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Borel extensions of Baire measures in ZFC

by

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Abstract. We prove:
(1) Every Baire measure on the Kojman–Shelah Dowker space admits a Borel extension.
(2) If the continuum is not real-valued-measurable then every Baire measure on M. E. Rudin's Dowker space admits a Borel extension.

Consequently, Balogh's space remains the only candidate to be a ZFC counterexample to the measure extension problem of the three presently known ZFC Dowker spaces.

1. Introduction. In the mid 1990s, after two decades in which M. E. Rudin's space [16] had been the only known absolute Dowker space, two new ones were discovered [3, 13]. At a workshop on general topology, held in Budapest in 1999, the new spaces were presented, and D. Fremlin seized the opportunity to remind the speakers that only Dowker spaces could provide counterexamples to the measure extension problem. He expressed the hope that one of the three absolute spaces would eventually prove to be an absolute counterexample to the problem.

Below we prove that two of the three potential candidates are not absolute counterexamples to the measure extension problem, in fact, that a large class of absolute Dowker spaces to which these two belong does not contain an absolute counterexample.

The *measure extension problem* is the following: given a normal topological space X and a probability measure μ on the minimal σ-algebra which makes all continuous real functions on X measurable, does μ admit an extension to the σ-algebra of all Borel subsets of X?

The σ-algebra which is generated over a topological space X by all zero-sets of continuous real-valued functions on X is called the *Baire algebra* of

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X and is denoted $\mathrm{Ba}(X)$. A probability measure on $\mathrm{Ba}(X)$ for a normal space X is called a *Baire measure*. All Baire measures are *regular*, that is, the measure of a set is the supremum of the measures of its measurable *closed* subsets [8, 412D].

In most normal spaces the measure extension problem is solved positively by Mařík's extension theorem [14]: if a normal topological space X is *countably paracompact* then every Baire measure on X admits a unique regular Borel extension. Mařík's theorem, then, restricts the measure extension problem to normal spaces which are not countably paracompact. By Dowker's theorem [6], these are exactly the normal spaces X for which $X \times [0, 1]$ is not normal. Such spaces are called *Dowker spaces*. Whether Dowker spaces existed or not had been an open problem for quite some time (see [17] for the history of the subject, which began with Borsuk's work on homotopy theory).

The existence of Dowker spaces has been established on the basis of various additional (independent) axioms to the standard Zermelo–Fraenkel with Choice axiom system, ZFC (see [17, 23]), mostly axioms in the direction of Gödel's constructibility axiom $V = L$. In 1970 also an absolute Dowker space was constructed, that is, proved to exist in ZFC. Three absolute Dowker spaces are known presently [16, 3, 13]. Prior to this paper, the measure extension problem has not been decided in any one of them.

It is customary to call a normal space in which every Baire measure admits an inner regular Borel extension a *Mařík space* (see [24]). Mařík's theorem says, then, that every normal non-Mařík space is Dowker. Ohta and Tamano [15] call a normal space X *quasi-Mařík* if every Baire measure on X admits *some* Borel extension. In this terminology, a counterexample to the measure extension problem is a non-quasi-Mařík Dowker space.

The existence of Dowker spaces with the following prescribed measure extendibility properties has been raised in the literature. Wheeler [24] asks if there are Dowker spaces that are Mařík. Ohta and Tamano [15] ask if quasi-Mařík non-Mařík Dowker spaces exist. And Fremlin [8] asks for a non-Mařík Dowker space (that is, for a counterexample to the measure extension problem).

Each of these three questions has been provided with a *consistent* positive answer. Fremlin [8] constructs a non-quasi-Mařík Dowker space of cardinality $\aleph_1$ from the axiom $\clubsuit(\aleph_1)$, and thus establishes the consistency of a counterexample to the measure extension problem. Aldaz [2] uses the same axiom to establish the consistent existence of a quasi-Mařík non-Mařík Dowker space. He also proves, using a construction of M. G. Bell [4], that under Martin's Axiom, or even under the weaker axiom $P(\mathfrak{c})$, there exists a Mařík Dowker space (thus showing that it is impossible to prove that all Dowker spaces are non-Mařík).

No absolute positive answers were known to any of these questions.

The following is a list of all presently known ZFC Dowker spaces:

(1) M. E. Rudin's space X^{R} [16], whose cardinality is $(\aleph_\omega)^{\aleph_0}$;
(2) Balogh's space [3], whose cardinality is $2^{\aleph_0}$;
(3) Kojman and Shelah's space [13], whose cardinality is $\aleph_{\omega+1}$.

P. Simon [20] proved that space (1) was not Mařík shortly after its discovery. Quasi-Maříkness prior to this paper has not been decided in any of the three spaces (in no extension of ZFC).

1.1. The results. We introduce an infinite class of normal spaces, the class of *Rudin spaces*. Spaces (1) and (3) belong to this class. We prove in ZFC that every Rudin space is a non-Mařík Dowker space and:

(A) Space (3) and any other space of cardinality $\aleph_{\omega+1}$ in this class are quasi-Mařík.
(B) If the class of Rudin spaces contains a non-quasi-Mařík member then the continuum is real-valued-measurable.

By (A), space (3) is an absolute quasi-Mařík non-Mařík space; this provides a positive ZFC solution to Ohta and Tamano's question. Being quasi-Mařík, space (3) is not an absolute counterexample to the measure extension problem.

By (B), also space (1) is not an absolute counterexample to the measure extension problem. It is impossible to prove in ZFC that the continuum is real-valued-measurable (unless ZFC is inconsistent). Therefore (B) implies that an absolute counterexample to the measure extension problem does not exist in the class of Rudin spaces. In particular, Rudin's space (1) is not such an example.

Furthermore, it follows from (B) and Solovay's theorem [22] that the consistency strength of the existence of a non-quasi-Mařík Rudin space is that of a measurable cardinal. This means that if the statement "there exists a non-quasi-Mařík Rudin space" is consistent with ZFC, then this formal consistency will have to be established from the assumption that the theory ZFC + "there exists a measurable cardinal" is consistent. This is a much stronger assumption than the assumption that ZFC is consistent.

Balogh's space (2) remains now the only known candidate to be a ZFC counterexample to the measure extension problem.

1.2. The method. The main tool we use is some further development of Shelah's PCF theory, which is then employed to analyze Baire measures on Rudin spaces. Nonextendible Baire measures are shown to necessarily come from a real-valued measure on the cardinality of the space. In Subsection 2.4 we indicate how to construct with PCF theory Rudin spaces whose cardinality is absolutely not real-valued-measurable. In these constructions no infinite products may be used: an infinite product of sets, each with at least two members, has cardinality which is greater than or equal to the continuum, which, by Solovay's work, can be real-valued-measurable.

1.3. Organization of the paper. In Section 2 we introduce the class of *Rudin spaces* and develop their PCF-theoretic properties. Then we prove that every Rudin space is Dowker and indicate how to prove that every Rudin space contains a (closed) Rudin subspace of cardinality $\aleph_{\omega+1}$. In Section 3 we prove that *cofinal Baire measures* on Rudin spaces do not admit *regular* Borel extensions, but always admit *some* Borel extensions, and prove the main Baire-measure decomposition theorem: if the cardinality of a Rudin space X is not real-valued-measurable then every Baire measure on X is a countable sum of measures concentrated on singletons and of cofinal Baire measures supported on pairwise disjoint Rudin subspaces. This suffices to prove that every Rudin space of non-real-valued-measurable cardinality is quasi-Mařík. We conclude with some open problems in Section 4. The PCF results contained in the Appendix are needed only in Section 2.4.

2. Rudin spaces. We define Rudin spaces and develop their properties.

By ON we denote the class of ordinal numbers. The ordinal ω is the set of natural numbers. For an ordinal α, $\operatorname{cf}\alpha$ is the cofinality of α. By ON^ω we denote the class of all functions from ω to ON. For $f, g \in \mathrm{ON}^\omega$ we write $f \le g$ if $f(n) \le g(n)$ for all $n \in \omega$.

Let

$$P = \prod_{n\in\omega}(\omega_{n+2}+1) = \{f : f \in \mathrm{ON}^\omega \text{ and } (\forall n)[f(n) \le \omega_{n+2}]\Big\}.$$

Let

$$T = \{f : f \in P \text{ and } (\forall n)[\operatorname{cf} f(n) > \aleph_0]\}.$$

Finally,

$$X^{\mathrm{R}} = \{f \in T : (\exists l)(\forall n)[\operatorname{cf} f(n) < \aleph_l]\}.$$

X^{R} is the underlying set of Rudin's space (1). The topology on X^{R} is defined in 2.10 below.

2.1. m-clubs and m-stationarity in X^g for $g \in T \setminus X^{\mathrm{R}}$. The topological properties of Rudin spaces follow from the PCF-theoretic properties of $(X^{\mathrm{R}}, \le)$. We establish the latter in this section.

The presentation is self-contained and no familiarity with PCF theory is needed to read it.

DEFINITION 2.1. Suppose that $g \in T$.

(1) For each $m \in \omega$ let

$$C^g_m = \{n : \operatorname{cf} g(n) = \aleph_m\}.$$

Let

$$C^g_{\le m} = \bigcup_{m' \le m} C^g_{m'} \quad \text{and} \quad C^g_{>m} = \bigcup_{m' > m} C^g_{m'}.$$

(2) Let

$$X^g := \{f \in X^{\mathrm{R}} : f \le g\}.$$

If $g \in X^{\mathrm{R}}$, then by the definition of X^{R}, there is some m such that $C^g_{\le m} = \omega$, or equivalently, $C^g_{>m} = \emptyset$ for all sufficiently large m. On the other hand, if $g \in T \setminus X^{\mathrm{R}}$, then $C^g_m \ne \emptyset$ for infinitely many $m \in \omega$.

CLAIM 2.2. *Suppose $g \in T \setminus X^{\mathrm{R}}$. The partially ordered set $(X^g, \le)$ is directed.*

Proof. Suppose $h_1, h_2 \in X^g$ and let $h = \max\{h_1, h_2\}$. For every n we have $\aleph_0 < \operatorname{cf} h(n) \le \max\{\operatorname{cf} h_1(n), \operatorname{cf} h_2(n)\}$, thus there is some ℓ such that $\operatorname{cf} h(n) < \aleph_\ell$ for all n and $h \in X^{\mathrm{R}}$. Also, $h \le g$, since $h_1 \le g$ and $h_2 \le g$. Thus $h \in X^g$ and $h_1, h_2 \le h$. ∎

Suppose $(Q, \le)$ is any directed poset. Then for every $p \in Q$ the set $\{q \in Q : p \le q\}$ is *cofinal* in $(Q, \le)$, that is, for every $t \in Q$ there is some $q \in Q$ such that $t, p \le q$. If $S \subseteq Q$ is not cofinal in $(Q, \le)$ then there is $p \in Q$ such that $S \cap \{q \in Q : p \le q\} = \emptyset$ and thus $Q \setminus S$ is cofinal in S. The following follows immediately from this observation and Claim 2.2:

CLAIM 2.3. *Suppose $g \in T \setminus X^{\mathrm{R}}$. Then for every subset $D \subseteq X^g$, at least one of the sets $\{D, X^g \setminus D\}$ is cofinal in $(X^g, \le)$.*

DEFINITION 2.4. Suppose $g \in T \setminus X^{\mathrm{R}}$. For $m \in \omega$ let

(i) $\quad D^g_m = \{h \in X^g : (\exists n \in C^g_{>m})[h(n) = g(n)]\}$,

(ii) $\quad X^g_m = X^g \setminus D^g_m = \{h \in X^g : (\forall n \in C^g_{>m})[h(n) < g(n)]\}$.

