FUNDAMENTA
MATHEMATICAE
224 (2014)

Complete sequences of coanalytic sets

by

Riccardo Camerlo (Torino)

Abstract. The notion of a complete sequence of pairwise disjoint coanalytic sets is investigated. Several examples are given and such sequences are characterised under analytic determinacy. The ideas are based on earlier results of Saint Raymond, and generalise them.

1. Introduction. Given a pointclass Γ in Polish spaces and a set B in $\Gamma(Y)$, B is said to be Γ-*complete* if for any zero-dimensional Polish space X and $A \in \Gamma(X)$ there is a continuous $f : X \to Y$ such that $A = f^{-1}(B)$. The set B is said to be *Borel* Γ-*complete* if the property holds with f Borel. The pointclass $\Gamma = \mathbf{\Pi}^1_1$ of coanalytic sets admits a complete member: several examples are provided in [3, §33]. Moreover, Kechris proved in [4] that for coanalytic sets the concepts of completeness and Borel completeness coincide.

In [5], Saint Raymond investigated the notion of complete pair—and, more generally, complete sequence—of disjoint coanalytic sets (see Section 2 for the definition). In particular, the members of such a complete pair (A, B) must be Borel inseparable: for no Borel $C \subseteq X$ does one have $A \subseteq C$, $C \cap B = \emptyset$. Actually he proved that under the axiom of analytic determinacy, completeness and Borel inseparability are equivalent conditions.

On the other hand, in [2] and [1] several uncountable families of pairwise disjoint, Borel inseparable coanalytic sets were constructed. These seem to be good candidates to provide concrete examples of complete sequences. The present paper elaborates on the techniques developed by Saint Raymond, and shows that this is indeed the case, extending ideas and results of the aforementioned articles.

After displaying in Section 2 definitions and some technical tools to be used later, it is shown in Section 3 that for the main examples of [1] every

2010 *Mathematics Subject Classification*: Primary 03E15.
Key words and phrases: coanalytic set, continuous reduction, inseparable sets.

DOI: 10.4064/fm224-1-1

countable subsequence of the families of coanalytic sets studied there is in fact complete. In Section 4 a similar result is obtained for the example of [2].

In Section 5, assuming analytic determinacy, a characterisation of complete sequences is given; it generalises the condition of completeness for pairs given in [5]. To do this, two classical notions are extended in a suitable way to families of sets: the concepts of a universal set and of a Borel inseparable pair of sets. The former gives rise to the notion of a universal sequence, the latter to the one of strongly inseparable sets. In order to get the desired characterisation, I prove the non-existence of universal sequences of Borel sets, and the existence of strongly inseparable coanalytic sets. These are strengthenings of the non-existence of a universal Borel set and the existence of a Borel inseparable pair of disjoint coanalytic sets, and might be of independent interest.

2. Some preliminary constructions. This section contains preliminary definitions, notations and some lemmas that will be applied to obtain the main results. Several of these lemmas are adaptations or modifications of ideas from [2], [1], or [5].

Definition 1. A (possibly finite) sequence $(Q_0, Q_1, \ldots)$ of pairwise disjoint coanalytic subsets of a Polish space Y is *complete* if, for every sequence of the same length $(C_0, C_1, \ldots)$ of pairwise disjoint coanalytic subsets of a zero-dimensional Polish space X, there is a continuous function $f : X \to Y$ such that $\forall n\ C_n = f^{-1}(Q_n)$. A function f as above is said to *reduce* $(C_0, C_1, \ldots)$ to $(Q_0, Q_1, \ldots)$.

One could state a similar definition by declaring that a sequence $(Q_0, Q_1, \ldots)$ is *Borel complete* if the above definition holds with f Borel. However this does not extend the classes of sequences under consideration.

Theorem 1. *If $(Q_0, Q_1, \ldots)$ is a Borel complete sequence of pairwise disjoint coanalytic sets, then it is actually complete.*

Proof. The proof is the same as the one in [4]. The only adaptation is to define sets $(B_0^*, B_1^*, \ldots)$, instead of one single set B^*, by letting

$$\langle a^0, c^0 \rangle \in B_h^* \ \Leftrightarrow\ \forall i \in \mathbb{N}\ (\langle a^i, c^i \rangle \text{ is good}) \wedge (m_i) \in B$$

and then notice that the coanalytic sets $B_0^*, B_1^*, \ldots$ are pairwise disjoint. ∎

Lemma 2. *Suppose $(Q_0, Q_1, \ldots)$ is a complete sequence of pairwise disjoint coanalytic subsets of a Polish space X. Then:*

1. *If $(Q'_0, Q'_1, \ldots)$ is a sequence of pairwise disjoint coanalytic subsets of a Polish space Y and $g : X \to Y$ is a Borel function such that $\forall n\ Q_n = g^{-1}(Q'_n)$, then $(Q'_0, Q'_1, \ldots)$ is complete as well.*
2. *Any subsequence $(Q_{n_0}, Q_{n_1}, \ldots)$ is complete as well.*

(3) *Any permutation of* $(Q_0, Q_1, \ldots)$ *is complete as well.*
(4) *The sets* $(Q_0, Q_1, \ldots)$ *are complete coanalytic and pairwise Borel inseparable.*

Proof. (1) If $(C_0, C_1 \ldots)$ are pairwise disjoint coanalytic subsets of a zero-dimensional Polish space Z and $f : Z \to X$ is continuous and such that $\forall n\ C_n = f^{-1}(Q_n)$, then gf is Borel and $\forall n\ C_n = (gf)^{-1}(Q'_n)$.

(2) Given a sequence $(C_0, C_1, \ldots)$ of pairwise disjoint coanalytic subsets of a zero-dimensional Polish space, apply the completeness of $(Q_0, Q_1, \ldots)$ to the sequence of coanalytic sets

$$D_m = \begin{cases} C_i & \text{if } m = n_i, \\ \emptyset & \text{otherwise.} \end{cases}$$

(3) This holds since the sequence of pairwise disjoint coanalytic subsets of a zero-dimensional Polish space to be reduced to a complete sequence is arbitrary.

(4) Reducing to $(Q_0, Q_1, \ldots)$ a sequence of pairwise disjoint, Borel inseparable, complete coanalytic sets shows that the sets $Q_0, Q_1, \ldots$ are Borel inseparable as well. ∎

The space of (descriptive set-theoretic) trees on $\mathbb{N}$ will be denoted by Tr, the subset of well-founded trees will be denoted by WF, while UB will stand for the subset of trees with a unique infinite branch.

The sets WF and UB are complete coanalytic and Borel inseparable (see [3]). This has been improved in [2, Theorem 3], where it is shown that $\mathrm{WF} \times \mathrm{UB}, \mathrm{UB} \times \mathrm{WF}$ are Borel inseparable in Tr^2. In fact a further extension of this result can be deduced. For $n \in \mathbb{N}$, let

$$\mathcal{A}_n = \mathrm{WF} \times \cdots \times \mathrm{WF} \times \mathrm{UB} \times \mathrm{WF} \times \cdots$$

be the set of sequences $(T_0, T_1, \ldots) \in \mathrm{Tr}^{\mathbb{N}}$ such that T_n has a unique branch, while T_m is well-founded for $m \neq n$.

