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Totally proper forcing and the Moore–Mrówka problem

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Abstract. We describe a totally proper notion of forcing that can be used to shoot uncountable free sequences through certain countably compact non-compact spaces. This is almost (but not quite!) enough to produce a model of ZFC + CH in which countably tight compact spaces are sequential—we still do not know if the notion of forcing described in the paper can be iterated without adding reals.

1. Introduction. The Moore–Mrówka problem addresses the question of whether or not countably tight compact spaces (i.e., compact spaces where the closure operator is determined by how it acts on countable sets) must be sequential (i.e., the closure operator is determined by iterating the process of taking limits of convergent sequences). The best introduction to what is known about this problem is probably Shakhmatov's article [12] but we will take a little time to outline the major results known. A famous example constructed by Ostaszewski in the 1970's (see [11]) showed that $\diamondsuit$ implies that there is a countably tight compact space that is not sequential, while Balogh [1] showed in the late 1980's that the Proper Forcing Axiom implies compact spaces of countable tightness are sequential. Other people have had significant results concerning the influence of the Proper Forcing Axiom on the structure of countably tight compact spaces. A good survey of these results can be found in the article [2].

The author and others have been involved in research concerning the influence of the Continuum Hypothesis on this problem. Over the years, several results in this vein have been obtained—see, e.g., the papers [6], [4], and [5]. This paper contains a partial result supporting the conjecture that CH is not enough to imply the existence of a non-sequential countably tight compact space. We make some comments at the end of the paper about what might be needed to completely resolve this question.

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We begin with the basic definitions involved in the statement of the problem.

Definition 1.1. Let X be a topological space. X is said to be *countably tight* ($t(X) = \aleph_0$) if whenever a point z is in the closure of a set A, we can find a countable $A_0 \subseteq A$ such that z is in the closure of A_0. X is said to be *sequential* if a set $A \subseteq X$ is closed if and only if A contains all limits of convergent sequences from A.

It is not hard to show that a sequential space is countably tight, and there are fairly easy examples of countably tight (non-compact!) spaces that are not sequential. The Moore–Mrówka problem arises when we ask if these concepts coincide in the class of compact Hausdorff spaces.

In this paper, we show that there is a notion of forcing that will destroy a fixed counterexample to the Moore–Mrówka problem while not adding reals. The notion of forcing is proper, but it is not clear if it can be iterated safely without adding reals. If it can be safely iterated, then we can build a model of ZFC + CH in which compact spaces of countable tightness are sequential—in the final section the paper we will show why this is true.

Our strategy is to follow the route of Balogh. In models where CH is true, a countably tight compact space is sequential if and only if every countably compact subspace of it is closed (this is a result of Ismail and Nyikos [9], and in fact only requires the assumption $2^{\aleph_0} < 2^{\aleph_1}$). Thus a potential counterexample would consist of a compact, countably tight space X and a countably compact $Y \subseteq X$ such that Y is not closed in X. This gives us the first bit of ammunition for our attack on the problem.

A well known result on cardinal functions due to Arkhangel'skiĭ tells us that a compact space is countably tight if and only if it does not contain an uncountable free sequence. Good references for this result are the monograph [10] of Juhász, and Hodel's survey [8]. We recall the definition for those unfamiliar with it.

Definition 1.2. Let X be a topological space. A sequence $\{x_\alpha : \alpha < \kappa\}$ is a *free sequence* (of length κ) if for each $\alpha < \kappa$,

$$\overline{\{x_\beta : \beta < \alpha\}} \cap \overline{\{x_\beta : \beta \geq \alpha\}} = \emptyset. \tag{1.1}$$

Our strategy is to take a potential counterexample and to "shoot" an uncountable free sequence through it, thereby wrecking its countable tightness. This is where the results of [6] and [4] come in—in both of these papers, it is shown that in certain circumstances one can take a countably compact (regular) space that does not contain an uncountable free sequence, and then shoot an uncountable free sequence through it without adding new reals to the ground model. A natural attack on the Moore–Mrówka problem would be to use the fact that X contains a countably compact, non-compact subspace Y and try and shoot an uncountable free sequence through Y and arrange that it will be an uncountable free sequence in X.

There are two obstacles to this approach. The first is that the results of [4] and [5] demand that the space under consideration be first countable, and counterexamples to the Moore–Mrówka problem have got to be far from first countable. The second obstacle is that we cannot hope to blindly generalize the methods of [5] because known examples prove that it is impossible—a result of Hajnal and Juhász [7] tells us that CH implies the existence of a countably compact, non-compact countably tight space that contains no uncountable free sequences. In fact, their space is hereditarily separable and even a topological group.

The space of Hajnal and Juhász does not come up in the study of the Moore–Mrówka problem, however. The key to this rests on another cardinal function from topology—hereditary π-character.

Definition 1.3. Let A be a subset of the topological space X. A family $\mathcal{B}$ of subsets of X is a π-*network* at A in X if every open neighborhood of A contains some $B \in \mathcal{B}$. If $\mathcal{B}$ consists of open subsets of X, we say $\mathcal{B}$ is a π-*base* for A in X. If A is a singleton $\{x\}$, we call $\mathcal{B}$ a π-network, respectively π-base, at x in X.

Definition 1.4. We say a point x has *countable π-character in X* ($\pi\chi(x, X) = \aleph_0$) if x has a countable π-base in X. If $\pi\chi(x, X) = \aleph_0$ for every $x \in X$, then we say X has *countable π-character* and denote this by $\pi\chi(X) = \aleph_0$. We say that X is *hereditarily of countable π-character* ($h\pi\chi(X) = \aleph_0$) if $\pi\chi(Y) = \aleph_0$ for every subspace Y of X.

