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Function spaces and shape theories

by

Jerzy Dydak (Knoxville, TN) and **Sławomir Nowak** (Warszawa)

Abstract. The purpose of this paper is to provide a geometric explanation of strong shape theory and to give a fairly simple way of introducing the strong shape category formally. Generally speaking, it is useful to introduce a shape theory as a localization at some class of "equivalences". We follow this principle and we extend the standard shape category Sh(HoTop) to Sh(pro-HoTop) by localizing pro-HoTop at shape equivalences. Similarly, we extend the strong shape category of Edwards–Hastings to sSh(pro-Top) by localizing pro-Top at strong shape equivalences. A map $f : X \to Y$ is a shape equivalence if and only if the induced function $f^* : [Y, P] \to [X, P]$ is a bijection for all $P \in \mathrm{ANR}$. A map $f : X \to Y$ of k-spaces is a strong shape equivalence if and only if the induced map $f^* : \mathrm{Map}(Y, P) \to \mathrm{Map}(X, P)$ is a weak homotopy equivalence for all $P \in \mathrm{ANR}$. One generalizes the concept of being a shape equivalence to morphisms of pro-HoTop without any problem and the only difficulty is to show that a localization of pro-HoTop at shape equivalences is a category (which amounts to showing that the morphisms form a set). Due to peculiarities of function spaces, extending the concept of strong shape equivalence to morphisms of pro-Top is more involved. However, it can be done and we show that the corresponding localization exists. One can introduce the concept of a super shape equivalence $f : X \to Y$ of topological spaces as a map such that the induced map $f^* : \mathrm{Map}(Y, P) \to \mathrm{Map}(X, P)$ is a homotopy equivalence for all $P \in \mathrm{ANR}$, and one can extend it to morphisms of pro-Top. However, the authors do not know if the corresponding localization exists. Here are applications of our methods:

Theorem. *A map $f : X \to Y$ of k-spaces is a strong shape equivalence if and only if $f \times \mathrm{id}_Q : X \times_k Q \to Y \times_k Q$ is a shape equivalence for each CW complex Q.*

Theorem. *Suppose $f : X \to Y$ is a map of topological spaces.*

(a) *f is a shape equivalence if and only if the induced function $f^* : [Y, M] \to [X, M]$ is a bijection for all $M = \mathrm{Map}(Q, P)$, where $P \in \mathrm{ANR}$ and Q is a finite CW complex.*

(b) *If f is a strong shape equivalence, then the induced function $f^* : [Y, M] \to [X, M]$ is a bijection for all $M = \mathrm{Map}(Q, P)$, where $P \in \mathrm{ANR}$ and Q is an arbitrary CW complex.*

(c) *If X, Y are k-spaces and the induced function $f^* : [Y, M] \to [X, M]$ is a bijection for all $M = \mathrm{Map}(Q, P)$, where $P \in \mathrm{ANR}$ and Q is an arbitrary CW complex, then f is a strong shape equivalence.*

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0. Introduction. K. Borsuk (see [Bo]) introduced the shape category of subcompacta of the Hilbert cube Q as follows: a morphism $f : X \to Y$ is the homotopy class of a sequence $\{f_n : X \to Q\}_{n\ge 1}$ of maps with the property that for any neighborhood U of Y there is m so that $f_n(X) \subset U$ for all $n > m$. Two sequences $\{f_n : X \to Q\}_{n\ge 1}$ and $\{g_n : X \to Q\}_{n\ge 1}$ are homotopic if, for each neighborhood U of Y, there is m so that $f_n(X) \cup g_n(X) \subset U$ for all $n > m$ and $f_n : X \to U$ is homotopic to $g_n : X \to U$ in U for all $n > m$. Morita and Mardešić generalized this construction to arbitrary topological spaces by looking at $f_s : X \to Y_s$, $s \in S$, where $\{Y \to Y_s\}_{s\in S}$ is a certain "resolution" of Y (see [M-S]). An alternative description of a shape morphism $\phi : X \to Y$ is as a natural transformation from $[Y, ?]$ to $[X, ?]$. That is, given a homotopy class $f : Y \to P \in \mathrm{ANR}$, $\phi(f) : X \to P$ is a homotopy class so that $\alpha \circ \phi(f) = \phi(g)$ whenever $\alpha : P \to Q \in \mathrm{ANR}$ and $g = \alpha \circ f$.

Shape theory has been very useful in tackling many geometrical problems (see [D-S$_1$] and [M-S]). However, it is plagued with some problems. For example, it is unknown if a map $f : X \to Y$ of compacta which induces a shape isomorphism, also induces a shape isomorphism $f : (X, x_0) \to (Y, f(x_0))$ in the pointed category (see [D$_2$]). This problem was remedied by Quigley (see [Q$_{1,2}$]) who introduced what is now known as the strong shape category of compacta. Instead of a sequence $\{f_n : X \to Q\}_{n\ge 1}$ of maps one considers a continuous family of maps $f_t : X \to Q$, $t \ge 1$, arising from a map $F : X \times [1,\infty) \to Q$ via $f_t(x) = F(x, t)$ for all $(x, t) \in X \times [1,\infty)$. One requires that for any neighborhood U of Y there is m so that $f_t(X) \subset U$ for all $t > m$. Two sequences arising from $F : X \times [1,\infty) \to Q$ and $G : X \times [1,\infty) \to Q$ are homotopic if there is a homotopy $H : X \times [1,\infty) \times I \to Q$ joining F and G so that for any neighborhood U of Y there is m so that $H(x, t, s) \subset U$ for all $t > m$, all $s \in I$, and all $x \in X$. There are ways to extend that procedure to arbitrary topological spaces (see [Gu$_1$]) but they are all quite complex. In this paper we offer a new way of looking at the strong shape category of topological spaces. Recall that a map $f : X \to Y$ is a shape equivalence if and only if the induced function $f^* : [Y, P] \to [X, P]$ is a bijection for all $P \in \mathrm{ANR}$ (see the alternative description of the shape category above). Since $[A, P] = \pi_0(P^A)$ for most spaces A (for example, if A is a k-space), one might consider the following generalizations of shape equivalences:

1. $f : X \to Y$ is a *strong shape equivalence* if $f^* : P^Y \to P^X$ is a weak homotopy equivalence for all $P \in \mathrm{ANR}$.

2. $f : X \to Y$ is a *super strong shape equivalence* if $f^* : P^Y \to P^X$ is a homotopy equivalence for all $P \in \mathrm{ANR}$.

In the case of compact spaces the above two concepts are identical and, in the case of compacta, one can create a new category by formally inverting all strong shape equivalences. This is known as localization (see [G-Z]) of the homotopy category of compacta at strong shape equivalences. It turns out (see 2.4) that this localization is isomorphic to the strong shape category of compacta. One might introduce strong shape equivalences between pro-spaces as in [D-N]. However, one faces the difficulty of proving that the corresponding localization exists (in the case of compacta one deals with a small category and any localization can be constructed as in [G-Z]). In this paper we offer a simplification of that step. Recall that the concept of a homotopy equivalence can be introduced in two steps:

(a) An inclusion $i : A \to X$ of topological spaces is called an *SDR map* (A is a strong deformation retract of X) if there is a homotopy $H_t : X \to X$ rel. A starting at id_X and ending at a retraction $r : X \to A$ (sometimes $i : A \to X$ is called a trivial cofibration in this case).

(b) $f : X \to Y$ is a homotopy equivalence if the inclusion $X \to M(f)$ from X to the mapping cylinder of f is an SDR map.

One can dualize (a) using trivial fibrations and arrive at the concept of an *SSDR map* $f : X \to Y$ (X is a strong shape deformation retract of Y) by requiring that the induced map $f^* : P^Y \to P^X$ be a trivial Serre fibration. An equivalent condition is to require that any commutative diagram

$$\begin{tikzcd} X \arrow[r, "a"] \arrow[d, "f"] & \mathrm{Map}(K,P) \arrow[d, "i^*"] \\ Y \arrow[r, "b"] & \mathrm{Map}(L,P) \end{tikzcd}$$

has a filler $Y \to \mathrm{Map}(K,P)$ provided K is a finite CW complex, L is a subcomplex of K, $i : L \to K$ is the inclusion, and $P \in \mathrm{ANR}$. It turns out that the above condition generalizes easily to pro-maps and one verifies that a level pro-map $f : X \to Y$ is a strong shape equivalence if and only if the inclusion $X \to M(f)$ is an SSDR pro-map. Finally, we construct a resolution $X \to R(X)$ for any pro-space X which is an SSDR pro-map. This seems to be a resolution which is stronger than those previously known (see [M-S]). Using that resolution one proves easily that the localization of π(pro-Top) at strong shape equivalences exists and is isomorphic (when restricted to topological spaces) to the strong shape category of topological spaces.

We offer another simplification of a concept from strong shape theory, namely, the strong homology groups. Following our approach to the strong shape category one deduces easily that, for any topological space X, there exists a strong shape morphism $s : K \to X$ so that K is a CW complex and s induces a bijection $s_* : \mathrm{Mor}(L,K) \to \mathrm{Mor}(L,X)$ of strong shape morphisms for any CW complex L. This is in direct analogy to the singular complex of a space X and we conjecture that the cellular homology of K represents the strong homology of X.

1. Localizations and adjoint functors. Given a small category C (i.e., a category whose objects and morphisms form a set) and a class Σ of morphisms of C, Gabriel and Zisman [G-Z] described a general way of constructing the localization $\Sigma^{-1}C$ of C at Σ together with a functor $l_\Sigma : C \to \Sigma^{-1}C$ so that $l_\Sigma(f)$ is an isomorphism for all $f \in \Sigma$. The functor l_Σ is universal for all functors inverting the elements of Σ. That means, given any functor $F : C \to D$ such that $F(f)$ is an isomorphism of D for all $f \in \Sigma$, there is a unique functor $G : \Sigma^{-1}C \to D$ with $F = G \circ l_\Sigma$. Here is a short description of the construction. First, one takes the diagram scheme of C (the directed graph with vertices being the objects of C and with directed edges corresponding to the morphisms of C) and enlarges it to T by adding the "inverses" of elements of Σ. Let f^{-1} be the "inverse" of $f \in \Sigma$. One takes the category $\mathrm{Pa}(T)$ of paths in T and $\Sigma^{-1}C$ is its quotient under obvious relations:

(a) the path (f, g) is equal to $(f \circ g)$ if the composition is defined,

(b) the path (f, f^{-1}) is equal to (id_Y) if $f \in \Sigma$ and $f : X \to Y$,

(c) the path (f^{-1}, f) is equal to (id_X) if $f \in \Sigma$ and $f : X \to Y$,

(d) the path (id_X) is equal to the identity of $\mathrm{Pa}(T)$ at X for all objects X of C.

Clearly, the assumption of C being small is needed only to conclude that the morphisms from X to Y in $\Sigma^{-1}C$ form a set. In this section we will show that the Gabriel–Zisman construction yields a category in the following special case: there is a full subcategory D of C and a functor $F : C \to D$ which is left-adjoint (respectively, right-adjoint) to the inclusion functor $i : D \to C$ so that $F(f)$ is an isomorphism of D if and only if $f \in \Sigma$. This is probably well known as it simply generalizes Proposition 1.3 of [G-Z] (see p. 7) but is of such importance to our construction of the strong shape category that we decided to provide the essential details.

1.1. Definition. Suppose D is a full subcategory of C and X is an object of C. A *D-reflection* $r_X : X \to F(X)$ (respectively, a *D-coreflection* $r_X : F(X) \to X$) is a morphism so that $F(X)$ is an object of D and the induced function $r_X^* : \mathrm{Mor}(F(X), P) \to \mathrm{Mor}(X, P)$ (respectively, $(r_X)_* : \mathrm{Mor}(P, F(X)) \to \mathrm{Mor}(P, X)$) is a bijection for all objects P of D.

1.2. Theorem. *Suppose D is a full subcategory of C and Σ is the class of all morphisms $f : X \to Y$ of C such that the induced function $f^* : \mathrm{Mor}(Y, Z) \to \mathrm{Mor}(X, Z)$ (respectively, $f_* : \mathrm{Mor}(Z, X) \to \mathrm{Mor}(Z, Y)$) is a bijection for all objects Z of D. If, for every object X of C, there is a D-reflection $r_X : X \to F(X)$ (respectively, a D-coreflection $r_X : F(X) \to X$), then*:

(a) *There is a functor* $F : C \to D$ *which is left-adjoint* (*respectively, right-adjoint*) *to the inclusion* $i : D \to C$.

(b) $r_X \in \Sigma$ *for each object* X *of* C, *and* $F(f)$ *is invertible if and only if* $f \in \Sigma$.

(c) *Any path from* X *to* Y *is equivalent to the path* $((r_Y)^{-1}, h, r_X)$ *for some morphism* $h : F(X) \to F(Y)$ *and that morphism is the same for equivalent paths.*

(d) *The localization* $\Sigma^{-1}C$ *exists.*

(e) *There is a functor* $F' : \Sigma^{-1}C \to D$ *so that* $F = l_\Sigma \circ F'$ *and* F' *is an equivalence of categories.*

Proof. (a) Given a morphism $f : X \to Y$ of C, one looks at $r_Y \circ f : X \to F(Y)$ (respectively, $f \circ r_X : F(X) \to Y$) and one picks the unique $F(f)$ so that $r_Y \circ f = F(f) \circ r_X$ (respectively, $f \circ r_X = r_Y \circ F(f)$). One easily checks that in that way one gets a functor which is left-adjoint (respectively, right-adjoint) to $i : D \to C$.