THEOREM 3. *The complete coanalytic sets* $\mathcal{A}_{n_0}, \mathcal{A}_{n_1}$ *are Borel inseparable for* $n_0 \neq n_1$.

Proof. Fix a well-founded tree W and define a continuous function $g : \mathrm{Tr}^2 \to \mathrm{Tr}^{\mathbb{N}}$ by

$$g(T, U)(m) = \begin{cases} T & \text{if } m = n_0, \\ U & \text{if } m = n_1, \\ W & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned} (T, U) \in \mathrm{UB} \times \mathrm{WF} &\Leftrightarrow g(T, U) \in \mathcal{A}_{n_0}, \\ (T, U) \in \mathrm{WF} \times \mathrm{UB} &\Leftrightarrow g(T, U) \in \mathcal{A}_{n_1}, \end{aligned}$$

and the result follows by the aforementioned [2, Theorem 3]. ∎

However, this fact can be further strengthened, providing a complete sequence that will be used to prove other completeness results.

If A is a subset of a product $X \times Y$, for $x \in X$ the vertical section of A above x, that is, the set $\{y \in Y \mid (x,y) \in A\}$, will be denoted $A(x)$. Similar notations will be used for longer products.

THEOREM 4. *The sequence* $(\mathrm{WF}^{\mathbb{N}}, \mathcal{A}_0, \mathcal{A}_1, \ldots)$ *is a complete sequence of pairwise disjoint coanalytic sets. In fact, given any sequence* $(C_0, C_1, \ldots)$ *of pairwise disjoint coanalytic subsets of some* $X \in \mathbf{\Pi}^0_1(\mathbb{N}^{\mathbb{N}})$*, there is a continuous function* $f : X \to \mathrm{Tr}^{\mathbb{N}}$ *such that, for* $x \in X$*,*

- *if* $x \in C_0$*, then* $f(x) \in \mathrm{WF}^{\mathbb{N}}$*;*
- *if* $x \in C_{n+1}$*, then* $f(x) \in \mathcal{A}_n$*;*
- *if* $x \notin \bigcup_{n\in\mathbb{N}} C_n$*, then all components of* $f(x)$ *have at least two infinite branches.*

Proof. Given $(C_0, C_1, \ldots)$, for each $n \geq 1$ let $D_n = \bigcup_{m\in\mathbb{N}\setminus\{n\}} C_m$. Applying [5, Lemma 26] to the sequence $(D_n, C_n, \emptyset, \emptyset, \ldots)$, let $F_n \in \mathbf{\Pi}^0_1(X \times \mathbb{N}^{\mathbb{N}})$ be such that

$$D_n = \{x \in X \mid F_n(x) = \emptyset\},$$
$$C_n = \{x \in X \mid F_n(x) \text{ is a singleton}\}.$$

Let T_n be the tree of F_n. The function $f_n : X \to \mathrm{Tr}$ assigning to each element of X the corresponding section tree of T_n is continuous. So the function $f : X \to \mathrm{Tr}^{\mathbb{N}}$ defined by $f(x) = (f_1(x), f_2(x), \ldots)$ is continuous too. Moreover, fix $x \in X$. Then:

- If $x \in C_0$, then $F_n(x) = \emptyset$ for all $n \geq 1$, so all $f_n(x)$ are well-founded.
- If $x \in C_n$ for some $n \geq 1$, then $F_n(x)$ is a singleton, while $F_m(x) = \emptyset$ for $m \neq n$. It follows that $f(x) \in \mathcal{A}_{n-1}$.
- If $x \notin \bigcup_{m\in\mathbb{N}} C_m$, then all $F_n(x)$ have at least two points, which are the elements of $[f_n(x)]$. ∎

If $K(X)$ denotes the set of (possibly empty) compact subsets of a topological space X, for $K \in K(\mathbb{N}^{\mathbb{N}})$ let $\mathcal{T}_K \subseteq \mathrm{Tr}$ be the set of trees whose body is homeomorphic to K.

Given $n \in \mathbb{N}$ and a set A of finite sequences of natural numbers, nA will denote the set obtained by adding a first term n to all sequences in A, that is,

$$(s_0, s_1, \ldots, s_m) \in nA \iff s_0 = n \wedge (s_1, \ldots, s_m) \in A.$$

The next lemmas describe techniques to modify in a continuous way a given tree to provide it with some desired features.

LEMMA 5. *Let* $K \in K(\mathbb{N}^{\mathbb{N}})$*. Then there is a continuous function* $g : \mathrm{Tr} \to \mathrm{Tr}$ *such that, given* $T \in \mathrm{Tr}$*:*

- *if T is well-founded, then $g(T)$ is well-founded;*
- *if T has a unique infinite branch, then $g(T) \in \mathcal{T}_K$;*
- *if T has more than one infinite branch, then $[g(T)]$ is not compact.*

Proof. Fix a continuous function $\gamma : \mathrm{Tr} \to \mathrm{Tr}$ reducing $\mathrm{WF} \cup \mathrm{UB}$ to WF. By [1, Lemma 1.3] there are continuous functions $\varphi, \psi : \mathrm{Tr} \to \mathrm{Tr}$ such that:

- if T is well-founded, then $\varphi(T), \psi(T)$ are well-founded;
- if T has a unique infinite branch, then $[\psi(T)]$ is homeomorphic to K;
- if T is ill-founded, then $[\varphi(T)]$ contains a closed subset homeomorphic to Baire space.

Now define $g(T) = \{\emptyset\} \cup 0\varphi\gamma(T) \cup 1\psi(T)$. Then:

- if T is well-founded, then both $\varphi\gamma(T), \psi(T)$ are well-founded, hence so is $g(T)$;
- if T has a unique infinite branch, then $\varphi\gamma(T)$ is well-founded while $[\psi(T)]$ is homeomorphic to K, so $[g(T)]$ is homeomorphic to K;
- if T has more than one infinite branch, then $[\varphi\gamma(T)]$ is not compact, so neither is $[g(T)]$. ∎

Lemma 6. *There is a continuous function $g : \mathrm{Tr} \to \mathrm{Tr}$ such that:*

- *if T is well-founded, then $g(T)$ is well-founded;*
- *if T has exactly one infinite branch, then $g(T)$ has exactly one infinite branch;*
- *if T has at least two infinite branches, then $g(T)$ has continuum many infinite branches.*

Proof. For $T \in \mathrm{Tr}$, $t \in \mathbb{N}^{<\omega}$, let $t \in h(T)$ if and only if

- $t = \emptyset$, or
- $t = (u_0, v_0, \ldots, u_{n-1}, v_{n-1}, u_n)$ for some $(u_0, \ldots, u_n) \in T$ and some $(v_0, \ldots, v_{n-1}) \in \mathbb{N}^{<\omega}$, or
- $t = (u_0, v_0, \ldots, u_n, v_n)$ for some $(u_0, \ldots, u_n) \in T, (v_0, \ldots, v_n) \in \mathbb{N}^{<\omega}$.

