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Quasi-orbit spaces associated to T_0-spaces

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Abstract. Let $G \subset \mathrm{Homeo}(E)$ be a group of homeomorphisms of a topological space E. The class of an orbit O of G is the union of all orbits having the same closure as O. Let $E/\widetilde{G}$ be the space of classes of orbits, called the quasi-orbit space. We show that every second countable T_0-space Y is a quasi-orbit space $E/\widetilde{G}$, where E is a second countable metric space.

The regular part X_0 of a T_0-space X is the union of open subsets homeomorphic to $\mathbb{R}$ or to $\mathbb{S}^1$. We give a characterization of the spaces X with finite singular part $X - X_0$ which are the quasi-orbit spaces of countable groups $G \subset \mathrm{Homeo}_+(\mathbb{R})$.

Finally we show that every finite T_0-space is the singular part of the quasi-leaf space of a codimension one foliation on a closed three-manifold.

1. Introduction

1.1. T_0-spaces and quasi-orbit spaces. In general, the orbit space of a dynamical system is not a Hausdorff space, even for simple and regular systems: just consider the example of an irrational rotation on the circle. This example shows that the orbit space does not even satisfy weaker separation axioms, like the T_0 separation axiom. However, if one considers a Hausdorff quotient of the orbit space, one looses most of the dynamical information on the initial system. For this reason, [HS] considers an intermediary quotient called the *quasi-orbit space* which is a T_0-space and keeps more information on the initial dynamical system.

Let E be a Hausdorff topological space and $\mathrm{Homeo}(E)$ its group of homeomorphisms. Consider a subgroup $G \subset \mathrm{Homeo}(E)$. The family of G orbits $G(x) = \{g(x) : g \in G\}$ determines an open equivalence relation on E. We define the *class* of an orbit O to be the union of all orbits O' having the same closure as O. In other words, we define on E a new equivalence relation $\widetilde{G}$ by

$$x\widetilde{G}y \quad \text{if} \quad \overline{G(x)} = \overline{G(y)}.$$

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Since the saturated sets, under G and under $\widetilde{G}$, of an open set are equal, $\widetilde{G}$ is an open equivalence relation.

We denote by $Z = E/G$ the space of orbits, and by $X = E/\widetilde{G}$ the space of classes of orbits, called the *quasi-orbit space*. We denote by $q : E \to Z$ and $p : E \to X$ the canonical projections. The map $\varphi : Z \to X$ which associates to each orbit its class is an onto quasi-homeomorphism ([1]). According to [HS], the quasi-orbit space X is a T_0-space.

The goal of this paper is to give some conditions for which a T_0-space X is homeomorphic to a quasi-orbit space. One of the main results of this paper is:

THEOREM 1.1. *Let X be a connected second countable T_0-space.*

(a) *There exist a connected second countable metric space E and an abelian subgroup G of* $\mathrm{Homeo}(E)$ *such that X is homeomorphic to the quasi-orbit space $E/\widetilde{G}$.*
(b) *Assume moreover that X is finite. Then there are an integer n and a finitely generated abelian subgroup $G \subset \mathrm{Diff}^\infty(\mathbb{T}^n)$, where $\mathbb{T}^n$ is the n-torus, and an invariant set $E \subset \mathbb{T}^n$ such that X is the quasi-orbit space of the action of G on E.*

1.2. Countable groups of homeomorphisms of $\mathbb{R}$. Our second main result gives a characterization of the quasi-orbit spaces of countable subgroups of the group $\mathrm{Homeo}_+(\mathbb{R})$ of increasing homeomorphisms of the line $\mathbb{R}$. More precisely, if X is a T_0-space, we define the *regular part* X_0 of X to be the union of all open subsets of X homeomorphic to $\mathbb{R}$ or $\mathbb{S}^1$. The *singular part* of X is the complement $X - X_0$. Given a T_0-space X whose singular part is finite, Theorem 1.8 below gives a necessary and sufficient condition for a T_0-space to be the quasi-orbit space $\mathbb{R}/\widetilde{G}$ of a countable subgroup $G \subset \mathrm{Homeo}_+(\mathbb{R})$.

Before stating our condition, note that the regular part X_0 of X has a simple interpretation if X is the quasi-orbit space $\mathbb{R}/\tilde{G}$ of a countable subgroup $G \subset \mathrm{Homeo}_+(\mathbb{R})$. An orbit $G(x)$ is *proper* if it is locally closed; a proper orbit is *stable* if the stabilizer G_x fixes every point in a neighborhood $]x-\varepsilon, x+\varepsilon[$, for some $\varepsilon > 0$. In this case the projection $p(]x-\varepsilon, x+\varepsilon[)$ is contained in the regular part X_0. According to [HS], $p^{-1}(X_0)$ is the union of all stable proper orbits.

Our description of the quasi-orbit spaces X of groups of homeomorphisms of $\mathbb{R}$ consists in decomposing X into levels given by the notion of height of an orbit (or more generally of height of a point in an ordered space). We

([1]) A continuous map $f : X \to Y$ between two topological spaces is called *a quasi-homeomorphism* if the map which assigns to each open set $V \subset Y$ the open set $f^{-1}(V)$ is a bijective map.

will show that, for every level k, the relative position of points of height $> k$ with respect to points of height k may be classified into five types. So, in order to state our main result, we will first recall the notion of height; then we will describe the five types. This will allow us to define the notion of *admissible* space. Finally, Theorem 1.8 states that admissibility is equivalent to realisability as a quasi-orbit space.

1.2.1. *Notion of height.* Let $(Y, \leq)$ be a partially ordered set. A *chain* of $(Y, \leq)$ is a totally ordered subset of Y. Given an integer $n \geq 0$, we say that a point $y \in Y$ *has height* $\mathrm{ht}(y) = n$ if $n+1$ is the upper bound of the cardinality of the chains of $(Y, \leq)$ bounded by y. So a point y is *minimal* for $\leq$ if $\mathrm{ht}(y) = 0$. We say that Y *has finite height* $\mathrm{ht}(Y) \in \mathbb{N}$ if $\mathrm{ht}(y)$ is defined for every $y \in Y$ and $\mathrm{ht}(Y) = \sup\{\mathrm{ht}(y) : y \in Y\}$.

Let X be a T_0-space partially ordered by $x \leq y$ if $\overline{\{x\}} \subset \overline{\{y\}}$. The *regular part*, X_0, of X is the interior of the subset of maximal elements for this order. Hence, if the singular part of X is finite, then the height of X is finite. For $0 \leq k \leq \mathrm{ht}(X)$, we set

$$X^k = \{x \in X : \mathrm{ht}(x) \leq k\}.$$

By convention we denote $X^{-1} = \emptyset$. Notice that each X^k is a closed subset of X.

For every $A \subset X^0$ we denote $X_A = \{x \in X - X^0 : \overline{\{x\}} \cap X^0 = A\}$. By definition the sets X_A are pairwise disjoint, so that the family of nonempty X_A induces a partition of $X - X^0$.

Notice that every subset of X is equipped with the induced order. In particular, two elements in the regular part are incomparable.

1.2.2. *Five types of position of $X - X^0$ with respect to X^0.*

- We say that X is *of type* $\mathcal{T}_0$ if $X = X^0$ and is the line $\mathbb{R}$.
- We say that X is *of type* $\mathcal{T}_1$ if $X = X^0$ and is the circle $\mathbb{S}^1$.
- We say that X is *of type* $\mathcal{T}_2$ if X^0 is a single point.

We say that X is of type $\mathcal{T}_3$ if $X^0 - X_0$ is a finite set which is cyclically ordered, each point being connected to the next one either by an interval in X_0 or by a connected component of $X - X^0$. A cyclic order on a set of two points has no real meaning, so we need to distinguish the case $\mathrm{card}(X - X_0) = 2$. For this reason we say that X is *of type* $\mathcal{T}_3$ if it is of type $\mathcal{T}_{3'}$ or $\mathcal{T}_{3''}$ defined as follows:

Definition 1.2. We say that X is *of type* $\mathcal{T}_{3'}$ if $\mathrm{card}(X^0 - X_0) = k \geq 3$ and there is a cyclic indexation $X^0 - X_0 = \{a_0, \ldots, a_i, \ldots, a_{k-1}\}$, $i \in \mathbb{Z}/k\mathbb{Z}$ such that, for every i, either

- $X_{\{a_i, a_{i+1}\}} \neq \emptyset$ and $X_{\{a_i, a_{i+1}\}}$ is a connected component of $X - X^0$, in which case we denote $I_{i,i+1} = \emptyset$; or

- $X_{\{a_i,a_{i+1}\}} = \emptyset$ and there is an interval $I_{i,i+1} \subset X^0 \cap X_0$ whose closure $\overline{I_{i,i+1}}$ is a segment with extremities a_i and a_{i+1}.

Furthermore,

$$X - X^0 = \bigcup_{i\in\mathbb{Z}/k\mathbb{Z}} X_{\{a_i,a_{i+1}\}} \quad \text{and} \quad X^0 \cap X_0 = \bigcup_{i\in\mathbb{Z}/k\mathbb{Z}} I_{i,i+1}.$$

Definition 1.3. We say that X is *of type* $\mathcal{T}_{3''}$ if $X^0 - X_0$ is a pair $\{a_0, a_1\}$ and either

- $X_{\{a_0,a_1\}}$ consists of two different connected components of $X - X^0$, in which case we denote $I = \emptyset$; or
- $X_{\{a_0,a_1\}}$ is a connected component of $X - X^0$ and there is an interval $I \subset X_0 \cap X^0$ whose closure is a segment with extremities a_0 and a_1.

Moreover, $X - X^0 = X_{\{a_0,a_1\}}$ and $X^0 \cap X_0 = I$.