A celebrated result of Shapirovskiĭ [13] tells us that in compact Hausdorff spaces, tightness and hereditary π-character coincide (again, see [8] for a nice proof of this). It is not hard to see that the space of Hajnal and Juhász is not hereditarily of countable π-character—the quickest way is to take advantage of the fact that the space is a topological group, for it is well known that character and π-character coincide in the class of topological groups (see Comfort's survey [3] for a proof of this).

The fact that the spaces of concern to us are all hereditarily of countable π-character is the key to our arguments—in a sense, countably compact spaces that are hereditarily of countable π-character behave enough like first countable spaces so that our methods generalize. The moral is that for countably compact, regular X, the assumption that $h\pi\chi(X) = \aleph_0$ has a tremendous impact on how badly behaved the space might be. We will see examples of this phenomenon when we analyze the notion of forcing that we define.

2. Elementary submodels. In this section, we investigate how elementary submodels interact with the topological spaces of interest to us. We will assume that X is a topological space satisfying the following:

- X is countably compact, non-compact T_3,
- $\mathcal{F}$ is a maximal free filter of closed subsets of X,
- $t(X) = \aleph_0$,
- $|X| = \aleph_1$.

Not all of the proofs will use all of the assumptions about our space; in particular, the tightness and cardinality restrictions are not needed in all cases.

Throughout the rest of the paper, we shall use the phrase "for almost all x" to mean "the set of such x is in $\mathcal{F}$".

The next batch of definitions has appeared in various guises in earlier work of the author, e.g., [4] and [5].

Let N be a countable elementary submodel of $H(\lambda)$ for some large regular λ, and assume $\{X, \mathcal{F}\} \in N$.

Definition 2.1. The *trace* of N, denoted $\mathrm{Tr}(N)$, is defined by

$$\mathrm{Tr}(N) = \bigcap_{A \in N \cap \mathcal{F}} \mathrm{cl}(N \cap A). \tag{2.1}$$

Proposition 2.2. $\mathrm{Tr}(N)$ *is a non-empty closed subset of* X.

Proof. We need only worry about showing that $\mathrm{Tr}(N)$ is non-empty. This follows from the countable compactness of X because the collection $\{\overline{N \cap A} : A \in N \cap \mathcal{F}\}$ is countable and centered. ■

There is another closed subset of X that is natural to consider in this context.

Definition 2.3. If N is a countable elementary submodel of $H(\lambda)$ containing X and $\mathcal{F}$, then we define the *weak trace* of N, denoted $\mathrm{wTr}(N)$, by

$$\mathrm{wTr}(N) = \bigcap \{A : A \in N \cap \mathcal{F}\}. \tag{2.2}$$

Note that $\mathrm{wTr}(N)$ is a countable intersection of elements of $\mathcal{F}$, and therefore $\mathrm{wTr}(N)$ is always an element of $\mathcal{F}$. We can certainly conclude that $\mathrm{Tr}(N) \subseteq \mathrm{wTr}(N)$, but in general these two sets need not be equal; the next definition and proposition will shed some light on the situation.

Definition 2.4. We say that $(X, \mathcal{F})$ is of *Type A* if $\mathcal{F}$ is generated by separable sets, i.e., if for every set $E \in \mathcal{F}$, there is a separable $E_0 \subseteq E$ such that $E_0 \in \mathcal{F}$. We say $(X, \mathcal{F})$ is of *Type B* if it is not of Type A.

Proposition 2.5. *The following statements are equivalent*:

(1) $\mathrm{Tr}(N) \in \mathcal{F}$,

(2) *$(X,\mathcal{F})$ is of Type A*,
(3) $\mathrm{Tr}(N) = \mathrm{wTr}(N)$.

Proof. Left to reader. ∎

Our next few definitions and propositions work toward developing a notion of diagonal intersection for the filter $\mathcal{F}$.

PROPOSITION 2.6. *Let $\mathfrak{M} = \langle M_\alpha : \alpha < \omega_1\rangle$ be an $\in$-increasing chain of countable elementary submodels of $H(\lambda)$, continuous at limit ordinals, such that $\{X,\mathcal{F}\} \in M_0$ and for $\alpha < \omega_1$, $\langle M_\beta : \beta < \alpha\rangle \in M_{\alpha+1}$. Then*

$$\mathrm{Tr}(\mathfrak{M}) := \bigcup_{\alpha<\omega_1} \mathrm{Tr}(M_\alpha) \textit{ is closed.} \tag{2.3}$$

Proof. If $(X,\mathcal{F})$ is of Type A, then this follows easily as the set in question is just equal to $\mathrm{Tr}(M_0)$. Thus assume that $(X,\mathcal{F})$ is of Type B.

We prove by induction on $\alpha < \omega_1$ that the set

$$A_\alpha = \bigcup_{\beta\le\alpha} \mathrm{Tr}(M_\beta) \tag{2.4}$$

is closed; this is sufficient as X is countably tight.

The cases where $\alpha = 0$ or α is a successor ordinal are already handled by the induction hypothesis, so assume that α is a limit ordinal.

Let $A_{<\alpha}$ denote $\bigcup_{\beta<\alpha}\mathrm{Tr}(M_\beta)$. To show that A_α is closed, it suffices to prove

$$\overline{A}_{<\alpha} \setminus A_{<\alpha} \subseteq \mathrm{Tr}(M_\alpha),$$

so assume that x is a member of $\overline{A}_{<\alpha} \setminus A_{<\alpha}$.