Then $h : \mathrm{Tr} \to \mathrm{Tr}$ continuously reduces well-founded trees to well-founded trees and ill-founded trees to trees, having continuum many infinite branches. Now take a continuous reduction γ of $\mathrm{WF} \cup \mathrm{UB}$ to WF and set $g(T) = \{\emptyset\} \cup 0T \cup 1h\gamma(T)$. ∎

Finally, the next couple of lemmas will take care of building closed subsets of $(\mathbb{N}^{\mathbb{N}})^2$ with vertical sections of prescribed order types, with respect to the lexicographic order on $\mathbb{N}^{\mathbb{N}}$.

Lemma 7. *Let C_0, C_1 be disjoint coanalytic subsets of a Polish space X, and α a countable ordinal. Then there is a closed $F \subseteq X \times \mathbb{N}^{\mathbb{N}}$ such that:*

- *if* $x \in C_0$, *then* $F(x) = \emptyset$;
- *if* $x \in C_1$, *then* $F(x)$ *has order type* α *with respect to the lexicographic order;*
- *if* $x \in X \setminus (C_0 \cup C_1)$, *then* $F(x)$ *is uncountable.*

Proof. We proceed by induction on α, borrowing some arguments from the proof of [5, Theorem 23]. Let $H \in \mathbf{\Pi}^0_1(X \times \mathbb{N}^\mathbb{N})$ be such that $\pi_1(H) = X \setminus (C_0 \cup C_1)$, where π_1 denotes the first projection on a Cartesian product. Then $K = H \times \mathbb{N}^\mathbb{N}$ is a closed subset of $X \times (\mathbb{N}^\mathbb{N})^2$ such that $\pi_1(K) = X \setminus (C_0 \cup C_1)$ and whose non-empty vertical fibers are uncountable. Using a homeomorphism of $\mathbb{N}^\mathbb{N}$ with $(\mathbb{N}^\mathbb{N})^2$, a closed $L \subseteq X \times \mathbb{N}^\mathbb{N}$ is found such that $\pi_1(L) = X \setminus (C_0 \cup C_1)$, whose non-empty vertical fibers are uncountable.

If $\alpha \in \mathbb{N}$, one can use the statement of [5, Lemma 26] to find $F' \in \mathbf{\Pi}^0_1(X \times \mathbb{N}^\mathbb{N})$ such that $F'(x)$ is empty if $x \in C_0$ and it has α points for $x \in C_1$ (for $\alpha > 0$ take C_m in the statement there to be C_0 if $m = 0$; C_1 if $m = \alpha$; empty otherwise). Then $F = F' \cup L$ does.

Assume the assertion for $\alpha \geq 1$. Let $F_0, F_1 \in \mathbf{\Pi}^0_1(X \times \mathbb{N}^\mathbb{N})$ be such that

- the vertical sections of F_0, F_1 at any point of C_0 are empty;
- the vertical section of F_0 at any point of C_1 has order type α under the lexicographic order;
- the vertical section of F_1 at any point of C_1 is a singleton;
- all other vertical sections of F_0, F_1 are uncountable.

Let $(x, y) \in F \Leftrightarrow (y(0) = 0 \wedge (x, (y(1), y(2), \ldots)) \in F_0) \vee (y(0) = 1 \wedge (x, (y(1), y(2), \ldots)) \in F_1)$. Notice that F is closed and the order type of its sections is the sum of the order types of the corresponding sections of F_0 and of F_1; so it is 0 for points in C_0, it is $\alpha + 1$ for points in C_1 and it is uncountable outside $C_0 \cup C_1$.

Finally, let α be limit and assume the assertion for all $\beta < \alpha$. Let γ_n be a sequence of ordinals such that $\sum_{n \in \mathbb{N}} \gamma_n = \alpha$. For each $n \in \mathbb{N}$ let $F_n \in \mathbf{\Pi}^0_1(X \times \mathbb{N}^\mathbb{N})$ satisfy the assertion for γ_n. Set $(x, y) \in F$ if and only if $(x, (y(1), y(2), \ldots)) \in F_n$ where $y(0) = n$. Thus F is closed and the order type of its sections is the sum of the order types of the corresponding sections of all F_n, so it is 0 for points in C_0, it is α for points in C_1 and it is uncountable outside $C_0 \cup C_1$. ∎

LEMMA 8. *Fix a countable ordinal* α. *Let* X *be a Polish space, and let* $\{C_\xi\}_{\xi \in \alpha}$ *be a family of pairwise disjoint coanalytic sets. Then there is a closed* $F \subseteq X \times \mathbb{N}^\mathbb{N}$ *such that:*

- *if* $x \in C_\xi$, *then* $F(x)$ *has order type* ξ;
- *if* $x \notin \bigcup_{\xi \in \alpha} C_\xi$, *then* $F(x)$ *is uncountable.*

Proof. If $\alpha = 0$, let $F = X \times \mathbb{N}^{\mathbb{N}}$. For $\alpha > 0$ let $\{\alpha_n\}_n$ be a (possibly finite) enumeration of α. By Lemma 7, for each n let $F_n \in \mathbf{\Pi}^0_1(X \times \mathbb{N}^{\mathbb{N}})$ have vertical sections at points of C_{α_n} of order type α_n, empty vertical sections at points of $\bigcup_{\beta\in\alpha\setminus\{\alpha_n\}} C_\beta$, all other vertical sections being uncountable. Put $(x, y) \in F$ if and only if $(x,(y(1), y(2), \ldots)) \in F_n$ where $y(0) = n$. Then F is closed. Moreover, the order type of any of its sections is the sum of the order types of the corresponding sections of the F_n, so it has the desired properties. ■

3. Complete sequences of classes of trees. In [1] the members of the following families of pairwise disjoint coanalytic sets were shown to be Borel inseparable:

- The family of all $\{\mathcal{T}_K\}_{K\in\mathcal{K}}$, where $\mathcal{K}$ consists of a representative from each homeomorphism class in $K(\mathbb{N}^{\mathbb{N}})$.
- The family $\{\mathcal{V}_\alpha\}_{\alpha\in\omega_1}$, where $\mathcal{V}_\alpha$ is the class of trees whose body is well-ordered in type α with respect to the lexicographic order of $\mathbb{N}^{\mathbb{N}}$.
- The family $\{\mathrm{UB}_A\}_{A\in\mathcal{A}}$, where $\mathcal{A}$ is any class of pairwise disjoint coanalytic subsets of $\mathbb{N}^{\mathbb{N}}$ each one containing a closed subset of $\mathbb{N}^{\mathbb{N}}$ homeomorphic to $\mathbb{N}^{\mathbb{N}}$, and UB_A is the set of trees with a unique infinite branch, such a branch belonging to A.

