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Taylor towers of symmetric and exterior powers

by

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Abstract. We study the Taylor towers of the nth symmetric and exterior power functors, Sp^n and Λ^n. We obtain a description of the layers of the Taylor towers, $D_k \mathrm{Sp}^n$ and $D_k\Lambda^n$, in terms of the first terms in the Taylor towers of Sp^t and Λ^t for $t < n$. The homology of these first terms is related to the stable derived functors (in the sense of Dold and Puppe) of Sp^t and Λ^t. We use stable derived functor calculations of Dold and Puppe to determine the lowest nontrivial homology groups for $D_k \mathrm{Sp}^n$ and $D_k\Lambda^n$.

0. Introduction. To any functor $F : \mathcal{C} \to \mathcal{A}$ (where $\mathcal{C}$ is a pointed category with finite coproducts and $\mathcal{A}$ is an abelian category) one can associate a sequence of functors known as its Taylor tower. The Taylor tower of F behaves much like the Taylor series for a real-valued function—it is a sequence of functors and natural transformations

$$\begin{tikzcd}
 & & F \arrow[dl, "p_{k+1}"'] \arrow[d, "p_k"] \arrow[dr, "p_{k-1}"] & & & \\
\cdots \arrow[r] & P_{k+1}F \arrow[r, "q_{k+1}"] & P_kF \arrow[r, "q_k"] & P_{k-1}F \arrow[r] & \cdots \arrow[r] & P_0F = F(*)
\end{tikzcd}$$

in which each P_kF can be treated as a degree k approximation to F. Typically, the approximation improves as k increases. At this point in time, we understand much about the formal structure of Taylor towers. A general method for constructing the functors P_kF was given in [JM2]. This method has been useful both in establishing properties of Taylor towers ([JM2]) and classifying degree k functors in terms of modules over a particular differential graded algebra ([JM3]).

As discussed in Section 6 of [JM2], the domain categories of the functors one wishes to study often carry more structure than is required for the

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construction of P_kF. To guarantee that P_kF behaves well with respect to this additional structure can require enhancing the standard construction of P_kF. For example, in this paper, we study functors of abelian groups. The first term in the Taylor tower of such functors was originally developed by Dold and Puppe [DP] to serve the role of derived functors for nonadditive functors. To ensure that P_1F takes short exact sequences to long exact sequences in homology, one replaces an abelian group with a projective resolution of that group. This necessitates extending F to a functor of chain complexes in such a way that it will preserve chain homotopy equivalences. One accomplishes this by using the prolongation of F as defined in Section 1.

With the formal theory of Taylor towers well-established, we focus our attention on identifying the Taylor tower of two specific classes of functors, the symmetric and exterior power functors. Our purposes in doing so are twofold. First, by determining the Taylor towers for particular examples, we would like to show how the general construction and properties for Taylor towers set up in [JM2] can be used to gather information about Taylor towers of specific functors. Secondly, we would like to improve our understanding of degree k functors, by better understanding the structure of the differential graded algebra over which the functors are classified in [JM3, §5]. The DGA is constructed from the kth term in the Taylor tower of the functor $\mathbb{Z}[\mathrm{Hom}(\bigvee_k C, -)]$ which is stably equivalent to the k-fold tensor product of the infinite symmetric power functor by work of Dold, Thom and Puppe ([DP], [DT]).

The present paper is a first step towards understanding the Taylor towers of the symmetric and exterior power functors. We produce a functorial description of the fibers of the natural transformations q_k in these Taylor towers and use some of the formal properties of Taylor towers in conjunction with stable derived functor calculations of Dold and Puppe to determine some homology of the towers. The calculations are driven by two results of earlier papers:

(1) As we mentioned above, in certain contexts the first term in the Taylor tower of F is equivalent to the Dold–Puppe stabilization of F ([JM1]).
(2) The kth layer in the Taylor tower of F, $D_kF = \mathrm{fiber}(P_kF \to P_{k-1}F)$, can be expressed in terms of homotopy orbits of the kth cross effect functor associated to F ([JM2]).