Let U be any neighborhood of x. Since x is not in $A_{<\alpha}$, our induction hypothesis implies that U must intersect A_β for arbitrarily large $\beta < \alpha$.

Now given $B \in M_\alpha \cap \mathcal{F}$, there is some $\beta_0 < \alpha$ with $B \in M_{\beta_0}$, and hence there is $\beta < \alpha$ such that $B \in M_\beta$ and $U \cap \mathrm{Tr}(M_\beta) \ne \emptyset$.

By the definition of $\mathrm{Tr}(M_\beta)$, we see that $U \cap M_\beta \cap B$ is non-empty, and hence $U \cap M_\alpha \cap B$ is non-empty. Since U was an arbitrary neighborhood of x and B was an arbitrary member of $M_\alpha \cap \mathcal{F}$, we have $x \in \mathrm{Tr}(M_\alpha)$ as required. ∎

THEOREM 1. *If $\mathfrak{M}$ is as in the previous proposition, then $\mathrm{Tr}(\mathfrak{M}) \in \mathcal{F}$.*

Proof. If $(X,\mathcal{F})$ is of Type A, then this is immediate by Proposition 2.5, so we assume that X is of Type B.

We know that $\mathrm{Tr}(\mathfrak{M})$ is closed, so it suffices (because of the maximality of $\mathcal{F}$) to show that it meets every set in $\mathcal{F}$. Let $B \in \mathcal{F}$ be arbitrary, and let N be a countable elementary submodel of $H(\lambda)$ that contains X, $\mathcal{F}$, $\mathfrak{M}$, and B. Note that if $\delta = N \cap \omega_1$, then $M_\delta \cap \omega_1 = \delta$ as well, and since $|X| = \aleph_1$, we have $N \cap X = M_\delta \cap X$.

For $\alpha < \delta$, $M_\alpha \in N$ and hence $M_\alpha \subseteq N$ as well. Thus $M_\delta \subseteq N$. Together with the fact that $M_\delta \cap X = N \cap X$, we have $\mathrm{Tr}(N) \subseteq \mathrm{Tr}(M_\delta)$. Since $\mathrm{Tr}(N)$ is a non-empty subset of B (as $B \in N \cap \mathcal{F}$), we have $B \cap \mathrm{Tr}(\mathfrak{M}) \neq \emptyset$, as required. ∎

We end this section with a corollary that summarizes the work we have done so far.

Corollary 2.7. *Almost every point of X is a member of* $\mathrm{Tr}(M)$ *for some appropriate M.*

3. Promises. In this section, we investigate promises, a combinatorial tool that we use to define side conditions for our notion of forcing.

Definition 3.1. Let us say that a subset A of X is *large* if it meets every set in $\mathcal{F}$; otherwise we say that A is *small*.

Note that since $\mathcal{F}$ is closed under countable intersections, any countable union of small sets is small.

Definition 3.2. A *promise* is a function f whose domain is a large subset of X such that for $x \in \operatorname{dom} f$, $f(x)$ is an open neighborhood of x, i.e., f is a neighborhood assignment for a large subset of X.

Definition 3.3. If f is a promise, then we say a point y is *banned by f* if

$$\{x \in \operatorname{dom} f : y \in f(x)\} \text{ is small.} \tag{3.1}$$

We let $\operatorname{Ban} f$ be the set of all $y \in X$ that are banned by f.

Claim 3.4. *If f is a promise, then* $\operatorname{Ban} f$ *is closed.*

Proof. Suppose not, and let y be a limit point of $\operatorname{Ban} f$ that is not banned by f. Since X is countably tight, there is a countable set $A = \{y_n : n \in \omega\} \subseteq \operatorname{Ban} f$ such that $y \in \overline{A}$. Now let

$$B = \{x \in \operatorname{dom} f : y \in f(x)\}. \tag{3.2}$$

Note that B is large as y is not banned by f.

For $n \in \omega$, we let

$$B_n = \{x \in B : y_n \in f(x)\}. \tag{3.3}$$

Each B_n is small as y_n is banned by f, but since $y \in \overline{A}$, we have

$$B = \bigcup_{n \in \omega} B_n, \tag{3.4}$$

which is a contradiction. ∎

Lemma 3.5. *If f is a promise and $(X, \mathcal{F})$ is of Type A, then* $\operatorname{Ban} f$ *is not in $\mathcal{F}$.*

Proof. Suppose $\operatorname{Ban} f \in \mathcal{F}$. Since $(X, \mathcal{F})$ is of Type A, there is a separable set $A \subseteq \operatorname{Ban} f$ such that $A \in \mathcal{F}$, say $A = \overline{\{y_n : n \in \omega\}}$. Let $B = A \cap \operatorname{dom} f$. Since $\operatorname{dom} f$ is large and $A \in \mathcal{F}$, we deduce that B is large as well.

Now let $B_n = \underline{\{x \in B : y_n} \in f(x)\}$. Each B_n is small as y_n is banned by f, but since $B \subseteq \overline{\{y_n : n \in \omega\}}$, we have

$$B = \bigcup_{n \in \omega} B_n, \tag{3.5}$$

a contradiction. ■

LEMMA 3.6. *If f is a promise and $(X, \mathcal{F})$ is of Type B, then* $\operatorname{Ban} f$ *is not in $\mathcal{F}$.*

Proof. Suppose $\operatorname{Ban} f$ is an element of $\mathcal{F}$. For each $y \in \operatorname{Ban} f$, let A_y be a set in $\mathcal{F}$ such that

$$A_y \cap \{x \in \operatorname{dom} f : y \in f(x)\} = \emptyset. \tag{3.6}$$

Now let $\mathfrak{M} = \langle M_\alpha : \alpha < \omega_1 \rangle$ be an $\in$-chain of countable elementary submodels as in the previous section such that both the promise f and the function $y \mapsto A_y$ are elements of M_0.