If $g \in T \setminus X^{\mathrm{R}}$ and $h \in X^g_m$ satisfies $f \le h$ then $f \in X^g_m$. Since $X^g \subseteq X^{\mathrm{R}}$, for every $h \in X^g$ we have $h(n) < g(n)$ for all $n \in C^g_{>m}$ for some m. Then, for every $g \in T \setminus X^{\mathrm{R}}$,

(iii) $$\bigcap_{m \in \omega} D^g_m = \emptyset,$$

(iv) $$\bigcup_{m \in \omega} X^g_m = X^g.$$

DEFINITION 2.5. Suppose that $g \in T \setminus X^{\mathrm{R}}$ and $m \in \omega$, $m > 0$. An element $f \in X^g$ is called *m-normal in X^g* if

(1) $f(c) = g(c)$ for all $c \in C^g_{\le m}$,
(2) $\operatorname{cf} f(c) = \omega_m$ for all $c \in C^g_{>m}$.

Claim 2.6. *Suppose that $g \in T \setminus X^{\mathrm{R}}$ and $m > 0$. Then:*

(1) *If $f \in X^g$ is m-normal in X^g then $f \in X^g_m$.*
(2) *For every $h \in X^g_m$ there is an m-normal $f \in X^g_m$ such that $h \le f$.*
(3) *For every m-normal $h \le g$ the cofinality of*
$$(\{f : f \in P \text{ and } f < h\}, \le)$$
is equal to $\aleph_m$.
(4) *The set of m-normal elements in X^g is $\le$-directed.*

Proof. Suppose $h \in X^g_m$. If f is an m-normal element in X^g then $f(c) < g(c)$ for all $c \in C^g_{>m}$ and hence $f \in X^g_m$.

To prove (2), suppose that $h \in X^g_m$. For every $n \in C^g_{>m}$ let $f(n) \in g(n) \setminus (h(n) + 1)$ be an ordinal of cofinality ω_m. Let $f(n) = g(n)$ for all $n \in C^g_{\le m}$. The element f is m-normal in X^g and $h \le f$.

To prove (3), observe first that if $C \subseteq \omega$ is any nonempty set, $m > 0$ and $h : C \to \mathrm{ON}$ satisfies $\operatorname{cf} h(n) = \aleph_m$ for all $n \in C$, then $\operatorname{cf}(\{f \in \mathrm{ON}^C : f < h\}, \le) = \aleph_m$. This can be seen by fixing, for every $n \in C$, a $<$-increasing sequence $\langle \zeta^n_\alpha : \alpha < \omega_m \rangle$ with $\sup\{\zeta^n_\alpha : \alpha < \omega_m\} = h(n)$, and for every $\alpha < \omega$ letting $f_\alpha(n) = \zeta^n_\alpha$. If $f < h$ is any ordinal function on C, then $\operatorname{cf} h(n) > \aleph_0$ for $n \in C$, and so there is some $\alpha < \omega_m$ such that $f(n) < f^n_\alpha$ for all $n \in C$.

Next observe that the cofinality of a product of finitely many posets, each with an infinite cofinality, is the maximum of their cofinalities. Now (3) follows from the fact that $(\{f : f \in P \text{ and } f < h\}, \le)$ is isomorphic to the product of $(\{(f{\restriction}A_i) : f \in P \text{ and } f < h\}, \le)$ over all $i \le m$ such that there is some n with $\operatorname{cf} h(n) = \aleph_i$, where $A_i = \{c \in \omega : \operatorname{cf} h(c) = \aleph_i\}$.

For part (4) it suffices to observe that $\max\{h_1, h_2\}$ is m-normal if h_1, h_2 are m-normal. ∎

Definition 2.7. Suppose $g \in T \setminus X^{\mathrm{R}}$ and $0 < m \in \omega$.

(1) An *m-club* in X^g is a subset $D \subseteq X^g$ which satisfies:
   (a) Every $h \in D$ is m-normal.
   (b) D is *cofinal* in $(X^g_m, \le)$, that is, for every $h \in X^g_m$ there is some $f \in D$ such that $h \le f$.
   (c) If $h_\zeta \in D$ for $\zeta < \omega_m$ and $\langle (h_\zeta{\restriction}C^g_{>m}) : \zeta < \omega_m \rangle$ is $<$-increasing, then $\sup\{h_\zeta : \zeta < \omega_m\} \in D$.
(2) A set $S \subseteq X^g$ is *m-stationary* in X^g if S has a nonempty intersection with every m-club in X^g.

Example. Suppose $g \in T \setminus X^{\mathrm{R}}$ and $0 < m \in \omega$. Assume that $h \in X^g_m$. Then the set

$$\{f \in X^g : f \text{ is } m\text{-normal and } h \le f\}$$

is an m-club in X^g.

CLAIM 2.8. *Suppose that $g \in T \setminus X^{\mathrm{R}}$ and $m > 0$.*

(1) *Every intersection of $\aleph_m$ m-clubs is an m-club.*
(2) *If $S \subseteq X^g$ is m-stationary then S is cofinal in $(X^g_m, \le)$.*
(3) *Suppose S is not cofinal in X^g_m. Then $X^g \setminus S$ contains an m-club of X^g.*

Proof. Suppose $m > 0$ and D_α is a given m-club for every $\alpha < \omega_m$. The intersection $D := \bigcap_{\alpha<\omega} D_\alpha$ clearly satisfies conditions (a) and (c). To see that it satisfies (b) let h be an arbitrary m-normal element in X^g. By induction on $\alpha < \omega_m$ choose an m-normal element h_α as follows. For $\alpha = 0$ let $h_0 = \sup\{h^0_\beta : \beta < \omega_m\}$ where $h \le h^0_0$, $h^0_\beta \in D_\beta$ and $\beta < \gamma \Rightarrow (h^0_\beta \restriction C^g_{>m}) < (h^0_\gamma \restriction C^g_{>m})$. This is clearly possible, as each D_β is an m-club. At limit $\alpha < \omega_m$ let $h_\alpha = \sup\{h_\beta : \beta < \alpha\}$, and for $\alpha + 1 < \omega_m$ let $h_{\alpha+1}$ be defined from h_α the same way h_0 is defined from h.

Let $h^* = \sup\{h_\alpha : \alpha < \omega_m\}$. Since $\langle h^\alpha_\beta : \alpha < \omega_m\rangle$ is $<$-increasing on $C_{>m}$, has supremum h^* and each h^α_β belongs to D_β, it follows by (b) that $h^* \in D_\beta$ for all $\beta < \omega_m$. Clearly, $h \le h^*$, so we are done.

Part (2) follows from the fact that for every $h \in X^g_m$, the set $\{f \in X^g : f \text{ is } m\text{-normal and } h \le f\}$ is an m-club.

(3) Let $S \subset X^g$ be a subset which is not cofinal in X^g_m. From (2) it follows that S is not m-stationary. In particular, there exists an m-club in $X^g \setminus S$. ∎

LEMMA 2.9 (Fodor lemma for m-clubs). *Suppose that $g \in T \setminus X^{\mathrm{R}}$, $0 < m \in \omega$ and D is an m-club in X^g. Suppose that $F : D \to P$ with $F(h) < h$ for all $h \in D$. Then there is some $f_0 \in P$ and an m-stationary $S \subseteq D$ such that $f_0 < g$ and that $F(h) < f_0$ for all $h \in S$.*

Proof. Suppose that, contrary to the statement, for every $f < g$ in P there is an m-club D_f such that $F(h) \not< f$ for any $h \in D_f$. By intersecting each D_f with D we assume that $D_f \subseteq D$ for all $f < g$ in P.

By induction on $\zeta < \omega_m$ define h_ζ and A_ζ so that the following hold:

(1) $A_\zeta \subseteq \{f \in P : f < h_\zeta\}$ is cofinal in $(\{f \in P : f < h_\zeta\}, \le)$, $|A_\zeta| = \aleph_m$ and $\xi < \zeta \Rightarrow A_\xi \subseteq A_\zeta$.
(2) $h_\zeta \in \bigcap\{D_f : f \in \bigcup_{\xi<\zeta} A_\xi\}$ and $\xi < \zeta \Rightarrow (h_\xi \restriction C^g_{>m}) < (h_\zeta \restriction C^g_{>m})$.

Suppose that $\zeta < \omega_m$, and h_ξ and A_ξ are defined for all $\xi < \zeta$. Pick $h_\zeta \in \bigcap\{D_f : f \in \bigcup_{\xi<\zeta} A_\xi\}$ such that $(h_\xi \restriction C^g_{>m}) < (h_\zeta \restriction C^g_{>m})$ and $h_\xi | C^g_{\le m} = h_0 | C^g_{\le m}$ for all $\xi < \zeta$. Since $|\bigcup_{\xi<\zeta} A_\xi| = \aleph_m$ and the intersection of $\aleph_m$ m-clubs is an m-club, it is possible to pick h_ζ. (For $\zeta = 0$ let $h_\zeta \in X^g_m$ be arbitrary.) Fix now a cofinal set $B \subseteq \{f \in P : f < h_\zeta\}$ satisfying $|B| = \aleph_m$ and let $A_\zeta = B \cup \bigcup_{\xi<\zeta} A_\xi$.

Let $h = \sup\{h_\zeta : \zeta < \omega_m\}$. Since $h_\zeta \in D$ for all $\zeta < \omega_m$ and D is an m-club, $h \in D$. Denote now $t := F(h) < h$. Since $\langle (h_\zeta \restriction C^g_{>m}) : \zeta < \omega_m \rangle$ is strictly increasing with supremum $(h \restriction C^g_{>m})$, there is some $\zeta < \aleph_m$ such that $(t \restriction C^g_{>m}) < (h_\zeta \restriction C^g_{>m})$. Notice that $(t \restriction C^g_{\le m}) < (h_\zeta \restriction C^g_{\le m}) = (h \restriction C^g_{\le m})$. This means that $t < h_\zeta$. Since A_ζ is cofinal in $\{f \in P : f < h_\zeta\}$ and $t \in P$ satisfies $t < h_\zeta$, there is some $f \in A_\zeta$ such that $t < f$. Finally, since $h_\xi \in D_f$ for all $\zeta < \xi < \omega_m$, it follows that $h \in D_f$. Now a contradiction follows, since $F(h) = t < f$. ∎

2.2. Topologically closed cofinal subsets of X^g for $g \in T \setminus X^{\mathrm{R}}$. For $f < g$ from P let

$$(f, g] = \{h : h \in P \text{ and } f < h \le g\}.$$

The family of all sets $(f, g]$ for $f < g$ in P constitutes a basis for the *box product topology* on P. In this topology, every basic open set $(f, g]$ for $f < g$ in P is actually clopen.