(b) Notice that $r_X \in \Sigma$ by the definition of D-reflections (respectively, D-coreflections). If $f \in \Sigma$, then $F(f)^* : \mathrm{Mor}(F(Y), Z) \to \mathrm{Mor}(F(X), Z)$ (respectively, $F(f)_* : \mathrm{Mor}(Z, F(X)) \to \mathrm{Mor}(Z, F(Y))$) is a bijection for all objects Z of D, which means that $F(f)$ is invertible. If $F(f)$ is invertible, then $f \in \Sigma$, as $r_Y \circ f = F(f) \circ r_X$ (respectively, $f \circ r_X = r_Y \circ F(f)$).

(c) Notice that one can create $F(p)$ for any path p in the diagram scheme T associated with C and Σ. Since $r_Y \circ g = F(g) \circ r_X$ for any morphism g of C, we see that the path (g^{-1}) is equivalent to $((r_X)^{-1}, F(g)^{-1}, r_Y)$ if $g \in \Sigma$, and the path g is equivalent to $((r_Y)^{-1}, F(g), r_X)$ for all $g : X \to Y$. Thus, any path from X to Y is equivalent to $((r_Y)^{-1}, h, r_X)$ for some morphism $h : F(X) \to F(Y)$ and that morphism is the same for equivalent paths. F' is created using $F(p)$ for any path p. ∎

Our main illustration of Theorem 1.2 is the case of weak homotopy equivalences: A map $f : X \to Y$ of topological spaces is defined to be a *weak homotopy equivalence* if it induces a bijection $f_* : [P, X] \to [P, Y]$ for all CW complexes P. For each space X there is a map $i_X : \mathrm{Sin}(X) \to X$ (we will call it the *singular complex* of X) from a CW complex $\mathrm{Sin}(X)$ which is a weak homotopy equivalence. Thus, there is a functor $\mathrm{Sin} : \mathrm{HoTop} \to \mathrm{HoCW}$ from the homotopy category of topological spaces to the homotopy category of CW complexes which is right-adjoint to the inclusion $\mathrm{HoCW} \to \mathrm{HoTop}$ and $\mathrm{Sin}(f)$ is an isomorphism if and only if f is a weak homotopy equivalence. Thus, one can localize HoTop at the class of weak homotopy equivalences and the resulting category Sing (see [Ed-H]) is equivalent to HoCW.

Here is a quick outline of a construction of the singular complex of X which we need in order to point out the differences and similarities with shape theory. First, the 0-cells of $\mathrm{Sin}(X)$ are declared to be points of X. Suppose the n-skeleton $\mathrm{Sin}(X)^{(n)}$ of $\mathrm{Sin}(X)$ has been constructed together with a map $i_{X,n} : \mathrm{Sin}(X)^{(n)} \to X$. Consider all commutative diagrams

(D)
$$\begin{tikzcd} \partial I^{n+1} \arrow[r, "f"] \arrow[d, "i"] & \mathrm{Sin}(X)^{(n)} \arrow[d, "i_{X,n}"] \\ I^{n+1} \arrow[r, "f'"] & X \end{tikzcd}$$

where f, f' are maps and i is the inclusion. For each diagram attach an $(n+1)$-cell to $\mathrm{Sin}(X)^{(n)}$ along f to obtain $\mathrm{Sin}(X)^{(n+1)}$ and extend $i_{X,n}$ to $i_{X,n+1} : \mathrm{Sin}(X)^{(n+1)} \to X$ using the map f' for the $(n+1)$-cell determined by the diagram (D). Thus, one gets $i_X : \mathrm{Sin}(X) \to X$ which is a trivial Serre fibration in the following sense: any commutative diagram

$$\begin{tikzcd} L \arrow[r, "f"] \arrow[d, "i"] & \mathrm{Sin}(X) \arrow[d, "i_X"] \\ K \arrow[r, "f'"] & X \end{tikzcd}$$

has a filler $K \to \mathrm{Sin}(X)$ provided K is a CW complex, L is a subcomplex of K, $i : L \to K$ is the inclusion, and f is a cellular map (i.e., it sends the n-skeleton of L to the n-skeleton of $\mathrm{Sin}(X)$ for each n). In particular, $i_X : \mathrm{Sin}(X) \to X$ is a weak homotopy equivalence.

Thus, the reason for existence of the singular complex of topological spaces is that CW complexes are preserved under direct limits in which bonding maps are inclusions.

The concept of shape equivalence is dual to that of weak homotopy equivalence. Thus, $f : X \to Y$ is a shape equivalence if and only if the induced function $f^* : [Y, Q] \to [X, Q]$ is a bijection for all CW complexes Q. Is there a dual concept to the singular complex? Well, it turns out that there are spaces X such that there is no map $f : X \to Q$ which is a shape equivalence and Q is a CW complex. One of the simplest examples is the dyadic solenoid DS. If $f : \mathrm{DS} \to Q$ were a shape equivalence of DS to a CW complex Q, then $f(\mathrm{DS})$ would be contained in a finite subcomplex K of Q so that $[K, P] \to [\mathrm{DS}, P]$ would be a surjection for all CW complexes P. One gets a contradiction by picking $P = S^1$ in which case $[Y, P]$ is the first cohomology of Y and it is well known that the first cohomology of DS is not finitely generated. The general reason for this failure is that if one tries to dualize the construction of the singular complex, then one faces inverse limits of CW complexes and the class of CW complexes is not preserved under inverse limits. It turns out that the best way to avoid this obstacle is to enlarge the category of CW complexes to a category which has inverse limits. The general construction is that of pro-categories (see [M-S]). Thus, any category C can be embedded as a full category into a category pro-C which has inverse limits. Now, if one dualizes the construction of the singular complex, one arrives at a morphism $X \to \mathrm{ShS}(X)$ of pro-HoANR (it is more convenient to switch to ANRs while doing shape theory, and every CW complex is homotopy equivalent to an ANR) which is a shape equivalence. That is how shape theory is described in [D-S$_1$] or [M-S]. In our paper we will make one more step: namely, we will construct shape equivalences $i_X : X \to \mathrm{ShS}(X)$ (called shape systems of X) for any object of pro-HoTop so that $\mathrm{ShS}(X)$ is an object of pro-HoANR. One reason is that if we allow the shape systems of topological spaces to be inverse systems, then it makes sense, for symmetry reasons, to enlarge the class of topological spaces to the class of inverse systems in HoTop. The second, and more important, reason is that we can apply Theorem 1.2 immediately and construct the localization Sh(pro-HoTop) (called the shape category) of pro-HoTop at shape equivalences which is equivalent to pro-HoANR.

So what is the strong shape category? We will show that a map $f : X \to Y$ of k-spaces is a strong shape equivalence if and only if the induced map $f^* : \mathrm{Map}(Y, P) \to \mathrm{Map}(X, P)$ is a weak homotopy equivalence for each $P \in \mathrm{ANR}$ (equivalently, for each CW complex P). Since $\pi_0(\mathrm{Map}(Z, P)) = [Z, P]$ for k-spaces Z, one sees that strong shape equivalences are indeed a subclass of shape equivalences. Since, outside of k-spaces, the compact-open topology on function spaces does not have good properties, one has to define strong shape equivalences in an alternative way. The general principle is to replace maps from K to $\mathrm{Map}(X, P)$ by maps from $K \times X$ to P and mimic the property of $f^* : \mathrm{Map}(Y, P) \to \mathrm{Map}(X, P)$ being a weak homotopy equivalence that way. This is done in the paper and we show that one can construct the strong shape category sSh(pro-Top) by localizing π(pro-Top) (the simplest homotopy category on pro-Top) at strong shape equivalences. This is done by constructing the strong shape system $i_X : X \to \mathrm{sShS}(X)$ of every object X of pro-Top with $\mathrm{sShS}(X)$ being an object of SSDR-FIBRANT which is a full subcategory of π(pro-Top) so that $i_X^* : \mathrm{Mor}(\mathrm{sShS}(X), P) \to \mathrm{Mor}(X, P)$ is a bijection for all objects P of SSDR-FIBRANT. There is a natural functor from sSh(pro-Top) to the homotopy category Ho(pro-Top) of [Ed-H]. We will provide an example of a shape equivalence which is not a strong shape equivalence in the form of a morphism of inverse sequences of CW complexes. In particular, one gets a morphism of tow(CW) which induces an isomorphism of tow(HoCW) but the induced morphism of Ho(tow(CW)) is not an isomorphism. This solves problems 5.28 (p. 175) and 9.3 (p. 279) of [Ed-H].

2. Strong shape category for compacta. In this section we assume the following definition (the function spaces are equipped with the compact-open topology):

2.1. Definition. A map $f : X \to Y$ of k-spaces is a *strong shape equivalence* if the induced map $f^* : \mathrm{Map}(Y, P) \to \mathrm{Map}(X, P)$ is a weak homotopy equivalence for each $P \in \mathrm{ANR}$ (equivalently, for each CW complex P).

We start with this definition as it is the easiest way to introduce strong shape theory to any topologist. It will be seen later that 2.1 is a special case of strong shape equivalences as defined in [D-N] (see 3.8 in this paper).

Using this definition, we will show that the homotopy category of compacta localized at strong shape equivalences yields a category equivalent to the strong shape category introduced by Edwards and Hastings [Ed-H]. First, let us prove the following:

2.2. Theorem. *Suppose A is a closed subset of a compactum X. The following conditions are equivalent*:

(A) *The inclusion $i : A \to X$ of compacta is a strong shape equivalence.*

(B) $i^* : \mathrm{Map}(X, P) \to \mathrm{Map}(A, P)$ *is a trivial Serre fibration for all* $P \in \mathrm{ANR}$.

(C) *Any map from A to* $P \in \mathrm{ANR}$ *extends over X, and any map from $X \times \{0, 1\} \cup A \times I$ to* $P \in \mathrm{ANR}$ *extends over $X \times I$.*

Proof. Notice that $i^* : \mathrm{Map}(X, P) \to \mathrm{Map}(A, P)$ is a Serre fibration for any $P \in \mathrm{ANR}$. It is simply a reformulation of the Homotopy Extension Theorem. Indeed, if

$$\begin{tikzcd} K \times \{0\} \arrow[r, "f"] \arrow[d, "j"] & \mathrm{Map}(X, P) \arrow[d, "i^*"] \\ K \times I \arrow[r, "g"] & \mathrm{Map}(A, P) \end{tikzcd}$$

is a commutative diagram, where K is a finite CW complex and j is the inclusion, then by switching to maps $f' : K \times \{0\} \times X \to P$ and $g' : K \times I \times A \to P$ one finds that $f'|K \times \{0\} \times A = g'|K \times \{0\} \times A$. Thus, by the Homotopy Extension Theorem, there is a map $h : K \times I \times X \to P$ extending both f' and g'. The map h induces $H : K \times I \to \mathrm{Map}(X, P)$ which is an extension of f and which is a lift of g. This proves (A)$\Rightarrow$(B) as a Serre fibration between metrizable spaces which is a weak homotopy equivalence must be a trivial Serre fibration (see 9.2).

Suppose any map from A to $P \in \mathrm{ANR}$ extends over X and any map from $X \times \{0,1\} \cup A \times I$ to $P \in \mathrm{ANR}$ extends over $X \times I$. Our goal is to show that, for any $P \in \mathrm{ANR}$, the induced map $i^* : \mathrm{Map}(X,P) \to \mathrm{Map}(A,P)$ is a trivial Serre fibration. That is, any commutative diagram

(D)
$$\begin{tikzcd} L \arrow[r, "f"] \arrow[d, "j"] & \mathrm{Map}(X,P) \arrow[d, "i^*"] \\ K \arrow[r, "g"] & \mathrm{Map}(A,P) \end{tikzcd}$$

where K is a finite CW complex, L is a subcomplex of K, and j is the inclusion, has a filler f' (i.e., an extension $f' : K \to \mathrm{Map}(X,P)$ of f so that $i^* \circ f' = g$). In particular, that implies that i^* is a weak homotopy equivalence. Notice that if every map from A to $P \in \mathrm{ANR}$ extends over X, then it means that $i^* : \mathrm{Map}(X,P) \to \mathrm{Map}(A,P)$ is a surjection, which is the same as verifying the existence of a filler in diagrams (D), where K is of dimension at most 0. In the same manner, if every map from $X \times \{0,1\} \cup A \times I$ to $P \in \mathrm{ANR}$ extends over $X \times I$, then it amounts to saying that there is a filler in diagram (D) if $K = I$ and $L = \partial I$. We can summarize both conditions:

(1) any map from A to $P \in \mathrm{ANR}$ extends over X,

(2) any map from $X \times \{0,1\} \cup A \times I$ to $P \in \mathrm{ANR}$ extends over $X \times I$

as equivalent to existence of a filler in diagrams (D), where K is a finite CW complex of dimension at most 1. In particular, (B)$\Rightarrow$(C).

(C)$\Rightarrow$(B). To show that i^* is a trivial Serre fibration it suffices to show that the homotopy groups of each fiber F of i^* are 0 (see 9.2). Suppose S is the n-sphere, $n \geq 1$, and $a : S \to F$ is a map. Let $c : S \to F$ be a constant map. We need to show that there is a homotopy $H : S \times I \to F$ joining a and c. Let $G : S \times I \to \mathrm{Map}(A,P)$ be the constant homotopy between $i^* \circ a$ and $i^* \circ c$. By switching to $\mathrm{Map}(X, \mathrm{Map}(S,P))$ and $\mathrm{Map}(A, \mathrm{Map}(S,P))$ one gets a map $f : \partial I \to \mathrm{Map}(X, \mathrm{Map}(S,P))$ and a map $g : I \to \mathrm{Map}(A, \mathrm{Map}(S,P))$ so that $g|\partial I = i^* \circ f$. Since $\mathrm{Map}(S,P) \in \mathrm{ANR}$, there is a lift $G : I \to \mathrm{Map}(X, \mathrm{Map}(S,P))$ of g which is an extension of f. The map G induces a map $H : S \times I \to \mathrm{Map}(X,P)$ which is a homotopy from a to c with values in F.