These classes, however, behave differently with respect to completeness. In this section it will be proved that any countable sequence taken from the first two classes is complete, while in the next section it will be shown that this is not the case for the last one.

THEOREM 9. *Fix any sequence $(K_0, K_1, \ldots)$ of pairwise non-homeomorphic compact subsets of $\mathbb{N}^{\mathbb{N}}$. Then the sequence $(\mathcal{T}_{K_0}, \mathcal{T}_{K_1}, \ldots)$ is a complete sequence of coanalytic sets.*

Proof. By Lemma 2 it can be assumed $K_0 = \emptyset$. Fix a sequence $(C_0, C_1, \ldots)$ of pairwise disjoint coanalytic subsets of some $X \in \mathbf{\Pi}^0_1(\mathbb{N}^{\mathbb{N}})$ and a continuous function $f : X \to \mathrm{Tr}^{\mathbb{N}}, x \mapsto (f_0(x), f_1(x), \ldots)$ with the properties given in Theorem 4. Using Lemma 5, for each $n \in \mathbb{N}$ let $g_n : \mathrm{Tr} \to \mathrm{Tr}$ be continuous and such that:

- if T is well-founded, then $g_n(T)$ is well-founded;
- if T has a unique infinite branch, then $g_n(T) \in \mathcal{T}_{K_{n+1}}$;
- if T has more than one infinite branch, then $[g_n(T)]$ is not compact.

Define $g : X \to \mathrm{Tr}$ by

$$g(x) = \{\emptyset\} \cup \bigcup_{n\in\mathbb{N}} n g_n f_n(x).$$

So, if $x \in C_0$, then all $f_n(x)$ are well-founded, thus $g(x)$ is well-founded as well. If $x \in C_n$ for some $n > 0$, then all $f_i(x)$ are well-founded, with the exception of $f_{n-1}(x)$, that have a unique branch; consequently, $[g(x)]$ is homeomorphic to K_n. Finally, if $x \notin \bigcup_{n\in\mathbb{N}} C_n$, then all $f_n(x)$ have at least two infinite branches, so that no $[g_n f_n(x)]$ is compact and neither is $[g(x)]$. ∎

In particular, the following result of [1, Section 1] is obtained.

Corollary 10. *The complete coanalytic sets $\mathcal{T}_K, \mathcal{T}_L$ are Borel inseparable whenever K, L are non-homeomorphic compact subsets of $\mathbb{N}^{\mathbb{N}}$.*

Theorem 11. *For any sequence $(\alpha_0, \alpha_1, \ldots)$ of distinct countable ordinals, the sequence $(\mathcal{V}_{\alpha_0}, \mathcal{V}_{\alpha_1}, \ldots)$ is a complete sequence of coanalytic subsets of* Tr.

Proof. Let X be a closed subset of $\mathbb{N}^{\mathbb{N}}$ and $(C_0, C_1, \ldots)$ a sequence of pairwise disjoint coanalytic subsets of X. Let $F \in \mathbf{\Pi}^0_1(X \times \mathbb{N}^{\mathbb{N}})$ be such that every C_n is the set of all points $x \in X$ whose section $F(x)$ has order type α_n (this can be done by applying Lemma 8 to $\sup\{\alpha_n + 1\}_n$). Let T be a tree on $\mathbb{N}^2$ whose body is F. The function $f : X \to \mathrm{Tr}$ assigning to each $x \in X$ the corresponding section tree of T is continuous and $\forall n\ f^{-1}(\mathcal{V}_{\alpha_n}) = C_n$. ∎

As a corollary, one can deduce the following result of [1, Section 2].

Corollary 12. *The members of the family of complete coanalytic sets $\{\mathcal{V}_\alpha\}_{\alpha\in\omega_1}$ are pairwise Borel inseparable.*

Theorems 9 and 11 are both extensions of [5, Theorem 27].

4. Complete sequences of classes of structures. The family of Borel inseparable coanalytic sets considered in [2] provides complete sequences in the framework of countable structures.

Let $L = \{f_n\}_{n\in\mathbb{N}}$ be a language with countably many unary function symbols, and let X_L be the set of (codes for) L-structures. For each countable group G let $\mathcal{B}_G$ be the set of (codes for) L-structures whose automorphism group is isomorphic to G.

Theorem 13. *Fix any sequence $\{G_n\}_{n\in\mathbb{N}}$ of pairwise non-isomorphic countable groups. Then the sequence $(\mathcal{B}_{G_0}, \mathcal{B}_{G_1}, \ldots)$ is complete.*

Proof. By Lemma 2 it can be assumed that $G_0 = \{1_{G_0}\}$. It will be shown that there is a Borel function $f : \mathrm{Tr}^{\mathbb{N}} \to X_L$ such that $\mathrm{WF}^{\mathbb{N}} = f^{-1}(\mathcal{B}_{G_0})$ and $\forall n \in \mathbb{N}\ \mathcal{A}_n = f^{-1}(\mathcal{B}_{G_{n+1}})$. This will be enough by Lemma 2(1).

By Theorem 4 let $g_0 : \mathrm{Tr}^{\mathbb{N}} \to \mathrm{Tr}^{\mathbb{N}}$ be a continuous function such that:

- $g_0^{-1}(\mathrm{WF}^{\mathbb{N}}) = \mathrm{WF}^{\mathbb{N}}$;
- $\forall n \in \mathbb{N}\ g_0^{-1}(\mathcal{A}_n) = \mathcal{A}_n$;

- if $(T_0, T_1, \ldots) \notin \mathrm{WF}^{\mathbb{N}} \cup \bigcup_{n\in\mathbb{N}} \mathcal{A}_n$, then all the components of $g_0(T_0, T_1, \ldots)$ have at least two infinite branches.

If g is the function granted by Lemma 6, let $g_1 = (g \times g \times \cdots)g_0$. Thus:

- $g_1^{-1}(\mathrm{WF}^{\mathbb{N}}) = \mathrm{WF}^{\mathbb{N}}$;
- $\forall n \in \mathbb{N}\ g_1^{-1}(\mathcal{A}_n) = \mathcal{A}_n$;
- if $(T_0, T_1, \ldots) \notin \mathrm{WF}^{\mathbb{N}} \cup \bigcup_{n\in\mathbb{N}} \mathcal{A}_n$, then all components of $g_1(T_0, T_1, \ldots)$ have continuum many infinite branches.

By [2], there are Borel functions $\Phi_n : \mathrm{Tr} \to X_L$ such that:

- if T is well-founded, then Φ_n is rigid;
- if T has exactly one infinite branch, then the automorphism group of $\Phi_n(T)$ is isomorphic to G_n;
- if T has continuum many infinite branches, then the automorphism group of $\Phi_n(T)$ is uncountable;
- if M is in the range of Φ_n, then there is no element of M that is fixed under all f_m^M, except if $G_n = \{1_{G_n}\}$, in which case there is just one such element (the one denoted e_0 there).

