beta \in M \cap \Lambda$ such that $K = M \cap \beta$. Then $\sigma_{M,N}(\beta) \in N \cap \Lambda$, and easily $K^* = N \cap \sigma_{M,N}(\beta)$. By the argument from the previous lemma, $\sigma_{M,N}[\mathrm{Sk}(K)] = \mathrm{Sk}(\sigma_{M,N}[K])$ $= \mathrm{Sk}(K^*)$, and $\sigma_{M,N} \restriction \mathrm{Sk}(K)$ is an isomorphism of $\mathrm{Sk}(K)$ to $\mathrm{Sk}(K^*)$. Hence K and K^* are isomorphic, and $\sigma_{M,N} \restriction \mathrm{Sk}(K) = \sigma_{K,K^*}$. ∎

Suppose that $M \cap \beta_{M,N} = N \cap \beta_{M,N}$, and M and N are isomorphic. Applying the previous lemma, $\sigma_{M,N} \restriction (M \cap \beta_{M,N})$ is an isomorphism of $M \cap \beta_{M,N}$ to the initial segment $\sigma_{M,N}[M \cap \beta_{M,N}]$ of N. But the latter initial segment has the same order type as the initial segment $N \cap \beta_{M,N}$, so it is equal to it. Hence $\sigma_{M,N} \restriction \mathrm{Sk}(M \cap \beta_{M,N})$ is an isomorphism of $\mathrm{Sk}(M \cap \beta_{M,N})$ to itself, and therefore it is the identity map. But $M \cap \beta_{M,N} = M \cap N$. In particular, we have proven the following lemma.

LEMMA 2.3. *Let $\{M, N\}$ be adequate, where M and N are isomorphic and $M \cap \beta_{M,N} = N \cap \beta_{M,N}$. Then $\sigma_{M,N} \restriction \mathrm{Sk}(M \cap N)$ is the identity function.*

We now introduce the most important idea of the paper.

DEFINITION 2.4. Let $A \subseteq \mathcal{X}$. Then A is a *coherent adequate set* if A is adequate and for all M and N in A:

(1) if $M \cap \beta_{M,N} = N \cap \beta_{M,N}$, then M and N are isomorphic;
(2) if $M \cap \beta_{M,N} \in \mathrm{Sk}(N)$, then there exists N' in A such that $M \in \mathrm{Sk}(N')$ and N and N' are isomorphic;
(3) if $M \cap \beta_{M,N} = N \cap \beta_{M,N}$ and $K \in A \cap \mathrm{Sk}(M)$, then $\sigma_{M,N}(K) \in A$.

Recall that if A is adequate and M and N are in A, then $M \cap \beta_{M,N} \in \mathrm{Sk}(N)$ iff $M \cap \omega_1 < N \cap \omega_1$, and $M \cap \beta_{M,N} = N \cap \beta_{M,N}$ iff $M \cap \omega_1 = N \cap \omega_1$. It follows that a finite adequate set A is coherent iff the set $\{\mathrm{Sk}(M) : M \in A\}$ is a nicely arranged family in the sense of Definition 3.3 of [1].

Also note that if M and N are in $\mathcal{X}$ and are isomorphic, then $M \cap \omega_1 = N \cap \omega_1$. For in that case $\sigma_{M,N}(\omega_1) = \omega_1$, and therefore $\sigma_{M,N}[M \cap \omega_1] = N \cap \omega_1$. But this implies that $M \cap \omega_1$ and $N \cap \omega_1$ have the same order type and thus are the same ordinal. Consequently, the following are equivalent for M and N in a coherent adequate set: (1) $M \cap \omega_1 = N \cap \omega_1$; (2) $M \cap \beta_{M,N} = N \cap \beta_{M,N}$; (3) M and N are isomorphic.

LEMMA 2.5. *Let A be a coherent adequate set. Let M and K be in A. If $K \cap \beta_{K,M} \in \mathrm{Sk}(M)$, then there is K^* in $A \cap \mathrm{Sk}(M)$ such that K and K^* are isomorphic and $K \cap \beta_{K,M} = K^* \cap \beta_{K,M}$.*

Proof. Since A is coherent, there exists M' in A such that $K \in \mathrm{Sk}(M')$ and M and M' are isomorphic. Let $K^* = \sigma_{M',M}(K)$. Since A is coherent, we have $K^* \in A$. By Lemma 2.1, $\sigma_{M',M} \restriction \mathrm{Sk}(K)$ is an isomorphism of $\mathrm{Sk}(K)$ to $\mathrm{Sk}(K^*)$ and is equal to σ_{K,K^*}. And $\sigma_{M',M}$ is the identity on $M' \cap M = M' \cap \beta_{M,M'} = M \cap \beta_{M,M'}$. Since $K \subseteq M'$, it follows that $\beta_{K,M} \leq \beta_{M',M}$.

As $\sigma_{M',M}{\restriction}M'\cap\beta_{M',M}$ is the identity, $\sigma_{M',M}(K\cap\beta_{K,M})=\sigma_{M',M}[K\cap\beta_{K,M}]$ $= K\cap\beta_{K,M}$. Since $\sigma_{M',M}{\restriction}\,\mathrm{Sk}(K) = \sigma_{K,K^*}$, Lemma 2.2 implies that $K\cap\beta_{K,M}$ is an initial segment of K^*. If γ is in $K^*\setminus K$ and $\gamma<\beta_{K,M}$, then $\gamma<\beta_{M',M}$ implies that $\gamma=\sigma_{M,M'}(\gamma)\in K$, which is a contradiction. So $K\cap\beta_{K,M} = K^*\cap\beta_{K,M}$. ∎

Lemma 2.6. *Suppose that A is a finite coherent adequate set, $N\in\mathcal{X}$, and $A\in\mathrm{Sk}(N)$. Then $A\cup\{N\}$ is a coherent adequate set.*

Proof. If $M\in A$ then since $M\in\mathrm{Sk}(N)$, $M\cap\beta_{M,N}=M$, which is in $\mathrm{Sk}(N)$. So $A\cup\{N\}$ is adequate, and the requirements of being coherent are trivially satisfied. ∎

Lemma 2.7. *Let A be a coherent adequate set and $N\in A$. Then $A\cap\mathrm{Sk}(N)$ is a coherent adequate set.*

Proof. Clearly $A\cap\mathrm{Sk}(N)$ is adequate, and (1) of Definition 2.4 is obvious. (3) is also straightforward. For (2), let M and K be in $A\cap\mathrm{Sk}(N)$ and suppose that $K\cap\beta_{K,M}\in\mathrm{Sk}(M)$. Since A is coherent, there exists M' in A such that $K\in\mathrm{Sk}(M')$ and M and M' are isomorphic. As $M\in\mathrm{Sk}(N)$, $M'\cap\omega_1 = M\cap\omega_1 < N\cap\omega_1$. Hence $M'\cap\beta_{M',N}\in\mathrm{Sk}(N)$. By Lemma 2.5 there exists M^* in $A\cap\mathrm{Sk}(N)$ such that M' and M^* are isomorphic and $M^*\cap\beta_{M',N} = M\cap\beta_{M',N}$. Now $K\in\mathrm{Sk}(M')\cap\mathrm{Sk}(N) = \mathrm{Sk}(M'\cap N) = \mathrm{Sk}(M'\cap\beta_{M',N}) = \mathrm{Sk}(M^*\cap\beta_{M',N})$. So $K\in\mathrm{Sk}(M^*)$, $M^*\in A\cap\mathrm{Sk}(N)$, and M^* and M are isomorphic. ∎

Lemma 2.8. *Let A be a coherent adequate set. Suppose that N, N', and N^* are in A and are isomorphic, where $N'\neq N^*$. Then $\sigma_{N',N}{\restriction}\,\mathrm{Sk}(N'\cap N^*) = \sigma_{N^*,N}{\restriction}(N'\cap N^*)$, and for some $\beta\in N\cap\Lambda$, this function is an isomorphism of $\mathrm{Sk}(N'\cap N^*)$ to $\mathrm{Sk}(N\cap\beta)$. Also $\sigma_{N,N'}{\restriction}\,\mathrm{Sk}(N\cap\beta) = \sigma_{N,N^*}{\restriction}\,\mathrm{Sk}(N\cap\beta)$.*

Proof. By Lemma 2.3, $\sigma_{N',N^*}{\restriction}\,\mathrm{Sk}(N'\cap N^*)$ is the identity function. Also $\sigma_{N',N} = \sigma_{N^*,N}\circ\sigma_{N',N^*}$. So for any $x\in\mathrm{Sk}(N'\cap N^*)$, $\sigma_{N',N}(x) = \sigma_{N^*,N}(\sigma_{N',N^*}(x)) = \sigma_{N^*,N}(x)$. This proves that $\sigma_{N',N}{\restriction}\,\mathrm{Sk}(N'\cap N^*) = \sigma_{N^*,N}{\restriction}(N'\cap N^*)$. Denote this map by σ.

Since $N'\neq N^*$, $N'\cap N^*$ is a proper initial segment of N' and of N^*. By Lemma 2.2, $\sigma[N'\cap N^*]$ is equal to $N\cap\beta$ for some $\beta\in N\cap\Lambda$, and σ is an isomorphism of $\mathrm{Sk}(N'\cap N^*)$ to $\mathrm{Sk}(N\cap\beta)$. The last statement of the lemma follows from the fact that $\sigma_{N,N'}{\restriction}\,\mathrm{Sk}(N\cap\beta)$ and $\sigma_{N,N^*}{\restriction}\,\mathrm{Sk}(N\cap\beta)$ are both the inverse of σ. ∎

3. Amalgamating coherent adequate sets. One of the main methods for preserving cardinals when forcing with models as side conditions is amalgamating conditions over elementary substructures. Proposition 3.5, which handles amalgamation over countable substructures, will be used to prove that the forcing poset in the next section is strongly proper and

hence preserves ω_1. Proposition 3.6 covers amalgamation over models of size ω_1 and will be used to prove that the forcing poset in the next section is ω_2-c.c.

The next four technical lemmas will be used to prove Proposition 3.5.