(B)$\Rightarrow$(A) is obvious. ∎

Since the homotopy category HoCM of compacta is equivalent to a small category, namely its full subcategory whose objects are closed subsets of the Hilbert cube, one can localize HoCM at any class of morphisms. In particular, the following definition makes sense.

2.3. Definition. The *strong shape category of compacta* sSh(CM) is defined to be the localization of the homotopy category HoCM of compacta at the class of all strong shape equivalences.

2.4. Theorem. sSh(CM) *is equivalent to the strong shape category introduced by Edwards–Hastings.*

Proof. Calder and Hastings [C-H] proved that the strong shape category of compacta $\mathrm{sSH}_{\mathrm{EH}}(\mathrm{CM})$ introduced by Edwards and Hastings [Ed-H] is equivalent to the localization of HoCM at all homotopy classes of inclusions $[i]$ such that $i : A \to X$ induces an isomorphism of $\mathrm{sSH}_{\mathrm{EH}}(\mathrm{CM})$. Dydak and Segal [D-S$_2$] proved that i induces an isomorphism of $\mathrm{sSH}_{\mathrm{EH}}(\mathrm{CM})$ if and only if every map $f : A \to P \in \mathrm{ANR}$ extends over X and every map $H : X \times \{0,1\} \cup A \times I \to P \in \mathrm{ANR}$ extends over $X \times I$. By Theorem 2.2, $\mathrm{sSH}_{\mathrm{EH}}(\mathrm{CM})$ is equivalent to sSH(CM). ∎

3. Strong shape equivalences in pro-Top.

One has a natural extension of the notion of the shape equivalence to morphisms of pro-HoTop:

3.1. Definition. A morphism $f : X \to Y$ of pro-HoTop is *a shape equivalence* if the induced function $f^* : \mathrm{Mor}(Y,P) \to \mathrm{Mor}(X,P)$ is a bijection for all $P \in \mathrm{ANR}$.

3.2. Proposition. $f : X \to Y$ *is a shape equivalence in* pro-HoTop *if and only if the induced function* $f^* : \mathrm{Mor}(Y,P) \to \mathrm{Mor}(X,P)$ *is a bijection for all objects* P *of* pro-HoANR.

Proof. This follows easily from the fact that any object $P = \{P_a, p_a^b, A\}$ of pro-HoANR is the inverse limit of projections $P \to P_a$, $a \in A$. ∎

To shorten the terminology we will make the following convention.

3.3. Definition. An object X of pro-Top is called a *pro-space* and a morphism of pro-Top is called a *pro-map*.

In the proof of Theorem 2.2 we could see that an inclusion $i : A \to X$ is a strong shape equivalence of compacta if and only if every diagram

(D)
$$\begin{tikzcd} L \arrow[r, "f"] \arrow[d, "j"] & \mathrm{Map}(X,P) \arrow[d, "i^*"] \\ K \arrow[r, "g"] & \mathrm{Map}(A,P) \end{tikzcd}$$

has a filler $f' : K \to \mathrm{Map}(X,P)$ for all finite CW complexes K and all subcomplexes L of K where j is the inclusion. The existence of a filler f' for

(D) is equivalent to the existence of a filler in the following adjoint diagram:

$$(\mathrm{D}')\qquad \begin{tikzcd} A \arrow[r, "g'"] \arrow[d, "i"] & \mathrm{Map}(K,P) \arrow[d, "j^*"] \\ X \arrow[r, "f'"] & \mathrm{Map}(L,P) \end{tikzcd}$$

Diagram (D′) is much more appropriate for pro-spaces than diagram (D).

3.4. Definition. A pro-map $f : X \to Y$ is called an *SSDR pro-map* provided any commutative diagram in pro-Top

$$\begin{tikzcd} X \arrow[r, "a"] \arrow[d, "f"] & \mathrm{Map}(K,P) \arrow[d, "i^*"] \\ Y \arrow[r, "b"] & \mathrm{Map}(L,P) \end{tikzcd}$$

has a filler $Y \to \mathrm{Map}(K,P)$ whenever K is a finite CW complex, L is a subcomplex of K, $i : L \to K$ is the inclusion, and $P \in \mathrm{ANR}$.

F. Cathey introduced SSDR inclusions of metrizable spaces in a way equivalent to the following statement (see [C_2], Theorem 1.2): a closed inclusion $f : X \to Y$ of metrizable spaces is an SSDR map if any commutative diagram

$$\begin{tikzcd} X \arrow[r, "a"] \arrow[d, "f"] & E \arrow[d, "p"] \\ Y \arrow[r, "b"] & B \end{tikzcd}$$

has a filler provided $p : E \to B$ is a Hurewicz fibration of ANRs. One is tempted to define SSDR pro-maps in an analogous way. However, the authors faced difficulties with constructing SSDR pro-maps $s_X : X \to X'$ so that X' is a pro-ANR for arbitrary pro-space X (see Section 4). Thus, instead of considering arbitrary Hurewicz fibrations $p : E \to B$, we restrict ourselves to fibrations $i^* : \mathrm{Map}(K,P) \to \mathrm{Map}(L,P)$, where K is a finite CW complex, L is a subcomplex of K, $i : L \to K$ is the inclusion, and $P \in \mathrm{ANR}$.

Notice that SSDR pro-maps generalize a variety of useful notions from general topology (see [D_4] and [$\mathrm{Se}_{1,2}$]). In particular, every SSDR map $f : X \to Y$ of paracompact spaces is an M-embedding (see 7.17).

3.5. Theorem. *A pro-map $f : X \to Y$ is an SSDR pro-map if and only if the following two conditions hold*:

(a) *For any pro-map $g : X \to P \in \mathrm{ANR}$ there is a pro-map $h : Y \to P$ such that $g = h \circ f$.*

(b) *For any two pro-maps* $u, v : Y \to P \in$ ANR *and any homotopy* $H : X \times I \to P$ *joining* $u \circ f$ *and* $v \circ f$, *there is a homotopy* $G : Y \times I \to P$ *joining* u *and* v *such that* $H = G \circ (f \times \mathrm{id}_I)$.

Proof. Suppose f is an SSDR pro-map. Given a pro-map $g : X \to P \in$ ANR one generates

$$\begin{tikzcd} X \arrow[r, "a"] \arrow[d, "f"] & \mathrm{Map}(*, P) \arrow[d, "i^*"] \\ Y \arrow[r, "b"] & \mathrm{Map}(\emptyset, P) \end{tikzcd}$$

where $a(x)$ is the pro-map sending $*$ to $g(x)$. A filler in the above diagram induces a pro-map $h : Y \to P$ so that $g = h \circ f$.

Suppose we have two pro-maps $u, v : Y \to P \in$ ANR and a homotopy $H : X \times I \to P$ joining $u \circ f$ and $v \circ f$. One generates

$$\begin{tikzcd} X \arrow[r, "a"] \arrow[d, "f"] & \mathrm{Map}(I, P) \arrow[d, "i^*"] \\ Y \arrow[r, "b"] & \mathrm{Map}(\{0,1\}, P) \end{tikzcd}$$

and a filler of this diagram induces a homotopy $G : Y \times I \to P$ joining u and v such that $H = G \circ (f \times \mathrm{id}_I)$.

Suppose $f : X \to Y$ is a pro-map satisfying conditions (a) and (b) of the theorem.

Our plan is to show that if $P \in$ ANR then any commutative diagram

$$\text{(D)} \qquad \begin{tikzcd} X \arrow[r, "a"] \arrow[d, "f"] & \mathrm{Map}(I^n, P) \arrow[d, "i^*"] \\ Y \arrow[r, "b"] & \mathrm{Map}(\partial I^n, P) \end{tikzcd}$$

has a filler for all $n \geq 1$. The case $n = 1$ follows as above. The adjoint diagram to (D) is

$$\text{(D}'\text{)} \qquad \begin{tikzcd} X \times \partial I^n \arrow[r, "j"] \arrow[d, "f \times \mathrm{id}"] & X \times I^n \arrow[d, "a'"] \\ Y \times \partial I^n \arrow[r, "b'"] & P \end{tikzcd}$$

Let $\pi : \partial I^n \times I \to I^n$ be the quotient map such that $\pi|\partial I^n \times \{0\} = \mathrm{id}$ and $I^n = \partial I^n \times I / \partial I^n \times \{1\}$. Then $a'|X \times \pi(\partial I^n \times \{1\})$ admits a pro-map

$u : Y \to P$ so that $u \circ f = a'|X$. Now, one generates a diagram

$$\begin{tikzcd} X \times \partial I^n \times \{0,1\} \arrow[r, "j"] \arrow[d, "f\times \mathrm{id}"] & X \times \partial I^n \times I \arrow[d, "a'"] \\ Y \times \partial I^n \times \{0,1\} \arrow[r, "b'"] & P \end{tikzcd}$$

with adjoint diagram being

$$\begin{tikzcd} X \arrow[r, "a"] \arrow[d, "f"] & \mathrm{Map}(I, \mathrm{Map}(\partial I^n, P)) \arrow[d, "i^*"] \\ Y \arrow[r, "b"] & \mathrm{Map}(\{0,1\}, \mathrm{Map}(\partial I^n, P)) \end{tikzcd}$$

The last diagram has a filler which induces a map $Y \times \partial I^n \times I \to P$. That map induces a map (see 9.3) $Y \times I^n \to P$, which is what we need. ∎

Since every pro-map can be replaced by a level pro-map (see [M-S]), some of the subsequent results deal with properties of level SSDR pro-maps.

3.6. Proposition. *Suppose* $i = \{i_a\}_{a\in A} : X = \{X_a, p^b_a, A\} \to Y = \{Y_d, q^b_a, A\}$ *is a level pro-inclusion such that* X_a *is a closed subset of* Y_a *for each* $a \in A$. *If* i *is an SSDR pro-map, then the induced pro-inclusion* $j : Y \times \partial I \cup X \times I \to Y \times I$ *is an SSDR pro-map.*

Proof. Suppose $g : Y \times \partial I \cup X \times I \to P \in \mathrm{ANR}$ is a pro-map. Notice that $g|X \times I$ is a homotopy joining $g|X \times \{0\}$ and $g|X \times \{1\}$. There is a homotopy $G : Y \times I \to P$ joining $g|Y \times \{0\}$ and $g|Y \times \{1\}$ so that $g|X \times I = G \circ (i \times \mathrm{id}_I)$. Notice that G is an extension of g.

Consider two copies I_1 and I_2 of the unit interval. Suppose $H : (Y \times \partial I_1 \cup X \times I_1) \times I_2 \to P \in \mathrm{ANR}$ is a homotopy joining $g_1|(Y \times \partial I_1 \cup X \times I_1) \times \{0\}$ and $g_2|(Y \times \partial I_1 \cup X \times I_1) \times \{1\}$ for some $g_1, g_2 : Y \times I_1 \to P$. One arrives at a commutative diagram

$$\text{(D)} \qquad \begin{tikzcd} X \arrow[r, "a"] \arrow[d, "f"] & \mathrm{Map}(I_1 \times I_2, P) \arrow[d, "i^*"] \\ Y \arrow[r, "b"] & \mathrm{Map}(I_1 \times \partial I_2 \cup \partial I_1 \times I_2, P) \end{tikzcd}$$

where a is induced by H, and b is induced by both H and g_1, g_2. The filler G' of diagram (D) induces a homotopy $G : (X \times I_1) \times I_2 \to P$ joining g_1 and g_2 so that G extends H. ∎

If $f : X \to Y$ is a level pro-map, then one easily constructs the mapping cylinder $M(f)$ of f and the inclusion pro-map $i : X \to M(f)$. The *double mapping cylinder* $DM(f)$ of f is defined as $X \times I \cup M(f) \times \partial I$ and one has an inclusion pro-map $J : DM(f) \to M(f) \times I$.

3.7. Proposition. *Suppose* $f = \{f_a\}_{a\in A} : X = \{X_a, p_a^b, A\} \to Y = \{Y_d, q_a^b, A\}$ *is a level pro-map.*

(1) *If* f *is an SSDR pro-map*, *then the inclusion* $i : X \to M(f)$ *is an SSDR pro-map.*

(2) *If the inclusion* $i : X \to M(f)$ *is an SSDR pro-map*, *then the inclusion* $j : DM(f) \to M(f) \times I$ *is an SSDR pro-map.*

Proof. It suffices to prove (1), as (2) follows from 3.6 and (1). Let $p : M(f) \to Y$ be the natural projection. If $g : X \to P \in \mathrm{ANR}$ is a pro-map, then there is $h' : Y \to P$ so that $g = h' \circ f$. Put $h = h' \circ p$ and notice that $h \circ i = h' \circ p \circ i = h' \circ f = g$. Suppose $G : X \times I \to P \in \mathrm{ANR}$ is a homotopy joining $g \circ f$ and $h \circ f$ for some $g, h : M(f) \to P$. Let $q : X \times I' \to M(f)$ be the natural pro-map, where I' is a copy of the unit interval I. Then g and h induce a pro-map $Y \times \{1\} \times \partial I \to P$, and G, $g \circ q$, and $h \circ q$ induce a pro-map $X \times (\{0\} \times I \cup I' \times \partial I) \to P$. Since there is a retraction $r : I' \times I \to \{0\} \times I \cup I' \times \partial I$ one gets a pro-map $X \times I' \times I \to P$ and, switching to function spaces, one gets a commutative diagram

$$\text{(D)}\qquad \begin{array}{ccc} X & \xrightarrow{a} & \mathrm{Map}(I' \times I, P) \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle i^*} \\ Y & \xrightarrow{b} & \mathrm{Map}(\{1\} \times \partial I, P) \end{array}$$

The filler of (D) induces $Y \times I' \times I \to P$, which gives rise to a homotopy $M(f) \times I \to P$ joining g and h and extending G. ■

Let us recall the definition of strong shape equivalences (see [D-N]).