For the nth symmetric power functor Sp^n and the nth exterior power functor Λ^n, the equivalence of (2) is used to express the layers of the Taylor towers of Sp^n and Λ^n in terms of homotopy orbits of tensor products of lower order symmetric and exterior powers. The main results are the following.

THEOREMS 2.9, 2.11. *Let $1 \leq k \leq n$. Then*

$$D_k \operatorname{Sp}^n \simeq \bigoplus_{[\mathbf{t}]=[(t_1,\dots,t_k)]\in[V_k(n)]} (D_1 \operatorname{Sp}^{t_1} \otimes \cdots \otimes D_1 \operatorname{Sp}^{t_k})_{h\mathrm{st}(\mathbf{t})}$$

and

$$D_k \Lambda^n \simeq \bigoplus_{[\mathbf{t}]=[(t_1,\dots,t_k)]\in[V_k(n)]} (D_1 \Lambda^{t_1} \otimes \cdots \otimes D_1 \Lambda^{t_k})^{\sigma}_{h\mathrm{st}(\mathbf{t})}$$

where $V_k(n) = \{(t_1,\dots,t_k) \in \mathbb{N}^{\times k} \mid t_1 + \cdots + t_k = n\}$, $[\mathbf{t}]$ is the equivalence class of $\mathbf{t}$ under the Σ_k action that permutes the coordinates of $(t_1,\dots,t_k)$, $[V_k(n)]$ is the set of all equivalence classes under this action, $\mathrm{st}(\mathbf{t})$ is the stabilizer group of $\mathbf{t}$ under this action, $h\mathrm{st}(\mathbf{t})$ denotes the homotopy orbits with respect to this subgroup of Σ_k, and $(-)^{\sigma}$ indicates that $\mathrm{st}(\mathbf{t})$ acts with signature.

The homology of $D_1 \operatorname{Sp}^n(P)$ and $D_1\Lambda^n(P)$, where P is a projective resolution of an abelian group A, is equivalent under (1) to the stable derived functors of Sp^n and Λ^n as defined by Dold and Puppe ([DP]). The stable derived functors have been studied extensively by Dold and Puppe ([DP]), Bousfield ([Bo]), Betley ([Be]), and Simson and Tyc ([ST], [S]). We use their computations and our Theorems 2.9 and 2.11 to determine the first nontrivial homology groups of $D_k \operatorname{Sp}^n(P)$ and $D_k\Lambda^n(P)$ for $k > 1$. This leads to the computations below.

THEOREM 4.1. *Let A be a finitely generated abelian group, P be a projective resolution of A, and $n, k > 1$. The lowest degree in which $D_k \operatorname{Sp}^n(P)$ can have nontrivial homology is $2n - 2k$. For $k = n$,*

$$H_{2n-2k} D_k \operatorname{Sp}^n(P) \cong \operatorname{Sp}^n(A).$$

For $k < n$,

$$H_{2n-2k} D_k \operatorname{Sp}^n(P) \cong \bigoplus_{\substack{a_1+a_2p^{r_2}+\cdots+a_sp^{r_s}=n \\ p \text{ is prime},\, 0<r_2<\cdots<r_s \\ a_1+\cdots+a_s=k \\ a_2,\dots,a_s\geq 1;\, a_1\geq 0}} \operatorname{Sp}^{a_1}(A) \otimes \bigotimes_{j=2}^{s} \operatorname{Sp}^{a_j}(A/p).$$

By A/p we mean $A \otimes_{\mathbb{Z}} \mathbb{Z}/p$. For $a_1 = 0$, we set $\operatorname{Sp}^{a_1}(A) = \mathbb{Z}$.