LEMMA 3.1. *Let M and N be in $\mathcal{X}$ and suppose that M and N are isomorphic. If $\alpha < \gamma$ are in M and $\Lambda \cap [\alpha, \gamma] = \emptyset$, then $\Lambda \cap [\sigma_{M,N}(\alpha), \sigma_{M,N}(\gamma)] = \emptyset$.*

Proof. Suppose for a contradiction that ζ is in $\Lambda \cap [\sigma_{M,N}(\alpha), \sigma_{M,N}(\gamma)]$. Let $\zeta^* = \min(N \setminus \zeta)$. Then $\zeta^* \in N \cap \Lambda \cap [\sigma_{M,N}(\alpha), \sigma_{M,N}(\gamma)]$. Therefore $\sigma_{N,M}(\zeta^*) \in \Lambda \cap [\alpha, \gamma]$, which contradicts $\Lambda \cap [\alpha, \gamma] = \emptyset$. ∎

LEMMA 3.2. *Let M and N be in $\mathcal{X}$. Let $\alpha \leq \gamma$ be ordinals, where $\alpha \in M \cup \lim(M)$ and $\gamma \in N \cup \lim(N)$. If $\Lambda \cap [\alpha, \gamma] = \emptyset$, then $\gamma < \beta_{M,N}$.*

Proof. Let $\beta = \min(\Lambda \setminus \gamma)$. Then $\gamma \leq \sup(N \cap \beta)$, so $\beta = \min(\Lambda \setminus \sup(N \cap \beta))$. Also $\alpha \leq \sup(M \cap \beta)$, and since $\Lambda \cap [\alpha, \gamma] = \emptyset$, $\beta = \min(\Lambda \setminus \sup(M \cap \beta))$. Therefore $\beta \in \Lambda_M \cap \Lambda_N$, which implies that $\beta \leq \beta_{M,N}$. Since γ is not in Λ, it follows that $\gamma < \beta_{M,N}$. ∎

LEMMA 3.3. *Let M, N, K, and P be in $\mathcal{X}$, where M and N are isomorphic and K and P are in* $\mathrm{Sk}(M)$. *Let $\sigma := \sigma_{M,N}$, $K^* := \sigma(K)$, and $P^* = \sigma(P)$. Suppose that $\beta = \min(M \setminus \beta_{K,P})$. Then $\sigma(\beta) = \min(N \setminus \beta_{K^*,P^*})$.*

Proof. Let $\alpha = \sup(K \cap \beta)$ and $\gamma = \sup(P \cap \beta)$. Without loss of generality assume that $\alpha \leq \gamma$. Since α and γ have cofinality ω, they are not in Λ. And as α and γ are in M and below β, we see that α and γ are less than $\beta_{K,P}$. Thus $\alpha = \sup(K \cap \beta_{K,P})$ and $\gamma = \sup(P \cap \beta_{K,P})$.

Since $\beta_{K,P} \in \Lambda_K \cap \Lambda_P$, we have $\beta_{K,P} = \min(\Lambda \setminus \alpha) = \min(\Lambda \setminus \gamma)$. So $\Lambda \cap [\alpha, \gamma] = \emptyset$. By Lemma 3.1 it follows that $\Lambda \cap [\sigma(\alpha), \sigma(\gamma)] = \emptyset$. Since $\sigma(\alpha) \in \lim(K^*)$ and $\sigma(\gamma) \in \lim(P^*)$, Lemma 3.2 implies that $\beta_{K^*,P^*} > \sigma(\gamma)$.

By the definition of β, $\sup(M \cap \beta) < \beta_{K,P}$. Since $\beta_{K,P} = \min(\Lambda \setminus \gamma)$, it follows that for all $\gamma' \in M \cap [\gamma, \beta)$, $\Lambda \cap [\gamma, \gamma'] = \emptyset$. Hence by Lemma 3.1, for all $\gamma^* \in N \cap [\sigma(\gamma), \sigma(\beta))$, $\Lambda \cap [\sigma(\gamma), \gamma^*] = \emptyset$. Therefore $\beta_{K^*,P^*} > \sup(N \cap \sigma(\beta))$.

We will be done if we can show that $\beta_{K^*,P^*} \leq \sigma(\beta)$. Suppose for a contradiction that $\beta_{K^*,P^*} > \sigma(\beta)$. Let $\tau = \sup(K^* \cap \beta_{K^*,P^*})$ and $\xi = \sup(P^* \cap \beta_{K^*,P^*})$. Without loss of generality assume that $\tau \leq \xi$, since the other case follows by a symmetric argument. So $\beta_{K^*,P^*} = \min(\Lambda \setminus \tau) = \min(\Lambda \setminus \xi)$. Since $\beta_{K^*,P^*} > \sigma(\beta)$ and $\sigma(\beta) \in \Lambda$, we find that τ and ξ are greater than $\sigma(\beta)$. Also clearly $\Lambda \cap [\tau, \xi] = \emptyset$. By Lemma 3.1, $\Lambda \cap [\sigma^{-1}(\tau), \sigma^{-1}(\xi)] = \emptyset$. Since $\sigma^{-1}(\tau) \in \lim(K)$ and $\sigma^{-1}(\xi) \in \lim(P)$, Lemma 3.2 implies that $\beta_{K,P} > \sigma^{-1}(\xi)$. But $\xi > \sigma(\beta)$ implies that $\sigma^{-1}(\xi) > \beta$. Hence $\beta_{K,P} > \beta$, which is a contradiction. ∎

LEMMA 3.4. *Let M, N, K, and P be in $\mathcal{X}$. Suppose that M and N are isomorphic, and K and P are in $\mathrm{Sk}(M)$. If $\{K, P\}$ is adequate, then $\{\sigma_{M,N}(K), \sigma_{M,N}(P)\}$ is adequate.*

Proof. Let $\sigma := \sigma_{M,N}$, $K^* := \sigma_{M,N}(K)$, and $P^* := \sigma_{M,N}(P)$. By symmetry it suffices to consider the cases when $K \cap \beta_{K,P} \in \mathrm{Sk}(P)$ and $K \cap \beta_{K,P} = P \cap \beta_{K,P}$. First assume that $\beta_{K,P} \geq \sup(M)$. Then $K \cap \beta_{K,P} = K$ and $P \cap \beta_{K,P} = P$. If $K \cap \beta_{K,P} \in \mathrm{Sk}(P)$, then $K \in \mathrm{Sk}(P)$. Therefore $\sigma(K) \in \sigma(\mathrm{Sk}(P)) = \mathrm{Sk}(\sigma(P))$. Also if $K \cap \beta_{K,P} = P \cap \beta_{K,P}$, then $K = P$, which implies that $\sigma(K) = \sigma(P)$.

Now assume that $\beta_{K,P} < \sup(M)$. Let $\beta := \min(M \setminus \beta_{K,P})$. Then $K \cap \beta = K \cap \beta_{K,P}$ and $P \cap \beta = P \cap \beta_{K,P}$. By Lemma 3.3, $\sigma(\beta) = \min(N \setminus \beta_{K^*,P^*})$. Therefore $K^* \cap \sigma(\beta) = K^* \cap \beta_{K^*,P^*}$ and $P^* \cap \sigma(\beta) = P^* \cap \beta_{K^*,P^*}$.

Suppose that $K \cap \beta_{K,P} \in \mathrm{Sk}(P)$. Then $K \cap \beta \in \mathrm{Sk}(P)$. Therefore $\sigma(K \cap \beta) = K^* \cap \sigma(\beta) \in \sigma(\mathrm{Sk}(P)) = \mathrm{Sk}(P^*)$. So $K^* \cap \beta_{K^*,P^*} \in \mathrm{Sk}(P^*)$. Now suppose that $K \cap \beta_{K,P} = P \cap \beta_{K,P}$. Then $K \cap \beta = P \cap \beta$. So $K^* \cap \sigma(\beta) = \sigma(K \cap \beta) = \sigma(P \cap \beta) = P^* \cap \sigma(\beta)$. Hence $K^* \cap \beta_{K^*,P^*} = P^* \cap \beta_{K^*,P^*}$. ∎

The following proposition describes amalgamation of coherent adequate sets over countable elementary substructures. It will be used to prove that the forcing poset in the next section is strongly proper.

PROPOSITION 3.5. *Let A be a coherent adequate set and $N \in A$. Suppose that B is a coherent adequate set and $A \cap \mathrm{Sk}(N) \subseteq B \subseteq \mathrm{Sk}(N)$. Let C be the set*

$$\{M \in A : N \cap \omega_1 \leq M \cap \omega_1\} \cup \{\sigma_{N,N'}(K) : N' \in A,\ N \cap \omega_1 = N' \cap \omega_1,\ K \in B\}.$$

Then C is a coherent adequate set which contains $A \cup B$.

Proof. First we prove that C is adequate. Obviously, any two sets in $\{M \in A : N \cap \omega_1 \leq M \cap \omega_1\}$ compare properly since A is adequate. Consider $M \in A$ with $N \cap \omega_1 \leq M \cap \omega_1$, and $L = \sigma_{N,N'}(K)$ for some $N' \in A$ with $N \cap \omega_1 = N' \cap \omega_1$ and some $K \in B$. Since $N' \cap \omega_1 = N \cap \omega_1 \leq M \cap \omega_1$, the set $N' \cap \beta_{M,N'}$ is either in $\mathrm{Sk}(M)$ or equal to $M \cap \beta_{M,N'}$. In either case, $\mathrm{Sk}(N' \cap \beta_{M,N'})$ is a subset of $\mathrm{Sk}(M)$. Since $L \subseteq N'$, we have $\beta_{L,M} \leq \beta_{M,N'}$. As L is in $\mathrm{Sk}(N')$, $L \cap \beta_{L,M}$ is in $\mathrm{Sk}(N') \cap \mathrm{Sk}(\beta_{M,N'}) = \mathrm{Sk}(N' \cap \beta_{M,N'})$. Hence $L \cap \beta_{L,M}$ is a member of $\mathrm{Sk}(M)$.

Now consider M and L such that $M = \sigma_{N,N'}(K)$ for some $N' \in A$ with $N \cap \omega_1 = N' \cap \omega_1$ and some $K \in B$, and $L = \sigma_{N,N^*}(P)$ for some $N^* \in A$ with $N \cap \omega_1 = N^* \cap \omega_1$ and some $P \in B$. Since B is adequate, K and P compare properly. If $N' = N^*$, then $\{M, L\}$ is adequate by Lemma 3.4. Suppose $N' \neq N^*$. By symmetry it suffices to consider the cases when $K \cap \beta_{K,P}$ either is in $\mathrm{Sk}(P)$ or is equal to $P \cap \beta_{K,P}$.

The sets N' and N^* are isomorphic, and $N' \cap \beta_{N',N^*} = N^* \cap \beta_{N',N^*} = N' \cap N^*$. By Lemma 2.8, $\sigma_{N',N} \restriction N' \cap N^* = \sigma_{N^*,N} \restriction N' \cap N^*$, and there exists $\beta \in N \cap \Lambda$ such that $N \cap \beta = \sigma_{N',N}[N' \cap N^*]$. Let $\sigma := \sigma_{N,N'} \restriction \mathrm{Sk}(N \cap \beta)$. By Lemma 2.8, $\sigma = \sigma_{N,N^*} \restriction \mathrm{Sk}(N \cap \beta)$ and σ is an isomorphism of $\mathrm{Sk}(N \cap \beta)$ to $\mathrm{Sk}(N' \cap N^*)$. Now $\sigma(K \cap \beta) = \sigma_{N,N'}[K \cap (N \cap \beta)] = \sigma_{N,N'}[K] \cap \sigma_{N,N'}[N \cap \beta] = M \cap (N' \cap \beta_{N',N^*}) = M \cap \beta_{N',N^*}$, and similarly $\sigma(P \cap \beta) = L \cap \beta_{N',N^*}$.

Since $\{K, P\}$ is adequate, so is $\{K \cap \beta, P \cap \beta\}$. By Lemma 3.4, it follows that $\{\sigma(K \cap \beta), \sigma(P \cap \beta)\}$ is adequate. In other words, $\{M \cap \beta_{N',N^*}, L \cap \beta_{N',N^*}\}$ is adequate. Since $M \subseteq N'$, we have $\beta_{L,M} \leq \beta_{L,N'}$, and since $L \subseteq N^*$, it follows that $\beta_{L,N'} \leq \beta_{N',N^*}$. Hence $\beta_{L,M} \leq \beta_{N',N^*}$. Therefore $\{M \cap \beta_{L,M}, L \cap \beta_{L,M}\}$ is adequate. By Lemma 1.3 it follows that $\{M, L\}$ is adequate.