We say that X is of type $\mathcal{T}_4$ if $X^0 - X_0$ is a finite ordered set such that each point is connected to the next one by either an interval in X_0 or by a connected component of $X - X^0$. Moreover, there is either an interval in X_0 or a connected component of $X - X^0$ which is attached to the first and to the last point. We have to distinguish the case where $X^0 - X_0$ is a single point, because the first and the last point are the same. For this reason we divide the type $\mathcal{T}_4$ into two subtypes $\mathcal{T}_{4'}$ and $\mathcal{T}_{4''}$:

Definition 1.4. We say that X is *of type* $\mathcal{T}_{4'}$ if $\mathrm{card}(X^0 - X_0) = k \geq 2$ and there is an indexation $X^0 - X_0 = \{a_1, \dots, a_i, \dots, a_k\}$, $i \in \{1, \dots, k\}$ such that:

- for every $i \in \{1, \dots, k-1\}$, either
  - $X_{\{a_i,a_{i+1}\}} \neq \emptyset$ and $X_{\{a_i,a_{i+1}\}}$ is a connected component of $X - X^0$, in which case we denote $I_{i,i+1} = \emptyset$; or
  - $X_{\{a_i,a_{i+1}\}} = \emptyset$ and there is an interval $I_{i,i+1} \subset X^0 \cap X_0$ whose closure $\overline{I_{i,i+1}}$ is a segment with extremities a_i and a_{i+1}.
- for every $j \in \{1, k\}$, either
  - $X_{\{a_j\}} \neq \emptyset$ and is a connected component of $X - X^0$, in which case we denote $I_j = \emptyset$; or
  - $X_{\{a_j\}} = \emptyset$ and there is an interval $I_j \subset X^0 \cap X_0$ whose closure $\overline{I_j}$ (in X) is a semi-open interval (homeomorphic to $[0, 1[$) whose extremity is a_j.

Furthermore,

$$X - X^0 = X_{\{a_1\}} \cup X_{\{a_k\}} \cup \bigcup_{i=1}^{k-1} X_{\{a_i,a_{i+1}\}}, \qquad X^0 \cap X_0 = I_1 \cup I_k \cup \bigcup_{i=1}^{k-1} I_{i,i+1}.$$

DEFINITION 1.5. We say that X is *of type $\mathcal{T}_{4''}$* if $X^0 - X_0$ is a singleton $\{a\}$, $X_{\{a\}}$ consists of connected components of $X - X^0$, and there is an interval $I \subset X^0 \cap X_0$ whose closure $\overline{I}$ in X is a semi-open interval whose extremity is a.

Moreover, $X - X^0 = X_{\{a\}}$, and $X^0 \cap X_0 = I$.

REMARK 1.6. If $X_{\{a_i,a_{i+1}\}} = \emptyset$ and $X_{\{a_{i-1},a_i\}} = \emptyset$, then $a_i \in X_0$. If $X_{\{a_1\}} = \emptyset$ and $X_{\{a_1,a_2\}} = \emptyset$, then $a_1 \in X_0$. If $X_{\{a_k\}} = \emptyset$ and $X_{\{a_{k-1},a_k\}} = \emptyset$, then $a_k \in X_0$. These three implications contradict the definitions of the points a_i, a_1 and a_k. So

$$\begin{aligned} X_{\{a_i,a_{i+1}\}} = \emptyset \ &\Rightarrow\ X_{\{a_{i-1},a_i\}} \neq \emptyset, \\ X_{\{a_1\}} = \emptyset \ &\Rightarrow\ X_{\{a_1,a_2\}} \neq \emptyset, \\ X_{\{a_k\}} = \emptyset \ &\Rightarrow\ X_{\{a_{k-1},a_k\}} \neq \emptyset. \end{aligned}$$

1.2.3. *Admissible T_0-spaces and quasi-orbit spaces*

DEFINITION 1.7. We say that a T_0-space X is *admissible* if:

- X is connected and second countable;
- the singular part $X - X_0$ is finite; in particular X has finite height;
- every connected component Y of $X - X^k$, for $-1 \le k \le \mathrm{ht}(X)$, is of type $\mathcal{T}_0$, $\mathcal{T}_1$, $\mathcal{T}_2$, $\mathcal{T}_3$, or $\mathcal{T}_4$;
- whenever a connected component Y of $X - X^k$, for $0 \le k \le \mathrm{ht}(X)$, is of type $\mathcal{T}_0$ or $\mathcal{T}_4$, and Z denotes the connected component of $X - X^{k-1}$ containing Y, then Z is of type $\mathcal{T}_2$.

We can now state precisely our main result.

THEOREM 1.8. *A T_0-space X is homeomorphic to the quasi-orbit space $\mathbb{R}/\widetilde{G}$ of a finitely generated subgroup G of* $\mathrm{Homeo}_+(\mathbb{R})$ *if and only if X is admissible. Moreover, if X is admissible, then one can choose the group G to be abelian.*

If G is a group of diffeomorphisms of $\mathbb{R}$, then we obtain the following results:

THEOREM 1.9. *If X is the quasi-orbit space of an abelian subgroup $G \subset$* $\mathrm{Diff}^2_+(\mathbb{R})$, *then* $\mathrm{ht}(X) \le 2$. *Moreover, if* $\mathrm{ht}(X) = 2$, *then G is not finitely generated and it has an exceptional minimal set.*

THEOREM 1.10. *If X is the quasi-orbit space of a subgroup $G \subset$* $\mathrm{Diff}^\omega_+(\mathbb{R})$, *then* $\mathrm{ht}(X) \le 2$.

1.3. Codimension one foliations. The study of codimension one transversally oriented foliations is closely related to the study of countable subgroups of $\mathrm{Homeo}_+(\mathbb{R})$.

If $\mathcal{F}$ is a transversally oriented codimension one foliation on a closed manifold M, then we define, in the same manner as for groups of homeomorphisms, the space $Z = M/\mathcal{F}$ of leaves and the space $X = M/\widetilde{\mathcal{F}}$ of quasi-leaves. As in the case of subgroups of $\mathrm{Homeo}_+(\mathbb{R})$, we define the regular part X_0 of X to be the union of all open subsets of X homeomorphic to $\mathbb{R}$ or to $\mathbb{S}^1$. According to [BES], $p^{-1}(X_0)$ is the union of stable proper leaves. Without any condition on the space Y, we obtain the following result:

THEOREM 1.11. *Every connected finite T_0-space Y is homeomorphic to the singular part $X - X_0$ of the quasi-leaf space X of a transversally oriented codimension one C^1-foliation $\mathcal{F}$ on a closed three-manifold M.*

We will prove a stronger statement, by induction on the height $\mathrm{ht}(Y)$:

THEOREM 1.12. *Every connected finite T_0-space Y is homeomorphic to the singular part $X - X_0$ of the quasi-leaf space X of a transversally oriented codimension one C^1-foliation $\mathcal{F}$ on a connected closed three-manifold M, satisfying the following condition: for each $x \in Y$, there is a closed transversal γ_x of $\mathcal{F}$ such that the class $y \in Y$ of a leaf intersects γ_x if and only if $x \subset \overline{y}$. Furthermore, any component c of X_0 is a circle which corresponds to stable proper leaves and there is a closed transversal γ_c of $\mathcal{F}$ which intersects every leaf of c in exactly one point.*

2. Preliminaries. We recall some general notions which we will use in the rest of this paper.

Let $\mathcal{R}$ be an equivalence relation on a topological space E. Throughout this paper we write $\mathcal{R}(x)$ for the equivalence class of x and we call it the *trajectory* of x.

(1) If $A \subset E$, the *saturation* $\mathrm{Sat}_{\mathcal{R}}(A)$ of A is the union of all trajectories meeting A. The subset A is called *saturated* (or *invariant*) if we have $A = \mathrm{Sat}_{\mathcal{R}}(A)$.

(2) The relation $\mathcal{R}$ is called *open* if the saturation $\mathrm{Sat}_{\mathcal{R}}(O)$ of every open subset O is open. Equivalently, the natural (continuous) projection $p\colon E \to E/\mathcal{R}$ is open.

(3) The *class* $\mathrm{Cl}(T)$ of a trajectory T is the union of all trajectories T' having the same closure as T. We denote by $\widetilde{\mathcal{R}}$ the equivalence relation on E defined by the classes of the trajectories of $\mathcal{R}$: $x\widetilde{\mathcal{R}}y$ if $\overline{\mathcal{R}(x)} = \overline{\mathcal{R}(y)}$.

Notice that the saturations of any open set under $\mathcal{R}$ and $\widetilde{\mathcal{R}}$ are equal, so that $\widetilde{\mathcal{R}}$ is open $\Leftrightarrow$ $\mathcal{R}$ is open.

(4) A *minimal* set is a minimal element of the family of nonempty saturated closed subsets (ordered by inclusion). Equivalently, a minimal set is a nonempty saturated subset $S \subset E$ such that the closure of every trajectory $T \subset S$ is equal to S.

(5) A trajectory T is called *proper* if it is locally closed with $\mathrm{int}(\overline{T}) = \emptyset$.
(6) A trajectory T is called a *maximal* (resp. *minimal*) *trajectory* if every trajectory T' such that $T \subset \overline{T'}$ (resp. $T' \subset \overline{T}$) has the same closure as T: $\overline{T} = \overline{T'}$.

A trajectory T is a minimal trajectory if and only if it is contained in a minimal set.
(7) Let $\mathcal{R}$ be the identity relation on a topological space E. We say that a point x has a property $\mathcal{P}$ if the trajectory $\mathcal{R}(x)$ has this property. For example the class of a point x is the subset $\mathrm{Cl}(x) = \{y : \overline{\{y\}} = \overline{\{x\}}\}$.

Let X be a topological space. Recall that a closed subset C of X is *irreducible* if it is not the union of two proper closed subsets (or equivalently if the intersection of two nonempty open subsets is nonempty). An element x of C is called *a generic point* if $\overline{\{x\}} = C$. The space X is said to be *quasi-compact* if it has the property of Borel–Lebesgue (every open cover admits a finite subcover) but it is not necessarily a Hausdorff space.