THEOREM 4.8. *Let A be a finitely generated abelian group, P be a projective resolution of A, and $n, k > 1$. The lowest degree in which $D_k\Lambda^n(P)$ can have nontrivial homology is $n - k$. For $k = n$,*

$$H_{n-k} D_k\Lambda^n(P) \cong \Lambda^n(A).$$

For $k < n$,

$$H_{n-k} D_k \Lambda^n(P) \cong \bigoplus_{\substack{a_1 + a_2 p^{r_2} + \cdots + a_s p^{r_s} = n \\ p \text{ is prime}, 0 < r_2 < \cdots < r_s \\ a_1 + \cdots + a_s = k \\ a_2, \ldots, a_s \geq 1;\, a_1 \geq 0}} \Lambda^{a_1}(A) \otimes \Lambda_{a_2 p^{r_2}, \ldots, a_s p^{r_s}}(A)$$

where

$$\Lambda_{a_2 p^{r_2}, \ldots, a_s p^{r_s}}(A) \cong \begin{cases} \Lambda^{a_2}(A/p) \otimes \cdots \otimes \Lambda^{a_s}(A/p) & \text{if } p \text{ is odd}, \\ Sp^{a_2}(A/2) \otimes \cdots \otimes Sp^{a_s}(A/2) & \text{if } p = 2. \end{cases}$$

The paper is organized as follows. In Section 1, we review basic facts about Taylor towers and degree k functors that are needed in this paper. In Section 2, we review the equivalence alluded to in (2) above and use it to prove Theorems 2.9 and 2.11. Section 3 is devoted to recovering the calculations of $H_* D_1 \operatorname{Sp}^n(P)$ and $H_* D_1 \Lambda^n(P)$ originally carried out by Dold and Puppe. The results of Sections 2 and 3 are combined in Section 4 to produce the first nontrivial homology of $D_k \operatorname{Sp}^n(P)$, $D_k \Lambda^n(P)$, and $P_k \operatorname{Sp}^n(P)$.

1. Cross effects and the Taylor tower. The method for constructing the Taylor tower of a functor F described in [JM2] relies on a collection of functors, called the cross effects, that can be associated to F. We use this section to review cross effects and the basic properties of the Taylor tower of use to us in this paper. Throughout this section, unless otherwise indicated, we work with a functor $F : \mathcal{C} \to \mathcal{A}$ or $F : \mathcal{C} \to \mathrm{Ch}_{\geq 0} \mathcal{A}$ where $\mathcal{C}$ is a pointed category (has an object that is both initial and final) with finite coproducts, and $\mathcal{A}$ is an abelian category.

The functors of interest to us are all examples of *prolongations*—functors of modules over a fixed ring R that have been extended to functors of chain complexes of R-modules by passing through the category of simplicial R-modules. Recall from the Dold–Kan theorem that $\mathrm{Ch}_{\geq 0} \mathcal{A}$, the category of chain complexes over an abelian category $\mathcal{A}$, concentrated in degrees greater than or equal to 0, is naturally equivalent to $\operatorname{Simp} \mathcal{A}$, the category of simplicial objects over $\mathcal{A}$. This equivalence is realized by the functors

$$C : \operatorname{Simp} \mathcal{A} \to \mathrm{Ch}_{\geq 0} \mathcal{A} \quad \text{and} \quad K : \mathrm{Ch}_{\geq 0} \mathcal{A} \to \operatorname{Simp} \mathcal{A}.$$

For more details about these functors and the Dold–Kan theorem, see [D], [K], [We, pp. 270–276] or Section 2 of [JM1]. The prolongation of a functor is defined as follows.

Definition 1.1. Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories and F be a functor from $\mathcal{A}$ to $\mathcal{B}$. The *prolongation* of F is the functor from $\mathrm{Ch}_{\geq 0} \mathcal{A}$ to $\mathrm{Ch}_{\geq 0} \mathcal{B}$ given by

$$C \circ F \circ K$$

where for an object A in $\mathrm{Ch}_{\geq 0}\,\mathcal{A}$, F is applied degreewise to the simplicial object KA. When restricted to objects in $\mathcal{A}$, the prolongation of F agrees with the original functor F.

At the risk of some confusion we will use F to denote both the original functor and its prolongation. For example, we will use Sp^n to denote the nth symmetric power functor both as a functor of modules and a functor of chain complexes. Using prolongations guarantees that chain homotopy equivalences will be preserved.