Now choose a point $x \in \operatorname{dom} f \cap \operatorname{Tr}(\mathfrak{M})$, say $x \in \operatorname{Tr}(M_\alpha) \cap \operatorname{dom} f$. By definition of $\operatorname{Tr}(M_\alpha)$, we can find a point

$$y \in f(x) \cap M_\alpha \cap \operatorname{Ban} f, \tag{3.7}$$

and this is a contradiction as $A_y \in M_\alpha \cap \mathcal{F}$ implies $x \in \operatorname{Tr}(M_\alpha) \subseteq A_y$. ■

Putting the two previous lemmas together, we come to the main point of this section.

THEOREM 2. *If f is a promise, then* $\operatorname{Ban} f$ *is a closed set that is not in $\mathcal{F}$.*

4. A notion of forcing. Armed with the results of the previous two sections, we are now ready to define our notion of forcing. From now on, we assume that X is a topological space such that

- X is a countably compact, non-compact T_3 space,
- $h\pi\chi(X) = \aleph_0$,
- $|X| = \aleph_1$,
- $\mathcal{F}$ is a maximal free filter of closed subsets of X,
- $\mathcal{C}$ is an open cover of X.

We will define a totally proper (see Definition 5.1) notion of forcing that will adjoin an uncountable free sequence $F = \{x_\alpha : \alpha < \omega_1\}$ to X. Furthermore, we can guarantee that each countable subset of F is covered by finitely many members of $\mathcal{C}$, and for each $A \in \mathcal{F}$, all but countably many members of F are contained in A.

Definition 4.1. A condition is a triple $p = (\sigma_p, A_p, \Phi_p)$ such that

(1) σ_p is a one-to-one function from some countable ordinal into X,
(2) $[p] := \operatorname{ran} \sigma_p$ is covered by finitely many members of $\mathcal{C}$,
(3) $\operatorname{cl}[p] \cap A_p = \emptyset$,
(4) $A_p \in \mathcal{F}$,
(5) Φ_p is a countable collection of promises.

We say that a condition q extends p (written $q \le p$) if

(6) $\sigma_q \supseteq \sigma_p$,
(7) $A_q \subseteq A_p$,
(8) $\Phi_q \supseteq \Phi_p$,
(9) $[q] \setminus [p] \subseteq A_p$,
(10) if $f \in \Phi_p$, then the set

$$Y(f,q,p) := \{x \in \operatorname{dom} f : [q] \setminus [p] \subseteq f(x)\}$$

is large, and $f \restriction Y(f,q,p) \in \Phi_q$.

We will have to postpone the proof that this notion of forcing is totally proper until the next section. For the rest of this section, we will be proving combinatorial lemmas to aid in the proof of total properness. We start with and *ad hoc* definition that will help us investigate how much freedom the first component of a given condition has to grow.

Definition 4.2. Let $p \in P$ be a condition. A point $z \in X$ is *eligible* for p if there is a condition $q \le p$ such that $z \in [q]$.

Note that if $\mathcal{F} \in \Phi_p$ and $z \in \operatorname{Ban} f$, then z is not eligible for p; the promises in Φ_p put some restrictions on how $[p]$ can grow. However, it turns out that relatively few points are excluded from membership in $[p]$.

Lemma 4.3. *Let $p \in P$ be a condition. Then there is a set $A \in \mathcal{F}$ such that every point in A is eligible for p.*

Proof. Let us define B to be the union of sets of the form $\operatorname{Ban} f$ for $f \in \Phi_p$. By Theorem 2 and the fact that Φ_p is countable, we know that B is a small set. Thus there is a set $A \in \mathcal{F}$ such that $A \cap B = \emptyset$ and furthermore, without loss of generality, $A \subseteq A_p$.

Take a point $x \in A$, and let A' be a subset of A in $\mathcal{F}$ that does not contain x. For each $f \in \Phi_p$, let $Y_f = \{y \in \operatorname{dom} f : x \in f(y)\}$. Each Y_f is a large set by the definitions involved. Let $\alpha = \operatorname{dom} \sigma_p$. We define a condition q by setting

$$\sigma_q = \sigma_p \cup \{\langle \alpha, x \rangle\}, \qquad A_q = A', \qquad \Phi_q = \Phi_p \cup \{f \restriction Y_f : f \in \Phi_p\}.$$

It is straightforward to verify that q is a condition in P that extends p and satisfies $x \in [q]$. ■

COROLLARY 4.4. *If $p \in P$, then there is a condition $q \leq p$ such that $[q] \setminus [p]$ is non-empty.*

LEMMA 4.5. *Let $p \in P$ be arbitrary, and let $D \subseteq P$ be dense open. For almost every $x \in X$, the collection*

$$\mathcal{D} = \{[q] \setminus [p] : q \leq p \text{ and } q \in D\} \tag{4.1}$$

is a π-network at $x \in X$.

Proof. Clearly, the set of such points x is closed, so it suffices to prove that the complement of this set is small. Let us define

$$E = \{x \in X : \mathcal{D} \text{ is not a } \pi\text{-network at } x \text{ in } X\},$$

and assume by way of contradiction that E is large.