3.8. Definition. A pro-map $f : X \to Y$ is called a *strong shape equivalence* provided the following two conditions hold:

(a) for any pro-map $g : X \to P \in \mathrm{ANR}$ there is a pro-map $h : Y \to P$ such that $g \approx h \circ f$,

(b) for any two pro-maps $u, v : Y \to P \in \mathrm{ANR}$ and any homotopy $H : X \times I \to P$ joining $u \circ f$ and $v \circ f$, there is a homotopy $G : Y \times I \to P$ joining u and v such that $H \approx G \circ (f \times \mathrm{id}_I)$ rel. $X \times \{0, 1\}$.

3.9. Theorem. *Suppose* $f = \{f_a\}_{a\in A} : X = \{X_a, p_a^b, A\} \to Y = \{Y_d, q_a^b, A\}$ *is a level pro-map. The following conditions are equivalent*:

(1) f *is a strong shape equivalence.*

(2) *The inclusion* $i : X \to M(f)$ *from* X *to the mapping cylinder of* f *is an SSDR pro-map.*

Proof. (1)⇒(2). Suppose f is a strong shape equivalence. Suppose $g : X \to P \in \mathrm{ANR}$. There is $h : Y \to P$ and a homotopy $H : X \times I \to P$ joining g and $h \circ f$. Notice that H induces an extension $H' : M(f) \to P$ of g.

Suppose $H : X \times I \to P$ is a homotopy joining $a|X$ and $b|X$ for some $a, b : M(f) \to P \in$ ANR. We can paste the three maps a, b, and H, to produce a homotopy F on $X \times I$ joining $(a|Y) \circ f$ and $(b|Y) \circ f$. There is a homotopy $G : Y \times I \to P$ so that $G \circ (f \times \mathrm{id})$ is homotopic to F rel. $X \times \{0,1\}$. This produces a homotopy on $M(f) \times I$ extending H and joining a and b.

(2)⇒(1). Suppose $i : X \to M(f)$ is an SSDR pro-map. Suppose $g : X \to P \in$ ANR. There is an extension $h' : M(f) \to P$ of g. Let $h = h'|Y$. Define $H : X \times I \to P$ by $H(x,t) = h'[x,t]$ for $t \in I$ and $x \in X$.

Suppose $H : X \times I \to P \in$ ANR is a homotopy joining $a \circ f$ and $b \circ f$ for some $a, b : Y \to P$. Let $A, B : M(f) \to P$ be $A = a \circ \pi$, $B = b \circ \pi$, where $\pi : M(f) \to Y$ is the projection. There is a homotopy $G : M(f) \times I \to P$ extending H and joining A and B. Notice that $(G|Y \times I) \circ (f \times \mathrm{id})$ is homotopic rel. $X \times \{0,1\}$ to H. ∎

Given two spaces X and Y, one can create the k-product $X \times_k Y$ as $k(X \times Y)$, where kZ is the universal k-space on the set Z so that $\mathrm{id} : kZ \to Z$ is continuous (see [D_4]). In the case of Hausdorff spaces Z, a subset U of Z is declared open in kZ if and only if $U \cap C$ is open in C for all compact subsets C of Z (see [Wh]).

3.10. Theorem. *A map $f : X \to Y$ of k-spaces is a strong shape equivalence if and only if $f \times \mathrm{id}_Q : X \times_k Q \to Y \times_k Q$ is a shape equivalence for each CW complex Q.*

Proof. Suppose $f : X \to Y$ is a strong shape equivalence. By switching to the inclusion from X to the mapping cylinder of f, we may assume that f is an SSDR map. Suppose Q is a CW complex and $g : X \times_k Q \to P \in$ ANR is a map. It induces $\mathrm{adj}(g) : Q \to \mathrm{Map}(X,P)$ which lifts to $h' : Q \to \mathrm{Map}(Y,P)$ as $f^* : \mathrm{Map}(Y,P) \to \mathrm{Map}(X,P)$ is a trivial Serre fibration (it is a Serre fibration by Theorem 7.8 (p. 31) of [Wh], it is a weak homotopy equivalence, and 9.2 says it is a trivial Serre fibration). Then h' induces $h : Y \times_k Q \to P$ so that $g = h \circ (f \times \mathrm{id}_Q)$.

Suppose $a, b : Y \times_k Q \to P \in$ ANR and $H : (X \times_k Q) \times I \to P$ is a homotopy joining $(f \times \mathrm{id}_Q) \circ a$ and $(f \times \mathrm{id}_Q) \circ b$. Dualize to $a', b' : Q \to \mathrm{Map}(Y,P)$ and $H' : Q \times I \to \mathrm{Map}(X,P)$. Lift H' to $\mathrm{Map}(Y,P)$, which gives a homotopy joining a and b.

Suppose $i : X \to M(f)$ has the property that $i \times \mathrm{id}_Q$ is a shape equivalence for all CW complexes Q.

Claim 1. *$\mathrm{Map}(M(f),P) \to \mathrm{Map}(X,P)$ is a weak homotopy equivalence for all $P \in$ ANR.*

Proof. Given $a : Q \to \mathrm{Map}(X,P)$ convert it to $a' : X \times_k Q \to P$ and extend it to $b' : M(f) \times_k Q \to P$. Switch to $b : Q \to \mathrm{Map}(M(f),P)$. Then b

is a lift of a. Suppose $a, b : Q \to \mathrm{Map}(M(f), P)$ are two maps so that $i^* \circ a$ and $i^* \circ b$ are joined by a homotopy $H : Q \times I \to \mathrm{Map}(X, P)$. By dualizing we have $a', b' : M(f) \times_k Q \to P$ so that $a'|X \times_k Q \approx b'|X \times_k Q$. Since $X \times_k Q \to M(f) \times_k Q$ is a shape equivalence, we have $a' \approx b'$, which implies $a \approx b$. ■

Claim 2. *$\mathrm{Map}(M(f), P) \to \mathrm{Map}(X, P)$ is a Serre fibration for all $P \in$ ANR.*

Proof. This follows from Theorem 7.8 (p. 31) of [Wh]. ■

By 9.2, $\mathrm{Map}(M(f), P) \to \mathrm{Map}(X, P)$ is a trivial Serre fibration for all $P \in$ ANR, which means that i is an SSDR map. $\blacksquare_{3.10}$

3.11. Theorem. *Suppose $f = \{f_n\}_{n\geq 1} : \{X_n, p_n^m\} \to \{Y_n, q_n^m\}$ is a level pro-map between inverse sequences of ANRs which is a strong shape equivalence. If there are points $x_n \in X_n$, $y_n \in Y_n$ for $n \geq 1$ such that $p_n^m(x_m) = x_n$, $q_n^m(y_m) = y_n$ for $m \geq n$ and $f_n(x_n) = y_n$ for all n, then $f = \{f_n\}_{n\geq 1} : \{(X_n, x_n), p_n^m\} \to \{(Y_n, y_n), q_n^m\}$ is a pointed shape equivalence.*

Proof. Fix $n \geq 1$. There is $m > n$ and a map $u : Y_m \to X_n$ such that $f_n \circ u \approx q_n^m$ and $u \circ f_m \approx p_n^m$. By homotoping u we may achieve that $u(y_m) = x_n$ and $u \circ f_m \approx p_n^m$ rel. x_m (see [D-G$_{1,2}$]). Notice that there is a homotopy H from $(f_n \circ u) \circ f_m$ to $q_n^m \circ f_m$ rel. x_m. Since f is a strong shape equivalence, there is a homotopy $G : Y_k \times I \to Y_n$ for some $k > m$ such that $G \circ (f_k \times \mathrm{id}_I)$ is homotopic to $H \circ (p_m^k \times \mathrm{id}_I)$ rel. $X_k \times \{0, 1\}$ and G joins $f_n \circ u \circ\ q_m^k$ and q_n^k. In particular, $G|\{y_k\} \times I$ is homotopic to the trivial loop, which means that $f_n \circ u \circ\ q_m^k \approx q_n^k$ rel. y_n. Thus, $f = \{f_n\}_{n\geq 1} : \{(X_n, x_n), p_n^m\} \to \{(Y_n, y_n), q_n^m\}$ is a pointed shape equivalence. ■

3.12. Corollary. *There is a pro-map $f : X \to Y$ of CW-sequences which is not a strong shape equivalence but $f \times \mathrm{id}_Q : X \times_k Q \to Y \times_k Q$ is a shape equivalence for each CW complex Q.*

Proof. In [D$_2$] an example of a pro-map $f : (X, x_0) \to (Y, y_0)$ of CW-sequences is given such that f is not a pointed shape equivalence but $f : X \to Y$ is a shape equivalence. Notice that $f \times \mathrm{id}_Q : X \times_k Q \to Y \times_k Q$ is a shape equivalence for each CW complex Q. ■

Notice that in the case of a map $F : X \to Y$ of k-spaces we have two definitions of being a strong shape equivalence: 2.1 and 3.8. The purpose of the next result is to confirm that the two definitions are equivalent.

3.13. Proposition. *Suppose $f : X \to Y$ is a map of k-spaces. The following conditions are equivalent.*

(a) *The induced map* $f^* : \mathrm{Map}(Y, P) \to \mathrm{Map}(X, P)$ *is a weak homotopy equivalence for each* $P \in \mathrm{ANR}$.

(b) *For any map* $g : X \to P \in \mathrm{ANR}$ *there is a map* $h : Y \to P$ *such that* $g \approx h \circ f$, *and for any two maps* $u, v : Y \to P \in \mathrm{ANR}$ *and any homotopy* $H : X \times I \to P$ *joining* $u \circ f$ *and* $v \circ f$, *there is a homotopy* $G : Y \times I \to P$ *joining* u *and* v *such that* $H \approx G \circ (f \times \mathrm{id}_I)$ *rel.* $X \times \{0, 1\}$.

Proof. Consider the inclusion $i : X \to M(f)$ from X to the mapping cylinder of f. Notice that the $M(f)$ is the quotient space of the disjoint union $X \times I \oplus Y$ which is a k-space. Therefore $M(f)$ is a k-space.

(a)⇒(b). Notice that $i^* : \mathrm{Map}(M(f), P) \to \mathrm{Map}(X)$ is a fibration for each $p \in \mathrm{ANR}$. Since i^* is a weak homotopy equivalence, it must be a trivial Serre fibration, which amounts to i being an SSDR map. Use 3.9.

(b)⇒(a). By 3.9, i is an SSDR map, which is equivalent to i^* being a trivial Serre fibration. Since $Y \to M(f)$ is a homotopy equivalence, we see that $f^* : \mathrm{Map}(Y, P) \to \mathrm{Map}(X, P)$ is a weak homotopy equivalence for each $P \in \mathrm{ANR}$. ∎

4. SSDR resolutions of pro-spaces. The purpose of this section is to provide a construction of SSDR pro-maps $X \to X'$ for every pro-space X so that X' is a pro-ANR.

4.1. Theorem. *For every pro-space* X *there is a pro-map* $r_X : X \to R(X)$ *such that* $R(X)$ *is an object of* pro-ANR *and any commutative diagram*

$$\begin{array}{ccc} X & \xrightarrow{u} & \mathrm{Map}(K, P) \\ \downarrow{\scriptstyle r_X} & & \downarrow{\scriptstyle i_*} \\ R(X) & \xrightarrow{v} & \mathrm{Map}(L, P) \end{array}$$

has a filler $R(X) \to \mathrm{Map}(K, P)$ *provided* $P \in \mathrm{ANR}$, K *is a compact metrizable space*, L *is a closed subspace of* K, *and* $i : L \to K$ *is the inclusion. In particular*, r_X *is an SSDR pro-map.*

Proof. Suppose $X = \{X_a, p_a^b, A\}$ and choose a cardinal number $m \geq \aleph_0$ such that the density of each X_a is at most m. Let $\mathrm{ANR}(m)$ be the set of all ANRs contained in the Tikhonov cube I^m. Thus, any ANR of density at most m is homeomorphic to an element of $\mathrm{ANR}(m)$. Given a pro-map $f : X \to Y$ from X to a topological space Y we say that U *contains an image of* f provided there is a representative $f_a : X_a \to Y$ of f so that $f_a(X_a) \subset U$. Also, given a finite sequence $f_i : X \to Y_i$, $i \leq n$, of pro-maps from X to topological spaces Y_i, one easily constructs the diagonal $\Delta_{i\leq n} f_i : X \to \prod_{i\leq n} Y_i$ of all f_i.

Consider the set $M = \bigcup_{P\in \mathrm{ANR}(m)} \mathrm{Mor}(X,P)$ and let Ω be the set of all pairs (S,U) such that S is a finite subset of M (in particular, elements of S are mutually different) and U is a neighborhood of an image of X in $\prod_{s\in S} P_s$ under the diagonal $\Delta_{s\in S} f_s$, where S consists of $f_s : X \to P_s$, $s \in S$. We declare $\lambda = (S,U) \le \mu = (T,V)$ provided $S \subset T$ and $q^\mu_\lambda(V) \subset U$, where $q^\mu_\lambda : \prod_{t\in T} P_t \to \prod_{s\in S} P_s$ is the projection.