Now we show that $A \cup B \subseteq C$ and C is coherent. This statement follows immediately from Lemmas 3.8 and 3.9 of [1]; we include a proof for completeness. If $K \in B$, then $K = \sigma_{N,N}(K)$ is in C by definition. Let $M \in A$. If $N \cap \omega_1 \leq M \cap \omega_1$, then $M \in C$ by definition. Otherwise $M \cap \omega_1 < N \cap \omega_1$. So there exists $N' \in A$ isomorphic to N such that $M \in \mathrm{Sk}(N')$. Let $K := \sigma_{N',N}(M)$, which is in $A \cap \mathrm{Sk}(N)$ and hence in B. Then $M = \sigma_{N,N'}(K)$ is in C.

Suppose that L and M are in C and $L \cap \omega_1 = M \cap \omega_1$. We will show that L and M are isomorphic. If $M \cap \omega_1 \geq N \cap \omega_1$, then L and M are in A and hence are isomorphic. Otherwise $M = \sigma_{N,N'}(M^*)$ and $L = \sigma_{N,N''}(L^*)$, where M^* and L^* are in B and N' and N'' are in A and are isomorphic to N. Then $M^* \cap \omega_1 = L^* \cap \omega_1$, which implies that M^* and L^* are isomorphic. It follows that M and L are isomorphic.

Assume that L and M are in C and $L \cap \omega_1 < M \cap \omega_1$. We will show that there is M' in C isomorphic to M such that $L \in \mathrm{Sk}(M')$. If $N \cap \omega_1 \leq L \cap \omega_1$, then L and M are in A and we are done. Suppose that $L \cap \omega_1 < N \cap \omega_1 \leq M \cap \omega_1$. Then $L = \sigma_{N,N'}(L^*)$ for some L^* in B and N' in A which is isomorphic to N. Fix M' in A which is isomorphic to M such that N' is either equal to M' or is a member of $\mathrm{Sk}(M')$. Then $L \in \mathrm{Sk}(M')$ and we are done.

Assume that $M \cap \omega_1 < N \cap \omega_1$. Then $L = \sigma_{N,N'}(L^*)$ and $M = \sigma_{N,N''}(M^*)$, where L^* and M^* are in B and N' and N'' are in A and are both isomorphic to N. Since $L^* \cap \omega_1 < M^* \cap \omega_1$, there is M^{**} in B isomorphic to M^* such that $L^* \in \mathrm{Sk}(M^{**})$. Then $\sigma_{N,N'}(M^{**})$ is in C, is isomorphic to M^{**} and hence to M, and its Skolem hull contains L.

Now assume that M, K, and L are in C, $M \cap \omega_1 = K \cap \omega_1$, and $L \in C \cap \mathrm{Sk}(M)$. We will show that $\sigma_{M,K}(L) \in C$. First assume that $N \cap \omega_1 \leq M \cap \omega_1$. Then M and K are in A. If $L \in A$ then we are done. So assume that $L = \sigma_{N,N'}(L^*)$ for some L^* in B and N' in A isomorphic to N. Fix J in

A isomorphic to M such that N' is either equal to J or a member of $\mathrm{Sk}(J)$. Let $N'' := \sigma_{J,M}(N')$ and let $N''' := \sigma_{M,K}(N'')$. Then N'' and N''' are in A. So $\sigma_{N,N'''}(L^*) \in C$. Since L is in $\mathrm{Sk}(J) \cap \mathrm{Sk}(M)$, we have $\sigma_{J,M}(L) = L$. Then $\sigma_{N,N'''}(L^*) = \sigma_{N'',N'''}(\sigma_{N',N''}(\sigma_{N,N'}(L^*))) = \sigma_{N'',N'''}(\sigma_{N',N''}(L)) = \sigma_{M,K}(\sigma_{J,M}(L)) = \sigma_{M,K}(L)$. So $\sigma_{M,K}(L) \in C$.

Finally, assume that $M \cap \omega_1 < N \cap \omega_1$. Then $M = \sigma_{N,N'}(M^*)$, $K = \sigma_{N,N''}(K^*)$, and $L = \sigma_{N,N'''}(L^*)$, where M^*, K^*, and L^* are in B, and N', N'', and N''' are in A and are isomorphic to N. Since $L \in \mathrm{Sk}(M)$, we have $L \in \mathrm{Sk}(N') \cap \mathrm{Sk}(N''')$. So $\sigma_{N',N}(L) = \sigma_{N''',N}(L) = L^*$. Therefore $\sigma_{M,M^*}(L) = \sigma_{N',N}(L) = L^*$. Then $\sigma_{M,K}(L) = \sigma_{K^*,K}(\sigma_{M^*,K^*}(\sigma_{M,M^*}(L))) = \sigma_{K^*,K}(\sigma_{M^*,K^*}(L^*)) = \sigma_{N,N''}(\sigma_{M^*,K^*}(L^*))$. Since $L^* \in B$, $\sigma_{M^*,K^*}(L^*) \in B$. Hence $\sigma_{N,N''}(\sigma_{M^*,K^*}(L^*)) \in C$. So $\sigma_{M,K}(L) \in C$. ∎

The next result describes amalgamation of coherent adequate sets over models of size ω_1. It will be used to show that the forcing poset in the next section is ω_2-c.c.

Proposition 3.6. *Let A be a coherent adequate set and $\beta \in \Lambda$. Let $A^+ := \{M \in A : M \setminus \beta \neq \emptyset\}$ and $A^- := \{M \in A : M \subseteq \beta\}$. Suppose that $\beta^* \in \beta \cap \Lambda$ and for all $M \in A$, $\sup(M \cap \beta) < \beta^*$. Assume that there exists a map $M \mapsto M'$ from A^+ into $\mathcal{X} \cap \mathrm{Sk}(\beta)$ such that for all M and K in A^+:*

(1) *M and M' are isomorphic and $M \cap \beta^* = M' \cap \beta^*$;*
(2) *$K \in \mathrm{Sk}(M)$ iff $K' \in \mathrm{Sk}(M')$;*
(3) *if $K \in \mathrm{Sk}(M)$ then $\sigma_{M,M'}(K) = K'$;*
(4) *$A^- \cup \{M' : M \in A^+\}$ is a coherent adequate set.*

Then $C := A \cup \{M' : M \in A^+\}$ is a coherent adequate set.

Proof. Note that by assumption (1), $\sigma_{M,M'} \restriction \beta^*$ is the identity function for all $M \in A^+$. Let us begin by proving that C is adequate. Note that if $M \in A^+$, then M and M' have the same order type, which is larger than the order type of $M \cap \beta^* = M' \cap \beta^*$; it follows that $M' \setminus \beta^*$ is nonempty. Therefore C is the union of the three disjoint sets A^-, A^+, and $\{M' : M \in A^+\}$. By (4) and the fact that A is adequate, it suffices to compare a set in A^+ with a set in $\{M' : M \in A^+\}$.

Let K and M be in A^+, and let us compare K and M'. Since $M' \subseteq \beta$, $\beta_{K,M'} \leq \beta$ by Lemma 1.2(1). Hence $\beta_{K,M'} = \beta_{K \cap \beta, M'}$ by Lemma 1.2(3). But $K \cap \beta = K \cap \beta^*$, which implies by Lemma 1.2(1, 4) that $\beta_{K,M'} = \beta_{K \cap \beta, M'} = \beta_{K \cap \beta^*, M' \cap \beta^*} \leq \beta^*$. Also $K \cap \beta^* = K' \cap \beta^*$ and $M' \cap \beta^* = M \cap \beta$. Now $\beta_{K,M'} = \beta_{K \cap \beta^*, M' \cap \beta^*}$, and since $K \cap \beta^* \subseteq K$ and $M' \cap \beta^* \subseteq M$, it follows that $\beta_{K,M'} \leq \beta_{K,M}$.

We split into cases depending on the comparison of K and M. Suppose that $K \cap \beta_{K,M} \in \mathrm{Sk}(M)$. Since $\beta_{K,M'} \leq \beta^*, \beta_{K,M}$, it follows that $K \cap \beta_{K,M'} \in \mathrm{Sk}(M) \cap \mathrm{Sk}(\beta^*) = \mathrm{Sk}(M \cap \beta^*) = \mathrm{Sk}(M' \cap \beta^*)$. Therefore $K \cap \beta_{K,M'} \in \mathrm{Sk}(M')$.

Now assume that $M \cap \beta_{K,M} \in \mathrm{Sk}(K)$. As $\beta_{K,M'} \leq \beta_{K,M}$, we have $M \cap \beta_{K,M'} \in \mathrm{Sk}(K)$. But $\beta_{K,M'} \leq \beta^*$ implies that $M \cap \beta_{K,M'} = M' \cap \beta_{K,M'}$. So $M' \cap \beta_{K,M'} \in \mathrm{Sk}(K)$. Finally, assume that $K \cap \beta_{K,M} = M \cap \beta_{K,M}$. Since $\beta_{K,M'} \leq \beta_{K,M}$, we have $K \cap \beta_{K,M'} = M \cap \beta_{K,M'}$. But $\beta_{K,M'} \leq \beta^*$, so $M \cap \beta_{K,M'} = M' \cap \beta_{K,M'}$. Hence $K \cap \beta_{K,M'} = M' \cap \beta_{K,M'}$.

Now we show that C is coherent. Recall that A is the union of the three disjoint sets A^+, A^-, and $\{M' : M \in A^+\}$. The union of the first and second set is equal to A, which is coherent, and the union of the second and third set is coherent by (4). Note that requirements (1) and (2) in the definition of coherence follow immediately from these facts, except for the case of a pair of models where one is in A^+ and the other is in $\{M' : M \in A^+\}$.

Let K and M be in A^+; we verify requirements (1) and (2) for K and M'. Suppose that $K \cap \beta_{K,M'} = M' \cap \beta_{K,M'}$. Then $K \cap \omega_1 = M \cap \omega_1$. Since A is coherent, K and M are isomorphic. Hence K and M' are isomorphic.

Suppose that $K \cap \beta_{K,M'} \in \mathrm{Sk}(M')$. Then $K \cap \omega_1 < M \cap \omega_1$, so $K \cap \beta_{K,M} \in \mathrm{Sk}(M)$. So there exists M^* in A such that $K \in \mathrm{Sk}(M^*)$ and M and M^* are isomorphic. Hence M^* and M' are isomorphic. Now assume that $M' \cap \beta_{K,M'} \in \mathrm{Sk}(K)$. Then $M' \cap \omega_1 < K' \cap \omega_1$, so $M' \cap \beta_{K',M'} \in \mathrm{Sk}(K')$. Since $A^- \cup \{L' : L \in A^+\}$ is coherent, there is K^* in C such that $M' \in \mathrm{Sk}(K^*)$ and K^* and K' are isomorphic. Then K^* and K are isomorphic.