If E is a locally compact second countable topological space, then the quasi-orbit space $X = E/\widetilde{G}$ has the following properties [HS]:

(a) Every irreducible closed subset of X has a generic point.
(b) Every totally ordered family $\{a_i : i \in I\}$ of X has a supremum a such that $\overline{\{a\}} = \overline{\{a_i : i \in I\}}$.
(c) Every orbit of G is contained in the closure of a maximal orbit.

3. Quasi-orbit spaces associated to some T_0-spaces. The goal of this section is to show that every second countable T_0-space X is homeomorphic to a quasi-orbit space $E/\widetilde{G}$ (where E is a second countable metric space) and to give some particular properties of the space E and of the group G under some additional hypotheses on X.

3.1. Product of Sierpiński spaces. The two-point set $S = \{0, 1\}$ equipped with the topology $\{\emptyset, S, \{1\}\}$ is called the *Sierpiński space*; it is a connected T_0-space but it is not a T_1-space. The order associated to this topology satisfies $0 \le 1$; indeed, we have $\overline{\{0\}} = \{0\}$ and $\overline{\{1\}} = S$.

Remark 3.1.

- For each integer $n \in \mathbb{N}$, we consider the set $\Gamma_n = \{0, 1\}^n$. We define an order on Γ_n by:

$$(\varepsilon_1, \ldots, \varepsilon_n) \le (\varepsilon'_1, \ldots, \varepsilon'_n) \quad \text{if } \varepsilon_i \le \varepsilon'_i \text{ for } 1 \le i \le n.$$

We endow Γ_n with the order-topology generated by the family $\{[\gamma, \nearrow[: \gamma \in \Gamma_n\}$ where $[\gamma, \nearrow[= \{\gamma' \in \Gamma_n : \gamma \le \gamma'\}$. The space Γ_n equipped with this order-topology is homeomorphic to the product of n copies of the Sierpiński space.

- For each $I \subset \{1, \ldots, n\}$, let $\gamma_I = (\varepsilon_1, \ldots, \varepsilon_n) \in \Gamma_n$, where $\varepsilon_i = 1$ if $i \in I$ and $\varepsilon_i = 0$ otherwise. So if I and J are two subsets of $\{1, \ldots, n\}$, then $\gamma_I \leq \gamma_J$ whenever $I \subset J$.

  This shows that Γ_n is homeomorphic to the set of subsets of $\{1, \ldots, n\}$ for the topology associated to the inclusion order.

We denote by $\mathrm{Diff}^r_+(M)$ the group of orientation-preserving C^r-diffeomorphisms of an oriented manifold M ($r \geq 0$). A C^0-diffeomorphism is a homeomorphism. In general, we take M to be the unit circle $\mathbb{S}^1$, the n-torus $\mathbb{T}^n$ or the n-Euclidean space $\mathbb{R}^n$.

LEMMA 3.2. *For each positive integer n, there exists a finitely generated abelian subgroup G_n of $\mathrm{Diff}^\infty_+(\mathbb{T}^n)$ such that its quasi-orbit space $\mathbb{T}^n/\widetilde{G}_n$ is homeomorphic to the product $S \times \cdots \times S$ of n copies of the Sierpiński space S (that is, to Γ_n).*

Proof. Let G_1 be a finitely generated abelian subgroup of $\mathrm{Diff}^\infty_+(\mathbb{S}^1)$ of finite rank $k \geq 2$ having only one fixed point $e \in \mathbb{S}^1$. Then all other orbits are everywhere dense (N. Kopell [Kop], G. Reeb [Reb]). Thus the quasi-orbit space $\mathbb{S}^1/\widetilde{G}_1$ is homeomorphic to the Sierpiński space S.

The product group G_n of n copies of G_1 is a finitely generated abelian subgroup of $\mathrm{Diff}^\infty_+(\mathbb{T}^n)$ of finite rank.

Since each $\widetilde{G}_i$ ($1 \leq i \leq n$) is an open equivalence relation, by applying [Bou, Chapitre I, p. 34, Corollaire], the quasi-orbit space $\mathbb{T}^n/\widetilde{G}_n$ of G_n is homeomorphic to the product $\prod_{i=1}^n \mathbb{S}^1/\widetilde{G}_1$, and hence to $S \times \cdots \times S$. ∎

3.2. Proof of Theorem 1.1. (a) Since X is a T_0-space, applying [Eng, Theorem 2.3.26, p. 84], there exists an embedding $\psi : X \to \prod_{i\in I} S_i$ (where S_i is the Sierpiński space $\{0, 1\}$). We can suppose that $I \subset \mathbb{N}$, as X is second countable. We know that for each $i \in I$ there is a homeomorphism $f_i : S_i \to \mathbb{S}^1_i/\widetilde{G}_i$, where $\mathbb{S}^1_i$ is the unit circle $\mathbb{S}^1$ and G_i is the group G_1 defined in the proof of Lemma 3.2. The product map $\prod_{i\in I} f_i : \prod_{i\in I} S_i \to \prod_{i\in I} \mathbb{S}^1_i/\widetilde{G}_i$ is also a homeomorphism. Moreover, $\prod_{i\in I} \mathbb{S}^1_i/\widetilde{G}_i$ is homeomorphic to $\prod_{i\in I} \mathbb{S}^1_i/\prod_{i\in I} \widetilde{G}_i$. The space $\mathbb{T}^I = \prod_{i\in I} \mathbb{S}^1_i$ is a compact second countable metric space. We put $G^I = \prod_{i\in I} G_i$. The group G^I is abelian. Thus we conclude that there exists an embedding $\varphi : X \to \mathbb{T}^I/\widetilde{G^I}$. Let $p : \mathbb{T}^I \to \mathbb{T}^I/\widetilde{G^I}$ be the canonical projection. We set $E = p^{-1}(\varphi(X))$ and denote by $G = G^I/E$ the subgroup induced by G^I on E. Since E is a saturated subset of $\mathbb{T}^I$, we have $G(x) = G^I(x)$ for each $x \in E$.

We will show that $E/\widetilde{G}$ is homeomorphic to $\varphi(X)$ and so to X. Let $f : E/\widetilde{G} \to \varphi(X) \subset \mathbb{T}^I/\widetilde{G^I}$ map the class of an orbit $G(x)$ to the class of the orbit $G^I(x)$. We prove now that this bijective map f is a homeomorphism.

Let V be an open subset of $\varphi(X)$, that is, $V = U \cap \varphi(X)$ where U is an open subset of $\mathbb{T}^I/\widetilde{G^I}$. Then $p^{-1}(V) = p^{-1}(U) \cap p^{-1}(\varphi(X)) = p^{-1}(U) \cap E$.

Since $p^{-1}(U)$ is an open subset of $\mathbb{T}^I$, $p^{-1}(V)$ is open in E. Thus V is open in $E/\widetilde{G}$ and so f is a continuous map.

Let $p_1 : E \to E/\widetilde{G}$ be the canonical projection and let V be an open subset of $E/\widetilde{G}$, that is, $p_1^{-1}(V)$ is open in E and so there exists an open subset U of $\mathbb{T}^I$ such that $p_1^{-1}(V) = U \cap E$. We have $V = p(p_1^{-1}(V)) = p(U \cap E)$. Since E is saturated, we deduce that $V = p(U) \cap p(E) = p(U) \cap \varphi(X)$. The fact that p is an open map implies that V is an open subset of $\varphi(X)$. Therefore f is an open map. We conclude that f is a homeomorphism.

It remains to show that E is connected. Let $x = (x_i, i \in I)$ be an element of $E \subset \mathbb{T}^I$. Then $\mathrm{Cl}(G(x)) = \prod_{i\in I} \mathrm{Cl}(G_i(x_i))$, where

$$\mathrm{Cl}(G_i(x_i)) = \mathrm{Cl}(G_1(x_i)) = \begin{cases} \{e\} & \text{if } x_i = e, \\ \mathbb{S}^1 - \{e\} & \text{if } x_i \in \mathbb{S}^1 - \{e\}. \end{cases}$$

Thus each $\mathrm{Cl}(G_1(x_i))$ is connected, and so $\mathrm{Cl}(G(x))$ is connected. On the other hand, $E/\widetilde{G}$ is homeomorphic to X and so it is connected, which implies that E is connected.

(b) Since X is finite, it can be embedded into a finite product of n copies of the Sierpiński space; see [Eng, Theorem 2.3.26, p. 84]. Hence by Lemma 3.2 there is an embedding $\varphi : X \to \mathbb{T}^n/\widetilde{G}_n$, where G_n is a finitely generated abelian subgroup of finite rank. We set $E = p^{-1}(\varphi(X))$ ($p : \mathbb{T}^n \to \mathbb{T}^n/\widetilde{G}_n$ is the canonical projection) and denote by G the subgroup induced by G_n on E. In this case E is a connected second countable metric subspace of $\mathbb{T}^n$ and G is a finitely generated abelian subgroup of finite rank. ∎

Remark 3.3. We have

$$E = \bigcup_{i=1}^{r} p^{-1}(\{\varphi(x_i)\}) = \bigcup_{i=1}^{r} \mathrm{Cl}(G(a_i)) \tag{$*$}$$

where $a_i = (a_i^1, \ldots, a_i^n) \in \mathbb{T}^n$ with $p(a_i) = \varphi(x_i)$.

For each $1 \le i \le r$ and $1 \le j \le n$, by the proof of (a) we have

$$\mathrm{Cl}(G_1(a_i^j)) = \begin{cases} \{e\} & \text{if } a_i^j = e, \\ \mathbb{S}^1 - \{e\} & \text{if } a_i^j \neq e. \end{cases}$$

Then

$$\mathrm{Cl}(G(a_i)) = \prod_{j=1}^{n} \mathrm{Cl}(G_1(a_i^j)).$$

This implies that $\mathrm{Cl}(G(a_i))$ is homeomorphic to some $\mathbb{R}^{p_i}$ where $0 \le p_i \le n$. So by $(*)$, E is a union of some Euclidean spaces.