The kth *cross effect* of the functor $F : \mathcal{C} \to \mathcal{A}$ is a functor of k variables, $\mathrm{cr}_k\, F : \mathcal{C}^{\times k} \to \mathcal{A}$, that can be defined inductively on objects $C_1, \dots, C_k$ as shown below. For the original definition, see [EM]. We use $\vee$ to denote the coproduct and $*$ to denote the initial/final object in $\mathcal{C}$. For $n = 1$ and $n = 2$, we have

$$F(C_1) \cong F(*) \oplus \mathrm{cr}_1\, F(C_1),$$

and

$$\mathrm{cr}_1\, F(C_1 \vee C_2) \cong \mathrm{cr}_1\, F(C_1) \oplus \mathrm{cr}_1\, F(C_2) \oplus \mathrm{cr}_2\, F(C_1, C_2).$$

In general,

$$\mathrm{cr}_{k-1}\, F(C_1 \vee C_2, C_3, \dots, C_k)$$

is isomorphic to

$$\mathrm{cr}_{k-1}\, F(C_1, C_3, \dots, C_k) \oplus \mathrm{cr}_{k-1}\, F(C_2, C_3, \dots, C_k) \oplus \mathrm{cr}_k\, F(C_1, C_2, \dots, C_k).$$

If F is the prolongation of a functor G, then it is straightforward to show that the cross effects of F are isomorphic to the prolongation of the cross effects of G. For more details see [JM1, §3].

The cross effect functors have the following properties.

PROPOSITION 1.2. *Let $F : \mathcal{C} \to \mathcal{A}$, and $C_1, \dots, C_k$ be objects in $\mathcal{C}$.*

(1) $\mathrm{cr}_k\, F$ *is symmetric with respect to its k variables. That is, for every $\sigma \in \Sigma_k$, the kth symmetric group,* $\mathrm{cr}_k\, F(C_1, \dots, C_k)$ *is isomorphic to* $\mathrm{cr}_k\, F(C_{\sigma(1)}, \dots, C_{\sigma(k)})$.
(2) $\mathrm{cr}_k\, F(C_1, \dots, C_k) \cong 0$ *if any* $C_i = *$.
(3) $F(C_1 \vee \cdots \vee C_k) \cong F(*) \oplus \bigoplus_{t=1}^{k} \bigoplus_{j_1 < \cdots < j_t} \mathrm{cr}_t\, F(C_{j_1}, \dots, C_{j_t})$.

Cross effects also determine the degree of a functor.

DEFINITION 1.3. A functor $F : \mathcal{C} \to \mathrm{Ch}_{\geq 0}\,\mathcal{A}$ is *degree k* iff $\mathrm{cr}_{k+1}\, F$ is quasi-isomorphic to 0.

EXAMPLES 1.4. Let R be a commutative ring and $\mathcal{M}_R$ denote the category of modules over R.

(1) For $n \geq 1$, the nth symmetric power functor, $\mathrm{Sp}^n : \mathcal{M}_R \to \mathcal{M}_R$, is defined for an object M in $\mathcal{M}_R$ by

$$\mathrm{Sp}^n(M) = \otimes^n(M)/U(M)$$

where $U(M)$ is the submodule of $\otimes^n(M)$ generated by all elements of the form $m_1 \otimes \cdots \otimes m_n - m_{\sigma(1)} \otimes \cdots \otimes m_{\sigma(n)}$ for $\sigma \in \Sigma_n$. When evaluated on a direct sum, the nth symmetric power functor decomposes as a sum of tensor products of symmetric powers. In particular, for R-modules $M_1, \ldots, M_k$,

$$\mathrm{Sp}^n(M_1 \oplus \cdots \oplus M_k) \cong \bigoplus_{t_1+\cdots+t_k=n} \mathrm{Sp}^{t_1}(M_1) \otimes \cdots \otimes \mathrm{Sp}^{t_k}(M_k).$$