All spaces that we shall consider are subspaces of T taken with the induced box product topology from P. The first space we consider is M. E. Rudin's Dowker space from [16]:

Definition 2.10. *The Rudin space* $X^{\mathrm{R}} \subseteq T$ is defined by

$$X^{\mathrm{R}} = \{f \in T : (\exists l)(\forall n)[\mathrm{cf}\, f(n) < \aleph_l]\}$$

and is equipped with the induced topology from the box product topology on P.

If $h \in X^{\mathrm{R}}$ and $X \subseteq X^{\mathrm{R}}$, then h belongs to the closure of X in X^{R} if and only if for all $t \in P$ that satisfies $t < h$ the set $X \cap (t, h]$ is nonempty.

Lemma 2.11. *Let $g \in T \setminus X^{\mathrm{R}}$ and $X \subseteq X^g$. Assume that for all $h < g$ there exists $f \in X$ such that $h \le f$. Then there exists $m_0 \in \omega$ such that for every $h < g$ there exists $f \in X \cap X^g_{m_0}$ such that $h < f$.*

Proof. Suppose that for each m there is some $h_m < g$ such that $h_m \not\le f$ for all $f \in X \cap X^g_m$. Let $h = \sup\{h_m : m \in \omega\}$. Since $\mathrm{cf}\, g(c) > \omega_0$ for all $c \in \mathbb{N}$, it follows that $h < g$. By the definition of h, $h \not\le f$ for all $f \in X \cap X^g_m$ and $m \in \omega$. Since $\bigcup_m X^g_m = X^g$, this contradicts the assumption on X. ∎

Lemma 2.12. *Suppose $g \in T \setminus X^{\mathrm{R}}$, $X \subseteq X^g$, and $m_0 > 0$ has the property that for all $h < g$ there is some $f \in X \cap X^g_{m_0}$ such that $h < f$. Then the set of m-normal elements in the closure of X is cofinal in X^g_m for all $m \ge m_0$.*

Proof. Let $m \ge m_0$ be given. Let $m' = \max\{n : n \le m \text{ and } C^g_n \ne \emptyset\}$ and fix a cofinal set $A \subseteq (\prod_{c \in C^g_{\le m}} g(c), \le)$ of cardinality $\aleph_{m'}$. Since $m' \le m$, we can fix an enumeration $\langle t_\alpha : \alpha < \omega_m \rangle$ of A in which every $t \in A$ appears

$\aleph_m$ times. Thus, the set $\{t_\alpha : \beta < \alpha < \omega_m\}$ is cofinal in $(\prod_{c \in C^g_{\le m}} g(c), \le)$ for every $\beta < \omega_m$.

Let $h \in X^g_m$ be given. By induction on $\alpha < \omega_m$ find $f_\alpha \in X \cap X^g_{m_0}$ such that $\alpha < \beta < \omega_m$ implies that $(f_\alpha \restriction C^g_{>m}) < (f_\beta \restriction C^g_{>m})$ and $t_\alpha < (f_\alpha \restriction C^g_{\le m})$. Let $f_0 \in D$ be chosen so that $t_0 \cup (h \restriction C^g_{>m}) < f_0$. This is possible by the assumption on X because $t_0 \cup (h \restriction C^g_{>m}) < g$. At stage $\alpha > 0$ let $h_\alpha = t_\alpha \cup \sup\{(f_\beta \restriction C^g_{>m}) : \beta < \alpha\}$ and find $f_\alpha \in D \cap X^g_m$ such that $h_\alpha < f_\alpha$.

Let $f := (g \restriction C^g_{\le m}) \cup \sup\{f_\alpha \restriction C^g_{>m} : \alpha < \omega_m\}$. Clearly, $f \in X_g$, is m-normal and $h \le f$. To see that f belongs to the closure of X let $t < f$ be arbitrary. Find some $\beta < \omega_m$ such that $(t \restriction C^g_{>m}) < (f_\beta \restriction C^g_{>m})$. Then find some $\beta < \alpha < \omega_m$ such that $(t \restriction C^g_{\le m}) < t_\alpha$. Now $f_\alpha \in (t, f]$. ∎

THEOREM 2.13. *Suppose $g \in T \setminus X^{\mathrm{R}}$ and $X \subseteq X^g$ is closed in X^{R}. Then X is cofinal in $(X^g, \le)$ if and only if there is some $m_0 > 0$ such that X contains an m-club of X^g for all $m \ge m_0$.*

Proof. If $X \subseteq X^g$ is any cofinal set in $(X^g, \le)$, then by Lemma 2.11 there exists some m_0 such that for all $f < g$ there is $h \in X \cap X^g_{m_0}$ such that $f < h$. If X is also closed in X^{R}, then by Lemma 2.12, for all $m \ge m_0$, the set of all m-normal elements in X—denote it by A_m—is cofinal in X^g_m. Hence A_m satisfies conditions (a) and (b) in Definition 2.7 of an m-club. It also satisfies (c), since X is closed, and therefore $A_m \subseteq X$ is an m-club.

Conversely, suppose $X \subseteq X^g$ is closed and contains an m-club for every $m \ge m_0$ for some $m_0 > 0$. Let $h \in X^g$ be given. Since $X^g = \bigcup_m X^g_m$, there exists some m such that $h \in X^g_m$. By increasing m, we may assume $m \ge m_0$. Since X contains an m-club, there is some m-normal $f \in X$ such that $h \le f$. ∎

REMARK. Taking $g(n) = \omega_{n+2}$ for all n, the space D^g_m for $m > 1$ is closed and cofinal in $X^g = X^{\mathrm{R}}$ but contains no m-clubs for $m' < m$. This shows that the restriction to $m \ge m_0$ for some m_0 is necessary in Theorem 2.13.

LEMMA 2.14. *Suppose $g \in T \setminus X^{\mathrm{R}}$ and $X \subseteq X^g$ is closed in X^{R} and cofinal in $(X^g, \le)$. Suppose $m > 0$ and X contains an m-club in X^g. Then for every closed $D \subseteq X$, either D or $X \setminus D$ contains an m-club of X^g.*

Proof. Suppose $m > 0$, $E \subseteq X$ is an m-club of X^g and $D \subseteq X$ is closed. As D is closed, $D \cap E$ satisfies condition (c) in the definition of m-club. Thus, if $D \cap E$ is cofinal in $(E, \le)$, then $D \cap E$ is an m-club.

Otherwise, there is some $f \in E$ such that $\{h : h \in E \text{ and } f \le h\} \cap D = \emptyset$, which implies that $X \setminus D$ contains an m-club. ∎

2.3. Rudin spaces

DEFINITION 2.15. A space X is a *Rudin space* if there exists $g \in T \setminus X^{\mathrm{R}}$ such that $X \subseteq X^g$ is closed in X^{R} and cofinal in $(X^g, \le)$.

Observe that X^g is clopen in X^{R} for every $g \in T \setminus X^{\mathrm{R}}$. This means that for $X \subseteq X^g$, X is closed in X^{R} iff X is closed in X^g. From now on we refer to this situation just by "$X \subseteq X^g$ is closed".

Claim 2.16. *X^{R} is a Rudin space and for every $g \in T \setminus X^{\mathrm{R}}$, X^g is a Rudin space.*

Proof. The second part is obvious as X^g is clopen in X^{R} and cofinal in $(X^g, \le)$. For the first part let g be defined by $g(n) = \omega_{n+2}$. Now $X^g = X^{\mathrm{R}}$. ∎

Claim 2.17 ([16, Lemma 4]). *X^{R} is a P-space, that is, every countable intersection of open subsets of X^{R} is open.*

Proof. Suppose $U_m \subseteq X^{\mathrm{R}}$ is open for each $m \in \mathbb{N}$ and suppose $f \in \bigcap U_m$. For each m there is some $h_m < f$ such that $(h_m, f] \subseteq U_f$. Since $\operatorname{cf} f(n) > \aleph_0$ for all $n \in \mathbb{N}$, it follows that $h = \sup\{h_m : m \in \mathbb{N}\}$ satisfies $h < f$ and clearly $(h, f] \subseteq \bigcap U_m$. ∎

Corollary 2.18. *Every Rudin space is a P-space.*

Definition 2.19.

(1) A topological space X is *collectionwise normal* if for every discrete family $\{H_j : j \in J\}$ of closed subsets of X there exists a family $\{U_j : j \in J\}$ of open pairwise disjoint subsets of X such that for every $j \in J$ there is $H_j \subseteq U_j$.
(2) A normal topological space X is *countably paracompact* if for every family $\{D_n : n \in \omega\}$ of closed subsets of X, if $\bigcap_{n\in\omega} D_n = \emptyset$, then there exists a family $\{U_n : n \in \omega\}$ of open subsets of X such that $D_n \subseteq U_n$ for every $n \in \omega$ and $\bigcap_{n\in\omega} U_n = \emptyset$.
(3) A topological space X is *Dowker* if it is normal and not countably paracompact.

Every collectionwise normal space is of course normal.

Theorem 2.20 (M. E. Rudin, [16]). *X^{R} is collectionwise normal.*

Every Rudin space is a closed subspace of X^{R}, therefore:

Corollary 2.21. *Every Rudin space is collectionwise normal.*

Lemma 2.22. *Suppose that $g \in T \setminus X^{\mathrm{R}}$ and that $X \subseteq X^g$ is closed and cofinal in $(X^g, \le)$. The collection of closed subsets of X which are cofinal in $(X, \le)$ has the finite intersection property, in fact, the intersection of two closed and cofinal subsets of X is closed and cofinal.*

Proof. Suppose $D_1, D_2 \subseteq X$ are closed and cofinal in $(X, \le)$. Then they are closed and cofinal in $(X^g, \le)$. By Theorem 2.13 applied to D_1 and D_2, there is some $m > 0$ such that both D_1 and D_2 contain m-clubs of X^g. Now by Claim 2.8, $D_1 \cap D_2$ contains an m-club of X^g and is therefore cofinal.

The intersection $D_1 \cap D_2$ is clearly closed. ∎

DEFINITION 2.23. Suppose $g \in T \setminus X^{\mathrm{R}}$ and $X \subseteq X^g$ is closed and cofinal in $(X^g, \leq)$. For every $m > 0$ we define, analogously to Definition 2.4,

(v) $D_m^X = X \cap D_m^g = \{h \in X : (\exists n \in C_{>m}^g)[h(n) = g(n)]\}$,

(vi) $X_m = X \cap X_m^g = X \setminus D_m^X = \{h \in X : (\forall n \in C_{>m}^g)[h(n) < g(n)]\}$.