Claim 1. *$(\Omega,\le)$ is a directed set.*

Proof. Clearly, $\lambda \le \mu$ and $\mu \le \nu$ implies $\lambda \le \nu$. We need to show that given $\lambda = (S,U)$ and $\mu = (T,V)$ there is $\nu = (R,W)$ so that $\nu \ge \lambda$ and $\nu \ge \mu$. Put $R = S\cup T$ and $W = (q^\nu_\lambda)^{-1}(U) \cap (q^\nu_\mu)^{-1}(V)$. ∎

Given $\lambda = (S,U)$ we put $Y_\lambda = U$ and $g_\lambda : X \to Y_\lambda$ is induced by the diagonal of $f_s : X \to P_s$, $s \in S$. Thus, we get a pro-ANR $Y = \{Y_\lambda, q^\mu_\lambda, \Omega\}$ and a pro-map $g : X \to Y$. Since Ω is not cofinite, we will have to adjust it.

Claim 2. *Any commutative diagram*

$$\text{(D)}\qquad \begin{array}{ccc} X & \xrightarrow{u} & E \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle p} \\ Y & \xrightarrow{v} & B \end{array}$$

has a filler provided $E, B \in \mathrm{ANR}(m)$ and p is a Hurewicz fibration.

Proof. Switch to a commutative diagram

$$\begin{array}{ccc} X_a & \xrightarrow{u'} & E \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle p} \\ Y_\lambda & \xrightarrow{v'} & B \end{array}$$

We may assume that E does not appear as the range of maps involved in the definition of Y_λ. Let Z be the subset of $Y_\lambda \times E$ consisting of all (y,e) so that $v'(y) = p(e)$. Then Z is a closed subset of $Y_\lambda \times E$ and is an ANR (see 6.5). Therefore, one has a neighborhood V of Z in $Y_\lambda \times Q$ and a retraction $r : V \to Z$. Let $\pi : Y_\lambda \times E \to Y_\lambda$ be the projection. Since $\pi\circ r|Z = \pi|Z$, one may assume that $\pi \circ r \approx \pi|V$ rel. Z (decrease V if necessary). Notice that $\pi|Z : Z \to Y_\lambda$ is a fibration for metrizable spaces (see 6.5). We have a commutative diagram

$$\begin{array}{ccc} V\times\{0\}\cup Z\times I & \xrightarrow{a} & Z \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle \pi|Z} \\ V\times I & \xrightarrow{H} & Y_\lambda \end{array}$$

where H is a homotopy rel. Z joining $\pi \circ r$ and π, and $a(v,t) = r(v)$ for all $(v,t) \in V \times \{0\} \cup Z \times I$. Since fibrations of metrizable spaces are regular, the above diagram has a filler $G : V \times I \to Z$. Notice that $r' : V \to Z$ defined by $r'(v) = G(v,1)$ is a new retraction of V onto Z so that $\pi \circ r' = \pi|V$. Now, V defines Y_ω, $\omega > \lambda$, so that $\pi' \circ r' : Y_\omega \to E$ induces a filler of (D), where $\pi' : Y_\lambda \times E \to E$ is the projection. ∎

CLAIM 3. *Any commutative diagram*

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "g"] & \mathrm{Map}(K,P) \arrow[d, "i_*"] \\ Y \arrow[r, "v"] & \mathrm{Map}(L,P) \end{tikzcd}$$

has a filler provided $P \in$ ANR, K *is a compact metrizable space*, L *is a closed subspace of* K, *and* $i : L \to K$ *is the inclusion.*

Proof. Switch to a commutative diagram

$$\begin{tikzcd} X_a \arrow[r, "u'"] \arrow[d, "h"] & \mathrm{Map}(K,P) \arrow[d, "i_*"] \\ Y_\lambda \arrow[r, "v'"] & \mathrm{Map}(L,P) \end{tikzcd}$$

and consider $\mathrm{adj}(u') : X_a \times K \to P$ and $\mathrm{adj}(v') : Y_\lambda \times L \to P$. The union of their images is of density at most m, so by following the proof of Theorem 5 in [M-S] on p. 39 one finds $P' \in \mathrm{ANR}(m)$ containing that union so that there is a map $z : P' \to P$ which is the identity on that union. Now, we have a commutative diagram

$$\begin{tikzcd} X \arrow[r, "u''"] \arrow[d, "h"] & \mathrm{Map}(K,P') \arrow[r, "z_*"] \arrow[d, "i_*"] & \mathrm{Map}(K,P) \arrow[d, "i_*"] \\ Y \arrow[r, "v''"] & \mathrm{Map}(L,P') \arrow[r, "z_*"] & \mathrm{Map}(L,P) \end{tikzcd}$$

where $u'' : X \to \mathrm{Map}(K,P')$ is constructed as follows: $\mathrm{adj}(u') : X_a \times K \to P$ factors through P' and induces $X_a \to \mathrm{Map}(K,P')$, which followed by the inverse of j gives u''. By Claim 2, the left part of the diagram has a filler as $i_* : \mathrm{Map}(K,P') \to \mathrm{Map}(L,P')$ is a fibration for metrizable spaces and both $\mathrm{Map}(K,P')$ and $\mathrm{Map}(L,P')$ are homeomorphic to elements of $\mathrm{ANR}(m)$. ∎

Finally, we have

CLAIM 4. *If* $j : Y \to R(X)$ *is the reindexing isomorphism so that the directed set of* $R(X)$ *is cofinite*, *then* j *is an SSDR pro-map and* $j \circ g$ *is an SSDR pro-map.*

Proof. Obvious, as lifting is not affected by composing with isomorphisms. ■ $■_{4.1}$

5. The shape category Sh(pro-HoTop)**.** In this section we construct a shape system $\mathrm{sh}_X : X \to \mathrm{ShS}(X)$ for any object X of pro-HoTop.

5.1. Definition. A *shape system* of an object X of pro-HoTop is a morphism to an object of pro-HoANR which is a shape equivalence.

5.2. Theorem. *For each object X of* pro-HoTop *there is a shape system* $\mathrm{sh}_X : X \to \mathrm{ShS}(X)$.

Proof. Notice that for any topological space X the morphism $r_X : X \to R(X)$ induces a shape system of X. If $X = \{X_a, [p^b_a], A\}$ is an object of pro-HoTop, then for each pair $b \geq a$ there is a unique morphism $q^b_a : R(X_b) \to R(X_a)$ of pro-HoTop so that $q^b_a \circ r_{X_b} = r_{X_a} \circ [p^b_a]$. Thus, $\{R(X_a), q^b_a, A\}$ is an inverse system in pro-HoTop and its inverse limit gives a shape system of X. ■

5.3. Corollary. *The localization* Sh(pro-HoTop) *of* pro-HoTop *at the class of shape equivalences exists and is equivalent to* pro-HoANR.

Proof. Use 1.2. ■

6. Fibrations and cofibrations. Part of our strategy is to follow Edwards–Hastings' [Ed-H] use of closed model categories in the sense of Quillen. However, we find it easier to avoid declaring up front which morphisms of pro-Top are fibrations and which are cofibrations. For us, it is more convenient to define fibrations (or cofibrations) depending on a given family of morphisms. Our definitions apply to any category C with initial object $\emptyset$ and terminal object $*$.

The following definition from [Ed-H] is useful:

6.1. Definition. An ordered pair (i, p) of morphisms $i : A \to X$, $p : Y \to B$ has the *lifting property* if for any diagram

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "i"] & Y \arrow[d, "p"] \\ X \arrow[r, "b"] & B \end{tikzcd}$$

a filler $f : X \to Y$ exists.

6.2. Definition. Given a class Σ of morphisms of a category C, we call $i : A \to X$ a *Σ-cofibration* if (i, p) has the lifting property for all $p \in \Sigma$.

A morphism $p : Y \to B$ is called a *Σ-fibration* if (i, p) has the lifting property for all $i \in \Sigma$.

An object Y of C is called *Σ-fibrant* if the morphism $Y \to *$ is a Σ-fibration.

6.3. Examples. 1. A map $p: Y \to B$ is a Hurewicz fibration (respectively, Serre fibration) if and only if it is a Σ-fibration for Σ being the class of all inclusions $X \times 0 \to X \times I$, where X is a topological space (respectively, a CW complex).

2. A map $p: Y \to B$ is a regular Hurewicz fibration if and only if it is a Σ-fibration for Σ being the class of all inclusions $X \times 0 \cup A \times I \to X \times I$, where X is a topological space and A is a closed subset of X.

3. A metrizable space Y is an AR if it is Σ-fibrant for Σ being the class of all inclusions $A \to X$, where X is a metrizable space and A is a closed subset of X.

4. Given a closed subset A of a metrizable space X let $N(A, X)$ be the object of pro-Top consisting of all neighborhoods of A in X bonded by inclusions. One has a natural pro-map $i_{A,X}: A \to N(A, X)$. Let Σ be the class of all such pro-maps. Then:

(a) a metrizable space Y is an ANR if and only if it is Σ-fibrant,

(b) every Hurewicz fibration of ANRs $p: Y \to B$ is a Σ-fibration.

Proof. Only 4(b) needs detailed justification. Suppose

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "i_{A,X}"] & Y \arrow[d, "p"] \\ N(A,X) \arrow[r, "b"] & B \end{tikzcd}$$

is a commutative diagram. Since A is a closed subset of a metrizable space X and Y is an ANR, there is an extension $a': U \to Y$ of $a: A \to Y$ over a neighborhood U of A in X. Choose a representative $b': V \to B$ of $b: N(A, X) \to B$ and notice that $b'|A = p \circ a$. We may assume that $V \subset U$ and that $p \circ a'|V$ is homotopic to b' rel. A. Since p is a regular fibration in the class of metrizable spaces, the relative homotopy joining $p \circ a'|V$ and b' can be lifted to Y starting from $a'|V$. The end of the lifted homotopy determines a filler $N(A, X) \to Y$ of diagram (D). ■

6.4. Proposition. *Suppose*

$$\begin{tikzcd} L \arrow[r, "a"] \arrow[d, "q"] & E \arrow[d, "p"] \\ B \arrow[r, "b"] & Y \end{tikzcd}$$

is a pull-back of

$$\begin{tikzcd} & E \arrow[d, "p"] \\ B \arrow[r, "b"] & Y \end{tikzcd}$$

in C. Suppose Σ is a class of morphisms of C such that p is a Σ-fibration. Then q is a Σ-fibration, and if B is Σ-fibrant, then so is L.

Proof. Suppose

(D)
$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"] & L \arrow[d, "q"] \\ X \arrow[r, "g"] & B \end{tikzcd}$$

is commutative and $i \in \Sigma$. Then

$$\begin{tikzcd} A \arrow[r, "a\circ f"] \arrow[d, "i"] & E \arrow[d, "p"] \\ X \arrow[r, "b\circ g"] & Y \end{tikzcd}$$

is commutative and, since p is a Σ-fibration, there is a filler $u : X \to E$. Now,

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "g"] & E \arrow[d, "p"] \\ B \arrow[r, "b"] & Y \end{tikzcd}$$

is commutative and, by the property of pull-backs, there is a unique morphism $v : X \to L$ so that $g = q \circ v$ and $u = a \circ v$. To check that v is a filler of (D) it remains to show that $v \circ i = f$ in view of $g = q \circ v$. Put $f' = v \circ i : A \to L$. Since L arises from a pull-back, it is sufficient to show that $a \circ f = a \circ f'$ and $q \circ f = q \circ f'$. Now, $q \circ f' = q \circ v \circ i = g \circ i = q \circ f$ and $a \circ f' = a \circ v \circ i = u \circ i = a \circ f$.

Suppose B is Σ-fibrant and $i : A \to X$ belongs to Σ. Suppose

(D′)
$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "i"] & L \arrow[d, "\mathrm{const}"] \\ X \arrow[r, "c"] & * \end{tikzcd}$$

is commutative. Then

$$\begin{tikzcd} A \arrow[r, "q\circ a"] \arrow[d, "i"] & B \arrow[d, "\mathrm{const}"] \\ X \arrow[r, "c"] & * \end{tikzcd}$$

is commutative. Since B is Σ-fibrant, a filler $g : X \to B$ exists. Now,

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "i"] & L \arrow[d, "q"] \\ X \arrow[r, "g"] & B \end{tikzcd}$$

is commutative and, as we just proved that q is a Σ-fibration, a filler in that diagram exists which is also a filler for (D′). ∎

6.5. Corollary. *Suppose*

$$\begin{tikzcd} L \arrow[r, "a"] \arrow[d, "q"] & C \arrow[d, "p"] \\ B \arrow[r, "b"] & Y \end{tikzcd}$$

is the pull-back of

$$\begin{tikzcd} & C \arrow[d, "p"] \\ B \arrow[r, "b"] & Y \end{tikzcd}$$

in Top, *where p is a Hurewicz fibration and $B, C, Y \in$* ANR*. Then q is a Hurewicz fibration and $L \in$* ANR*.*

Proof. Let Σ be the class of all inclusions $Z \times 0 \to Z \times I$, where Z is a metrizable space. 6.3 implies that q is a Σ-fibration, i.e., a fibration for metrizable spaces. Let Σ' be the class of all pro-maps $i_{A,X} : A \to N(A, X)$, where A is a closed subset of a metrizable space X. Examples 6.3.4(a),(b) say that p is a Σ'-fibration and B is Σ'-fibrant. Thus, L is Σ'-fibrant, which means that $L \in$ ANR by 6.3.4(a). ∎

Suppose D is a diagram in a category C. Given a vertex a of D, by $D(a)$ we denote the object of C at that vertex. Given two vertices a and b of D, $b < a$ means that there is an arrow from a to b in D. The morphism corresponding to that arrow is denoted by $D(a, b)$.