It follows by induction that

$$\mathrm{cr}_k \, \mathrm{Sp}^n(M_1, \ldots, M_k) \cong \bigoplus_{\substack{t_1+\cdots+t_k=n \\ t_1,\ldots,t_k>0}} \mathrm{Sp}^{t_1}(M_1) \otimes \cdots \otimes \mathrm{Sp}^{t_k}(M_k).$$

If $k > n$, then $\mathrm{cr}_k \, \mathrm{Sp}^n \cong 0$ and Sp^n is a degree n functor. When $M_1 = \cdots = M_k$, the symmetric group Σ_k acts on this cross effect by $\sigma(t_1, \ldots, t_k; m_1, \ldots, m_k) = (t_{\sigma(1)}, \ldots, t_{\sigma(k)}; m_{\sigma(1)}, \ldots, m_{\sigma(k)})$. Here we are treating σ as an endomorphism of the set $\{1, \ldots, k\}$ so that for $\tau, \sigma \in \Sigma_k$, $\tau\sigma = \tau \circ \sigma$. These results hold for the prolongation of Sp^n as well.

(2) The nth exterior power functor $\Lambda^n : \mathcal{M}_R \to \mathcal{M}_R$ is defined for an object M in $\mathcal{M}_R$ by

$$\Lambda^n(M) = \otimes^n(M)/V(M)$$

where $V(M)$ is the submodule of $\otimes^n(M)$ generated by all elements of the form $m_1 \otimes \cdots \otimes m_n$ with $m_i = m_j$ for some $1 \leq i < j \leq n$. In a manner similar to that of the previous example, one can show that

$$\mathrm{cr}_k \, \Lambda^n(M_1, \ldots, M_k) \cong \bigoplus_{\substack{t_1+\cdots+t_k=n \\ t_1,\ldots,t_k>0}} \Lambda^{t_1}(M_1) \otimes \cdots \otimes \Lambda^{t_k}(M_k)$$

for R-modules $M_1, \ldots, M_k$. However, when $M_1 = \cdots = M_k$ the action of Σ_k on this decomposition is now twisted by the signature. Hence

$$\sigma(t_1, \ldots, t_k; m_1, \ldots, m_k) = \mathrm{sgn}(\sigma) \cdot (t_{\sigma(1)}, \ldots, t_{\sigma(k)}; m_{\sigma(1)}, \ldots, m_{\sigma(k)}).$$

This holds for the prolongation of Λ^n as well.

For more examples of cross effects, see Section 1 of [JM2].

The cross effects play a central role in the construction of the Taylor tower of F. We review the basic features of the Taylor tower here and refer the reader to [JM2] for the details of the construction.

THEOREM 1.5 ([JM2, 2.12]). *Given a functor F from $\mathcal{C}$ to $\mathcal{A}$ or to* $\mathrm{Ch}_{\geq 0}\,\mathcal{A}$, *there is a natural tower of functors*:

$$\begin{tikzcd} & & F \arrow[dl, "p_{k+1}"'] \arrow[d, "p_k"] \arrow[dr, "p_{k-1}"] & & & \\ \cdots \arrow[r] & P_{k+1}F \arrow[r, "q_{k+1}"] & P_kF \arrow[r, "q_k"] & P_{k-1}F \arrow[r] & \cdots \arrow[r] & P_0F = F(*). \end{tikzcd}$$

For each k, the functor P_kF is a degree k functor. Moreover, the pairs (P_k, p_k) are universal (up to natural quasi-isomorphism) with respect to maps from F to degree k functors.

The following properties of the Taylor tower functors will be of use to us.