Fix a bijection $\mathbb{N}^2 \to \mathbb{N}, (h,k) \mapsto \langle h,k\rangle$. Given a sequence M_n of L-structures, define their direct sum $\bigoplus_{n\in\mathbb{N}} M_n$ whose universe is the disjoint union of the M_n and where $f_{\langle h,k\rangle}^{\bigoplus_{n\in\mathbb{N}} M_n}$ acts as $f_k^{M_h}$ on M_h and it is the identity elsewhere. Finally define

$$f(T_0, T_1, \ldots) = \bigoplus_{n\in\mathbb{N}} \Phi_n \pi_n g_1(T_0, T_1, \ldots),$$

where π_n denotes the projection on the nth component.

Now notice that every automorphism φ of $f(T_0, T_1, \ldots)$ is invariant on the summands. Indeed, if u is an element of $\Phi_n \pi_n g_1(T_0, T_1, \ldots)$ and v is an element of $\Phi_m \pi_m g_1(T_0, T_1, \ldots)$ with $0 \neq n \neq m$, then $f_{\langle n,k\rangle}(u) \neq u$ for some $k \in \mathbb{N}$, while $f_{\langle n,k\rangle}(v) = v$, so $\varphi(u) \neq v$. If $0 = n \neq m$, apply the argument to φ^{-1}.

Thus using the properties of functions g_1, Φ_n, the asserted properties for f follow. Indeed, suppose first that $(T_0, T_1, \ldots) \in \mathrm{WF}^{\mathbb{N}}$. Then $\pi_n g_1(T_0, T_1, \ldots)$ is well-founded for all $n \in \mathbb{N}$, so all structures $\Phi_n \pi_n g_1(T_0, T_1, \ldots)$ are rigid; by the remark above, $f(T_0, T_1, \ldots)$ is rigid as well. Suppose now $(T_0, T_1, \ldots) \in \mathcal{A}_n$. Then all $\pi_m g_1(T_0, T_1, \ldots)$ are well-founded, except for $\pi_n g_1(T_0, T_1, \ldots) \in \mathrm{UB}$. Consequently, the group of automorphisms of $\Phi_n \pi_n g_1(T_0, T_1, \ldots)$ is isomorphic to G_n and the same is true for $f(T_0, T_1, \ldots)$. Finally, if among $T_0, T_1, \ldots$ there are at least two ill-founded trees or there is a tree with at least two infinite branches, then $\pi_n g_1(T_0, T_1, \ldots)$ has continuum many infinite branches for all $n \in \mathbb{N}$. This implies that all $\Phi_n \pi_n g_1(T_0, T_1, \ldots)$ have an uncountable group of automorphisms and the same holds for $f(T_0, T_1, \ldots)$. ∎

The statement of [2, Theorem 4] becomes now a corollary.

Corollary 14. *Given non-isomorphic, countable groups G, H, the sets $\mathcal{B}_G, \mathcal{B}_H$ are Borel inseparable.*

5. A characterisation of completeness. [5, Theorem 9] states that under analytic determinacy all pairs of disjoint Borel inseparable coanalytic sets are complete. This does not hold for longer sequences.

Theorem 15.

(1) *For every $n \geq 2$ there is an $n+1$-tuple $(C_0, \ldots, C_n)$ of pairwise disjoint, Borel inseparable, complete coanalytic subsets of $\mathbb{N}^{\mathbb{N}}$ that is not complete. Moreover, $(C_0, \ldots, C_n)$ can be built so that any subsequence of length n obtained by omitting one of its terms is complete.*
(2) *There is a sequence $(C_0, C_1, \ldots)$ of pairwise disjoint, Borel inseparable, complete coanalytic subsets of $\mathbb{N}^{\mathbb{N}}$ that is not complete, but such that every finite sequence $(C_0, \ldots, C_n)$ is complete.*

Proof. (1) Write $\mathbb{N}^{\mathbb{N}} = \bigcup_{i=0}^{n} X_i$, where $X_0, \ldots, X_n$ are pairwise disjoint, clopen and non-empty. For every i, let $(Q_{i0}, \ldots, Q_{i,n-1})$ be a complete n-tuple of coanalytic subsets of X_i. Set $C_i = \bigcup_{j=0}^{n-1} Q_{i \oplus j, j}$, where $\oplus$ is sum modulo $n+1$ (so each C_i intersects all X_h except $X_{i \oplus n}$).

The inclusion map continuously reduces the sequence $(Q_{i0}, \ldots, Q_{i,n-1})$ to the one obtained from $(C_0, \ldots, C_n)$ by omitting $C_{i \oplus 1}$, so the latter is complete.

Nevertheless, $C_0, \ldots, C_n$ is not complete. Indeed, let $x^0, \ldots, x^n$ be sequences in $\mathbb{N}^{\mathbb{N}}$ converging to a same limit such that the sets of their terms are pairwise disjoint. If $D_i = \{x^i(m)\}_{m \in \mathbb{N}}$, no continuous function $g : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ can reduce $(D_0, \ldots, D_n)$ to $(C_0, \ldots, C_n)$, since for such a function the sequences $gx^0(m), \ldots, gx^n(m)$ would eventually belong to a same X_i, but $X_i \cap C_{i \oplus 1} = \emptyset$.

(2) Let $\mathbb{N}^{\mathbb{N}} = \bigcup_{n \in \mathbb{N}} X_n$, where the X_n are pairwise disjoint, clopen, non-empty subsets. For each $n \in \mathbb{N}$, let $(Q_{n0}, \ldots, Q_{nn})$ be a complete sequence of pairwise disjoint, Borel inseparable, coanalytic subsets of X_n (for $n = 0$, this amounts to a single complete coanalytic $Q_{00} \subseteq X_0$). Set $C_i = \bigcup_{n=i}^{\infty} Q_{ni}$.

Then $(C_0, \ldots, C_n)$ is complete. Indeed, given any sequence $(D_0, \ldots, D_n)$ of pairwise disjoint coanalytic sets of $\mathbb{N}^{\mathbb{N}}$, any function $g : \mathbb{N}^{\mathbb{N}} \to X_n$ reducing $(D_0, \ldots, D_n)$ to $(Q_{n0}, \ldots, Q_{nn})$ reduces $(D_0, \ldots, D_n)$ to $(C_0, \ldots, C_n)$ too. On the other hand, if x_n is an injective converging sequence in $\mathbb{N}^{\mathbb{N}}$, no continuous reduction of $(\{x_0\}, \{x_1\}, \ldots)$ to $(C_0, C_1, \ldots)$ can exist, since for such a reduction g the sequence $g(x_n)$ cannot converge. ∎

An example of a sequence satisfying Theorem 15(2) can be extracted from [1, Theorem 3.1], where it is stated, in the notations of the preceding section, that if A, B are disjoint coanalytic subsets of Baire space, each one containing a closed subset of $\mathbb{N}^{\mathbb{N}}$ homeomorphic to $\mathbb{N}^{\mathbb{N}}$, then $\mathrm{UB}_A, \mathrm{UB}_B$ are disjoint, Borel inseparable, complete coanalytic subsets of Tr.