A *cone* over D is a diagram D' containing D with one additional vertex v so that there is a unique arrow from v to every vertex of D. The inverse limit $\lim(D)$ of D is a terminal cone over D. By abusing notation, the object at the initial vertex of $\lim(D)$ will also be denoted by $\lim(D)$.

A subdiagram D' of D is called *full* provided every arrow of D connecting two vertices of D' is present in D'.

By the *cardinality* of a finite diagram D we mean the number of its vertices.

6.6. Proposition. *Suppose C is a full subcategory of C', Σ is a class of morphisms of C', and every finite diagram in C has an inverse limit.*

Suppose D is a finite diagram in C so that no two vertices of D are connected by more than one arrow. Define the following statements for $n \geq 1$:

$P(n)$: *Given two full subdiagrams $E \subset F$ of D of cardinality at most n, the projection* $\lim F \to \lim E$ *is a Σ-fibration.*

$Q(n)$: *The projection* $D(a) \to \lim_{b<a} D(b)$ *is a Σ-fibration for all a such that $\{b \mid b < a\}$ has cardinality at most n.*

$R(n)$: $\lim E$ *is Σ-fibrant for all full subdiagrams E of D of cardinality at most n.*

The following implications hold for all $n \geq 1$:

(a) $(P(n)$ *and* $Q(n)) \Rightarrow P(n+1)$,
(b) $(P(n), Q(n)$ *and* $R(n)) \Rightarrow R(n+1)$.

Proof. (a) Suppose $E \subset F$ are full subdiagrams of D so that $\operatorname{card}(F) \leq n+1$. Choose $a \in F - E$ (if $F = E$, then there is no work needed) so that there is no $b \in F$ with $b > a$. Let $G = \{b \mid b < a\}$ and $F' = F - \{a\}$. Notice that the pull-back of

$$\text{(D)} \qquad \begin{array}{ccc} & & \lim F' \\ & & \downarrow \\ D(a) & \longrightarrow & \lim G \end{array}$$

is $\lim F$. Since $D(a) \to \lim G$ is a Σ-fibration, we infer that $\lim F \to \lim F'$ is a Σ-fibration and the composition $\lim F \to \lim F' \to \lim E$ is a Σ-fibration.

(b) Suppose F is a full subdiagram of D so that $\operatorname{card}(F) \leq n+1$. Choose $a \in F$ such that there is no $b \in F$ with $b > a$. Define G and F' as in the proof of (a). Again, the pull-back of (D) is $\lim F$. By Proposition 6.4, $\lim F$ is Σ-fibrant. ∎

6.7. Theorem. *Suppose C is a category with inverse limits of finite diagrams and Σ is a class of morphisms of* pro-C. *Suppose $Y = \{Y_a, p_a^b, A\}$ is an object of* pro-C *such that A is cofinite and the projection* $Y_a \to \lim_{b<a} Y_b$ *is a Σ-fibration for all $a \in A$. Let Σ_C be the class of all morphisms $f : X \to Z$ of Σ such that Z is an object of C. Then*:

(a) *If Y_a is Σ-fibrant for all $a \in A$, then Y is Σ-fibrant.*

(b) *If Y_a is Σ_C-fibrant for all $a \in A$ and C has inverse limits, then* $\lim Y$ *is Σ_C-fibrant.*

Proof. Suppose $i : B \to X$ is a morphism of Σ and $g : B \to Y$ is a morphism of pro-C. Given $a \in A$ let $n(a)$ be the cardinality of $\{b \mid b < a\}$. Let us construct, by induction on $n(a)$, morphisms $h_a : X \to Y_a$ such that $h_a \circ i = p_a \circ g$ and $p_a^b \circ h_b = h_a$ for $a < b$, where $p_a : Y \to Y_a$ is the projection.

The morphisms h_a generate $h : X \to Y$ so that $h \circ i = g$, which proves that Y is Σ-fibrant.

If $n(a) = 0$, then we choose any $h_a : X \to Y_a$ so that $h_a \circ i = p_a \circ g$. The existence of h_a is guaranteed by the fact that Y_a is Σ-fibrant. Suppose h_a exists for all a with $n(a) \le n$. Given $a \in A$ with $n(a) = n + 1$ one has a morphism v from X to $\lim_{b<a} Y_b$ so that

$$\begin{tikzcd} B \arrow[r, "p_a \circ g"] \arrow[d, "i"] & Y_a \arrow[d, "p"] \\ X \arrow[r, "v"] & \lim_{b<a} Y_b \end{tikzcd}$$

is commutative. The filler of that diagram is chosen as h_a.

Suppose X is an object of C, $i : B \to X$ belongs to Σ, and $g : B \to \lim Y$. By the same construction as above, there is $h : X \to Y$ so that $h \circ i = p \circ g$, where $p : \lim Y \to Y$ is the projection morphism. Then h induces $\lim h : X \to \lim Y$ so that $(\lim h) \circ i = g$. ∎

6.8. PROPOSITION. *Given a map $f : X \to Y$ of metrizable ANRs there is a metrizable ANR Z containing X as a strong deformation retract and an extension $f' : Z \to Y$ of f so that f' is an SSDR-fibration.*

Proof. This follows the standard way of replacing a map by a Hurewicz fibration (see [Sp], Theorem 9 on p. 99). Define Z as pairs (ω, x), where $x \in X$ and ω is a path in Y starting at $f(x)$. In other words, Z is the pull-back of

$$\begin{tikzcd} & \mathrm{Map}(I, Y) \arrow[d, "p"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

where $p(\omega) = \omega(0)$. By 6.4, Z is an ANR and f' is an SSDR-fibration as p is an SSDR-fibration (that follows from the definition of SSDR pro-maps). As in [Sp], Theorem 9 on p. 99, X is a strong deformation retract of Z. ∎

6.9. THEOREM. *Suppose $X = \{X_a, p^b_a, A\}$ is a pro-ANR and A is cofinite. There is a pro-ANR $Y = \{Y_a, p^b_a, A\}$ and a level pro-map $f = \{f_a\}_{a \in A}$ from X to Y such that f is an SSDR pro-map and Y is SSDR-fibrant.*

Proof. Given $a \in A$ let $n(a)$ be the cardinality of $\{b \mid b < a\}$. Let us construct, by induction on $n(a)$, an ANR-space Y_a, inclusion $f_a : X_a \to Y_a$, and maps $q^a_b : Y_a \to Y_b$, $b < a$, such that $f_b \circ p^a_b = q^a_b \circ f_a$ for $b < a$.

If $n(a) = 0$, then we put $Y_a = X_a$ and $f_a = \mathrm{id}$. Suppose the objects are constructed for all a with $n(a) \le n$ so that $P(n-1)$, $Q(n-1)$, and $R(n-1)$

of 6.6 are satisfied. Given $a \in A$ with $n(a) = n+1$, $\lim_{b<a} Y_b$ is an ANR by 6.5–6.6 and we replace the map $p_a : X_a \to \lim_{b<a} Y_b$ by an inclusion $f_a : X_a \to Y_a$ so that p_a extends to an SSDR-fibration $q_a : Y_a \to \lim_{b<a} Y_b$. Then $q_c^a : Y_a \to Y_c$, $c < a$, is defined as the composition of SSDR-fibrations $Y_a \to \lim_{b<a} Y_b \to Y_c$. This way, we ensure $Q(n)$ of 6.6, which allows us to continue the induction process. 6.7 verifies that Y is SSDR-fibrant, and f is clearly an SSDR pro-map. ■

7. Strong shape category sSh(pro-Top)**.** Let π(pro-Top) be the basic homotopy category of pro-Top. Its classes of morphisms will be denoted by $[X,Y]$. Let SSDR-FIBRANT be the full subcategory of π(pro-Top) whose objects are SSDR-fibrant.

7.1. THEOREM. *For each pro-space X there is an SSDR pro-map $s_X : X \to \mathrm{sShS}(X)$ such that* sShS(X) *is SSDR-fibrant and is an object of* pro-ANR.

Proof. Use 4.1 and 6.9. ■

7.2. DEFINITION. The morphism constructed above is called *the strong shape system of X*. Any strong shape equivalence $f : X \to Z$ such that Z is an SSDR-fibrant pro-ANR is called *a strong shape system of X*.

7.3. THEOREM. *A pro-map $f : X \to Y$ is a strong shape equivalence if and only if $f^* : [Y,Z] \to [X,Z]$ is a bijection for all Z which are SSDR-fibrant.*

Proof. It suffices to consider f which is a level morphism. Let $i : X \to M(f)$ be the inclusion pro-map. Suppose f is a strong shape equivalence. By 3.9, i is an SSDR pro-map and, given $g : X \to Z$ with Z being SSDR-fibrant, there is $h : M(f) \to Y$ with $h|X = g$. Thus $f^* : [Y,Z] \to [X,Z]$ is a surjection for all strong shape equivalences f and all Z which are SSDR-fibrant. Since $DM(f) \to M(f) \times I$ is an SSDR pro-map if f is a strong shape equivalence (see 3.6 and 3.9), we see that $f^* : [Y,Z] \to [X,Z]$ is a injection. Suppose $f^* : [Y,Z] \to [X,Z]$ is a bijection for all Z which are SSDR-fibrant. That implies the existence of a commutative diagram in π(pro-Top)

$$\begin{array}{ccc} X & \xrightarrow{s_X} & \mathrm{sShS}(X) \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} \\ Y & \xrightarrow{s_Y} & \mathrm{sShS}(Y) \end{array}$$

in which f' is a homotopy equivalence. From this one concludes easily that f is a strong shape equivalence. ■

7.4. COROLLARY. (a) *There is a functor* π(pro-Top) $\to$ SSDR-FIBRANT *which is left-adjoint to the inclusion* SSDR-FIBRANT $\to \pi$(pro-Top).

(b) *The localization* sSh(pro-Top) *of* π(pro-Top) *at strong shape equivalences exists and is equivalent to* SSDR-FIBRANT.

Proof. Use 2.1 and 7.3. ∎

REMARK. A. V. Prasolov [P] defines the strong shape category as π(pro-Top) localized at strong shape equivalences. His definition requires the usage of the Axiom of Universe plus the usual ZFC. Corollary 7.4 avoids a deeper understanding of set theory and shows that the strong shape theory constructed in this paper is equivalent to that of A. V. Prasolov.

7.5. PROPOSITION. *If X is a space and Z is a pro-space which is SSDR-fibrant, then the natural functions $[X, \lim Z] \to [X, Z]$, $[X, Z] \to \mathrm{Mor}_{\mathrm{sSh}}(X, Z)$ are bijections.*

Proof. $[X, \lim Z] \sim [X, Z]$ holds for all pro-spaces Z and all spaces X. Suppose Z is SSDR-fibrant and $f, g : X \to Z$ are equal in sSh(pro-Top). That means $i_Z \circ f \sim i_Z \circ g$, where $i_Z : Z \to \mathrm{sShS}(Z)$ is the strong shape system of Z. Notice that i_Z is a homotopy equivalence (see 7.3), which proves $[X, Z] \sim \mathrm{Mor}_{\mathrm{sSh}}(X, Z)$. ∎

7.6. THEOREM. *Suppose $f : X \to Y$ is a map of topological spaces.*

(a) *If $f^* : [Y, \lim Z] \to [X, \lim Z]$ is a bijection for all SSDR-fibrant pro-spaces Z, then f is a strong shape equivalence.*

(b) *f is a strong shape equivalence if and only if $f^* : [Y, Z] \to [X, Z]$ is a bijection for all spaces Z which are $SSDR_{\mathrm{Top}}$-fibrant.*

Proof. (a) Follows from 7.5 and 7.3 as $[A, \lim Z] \sim [A, Z]$.

(b) Suppose f is a strong shape equivalence. We may assume that f is an SSDR map, which implies that $[Y, Z] \to [X, Z]$ is epi. Since $DM(f) \to M(f) \times I$ is an SSDR map, $[Y, Z] \to [X, Z]$ is mono. ∎

7.7. THEOREM. *Suppose $G : X \to Y$ is a morphism of* sSh(pro-Top) *and X, Y are topological spaces. Given two strong shape systems $s_X : X \to F(X)$ and $s_Y : Y \to F(Y)$, there is a homotopy class $f : \lim F(X) \to \lim F(Y)$ and a pro-map $G' : F(X) \to F(Y)$ so that*

$$\begin{tikzcd}
X \arrow[r, "G"] \arrow[d, "\lim s_X"] & Y \arrow[d, "\lim s_Y"] \\
\lim F(X) \arrow[r, "f"] \arrow[d, "j_X"] & \lim F(Y) \arrow[d, "j_Y"] \\
F(X) \arrow[r, "G'"] & F(Y)
\end{tikzcd}$$

is commutative in sSh(pro-Top). *If*

$$\begin{tikzcd} X \arrow[r, "G"] \arrow[d, "\lim s_X"] & Y \arrow[d, "\lim s_Y"] \\ \lim F(X) \arrow[r, "f'"] \arrow[d, "j_X"] & \lim F(Y) \arrow[d, "j_Y"] \\ F(X) \arrow[r, "G''"] & F(Y) \end{tikzcd}$$

is another commutative diagram in sSh(pro-Top), *then* $f = f'$ *and* $G' \approx G''$.