Now we prove that requirement (3) holds of C. Let M_1 and M_2 be in C with $M_1 \cap \beta_{M_1,M_2} = M_2 \cap \beta_{M_1,M_2}$, and let $K \in C \cap \mathrm{Sk}(M_1)$. We will prove that $\sigma_{M_1,M_2}(K)$ is in C. Note that if M_1 and M_2 are both in A, then so is K, and if M_1 and M_2 are both in $A^- \cup \{M' : M \in A^+\}$, then so is K. Since A and $A^- \cup \{M' : M \in A^+\}$ are both coherent, we are done in these cases. So again it suffices to prove (3) in the case of two sets where one is in A^+ and the other is in $\{M' : M \in A^+\}$.

Assume that M_1 is in A^+ and $M_2 = M'$ for some $M \in A^+$. Then M_1 and M are isomorphic. Since $K \in \mathrm{Sk}(M_1)$ we have $K \cap \beta \subseteq \beta^*$, and hence K is in A. As A is coherent, $P := \sigma_{M_1,M}(K) \in A \cap \mathrm{Sk}(M)$. If $P \in A^-$, then since $\sigma_{M,M'} \restriction \beta^*$ is the identity, $\sigma_{M,M'}(P) = P$. Hence $\sigma_{M_1,M'}(K) = \sigma_{M,M'}(P) = P$ is in A. Otherwise $P \in A^+$, and by assumption (3), $\sigma_{M,M'}(P) = P'$. So $\sigma_{M_1,M'}(K) = \sigma_{M,M'}(\sigma_{M_1,M}(K)) = \sigma_{M,M'}(P) = P' \in C$.

In the last case assume that $M_1 = M'$ for some $M \in A^+$ and $M_2 \in A^+$. Since $K \in \mathrm{Sk}(M')$, we have $K \subseteq \beta$, so K is not in A^+. Suppose that K is in A^-. Then K is a subset of β^*, so $\sigma_{M',M}(K) = K$. Hence K is in $\mathrm{Sk}(M) \cap A$, and therefore $\sigma_{M,M_2}(K) \in A$ since A is coherent. But $\sigma_{M',M_2}(K) = \sigma_{M,M_2}(\sigma_{M',M}(K)) = \sigma_{M,M_2}(K) \in C$. Otherwise K is equal to P' for some $P \in A^+$. So $P' \in \mathrm{Sk}(M')$. By assumptions (3) and (4), $P \in \mathrm{Sk}(M)$ and $\sigma_{M,M'}(P) = P'$. Since P is in A and A is coherent, we have $\sigma_{M,M_2}(P) \in A$. So $\sigma_{M',M_2}(K) = \sigma_{M',M_2}(P') = \sigma_{M',M_2}(\sigma_{M,M'}(P)) = \sigma_{M,M_2}(P) \in C$. ∎

4. Forcing square with finite conditions. We define a forcing poset which adds a square sequence with finite conditions, using coherent adequate sets as side conditions.

By a *triple* we mean a sequence $\langle \alpha, \gamma, \beta \rangle$, where $\alpha \in \Lambda$ and $\gamma < \beta < \alpha$. Given distinct triples $\langle \alpha, \gamma, \beta \rangle$ and $\langle \alpha', \gamma', \beta' \rangle$, we say that they are *nonoverlapping* if either $\alpha \neq \alpha'$, or $\alpha = \alpha'$ and $[\gamma, \beta) \cap [\gamma', \beta') = \emptyset$; otherwise they are *overlapping*. Given a triple $\langle \alpha, \gamma, \beta \rangle$ and $M \in \mathcal{X}$, we say that $\langle \alpha, \gamma, \beta \rangle$ and M are *nonoverlapping* if $\alpha \in M$ implies that either γ and β are in M or $\sup(M \cap \alpha) < \gamma$; otherwise they are *overlapping*.

Clearly if M and N are isomorphic and a and b are nonoverlapping triples in $\mathrm{Sk}(M)$, then $\sigma_{M,N}(a)$ and $\sigma_{M,N}(b)$ are nonoverlapping triples. And if $K \in \mathrm{Sk}(M) \cap \mathcal{X}$ and a and K are nonoverlapping, then $\sigma_{M,N}(a)$ and $\sigma_{M,N}(K)$ are nonoverlapping.

Definition 4.1. Let $\mathbb{P}$ be the forcing poset whose conditions are pairs (x, A) satisfying:

(1) x is a finite pairwise nonoverlapping set of triples;
(2) A is a finite coherent adequate set;
(3) for all $M \in A$ and $\langle \alpha, \gamma, \beta \rangle \in x$, M and $\langle \alpha, \gamma, \beta \rangle$ are nonoverlapping;
(4) if M and M' are in A and $M \cap \beta_{M,M'} = M' \cap \beta_{M,M'}$, then for any triple $\langle \alpha, \gamma, \beta \rangle \in \mathrm{Sk}(M) \cap x$ we have $\sigma_{M,M'}(\langle \alpha, \gamma, \beta \rangle) \in x$.

Let $(y, B) \leq (x, A)$ if $x \subseteq y$ and $A \subseteq B$.

If $p = (x, A)$, we write $x_p := x$ and $A_p := A$.

We will prove that $\mathbb{P}$ preserves all cardinals. For each $\alpha \in \Lambda$, let $\dot{c}_\alpha$ be a $\mathbb{P}$-name for the set

$$\{\gamma : \exists p \in \dot{G}\ \exists \beta\ (\langle \alpha, \gamma, \beta \rangle \in x_p)\}.$$

We will show that each $\dot{c}_\alpha$ is a cofinal subset of α with order type ω_1, and whenever ξ is a common limit point of $\dot{c}_\alpha$ and $\dot{c}_{\alpha'}$, $\dot{c}_\alpha \cap \xi = \dot{c}_{\alpha'} \cap \xi$.

Lemma 4.2. *Let A be a coherent adequate set and x a set of triples. Let y be the set*

$$x \cup \{\sigma_{M,M'}(a) : M, M' \in A,\ M \cap \omega_1 = M' \cap \omega_1,\ a \in x \cap \mathrm{Sk}(M)\}.$$

Then for all N and N' in A which are isomorphic and any $a \in y$, $\sigma_{N,N'}(a) \in y$.

Proof. Let N and N' be isomorphic sets in A, and $a \in y$. If $a \in x$, then $\sigma_{N,N'}(a) \in y$ by definition. Otherwise there are M and M' in A which are isomorphic and b in x such that $a = \sigma_{M,M'}(b)$. So a is in $\mathrm{Sk}(M') \cap \mathrm{Sk}(N) = \mathrm{Sk}(M' \cap N)$.

First assume that $M' \cap \beta_{M',N} \in \mathrm{Sk}(N)$. By Lemma 2.5 there is M^* in $\mathrm{Sk}(N)$ which is isomorphic to M' such that $M' \cap \beta_{M',N} = M^* \cap \beta_{M',N}$. In particular, $a \in \mathrm{Sk}(M' \cap N) = \mathrm{Sk}(M' \cap \beta_{M',N}) = \mathrm{Sk}(M^* \cap \beta_{M',M})$. By

Lemma 2.8, $\sigma_{M^*,M}(a) = \sigma_{M',M}(a) = b$. So $\sigma_{M,M^*}(b) = \sigma_{M,M'}(b) = a$. Let $P := \sigma_{N,N'}(M^*)$. Then $\sigma_{N,N'} \restriction \mathrm{Sk}(M^*) = \sigma_{M^*,P}$. By Lemma 2.8, $\sigma_{M',P} \restriction \mathrm{Sk}(M' \cap M^*) = \sigma_{M^*,P} \restriction \mathrm{Sk}(M' \cap M^*)$. Hence $\sigma_{M',P}(a) = \sigma_{M^*,P}(a) = \sigma_{N,N'}(a)$. So $\sigma_{M,P}(b) = \sigma_{M^*,P}(\sigma_{M,M^*}(b)) = \sigma_{M^*,P}(a) = \sigma_{N,N'}(a)$. Since $b \in x$, we have $\sigma_{M,P}(b) \in y$ by definition. So $\sigma_{N,N'}(a) \in y$.

Now suppose that $M' \cap \beta_{M',N} = N \cap \beta_{M',N}$. Then by Lemma 2.8, $\sigma_{M',N'} \restriction \mathrm{Sk}(M' \cap N) = \sigma_{N,N'} \restriction \mathrm{Sk}(M' \cap N)$. Since a is in $\mathrm{Sk}(M' \cap N)$, we have $\sigma_{N,N'}(a) = \sigma_{M',N'}(a) = \sigma_{M',N'}(\sigma_{M,M'}(b)) = \sigma_{M,N'}(b)$, which is in y since $b \in x$.

Finally assume that $N \cap \beta_{M',N} \in \mathrm{Sk}(M')$. Fix $N^* \in \mathrm{Sk}(M')$ which is isomorphic to N such that $N \cap \beta_{M',N} = N^* \cap \beta_{M',N}$. Let $L := \sigma_{M',M}(N^*)$. Then $a \in \mathrm{Sk}(M' \cap N) = \mathrm{Sk}(N \cap \beta_{M',N}) = \mathrm{Sk}(N^* \cap \beta_{M',N})$, so $a \in \mathrm{Sk}(N^*)$. Also $\sigma_{M',M} \restriction N^* = \sigma_{N^*,L}$. Hence $\sigma_{N^*,L}(a) = \sigma_{M',M}(a) = b$. By Lemma 2.8, $\sigma_{N,N'} \restriction \mathrm{Sk}(N \cap N^*) = \sigma_{N^*,N'} \restriction \mathrm{Sk}(N \cap N^*)$. Therefore $\sigma_{N,N'}(a) = \sigma_{N^*,N'}(a)$. So $\sigma_{N,N'}(a) = \sigma_{N^*,N'}(a) = \sigma_{N^*,N'}(\sigma_{M,M'}(b)) = \sigma_{N^*,N'}(\sigma_{L,N^*}(b)) = \sigma_{L,N'}(b)$, which is in y since $b \in x$. ∎

Recall that a forcing poset $\mathbb{Q}$ is *strongly proper* if for all sufficiently large regular cardinals θ with $\mathbb{Q} \in H(\theta)$, there are club many sets N in $P_{\omega_1}(H(\theta))$ such that for all $p \in N \cap \mathbb{Q}$ there exists $q \leq p$ which is *strongly N-generic*, which means that for any dense subset D of the forcing poset $\mathbb{Q} \cap N$, D is predense below q in $\mathbb{Q}$ ([5]). Strong properness implies properness, which in turn implies that ω_1 is preserved.

Proposition 4.3. *The forcing poset $\mathbb{P}$ is strongly proper.*

Proof. Fix a regular cardinal $\theta > \omega_2$, and let N^* be a countable elementary substructure of $H(\theta)$ such that $\mathbb{P}$ and π are in N^* and $N := N^* \cap \omega_2 \in \mathcal{X}$. Clearly there are club many such sets N^*. Note that since $\pi \in N^*$, $\mathrm{Sk}(N) = \pi[N] = N^* \cap H(\omega_2)$. In particular, $\mathbb{P} \cap N^* \subseteq \mathrm{Sk}(N)$.