THEOREM 16. *Let $A_n = 0^n 1\mathbb{N}^{\mathbb{N}}$. Then the sequence* $(\mathrm{UB}_{A_0}, \mathrm{UB}_{A_1}, \ldots)$ *is not complete, while all finite sequences* $(\mathrm{UB}_{A_0}, \ldots, \mathrm{UB}_{A_m})$ *are.*

Proof. Let x_n be a sequence of distinct points in $\mathbb{N}^{\mathbb{N}}$ converging to x_0. Then no continuous function $\mathbb{N}^{\mathbb{N}} \to \mathrm{Tr}$ reduces the sequence $(\{x_0\}, \{x_1\}, \ldots)$ to the sequence $(\mathrm{UB}_{A_0}, \mathrm{UB}_{A_1}, \ldots)$. Indeed, suppose g is such a reduction and let $T_n = g(x_n)$, thus $\lim_{n\to\infty} T_n = T_0 \in \mathrm{UB}_{A_0}$. Then $0^n 1 \in T_n$, which implies that the branch of constant value 0 is in T_0, a contradiction.

On the other hand, any finite $(\mathrm{UB}_{A_0}, \ldots, \mathrm{UB}_{A_m})$ is complete. This can be shown with an argument similar to that used for the proof of Theorem 4. First, for $0 \le i \le m$, let $\mathcal{A}'_i \subseteq \mathrm{Tr}^{m+1}$ be the coanalytic set of those sequences $(T_0, \ldots, T_m)$ with T_i having a unique infinite branch and all other T_j being well-founded. Given a sequence $(C_0, \ldots, C_m)$ of pairwise disjoint coanalytic subsets of some $X \in \mathbf{\Pi}^0_1(\mathbb{N}^{\mathbb{N}})$, let $D_i = \bigcup_{j\ne i} C_j$. By Lemma 8, let $F_i \in \mathbf{\Pi}^0_1(X \times \mathbb{N}^{\mathbb{N}})$ be such that for $0 \le i \le m$:

- $D_i = \{x \in X \mid F_i(x) = \emptyset\}$,
- $C_i = \{x \in X \mid F_i(x) \text{ is a singleton}\}$.

Let S_i be the tree of F_i. The function $f_i : X \to \mathrm{Tr}$ assigning to each element of X the corresponding section tree of S_i is continuous, so $f : X \to \mathrm{Tr}^{m+1}$ defined by letting $f(x) = (f_0(x), \ldots, f_m(x))$ is continuous as well. Fix $x \in X$. Then:

- If $x \in C_i$, then $F_i(x)$ is a singleton, while $F_j(x) = \emptyset$ for $j \ne i$. So $f(x) \in \mathcal{A}'_i$.
- If $x \notin \bigcup_{i=0}^m C_i$, then any $F_i(x)$ contains at least two points, thus $f(x) \notin \bigcup_{i=0}^m \mathcal{A}'_i$.

This shows that $(\mathcal{A}'_0, \ldots, \mathcal{A}'_m)$ is a complete sequence of coanalytic sets.

Now define $g : \mathrm{Tr}^{m+1} \to \mathrm{Tr}$ by letting $g(T_0, \ldots, T_m)$ be the tree generated by $\bigcup_{i=0}^m 0^i 1 T_i$. Then g is continuous and reduces $(\mathcal{A}'_0, \ldots, \mathcal{A}'_m)$ to $(\mathrm{UB}_{A_0}, \ldots, \mathrm{UB}_{A_m})$, establishing the completeness of this latter sequence. ∎

To find the right generalisation of [5, Theorem 9] to longer sequences, a suitable extension of the notion of a Borel inseparable pair of coanalytic sets is to be isolated. As the existence of such a pair relies on the non-existence of a universal Borel set, the extension of this concept is also to be worked out. This is done as follows.

DEFINITION 2. Let X be a Polish space, Γ a class of subsets in Polish spaces. For $2 \le n \le \aleph_0$, an n-sequence $(A_0, A_1, \ldots)$ of members of $\Gamma(X^2)$ is *universal* for $\Gamma(X)$ if

- $\{A_\alpha \mid \alpha < n\}$ is a partition of X^2, and
- for every partition of X of the same cardinality $\{B_\alpha \mid \alpha < n\}$ with each $B_\alpha \in \Gamma(X)$, there is $x \in X$ such that $\forall \alpha < n\ B_\alpha \cap A_\alpha(x) = \emptyset$ (equivalently, $\forall \alpha < n\ B_\alpha \subseteq \bigcup_{\beta \neq \alpha} A_\beta(x)$).

When $n = 2$ and Γ is the class of Borel sets, for a pair $(A_0, X^2 \setminus A_0)$ being universal means that A_0 (equivalently, $X^2 \setminus A_0$) is a universal Borel set. In fact, a universal Borel set does not exist (see [3]). This is generalised by the following.

Theorem 17. *For X any Polish space, there is no universal sequence for $B(X)$.*

Proof. If $\{A_\alpha \mid \alpha < n\}$ is a partition of X^2, define $B_\alpha = \{x \in X \mid (x, x) \in A_\alpha\}$. Suppose there is $x \in X$ such that $\forall \alpha < n\ B_\alpha \cap A_\alpha(x) = \emptyset$. If α is such that $(x, x) \in A_\alpha$, then $x \in A_\alpha(x) \cap B_\alpha$, a contradiction. So $\{A_\alpha \mid \alpha < n\}$ is not universal. ∎

Definition 3. Suppose $2 \le n \le \aleph_0$ and let $\{C_k \mid k < n\}$ be a collection of pairwise disjoint coanalytic subsets of a Polish space X. Then sets $C_0, C_1, \ldots$ are *strongly inseparable* if for any partition $\{B_0, B_1, \ldots\}$ of X into n Borel pieces

$$\exists h\ \forall k\ B_h \cap C_k \neq \emptyset.$$

Notice that for $n = 2$ one recovers the definition of a Borel inseparable pair of coanalytic sets. Moreover, if $C_0, C_1, \ldots$ are strongly inseparable, then any subsequence $C_{h_0}, C_{h_1}, \ldots$ is strongly inseparable as well. Indeed, consider a Borel partition in the appropriate number of subsets, which can be indexed as $\{B_{h_0}, B_{h_1}, \ldots\}$. Suppose $\forall i\ \exists j\ B_{h_i} \cap C_{k_j} = \emptyset$. Then build a new partition $\{B'_0, B'_1, \ldots\}$ with B'_h a singleton whenever $h \neq h_i$ for all i, and $B'_{h_i} \subseteq B_{h_i}$ (it is enough to excise a countable proper subset from the first infinite B_{h_i} and use it to build all the required singletons). So, $\forall h\ \exists k\ B'_h \cap C_k = \emptyset$, contradicting the strong inseparability of $C_0, C_1, \ldots$.

Under analytic determinacy, strong inseparability will yield a necessary and sufficient condition for the completeness of a sequence, provided that strongly inseparable families of coanalytic sets actually exist. The latter is granted by the next theorem.