If $X \subseteq X^g$ is closed and cofinal in $(X^g, \leq)$ and $h \in X_m$, then for every $t \in X$ such that $t \leq h$ we have $t \in X_m$. This makes X_n an open subset of X, and D_m a closed subset of X for all $m > 0$. The set D_m^X is clearly cofinal in X for all $m > 0$. Finally, $\bigcap_m D_m^X = \emptyset$ and $X = \bigcup_m X_m$.

CLAIM 2.24. *Suppose $g \in T \setminus X^{\mathrm{R}}$ and $X \subseteq X^g$ is closed and cofinal in $(X^g, \leq)$. The collection of closed and cofinal subsets in $(X, \leq)$ does not have the countable intersection property.*

Proof. For every $m > 0$ the set D_m^X is closed and cofinal in $(X, \leq)$ and $\bigcap_{m>0} D_m^X = \emptyset$. ■

LEMMA 2.25. *Suppose that $g \in T \setminus X^{\mathrm{R}}$ and $D \subseteq X^g$ is closed and cofinal in $(X^g, \leq)$. Then for every open $U \subset X^g$ such that $D \subseteq U$ there is $f \in P$ such that $f < g$ and $(f, g] \cap X^g \subseteq U$.*

Proof. Suppose that $U \subseteq X^g$ is open and $D \subseteq U$ is closed and cofinal in $(X^g, \leq)$. For every $h \in D$, fix $F(h) \in P$ so that $F(h) < h$ and $(F(h), h] \cap X^g \subseteq U$. This is possible because $h \in U$ and U is open.

By Theorem 2.13 there exists some $m_0 > 0$ and, for every $m \geq m_0$, an m-club of X^g, $D_m \subseteq D$. By the Fodor Lemma for m-clubs, there is an m-stationary $S_m \subseteq D_m$ and $f_m < g$ such that for all $h \in S_m$ we have $F(h) < f_m$. Let $f = \sup\{f_m : m \geq m_0\}$. Since $\operatorname{cf} g(n) > \aleph_0$ for all n, it follows that $f(n) < g(n)$ for all n, that is, $f < g$.

Suppose now that $h \in (f, g] \cap X^g$ and we shall show that $h \in U$. There is some $m \geq m_0$ such that $h \in X_m^g$, and since S_m is cofinal in X_m^g, there is some $t \in S_m$ such that $h \leq t$. Now

$$h \in (f, t] \cap X^g \subseteq (f_m, t] \cap X^g \subseteq U.$$

Thus $X^g \cap (t, g] \subseteq U$. ■

THEOREM 2.26. *Every Rudin space is Dowker.*

Proof. Every Rudin space is normal by 2.21.

Given a closed and cofinal $X \subseteq X^g$ for some $g \in T \setminus X^{\mathrm{R}}$, it follows that $\bigcap_m D_m^X = \emptyset$ and each D_m^X is closed and cofinal in $(X, \leq)$. Therefore, each D_m^X is also closed and cofinal in $(X^g, \leq)$. Suppose $U_m \subseteq X$ is given for each m so that U_m is open and $D_m^X \subseteq U_m$, and let $U_m^* \subseteq X^g$ be open in X^g such that $U_m = U_m^* \cap X$.

By Lemma 2.25, for every $m \in \omega$ there is $f_m \in P$ such that $f_m < g$ and $(f_m, g] \cap X^g \subseteq U_m^*$. So $(f_m, g] \cap X \subseteq U_m$. Let $f = \sup_{m \in \omega} f_m$. We have $f < g$ and $X \cap (f, g] \subseteq \bigcap_{m\in\omega} U_m$. Thus, by cofinality of X in $(X^g, \leq)$, $\bigcap_{m\in\omega} U_m \neq \emptyset$. This shows that X is not countably paracompact. ∎

THEOREM 2.27. *Suppose $g \in T \setminus X^{\mathrm{R}}$ and $X \subseteq X^g$ is closed and cofinal in $(X^g, \leq)$. Then the collection of clopen and cofinal subsets of $(X, \leq)$ is a σ-ultrafilter of clopen sets.*

Proof. Suppose that for each $i \in \omega$ the set $D_i \subseteq X$ is clopen and cofinal in $(X, \leq)$. By Lemma 2.25, for each i there is some $f_i < g$ such that $X \cap (f_i, g] \subseteq D_i$. Now $f = \sup\{f_i : i \in \omega\} < g$ and $X \cap (f, g] \subseteq \bigcap_i D_i$. This shows that a countable intersection of clopen and cofinal subsets of X is cofinal in $(X, \leq)$. It is also clopen, since X is a P-space.

From Claim 2.3 it follows that for every clopen set $D \subseteq X$ either D or $X \setminus D$ is cofinal in $(X, \leq)$. Thus the clopen and cofinal sets form a σ-ultrafilter of clopen sets in X. ∎

2.4. Rudin spaces of bounded cardinality. In this section we prove that every Rudin space contains a closed Rudin subspace of cardinality $\aleph_{\omega+1}$. Certain statements in this subsection are sufficiently close to results of [13] to be stated here without proofs. The main difference from [13] is that the ideal at work there is the ideal of finite subsets of ω, whereas here we must consider ideals which contain infinite sets. The operation of closing a set $X \subset X^{\mathrm{R}}$ under all possible modifications of a function $f \in X^{\mathrm{R}}$ modulo the ideal of finite sets does not increase the cardinality of the set beyond $\aleph_\omega$, but an uncontrolled closure under all modifications modulo an ideal with infinite sets increases cardinality up to $(\aleph_\omega)^{\aleph_0}$. So, to handle all Rudin spaces, a new approach has to be taken.

We assume familiarity with the following concepts: exact upper bound (eub) of a given sequence of elements of ON^ω with respect to a given ideal on ω, flatness of a given sequence of elements of ON^ω with respect to a given ideal on ω, true cofinality and boundedness of a given subset of ON^ω with respect to a given ideal on ω. All these concepts are defined in the Appendix. See Definitions 5.1, 5.2, 5.5, together with formulations of Claim 5.3 and Lemma 5.4.

DEFINITION 2.28. Suppose $g \in T \setminus X^{\mathrm{R}}$ and $X \subseteq X^g$ is closed and cofinal in X^g.

(1) Let I_g be the ideal generated over ω by $\{C^g_{\leq m} : m \in \mathbb{N}\}$.
(2) For each m with $C^g_m \neq \emptyset$ fix a strictly increasing and continuous sequence $\langle t^m_\alpha : \alpha \leq \omega_m\rangle$ of functions on C^g_m with $t^m_{\omega_m} = (g\restriction C^g_m)$ and such that for every $\alpha \leq \omega_m$ and for every $n_1, n_2 \in C^g_m$ we have $\mathrm{cf}\, t^m_\alpha(n_1) = \mathrm{cf}\, t^m_\alpha(n_2)$. Let

$$D_g = \{h \in X^g : (\forall m)(\exists \alpha \leq \omega_m)[(h\restriction C^g_m) = t^m_\alpha]\}.$$

Claim 2.29. *D_g is closed and cofinal in X^g.*

Proof. Assume that $f \in X^g \setminus D_g$. Find $m \in \omega$ and $n \in C^g_m$ such that

$$(\forall \alpha \le \aleph_m)[f(n) \ne t^m_\alpha(n)]. \tag{vii}$$

By continuity of $\langle t^m_\alpha(n) : \alpha \le \omega_m\rangle$ there exists a largest $\alpha_0 < \omega_m$ satisfying $t^m_{\alpha_0}(n) < f(n)$. The set $\{h : h \in X^g \text{ and } t^m_{\alpha_0}(n) < h(n) \le f(n)\}$ is open, disjoint from D_g and contains f. This shows that D_g is closed.

To see that D_g is cofinal in $(X^g, \le)$ suppose $f \in X^g$ is arbitrary. Suppose $m \in \omega$ is such that $C^g_m \ne \emptyset$ and let $t^m = (f{\restriction}C^g_m)$. Since $f \in X^g$, for all but finitely many $n \in \omega$ we have $t^m < (g{\restriction}C^g_m)$. Fix $m_0 \in \omega$ so that $m_0 > 0$ and for all $m > m_0$, if t^m is defined, then $t^m < (g{\restriction}C^g_m)$. For each $m > m_0$ for which $C^g_m \ne \emptyset$ find $\alpha_m < \omega_m$ such that $t^m < t^m_{\alpha_m}$. Since $m \ge 2$ it follows that $\operatorname{cf} g(n) = \omega_m \ge \omega_2$, so we may increase $\alpha_m < \omega_m$ and assume that $\operatorname{cf} \alpha_m = \omega_1$. For each $m \le m_0$ let $\alpha_m = \omega_m$, so $t^m_{\alpha_m} = (g{\restriction}C^g_m)$. Now let $h = \bigcup_{m\in\omega_m,\, C^g_m\ne\emptyset} t^m_{\alpha_m}$. Then $f \le h \le g$ and since $\operatorname{cf} h(n) \ge \omega_1$ for all n, it follows that $h \in X^g$ and by definition of h, also $h \in D_g$. ∎

Claim 2.30. *Suppose $g \in T\setminus X^{\mathrm{R}}$. Then every subset of X^g of cardinality $\aleph_\omega$ is bounded in $(X^g, \le_{I_g})$ and the least cardinality of an unbounded subset of $(X^g, \le_{I_g})$, denoted by $\mathfrak{b}(X, \le_{I_g})$, is a regular cardinal.*

Proof. Let $B = \{f_\alpha : \alpha < \aleph_\omega\} \subseteq X^g$ be given. For every $m > 0$ and $n \in C^g_m$ let

$$f'(n) = \sup\{f_\alpha(n) : \alpha < \omega_{m-1} \text{ and } f_\alpha(n) < g(n)\}.$$

Since for $n \in C^g_m$ we have $\operatorname{cf} g(n) = \omega_m$, the definition of f' implies that $f' < g$.

Let $\alpha < \aleph_\omega$ be given. Then for some m_α, which, without loss of generality, satisfies $\alpha < \omega_{m_\alpha}$, for all $m > m_\alpha$ and $n \in C^g_m$ we have $f_\alpha(n) < g(n)$; therefore $f_\alpha(n) \le f'(n)$.

We showed that for every $\alpha < \aleph_\omega$ there is some m_α such that $(f_\alpha{\restriction}C^g_m) < (f'{\restriction}C^g_m) < (g{\restriction}C^g_m)$ for all $m \ge m_\alpha$. Find some $f \in X^g$ such that $f' \le f$; then $f_\alpha <_{I_g} f$ for all $\alpha < \aleph_\omega$ as required.

The proof that $\mathfrak{b}(X^g, \le_{I_g})$ is regular is straightforward. ∎

Theorem 2.31. *Suppose $g \in T \setminus X^{\mathrm{R}}$, X is closed and cofinal in $(X^g, \le)$ and $\operatorname{tcf}(X, \le_{I_g}) = \lambda$. Then there is a cofinal Rudin subspace $Y \subseteq X$ of cardinality λ.*

Proof (sketch). Since we here generalize the proof from [13], we just give the definition of a cofinal Rudin subspace $Y \subseteq X$ and leave the proof that Y is a closed subspace of X to the reader.