Let p be a condition in $N^* \cap \mathbb{P}$. Define $q = (x_p, A_p \cup \{N\})$. Then q is a condition and $q \leq p$. We will prove that q is strongly N^*-generic. So let D be a dense subset of $N^* \cap \mathbb{P}$; we will show that D is predense below q.

Fix $r \leq q$; we will find a condition w in D which is compatible with r. Since $N \in A_r$, $A_r \cap \mathrm{Sk}(N)$ is a coherent adequate set by Lemma 2.7. Let $v = (x_r \cap \mathrm{Sk}(N), A_r \cap \mathrm{Sk}(N))$. Then v is a condition in $\mathbb{P}$. Since D is dense in $N^* \cap \mathbb{P}$, we can fix w which is an extension of v in D. Then $A_r \cap \mathrm{Sk}(N) \subseteq A_w \subseteq \mathrm{Sk}(N)$.

Let C be the set

$$\{M \in A_r : N \cap \omega_1 \leq M \cap \omega_1\} \cup \{\sigma_{N,N'}(K) : N' \in A_r,\ N \cap \omega_1 = N' \cap \omega_1,\ K \in A_w\}.$$

By Proposition 3.5, C is a coherent adequate set which contains $A_r \cup A_w$.

Let y be the set

$$(x_r \setminus \mathrm{Sk}(N)) \cup \{\sigma_{N,N'}(a) : N' \in A_r,\ N \cap \omega_1 = N' \cap \omega_1,\ a \in x_w\}.$$

Let $s := (y, C)$.

We claim that s is a condition and $s \leq r, w$, which completes the proof since w is in D. If a is in x_w, then $\sigma_{N,N}(a) = a$ is in y. And if a is in x_r, then either a is in $x_r \setminus \mathrm{Sk}(N)$, and hence is in y by definition, or else a is in x_w, and hence is in y as just noted. So x_r and x_w are subsets of y. Also A_r and A_w are subsets of C. Thus if s is a condition then $s \leq r, w$.

(1) We show that y is a set of nonoverlapping triples. So let a_0 and a_1 be in y. Let $a_0 = \langle \alpha_0, \gamma_0, \beta_0 \rangle$ and $a_1 = \langle \alpha_1, \gamma_1, \beta_1 \rangle$. If $\alpha_0 \neq \alpha_1$, then a_0 and a_1 are nonoverlapping, so assume that $\alpha_0 = \alpha_1$. If a_0 and a_1 are both in $x_r \setminus \mathrm{Sk}(N)$, then they are nonoverlapping since r is a condition.

Suppose that $a_0 \in x_r \setminus \mathrm{Sk}(N)$ and $a_1 = \sigma_{N,N'}(a)$ for some $a \in x_w$ and N' in A_r which is isomorphic to N. Since $\alpha_0 \in N'$, either γ_0 and β_0 are in N', or $\sup(N' \cap \alpha_0) < \gamma_0$. In the latter case, $\beta_1 < \gamma_0$ and hence a_0 and a_1 are nonoverlapping. In the former case, a_0 is in $\mathrm{Sk}(N') \cap x_r$. Hence $a^* := \sigma_{N',N}(a_0)$ is in $\mathrm{Sk}(N) \cap x_r \subseteq x_w$. So a^* and a are nonoverlapping. Therefore their images under $\sigma_{N,N'}$, namely a_0 and a_1, are nonoverlapping.

Now suppose that $a_0 = \sigma_{N,N'}(a_0^*)$ and $a_1 = \sigma_{N,N^*}(a_1^*)$, where a_0^* and a_1^* are in x_w and N' and N^* are isomorphic in A_w. If $N' = N^*$, then since a_0^* and a_1^* are nonoverlapping, so are their images under σ_{N,N^*}, namely a_0 and a_1. Suppose $N \neq N'$. By Lemma 2.8, fix $\beta \in N \cap \Lambda$ such that $\sigma_{N,N'} \restriction \mathrm{Sk}(N \cap \beta) = \sigma_{N,N^*} \restriction \mathrm{Sk}(N \cap \beta)$ is an isomorphism of $N \cap \beta$ to $N' \cap N^*$. But $\alpha_0 = \alpha_1$ implies that $\beta_{N',N^*} > \alpha_0$. Hence a_0 and a_1 are in $\mathrm{Sk}(N' \cap N^*)$. Since a_0^* and a_1^* are nonoverlapping, their images under $\sigma_{N,N'} \restriction \mathrm{Sk}(N \cap \beta)$, namely a_0 and a_1, are also nonoverlapping.

(2) We already noted that C is a finite coherent adequate set.

(3) Let M be in C and a in y; we will show that M and a are nonoverlapping. If $M \cap \omega_1 \geq N \cap \omega_1$ and a is in $x_r \setminus \mathrm{Sk}(N)$, then we are done since r is a condition. Let $a = \langle \alpha, \gamma, \beta \rangle$. If $\alpha \notin M$, then a and M are nonoverlapping, so assume that $\alpha \in M$. We will show that either γ and β are in M or $\sup(M \cap \alpha) < \gamma$.

Suppose that $M \cap \omega_1 \geq N \cap \omega_1$ and $a = \sigma_{N,N'}(a^*)$ for some N' in A_r isomorphic to N and some a^* in x_w. Since $M \cap \omega_1 \geq N' \cap \omega_1$, either $N' \cap \beta_{N',M} \in \mathrm{Sk}(M)$ or $N' \cap \beta_{N',M} = M \cap \beta_{N',M}$. But $\alpha \in M \cap N'$, so $\beta_{N',M} > \alpha$. Thus γ and β are in $N \cap \beta_{N',M}$ and hence in M.

Assume that $M = \sigma_{N,N'}(K)$, where $N' \in A_r$ is isomorphic to N and $K \in A_w$, and $a \in x_r \setminus \mathrm{Sk}(N)$. Since $M \subseteq N'$, it follows that $\alpha \in N'$. So either γ and β are in N' or $\sup(N' \cap \alpha) < \gamma$. In the latter case, clearly $\sup(M \cap \alpha) < \gamma$ and we are done. Otherwise a is a member of $\mathrm{Sk}(N')$. So

$b := \sigma_{N',N}(a) \in x_r \cap \mathrm{Sk}(N) \subseteq x_w$. Therefore K and b are nonoverlapping. Hence their images under $\sigma_{N,N'}$, namely M and a, are nonoverlapping.

In the final case, suppose that $M = \sigma_{N,N'}(K)$, where $N' \in A_r$ is isomorphic to N and $K \in A_w$, and $a = \sigma_{N,N^*}(b)$ for some N^* in A_r isomorphic to N and some b in x_w. So K and b are nonoverlapping. If $N' = N^*$, then the images of K and b under $\sigma_{N,N'}$, namely M and a, are nonoverlapping. Otherwise by Lemma 2.8 we can fix $\beta \in N \cap \Lambda$ such that $\sigma_{N,N'} \restriction \mathrm{Sk}(N \cap \beta) = \sigma_{N,N^*} \restriction \mathrm{Sk}(N \cap \beta)$ is an isomorphism of $N \cap \beta$ to $N' \cap N^*$. As $\alpha \in M$, α is in $N' \cap N^*$. Since $N' \cap N^*$ is an initial segment of N' and N^*, it follows that $a \in \mathrm{Sk}(N' \cap N^*)$. Hence b is in $\mathrm{Sk}(N \cap \beta)$. Therefore $a = \sigma_{N,N^*}(b) = \sigma_{N,N'}(b)$. So a and M are the images of b and K under $\sigma_{N,N'}$, and b and K are nonoverlapping. Thus a and M are nonoverlapping.

(4) By Lemma 4.2 it suffices to show that y is equal to the set

$$x_r \cup x_w \cup \{\sigma_{M,M'}(a) : M, M' \in C,\ M \cap \omega_1 = M' \cap \omega_1,\ a \in (x_r \cup x_w) \cap \mathrm{Sk}(M)\}.$$

Clearly y is a subset of this set. It was noted above that $x_r \cup x_w \subseteq y$. Suppose that M and M' are isomorphic sets in C and $a \in (x_r \cup x_w) \cap \mathrm{Sk}(M)$. We will show that $a^* := \sigma_{M,M'}(a) \in y$.

Suppose that $M \cap \omega_1 > N \cap \omega_1$. Then also $M' \cap \omega_1 > N \cap \omega_1$. If a is in x_r, then we are done since r is a condition. Suppose that a is in x_w. Fix N^* in $\mathrm{Sk}(M)$ which is isomorphic to N and such that $N \cap \beta_{M,N} = N^* \cap \beta_{M,N}$. Then $a \in \mathrm{Sk}(N \cap \beta_{M,N}) = \mathrm{Sk}(N^* \cap \beta_{M,N})$. Let $P := \sigma_{M,M'}(N^*)$. So $\sigma_{M,M'} \restriction \mathrm{Sk}(N^*) = \sigma_{N^*,P}$. By Lemma 2.8, $\sigma_{M,M'}(a) = \sigma_{N^*,P}(a) = \sigma_{N,P}(a)$, which is in y by definition.

Now assume that $M \cap \omega_1 = N \cap \omega_1$. Then M, M', and N are all isomorphic. If $a \in x_r$ then we are done since r is a condition. Suppose that $a \in x_w$. Since $a \in \mathrm{Sk}(M) \cap \mathrm{Sk}(N) = \mathrm{Sk}(M \cap N)$, by Lemma 2.8 we have $\sigma_{M,M'}(a) = \sigma_{N,M'}(a)$, which is in y by definition.

Finally, suppose that $M \cap \omega_1 < N \cap \omega_1$. By the definition of C, $M = \sigma_{N,N'}(K)$ for some N' in A_r which is isomorphic to N and some $K \in A_w$. Then also $M' = \sigma_{N,N^*}(P)$ for some N^* in A_r which is isomorphic to N and some $P \in A_w$. Since a is in $\mathrm{Sk}(M)$, a is in $\mathrm{Sk}(N')$. We claim that $b := \sigma_{N',N}(a)$ is in x_w. If $a \in x_r$, then since r is a condition, b is in $x_r \cap \mathrm{Sk}(N)$ and hence in x_w. Otherwise a is in x_w and hence in $\mathrm{Sk}(N') \cap \mathrm{Sk}(N) = \mathrm{Sk}(N' \cap N)$. But $\sigma_{N',N} \restriction \mathrm{Sk}(N' \cap N)$ is the identity, so $b = a$.