Theorem 18. *For any $2 \le n \le \aleph_0$ there exists a family $\{C_k \mid k < n\}$ of strongly inseparable coanalytic subsets of $\mathbb{N}^{\mathbb{N}}$.*

Proof. The property will be proved for $(\mathbb{N}^{\mathbb{N}})^2$ instead of $\mathbb{N}^{\mathbb{N}}$. For $n = 2$, it states the existence of a pair of Borel inseparable coanalytic sets. However, in order to start out the induction, some specific pairs will be used. Let $U \in \mathbf{\Pi}^0_1((\mathbb{N}^{\mathbb{N}})^3)$ be universal for $\mathbf{\Pi}^0_1((\mathbb{N}^{\mathbb{N}})^2)$, meaning that for every closed

$C \subseteq (\mathbb{N}^{\mathbb{N}})^2$ there is $w \in \mathbb{N}^{\mathbb{N}}$ such that $C = U(w)$. For each $k \in \mathbb{N}$, let

$$U_k = \{(w,x) \in (\mathbb{N}^{\mathbb{N}})^2 \mid U(w,x) \text{ has exactly } k \text{ elements}\}.$$

By induction on $n \geq 2$ it will be shown that for any pairwise distinct $i_0, \ldots, i_{n-1} \in \mathbb{N}$, the sets $U_{i_0}, \ldots, U_{i_{n-1}}$ are strongly inseparable.

Let $n = 2$. Assume for contradiction that there exist Borel sets $B_0, B_1 \subseteq (\mathbb{N}^{\mathbb{N}})^2$ providing a counterexample. This amounts to the existence of $B \in B((\mathbb{N}^{\mathbb{N}})^2)$ such that

$$B \cap U_{i_0} = \emptyset, \qquad U_{i_1} \subseteq B.$$

So, for any $w \in \mathbb{N}^{\mathbb{N}}$,

$$B(w) \cap U_{i_0}(w) = \emptyset, \qquad U_{i_1}(w) \subseteq B(w).$$

Let V be an arbitrary Borel subset of $\mathbb{N}^{\mathbb{N}}$. By Lemma 8, there is $F \in \mathbf{\Pi}^0_1((\mathbb{N}^{\mathbb{N}})^2)$ such that

$$V_0 = \mathbb{N}^{\mathbb{N}} \setminus V = \{x \in \mathbb{N}^{\mathbb{N}} \mid F(x) \text{ has exactly } i_0 \text{ points}\},$$
$$V_1 = V = \{x \in \mathbb{N}^{\mathbb{N}} \mid F(x) \text{ has exactly } i_1 \text{ points}\}.$$

By the universality of U, there is $\bar{w} \in \mathbb{N}^{\mathbb{N}}$ such that $F = U(\bar{w})$. Hence

$$V_0 = \{x \in \mathbb{N}^{\mathbb{N}} \mid U(\bar{w},x) \text{ has exactly } i_0 \text{ points}\} \subseteq \mathbb{N}^{\mathbb{N}} \setminus B(\bar{w}),$$
$$V_1 = \{x \in \mathbb{N}^{\mathbb{N}} \mid U(\bar{w},x) \text{ has exactly } i_1 \text{ points}\} \subseteq B(\bar{w}),$$

implying $V = B(\bar{w})$. So $B \in B((\mathbb{N}^{\mathbb{N}})^2)$ would be universal for $B(\mathbb{N}^{\mathbb{N}})$, but such a set does not exist.

Assume now the assertion for n. Fix distinct $i_0, \ldots, i_n \in \mathbb{N}$ and Borel sets $B_0, \ldots, B_n$ partitioning $(\mathbb{N}^{\mathbb{N}})^2$. Assume, towards a contradiction, that

$$\forall h \in \{0, \ldots, n\}\ \exists k \in \{0, \ldots, n\}\ B_h \cap U_{i_k} = \emptyset.$$

Claim. *There is a bijection $f : \{0, \ldots, n\} \to \{0, \ldots, n\}$ such that $\forall h \in \{0, \ldots, n\}$ $B_h \cap U_{i_{f(h)}} = \emptyset$.*

Proof. First, build a partition $\{B'_0, \ldots, B'_n\}$ of $(\mathbb{N}^{\mathbb{N}})^2$ in $n+1$ Borel sets with the following properties:

- for all h and k, if $B_h \cap U_{i_k} \neq \emptyset$, then $B'_h \cap U_{i_k} \neq \emptyset$,
- each B'_h intersects exactly n sets among $U_{i_0}, \ldots, U_{i_n}$.

To see that this is possible, suppose $\bar{h}$ is least such that $B_{\bar{h}}$ meets less than n sets, say just m, among $U_{i_0}, \ldots, U_{i_n}$. Using the fact that each U_{i_k} is infinite, it is possible to add to $B_{\bar{h}}$ some $n - m$ points, taken away from some other B_h without affecting the non-emptiness of any $B_h \cap U_{i_k}$, to make $B_{\bar{h}}$ meet n sets among $U_{i_0}, \ldots, U_{i_n}$. Now proceed inductively until a partition with the required properties is built.

So, for each h, there is $f(h)$ such that $B'_h \cap U_{i_k} \neq \emptyset \iff k \neq f(h)$. In particular, this implies $\forall h \in \{0, \ldots, n\}$ $B_h \cap U_{i_{f(h)}} = \emptyset$.

It remains to show that $f : \{0,\dots,n\} \to \{0,\dots,n\}$ is injective. If not, suppose $f(h') = f(h'')$ for distinct h', h'' and consider the partition $\mathcal{B} = \{B_{h'} \cup B_{h''}, B_h \mid h \notin \{h', h''\}\}$ consisting of n Borel sets. By the inductive hypothesis, for each $\bar{k}$ there is a member of $\mathcal{B}$ intersecting all U_{i_k} with $k \neq \bar{k}$, different values of $\bar{k}$ giving rise to different elements of $\mathcal{B}$ since no element of $\mathcal{B}$ intersects all $U_{i_0}, \dots, U_{i_n}$. But this is impossible, as $\bar{k}$ ranges over $n+1$ values, while $\mathcal{B}$ has n elements. ∎

By the Claim it is possible to reindex the sets B_h so that f becomes the identity, that is,

$$\forall h \in \{0,\dots,n\}\ B_h \cap U_{i_h} = \emptyset.$$

Let now $V_0, \dots, V_n$ be arbitrary Borel sets partitioning $\mathbb{N}^{\mathbb{N}}$. By Lemma 8 there is a closed set $F \subseteq (\mathbb{N}^{\mathbb{N}})^2$ such that

$$V_k = \{x \in \mathbb{N}^{\mathbb{N}} \mid F(x) \text{ has exactly } i_k \text{ elements}\}.$$

By the universality of U there is $\bar{w} \in \mathbb{N}^{\mathbb{N}}$ such that $F = U(\bar{w})$. Hence

$$V_k = \{x \in \mathbb{N}^{\mathbb{N}} \mid U(\bar{w}, x) \text{ has exactly } i_k \text{ elements}\} \subseteq \mathbb{N}^{\mathbb{N}} \setminus B_k(\bar{w}),$$

as $(\bar{w}, x) \in U_{i_k} \Rightarrow (\bar{w}, x) \notin B_k$. So $(B_0, \dots, B_n)$ is a universal sequence, contradicting Theorem 17.