PROPOSITION 1.6. *Let F be a functor from $\mathcal{C}$ to $\mathcal{A}$ or* $\mathrm{Ch}_{\geq 0}\,\mathcal{A}$ *and let $k \geq 1$.*

(1) *P_k is an exact functor of functors. That is, if $F \to G \to H$ is an exact sequence of functors, then $P_kF \to P_kG \to P_kH$ is an exact sequence of functors or yields a long exact sequence in homology. In particular, $P_k(F \oplus G) \cong P_kF \oplus P_kG$.*
(2) *If F is degree k, then $P_kF \simeq F$.*

2. The layers of the Taylor tower. As a first step towards understanding the kth term in the Taylor towers of Sp^n and Λ^n, we study the kth layers of the Taylor towers of Sp^n and Λ^n. In this section we review the definition of the layers of the Taylor tower and recall a result of Goodwillie's, reproduced in our context in [JM2], that leads to descriptions of the kth layers of the Taylor towers of Sp^n and Λ^n in terms of lower symmetric and exterior powers. As in Section 1, $\mathcal{C}$ is a pointed category with finite coproducts and $\mathcal{A}$ is an abelian category.

DEFINITION 2.1. Let $F : \mathcal{C} \to \mathcal{A}$ and $k \geq 1$. The *kth layer* of the Taylor tower of F is the functor D_kF given by

$$D_kF := \mathrm{fiber}(P_kF \xrightarrow{q_kF} P_{k-1}F)$$

where fiber is the homotopy fiber of q_kF, i.e., the mapping cone of q_kF shifted down one degree.

As an immediate consequence of this definition we note that the sequence $D_kF \to P_kF \to P_{k-1}F$ gives rise to a long exact sequence in homology:

$$\cdots \to H_{*+1}P_{k-1}F \to H_*D_kF \to H_*P_kF \to H_*P_{k-1}F \to \cdots . \tag{2.2}$$

In [JM2], we showed that D_kF can be expressed as the multilinearization of the kth cross effect of F. This result, stated below, was inspired by Goodwillie's description [G] of the layers of the Taylor towers for functors of spaces.

Proposition 2.3 ([JM2, 3.9]). *Let $F : \mathcal{C} \to \mathrm{Ch}_{\geq 0}\,\mathcal{A}$. Then*

$$D_k F \simeq (D_1^{(k)} \operatorname{cr}_k F)_{h\Sigma_k}.$$

Here $D_1^{(k)}$ denotes the multilinearization of $\operatorname{cr}_k F$, which is the functor $D_1^k D_1^{k-1} \dots D_1^1 \operatorname{cr}_k F$ where $D_1^i \operatorname{cr}_k F$ is the functor obtained by holding all but the ith variable of $\operatorname{cr}_k F$ constant and applying D_1 to the resulting functor of one variable. By $h\Sigma_k$ we mean the homotopy orbits of $D_1^{(k)} \operatorname{cr}_k F$, in other words, the total complex of $\mathbb{Z}[E\Sigma_k] \otimes_{\mathbb{Z}[\Sigma_k]} D_1^{(k)} \operatorname{cr}_k F$ where $E\Sigma_k$ is a free Σ_k-resolution of $\mathbb{Z}$.

In Example 1.4, we saw that

$$\operatorname{cr}_k \mathrm{Sp}^n \cong \bigoplus_{\substack{t_1+\dots+t_k=n \\ t_1,\dots,t_k>0}} \mathrm{Sp}^{t_1} \otimes \dots \otimes \mathrm{Sp}^{t_k}. \tag{2.4}$$

By applying Proposition 2.3 to this formula, we determine $D_k \mathrm{Sp}^n$ in Theorem 2.9 below. Before doing so, we introduce some notation and terminology for working with the partitions $t_1 + \dots + t_k = n$ that index the summands in the cross effect formula (2.4).

Definition 2.5. For $1 \leq k \leq n$, let

$$V_k(n) = \{(t_1, \dots, t_k) \in \mathbb{N}^{\times k} \mid t_1 + \dots + t_k = n\}.$$

In other words, $V_k(n)$ is the set of partitions of n into k positive integers.