For each $x \in E$, there is an open set U_x such that $x \in U_x$ and there is no $q \leq p$ such that $q \in D$ and $[q] \setminus [p]$ is a non-empty subset of U_x. The function f with domain E defined by $f(x) = U_x$ is a promise (as E is large), and

$$p' = (\sigma_p, A_p, \Phi_p \cup \{f\})$$

is a condition in P. Since D is dense in P, we can find an extension q of p' that lies in D. By Corollary 4.4, without loss of generality $[q] \setminus [p']$ is non-empty.

By the definition of extension, the set

$$Y(f, q, p') = \{x \in \operatorname{dom} f : [q] \setminus [p'] \subseteq f(x)\}$$

is large, hence non-empty. Choose $x \in Y(f, q, p')$. For this particular x, we have

$$[q] \setminus [p] = [q] \setminus [p'] \subseteq f(x) = U_x, \tag{4.2}$$

and this contradicts the choice of U_x. ■

We need to sharpen the previous lemma a bit. We again make a rather *ad hoc* definition.

DEFINITION 4.6. Assume $p \in P$, $D \subseteq P$ is dense open, and $A \in \mathcal{F}$. We say that a point x is *good to p, D, and A* if the set

$$\mathcal{D}_A = \{[q] \setminus [p] : q \leq p,\ q \in D, \text{ and } [q] \setminus [p] \subseteq A\}$$

is a π-network at $x \in X$. We let $\operatorname{Good}(p, D, A)$ denote the set of points that are good to p, D, and A.

LEMMA 4.7. *Given $p \in P$, $D \subseteq P$ dense open, and $A \in \mathcal{F}$, almost every point is good to p, D, and A.*

Proof. Again, the set of points that are good to p, D, and A is closed, so it suffices to prove that the set of such points is in $\mathcal{F}$. Suppose this fails, and fix a set $B \in \mathcal{F}$ such that no $x \in B$ is good to p, D, and A. Let N be a countable elementary submodel of $H(\lambda)$ that contains $\{X, \mathcal{F}, P, p, D, A, B\}$.

Fix a point $x \in \mathrm{wTr}(N)$. Since $\mathrm{wTr}(N) \subseteq B$ we know that x has a neighborhood U such that there is no $q \leq p$ in D such that $[q] \setminus [p]$ is a non-empty subset of $U \cap A$. Now let us define

$$p' = (\sigma_p, A_p \cap A, \Phi_p).$$

Then p' is a condition in P that extends p and, more importantly for our purposes, $p' \in N$. The set of points $y \in X$ such that $\{[q] \setminus [p'] : q \leq p'$ and $q \in D\}$ is a π-network at y in X is an element of $\mathcal{F}$, and since all parameters required to define this set are in N, it is a set in $N \cap \mathcal{F}$. Since $x \in \mathrm{wTr}(N)$, this means that there is a $q \leq p'$ in D such that $[q] \setminus [p']$ is a non-empty subset of U. This is a contradiction as q is an extension of p in D, and

$$[q] \setminus [p] = [q] \setminus [p'] \subseteq A_p \cap A \subseteq A. \ ■$$

The proof of the next theorem is where we finally use our hypothesis that $h\pi\chi(X) = \aleph_0$—it shows that our space is in some sense nicely organized.

Definition 4.8. Assume $p \in P$, $D \subseteq P$ is dense open, and $A \in \mathcal{F}$. A point $x \in X$ is *nice to* p, D, *and* A if there is a countable family of conditions $\{q_n : n \in \omega\}$ such that

- $q_n \leq p$,
- $q_n \in D$,
- $\{[q_n] \setminus [p] : n \in \omega\}$ forms a π-network at x in A.

Theorem 3. *If* p, D, *and* A *are as in the previous definition*, *then almost every point* x *is nice to* p, D, *and* A.

We will prove this theorem shortly, but first we need a key lemma.

Lemma 4.9. *Let* X *be a countably compact space*, *and let* $\{A_n : n \in \omega\}$ *be a decreasing family of closed sets. Let* U *be an open set that meets* $K := \bigcap_{n\in\omega} A_n$. *Then* $\{U \cap A_n : n \in \omega\}$ *is a* π*-network at* $\overline{U} \cap K$ *in* X.

Proof. Let V be an open neighborhood of $\overline{U} \cap K$. It suffices to show that there is an n such that $(U \cap A_n) \setminus V$ is finite, because if $y \in (U \cap A_n) \setminus V$ then $y \notin K$ and hence (since the sequence is decreasing) there is an $m > n$ such that $y \notin A_m$. Given that $(U \cap A_n) \setminus V$ is finite, we can simply increase n to ensure that $(U \cap A_n) \setminus V$ is empty, i.e., $U \cap A_n \subseteq V$.

Suppose no such n exists. We can then choose distinct points x_n for $n \in \omega$ such that $x_n \in (U \cap A_n) \setminus V$. Since X is countably compact, the infinite set $\{x_n : n \in \omega\}$ has a limit point x.

Since each x_n is in U, we know that $x \in \overline{U}$. Since $x_n \in A_n$ and the A_n's are decreasing, we have $x \in K$. Thus

$$x \in \overline{U} \cap K \subseteq V. \tag{4.3}$$

This is a contradiction, as we have made sure that no x_n is in V, and hence

$$\overline{\{x_n : n \in \omega\}} \cap V = \emptyset. \tag{4.4}$$ ∎

Proof of Theorem 3. Since X is countably tight, the set of points that are nice to p, D, and A is closed. Let N be a countable elementary submodel of $H(\lambda)$ containing p, D, A, and the other relevant parameters. We show that any point in $\mathrm{wTr}(N)$ is nice to p, D, and A.