First, since $D_g = D^X_g$ is a closed and cofinal subset of X, by intersecting X with D_g, we may assume that $X \subseteq D_g$ (as this intersection also has true cofinality λ). Since $\lambda = \operatorname{tcf}(X, \le_{I_g})$, λ is regular by Claim 2.30, and greater than $\aleph_\omega$. We can fix a sequence $\langle h_\alpha : \alpha < \lambda\rangle$ of elements of X which is $<_{I_g}$-increasing and $<_{I_g}$-cofinal in X such that for every $\alpha < \lambda$ with $\mathrm{cf}\,\alpha = \aleph_n$, if $\overline{h}\restriction\alpha$ is flat, then h_α is an eub of $\overline{h}\restriction\alpha$. Let

(viii) $$Y = \{h \in X : (\exists\alpha < \lambda)[h =_{I_g} h_\alpha]\}.$$

It is clear from the definition that Y is a cofinal subspace.

Claim 2.32. $|Y| = \lambda$.

Proof of the Claim. For every $h \in Y$ there exists a (unique) $\alpha < \lambda$ such that $h =_{I_g} h_\alpha$. This means that there is some m such that $h_\alpha(n) \neq h(n) \Rightarrow n \in C^g_{\le m}$. Since $X \subseteq D_g$, for every $m' \le m$ the restriction $(h\restriction C^g_{m'})$ is one of $\omega_{m'}$ fixed functions. In total, the number of possible $h \in X$ which satisfy $h =_{I_g} h_\alpha$, for a given $\alpha < \lambda$, is $\le \aleph_\omega$. This shows that $|Y| \le \lambda \times \aleph_\omega = \lambda$. The reverse inequality holds too, since $h_\alpha \in Y$ for all $\alpha < \lambda$. ∎

The methods of [13] are sufficient to prove that Y is a closed subset of X, so we omit the proof of this fact. ∎

Below we shall need the following:

Lemma 2.33. *Suppose $g \in T \setminus X^{\mathrm{R}}$, $Y \subseteq D_g$ is nonempty and $g^* \in \mathrm{ON}^\omega$ is an eub of Y modulo I_g. Then there exists a sequence $\langle\alpha(m) : m \in \omega\rangle$ such that $\alpha(m) \le \omega_m$ for all m and there exists some $m_0 \in \omega$ such that for every $m \ge m_0$,*

$$(g^*\restriction C^g_m) = t^m_{\alpha(m)}.$$

Proof. For every $m \in \omega$ we define

$$\alpha(m) = \sup\{\beta : t^\beta_m \le (g^*\restriction C^g_m)\}.$$

From Definition 2.28 and $Y \neq \emptyset$ it follows that

$$t^{\alpha(m)}_m \le (g^*\restriction C^g_m).$$

For $n \in C^g_m$ define

$$h(n) = t^{\alpha(m)}_m(n) \le g^*(n).$$

The proof will be finished when we establish the equality

$$g^* =_{I_g} h.$$

From Claim 5.3 in the Appendix, in order to prove that $g^* =_{I_g} h$ it is enough to check that h is an upper bound of Y modulo I_g. Assume, then, to the contrary, that there exists $f \in Y$ such that $f \not\le_{I_g} h$. Let

$$L = \{m \in \omega : (f\restriction C^g_m) \not\le (h\restriction C^g_m)\} = \{m \in \omega : (f\restriction C^g_m) > (h\restriction C^g_m)\}.$$

Since $f \not\le_{I_g} h$, the set L is infinite. The function g^* is an upper bound of Y, hence there exists $M_0 \in \omega$ such that

(ix) $$(f\restriction C^{>M_0}_g) \le (g^*\restriction C^{>M_0}_g).$$

Let $m \in L$, $m > M_0$. Fix $\beta \leq \aleph_m$ such that $(f{\restriction}C^g_m) = t^\beta_m$ and observe that from inequality (ix) it follows that $\beta \leq \alpha(m)$, in particular

$$(h{\restriction}C^g_m) = t^{\alpha(m)}_m \geq t^\beta_m = (f{\restriction}C^g_m),$$

but $m \in L$, and from the definition of L it follows that

$$(f{\restriction}C^g_m) > (h{\restriction}C^g_m),$$

a contradiction. ∎

The following is a Rudin space analog of Shelah's existence theorem for an $\aleph_{\omega+1}$-scale [18, Theorem 2.5, p. 50].

THEOREM 2.34. *For every $g \in T \setminus X^{\mathrm{R}}$ and a Rudin space X closed and cofinal in X^g there is some $g^* \leq g$ in $T \setminus X^{\mathrm{R}}$ such that*

(1) $I_{g^*} = I_g$,
(2) $\operatorname{tcf}(X \cap X^{g^*}, \leq_{I_{g^*}}) = \aleph_{\omega+1}$.

Proof.

We assume that $X \subseteq D^g$.

First we shall define a $<_{I_g}$-increasing sequence $\overline{h} = \langle h_\alpha : \alpha < \aleph_{\omega+1}\rangle$ of members of X with an eub. In order to obtain an eub we will apply Theorem 5.6 from the Appendix. In particular it says that a $<_{I_g}$-sequence $\overline{h}$ has an eub if for all $k > 0$ the set

$$\{\alpha < \aleph_{\omega+1} : \operatorname{cf}\alpha = \aleph_k \wedge \overline{h}{\restriction}\alpha \text{ is flat modulo } I_g\}$$

is stationary in $\aleph_{\omega+1}$.

Let $S = \bigcup_{n>0} S_n \subseteq \aleph_{\omega+1}$ be fixed so that $S \in I[\aleph_{\omega+1}]$ and $S_n \subseteq \{\alpha < \aleph_{\omega+1} : \operatorname{cf}\alpha = \aleph_n\}$ is stationary. This is possible due to Shelah's Theorem 5.8 (see the Appendix). Let $P_\alpha \subseteq \mathcal{P}(\alpha)$ be fixed for all $\alpha < \aleph_{\omega+1}$ so that $|P_\alpha| \leq \aleph_\omega$, $\alpha < \beta < \aleph_{\omega+1}$ implies $P_\alpha \subseteq P_\beta$, $\operatorname{otp} c < \aleph_\omega$ for all $c \in P_\alpha$ and such that for all $\delta \in S_n$ there is $c \subseteq \delta$ of ordertype ω_n such that for all $\beta < \delta$ we have $c \cap \beta \in \bigcup_{\alpha<\delta} P_\alpha$.

By induction on $\alpha < \aleph_{\omega+1}$ we now construct a $<_{I_g}$-increasing sequence $\langle h_\alpha : \alpha < \aleph_{\omega+1}\rangle$ of members of X.

Let $h_0 \in X$ be arbitrary.

Suppose $\alpha < \aleph_{\omega+1}$ and h_β is defined for all $\beta < \alpha$. For all $c \in P_\alpha$ let $t_c = \sup\{h_{\beta+1} : \beta \in c\}$. Since $\operatorname{otp} c < \aleph_\omega$, it follows that $t_c <_{I_g} g$. Since $|P_\alpha| \leq \aleph_\omega$ and $|\alpha| \leq \aleph_\omega$, by Claim 2.30 it is possible to choose $h_\alpha \in X$ so that $t_c <_{I_g} h_\alpha$ for all $c \in P_\alpha$ and $h_\beta <_{I_g} h_\alpha$ for all $\beta < \alpha$.

CLAIM 2.35. *For $n > 0$ and every $\delta \in S_n$, $\overline{h}{\restriction}\delta$ is flat.*

Proof. Let $\delta \in S_n$ and fix $c \subseteq \delta$ cofinal in δ with $\operatorname{otp} c = \omega_n$ such that for all $\beta \in c$ there is some $\gamma < \delta$ such that $c \cap \beta \in P_\gamma$. According to Lemma 5.4 from the Appendix it suffices to show that $\overline{h}{\restriction}\delta$ is equivalent modulo I_g to $\langle t_{c\cap\beta} : \beta \in c\rangle$ where, according to the definition, $t_{c\cap\beta} = \sup\{h_\alpha : \alpha \in c \cap \beta\}$.

First, if $\alpha < \delta$ is arbitrary, find $\beta \in c$ and $\eta \in c \cap \beta$ satisfying

$$\alpha < \eta < \beta.$$

Clearly,

$$h_\alpha \leq_{I_g} h_\eta \leq t_{c\cap\beta},$$

thus we see that for all $\alpha < \delta$ there is some $\beta \in c$ such that $h_\alpha \leq_{I_g} t_{c\cap\beta}$.

Conversely, suppose $\beta \in c$ is given. There is some $\alpha < \delta$ so that $c\cap\beta \in P_\alpha$, therefore by the inductive construction of h_α we have $t_{c\cap\beta} <_{I_g} h_\alpha$. ■

By Theorem 5.6 from the Appendix, there exists an exact upper bound g' of the sequence $\overline{h} = \langle h_\alpha : \alpha < \aleph_{\omega+1}\rangle$ such that

$$(\forall k)[\{n : \operatorname{cf} g'(n) < \aleph_k\} \in I_g], \tag{x}$$

that is, the set $\{n : \operatorname{cf} g'(n) < \aleph_k\}$ is contained in some $C^g_{\leq m}$. Clearly, we may assume that $g' \leq g$ and that $\operatorname{cf} g'(n) > \aleph_0$ for all n.

For each $\alpha < \aleph_{\omega+1}$ there is some m such that $h_\alpha(n) < g'(n)$ for all $n \in C^g_{<m}$, so there is some m such that

$$\{h \in X : (\forall n \in C^g_{>m})[h(n) < g'(n)]\}$$

is cofinal for $<_{I_g}$ below g'. Let now $g^* = (g{\restriction}C^g_{\leq m}) \cup (g'{\restriction}C^g_{>m})$. Since g^* differs from g' only on some $C^g_{\leq m}$, g^* is also an eub of $\overline{h}$ modulo I_g.

Claim 2.36. $I_{g^*} = I_g$.

Proof. We have to prove that

(1) for every $k \in \omega$ we have $C^g_k \in I_{g^*}$, or equivalently for every $k \in \omega$ there exists $m \in \omega$ such that $C^g_k \subseteq C^{g^*}_{\leq m}$,

(2) for every $k \in \omega$ we have $C^{g^*}_k \in I_g$, or equivalently for every $k \in \omega$ there exists $m \in \omega$ such that $C^{g^*}_k \subseteq C^g_{\leq m}$.

First we prove (1). From Lemma 2.33 we may assume that for every $k \in \omega$ there exists $\alpha(k) \leq \aleph_k$ such that

$$(g^*{\restriction}C^g_k) = t^{\alpha(k)}_k.$$

We fix $k \in \omega$. Since from Definition 2.28 the cofinality of $t^{\alpha(k)}_k$ is fixed, equal to some $\aleph_m$, we have $C^g_k \subseteq C^{g^*}_m$.