Examples 2.6. (1) For $n = 6$ and $k = 3$, $V_k(n)$ is the set

$$\{(1,1,4),(1,4,1),(4,1,1),(1,2,3),(1,3,2),$$
$$(2,1,3),(2,3,1),(3,1,2),(3,2,1),(2,2,2)\}.$$

(2) Note that under the notation of Definition 2.5, (2.4) becomes

$$\operatorname{cr}_k \mathrm{Sp}^n \cong \bigoplus_{\mathbf{t}=(t_1,\dots,t_k)\in V_k(n)} \mathrm{Sp}^{t_1} \otimes \dots \otimes \mathrm{Sp}^{t_k}.$$

The kth symmetric group, Σ_k, acts on $V_k(n)$ by permuting the terms within a partition. Associated with this action, we have the following.

Definition 2.7. Let $1 \leq k \leq n$ and $\mathbf{t} = (t_1, \dots, t_k) \in V_k(n)$.

(1) The *stabilizer* of $\mathbf{t}$ is the group

$$\mathrm{st}(\mathbf{t}) = \{\sigma \in \Sigma_k \mid \sigma(\mathbf{t}) = \mathbf{t}\}.$$

(2) The *orbit* of $\mathbf{t}$ is the set

$$[\mathbf{t}] = \{\sigma(\mathbf{t}) \mid \sigma \in \Sigma_k\}.$$

(3) Recall that the orbits form a partition of $V_k(n)$. We use $[V_k(n)]$ to denote the set of orbits of $V_k(n)$ under the Σ_k action.

EXAMPLE 2.8. For $n = 6$ and $k = 3$, the orbits of the Σ_k action on $V_k(n)$ are

$$[(1,1,4)] = \{(1,1,4),(1,4,1),(4,1,1)\},$$
$$[(1,2,3)] = \{(1,2,3),(1,3,2),(2,3,1),(2,1,3),(3,1,2),(3,2,1)\},$$
$$[(2,2,2)] = \{(2,2,2)\}.$$

The stabilizer subgroups are

$$\mathrm{st}(1,1,4) = \{(1),(12)\}, \quad \mathrm{st}(1,4,1) = \{(1),(13)\}, \quad \mathrm{st}(4,1,1) = \{(1),(23)\},$$
$$\mathrm{st}(1,2,3) = \mathrm{st}(1,3,2) = \mathrm{st}(2,1,3) = \mathrm{st}(2,3,1) = \mathrm{st}(3,1,2)$$
$$= \mathrm{st}(3,2,1) = \{(1)\},$$
$$\mathrm{st}(2,2,2) = \Sigma_3.$$

With these ideas, we obtain the following.

THEOREM 2.9. *Let $1 \le k \le n$. Then*

$$D_k \operatorname{Sp}^n \simeq \bigoplus_{[\mathbf{t}]\in[V_k(n)]} (D\operatorname{Sp}^{\mathbf{t}})_{h\mathrm{st}(\mathbf{t})}$$

where for $\mathbf{t} = (t_1,\ldots,t_k)$,

$$D\operatorname{Sp}^{\mathbf{t}} = D_1\operatorname{Sp}^{t_1} \otimes \cdots \otimes D_1 \operatorname{Sp}^{t_k}.$$

Proof. We prove this result by first showing that for each $\mathbf{t} \in V_k(n)$ there is a Σ_k-equivariant isomorphism

$$\Phi : D\operatorname{Sp}^{\mathbf{t}} \times_{\mathrm{st}(\mathbf{t})} \Sigma_k \xrightarrow{\cong} \bigoplus_{\mathbf{v}\in[\mathbf{t}]} D\operatorname{Sp}^{\mathbf{v}}.$$

The map Φ takes $D\operatorname{Sp}^{\mathbf{t}} \times \sigma$ (for $\sigma \in \Sigma_k$) isomorphically to $D\operatorname{Sp}^{\sigma(\mathbf{t})}$. That Φ is well-defined follows from the observation that $\alpha\circ\sigma(\mathbf{t}) = \sigma(\mathbf{t})$ for $\alpha \in \mathrm{st}(\mathbf{t})$. That Φ is an isomorphism comes from the facts that $|[\mathbf{t}]| = |\Sigma_k|/|\mathrm{st}(\mathbf{t})|$ and that $D\operatorname{Sp}^{\sigma(\mathbf{t})} \ne D\operatorname{Sp}^{\tau(\mathbf{t})}$ for $\sigma,\tau \in \Sigma_k$ with $\sigma\tau^{-1}, \tau\sigma^{-1} \notin \mathrm{st}(\mathbf{t})$.