Proof. By 7.3 and 7.5, $s_Y \circ G$ is generated by a unique $G' : F(X) \to F(Y)$ with $G' \circ s_X = s_Y \circ G$. In particular, $G'' \approx G'$. There is a unique $f : \lim F(X) \to \lim F(Y)$ with $G' \circ j_X = j_Y \circ f$. ■

7.8. Corollary. *Given two strong shape systems* $s_1 : X \to Y$ *and* $s_2 : X \to Z$ *of a topological space* X, *there is a homotopy equivalence* $f : \lim Y \to \lim Z$ *so that*

$$\begin{tikzcd} X \arrow[r, "\mathrm{id}_X"] \arrow[d, "\lim s_1"] & X \arrow[d, "\lim s_2"] \\ \lim Y \arrow[r, "f"] & \lim Z \end{tikzcd}$$

is commutative in HoTop.

Proof. Apply 7.7. ■

7.9. Corollary. *If* X *is a compact metrizable space and* $s : X \to F(X)$ *is a strong shape system of* X, *then* $\lim s : X \to \lim F(X)$ *is a strong shape equivalence.*

Proof. Embed X in the Hilbert cube Q and consider the inclusion $X \to N(X, Q)$ which is an SSDR pro-map by 6.3.4(b). One may choose a cofinal subsequence X_n of $N(X, Q)$ consisting of closed neighborhoods of X which are ANRs. Apply 6.9 to $\{X_n\}$ and obtain an SSDR-fibrant sequence $\{X'_n\}$ with bonding maps being Hurewicz fibrations. F. Cathey [C_2] proved that the inclusion $X \to \lim\{X'_n\}$ is an SSDR map. Use 7.8 to see that $\lim s : X \to \lim F(X)$ is a strong shape equivalence. ■

Notice that 7.5 and 7.9 say that every compactum is strong shape equivalent to a space Z so that $[Y, Z] \to \mathrm{Mor}_{\mathrm{sSh}}(Y, Z)$ is a bijection for every space Y. This can be extended to compact Hausdorff spaces as in [Gu_2]. It would be interesting to find out if that property holds for separable metric spaces (or for general metrizable spaces) (see [I-U] for related material).

7.10. Corollary. *There is a functor* $\lim_{\mathrm{sSh}} : \mathrm{sSh(Top)} \to \mathrm{HoTop}$ *so that* $\lim_{\mathrm{sSh}}(X) = \lim \mathrm{sShS}(X)$ *for each* X. *In particular, the singular homology groups of* $\lim_{\mathrm{sSh}}(X)$ *are a strong shape invariant of* X.

Proof. Use 7.7. ∎

7.11. Theorem. *For each topological space* X *there is a CW complex* X_{sSh} *and a strong shape morphism* $j_X : X_{\mathrm{sSh}} \to X$ *such that* $(j_X)_* : \mathrm{Mor}_{\mathrm{sSh}}(K, X_{\mathrm{sSh}}) \to \mathrm{Mor}_{\mathrm{sSh}}(K, X)$ *is a bijection for any CW complex* K.

Proof. It is the singular complex of the limit of the strong shape system of X. ∎

Remark. As in the case of singular homology of topological spaces, the cellular homology of X_{sSh} should be the strong homology of X (see 8.3).

7.12. Theorem. *Let* FIB *be the class of all topological spaces* X *such that there is a pro-map* $s : X \to Z$ *so that* $\lim s : X \to \lim Z$ *is a strong shape equivalence and* Z *is SSDR-fibrant. The localization* sSh(FIB) *of* Ho(FIB) *at strong shape equivalences exists and is equivalent to the full subcategory of* sSh(pro-Top) *whose objects are those in* FIB.

Proof. Consider the class F of all topological spaces Y equal to $\lim Y'$ for some pro-space Y' which is SSDR-fibrant. If $s_Y : Y \to s(Y)$ is the strong shape system of Y, then it is an SSDR pro-map (see 4.1) and there is $r : s(Y) \to Y'$ with $r \circ s_Y = \pi$, where $\pi : Y \to Y'$ is the projection. Any strong shape morphism $f : X \to Y$ from a topological space to Y can be identified with the unique homotopy class $f' : s(X) \to s(Y)$. Composing a representative $f'' : X \to s(Y)$ of $f' \circ [s_X]$ with r generates $X \to Y'$, and by passing to the inverse limit one gets a map $g : X \to Y$. That map is unique up to homotopy as it is unique up to homotopy in Y'. We can sum up as follows: the full subcategory $\mathrm{sSh}(F)$ of sSh(pro-Top) whose objects are in F is naturally isomorphic to $\mathrm{Ho}(F)$. Notice that sSh(FIB) is isomorphic to $\mathrm{sSh}(F)$ and any functor $\phi : \mathrm{Ho(FIB)} \to C$ inverting all strong shape equivalences factors through $\mathrm{sSh}(F) = \mathrm{Ho}(F)$. ∎

7.13. Problems. (a) *Characterize spaces* X *such that there is a pro-map* $s : X \to Z$ *so that* $\lim s : X \to \lim Z$ *is a strong shape equivalence and* Z *is SSDR-fibrant.*

(b) *Characterize spaces* X *such that there is a strong shape system* $s : X \to Z$ *of* X *so that* $\lim s : X \to \lim Z$ *is a strong shape equivalence.*

(c) *Characterize spaces* X *such that there is a strong shape equivalence* $s : X \to Z$ *of spaces so that* Z *is* $\mathit{SSDR}_{\mathrm{Top}}$*-fibrant.*

FIB in 7.12 contains all compacta and all CW complexes. Does it contain metrizable spaces? Does it contain compact Hausdorff spaces?

As we mentioned before, F. Cathey introduced SSDR inclusions of metrizable spaces in a way equivalent to the following statement (see [C_2], Theorem 1.2): a closed inclusion $f : X \to Y$ of metrizable spaces is an SSDR map if any commutative diagram

$$\begin{tikzcd} X \arrow[r, "a"] \arrow[d, "f"] & E \arrow[d, "p"] \\ Y \arrow[r, "b"] & B \end{tikzcd}$$

has a filler provided $p : E \to B$ is a Hurewicz fibration of ANRs.

Let us show that our definition of SSDR pro-maps extends Cathey's concept of SSDR inclusions.

7.14. Proposition. *Suppose* $i = \{i_a\}_{a\in A} : X = \{X_a, p_a^b, A\} \to Y = \{Y_d, q_a^b, A\}$ *is a level pro-inclusion such that* X_a *is a closed subset of* Y_a *for each* $a \in A$. *If* i *is an SSDR pro-map*, *then every commutative diagram*

$$\begin{tikzcd} X \arrow[r, "a"] \arrow[d, "i"] & E \arrow[d, "p"] \\ Y \arrow[r, "b"] & B \end{tikzcd}$$

has a filler if p *is a Hurewicz fibration of ANRs.*

Proof. We will show the details in the case of A being a one-point set (i.e., X is a closed subset of a space Y). The general case is similar. Pick a map $g : Y \to E$ so that $a = g|X$. Now, $p \circ g|X = p \circ b|X$ and there is a homotopy $H : Y \times I \to B$ joining $g \circ p$ and b so that $H|X \times I$ is the constant homotopy joining $p \circ a$ to itself. Since $p : E \to B$ is a regular fibration, there is a lift $H' : Y \times I \to E$ of H starting at g so that $H'|X \times I$ is the constant homotopy joining a to itself. The map $g' : Y \to E$ defined by $g'(y) = H'(y, 1)$ for $y \in Y$ is a filler of the diagram. ■

Ideally, a good way to generalize Cathey's SSDR inclusions to SSDR pro-maps would be to require that they have the property stated in 7.14. However, the following problem remains open.

7.15. Problem. *Suppose* X *is a pro-space. Is there a pro-map* $r : X \to Y$ *so that* Y *is a pro-ANR and every commutative diagram*

$$\begin{tikzcd} X \arrow[r, "a"] \arrow[d, "r"] & E \arrow[d, "p"] \\ Y \arrow[r, "b"] & B \end{tikzcd}$$

has a filler $Y \to E$ *if* p *is a Hurewicz fibration of ANRs?*

A related problem is to identify Hurewicz fibrations of ANRs which are SSDR-fibrations.

7.16. Problem. *Characterize all Hurewicz fibrations $p : E \to B$ of ANRs so that every commutative diagram*

$$\begin{tikzcd} X \arrow[r, "a"] \arrow[d, "r"] & E \arrow[d, "p"] \\ Y \arrow[r, "b"] & B \end{tikzcd}$$

has a filler $Y \to E$ if r is an SSDR pro-map.

The authors do not know of any example of a Hurewicz fibrations of ANRs which is not an SSDR-fibration.

Let us concentrate on SSDR maps of topological spaces.

7.17. Proposition. *Suppose $f : X \to Y$ is an SSDR map of topological spaces.*

(a) *If X is functionally Hausdorff, then f is one-to-one.*

(b) *If X is Tikhonov, then f is an embedding.*

(c) *If X is paracompact and Y is Hausdorff, then f is a closed embedding.*

Proof. (a) X being functionally Hausdorff means that for any $x \neq y$ in X there is a map $g : X \to I$ so that $g(x) \neq g(y)$. Since g factors as $h \circ f$ for some $h : Y \to I$, we have $f(x) \neq f(y)$.

(b) X being Tikhonov means that $\{g^{-1}(U) \mid g : X \to I, U = \mathrm{Int}(U) \subset I\}$ forms a basis of X and X is functionally Hausdorff. Since every g factors as $h \circ f$ for some $h : Y \to I$, we have $f(x) \neq f(y)$, $\{f^{-1}(V) \mid V = \mathrm{Int}(V) \subset Y\}$ forms a basis of X and f is an embedding.

(c) Since f is an embedding, we may assume $X \subset Y$ and $f = i$ is the inclusion. Suppose $y_0 \in Y - X$ belongs to the closure of X in Y. For each $x \in X$ choose a pair of disjoint open sets U_x and V_x in Y so that $x \in U_x$ and $y_0 \in V_x$. Choose a partition of unity on X subordinate to the covering $\{U_x\}_{x\in X}$ of X. As seen in [D_3], that partition of unity can be viewed as a map $\pi : X \to K$, where K is the full simplicial complex with vertices $\{v_x\}_{x\in X}$, so that the point-inverse of the star of vertex v_x is contained in U_x. Extend π over Y and assume $\pi(y_0) = \sum c_x \cdot v_x$. Since $\sum c_x = 1$, there is $z \in X$ so that $c_z \neq 0$. Since y_0 is in the closure of X, there is $x' \in X \cap V_z$ with $\pi(x') = \sum d_x \cdot v_x$ so that $d_z \neq 0$. That implies $x' \in U_z$, contradicting $x' \in V_z$. ■

7.17 and 7.14 imply that our definition of SSDR maps extends Cathey's definition of SSDR inclusions of metrizable spaces.

Let us point out differences between fibrant spaces as defined by Cathey [C_2] and SSDR-fibrant spaces. Recall that E is fibrant in the sense of Cathey if, for any SSDR inclusion $i : A \to X$ of metrizable spaces, any map $g : A \to E$ extends over X. Clearly, these are our $\mathrm{SSDR}_{\mathrm{Top}}$-fibrant spaces. As is pointed out in [Gu_2], all Hilbert cubes are fibrant (see also 7.19 here and use Q discrete, $P = I$). See [I-U] for examples related to fibrant spaces.

7.18. PROPOSITION. *If a space E is SSDR-fibrant, then E is a metrizable ANR.*

Proof. Pick an SSDR pro-map $r : E \to s(E)$ so that $s(E)$ is a pro-ANR (see 4.1). Since E is SSDR-fibrant, there is a pro-map $t : s(E) \to E$ so that $s \circ r = \mathrm{id}_E$ and it is clear E is a retract of a term of $s(E)$. ■

7.19. PROPOSITION. *If $P \in$ ANR and Q is a CW complex, then $\mathrm{Map}(Q, P)$ is $\mathrm{SSDR}_{\mathrm{Top}}$-fibrant.*

Proof. If Q is finite, then $\mathrm{Map}(Q, P)$ is SSDR-fibrant. Consider all finite subcomplexes K of Q ordered by inclusion. We form the inverse system $\{\mathrm{Map}(K, P) \mid K \subset Q\}$ whose inverse limit is $\mathrm{Map}(Q, P)$. If we prove that the inverse system satisfies the conditions of 6.7(b), we are done. Given a finite subcomplex K of Q, consider the union L of all proper subcomplexes of K. Notice that $\lim_{K'<K} \mathrm{Map}(K', P) = \mathrm{Map}(L, P)$ and the projection $\mathrm{Map}(K, P) \to \lim_{K'<K} \mathrm{Map}(K', P)$ is the restriction operator $f \mapsto f|L$, which is an SSDR-fibration. ■

We can improve 3.10 as follows.

7.20. THEOREM. *Suppose $f : X \to Y$ is a map of topological spaces.*

(a) *f is a shape equivalence if and only if the induced function $f^* : [Y, M] \to [X, M]$ is a bijection for all $M = \mathrm{Map}(Q, P)$, where $P \in$ ANR and Q is a finite CW complex.*

(b) *If f is a strong shape equivalence, then the induced function $f^* : [Y, M] \to [X, M]$ is a bijection for all $M = \mathrm{Map}(Q, P)$, where $P \in$ ANR and Q is an arbitrary CW complex.*

(c) *If X, Y are k-spaces and the induced function $f^* : [Y, M] \to [X, M]$ is a bijection for all $M = \mathrm{Map}(Q, P)$, where $P \in$ ANR and Q is an arbitrary CW complex, then f is a strong shape equivalence.*

Proof. (a) is obvious.