Choose $x \in \mathrm{wTr}(N)$. By assumption, $\pi\chi(x, \mathrm{wTr}(N)) = \aleph_0$, and since X is regular this implies that we can find a family $\{U_m : m \in \omega\}$ of open sets in X such that

- $U_m \cap \mathrm{wTr}(N) \neq \emptyset$,
- if V is an open neighborhood of x, then there is an m such that $\overline{U}_m \cap \mathrm{wTr}(N) \subseteq V$.

Let $\{B_i : i \in \omega\}$ be a decreasing family of closed sets in $N \cap \mathcal{F}$ generating $N \cap \mathcal{F}$ and also satisfying $B_0 \subseteq A$. Note that this implies

$$\mathrm{wTr}(N) = \bigcap_{i\in\omega} B_i. \tag{4.5}$$

For each $m < \omega$, we can apply Lemma 4.9 to conclude that

$$\mathcal{B}_m := \{B_i \cap U_m : i \in \omega\} \tag{4.6}$$

is a π-network at $\overline{U}_m \cap \mathrm{wTr}(N)$.

Now fix i and m. The set B_i is in $N \cap \mathcal{F}$, so by Lemma 4.7 applied in N, there is an $i' > i$ such that $B_{i'} \subseteq \mathrm{Good}(p, D, B_i)$. Since U_m meets $\mathrm{wTr}(N)$ and $\mathrm{wTr}(N) \subseteq B_{i'}$, we can find a point $y_{m,i} \in U_m \cap \mathrm{Good}(p, D, B_i)$. By definition, this means that there is some condition $q_{m,i}$ such that

- $q_{m,i} \leq p$,
- $[q_{m,i}] \setminus [p] \neq \emptyset$,
- $q_{m,i} \in D$,
- $[q_{m,i}] \setminus [p] \subseteq U_m \cap B_i$.

To finish, we show that the family $\{q_{m,i} : m, i < \omega\}$ witnesses that x is nice to p, D, and A. For this, we must take an arbitrary neighborhood V of x and show that for some m and i,

$$[q_{m,i}] \setminus [p] \subseteq V \cap A. \tag{4.7}$$

Given V, there is an m such that $\overline{U}_m \cap \mathrm{wTr}(N) \subseteq V$. Our definition of $\mathcal{B}_m$ implies that there is an i such that $U_m \cap B_i \subseteq V$. Our choice of $q_{m,i}$ means

$$[q_{m,i}] \setminus [p] \subseteq B_i \cap U_m \subseteq V, \tag{4.8}$$

and since $B_i \subseteq A$, we deduce that $q_{m,i}$ is as required. ∎

5. Total properness. In this section, we prove that the notion of forcing defined in the last section is totally proper, i.e., it is proper and forcing with it adds no new reals. We recall the definition for those who are not familiar with previous work in this area.

DEFINITION 5.1. A notion of forcing P is *totally proper* if whenever we are given $N \prec H(\lambda)$ countable (with λ "large enough") such that $P \in N$, and $p \in N \cap P$, we can find $q \leq p$ such that for every dense open subset D of P that is in N, there is some $p' \in N \cap D$ with $q \leq p'$. Such a q is said to be *totally* (N, P)*-generic.*

It is shown in [6] that a notion of forcing is totally proper if and only if it is proper and forcing with it adds no new countable subsets to the ground model.

Let us fix a countable elementary submodel N of $H(\lambda)$ and assume that N contains P. Note that N will contain X and $\mathcal{F}$ as well—N knows that P was built from such objects so we can find such objects in N.

LEMMA 5.2 (Extension Lemma). *Let $p \in N \cap P$, and let $D \in N$ be a dense subset of P. Given $A \in N \cap \mathcal{F}$ and an open set U such that $U \cap \mathrm{Tr}(N) \neq \emptyset$, we can find $q \leq p$ such that $q \in N \cap D$ and $[q] \setminus [p] \subseteq N \cap U \cap A$.*

Proof. By Theorem 3, the set B of points that are nice to p, D, and A is a member of $\mathcal{F}$, and since B is definable from parameters in N, it is a member of N as well. Since $U \cap \mathrm{Tr}(N) \neq \emptyset$, there is a point $y \in N \cap U \cap B$.

Since this y is nice to p, D, and A, there is a family $\{q_n : n \in \omega\}$ that witnesses this. By elementarity, we can assume this collection is in N and hence $\{q_n : n \in \omega\} \subseteq N$.

Fix an n such that $[q_n] \setminus [p] \subseteq U \cap A$. Since $[q_n]$ is countable and $[q_n] \setminus [p]$ is an element of N, we see that $[q_n] \setminus [p]$ is a subset of N. Thus we have produced $q_n \leq p$ in $N \cap D$ such that $[q_n] \setminus [p] \subseteq N \cap U \cap A$, as required. ∎

LEMMA 5.3 (Target Lemma). *Let $f \in N$ be a promise, and let U be an open set that meets $\mathrm{Tr}(N)$. Then there is an $A \in N \cap \mathcal{F}$ and an open $V \subseteq U$ such that*

- *V meets $\mathrm{Tr}(N)$,*
- *$\{x \in \mathrm{dom}\, f : N \cap V \cap A \subseteq f(x)\}$ is large.*

Proof. Choose $z \in U \cap \mathrm{Tr}(N)$. Since X is regular and $\pi\chi(z, \mathrm{Tr}(N)) = \aleph_0$, we can find a family $\{U_n : n \in \omega\}$ of open sets such that

- $U_n \subseteq U$,
- $U_n \cap \mathrm{Tr}(N) \neq \emptyset$,
- if W is an open neighborhood of z, there is an N such that $\overline{U}_n \cap \mathrm{Tr}(N) \subseteq W$.