Statement (2) follows from (x). ■

As $\overline{h}$ demonstrates, $\operatorname{tcf}(\{h \in X : h \leq g^*\}, <_{I_g}) = \aleph_{\omega+1}$. ■

Theorem 2.37. *For every Rudin space X there is a Rudin space $Y \subseteq X$ with $|Y| = \aleph_{\omega+1}$.*

Proof. Suppose $g \in T\setminus X^{\mathrm{R}}$ and $X \subseteq X^g$ is a Rudin space. By 2.34 there is some $g^* \leq g$ in $T\setminus X^{\mathrm{R}}$ such that $I_{g^*} = I_g$ and $\operatorname{tcf}(X \cap X^{g^*}, \leq_{I_{g^*}}) = \aleph_{\omega+1}$. By 2.31, $X \cap X^{g^*}$ contains a closed and cofinal subspace Y of cardinality $\aleph_{\omega+1}$. ■

3. Baire measures and their Borel extensions in Rudin spaces. In this section we prove that all Rudin spaces are non-Mařík and that every Rudin space whose cardinality is not real-valued-measurable is quasi-Mařík.

DEFINITION 3.1. For a given space X and a σ-field $\sigma \subset \mathcal{P}(X)$, a function $\mu : \Sigma \to \mathbb{R}$ is a *finite measure* if

(1) $\emptyset, X \in \Sigma$, $\mu(\emptyset) = 0$, $\mu(X) > 0$,
(2) for every $A, B \in \Sigma$, if $A \subseteq B$ then $\mu(A) \le \mu(B)$,
(3) for every pairwise disjoint family $\{A_n : n \in \omega\} \subseteq \Sigma$,

$$\mu\Big(\bigcup_{n\in\omega} A_n\Big) = \sum_{n\in\omega} \mu(A_n).$$

DEFINITION 3.2. For a given space X and a measure μ on a σ-field Σ of subsets of X, we say that μ is *concentrated on a singleton* if there exists $x \in X$ such that for every $A \in \Sigma$ we have $\mu(A) = \mu(X)$ if and only if $x \in A$.

DEFINITION 3.3. Let X be a topological space.

(1) A set $A \subseteq X$ is *functionally closed* if there exists a continuous function $f : X \to [0,1]$ such that $A = f^{-1}[\{0\}]$.
(2) The *Baire σ-field* $\mathrm{Ba}(X)$ on X is the σ-field generated by all functionally closed sets.
(3) A finite measure defined on $\mathrm{Ba}(X)$ is called a *Baire measure*.

Recall that in a normal space X a closed set $D \subseteq X$ is functionally closed if and only if D is G_δ in X, and that X is called *perfectly normal* if all closed subsets of X are functionally closed. All perfectly normal spaces are countably paracompact, hence Dowker spaces are never perfectly normal.

CLAIM 3.4. *Let X be a Rudin space. The Baire σ-field* $\mathrm{Ba}(X)$ *is equal to the field of all clopen subsets of X.*

Proof. By Claim 2.17 every countable intersection of clopen sets is clopen. Thus, the field of all clopen subsets of X is a σ-field of sets.

Each clopen set is functionally closed trivially. Conversely, a functionally closed set is closed and G_δ, hence clopen. ∎

DEFINITION 3.5. Let X be a topological space.

(1) The *Borel σ-field* $\mathrm{Bor}(X)$ on X is the σ-field generated by all closed subsets of X.
(2) A measure on $\mathrm{Bor}(X)$ is called a *Borel measure*.

DEFINITION 3.6. Let X be a topological space and let μ be a measure on $\mathrm{Bor}(X)$. A Borel measure μ is called *regular* if for every $A \in \mathrm{Bor}(X)$,

$$\mu(A) = \sup\{\mu(F) : F \subseteq A,\ F \text{ closed}\}.$$

Definition 3.7. Let X be a normal topological space. We call X a *Mařík* space if for every Baire measure $\mu : \mathrm{Ba}(X) \to [0,1]$ there exists a regular Borel measure which extends μ.

Definition 3.8. Suppose that X is a Rudin space. A Baire measure μ on X is called *cofinal* if there exists $0 < r \in \mathbb{R}$ such that for every set $A \in \mathrm{Ba}(X)$,

$$\mu(A) = r \quad \text{iff} \quad A \text{ is cofinal in } X.$$

By Theorem 2.27 the collection of all clopen and cofinal subsets of a Rudin space X forms a σ-ultrafilter of clopen sets. Since the family of all clopen subsets of X coincides with the σ-field $\mathrm{Ba}(X)$, the clopen and cofinal subsets of X form a σ-ultrafilter of Baire sets. A cofinal Baire measure is, then, a two-valued measure that assigns the value $r > 0$ to all sets in this σ-ultrafilter and the value 0 to all Baire sets which are not in this σ-ultrafilter.

The next theorem generalizes Simon's Theorem [20] that X^{R} is not Mařík. See also Wheeler [24].

Theorem 3.9. *Suppose $g \in T \setminus X^{\mathrm{R}}$ and $X \subseteq X^g$ closed and cofinal in X^g. If μ is a cofinal Baire measure on X then μ does not admit a regular Borel extension. In particular, no Rudin space is Mařík.*

Proof. Let μ be a cofinal Baire measure. We assume for simplicity that $\mu(X) = 1$. Assume to the contrary that there exists a regular Borel extension of μ and denote this extension also by μ.

According to Definition 2.23 for every $n \in \omega$ we have

$$D_m^X = X \cap D_m^g = \{h \in X : (\exists n \in C_{>m}^g)[h(n) = g(n)]\}.$$

Recall that for every $m \in \omega$ the set D_m^X is closed and $\bigcap_{m\in\omega} D_m^X = \emptyset$. In particular, $\lim_{m\to\infty} \mu(D_m^X) = 0$. We fix $m_0 \in \omega$ such that $\mu(D_{m_0}^X) < 1/2$.

By Definition 2.23, $X_{m_0} = X \setminus D_{m_0}^X$ and by our choice of m_0 we have $\mu(X_{m_0}) > 1/2$. The set X_{m_0} is open and from the regularity of μ we conclude that there exists a closed subset $F \subseteq X_{m_0}$ such that $\mu(F) \geq 1/2$. By normality of X there are open sets U and W such that $D_{m_0}^X \subseteq U$, $F \subseteq W$ and $U \cap W = \emptyset$.

Since $D_{m_0}^X$ is closed and cofinal and $D_{m_0}^X \subseteq U$, and U is open, by Lemma 2.25 there exists $f \in P$ so that $f < g$ and $X \cap (f, g] \subseteq U$. Thus U contains a clopen and cofinal set $X \cap (f, g]$ whose measure is 1. Consequently, $\mu(W) = 0$ and as $F \subseteq W$ also $\mu(F) = 0$, contrary to $\mu(F) \geq 1/2$. ∎

Theorem 3.10. *Suppose $g \in T \setminus X^{\mathrm{R}}$ and $X \subseteq X^g$ is closed and cofinal in $(X^g, \leq)$. Let μ be a cofinal Baire measure on X. Then there is some $m_0 > 0$ such that for all $m \geq m_0$, μ extends to a Borel measure μ^m via the* *definition*

(i) $$\mu^m(B) = \begin{cases} \mu(X) & \textit{if } B \textit{ contains an } m\textit{-club,} \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. Since $X \subseteq X^g$ is closed and cofinal in $(X^g, \leq)$, by Claim 2.13 there exists $m_0 > 0$ such that X contains an m-club in X^g for all $m \geq m_0$. We show that for all $m \geq m_0$ the condition (i) above defines a Borel measure μ^m which extends μ.

By Lemma 2.25 and the Example after Definition 2.7, we know that every clopen and cofinal subset B of X has the property that for every $m \geq m_0$ the set B contains an m-club. This shows that

$$\mu^m(B) = \mu(X) = \mu(B)$$

for every set B belonging to the σ-ultrafilter in $\mathrm{Ba}(X)$ of all clopen and cofinal subsets. According to Definition 3.8, this implies that $\mu^m(B) = \mu(B)$ for every $B \in \mathrm{Ba}(X)$.

Let

$$\mathcal{C} = \{B \subseteq X : (\forall m \geq m_0)[B \text{ or } X \setminus B \text{ contains an } m\text{-club of } X^g]\}.$$

To finish the proof it is enough to show that $\mathcal{C}$ contains all Borel sets. We prove this by showing that $\mathcal{C}$ is a σ-field to which all closed subsets of X belong.

Suppose that $D \subseteq X$ is closed. By Lemma 2.14, either D or $X \setminus D$ contains an m-club of X^g for every $m \geq m_0$, hence $D \in \mathcal{C}$.

Obviously, if $B \in \mathcal{C}$, then $X \setminus B \in \mathcal{C}$. To see that $\mathcal{C}$ is closed under countable intersections, suppose we are given $B_n \in \mathcal{C}$ for each $n \in \omega$ and that $m \geq m_0$ is fixed. Either for every $n \in \omega$ the set B_n contains an m-club, and then by Claim 2.8, the intersection $\bigcap_n B_n$ also contains an m-club; or for some $n \in \omega$ the set B_n does not contain an m-club, and then $X \setminus B_n$ contains an m-club, and since

$$X \setminus B_n \subseteq X \setminus \bigcap_{n\in\omega} B_n,$$

also $X \setminus \bigcap_{n\in\omega} B_n$ contains an m-club. ∎

If $m_2 > m_1 \geq m_0$ then $\mu^{m_1} \neq \mu^{m_2}$, since any m_1-club is disjoint from any m_2-club. The extensions described in Theorem 3.10 for all $m \geq m_0$ are not all possible extensions of μ. For every sequence $\overline{r} = \{r_m\}_{m\geq m_0}$ of nonnegative reals satisfying $\sum_{m\geq m_0} r_m = 1$ one can define a Borel extension $\mu_{\overline{r}}$ of μ via $\mu_{\overline{r}}(B) = \sum_{m\geq m_0} r_m \cdot \mu^m(B)$, and this correspondence is clearly injective. This shows that for a given Baire measure μ there are at least $2^{\aleph_0}$ different Borel extensions.

Definition 3.11. A cardinal κ is *real-valued-measurable* if there exists a measure $\mu : P(\kappa) \to [0,1]$ which is 0 on singletons and such that $\mu(\kappa) = 1$.

If $\kappa < \lambda$ are cardinals and κ is real-valued-measurable, then clearly also λ is real-valued-measurable.