We see that $\sigma_{N',N} \restriction \mathrm{Sk}(M) = \sigma_{M,K}$ and $\sigma_{N^*,N} \restriction \mathrm{Sk}(M') = \sigma_{M',P}$. And $\sigma_{M,M'} = \sigma_{P,M'} \circ \sigma_{K,P} \circ \sigma_{M,K} = \sigma_{N,N^*} \circ \sigma_{K,P} \circ (\sigma_{N',N} \restriction \mathrm{Sk}(M))$. So $\sigma_{M,M'}(a) = \sigma_{N,N^*}(\sigma_{K,P}(\sigma_{N',N}(a))) = \sigma_{N,N^*}(\sigma_{K,P}(b))$. Since $b \in x_w$ and K and P are in A_w, $\sigma_{K,P}(b)$ is in x_w. Hence $\sigma_{M,M'}(a) = \sigma_{N,N^*}(\sigma_{K,P}(b))$ is in y by definition. ∎

PROPOSITION 4.4. *The forcing poset $\mathbb{P}$ is ω_2-c.c.*

Proof. Fix $\theta > \omega_2$ regular and let N^* be an elementary substructure of $H(\theta)$ of size ω_1 such that π, $\mathcal{X}$, Λ, and $\mathbb{P}$ are in N^* and $\beta := N^* \cap \omega_2 \in \Lambda$. Since $\pi \in N^*$, we have $N^* \cap H(\omega_2) = \pi[N^* \cap \omega_2] = \pi[\beta] = \mathrm{Sk}(\beta)$. In particular, $N^* \cap \mathbb{P} \subseteq \mathrm{Sk}(\beta)$. Note that since $\mathcal{X} \cap P(\beta) \subseteq \mathrm{Sk}(\beta)$, it follows that $N^* \cap \mathcal{X} = P(\beta) \cap \mathcal{X} = \mathrm{Sk}(\beta) \cap \mathcal{X}$.

We will prove that the empty condition is N^*-generic. This implies that $\mathbb{P}$ is ω_2-c.c. by the following argument. Suppose for a contradiction that $\mathbb{P}$ has a maximal antichain S of size at least ω_2. By elementarity we may assume that S is in N^*. Since N^* has size ω_1, we can fix a condition $s \in S \setminus N^*$. Let D be the set of conditions which are below some member of S. Then D is dense and lies in N^*. Since the empty condition is N^*-generic, $N^* \cap D$ is predense in $\mathbb{P}$. So s is compatible with some member of $N^* \cap D$. By elementarity and the definition of D, s is compatible with some member of $N^* \cap S$, which contradicts the assumption that S is an antichain.

Note that since $2^\omega = \omega_1$ and $\omega_1 \subseteq N^*$, we have $H(\omega_1) \subseteq N^*$. Fix a dense open set D in N^*; we will show that $D \cap N^*$ is predense in $\mathbb{P}$. Let p be a given condition. Extend p to q which is in D.

Let $A^- := \{M \in A_q : M \subseteq \beta\}$. Let $A^+ := \{M \in A_q : M \setminus \beta \neq \emptyset\} = \{M_0, \dots, M_k\}$. Since $\Lambda \in N^*$, the set $\Lambda \cap \beta$ is cofinal in β. Fix β^* in $\Lambda \cap \beta$ such that for all $M \in A_q$, $\sup(M \cap \beta) < \beta^*$, and for all $\langle \alpha, \gamma, \zeta \rangle$ in $x_r \cap \mathrm{Sk}(\beta)$, $\alpha < \beta^*$. Let R be the set of pairs $\langle i, j \rangle$ in $k+1$ such that $M_i \in \mathrm{Sk}(M_j)$. Note that the objects A^-, $M_0 \cap \beta, \dots, M_k \cap \beta$, β^*, and R are in N^*.

For each $i = 0, \dots, k$, let $\overline{\mathfrak{M}}_i$ denote the transitive collapse of the structure $\mathfrak{M}_i = (\mathrm{Sk}(M_i), \in, \pi_{M_i}, \mathcal{X}_{M_i}, \Lambda_{M_i})$. And for each $\langle i, j \rangle$ in R, let $J_{\langle i,j \rangle} := \sigma_{M_j}(M_i)$. Note that each $\overline{\mathfrak{M}}_i$ is in $H(\omega_1)$ and hence in N^*, and therefore each $J_{\langle i,j \rangle}$ is in N^*.

Let $a_0, \dots, a_m$ enumerate the triples in x_q whose first component is larger than β. Let S be the set of pairs $\langle i, j \rangle$ where $i \leq m$, $j \leq k$, and $a_i \in \mathrm{Sk}(M_j)$. For each $\langle i, j \rangle$ in S, let $b_{\langle i,j \rangle} = \sigma_{M_j}(a_i)$.

As noted above, the following parameters all belong to N^*: $x_q \cap \mathrm{Sk}(\beta)$, A^-, D, $M_0 \cap \beta, \dots, M_k \cap \beta$, π, $\mathcal{X}$, Λ, $\mathfrak{M}_0, \dots, \mathfrak{M}_k$, R, $J_{\langle i,j \rangle}$ for each $\langle i, j \rangle \in R$, β^*, S, and $b_{\langle i,j \rangle}$ for each $\langle i, j \rangle \in S$. Let $\varphi_{x_0,\dots,x_k,y_0,\dots,y_m}$ be the formula in the language of set theory with constants for these parameters which expresses the following:

(i) the pair
$$\big((x_q \cap \mathrm{Sk}(\beta)) \cup \{y_0, \dots, y_m\}, A^- \cup \{x_0, \dots, x_k\}\big)$$
is in D;

(ii) for each $i = 0, \dots, k$, $x_i \cap \beta^* = M_i \cap \beta$;

(iii) for each $i = 0, \dots, k$, the transitive collapse of $(\mathrm{Sk}(x_i), \in, \pi_{x_i}, \mathcal{X}_{x_i}, \Lambda_{x_i})$ is equal to $\overline{\mathfrak{M}}_i$;

(iv) for any $i, j < k+1$, $x_i \in \mathrm{Sk}(x_j)$ iff $\langle i, j\rangle \in R$, and in that case, $\sigma_{x_j}(x_i) = J_{\langle i,j\rangle}$;
(v) for each $i = 0, \dots, m$, the first component of y_i is above β^*;
(vi) for each $i \le m$ and $j \le k$, $y_i \in \mathrm{Sk}(x_j)$ iff $\langle i, j\rangle \in S$, and in that case, $\sigma_{x_j}(y_i) = b_{\langle i,j\rangle}$.

Note that $H(\theta) \models \varphi[M_0, \dots, M_k, a_0, \dots, a_m]$. By elementarity we can find $M'_0, \dots, M'_k$ and $a'_0, \dots, a'_m$ in N^* such that $H(\theta) \models \varphi[M'_0, \dots, M'_k, a'_0, \dots, a'_m]$.

Let w denote the pair

$$\big((x_q \cap \mathrm{Sk}(\beta)) \cup \{a'_0, \dots, a'_m\}, A^- \cup \{M'_0, \dots, M'_k\}\big).$$

Then w is in D by (i).

Let us verify that the assumptions of Proposition 3.6 hold for the map which sends M to M' for each $M \in A^+$. Let M and K be in A^+. Then (iii) implies that $\mathfrak{M}$ and $\mathfrak{M}'$ have the same transitive collapse and hence are isomorphic, and (ii) implies that $M' \cap \beta^* = M \cap \beta = M \cap \beta^*$. Let $M = M_j$ and $K = M_i$ for $i, j \le k$. By (iv), $K \in \mathrm{Sk}(M)$ iff $\langle i, j\rangle \in R$ iff $K' \in \mathrm{Sk}(M')$, and in that case, $\sigma_M(K) = J_{\langle i,j\rangle}$ by definition and $\sigma_{M'}(K') = J_{\langle i,j\rangle}$ by (iv). But $\sigma_{M,M'} = \sigma_{M'}^{-1} \circ \sigma_M$. So $\sigma_{M,M'}(K) = \sigma_{M'}^{-1}(\sigma_M(K)) = \sigma_{M'}^{-1}(J_{\langle i,j\rangle}) = K'$. Finally, $A^- \cup \{M'_0, \dots, M'_k\}$ is a coherent adequate set by (i). It follows by Proposition 3.6 that

$$C := A_q \cup \{M' : M \in A^+\}$$

is a coherent adequate set.

By (vi), for each $i \le m$ and $j \le k$, $a_i \in \mathrm{Sk}(M_j)$ iff $\langle i, j\rangle \in J$ iff $a'_i \in \mathrm{Sk}(M'_j)$. Also, if $a_i \in \mathrm{Sk}(M_j)$, then $\sigma_{M_j,M'_j}(a_i) = \sigma_{M_j^{-1}}(\sigma_{M_j}(a_i)) = \sigma_{M_j^{-1}}(b_{\langle i,j\rangle}) = a'_j$. So $\sigma_{M_i,M'_i}(a_j) = a'_j$. Let

$$y := x_q \cup \{a'_j : j = 0, \dots, m\}.$$

By (v) the first component of each a'_j is above β^*. Hence any element of y is in $x_q \cap \mathrm{Sk}(\beta)$, $\{a'_j : j = 0, \dots, m\}$, or $x_q \setminus \mathrm{Sk}(\beta)$ depending on whether its first component is in $[0, \beta^*)$, $[\beta^*, \beta)$, or $[\beta^*, \omega_2)$.

We claim that $s = (y, C)$ is a condition. Then clearly $s \le r, w$, and since w is in D, we are done.

(1) Let $\langle \alpha, \gamma, \zeta\rangle$ and $\langle \alpha', \gamma', \zeta'\rangle$ be in y; we will show that they are nonoverlapping. If these triples are either both in x_q or both in x_w, then we are done. Otherwise we may assume that $\langle \alpha, \gamma, \zeta\rangle$ is equal to a_i for some $i = 0, \dots, m$ and $\langle \alpha', \gamma', \zeta'\rangle$ is equal to a'_j for some $j = 0, \dots, m$. Then $\alpha' < \beta \le \alpha$, so these triples are nonoverlapping.

(2) The set C is a finite coherent adequate set as previously noted.

(3) Let M be in C and $\langle \alpha, \gamma, \zeta\rangle$ in y; we will show that they are nonoverlapping. If α is not in M, then we are done, so assume that $\alpha \in M$. If these objects are either both in q or both in w, then we are done. Assume

that $M \in C \setminus \mathrm{Sk}(\beta)$ and $\langle \alpha, \gamma, \zeta \rangle \in y \cap \mathrm{Sk}(\beta)$. Since α is in $M \cap \beta$, it is in $M' \cap \beta^*$. But the triple and M' are nonoverlapping, and since $\alpha < \beta^*$ this clearly implies that the triple and M are nonoverlapping. Next assume that $M \in C \cap \mathrm{Sk}(\beta)$ and $\langle \alpha, \gamma, \zeta \rangle \in y \setminus \mathrm{Sk}(\beta)$. Then $\alpha \geq \beta$. But this is impossible since $M \subseteq \beta$.

(4) Let M and K be isomorphic sets in C and $a \in y \cap \mathrm{Sk}(M)$. We will show that $\sigma_{M,K}(a) \in y$. Let $a = \langle \alpha, \gamma, \zeta \rangle$.

Suppose that $M \in A_q$. Then $\alpha \notin [\beta^*, \beta)$, hence $a \in x_q$. If K is in A_q, then we are done; otherwise $K = P'$ for some $P \in A^+$. Then $\sigma_{M,P}(a) \in x_q \cap \mathrm{Sk}(P)$. Assume that $\sigma_{M,P}(\alpha) \geq \beta$. Then $\sigma_{M,P}(a) = a_i$ for some $i \leq m$. So $\sigma_{P,P'}(a) = a'_i$. Hence $\sigma_{M,K}(a) = \sigma_{P,P'}(\sigma_{M,P}(a)) = a'_i \in y$. Now assume that $\sigma_{M,P}(\alpha) < \beta^*$. Then $\sigma_{P,P'}(\sigma_{M,P}(a)) = \sigma_{M,P}(a)$ since $\sigma_{P,P'} \restriction \beta^*$ is the identity. So $\sigma_{M,K}(a) = \sigma_{P,P'}(\sigma_{M,P}(a)) = \sigma_{M,P}(a)$, which is in y.