4. Countable subgroups of Homeo$_+(\mathbb{R})$ and finite T_0-spaces. The goal of this section is to prove Theorems 1.8–1.10.

4.1. Elementary intervals, minimal sets. Let G be a countable subgroup of Homeo$_+(\mathbb{R})$.

For every $g \in G$ we denote by $\mathrm{Fix}(g)$ the set of fixed points of g and by G_{Fix} the subset $\{g \in G : \mathrm{Fix}(g) \neq \emptyset\}$. If A is a subset of G, we denote by $\mathrm{Fix}\,A$ the subset $\{x \in \mathbb{R} : g(x) = x, \forall g \in A\}$. Notice that $\mathrm{Fix}\,A = \bigcap_{f\in A} \mathrm{Fix}(f)$, hence it is a closed subset of $\mathbb{R}$. Notice that $\mathrm{Fix}\,G_{\mathrm{Fix}}$ is a G-invariant closed subset.

As G is countable, every orbit O is a discrete subset of $\mathbb{R}$. The *limit set* of an orbit O is $\lim(O) = \overline{\overline{O} - O}$; thus $\lim(O) = \overline{O} - O$ if O is proper and $\lim(O) = \overline{O}$ if it is not proper. An orbit O is called *exceptional* if it is nowhere dense and is not proper.

An interval $]a, b[$ is called an *elementary interval* of G if $G(a) \cap\,]a, b[\, = G(b) \cap\,]a, b[\, = \emptyset$.

Remark 4.1. An interval is elementary if and only if it is a bounded connected component of the complement of an invariant closed set.

Notice that if $]a, b[$ is an elementary interval, then $g(]a, b[) \cap\,]a, b[\, \neq \emptyset \Rightarrow g(]a, b[) = \,]a, b[$. One deduces immediately that every elementary interval $]a, b[$ has the following obvious properties:

Lemma 4.2.

(1) *The points a and b have the same stabilizer: $G_a = G_b$; we will call it the stabilizer of the interval and denote it by $G_{]a,b[}$.*
(2) *The orbits $G(a)$ and $G(b)$ have the same limit set:* $\lim(G(a)) = \lim(G(b))$.
(3) *For all $x \in\,]a, b[$ we have:*
   (i) *$G_x \subset G_a$ and* $\lim(G(a)) \subset \lim(G(x))$.
   (ii) *$G_a(x) = G(x) \cap\,]a, b[$.*

We say that two elementary intervals J and K are *equivalent* if $\mathrm{Sat}(J) = \mathrm{Sat}(K)$. Notice that J and K are equivalent if and only if there is $g \in G$ such that $g(J) = K$.

We say that an elementary interval $]a, b[$ is *wandering* if each orbit of G meets it in at most one point, equivalently $G_a(x) = \{x\}$ for all $x \in\,]a, b[$.

The importance of elementary intervals for our study is shown by the following straightforward lemma:

Lemma 4.3. *Let I be an elementary interval of a countable subgroup $G \subset$ Homeo$_+(\mathbb{R})$, and G_I be its stabilizer. Then $p(I) \subset \mathbb{R}/\widetilde{G}$ is an open subset homeomorphic to $I/\widetilde{G}_I$ via the canonical projection: to the class* $\mathrm{Cl}(G(x))$ *we associate the class* $\mathrm{Cl}(G_I(x))$.

Proof. Let $x, y \in I$. By Lemma 4.2 we have $G_I(x) = G(x) \cap I$ and $G_I(y) = G(y) \cap I$. The canonical projection is well defined: if $G(x)$ and $G(y)$ have the same closure then $G_I(x)$ and $G_I(y)$ have the same closure. This map is clearly surjective, let us show that it is injective: if $\overline{G_I(x)}$ and $\overline{G_I(y)}$ have the same closure in I then $y \in \overline{G_I(x)} \subset \overline{G(x)}$ and $x \in \overline{G_I(y)} \subset \overline{G(y)}$ so that $G(x)$ and $G(y)$ have the same closure.

To prove that this identification induces a homeomorphism, it is enough to see that the open subsets of $p(I)$ correspond to the G-saturated open subsets of $\mathrm{Sat}_G(I)$, which are in bijection, via intersection with I, to the G_I-saturated open subsets of I. ∎

Lemma 4.4 ([Sal2]).

(1) *If G has an exceptional minimal set m, then $m \subset \overline{G(x)}$ for all $x \in \mathbb{R} - m$. So m is the only minimal set of G.*
(2) *If G has an infinite closed orbit, then the closure of each orbit contains a closed orbit and the group G contains an element g_0 without fixed point.*
(3) *If G has two distinct minimal sets which are not fixed points, then the closure of each nonclosed orbit contains exactly two closed orbits.*

4.2. Description of the quasi-orbit space when the singular part is finite. Let $X = \mathbb{R}/\widetilde{G}$ be the quasi-orbit space of a countable group $G \subset \mathrm{Homeo}_+(\mathbb{R})$ and let $p\colon \mathbb{R} \to X$ be the canonical projection.

In the following we will describe X when the irregular part $X - X_0$ is finite.

4.2.1. *Suppose* $\mathrm{ht}(G) = 0$. This means that every orbit belongs to a minimal set. According to Lemma 4.4, this minimal set cannot be exceptional. So either every orbit is closed, or every orbit is everywhere dense. Thus, X reduces to a singleton or is homeomorphic to $\mathbb{R}$ (if $G = \{\mathrm{id}\}$) or to the circle $\mathbb{S}^1$. In particular, if $G = \{\mathrm{id}\}$ then X is of type $\mathcal{T}_0$, and if $G \neq \{\mathrm{id}\}$ then $X = X^0$ is of type $\mathcal{T}_1$ or $\mathcal{T}_2$, so it is always admissible.

We assume now that $\mathrm{ht}(G) \geq 1$.

4.2.2. *Suppose G has a fixed point $G(x_0) = \{x_0\}$ and* $\mathrm{ht}(G) \geq 1$. Denote $A = \mathrm{Fix}(G)$. Notice that every point in the boundary $A - \mathring{A}$ of A corresponds (bijectively) to a point in $X^0 - X_0$. As we assume $X - X_0$ to be finite, the set $A - \mathring{A}$ is finite. Let $a_1 < \cdots < a_k$ be the points in $A - \mathring{A}$. Let $J_0 =]-\infty, a_1[, \ldots, J_i =]a_i, a_{i+1}[, \ldots, J_k =]a_k, +\infty[$ be the connected components of $\mathbb{R} - (A - \mathring{A})$. Notice that each J_i is an elementary interval. We denote by H_i the group of homeomorphisms of J_i obtained by restriction of the elements of G; in particular $H_i = G_{a_i}/J_i$.

Notice that:

- if $J_i \subset \mathring{A}$, then $p(J_i)$ is an interval contained in X_0, and its points are maximal and minimal in X;
- if $J_i \not\subset \mathring{A}$, then H_i has no fixed point. Furthermore, the closure of $x \in p(J_i)$ contains x and the projections of the extremities of J_i. So $p(J_i) \subset X - X^0$. Moreover $p(J_i)$ is open (as J_i is open and the projection p is open) and connected. So $p(J_i)$ is a connected component of $X - X^0$.

The properties above show that:

(1) if G has a unique fixed point, then X is of type $\mathcal{T}_2$;
(2) if $\mathrm{Fix}(G)$ is an interval $]-\infty, a]$ or $[a, \infty[$, then X is of type $\mathcal{T}_{4''}$;
(3) if the boundary of $\mathrm{Fix}(G)$ contains at least two points, then X is of type $\mathcal{T}_{4'}$.

Finally, notice that every connected component of $X - X^0$ is one of the $p(J_i)$, where H_i has no fixed point. According to Lemma 4.3, $p(J_i)$ is homeomorphic to $J_i/\widetilde{H}_i$.

4.2.3. *Suppose G has no fixed point but has a closed orbit $O_0 = G(x_0)$ and* $\mathrm{ht}(G) \geq 1$. From Lemma 4.4, there exists $g_0 \in G$ without fixed point such that $]x_0, g_0(x_0)[\cap O_0 = \emptyset$ and $\mathbb{R} = \mathrm{Sat}([x_0, g_0(x_0)[)$. So every closed orbit meets the interval $[x_0, g_0(x_0)[$ in one point.

We set $A = [x_0, g_0(x_0)] \cap m_0$, where m_0 is the union of all closed orbits. Since $X - X_0$ is finite, so is $A - \mathring{A}$. We write $A - \mathring{A} = \{x_i : 0 \leq i \leq n\}$ with

$$x_0 < x_1 < \cdots < x_n = g_0(x_0).$$

Notice that every point in m_0 is fixed by every $g \in G_{\mathrm{Fix}}$; in particular $G_{\mathrm{Fix}} = G_{x_i}$. We denote by $H_i = G_{\mathrm{Fix}}/]x_i, x_{i+1}[$ the group of homeomorphisms induced by G_{Fix} on $]x_i, x_{i+1}[$ for $0 \leq i \leq n-1$.