Finally, it will be shown that $U_0, U_1, \dots$ form an infinite sequence of strongly inseparable coanalytic sets. Let $B_0, B_1, \dots \in B((\mathbb{N}^{\mathbb{N}})^2)$ partition $(\mathbb{N}^{\mathbb{N}})^2$. Assume, towards a contradiction, $\forall h \in \mathbb{N}\ \exists k \in \mathbb{N}\ B_h \cap U_k = \emptyset$. Given h, let $\varphi(h)$ be the least such k. Define $B'_k = \bigcup_{\varphi(h)=k} B_h$. Throwing away those B'_k that are empty, one gets a new Borel partition $\{B'_{k_0}, B'_{k_1}, \dots\}$ of $(\mathbb{N}^{\mathbb{N}})^2$, with the property that $B'_{k_l} \cap U_{k_l} = \emptyset$ for all l. If the sequence $k_0, k_1, \dots$ is finite, a contradiction is reached with the already established strong inseparability of $U_{k_0}, U_{k_1}, \dots$. If it is infinite, let $\{V_0, V_1, \dots\}$ be a Borel partition of $\mathbb{N}^{\mathbb{N}}$. As above, use Lemma 8 to find a closed set $F \subseteq (\mathbb{N}^{\mathbb{N}})^2$ such that

$$V_l = \{x \in \mathbb{N}^{\mathbb{N}} \mid F(x) \text{ has exactly } k_l \text{ elements}\}.$$

By the universality of U there is $\bar{w} \in \mathbb{N}^{\mathbb{N}}$ such that $F = U(\bar{w})$. Hence

$$V_l = \{x \in \mathbb{N}^{\mathbb{N}} \mid U(\bar{w}, x) \text{ has exactly } k_l \text{ elements}\} \subseteq \mathbb{N}^{\mathbb{N}} \setminus B'_{k_l}(\bar{w}).$$

So $(B'_{k_0}, B'_{k_1}, \dots)$ is a universal sequence, contrary to Theorem 17. ∎

THEOREM 19. *Assume analytic determinacy. Let $2 \le n \le \aleph_0$ and let $(C_0, C_1, \dots)$ be a sequence of length n of pairwise disjoint coanalytic subsets of $\mathbb{N}^{\mathbb{N}}$. The following are equivalent:*

(1) *the sequence $(C_0, C_1, \dots)$ is complete;*
(2) *$C_0, C_1, \dots$ are strongly inseparable.*

Proof. (1)⇒(2). Suppose $(C_0, C_1, \dots)$ is a complete sequence. By Theorem 18, let $(Q_0, Q_1, \dots)$ be an n-sequence of strongly inseparable coanalytic sets. Any reduction of $(Q_0, Q_1, \ldots)$ to $(C_0, C_1, \ldots)$ witnesses the strong inseparability of $(C_0, C_1, \ldots)$.

(2)$\Rightarrow$(1). Let $C_0, C_1, \ldots$ be strongly inseparable and $Q_0, Q_1, \ldots$ be pairwise disjoint coanalytic subsets of $\mathbb{N}^{\mathbb{N}}$. Consider the game

$$\begin{array}{llllllll} \mathbf{1} & a_0 & & a_1 & & a_2 & & \ldots \\ \mathbf{2} & & b_0 & & b_1 & & b_2 & \ldots \end{array}$$

where the two players play natural numbers and construct, respectively, $\alpha, \beta \in \mathbb{N}^{\mathbb{N}}$. Player **2** wins the game if and only if

$$\exists i < n \ (\alpha \in Q_i \wedge \beta \in C_i) \vee \Big(\alpha \notin \bigcup_{i<n} Q_i \wedge \beta \notin \bigcup_{i<n} C_i\Big),$$

so this game is determined.

If player **1** had a winning strategy, this would define a continuous function $f : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}},\ \beta \mapsto \alpha$ such that

$$\forall i < n \ (\alpha \notin Q_i \vee \beta \notin C_i) \wedge \Big(\alpha \in \bigcup_{i<n} Q_i \vee \beta \in \bigcup_{i<n} C_i\Big).$$

This means that, denoting $Q'_i = f^{-1}(Q_i)$,

$$\forall i \ C_i \cap Q'_i = \emptyset \quad \text{and} \quad \bigcup_{i<n} C_i \cup \bigcup_{i<n} Q'_i = \mathbb{N}^{\mathbb{N}}.$$

Let $B \in B(\mathbb{N}^{\mathbb{N}})$ separate the disjoint analytic sets $\mathbb{N}^{\mathbb{N}} \setminus \bigcup_i C_i$, $\mathbb{N}^{\mathbb{N}} \setminus \bigcup_i Q'_i$. Notice that all $Q'_i \cap B, C_i \setminus B$ are actually Borel. Let g be a bijection on the index set of the sequence such that $\forall i \ g(i) \neq i$ and set $B_i = (B \cap Q'_i) \cup (C_{g(i)} \setminus B)$. Then $\{B_0, B_1, \ldots\}$ is a Borel partition of $\mathbb{N}^{\mathbb{N}}$ such that $\forall i \ B_i \cap C_i = \emptyset$, in contradiction with the strong inseparability of $C_0, C_1, \ldots$.

So, player **2** has a winning strategy, which induces a continuous function $\mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$, $\alpha \mapsto \beta$ reducing $(Q_0, Q_1, \ldots)$ to $(C_0, C_1, \ldots)$. ∎

Acknowledgments. I wish to thank an anonymous referee, who suggested that Kechris' theorem in [4] could hold in the context of sequences as well, which was indeed the case.

References

[1] R. Camerlo and U. B. Darji, *Construction of Borel inseparable coanalytic sets*, Real Anal. Exchange 28 (2002/03), 163–180.

[2] R. Camerlo and A. S. Kechris, *Countable structures with a fixed group of automorphisms*, Israel J. Math. 117 (2000), 105–124.

[3] A. S. Kechris, *Classical Descriptive Set Theory*, Springer, 1995.

[4] A. S. Kechris, *On the concept of* Π^1_1*-completeness*, Proc. Amer. Math. Soc. 125 (1997), 1811–1814.

[5] J. Saint Raymond, *Complete pairs of coanalytic sets*, Fund. Math. 194 (2007), 267–281.

Riccardo Camerlo
Dipartimento di Scienze Matematiche
"Joseph-Louis Lagrange"
Politecnico di Torino
Corso Duca degli Abruzzi 24
10129 Torino, Italy
E-mail: camerlo@calvino.polito.it

Received 28 July 2012;
in revised form 19 November 2013