Now suppose that $M = L'$ for some $L \in A^+$. Then $M \in A_w$. So a is in $(x_q \cap \mathrm{Sk}(\beta)) \cup \{a'_0, \dots, a'_m\} = x_w$. If $K \in A_w$, then we are done since w is a condition. Otherwise $K \in C \setminus \mathrm{Sk}(\beta)$. Then $K' \in A_w$, so $\sigma_{M,K'}(a) \in x_w$. If $\sigma_{M,K'}(a) < \beta^*$, then $\sigma_{K',K}(\sigma_{M,K'}(a)) = \sigma_{M,K'}(a)$ since $\sigma_{K',K} \restriction \beta^*$ is the identity. Hence $\sigma_{M,K}(a) = \sigma_{K',K}(\sigma_{M,K'}(a)) = \sigma_{M,K'}(a)$, which is in y. Otherwise $\sigma_{M,K'}(a)$ is equal to a'_i for some $i = 0, \dots, m$. Thus $a'_i \in \mathrm{Sk}(K')$, which implies that $a_i \in \mathrm{Sk}(K)$ and $\sigma_{K,K'}(a_i) = a'_i$. Hence $\sigma_{M,K}(a) = \sigma_{K',K}(\sigma_{M,K'}(a)) = \sigma_{K',K}(a'_i) = a_i$, which is in y. ∎

This completes the proof that $\mathbb{P}$ preserves cardinals.

Recall that for each $\alpha \in \Lambda$, $\dot{c}_\alpha$ is a $\mathbb{P}$-name such that $\mathbb{P}$ forces

$$\dot{c}_\alpha = \{\gamma : \exists p \in \dot{G}\ \exists \beta\ \langle \alpha, \gamma, \beta \rangle \in x_p\}.$$

We will show that $\mathbb{P}$ forces that $\dot{c}_\alpha$ is a cofinal subset of α. Property (3) in the definition of $\mathbb{P}$ will imply that $\dot{c}_\alpha$ is forced to have order type ω_1. Property (4) will imply that $\mathbb{P}$ forces that whenever ξ is a common limit point of $\dot{c}_\alpha$ and $\dot{c}_{\alpha'}$, then $\dot{c}_\alpha \cap \xi = \dot{c}_{\alpha'} \cap \xi$.

LEMMA 4.5. *For each $\alpha \in \Lambda$, $\mathbb{P}$ forces that $\dot{c}_\alpha$ is a cofinal subset of α with order type ω_1.*

Proof. First we show that $\dot{c}_\alpha$ is forced to be a cofinal subset of α. Let p be a condition and $\delta < \alpha$. Choose an ordinal γ with $\delta < \gamma < \alpha$ such that for all $M \in A_p$, $\sup(M \cap \alpha) < \gamma$, and for all triples in x_p of the form $\langle \alpha, \tau, \beta \rangle$, τ and β are less than γ. Define $q = (x_p \cup \{\langle \alpha, \gamma, \gamma + 1 \rangle\}, A_p)$. It is easy to check that q is a condition, and clearly $q \leq p$. Also, q forces that $\dot{c}_\alpha \setminus \delta$ is nonempty. Thus $\mathbb{P}$ forces that $\dot{c}_\alpha$ is a cofinal subset of α.

Suppose for a contradiction that a condition p forces that $\dot{c}_\alpha$ has order type greater than ω_1. Extending p if necessary, assume that for some $\delta < \alpha$, p forces that $\dot{c}_\alpha \cap \delta$ has size ω_1. Fix M in $\mathcal{X}$ such that p, α, and δ are in $\mathrm{Sk}(M)$. Then easily $q = (x_p, A_p \cup \{M\})$ is a condition. Since q forces that

$\dot{c}_\alpha \cap \delta$ is uncountable, we can extend q to r such that for some triple $\langle \alpha, \gamma, \beta \rangle$ in x_r, γ is $\delta \setminus M$. Since $M \in A_r$ and $\alpha \in M$, we have $\sup(M \cap \alpha) < \gamma$, which contradicts $\delta \in M$. ∎

Now we prove that the sequence of $\dot{c}_\alpha$'s is coherent. Namely, we will show that $\mathbb{P}$ forces that whenever ξ is a common limit point of $\dot{c}_\alpha$ and $\dot{c}_{\alpha'}$, then $\dot{c}_\alpha \cap \xi = \dot{c}_{\alpha'} \cap \xi$.

LEMMA 4.6. *Let α be in Λ, $\xi < \alpha$, and suppose that p is a condition which forces that ξ is a limit point of $\dot{c}_\alpha$. Then there is $M \in A_p$ such that $\alpha \in M$ and $\sup(M \cap \alpha) = \xi$.*

Proof. Note that for all $q \leq p$, since q forces that ξ is a limit point of $\dot{c}_\alpha$, if $\langle \alpha, \gamma, \beta \rangle \in x_q$ and $\gamma < \xi$, then $\beta < \xi$. Suppose for a contradiction that for all $M \in A_p$, if $\alpha \in M$ then $\sup(M \cap \alpha) \neq \xi$.

We claim that if $M \in A_p$, $\alpha \in M$, and $\sup(M \cap \xi) < \xi$, then $\sup(M \cap \alpha) < \xi$. Otherwise fix a counterexample M. Then $\alpha \in M$, $\sup(M \cap \xi) < \xi$, and $\sup(M \cap \alpha) \geq \xi$. Since ξ is forced to be a limit point of $\dot{c}_\alpha$, we can find $q \leq p$ and $\gamma, \beta < \xi$ such that $\langle \alpha, \gamma, \beta \rangle \in x_q$ and $\sup(M \cap \xi) < \gamma$. Then γ and β are not in M, but $\sup(M \cap \alpha) \geq \xi > \gamma$, which contradicts the fact that q is a condition.

It follows from the claim that A is the union of the sets A_0, A_1, and A_2 defined by

$$A_0 = \{M \in A_p : \alpha \notin M\},$$
$$A_1 = \{M \in A_p : \alpha \in M, \sup(M \cap \alpha) < \xi\},$$
$$A_2 = \{M \in A_p : \alpha \in M, \sup(M \cap \xi) = \xi\}.$$

Since we are assuming that there is no M in A_p with $\alpha \in M$ and $\sup(M \cap \alpha) = \xi$, every set in A_2 meets the interval $[\xi, \alpha)$. Observe that if $N \in A_1$ and $M \in A_2$, then since $\alpha \in M \cap N$, we have $\beta_{M,N} > \alpha$; hence $\sup(N \cap \alpha) < \xi < \sup(M \cap \alpha)$ implies that $N \cap \beta_{M,N} \in \mathrm{Sk}(M)$.

Fix M in A_2 such that $M \cap \omega_1$ is minimal. Let $\tau = \min(M \setminus \xi)$. Then $\xi \leq \tau < \alpha$. Since $\sup(M \cap \xi) = \xi$, we can fix $\gamma < \xi$ in M such that for all $N \in A_1$, $\sup(N \cap \alpha) < \gamma$, and for all $\langle \alpha, \zeta, \beta \rangle \in x_p$, if $\zeta < \xi$ then $\zeta, \beta < \gamma$.

Let y be the set of triples of the form $\sigma_{N,N'}(\langle \alpha, \gamma, \tau \rangle)$, where N and N' are isomorphic sets in A_p and $\langle \alpha, \gamma, \tau \rangle \in \mathrm{Sk}(N)$. Let $q = (x_p \cup y, A_p)$. We claim that q is a condition. Then clearly $q \leq p$ and q forces that ξ is not a limit point of $\dot{c}_\alpha$, which is a contradiction.

Let us note that $\langle \alpha, \gamma, \tau \rangle$ is nonoverlapping with every triple in x_p. Let $\langle \alpha, \gamma', \beta' \rangle$ be in x_p. If $\gamma' < \xi$, then γ' and β' are below γ, so we are done. Suppose that $\gamma' \geq \xi$. Since $M \in A_p$, either γ' and β' are in M or $\sup(M \cap \alpha) < \gamma'$. In the former case, $\tau = \min(M \setminus \xi) \leq \gamma'$. In the latter case, $\tau < \sup(M \cap \alpha) < \gamma'$. In either case, $\tau \leq \gamma'$, which implies that $[\gamma, \tau) \cap [\gamma', \beta') = \emptyset$.

Next we claim that if $K \in A_p$ then K and $\langle \alpha, \gamma, \tau \rangle$ are nonoverlapping. If α is not in K then we are done, so assume that $\alpha \in K$. Then either $K \in A_1$ or $K \in A_2$. If $K \in A_1$, then $\sup(K \cap \alpha) < \gamma$ by the choice of γ. If $K \in A_2$, then since $M \cap \omega_1 \leq K \cap \omega_1$, either $M \cap \beta_{K,M} \in \mathrm{Sk}(K)$ or $M \cap \beta_{K,M} = K \cap \beta_{K,M}$. In either case, $M \cap \beta_{K,M} \subseteq K$. But since $\alpha \in K \cap M$, we have $\beta_{K,M} > \alpha$. So γ and τ are in K.

Now we prove that q is a condition.

(1) Consider a triple $\langle \alpha', \gamma', \beta' \rangle$ in x_p and a triple $\sigma_{N,N'}(\langle \alpha, \gamma, \tau \rangle)$, where N and N' are isomorphic in A_p and $\langle \alpha, \gamma, \tau \rangle \in \mathrm{Sk}(N)$. If $\alpha' \neq \sigma_{N,N'}(\alpha)$ then we are done, so assume that $\alpha' = \sigma_{N,N'}(\alpha)$. If γ' and β' are not in $\mathrm{Sk}(N')$, then $\sup(N' \cap \alpha') < \gamma'$, so clearly the triples are nonoverlapping. Otherwise γ' and β' are both in $\mathrm{Sk}(N')$. Then $\langle \alpha', \gamma', \beta' \rangle \in x_p \cap \mathrm{Sk}(N')$, so $\sigma_{N',N}(\langle \alpha', \gamma', \beta' \rangle)$ is in x_p. By the comments above, $\sigma_{N',N}(\langle \alpha', \gamma', \beta' \rangle)$ and $\langle \alpha, \gamma, \tau \rangle$ are nonoverlapping. Hence the images of these triples under $\sigma_{N,N'}$ are nonoverlapping and we are done.

Now consider $\sigma_{N_0,N'}(\langle \alpha, \gamma, \tau \rangle)$ and $\sigma_{N_1,N^*}(\langle \alpha, \gamma, \tau \rangle)$, where N_0 and N' are isomorphic in A_p and $\langle \alpha, \gamma, \tau \rangle \in \mathrm{Sk}(N_0)$, and N_1 and N^* are isomorphic in A_p and $\langle \alpha, \gamma, \tau \rangle \in \mathrm{Sk}(N_1)$. If $\sigma_{N_0,N'}(\alpha) \neq \sigma_{N_1,N^*}(\alpha)$ then the triples are nonoverlapping, so assume that $\alpha^* := \sigma_{N_0,N'}(\alpha) = \sigma_{N_1,N^*}(\alpha)$. Then $\beta_{N_0,N_1} > \alpha$ and $\beta_{N',N^*} > \alpha^*$.