The subset $Z_i = p(]x_i, x_{i+1}[)$ of X is the quasi-orbit space of the group H_i. By Lemma 4.3, Z_i is homeomorphic to $]x_i, x_{i+1}[/\widetilde{H}_i$. We have

$$X - X_0 = \{p(x_0), \ldots, p(x_{n-1})\} \cup \bigcup_{i=0}^{n} (Z_i - X_0).$$

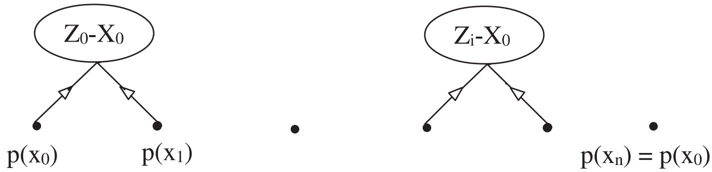

Fig. 1

For $0 \leq i \leq n-1$, either

- $H_i = \{\mathrm{Id}_{]x_i, x_{i+1}[}\}$, that is, $]x_i, x_{i+1}[\subset \mathring{A}$ and Z_i is homeomorphic to an open interval in $X^0 \cap X_0$ with extremities $p(x_i)$ and $p(x_{i+1})$; or

- H_i is a group without fixed point in $]x_i, x_{i+1}[$ and for every $y \in]x_i, x_{i+1}[$ one has $\{x_i, x_{i+1}\} \subset \overline{G(y)}$. More precisely, $\overline{\{p(y)\}} \cap X^0 = \{p(x_i), p(x_{i+1})\}$. Finally, Z_i is a connected component of $X - X^0$.

The properties above show that:

- if G has a unique closed orbit, then X is of type $\mathcal{T}_2$;
- if $A - \mathring{A}$ is a pair $\{x_0, x_1\}$ (which means that $X^0 - X_0$ is a pair), then X is of type $\mathcal{T}_{3''}$;
- if $\mathrm{card}(X^0 - X_0) \geq 3$, then X is of type $\mathcal{T}_{3'}$.

Furthermore, every connected component of $X - X^0$ is one of the $p(]x_i, x_{i+1}[)$, which is homeomorphic to $]x_i, x_{i+1}[/\widetilde{H}_i$, where H_i has no fixed point.

4.2.4. *Suppose G has an exceptional minimal set m.* Every orbit $O = G(x) \subset \mathbb{R} - m$ satisfies $m \subset \overline{O}$ and $m \neq \overline{O}$. Let $]a, b[$ be the connected component of $\mathbb{R} - m$ containing x. It is an elementary interval. Consider the group $H = G_a/]a, b[$. It is a countable subgroup of $\mathrm{Homeo}_+(]a, b[)$. From Lemma 4.3 we know that $p(]a, b[)$ is homeomorphic to $]a, b[/\widetilde{H}$.

It is easy to see that $\mathbb{R} - m$ is a union of countably many elementary intervals $]a_n, b_n[$, $n \in \mathbb{N}$. Thus, $X = \{p(m)\} \cup \bigcup_{n\in\mathbb{N}} p(]a_n, b_n[)$. Two cases are possible:

- $H_n = G_{a_n}/]a_n, b_n[= \{\mathrm{Id}_{]a_n,b_n[}\}$ and $p\colon]a_n, b_n[\to p(]a_n, b_n[) \subset X_0$ is a homeomorphism, in which case $p(]a_n, b_n[)$ is a connected component of $X - X^0$ contained in $X^1 \cap X_0$;
- $H_n = G_{a_n}/]a_n, b_n[$ is a group in $]a_n, b_n[$ and $p(]a_n, b_n[)$ is a connected component of $X - X^0$.

Since $X - X_0$ is finite, there are only finitely many orbits of elementary intervals $I_i =]a_i, b_i[$ such that $p(I_i)$ is not contained in X_0.

Notice that in this case, $X^0 = \{p(m)\}$ so that X is of type $\mathcal{T}_2$.

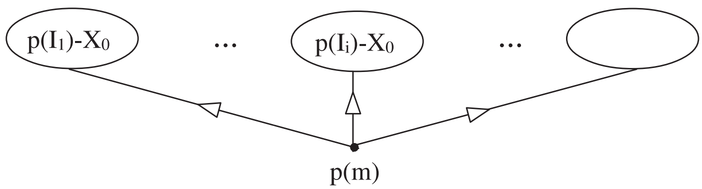

Fig. 2

4.3. Proof of Theorem 1.8. Let $G \subset \mathrm{Homeo}_+(\mathbb{R})$ be a countable subgroup, and X its quasi-orbit space. We assume that $X - X_0$ is finite. Then X is a connected second countable T_0-space. Let us show that X is admissible.

We have seen that X has finite height. So G has at least one minimal set. Then Section 4.2 shows that X is of type $\mathcal{T}_i$, $i \in \{0,1,2,3,4\}$, according to what are the minimal sets (fixed points, the line $\mathbb{R}$, proper obits or an exceptional minimal set).

Furthermore, Section 4.2 shows that each connected component of the set $X - X^0$ is homeomorphic to $I/\widetilde{G}_I$ where I is an elementary interval and G_I its stabilizer. Finally, the unique case allowing G_I to have a fixed point is G has an exceptional minimal set which implies that X is of type $\mathcal{T}_2$.

So a straightforward induction leads to the following statement:

Let Y be a connected component of $X - X^k$, $k \geq 0$, and Z be the connected component of $X - X^{k-1}$ containing Y. Then:

- *there is an elementary interval J such that Z is the quasi-orbit space $J/\widetilde{G}_J$;*
- *there is an elementary interval $I \subset J$ such that Y is the quasi-orbit space $I/\widetilde{(G_J)_I}$ where $(G_J)_I$ is the stabilizer of I for the group G_J;*
- *Y is a connected component of $Z - Z^0$.*

As a consequence, Y is of type $\mathcal{T}_i$ for some $i \in \{0,\dots,4\}$. Furthermore, $i \in \{0,4\}$ if and only if $(G_J)_I$ has a fixed point in I, and this implies that G_J has an exceptional minimal set in J, so that Z is of type $\mathcal{T}_2$.

This proves that X is admissible.

Conversely, assume that X is a connected second countable T_0-space whose singular part $X - X_0$ is finite. Assume furthermore that X is admissible. The aim of this section is to build a finitely generated abelian subgroup $G \subset \mathrm{Homeo}_+(\mathbb{R})$ such that X is homeomorphic to the quasi-orbit space of G.

We proceed inductively on $\mathrm{ht}(X)$.

First, notice that the admissible spaces X of height 0 are of type $\mathcal{T}_i$, $i \in \{0,1,2\}$, that is, either

- $X = \mathbb{R}$ corresponding to $G = \{\mathrm{id}\}$; or
- $X = \mathbb{S}^1$ corresponding to the cyclic group generated by a translation $x \mapsto x+1$; or
- X is a single point corresponding to G generated by two translations $x \mapsto x+1$ and $x \mapsto x+\alpha$ with $\alpha \notin \mathbb{Q}$.

All of them are quasi-orbit spaces.

We assume now, as our induction hypothesis, that every admissible space Y with $\mathrm{ht}(Y) \leq k$ is the quasi-orbit space of a finitely generated abelian group $\langle Y \rangle$. So Theorem 1.8 is a consequence of the following statement: *Every admissible space X with* $\mathrm{ht}(X) = k+1$ *is the quasi-orbit space of a finitely generated abelian group G.*

The proof of this statement is the aim of the rest of this section. Each connected component Y of $X - X^0$ is an admissible space with $\mathrm{ht}(Y) \leq k$ so it is the quasi-orbit space of a finitely generated group $\langle Y \rangle$. Moreover, X is admissible. Our construction will depend on the type of X.

The types $\mathcal{T}_0$ and $\mathcal{T}_1$ correspond to spaces of height 0, so do not correspond to X.

Case 1: *X is of type $\mathcal{T}_2$.* Notice that this type is the unique which allows $X - X^0$ to have infinitely (but countably) many connected components. However, only finitely many of these components may contain points of the singular part $X - X_0$. So all but finitely many components are of type $\mathcal{T}_0$ or $\mathcal{T}_1$. Let $\alpha \in \mathbb{N} \cup \{\infty\}$ be the number of connected components of $X - X^0$; as $\mathrm{ht}(X) \geq 1$ by hypothesis, one gets $\alpha \geq 1$. We choose an indexation, $Y_1, \ldots, Y_i, \ldots$, of these components.

Recall that there are homeomorphisms of the circle having an arbitrary number $\alpha \geq 1$ (finite or infinite) of wandering intervals. Hence there is an abelian group G_α generated by $f\colon x \mapsto x+1$ and g commuting with f, such that G_α has an exceptional minimal set m and $\mathbb{R} - m$ is the disjoint union of the orbits of α wandering intervals $I_1, \ldots, I_i, \ldots$.

Consider $T_1 = \{i \in \{1, \ldots, \alpha\} : Y_i \text{ is of type } \mathcal{T}_1\}$. We choose a homeomorphism $g_{T_1}\colon \mathbb{R} \to \mathbb{R}$ with the following properties:

- $g_{T_1}(x) = x$ if $x \notin \mathrm{Sat}_{G_\alpha}(\bigcup_{i \in T_1} I_i)$;
- the restriction of g_{T_1} to I_i is a homeomorphism $g_{T_1,i}\colon I_i \to I_i$ such that $g_{T_1,i}(x) > x$;
- any component of $\mathrm{Sat}_{G_\alpha}(I_i)$ is the image $h(I_i)$ for a unique $h \in G_\alpha$ (because I_i is wandering, g_{T_1} coincides with $h \circ g_{T_1,i} \circ h^{-1}$ on $h(I_i)$).

Notice that, by construction, g_{T_1} commutes with f and g.

Consider now the set $T_{\geq 2} = \{i \in \{1, \ldots, \alpha\} : Y_i \text{ is of type } \mathcal{T}_i, \text{ for } i \geq 2\}$. For every $i \in T_{\geq 2}$ we consider the corresponding group $H_i = \langle Y_i \rangle \subset \mathrm{Homeo}_+(\mathbb{R})$.