Let $E_0 = \{x \in \operatorname{dom} f : z \in f(x)\}$. Note that E_0 is large as no element of $\mathrm{wTr}(N)$ is banned by the promise f. If $x \in E_0$, then there is an n such that $\overline{U}_n \cap \mathrm{Tr}(N) \subseteq f(x)$. Since a countable union of small sets is small, there must be an n for which

$$E_1 := \{x \in E_0 : \overline{U}_n \cap \mathrm{Tr}(N) \subseteq f(x)\} \text{ is large.}$$

Choose such an n, and define $V = U_n$.

Now let $\{A_i : i \in \omega\}$ be a decreasing family in $N \cap \mathcal{F}$ that generates $N \cap \mathcal{F}$, and let $B_i = \overline{N \cap A_i}$. Note that $\mathrm{Tr}(N) = \bigcap_{i<\omega} B_i$. By Lemma 4.9, the sets $\{V \cap B_i : i \in \omega\}$ form a π-network at $\overline{V} \cap \mathrm{Tr}(N)$. Thus if $x \in E_1$, there is an i such that $V \cap B_i \subseteq f(x)$. Hence there must be a single i such that

$$E_2 := \{x \in E_1 : V \cap B_i \subseteq f(x)\} \text{ is large.}$$

If we let $A = A_i$, then we have $N \cap V \cap A \subseteq N \cap V \cap B_i$, and therefore

$$E_2 \subseteq \{x \in \operatorname{dom} f : N \cap V \cap A \subseteq f(x)\}, \tag{5.1}$$

as required. ∎

THEOREM 4. *The notion of forcing P is totally proper.*

Proof. Given $p \in N \cap P$, we must produce a totally (N, P)-generic $q \leq p$. Let $\{D_n : n \in \omega\}$ list the dense subsets of P that are elements of N. In ω stages we construct objects p_n, U_n, and A_n such that

(1) $p_0 = p$, $A_0 = X$,

(2) U_0 is some open set contained in a member of $\mathcal{C}$ that meets $\mathrm{Tr}(N)$ and satisfies $\overline{U}_0 \notin \mathcal{F}$,

(3) $p_{n+1} \in N \cap D_n$,

(4) $p_{n+1} \leq p_n$,

(5) U_n is an open set that meets $\mathrm{Tr}(N)$,

(6) $U_{n+1} \subseteq U_n$,

(7) A_n is a member of $N \cap \mathcal{F}$,

(8) $A_{n+1} \subseteq A_n$,

(9) $[p_{n+1}] \setminus [p_n] \subseteq N \cap U_{n+1} \cap A_{n+1}$,

(10) for each n and $f \in \Phi_{p_n}$, there is a stage $m \geq n$ for which

$$\{x \in Y(f, p_m, p_n) : N \cap A_{m+1} \cap U_{m+1} \subseteq f(x)\} \text{ is large} \tag{5.2}$$

(we say that the promise f is *taken care of at stage* $m + 1$).

At stage $n + 1$, we are handed p_n, D_n, U_n, and A_n, as well as some promise f appearing in some earlier Φ_{p_i} that must be taken care of at this stage.

By the definition of extension for our partial order, we know that $f' := f \restriction Y(f, p_n, p_i)$ is an element of Φ_{p_n} and hence an element of N as well. By the Target Lemma (Lemma 5.3) we can find an open $U_{n+1} \subseteq U_n$ that meets $\mathrm{Tr}(N)$, as well as an $A_{n+1} \subseteq A_n$ in $N \cap \mathcal{F}$ such that

$$\{x \in \operatorname{dom} f' : N \cap A_{n+1} \cap U_{n+1} \subseteq f(x)\} \text{ is large.} \tag{5.3}$$

By the Extension Lemma (Lemma 5.2) we can find $p_{n+1} \leq p_n$ in $N \cap D_n$ such that

$$[p_{n+1}] \setminus [p_n] \subseteq N \cap A_{n+1} \cap U_{n+1}. \tag{5.4}$$

To finish, we need only prove that the sequence $\{p_n : n \in \omega\}$ has a lower bound q in P. We define q "one piece at a time".

First, let $\sigma_q = \bigcup_{n\in\omega} \sigma_{p_n}$. Clearly σ_q is a one-to-one function from a countable ordinal into X as each σ_p is such a function. Let $[q]$ denote the range of σ_q.

We know that $[p]$ is covered by finitely many members of the open cover $\mathcal{C}$. By construction

$$[q] \setminus [p] \subseteq U_0 \tag{5.5}$$

and U_0 is contained in a member of $\mathcal{C}$. Thus $[q]$ is covered by finitely many members of $\mathcal{C}$. Since $\mathrm{cl}[p] \notin \mathcal{F}$ and $\overline{U}_0 \notin \mathcal{F}$, we have $\mathrm{cl}[q] \notin \mathcal{F}$.

Let A_q be some member of $\mathcal{F}$ that is a subset of $\mathrm{wTr}(N)$ and disjoint from $\mathrm{cl}[q]$. Clearly A_q is a subset of A_{p_n} for each n.