THEOREM 3.12 (S. Ulam, [9, Theorem 1D]). *If κ is the smallest real–valued-measurable cardinal then κ is weakly inaccessible, that is, a regular limit cardinal, and if the continuum is not real-valued-measurable, then κ is 2-valued-measurable, hence is strongly inaccessible, that is, regular and $\alpha, \beta < \kappa \Rightarrow \alpha^\beta < \kappa$.*

From Ulam's theorem the following follows immediately:

CLAIM 3.13. *Suppose α, β are infinite cardinals below the least real-valued-measurable cardinal. Then α^β is real-valued-measurable iff $2^{\aleph_0}$ is.*

Proof. Let α, β be infinite. So $2^{\aleph_0} \le \alpha^\beta$ and hence if $2^{\aleph_0}$ is real-valued-measurable so is α^β. Conversely, if $2^{\aleph_0}$ is not real-valued-measurable, the least real-valued-measurable κ is strongly inaccessible by Ulam's theorem. Thus, if $\alpha, \beta < \kappa$ also $\alpha^\beta < \kappa$. ∎

In the next theorem the reason for working with the full class of Rudin spaces becomes clear. In the generality of this class we can prove a structure theorem for all Baire measures on sufficiently small Rudin spaces.

THEOREM 3.14. *Suppose $g \in T \setminus X^{\mathrm{R}}$ and $X \subseteq X^g$ is closed and cofinal in $(X^g, \le)$. Suppose that $|X|$ is not real-valued-measurable. Suppose μ is a Baire measure on X. Then there are countable sets I and J, elements $f_i \in X$ for all $i \in I$, clopen Rudin subspaces $X_j \subseteq X$ for all $j \in J$ and measures μ_i for $i \in I$ and μ_j for $j \in J$ such that:*

(1) *for every $i \in I$, μ_i is a measure on X concentrated on the singleton $\{f_i\}$,*
(2) *if $j_1, j_2 \in J$ and $j_1 \ne j_2$ then $X_{j_1} \cap X_{j_2} = \emptyset$,*
(3) *for every $j \in J$, μ_j is a cofinal Baire measure on X_j.*

Finally,

$$\mu = \sum_{i \in I} \mu_i + \sum_{j \in J} \mu_j.$$

Proof. Fix a Baire measure μ on X and assume for simplicity that $\mu(X) = 1$.

For $n \in \omega$ and $\alpha \le g(n)$ let $U_{n,\alpha} := \{f \in X : f(n) \le \alpha\}$. This is a clopen set in X, and therefore belongs to $\mathrm{Ba}(X)$.

For each $n \in \omega$ we define by induction on $\xi < \xi_n$, for some ordinal $\xi_n < \omega_1$ which will be specified below, a strictly increasing and continuous countable sequence of ordinals $\alpha^n_\xi \le g(n)$. Assuming that α^n_ξ is defined, we define the real number $r^n_\xi \in [0,1]$ by

$$r^n_\xi := \mu(U_{n,\alpha^n_\xi}). \tag{ii}$$

Let $\alpha_0^n = 0$. Since $f(n) > 0$ for all $f \in X$ we have $U_{n,0} = \emptyset$ and $r_0^n := \mu(U_{n,\alpha_0}) = 0$.

For limit $\xi < \omega_1$ let $\alpha_\xi^n = \sup\{\alpha_\zeta^n : \zeta < \xi\}$. Since $\operatorname{cf} \alpha_\xi^n = \aleph_0$, it follows that $f(n) \neq \alpha_\xi^n$ for all $f \in X$ and therefore $\bigcup_{\zeta<\xi} U_{n,\alpha_\zeta^n} = U_{n,\alpha_\xi^n}$. Hence, $r_\xi^n = \sup\{r_\zeta^n : \zeta < \xi\}$.

If r_ξ^n is defined and $r_\xi^n < 1$, then necessarily $\alpha_\xi^n < g(n)$, as $U_{n,g(n)} = X$. Let

$$\alpha_{\xi+1}^n = \min\{\alpha \leq g(n) : \mu(U_{n,\alpha}) > r_\xi^n\}. \tag{iii}$$

If $r_\xi^n = 1$ we stop the induction and put $\xi_n = \xi + 1$.

The induction has to terminate at some $\xi_n < \omega_1$, or else $\{r_\zeta^n : \zeta < \omega_1\} \subseteq [0,1]$ would be order isomorphic to ω_1, which is impossible.

Claim 3.15. *For each $n \in \omega$ and $\xi < \xi_n$, $\operatorname{cf} \alpha_\xi^n > \aleph_0$ if and only if ξ is a successor ordinal.*

Proof. If $\xi < \xi_n$ is limit, then by continuity α_ξ^n has cofinality $\aleph_0$.

Suppose that $\xi = \zeta + 1 < \xi_n$. First we observe that $\alpha_{\zeta+1}^n$ cannot be a successor, since if $\alpha_{\zeta+1}^n = \beta + 1$ then $U_{n,\beta+1} = U_{n,\beta}$, contrary to the minimality of $\alpha_{\zeta+1}^n$. We know then that $\alpha_{\zeta+1}^n$ is limit, and need only prove that its cofinality is uncountable. Suppose to the contrary that $\langle \beta_i : i \in \omega\rangle$ is strictly increasing with limit $\alpha_{\zeta+1}^n$ and that $\beta_0 > \alpha_\zeta^n$. By the definition of $\alpha_{\zeta+1}^n$ (see formula (iii)),

$$\mu(U_{n,\beta_i}) = \mu(U_{n,\alpha_\zeta^n}) \quad \text{for all } i \in \omega.$$

Since μ is σ-additive,

$$\mu\Big(\bigcup_i U_{n,\beta_i}\Big) = \mu(U_{n,\alpha_\zeta^n}),$$

and since $\operatorname{cf} \alpha_{\zeta+1}^n = \aleph_0$,

$$\bigcup_i U_{n,\beta_i} = U_{n,\alpha_{\zeta+1}^n}.$$

This contradicts $\mu(U_{n,\alpha_{\zeta+1}^n}) > \mu(U_{n,\alpha_\zeta^n})$. ∎

For each $n \in \omega$ let $S_n = \{\alpha_\xi^n : \xi < \xi_n\}$.

Suppose that $x \in \prod_{n\in\omega}(S_n \setminus \sup\{S_n\})$; then for every $n \in \omega$, $\min\{S_n \setminus (x(n)+1)\}$ is well defined. Denote by x^s the function in $\prod_{n\in\omega} S_n$ that satisfies

$$x^s(n) = \min\{S_n \setminus (x(n)+1)\} \quad \text{for all } n \in \omega.$$

For every $x \in \prod_{n\in\omega}(S_n \setminus \{\sup S_n\})$ let

$$U_x = (x, x^s] = \{f \in X : (\forall n \in \omega)[x(n) < f(n) \leq x^s(n)]\}.$$

Let $x \in \prod_{n\in\omega} S_n$ and suppose that for some $n \in \omega$ we have $x(n) = \max S_n$. In that case let

$$U_x = (x, g].$$

If $x(n) = g(n)$ for some $n \in \omega$ then $U_x = \emptyset$. Thus U_x is a basic clopen subset of X for all $x \in \prod_{n\in\omega} S_n$.

In the case that $x(n) = \max S_n$ and $\max S_n < g(n)$ we see that $U_x \subseteq \{f \in X : f(n) > \max S_n\}$ and since $\mu(U_{n,\max S_n}) = 1$ it follows that $\mu(U_x) = 0$. Hence,

Claim 3.16. *If $x \in \prod_{n\in\omega} S_n$ and for some $n \in \omega$ we have $x(n) = \max S_n$ then $\mu(U_x) = 0$.*

It is obvious that $X = \bigcup_{x\in\prod_{n\in\omega} S_n} U_x$. If $x \neq y$ and $x, y \in \prod_{n\in\omega} S_n$ then clearly $U_x \cap U_y = \emptyset$.

Since $|X|$ is not real-valued-measurable, also the cardinality of the set

$$A = \Big\{x \in \prod_{n\in\omega} S_n : U_x \neq \emptyset\Big\}$$

is not real-valued-measurable. Given an arbitrary $D \subseteq A$, both $\bigcup_{x\in D} U_x$ and $\bigcup_{x\in A\setminus D} U_x$ are open, hence each of them is also clopen and μ-measurable.

By letting $\mu'(D) = \mu(\bigcup_{x\in D} U_x)$ we define a measure μ' on $\mathcal{P}(A)$. Since $|A|$ is not real-valued-measurable, according to [8, Lemma 438Bb], we conclude that μ' is a countable sum of measures concentrated on singletons, in particular there exists a countable subset $H \subseteq A$ such that

$$\mu(U_x) > 0 \quad \text{iff} \quad x \in H$$

and

$$\mu\Big(\bigcup\{U_x : x \in A \setminus H\}\Big) = 0.$$

From now on we work with a fixed x, assuming that:

- $\mu((x, x^s]) = \mu(U_x) = \epsilon > 0$,
- for every $n \in \omega$ and $\alpha \in (x(n), x^s(n))$ we have $\mu(\{t \in X : x(n) < t(n) \leq \alpha\}) = 0$.

The second item follows from the minimality of $x^s(n)$. From this and from the countable additivity of μ we obtain:

Claim 3.17. *For every function h which satisfies $x < h < x^s$ we have $\mu((h, x^s]) = \epsilon$; in particular, $X \cap (h, x^s] \neq \emptyset$.*

The set H is countable and now we decompose it into two subsets I and J, as stated in the Theorem, according to the following two cases:

Case 1: $x^s \in X^{\mathrm{R}}$. In this case, as X is closed in X^{R}, Claim 3.17 shows that $X \cap (h, x^s] \neq \emptyset$ for each $h < x^s$, in particular $x^s \in X$. Therefore, for every clopen and cofinal $U \subseteq (x, x^s]$ we have $x^s \in U$ and μ restricted to

$(x, x^s]$ is concentrated on a singleton, because on a clopen $U \subseteq (x, x^s]$ we have $\mu(U) = \epsilon$ if and only if $x^s \in U$.

CASE 2: $x^s \notin X^{\mathrm{R}}$.

Since $\aleph_0 < \operatorname{cf} x^s(n) < \aleph_\omega$ for all $n \in \omega$, it follows that $x^s \in T \setminus X^{\mathrm{R}}$. From Claim 3.17 we conclude that $\mu \restriction \mathrm{Ba}(X \cap U_x)$ is a cofinal Baire measure on the Rudin space $X \cap U_x = X \cap (x, x^s]$.

We define

$$I = \{x : x^s \in H \cap X^{\mathrm{R}}\}, \tag{iv}$$

$$J = \{x : x^s \in H \setminus X^{\mathrm{R}}\}. \tag{v}$$

For every $i = x \in I$ let $\mu_i := \mu \restriction \mathrm{Ba}(X \cap (x, x^s]$. This is indeed a measure concentrated on the singleton $\{f_i\} := \{x^s\}$.

For each $j = x \in J$ let $\mu_j = \mu \restriction \mathrm{Ba}(X \cap (x, x^s])$. This is indeed a cofinal Baire measure on the Rudin space $X_j := X \cap (x, x^s]$. We have already established that $X_{j_1} \cap X_{j_2} = \emptyset$ for $j_1 \neq j_2$ in J.