We fix a homeomorphism $\varphi_i\colon I_i \to \mathbb{R}$. For every $g \in \mathrm{Homeo}_+(\mathbb{R})$ we denote by g_{φ_i} the homeomorphism defined as follows:

- $g_{\varphi_i}(x) = x$ if $x \notin \mathrm{Sat}_{G_\alpha}(I_i)$;
- any component of $\mathrm{Sat}_{G_\alpha}(I_i)$ is the image $h(I_i)$ for a unique $h \in G_\alpha$. Then g_{φ_i} coincides with $h \circ \varphi_i^{-1} \circ g \circ \varphi_i \circ h^{-1}$ on $h(I_i)$.

We define $G_i = \{g_{\varphi_i} : g \in H_i\} \subset \mathrm{Homeo}_+(\mathbb{R})$. It is a finitely generated abelian subgroup. By construction, its elements commute with f and g. Furthermore, they are the identity map on $\mathrm{Sat}_{G_\alpha}(I_j)$ for $j \neq i$, so they commute with the elements of G_j for $j \in T_{\geq 2}$, $j \neq i$ and commute with g_{T_1}.

So $G_\alpha \cup \bigcup_{i \in T_{\geq 2}} G_i \cup \{g_{T_1}\}$ generates a finitely generated abelian subgroup G of $\mathrm{Homeo}_+(\mathbb{R})$.

We end this case by showing that X *is homeomorphic to the quasi-orbit space of* G.

Let X_G be the quasi-orbit space of G. Notice that the minimal set m of G_α is invariant under G, hence it is an exceptional minimal set. So X_G is of type $\mathcal{T}_2$, that is, $(X_G)^0 = p(m)$. Furthermore, for every orbit of an elementary interval I_i, the stabilizer of I_i is conjugate to the group $\langle Y_i \rangle$. So each connected component of $X_G - (X_G)^0 = X_G - p(m)$, corresponding to an elementary interval I_i, is homeomorphic to Y_i (a connected component of $X - X^0$). As X^0 is a point, we get a natural bijection from X_G to X. To prove that this bijection is a homeomorphism it is enough to see that the unique open set containing the point $(X_G)^0$ (resp. X^0) is the whole set X_G (resp. X).

Case 2: X *is of type* $\mathcal{T}_3$. The construction is completely analogous to the case of type $\mathcal{T}_2$ but simpler. So let us explain it more briefly.

Let f be the translation $x \mapsto x + 1$. As X is of type $\mathcal{T}_3$, $X^0 - X_0$ is a finite set cyclically ordered $x_0, \dots, x_i, \dots, x_{n-1}$, $i \in \mathbb{Z}/n\mathbb{Z}$, each point x_i being connected to the next either by a segment $I_i \subset X_0 \cap X^0$ or by a connected component $X_{\{x_i, x_{i+1}\}}$ of $X - X_0$. Notice that each space $X_{\{x_i, x_{i+1}\}}$ is admissible and of height less than k so that it is the quasi-orbit space of a finitely generated abelian group H_i. For simplicity, let $H_i = \{\mathrm{id}\}$ if x_i, x_{i+1} are the extremities of a segment I_i.

Choose $a_0 \in \mathbb{R}$ and a sequence $a_0 < a_1 < \cdots < a_{n-1} < a_n = f(a_0)$. The intervals $I_i =]a_i, a_{i+1}[$, $i \in \{0, \dots, n-1\}$, are elementary intervals whose saturations are pairwise disjoint. We fix a homeomorphism $\varphi_i \colon I_i \to \mathbb{R}$. For every $g \in \mathrm{Homeo}_+(\mathbb{R})$ we denote by g_{φ_i} the homeomorphism commuting with f, equal to the identity off $\mathrm{Sat}_f(I_i)$ and coinciding with $\varphi_i^{-1} \circ g \circ \varphi_i$ on I_i. We set $G_i = \{g_{\varphi_i} : g \in H_i\} \subset \mathrm{Homeo}_+(\mathbb{R})$. It is a finitely generated abelian group of homeomorphisms commuting with f and with the homeomorphisms of the other G_j. Let G be the finitely generated abelian group generated by the union $\{f\} \cup \bigcup G_i$.

To prove that the quasi-orbit space of G is homeomorphic to X, the point is to see that for $G_i \neq \{\mathrm{id}\}$ every point $x \in I_i$ satisfies $\overline{\{x\}} \cap X^0 = \{a_i, a_{i+1}\}$. Let us prove this claim. As X is admissible of type $\mathcal{T}_3$, every connected component $X_{\{x_i, x_{i+1}\}}$ of $X - X^0$ is of type $\mathcal{T}_1$ or $\mathcal{T}_2$. This implies that G_i has no fixed point, hence H_i has no fixed point in I_i. As a consequence, the closure of every orbit of $x \in I_i$ contains $\{a_i, a_{i+1}\}$. The orbit of x is contained in the f-orbit of I_i whose closure is disjoint from a_j, for $j \notin \{i, i+1\}$; so the claim is proved.

Case 3: X *is of type* $\mathcal{T}_4$. The construction is completely analogous to the previous case, and even simpler. As X is of type $\mathcal{T}_4$, $X^0 - X_0$ is an ordered finite set $\{x_1, \dots, x_i, \dots, x_n\}$. Each point x_i is connected to the next either by a segment $I_{i,i+1} \subset X_0 \cap X^0$ or by a connected component

$X_{\{x_i,x_{i+1}\}}$ of $X - X^0$. Furthermore, to x_1 (resp. x_n) is associated either a segment $I_1 \subset X_0 \cap X^0$ or a connected component $X_{\{x_i\}}$ of $X - X^0$. The spaces $X_{\{x_i,x_{i+1}\}}$, $X_{\{x_1\}}$ and $X_{\{x_n\}}$ are all admissible and of height less than k so that they are quasi-orbit spaces of finitely generated abelian groups $H_{i,i+1}$, H_1 and H_n respectively.

We choose n points $a_1 < \cdots < a_n$, and for every i we realize $H_{i,i+1}$ as a group of homeomorphisms of $\mathbb{R}$ which is the identity off $]a_i, a_{i+1}[$ and whose restriction to $]a_i, a_{i+1}[$ is obtained by conjugacy by a homeomorphism $\varphi_i\colon]a_i, a_{i+1}[\to \mathbb{R}$. In the same way we realize H_1 on $]-\infty, a_1[$ and H_n on $]a_n, \infty[$. All the groups G_i obtained are finitely generated abelian and are pairwise commuting. All these groups together generate a finitely generated abelian group whose quasi-orbit space is X. ∎

4.4. Proofs of Theorems 1.9 and 1.10; Examples. Let $G \subset \mathrm{Diff}^2_+(\mathbb{R})$ be an abelian subgroup and X be its quasi-orbit space.

The basic tools for studying the orbits of G are Denjoy's theory [Den] and Kopell's lemma [Kop]. Let us recall some basic properties that we will use here:

- (P_1) every abelian subgroup $G \subset \mathrm{Diff}^2_+(\mathbb{R})$ admits a minimal set (see [Sal1] and [God, Exercise 4.12]);
- (P_2) for every abelian subgroup $G \subset \mathrm{Diff}^2_+([0,1[)$ (see [Kop]), either
  - – G has a fixed point in $]0,1[$; or
  - – G is a cyclic group; or
  - – every orbit of $]0,1[$ is dense in $]0,1[$;
- (P_3) if $G \subset \mathrm{Diff}^2_+(\mathbb{R})$ is finitely generated then G has no exceptional minimal set;
- (P_4) in [Ima], Imanishi exhibits countable abelian subgroups $G \subset \mathrm{Diff}^2_+(\mathbb{R})$ having an exceptional minimal set.

Proof of Theorem 1.9. Since G is abelian, according to (P_1), it has a minimal set. So it suffices to sketch a proof which consists in considering all the possible minimal sets.

Case 1: *The interior of a minimal set m is nonempty.* In that case $m = \mathbb{R}$ and X is a single point.

Case 2: $\mathrm{Fix}(G) \neq \emptyset$. Let I be a connected component of $\mathbb{R} - \mathrm{Fix(G)}$. Let G_I be the group of diffeomorphisms of I induced by restricting the elements of G to I. Notice that G has no fixed point in I. Then, according to (P_2), either

- G_I is a cyclic group, and then all orbits of I are stable proper and $p(I)$ is a circle in $X^1 \cap X_0$; or
- every orbit of G_I is dense in I, and then $p(I)$ is a maximal point in X^1.

CASE 3: $\mathrm{Fix}(G) = \emptyset$ *and* G *admits a closed orbit* $G(x)$. In that case there is $g \in G$ such that $g(x) > x$ and $]x, g(x)[$ is an elementary interval. Let G_x be the stabilizer of x. Then $\mathrm{Fix}(G_x)$ corresponds to the union of all closed orbits of G. Let I be a connected component of $\mathbb{R} - \mathrm{Fix}(G_x)$ and let G_I denote the group of diffeomorphisms of I induced by G_x. Then G_I has no fixed point in I and exactly as in the previous case, either

- G_I is a cyclic group, and then all orbits of I are stable proper and $p(I)$ is a circle in $X^1 \cap X_0$; or
- every orbit of G_I is dense in I, and $p(I)$ is a maximal point in X^1.

CASE 4: *G admits an exceptional minimal set.* Denjoy's argument proves that, if G is finitely generated, then it has no exceptional minimal set. However, according to (P_4), there are examples of abelian groups of C^2-diffeomorphisms of $\mathbb{R}$ having an exceptional minimal set m. Each connected component of $\mathbb{R} - m$ is an elementary interval I, and we consider the group G_I induced by the stabilizer of I. Then either

- $G_I = \{\mathrm{id}_I\}$ and $p(I)$ is an interval in $X^1 \cap X_0$; or
- G_I is a cyclic group without fixed point, and $p(I)$ is a circle in $X^1 \cap X_0$; or
- G_I is an abelian group of rank larger than 2, and all orbits of G_I are dense in I so that $p(I)$ is a maximal point in X^1; or
- G_I admits a fixed point in I, and $p(I)$ is homeomorphic to $I/\widetilde{G}_I$ which has height 1, according to Case 2 above. ■

The paper [Sal1] builds a nonabelian countable subgroup G of $\mathrm{Diff}^\infty_+(\mathbb{R})$ without minimal set. Thus, in Theorem 1.9 we need the assumption that G is abelian.