If f is a promise appearing in Φ_{p_n} for some n, there is a stage $m \geq n$ such that we take care of f at stage $m+1$. Recall that this means we ensure

$$E := \{x \in Y(f, p_m, p_n) : U_{m+1} \cap A_{m+1} \cap N \subseteq f(x)\} \text{ is large.} \tag{5.6}$$

Since our sequences of U_n's and A_n's are decreasing, we know

$$[q] \setminus [p_m] \subseteq N \cap U_{m+1} \cap A_{m+1}, \tag{5.7}$$

and this means

$$Y(f, q, p_n) := \{x \in \operatorname{dom} f : [q] \setminus [p_n] \subseteq f(x)\} \text{ is large.} \tag{5.8}$$

Thus if we define

$$\Phi_q = \bigcup_{n\in\omega} \Phi_{p_n} \cup \bigcup_{n\in\omega} \{f \restriction Y(f, q, p_n) : f \in \Phi_{p_n}\}, \tag{5.9}$$

it is straightforward to verify that $q = (\sigma_q, A_q, \Phi_q)$ is a condition in P that is a lower bound for the sequence $\{p_n : n \in \omega\}$. Thus q is a totally (N, P)-generic extension of p and P is totally proper. ∎

6. How close have we come? In this final section, we take a look at how close we have come to producing a model where the Continuum Hypothesis holds and in which compact spaces of countable tightness are sequential.

Our first task is to show that given a triple $(X, \mathcal{F}, \mathcal{C})$, the forcing actually adjoins an uncountable free sequence through X with the property that every initial segment of the free sequence is covered by finitely many members of the given open cover $\mathcal{C}$.

To see this, suppose that $G \subseteq P$ is generic. Let us define

$$\sigma = \bigcup_{p \in G} \sigma_p. \tag{6.10}$$

Since G is a generic filter, it is not hard to see that σ is a function, and Corollary 4.4 combined with the genericity of G tells us that σ is a one-to-one function from ω_1 into X. Let $F = \{x_\alpha : \alpha < \omega_1\}$, where $x_\alpha = \sigma(\alpha)$. Given $\alpha < \omega_1$, there is a condition $p \in G$ such that $\alpha \subseteq \operatorname{dom} \sigma_p$. By definition, we know that $\mathrm{cl}[p] \cap A_p = \emptyset$. Thus the closure of every initial segment of F is a closed set that is not in $\mathcal{F}$. Given $A \in \mathcal{F}$, by genericity we can find a condition $q \in G$ such that $A_q \subseteq A$. Now for any $q' \leq q$, we know $[q'] \setminus [q] \subseteq A_q \subseteq A$, hence all but countably many members of F are contained in A.

These two facts taken together allow us to "thin out" the sequence F to an uncountable free sequence, each initial segment of which is covered by finitely many members of $\mathcal{C}$.

Now how does this relate to the Moore–Mrówka problem? Assume that the Continuum Hypothesis holds, and that K is a compact space of countable tight- ness that is not sequential. By a result of Ismail and Nyikos [9], we know that K contains a countably compact subspace X that is not closed in K. Another application of the Continuum Hypothesis tells us that there is such a subspace of size $\aleph_1$. We let $\mathcal{F}$ be a maximal filter of closed subsets of X that is not fixed.

Now for every point $x \in X$, we can find a set $A \in \mathcal{F}$ such that $x \notin A$. Note that this means $x \notin \mathrm{cl}_K(A)$, and since K is regular there is an open neighborhood U of x such that $\mathrm{cl}_K(U) \cap \mathrm{cl}_K(A) = \emptyset$.

For each $x \in X$, fix a neighborhood U_x as above, so $\mathrm{cl}_K(U_x)$ is disjoint from the closure (in K) of a set in $\mathcal{F}$. Define an open cover $\mathcal{C}$ of X by

$$\mathcal{C} = \{U_x \cap X : x \in X\}. \tag{6.11}$$

The triple $(X, \mathcal{F}, \mathcal{C})$ is "vulnerable" to our notion of forcing. Forcing with P adjoins an uncountable free sequence F through X with the property that every initial segment of P is covered by finitely many elements of $\mathcal{C}$. By the definition of $\mathcal{C}$, we can thin out F to an uncountable sequence that forms a free sequence in K, contradicting the countable tightness of K.

The upshot of this is that if we have a model of CH in which "PFA restricted to our notion of forcing" holds, then compact spaces of countable tightness are sequential. Standard arguments tell us that we ought to be able to achieve this by a countable support iteration of length ω_2, but we run into trouble because we cannot at present guarantee that the final model will satisfy the Continuum Hypothesis—the iteration may add new reals at a limit stage.

There are several conditions known that guarantee an iteration of totally proper notions of forcing remains totally proper (see the papers [6], [5], and [14]); however, the notion of forcing described here does not seem to fall under any of these known frameworks without additional restrictions being placed on the spaces under consideration. In the language of [14], we can show that the forcing possesses "medicine against weak diamond" but we do not know if it possesses "medicine against almost disjoint clubs".

There are some natural ways of trying to build on the work of this paper in order to resolve the question of Moore–Mrówka and CH. First, one may prove that the partial order described in this paper does actually fall under previously established iteration frameworks—the most likely scenario in this case would be to prove some topological facts along the lines of Theorem 3 that will allow one to prove the forcing is weakly $<\omega_1$-proper (see [5] for the definition). Another possibility is that an advance in iteration technology might make it clear that this notion of forcing can be iterated without adding new reals. Of course, there is always the possibility that the Continuum Hypothesis *does* imply the existence of a compact countably tight space that is not sequential; such a result, when combined with the results of this paper, would solve a long-standing open question of Shelah on weak diamonds (see [15]).

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