(b) Suppose f is a strong shape equivalence. By 3.7–3.9 both $i : X \to M(f)$ and $j : DM(f) \to M(f) \times I$ are SSDR maps. Since $M = \mathrm{Map}(Q, P)$ is $\mathrm{SSDR}_{\mathrm{Top}}$-fibrant (see 7.19), any map $g : X \to M$ extends over $M(f)$, which proves that $f^* : [Y, M] \to [X, M]$ is onto. Given two maps $g, h : Y \to M$ and a homotopy $H : X \times I \to M$ joining $g \circ f$ and $h \circ f$ one forms $G : DM(f) \to M$

which is extendible over $M(f)\times I$. The restriction of that extension to $Y\times I$ provides a homotopy joining g and h.

(c) Suppose the induced function $f^*:[Y,M]\to[X,M]$ is a bijection for all $M=\mathrm{Map}(Q,P)$, where $P\in\mathrm{ANR}$ and Q is an arbitrary CW complex. One can easily see that $f\times\mathrm{id}_Q:X\times_k Q\to Y\times_k Q$ is a shape equivalence for all CW complexes Q. By 3.10, f is a strong shape equivalence. ∎

7.20(c) fails if spaces are replaced by towers of spaces (see 3.12), which means that one cannot introduce strong shape category of k-spaces as shape category with ANRs being replaced by $\mathrm{Map}(Q,P)$, where $P\in\mathrm{ANR}$ and Q is an arbitrary CW complex.

By dualizing 7.20 one easily gets the following.

7.21. Corollary. *Suppose $f:X\to Y$ is a map of k-spaces.*

(a) *f is a shape equivalence if and only if the induced function $f^*:[Q,\mathrm{Map}(Y,P)]\to[Q,\mathrm{Map}(X,P)]$ is a bijection for all finite CW complexes Q and all $P\in\mathrm{ANR}$.*

(b) *f is a strong shape equivalence if and only if the induced function $f^*:[Q,\mathrm{Map}(Y,P)]\to[Q,\mathrm{Map}(X,P)]$ is a bijection for all CW complexes Q and all $P\in\mathrm{ANR}$.*

We do not know of any map f satisfying (a) and not satisfying (b) in 7.21. See [C-R] for results on detecting weak homotopy equivalences between function spaces $\mathrm{Map}(X,P)$ and $\mathrm{Map}(Y,Q)$, where X and Y are CW complexes.

8. Explaining and correcting errors in [D-N]. Much of the motivation for this paper came from the desire to correct errors in [D-N] which were first noticed by A. V. Prasolov. Namely, Theorem 4.6 of [D-N] has errors in its proof. Notice that 4.1 and 6.9 of the present paper can be used to give a different (and correct) proof of 4.6 of [D-N]. However, the authors' perspective on strong shape theory has changed since then, so this section of the paper is devoted to proofs of those results in [D-N] which depend on Theorem 4.6. First, let us state that all results in [D-N] prior to 4.6 are correct. The basic new idea of [D-N] was to introduce strong shape equivalences first and use them to construct the strong shape category. 4.6 of [D-N] attempted to solve 7.15 in the case of E,B having density bounded by a fixed cardinal number m. The construction in 4.6 of [D-N] is too abstract to be true, and 4.1 in the present paper provides a correct way to solve 7.15 in the case of E,B having density bounded by a fixed cardinal number m. Problem 7.15 is still of interest. However, the authors now believe that the right framework for the strong shape category is based on function spaces, and the fibrations $p:E\to B$ which matter are of the form

$i^* : \mathrm{Map}(K, P) \to \mathrm{Map}(L, P)$, where $P \in \mathrm{ANR}$, K is a finite CW complex, L is a subcomplex of K, and $i : L \to K$ is the inclusion.

Let us prove two most relevant results of Section 5 of [D-N].

8.1. THEOREM (5.13 of [D-N]). *Suppose X and Y are non-discrete, shape equivalent compacta. If $P \in \mathrm{ANR}$ is separable with no isolated points, then $\mathrm{Map}(X, P)$ and $\mathrm{Map}(Y, P)$ are homeomorphic.*

Proof. By [D-S$_2$] both X and Y are contained in a compactum Z so that the inclusions $X \to Z$ and $Y \to Z$ are strong shape equivalences. It suffices to prove that $\mathrm{Map}(X, P)$ is homeomorphic to $\mathrm{Map}(Z, P)$. By 2.2, they are homotopy equivalent. Sakai [Sak] showed that they are l_2-manifolds, and [B-P] (p. 316) shows they are homeomorphic. ∎

8.2. THEOREM (5.15 of [D-N]). *Suppose $i : X \to Y$ is an inclusion of compacta, $x_0 \in X$, and $k \geq 1$ is an integer so that $i^* : \mathrm{Map}((Y, x_0), (S^k, s)) \to \mathrm{Map}((X, x_0), (S^k, s))$ is a homotopy equivalence. Then i is a strong shape equivalence in the following cases*:

(a) *Y is a subset of the plane and $k \geq 2$.*

(b) *$k > 2\dim(Y)$, and both X and Y are shape 1-connected.*

Proof. Since $H_q(\mathrm{Map}((A, x_0), (S^k, s)))$ is naturally isomorphic to the reduced cohomology group $H^{k-q}(A)$ (see [Mo]), $i^* : H^p(Y) \to H^p(X)$ is an isomorphism for $p \leq 1$ in case (a), and for $p \leq \dim(Y)$ in case (b). Now, i is a shape equivalence; use [Bo], p. 221, in case (a), and the cohomological version of the Whitehead Theorem in [M-S] on p. 155 in case (b). Theorem 1.13 of [D-N] says that i is a strong shape equivalence. ∎

Theorems 5.10 and 5.12 of [D-N] seem to require a more involved treatment and the authors plan to do that in another paper. Section 6 of [D-N] deals mostly with strong homology groups. The most relevant issue now is the following.

8.3. CONJECTURE. *Suppose $f : X \to Y$ is a strong shape morphism of topological spaces. If $f_* : \mathrm{Mor}_{\mathrm{sSh}}(L, X) \to \mathrm{Mor}_{\mathrm{sSh}}(L, Y)$ is a bijection for all CW complexes L, then f induces isomorphisms of strong homology groups of X and Y.*

If 8.3 is verified, it would complete the analogy between homotopy and strong shape. Notice that 6.7 of [D-N] says that any shape equivalence $f : X \to Y$ of paracompact spaces induces isomorphisms of strong shape groups. 8.3 seems to be related to that result. For properties of strong homology groups see [L-M].

9. Appendix. For the convenience of the reader, we provide detailed proofs of some results which may be difficult to locate in the existing literature.

9.1. PROPOSITION. *Any unpointed weak homotopy equivalence $p: E \to B$ induces a pointed weak homotopy equivalence $p : (E,e) \to (B,p(e))$ for all $e \in E$.*

Proof. Clearly, $\pi_n(p) : \pi_n(E,e) \to \pi_n(B,p(e))$ is a monomorphism for each $n \geq 1$. We need to show that $\pi_n(p) : \pi_n(E,e) \to \pi_n(B,p(e))$ is an epimorphism for each $n \geq 1$. First, consider $n = 1$. Consider the wedge $\bigvee_{\alpha\in\pi_1(B,p(e))} S_\alpha$, where each S_α is a copy of the unit circle. There is a canonical map $q : \bigvee_{\alpha\in\pi_1(B,p(e))} S_\alpha \to B$ sending each S_α to a representative of α. Pick a map $q' : \bigvee_{\alpha\in\pi_1(B,p(e))} S_\alpha \to E$ so that $p \circ q' \approx q$. Pick a homotopy H joining $p \circ q'$ and q. Notice that $H(1,t)$, $0 \leq t \leq 1$, defines $\beta \in \pi_1(B,p(e))$. By looking at $H|S_\beta$ one sees that $p \circ q'|S_\beta$ is homotopic to β relative the base point. Find $q'' : \bigvee_{\alpha\in\pi_1(B,p(e))} S_\alpha \to E$ so that there is a homotopy from q'' to q' which moves the base point along the inverse of $q'|S_\beta$. Now, $p \circ q''$ is homotopic to q relative the base point, which proves that $\pi_1(p) : \pi_1(E,e) \to \pi_1(B,p(e))$ is an epimorphism.

For $n > 1$ the proof relies on the case for fundamental groups as follows: Consider the wedge $\bigvee_{\alpha\in\pi_n(B,p(e))} S^n_\alpha$, where each S^n_α is a copy of the n-sphere. There is a canonical map $q : \bigvee_{\alpha\in\pi_n(B,p(e))} S^n_\alpha \to B$ sending each S^n_α to a representative of α. Pick a map $q' : \bigvee_{\alpha\in\pi_n(B,p(e))} S^n_\alpha \to E$ so that $p \circ q' \approx q$. Pick a homotopy H joining $p \circ q'$ and q. Notice that $H(1,t)$, $0 \leq t \leq 1$, defines $\beta \in \pi_1((B,p(e))$. Pick $\gamma \in \pi_1(E,e)$ so that $p_*(\gamma) = \beta$. Find $q'' : \bigvee_{\alpha\in\pi_n(B,p(e))} S^n_\alpha \to E$ so that there is a homotopy from q'' to q' which moves the base point along the inverse of γ. Now, $p\circ q''$ is homotopic to q relative the base point, which proves that $\pi_n(p) : \pi_n(E,e) \to \pi_n(B,p(e))$ is an epimorphism. ■

The following result is well known in the pointed category. We need it in the unpointed category.

9.2. PROPOSITION. *Suppose $p : E \to B$ is a Serre fibration. The following conditions are equivalent*:

(a) *p is a weak homotopy equivalence.*
(b) *The homotopy groups of each fiber of p are trivial.*
(c) *p is a trivial Serre fibration.*

Proof. (a)⇒(b). By 9.1, p induces a pointed weak homotopy equivalence $p : (E,e) \to (B,p(e))$ for each $e \in E$. From the homotopy exact sequence of p one sees that each fiber is weak homotopy equivalent to a point.

(b)⇒(c). Suppose

(D)
$$\begin{tikzcd} S^n \arrow[r, "f"] \arrow[d, "j"] & E \arrow[d, "p"] \\ I^{n+1} \arrow[r, "g"] & B \end{tikzcd}$$

is a commutative diagram where j is the inclusion. We need to show that (D) has a filler f', i.e., an extension $f' : I^{n+1} \to E$ of f so that $p \circ f' = g$. First, consider a special case of S^n having a collar C in I^{n+1} so that $g|I^{n+1} - \operatorname{Int} C$ is constant. Then $g|C$ can be lifted to E so that the lift $g' : C \to E$ is an extension of f. Let S be the component of ∂C different from S^n. Notice that $g'(S)$ is contained in a fiber of p. Since the homotopy groups of fibers are trivial, g' can be extended over I^{n+1} so that the extension f' is a lift of g. In the general case, we homotope g rel. S^n to h so that $h|I^{n+1} - \operatorname{Int} C$ is constant for some collar C of S^n. Let $h' : I^{n+1} \to E$ be a lift of h so that $h'|S^n = f$. There is a homotopy $H : I^{n+1} \times I \to B$ rel. S^n joining g and $p \circ h'$. Then $H|(S^n \times I \cup I^{n+1} \times \{1\})$ has a lift $G : S^n \times I \cup I^{n+1} \times \{1\} \to E$ so that $G|I^{n+1} \times \{1\} = h'$ and $G|S^n \times I = f \times \mathrm{id}_I$. Since the pair $(I^{n+1} \times I,$ $S^n \times I \cup I^{n+1} \times \{1\})$ is topologically equivalent to $(I^{n+1} \times I, I^{n+1} \times \{0\})$, G extends to $G' : I^{n+1} \times I \to E$ so that $p \circ G' = H$. Notice that $G'|I^{n+1} \times \{0\}$ is a filler of (D) we were looking for.

(c)⇒(a). We know that every commutative diagram

(D)
$$\begin{tikzcd} K \arrow[r, "f"] \arrow[d, "j"] & E \arrow[d, "p"] \\ L \arrow[r, "g"] & B \end{tikzcd}$$

has a filler provided L is a CW complex, K is a subcomplex of L, and $j : K \to L$ is the inclusion map. In the case of $K = \emptyset$ we find that $p_* : [L, E] \to [L, B]$ is a surjection. For $L = M \times I$ and $K = M \times \partial I$ we see that $p_* : [M, E] \to [M, B]$ is a injection for every CW complex M. ■

9.3. Proposition. *If $f : K \to L$ is a quotient map which is proper (i.e., it is closed and point-inverses are compact), then $f \times \mathrm{id}_X : K \times X \to L \times X$ is a quotient map for any space X.*

Proof. Suppose $S \subset L \times X$ and $(f \times \mathrm{id}_X)^{-1}(S) = U$ is open in $K \times X$. Given $(a, x) \in S$ there are open subsets V of K and W of X so that $f^{-1}(x) \times \{x\} \subset V \times W \subset U$. Let $V' = f^{-1}(L - f(K - V))$. Then V' is open, $f(V')$ is open, and $f^{-1}(x) \times \{x\} \subset V' \times W \subset U$. Notice that $f(V') \times W$ is a neighborhood of (a, x) in S. ■

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Department of Mathematics
University of Tennessee
Knoxville, TN 37996, U.S.A.
E-mail: dydak@math.utk.edu

Institute of Mathematics
University of Warsaw
Banacha 2
02-097 Warszawa, Poland
E-mail: snowak@mimuw.edu.pl

and

Institute of Mathematics
Polish Academy of Sciences
Śniadeckich 8
00-950 Warszawa, Poland

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