If in Theorem 1.9 we assume that G is finitely generated, then we obtain the following result:

PROPOSITION 4.5. *Let X be an admissible T_0-space. Then X is the quasi-orbit space of a finitely generated abelian subgroup $G \subset \mathrm{Diff}^2_+(\mathbb{R})$ if and only if* $\mathrm{ht}(X) \le 1$ *and either*

- *X is not of type $\mathcal{T}_2$ (i.e. X^0 has more than 1 point); or*
- *X is of type $\mathcal{T}_2$ and $X^1 = X - X^0$ consists of at most two connected components, which are of type $\mathcal{T}_1$, $\mathcal{T}_2$, or $\mathcal{T}_3$.*

Proof. For the "only if" part, notice that a finitely generated abelian group of C^2-diffeomorphisms has no exceptional minimal set (see (P_3)). So if X is of type $\mathcal{T}_2$, it corresponds to a unique fixed point or a unique closed orbit. In the first case, X^1 has two connected components, and in the second case, it is connected. Furthermore, the group G_I corresponding to each connected component has no fixed point, so that the connected components of X^1 cannot be of type $\mathcal{T}_0$ or $\mathcal{T}_4$.

Conversely, if X is an admissible space satisfying the extra condition above, one easily realizes it as the quasi-orbit space of a finitely generated group of C^∞-diffeomorphisms of $\mathbb{R}$. This construction is analogous to the one for Theorem 1.8. We just need to choose, on each elementary interval, a diffeomorphism which extends smoothly on the extremities of the interval, the extremities being C^∞-tangent to the identity: this allows us to glue the dynamics on one elementary interval with the dynamics outside the interval. We leave the precise construction to the reader. ∎

Theorem 1.10 describes the quasi-orbit spaces of subgroups $G\subset \mathrm{Diff}^\omega_+(\mathbb{R})$ of analytic diffeomorphisms. Its proof is identical to the proof of Theorem 1.9: it consists in considering all the possible minimal sets for the group G, using the following properties (which are analogous to the case of an abelian group of C^2-diffeomorphisms):

- G has at least one minimal set (see [God, Exercice A.12]);
- (an unpublished result of G. Hector, see for instance [God, Theorem A.6]) if G is a group of analytic diffeomorphisms of $[0,\infty[$ without fixed point in $]0,\infty[$, then either
  - every orbit of $x \in\,]0,\infty[$ is dense in $]0,\infty[$, or
  - G is a cyclic group.

Remark 4.6. Notice that some admissible T_0-spaces X with $\mathrm{ht}(X) \le 2$ are not the quasi-orbit space of any subgroup $G \subset \mathrm{Diff}^\omega_+(\mathbb{R})$. For example, if X_0 contains an interval, this means that all the elements of the group coincide with the identity on that interval. As they are analytic, they are equal to the identity map and $X = X_0 = \mathbb{R}$. We do not know if there are other restrictions.

Examples. The following example shows that some dynamical properties of the group can be read off from the topology of the admissible space.

1) Let $Y = \{a, b_1, b_2, c_1, c_2\}$, equipped with the topology defined by $\overline{\{a\}} = \{a\}$, and for $i = 1, 2$, $\overline{\{b_i\}} = \{a, b_i\}$, $\overline{\{c_i\}} = \{a, b_i, c_i\}$ (Fig 3). Then Y is the singular part of an admissible space of type $\mathcal{T}_2$ or $\mathcal{T}_{4''}$. A group G such that $X - X_0 = Y$ has necessarily a fixed point or an exceptional minimal set. Let us show this claim. Assume towards a contradiction that the point a corresponds to a closed (nonfixed) orbit. As in Section 4.2.3 we can take $g \in G$ such that $I = \,]x_0, g(x_0)[$ is an elementary interval. We consider the group H induced by restricting the elements of G_{x_0} to I. The group H has no fixed point, but has two distinct minimal sets corresponding respectively to b_1 and b_2. So these minimal sets correspond to closed orbits.

So by Lemma 4.4, $\overline{\{c_1\}}$ contains two minimal elements of $Y - \{a\}$, which contradicts the hypotheses.

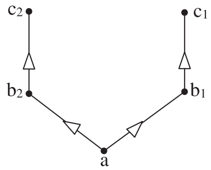

Fig. 3

We now present some examples of spaces Y which cannot be the singular part of an admissible space.

2) The fact that a nonminimal orbit contains in its closure at most two minimal sets implies that the set $Y = \{a_1, a_2, a_3, b\}$ equipped with the topology defined by $\overline{\{a_i\}} = \{a_i\}$ and $\overline{\{b\}} = Y$ cannot be the singular part of an admissible space.

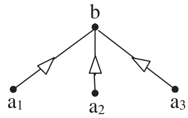

Fig. 4

3) The fact that a nonminimal orbit of a group having closed orbits contains in its closure exactly two closed orbits implies that the set $Y = \{a_1, a_2, b_1, b_2\}$ equipped with the topology defined by $\overline{\{a_i\}} = \{a_i\}$ and $\overline{\{b_1\}} = \{a_1, a_2, b_1\}$ and $\overline{\{b_2\}} = \{a_2, b_2\}$ cannot be the singular part of the quasi-orbit space of a group without fixed point.

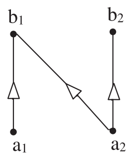

Fig. 5

4) Let $Y = \{a_1, a_2, a_3, a_4, b_1, b_2, b_3\}$, equipped with the topology defined by $\overline{\{a_i\}} = \{a_i\}$, $\overline{\{b_1\}} = \{a_1, a_2, b_1\}$, $\overline{\{b_2\}} = \{a_2, a_3, b_2\}$ and $\overline{\{b_3\}} = \{a_2, a_4, b_3\}$. Since $Y_{\{a_2, a_4\}} \neq \emptyset$ $(= \{b_3\})$, Y is not the singular part of an admissible space.

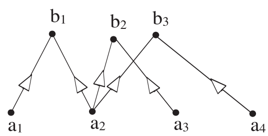

Fig. 6

5. Codimension one foliations and finite T_0-spaces. We recall that a codimension one foliation on an m-manifold M is in particular an open equivalence relation on M such that any equivalence class, called a leaf, is a weakly embedded submanifold of dimension $m-1$. Then we can define a proper leaf, a locally dense leaf, a class of leaves, the leaf space $M/\mathcal{F}$, and the quasi-leaf space $M/\widetilde{\mathcal{F}}$.

We can exhibit an infinite countable well ordered T_0-space with only one maximal element such that every irreducible closed subset has a generic point but it cannot be the quasi-leaf space of a transversally codimension-one foliation on a closed manifold [BES, Example 3.10]. Yet for finite T_0-spaces, we obtain Theorem 1.11.

In this section we will prove Theorem 1.12. We first consider the case of a space Y with height 0, that is, a discrete space. We will use two lemmas:

LEMMA 5.1. *There is a compact connected oriented three-manifold N such that, for any $k \in \{1, 2, \ldots, \infty\}$, there is a codimension one oriented C^1-foliation $\mathcal{F}_k$ on N with the following properties:*

- *$\mathcal{F}_k$ has a unique minimal set m_k;*
- *any leaf $L \subset N - m_k$ is proper and $\overline{L} - L = m_k$. In particular* $\mathrm{ht}(\mathcal{F}_k) = 1$*;*
- *if $X(\mathcal{F}_k)$ is the quasi-leaf space of $\mathcal{F}_k$ then the regular part X_0 of X consists of k circles;*
- *for any connected component c of X_0 there is a closed transversal curve γ_c embedded in N such that γ_c intersects each leaf $L \subset p^{-1}(c)$ in exactly one point, and is disjoint from the minimal set m_k and from the leaves corresponding to the other components $c' \neq c$ of X_0;*
- *there is a closed transversal γ_{m_k} which intersects every leaf of $\mathcal{F}_k$.*

Proof. The manifold N is the product $S \times \mathbb{S}^1$ where S is a closed connected oriented surface of genus 2. Recall that there is an onto (surjective) homomorphism ρ from the fundamental group $\pi_1(S)$ onto the free group $\mathbf{F}^2$. The foliation $\mathcal{F}_k$ is obtained as a suspension of a morphism of $\pi_1(S)$ in the set of diffeomorphisms of the circle $\mathbb{S}^1$ (which factorizes through ρ), whose image is the free group generated by:

- a diffeomorphism $f\colon \mathbb{S}^1 \to \mathbb{S}^1$ having a Denjoy minimal set m_f and precisely k orbits of wandering intervals (connected components of $\mathbb{S}^1 - m_f$); consider k wandering intervals $I_1, \ldots, I_k$ which are connected components of $\mathbb{S}^1 - m_f$, not pairwise equivalent; hence every orbit of $\mathbb{S}^1 - m_f$ meets $\bigcup_{i=1}^k I_i$ in precisely one point.
- g is a diffeomorphism of $\mathbb{S}^1$ which is the identity map off $\bigcup_{i=1}^k I_i$ and satisfies $g(x) > x$ for $x \in \bigcup_{i=1}^k I_i$.