It remains to show that $\mu = \sum_{i \in I} \mu_i + \sum_{j \in J} \mu_j$. Let $B \subseteq X$ be an arbitrary Baire set. As $\mu(\bigcup_{x \in A \setminus H} U_x) = 0$, it follows that $\mu(B) = \mu(B \cap \bigcup_{x \in H} U_x)$. Since $\{U_x : x \in H\}$ is a pairwise disjoint family of sets from $\mathrm{Ba}(X)$, we find that $\mu(B) = \sum_{x \in H} \mu(B \cap U_x)$, which is exactly what we need. ∎

THEOREM 3.18. *If X is a Rudin space and $|X|$ is not a real-valued-measurable cardinal, then X is quasi-Mařík.*

Proof. Let I, J, μ_i, μ_j and X^i, X^j for $i \in I, j \in J$ and f_i for $i \in I$ be as in Theorem 3.14. It is enough to extend each of the measures μ_i, μ_j to Borel measures on X^i and X^j respectively.

If $i \in I$ then μ_i is concentrated on the singleton $\{f_i\}$. We define the extension of μ_i on the whole $\mathcal{P}(X)$ by setting $\mu_i(A) = \mu_i(X)$ if and only if $f_i \in A$, for each $A \in \mathcal{P}(X)$.

For $j \in J$ many extensions of μ_j to a Borel measures on X_j exist by Theorem 3.10. ∎

COROLLARY 3.19 (ZFC). *There are quasi-Mařík non-Mařík Dowker spaces. In fact, every Rudin space contains a quasi-Mařík non-Mařík Rudin subspace of cardinality $\aleph_{\omega+1}$.*

Proof. Suppose $X \subseteq X^{\mathrm{R}}$ is a Rudin space. By Theorem 2.37 fix a Rudin subspace $Y \subseteq X$ with $|Y| = \aleph_{\omega+1}$. Then Y is not Mařík by Theorem 3.9. From Theorem 3.12 the cardinal $|Y|$ is not real-valued-measurable, so Y is quasi-Mařík by Theorem 3.18. ∎

THEOREM 3.20. *If the continuum is not real-valued-measurable then every Rudin space is quasi-Mařík.*

Proof. Every Rudin space has cardinality at most $|X^{\mathrm{R}}| = (\aleph_\omega)^{\aleph_0}$. As both $\aleph_\omega$ and ω are smaller than the least real-valued-measurable, Claim 3.13 concludes the proof. ∎

We conclude by commenting that in Claim 3.13 real-valued-measurability can be replaced by weak inaccessibility for the special case $\alpha^\beta = (\aleph_\omega)^{\aleph_0}$. Shelah's upper bound for $(\aleph_\omega)^{\aleph_0}$ states

$$(\aleph_\omega)^{\aleph_0} = \max\{2^{\aleph_0}, \max \mathrm{pcf}\{\aleph_n\}_{n\in\omega}\}, \qquad \max \mathrm{pcf}\{\aleph_n\}_{n\in\omega} < \aleph_{\omega_4}.$$

As $\max \mathrm{pcf}\{\aleph_n\}_{n\in\omega} < \aleph_{\omega_4}$, it is smaller than the least weakly inaccessible cardinal. Hence, from the equality above it follows that $(\aleph_\omega)^{\aleph_0}$ is weakly inaccessible if and only if $2^{\aleph_0}$ is.

Actually, from Shelah's general theory of upper bounds it follows that for every singular μ below the least fixed point, $\mu^{\mathrm{cf}(\mu)}$ is weakly inaccessible if and only if $2^{\mathrm{cf}(\mu)}$ is. By Gitik's result [10] it is consistent for the least fixed point μ itself that $\mu^{\mathrm{cf}(\mu)}$ is weakly inaccessible while $2^{\mathrm{cf}(\mu)} = 2^{\aleph_0}$ is accessible.

4. Problems. Let us conclude with three problems. The first problem is old and well known. The second is a weaker form of the first, localized to the measure-theoretic context.

PROBLEM 4.1. *Is it consistent with ZFC that there are no Dowker spaces of cardinality $\aleph_1$?*

PROBLEM 4.2. *Is it consistent that there is no counterexample to the measure extension problem of cardinality $\aleph_1$?*

PROBLEM 4.3. *Is it provable in ZFC that every Rudin space is quasi-Mařík?*

5. Appendix: PCF preliminaries. We set some standard PCF terminology and facts. The reader may consult [1, 5, 12, 18] for additional details.

Recall that ON^ω denotes the class of all functions from ω to the Ordinal Numbers. Let $0 \in \mathrm{ON}^\omega$ stand for the constant function 0.

DEFINITION 5.1. Suppose $I \subseteq \mathcal{P}(\omega)$ is an ideal on ω.

(1) For $f, g \in \mathrm{ON}^\omega$ we write

$$\begin{aligned} f \le_I g &\quad \text{if } \{n : f(n) > g(n)\} \in I,\\ f <_I g &\quad \text{if } \{n : f(n) \ge g(n)\} \in I,\\ f =_I g &\quad \text{if } \{n : f(n) \ne g(n)\} \in I. \end{aligned}$$

In the case $I = \{\emptyset\}$ we omit I from the notation and write just $<, \le$ and $=$.

(2) A function $h \in \mathrm{ON}^\omega$ is an *upper bound* of a set $A \subseteq \mathrm{ON}^\omega$ with respect to $<_I$ (modulo $<_I$, mod $<_I$, modulo I) if for every $f \in I$ we have $f \leq_I h$. For $A \subseteq X \subseteq \mathrm{ON}^\omega$ we say that X is *unbounded in* X with respect to $\leq_I$ there is no upper bound of A in X with respect to $\leq_I$.

(3) For $X \subset \mathrm{ON}^\omega$ we define $\mathfrak{b}(X, \leq_I)$ to be the smallest cardinality of a $\leq_I$-unbounded subset of X, if X has no maximum, and ∞ otherwise, where ∞ is taken to be larger than every cardinal.

(4) For $A \subseteq X \subset \mathrm{ON}^\omega$, A is *cofinal* in $(X, \leq_I)$ if for every $f \in X$ there exists $h \in A$ such that $f \leq_I h$. The *cofinality* of $(X, \leq_I)$, denoted $\mathrm{cf}(X, \leq_I)$, is the smallest cardinality of $A \subseteq X$ which is cofinal in $(X, \leq_I)$.

(5) For $X \subseteq \mathrm{ON}^\omega$ we say that $(X, \leq_I)$ *has true cofinality* if $\mathfrak{b}(X, \leq_I) = \mathrm{cf}(X, \leq_I)$. If $(X, \leq_I)$ has true cofinality we define the *true cofinality* of $(X, \leq_I)$, denoted $\mathrm{tcf}(X, \leq_I)$, by

$$\mathrm{tcf}(X, \leq_I) = \mathfrak{b}(X, \leq_I).$$

We remark that unless $\mathfrak{b}(X, \leq_I) = \infty$ we have $\mathfrak{b}(X, \leq_I) \leq \mathrm{cf}(X, \leq_I)$ and unless $\mathfrak{b}(X, \leq_I)$ is finite, it is an infinite *regular* cardinal.

Let $I \subseteq \mathcal{P}(\omega)$ be an ideal over ω which contains all finite subsets of ω.

Definition 5.2. Let $I \subseteq \mathcal{P}(\omega)$ be an ideal over ω. A function $h \in \mathrm{ON}^\omega$ is an exact upper bound (eub) of $A \subseteq \mathrm{ON}^\omega$ with respect to I if

(1) h is an upper bound of A with respect to $\leq_I$,
(2) for every $w \in \mathrm{ON}^\omega$, if $w <_I h$ then there exists $f \in A$ such that $w <_I f <_I h$.

Claim 5.3. *Let $I \subseteq \mathcal{P}(\omega)$ be an ideal over ω. If $A \subseteq \mathrm{ON}^\omega$ contains some h such that $0 <_I h$ and both $g, h \in \mathrm{ON}^\omega$ are eubs of A with respect to $<_I$ then $g =_I h$.*

Lemma 5.4 (see [11, Claim 5]). *Let $I \subseteq \mathcal{P}(\omega)$ be an ideal over ω. Let κ be a regular uncountable cardinal. Let $\overline{f} = \langle f_\alpha \in \mathrm{ON}^\omega : \alpha < \delta\rangle$ be a $<_I$-increasing sequence of functions. The following conditions are equivalent:*

(1) *There is an eub $f \in \mathrm{ON}^\omega$ of $\overline{f}$ such that $\{n \in \omega : \mathrm{cf}\, f(n) \neq \kappa\} \in I$.*
(2) *There exists a sequence $\overline{h} = \langle h_\alpha \in \mathrm{ON}^\omega\rangle$ such that the sequence $\langle h_\beta : \beta < \kappa\rangle$ is $<$-increasing and*
   (a) *for every $\alpha < \delta$ there exists $\beta < \kappa$ such that $f_\alpha <_I h_\beta$,*
   (b) *for every $\beta < \kappa$ there exists $\alpha < \delta$ such that $h_\beta <_I f_\alpha$.*

Definition 5.5. A given $<_I$-increasing sequence of functions $\langle f_\alpha \in \mathrm{ON}^\omega : \alpha < \lambda\rangle$ is *flat of cofinality* κ if one of the equivalent conditions of Lemma 5.4 is satisfied.

Theorem 5.6 ([11], [12, Theorem 20]). *Let $I \subseteq \mathcal{P}(\omega)$ be an ideal over ω. Let $\lambda > \aleph_1$ be a regular cardinal and let $\bar{f} = \langle f_\alpha \in \mathrm{ON}^\omega : \alpha < \lambda\rangle$ be a $<_I$-increasing sequence of functions. For every regular κ such that $\omega < \kappa < \lambda$ the following conditions are equivalent:*

(1) *The sequence $\bar{f}$ has an eub f and $\{n \in \omega : \mathrm{cf}\, f(n) \le \kappa\} \in I$.*
(2) *The set $\{\delta < \lambda : \mathrm{cf}\,\delta = \kappa,\ \bar{f}\restriction\delta$ is flat of cofinality $\kappa\}$ is stationary in λ.*

Definition 5.7 (Shelah's $I[\lambda]$ ideal, see [18, Definition 2.3, p. 14]). For a regular uncountable cardinal λ we define an ideal $I[\lambda]$ as the family of all $S \subseteq \lambda$ such that there exists a sequence $\langle P_\alpha : \alpha < \lambda\rangle$ of sets and a club $E \subseteq \lambda$ with the following properties:

(1) $P_\alpha \subseteq P(\alpha)$, $|P_\alpha| < \lambda$,
(2) for every $\delta \in E \cap S$ there exists $c \subseteq \delta$ such that $\sup(c) = \delta$, $\mathrm{otp}(c) = \mathrm{cf}\,\delta < \delta$ and for every $\beta \in c$ we have $c \cap \beta \in \bigcup_{\beta<\delta} P_\delta$.

Theorem 5.8 (Shelah, [19]). *For any two regular cardinals κ and λ such that $\kappa^+ < \lambda$ there exists a stationary set $S \subseteq \{\alpha < \lambda : \mathrm{cf}\,\alpha = \kappa\}$ such that $S \in I[\lambda]$.*

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