We will show that $\sigma_{N_0,N'}(\langle \alpha, \gamma, \tau \rangle) = \sigma_{N_1,N^*}(\langle \alpha, \gamma, \tau \rangle)$. By symmetry it suffices to consider the cases when $N_0 \cap \beta_{N_0,N_1} \in \mathrm{Sk}(N_1)$ and $N_0 \cap \beta_{N_0,N_1} = N_1 \cap \beta_{N_0,N_1}$. Suppose the former case. Then also $N' \cap \beta_{N',N^*} \in \mathrm{Sk}(N^*)$. Fix N_0^* in $\mathrm{Sk}(N_1) \cap A_p$ which is isomorphic to N_0 such that $N_0 \cap \beta_{N_0,N_1} = N_0^* \cap \beta_{N_0,N_1}$. Then $\langle \alpha, \gamma, \tau \rangle \in \mathrm{Sk}(N_0^*)$. Also fix $P \in \mathrm{Sk}(N^*) \cap A_p$ which is isomorphic to N' such that $N' \cap \beta_{N',N^*} = P \cap \beta_{N',N^*}$. Since $\beta_{N',N^*} > \alpha^*$, we have $\alpha^* \in P$.

Since $\sigma_{N_1,N^*}(\alpha) = \alpha^*$, we have $\alpha^* \in P \cap \sigma_{N_1,N^*}(N_0^*)$. As P and $\sigma_{N_1,N^*}(N_0^*)$ are isomorphic and are in the adequate set A_p, it follows that $P \cap \alpha^* = \sigma_{N_1,N^*}(N_0^*) \cap \alpha^*$. Now $\sigma_{N_0,N'} \restriction \alpha$ is the unique order preserving map from $N_0 \cap \alpha = N_0^* \cap \alpha$ onto $N' \cap \alpha^* = P \cap \alpha^* = \sigma_{N_1,N^*}(N_0^*) \cap \alpha^*$. But also $\sigma_{N_1,N^*} \restriction (N_0^* \cap \alpha)$ is an order preserving map from $N_0^* \cap \alpha$ onto $\sigma_{N_1,N^*}(N_0^*) \cap \alpha^*$. It follows that $\sigma_{N_0,N'} \restriction \alpha = \sigma_{N_1,N^*} \restriction (N_0^* \cap \alpha)$. In particular, $\sigma_{N_0,N'}(\langle \alpha, \gamma, \tau \rangle) = \sigma_{N_1,N^*}(\langle \alpha, \gamma, \tau \rangle)$.

Now suppose that $N_0 \cap \beta_{N_0,N_1} = N_1 \cap \beta_{N_0,N_1}$. Then also $N' \cap \beta_{N',N^*} = N^* \cap \beta_{N',N^*}$. In particular, $N_0 \cap \alpha = N_1 \cap \alpha$ and $N' \cap \alpha^* = N^* \cap \alpha^*$. But $\sigma_{N_0,N'} \restriction \alpha$ is the unique order preserving map from $N_0 \cap \alpha$ onto $N' \cap \alpha^*$, and $\sigma_{N_1,N^*} \restriction \alpha$ is the unique order preserving map from $N_1 \cap \alpha$ onto $N^* \cap \alpha$. Hence $\sigma_{N_0,N'} \restriction \alpha = \sigma_{N_1,N^*} \restriction \alpha$. So $\sigma_{N_0,N'}(\langle \alpha, \gamma, \tau \rangle) = \sigma_{N_1,N^*}(\langle \alpha, \gamma, \tau \rangle)$.

(2) is immediate.

(3) Let K be in A_p and consider $\langle \alpha^*, \gamma^*, \tau^* \rangle := \sigma_{N,N'}(\langle \alpha, \gamma, \tau \rangle)$, where N and N' are isomorphic sets in A_p and $\langle \alpha, \gamma, \tau \rangle$ is in $\mathrm{Sk}(N)$. We will prove that K and $\langle \alpha^*, \gamma^*, \tau^* \rangle$ are nonoverlapping. If α^* is not in K, then we are done, so assume that $\alpha^* \in K$. Then $\beta_{K,N'} > \alpha^*$.

If $N' \cap \beta_{K,N'}$ is either in $\mathrm{Sk}(K)$ or equal to $K \cap \beta_{K,N'}$, then γ' and τ' are in K and we are done. So assume that $K \cap \beta_{K,N'} \in \mathrm{Sk}(N')$. Then there is K^* in $\mathrm{Sk}(N') \cap A_p$ which is isomorphic to K such that $K^* \cap \beta_{K,N'} = K \cap \beta_{K,N'}$. Since $\alpha^* < \beta_{K,N'}$, it suffices to show that K^* and $\langle \alpha^*, \gamma^*, \tau^* \rangle$ are nonoverlapping. But $L := \sigma_{N',N}(K^*)$ is in A_p, and we showed above that L is nonoverlapping with $\langle \alpha, \gamma, \tau \rangle$. Therefore the images of L and $\langle \alpha, \gamma, \tau \rangle$ under $\sigma_{N,N'}$, namely K^* and $\langle \alpha^*, \gamma^*, \tau^* \rangle$, are nonoverlapping.

(4) By Lemma 4.2 it suffices to show that $x_p \cup y$ is equal to

$$x_p^* \cup \{\sigma_{N,N'}(a) : N, N' \in A_p,\ N \cap \omega_1 = N' \cap \omega_1,\ a \in x_p^* \cap \mathrm{Sk}(N)\},$$

where $x_p^* = x_p \cup \{\langle \alpha, \gamma, \tau \rangle\}$. Clearly $x_p \cup y$ is included in the second set by definition, and $x_p^* \subseteq x_p \cup y$. Consider $a \in x_p \cup \{\langle \alpha, \gamma, \tau \rangle\}$ and isomorphic N and N' in A_p with $a \in \mathrm{Sk}(N)$. If $a \in x_p$ then $\sigma_{N,N'}(a) \in x_p$ since p is a condition. Otherwise $a = \langle \alpha, \gamma, \beta \rangle$, and $\sigma_{N,N'}(a) \in y$ by the definition of y. ∎

PROPOSITION 4.7. *Let α and α' be distinct ordinals in Λ. Then $\mathbb{P}$ forces that whenever ξ is a common limit point of $\dot{c}_\alpha$ and $\dot{c}_{\alpha'}$, $\dot{c}_\alpha \cap \xi = \dot{c}_{\alpha'} \cap \xi$.*

Proof. Let p be a condition which forces that ξ is a common limit point of $\dot{c}_\alpha$ and $\dot{c}_{\alpha'}$. Then by the previous lemma, there are M and M' in A_p such that $\alpha \in M$ and $\sup(M \cap \alpha) = \xi$, and $\alpha' \in M'$ and $\sup(M' \cap \alpha') = \xi$. Since ξ is a common limit point of M and M', it follows that $\xi < \beta_{M,M'}$. It is not possible that $M \cap \beta_{M,M'} \in \mathrm{Sk}(M')$, since in that case ξ, which is a limit point of $M \cap \beta_{M,M'}$, would be in M'. Similarly, $M' \cap \beta_{M,M'}$ is not in $\mathrm{Sk}(M)$. So $M \cap \beta_{M,M'} = M' \cap \beta_{M,M'}$. It follows that M and M' are isomorphic. Also $\sigma_{M,M'} \restriction M \cap \beta_{M,M'}$ is the identity and $\sigma_{M,M'}(\alpha) = \alpha'$.

Suppose that $q \leq p$ and q forces that γ is in $\dot{c}_\alpha \cap \xi$. Extending q if necessary, assume that $\langle \alpha, \gamma, \beta \rangle \in x_q$ for some β. Since $\gamma < \xi = \sup(M \cap \alpha)$, we see that γ and β are in M. So $\sigma_{M,M'}(\langle \alpha, \gamma, \beta \rangle) = \langle \alpha', \gamma, \beta \rangle$ is in x_q. Hence q forces that γ is in $\dot{c}_{\alpha'}$. This proves that p forces that $\dot{c}_\alpha \cap \xi \subseteq \dot{c}_{\alpha'}$. The other inclusion is proved using a symmetric argument. ∎

Let us show that $\square_{\omega_1}$ holds in any generic extension by $\mathbb{P}$. This follows from well-known arguments which we review for completeness. First note that it suffices to find a sequence $\langle d_\alpha : \alpha \in \omega_2 \cap \mathrm{cof}(\omega_1) \rangle$ such that each d_α is a club subset of α with order type ω_1, and for any $\alpha < \alpha'$ and ξ a common limit point of d_α and $d_{\alpha'}$, $d_\alpha \cap \xi = d_{\alpha'} \cap \xi$. For then we can extend this sequence to a square sequence by defining d_γ for $\gamma \in \omega_2 \cap \mathrm{cof}(\omega)$ by letting $d_\gamma = d_\alpha \cap \gamma$ for some (any) α in $\omega_2 \cap \mathrm{cof}(\omega_1)$ such that γ is a limit point of d_α, and if no such α exist, letting d_γ be a cofinal subset of γ of order type ω.

Recall that each α in Λ is in $C^* \cap \mathrm{cof}(\omega_1)$ and is a limit point of C^*. For each $\alpha \in \Lambda$ let $d_\alpha = \lim(c_\alpha) \cap C^* \cap \alpha$. Then by Lemma 4.5 and Proposition 4.7, the sequence $\langle d_\alpha : \alpha \in \Lambda \rangle$ is such that each d_α is a club subset of α with order type ω_1, and for all ξ in $d_\alpha \cap d_{\alpha'}$, $d_\alpha \cap \xi = d_{\alpha'} \cap \xi$.

One can easily prove by induction that for any $\xi < \omega_2$, there exists a sequence $\langle e_\beta : \beta \in \xi \cap \mathrm{cof}(\omega_1) \rangle$ such that each e_β is a club subset of β of order type ω_1 and any e_β and $e_{\beta'}$ share no common limit points. Consider $\beta_0 < \beta_1$ which are consecutive elements of $C^* \cup \{0\}$. Using the fact just mentioned, we can transfer a sequence of clubs defined on $\mathrm{ot}(\beta_1 \setminus \beta_0) \cap \mathrm{cof}(\omega_1)$ to a sequence $\langle d_\alpha : \alpha \in (\beta_0, \beta_1) \cap \mathrm{cof}(\omega_1) \rangle$ so that each d_α is a club subset of α with minimum element greater than β_0 and order type ω_1, such that any d_α and $d_{\alpha'}$ share no common limit points. But any ordinal in $\omega_2 \cap \mathrm{cof}(\omega_1)$ which is not in C^* belongs to such an interval. So we have defined d_α for all $\alpha \in \omega_2 \cap \mathrm{cof}(\omega_1)$. It is straightforward to check that the extended sequence is as required.

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