One easily verifies that the foliation $\mathcal{F}_k$ has all the announced properties. ∎

Lemma 5.2. *For any integers $l, k > 0$ there is an oriented codimension one C^1-foliation $\mathcal{F}_{l,k}$ on a closed oriented connected three-manifold $M_{l,k}$ with the following properties:*

(1) *$\mathcal{F}_{l,k}$ has exactly l minimal sets, $m_1, \ldots, m_l$;*
(2) *every leaf L contained in $M_{l,k}$ is a proper leaf and $\overline{L} - L$ is contained in $\bigcup_{j=1}^{l} m_j$, so the foliation $\mathcal{F}_{l,k}$ has height 1;*
(3) *the regular part $X_0(\mathcal{F}_{l,k})$ of the quasi-leaf space of $\mathcal{F}_{l,k}$ consists of precisely $kl + 1$ circles;*
(4) *for any connected component c of $X_0(\mathcal{F}_{l,k})$, there is a closed transversal curve γ_c disjoint from the leaves not in c, and intersecting each leaf in c in exactly one point;*
(5) *for every j, there are k components of $X_0(\mathcal{F}_{l,k})$ corresponding to leaves L such that $\overline{L} - L = m_j$;*
(6) *there is one component of $X_0(\mathcal{F}_{l,k})$ corresponding to leaves L such that $\overline{L} - L = \bigcup_{j=1}^{l} m_j$;*
(7) *for every j there is a closed transversal γ_j intersecting m_j and disjoint from any $m_{j'}$ for $j' \neq j$; furthermore, γ_j is disjoint from every leaf L whose closure is disjoint from m_j.*

Proof. One builds $\mathcal{F}_{l,k}$ as follows: one considers l disjoint copies $N_1, \ldots, N_l$ of the manifold N endowed with the foliation $\mathcal{F}_{k+1}$ given by Lemma 5.1. In N_1 we remove $l-1$ disjoint solid tori which are small tubular neighborhoods of closed transversal curves parallel to the curve $\gamma_{c_{k+1}}$. In each N_j, $j \in \{2, \ldots, l\}$, we remove a solid torus parallel to the curve $\gamma_{c_{k+1}}$. Hence we get manifolds with boundary $\tilde{N}_1$ (with $l-1$ boundary components diffeomorphic to a torus $\mathbb{T}^2$, transverse to the foliation) and $\tilde{N}_2, \ldots, \tilde{N}_l$ (with one boundary component diffeomorphic to $\mathbb{T}^2$, transverse to the foliation).

Now we glue each torus $\partial\tilde{N}_j$, $j \geq 2$, to a component of ∂N_1, in order to glue the induced foliation. One gets a closed connected orientable three-manifold $M_{l,k}$ endowed with an orientable codimension one foliation $\mathcal{F}_{l,k}$, which has all the announced properties (the transversal γ_j corresponds to the transversal curves in N_j meeting every leaf of the foliation $\mathcal{F}_{k+1}$; this closed curve induces on $M_{l,k}$ a closed transversal intersecting the minimal set m_j and disjoint from any leaf L whose closure is disjoint from m_j). ∎

We are now ready to prove Theorem 1.12.

Proof. The step $\mathrm{ht}(Y) = h = 0$ of this induction argument is given by Lemma 5.2.

We assume (induction hypothesis) that Theorem 1.12 is proved for finite set Y' with $\mathrm{ht}(Y') \leq h - 1$ and we consider a finite T_0-space Y with $\mathrm{ht}(Y) = h$. We denote Y^0 the set of height 0 points in Y, and $Y' = Y - Y^0$. Notice that Y' is a finite T_0-space with $\mathrm{ht}(Y') \leq h - 1$ so that one may apply the inductive hypothesis. We denote by $M', \mathcal{F}'$ the closed connected oriented three-manifold endowed with an oriented codimension one foliation, associated to Y' by the inductive hypothesis.

Let l denote the cardinality of $Y^0 = \{x_1, \ldots, x_l\}$ and k the cardinality of $Y' = \{y_1, \ldots, y_k\}$. We consider the foliation $\mathcal{F}_{l,k}$ on the manifold $M_{l,k}$ given by Lemma 5.2. We need to consider some closed transversal curves of the foliations $\mathcal{F}'$ and $\mathcal{F}_{l,k}$:

- for every $i \in \{1, \ldots, k\}$ we denote by σ_i a closed transversal of the foliation $\mathcal{F}'$ such that a class of leaves $y \in Y'$ meets σ_i if and only if $y_i \subset \overline{y}$; for $j \in \{1, \ldots, l\}$ we denote by $\sigma_{j,i}$ a family of disjoint l circles parallel to σ_i; hence each $\sigma_{j,i}$ is a closed transversal of $\mathcal{F}'$ such that a class of leaves $y \in Y'$ meets $\sigma_{j,i}$ if and only if $y_i \subset \overline{y}$;
- for every $j \in \{1, \ldots, l\}$, we denote by m_j the minimal sets of $\mathcal{F}_{l,k}$, and by $c_{j,i}$, $i \in \{1, \ldots, k\}$, the k connected components of $M_{l,k} - \bigcup_{j=1}^{l} m_j$ whose leaves L (of the foliation $\mathcal{F}_{l,k}$) satisfy $\overline{L} - L = m_j$. We denote by $\gamma_{j,k}$ the closed transversal curves, given by Lemma 5.2, contained in the component $c_{j,k}$ and meeting each leaf $L \subset c_{j,k}$ in exactly one point.

One gets the announced foliation $\mathcal{F}$ by gluing $\mathcal{F}'$ to $\mathcal{F}_{l,k}$ according to the following rules: for any (j,i) such that the point $y_i \in Y' \subset Y$ contains $x_j \in Y^0 \subset Y$ in its closure ($x_j \in \overline{y_i}$) one removes two solid tori which are small tubular neighborhoods of $\sigma_{j,i}$ and of $\gamma_{j,i}$; hence we create two boundary components diffeomorphic to $\mathbb{T}^2$ and we glue one component to the other in order to glue $\mathcal{F}'$ to $\mathcal{F}_{l,k}$. ∎

6. Open problems

6.1. Quasi-orbit spaces associated to some T_0-spaces. Let Y be a T_0-space. Theorem 1.1 allows us to state the following problem:

PROBLEM 6.1. *Is it possible to choose E to be compact or at least locally compact such that Y is homeomorphic to a quasi-orbit space $E/\widetilde{G}$?*

EXAMPLE 6.2. Let $Y = \mathbb{N} \cup \{\omega\}$ equipped with the right topology (a basis for this topology is the family $\{\{x, x+1, \ldots\} : x \in \mathbb{N}\} \cup \{\omega\}$ [Bou, Chapitre I p. 89, Exercice 2]). From Theorem 1.1(a) there are a connected second countable metric space E and an abelian subgroup G of Homeo(E) such that Y is homeomorphic to the quasi-orbit space $E/\widetilde{G}$. [HS, Lemma 2.2] shows that E need not be a locally compact space; indeed, $\mathbb{N} = Y - \{\omega\}$ is an irreducible closed subset without generic point.

According to Example 6.2, if Y is infinite then the answer to Problem 6.1 is "no". It remains to study the case of Y finite.

We would like to know if Theorem 1.8 remains true if G is a group of diffeomorphisms; more precisely:

Problem 6.3. *Let X be an admissible space. Does there exist a finitely generated abelian subgroup $G \subset \mathrm{Diff}^1_+(\mathbb{R})$ such that X is the quasi-orbit space of G?*

We know that the answer to this problem is "no" if we replace $\mathrm{Diff}^1_+(\mathbb{R})$ by $\mathrm{Diff}^2_+(\mathbb{R})$. In fact the quasi-orbit space X of a subgroup of $\mathrm{Diff}^2_+(\mathbb{R})$ always has $\mathrm{ht}(X) \leq 2$ (see Theorem 1.9).

Problem 6.4. *Let X be an admissible space. Does there exist a finitely generated subgroup $G \subset \mathrm{Diff}^r_+(\mathbb{R})$, $r > 1$, such that X is the quasi-orbit space of G?*

If the answer is "no" characterize the admissible spaces which are the quasi-orbit spaces of a subgroup $G \subset \mathrm{Diff}^r_+(\mathbb{R})$, $r > 1$.

We know that the answer to this problem is "no" if we replace $\mathrm{Diff}^r_+(\mathbb{R})$ by $\mathrm{Diff}^\omega_+(\mathbb{R})$ (the group of analytic diffeomorphisms). In fact the quasi-orbit space X of a subgroup of $\mathrm{Diff}^\omega_+(\mathbb{R})$ always has $\mathrm{ht}(X) \leq 2$ (see Theorem 1.10).

Theorem 1.9 asserts that an abelian group $G \subset \mathrm{Diff}^2_+(\mathbb{R})$ has $\mathrm{ht}(\mathbb{R}/\widetilde{G}) \leq 2$. However, a group G with $\mathrm{ht}(\mathbb{R}/\widetilde{G}) = 2$ necessarily has an exceptional minimal set and an elementary interval I such that the induced group G_I has a fixed point. As far as we know, there are no examples available.

Problem 6.5. *Does there exist a group $G \subset \mathrm{Diff}^2_+(\mathbb{R})$ with $\mathrm{ht}(\mathbb{R}/\widetilde{G}) = 2$?*

More generally, which admissible spaces X with $\mathrm{ht}(X) \leq 2$ are quasi-orbit spaces of groups $G \subset \mathrm{Diff}^2_+(\mathbb{R})$?

Problem 6.6. *Which admissible spaces X with $\mathrm{ht}(X) \leq 2$ are quasi-orbit spaces of (finitely generated or not) subgroups $G \subset \mathrm{Diff}^\omega_+(\mathbb{R})$?*

6.2. Codimension one foliations and T_0-spaces. Theorem 1.11 shows that every finite T_0-space is always the singular part of the quasi-leaf space of a codimension one C^1-foliation.

Problem 6.7. *Characterize the finite T_0-spaces which are the singular part of the quasi-leaf space of a codimension one C^r-foliation ($r \geq 2$).*

Problem 6.8. *Characterize the T_0-spaces with finite singular part which are the quasi-leaf space of a codimension one foliation.*

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