Furthermore, for $\tau \in \Sigma_k$, we have

$$\Phi(\tau(D\operatorname{Sp}^{\mathbf{t}} \times\sigma)) = \Phi(D\operatorname{Sp}^{\mathbf{t}} \times \tau\circ\sigma) = D\operatorname{Sp}^{\tau(\sigma(\mathbf{t}))} = \tau(D\operatorname{Sp}^{\sigma(\mathbf{t})}).$$

Hence, Φ is Σ_k-equivariant. It follows that

$$\Big(\bigoplus_{[\mathbf{t}]\in[V_k(n)]}\bigoplus_{\mathbf{v}\in[\mathbf{t}]} D\operatorname{Sp}^{\mathbf{v}}\Big)_{h\Sigma_k} \cong \Big(\bigoplus_{[\mathbf{t}]\in[V_k(n)]} D\operatorname{Sp}^{\mathbf{t}} \times_{\mathrm{st}(\mathbf{t})}\Sigma_k\Big)_{h\Sigma_k} \tag{2.10}$$
$$= \bigoplus_{[\mathbf{t}]\in[V_k(n)]} D\operatorname{Sp}^{\mathbf{t}} \times_{\mathrm{st}(\mathbf{t})}\Sigma_k \otimes_{\mathbb{Z}[\Sigma k]} E\Sigma_k$$
$$\cong \bigoplus_{[\mathbf{t}]\in[V_k(n)]} D\operatorname{Sp}^{\mathbf{t}} \otimes_{\mathbb{Z}[\mathrm{st}(\mathbf{t})]} E\Sigma_k \cong \bigoplus_{[\mathbf{t}]\in[V_k(n)]} (D\operatorname{Sp}^{\mathbf{t}})_{h\mathrm{st}(\mathbf{t})}$$

where the last isomorphism comes from the fact that a free Σ_k resolution of $\mathbb{Z}$ is also a free $\mathrm{st}(\mathbf{t})$ resolution of $\mathbb{Z}$.

By Proposition 2.3,

$$\begin{aligned} D_k \operatorname{Sp}^n &\simeq (D_1^{(k)} \operatorname{cr}_k \operatorname{Sp}^n)_{h\Sigma_k} \\ &\cong \Big(D_1^{(k)} \bigoplus_{\mathbf{t}=(t_1,\dots,t_k)\in V_k(n)} \operatorname{Sp}^{t_1}\otimes\cdots\otimes \operatorname{Sp}^{t_k}\Big)_{h\Sigma_k}. \end{aligned}$$

It follows from the definition of P_1F in [JM2] that $D_1^i(\operatorname{Sp}^{t_1}\otimes\cdots\otimes\operatorname{Sp}^{t_k}) \cong \operatorname{Sp}^{t_1}\otimes\cdots\otimes D_1\operatorname{Sp}^{t_i}\otimes\cdots\otimes\operatorname{Sp}^{t_k}$. By Proposition 1.6, we know that D_1^i commutes with direct sums. From this and (2.10), we have

$$\begin{aligned} D_k \operatorname{Sp}^n &\simeq \Big(\bigoplus_{\mathbf{t}=(t_1,\dots,t_k)\in V_k(n)} D_1\operatorname{Sp}^{t_1}\otimes\cdots\otimes D_1\operatorname{Sp}^{t_k}\Big)_{h\Sigma_k} \\ &= \Big(\bigoplus_{[t]\in[V_k(n)]}\bigoplus_{\mathbf{v}\in[t]} D\operatorname{Sp}^{\mathbf{v}}\Big)_{h\Sigma_k} \cong \bigoplus_{[\mathbf{t}]\in[V_k(n)]} \big(D\operatorname{Sp}^{\mathbf{t}}\big)_{h\operatorname{st}(\mathbf{t})}, \end{aligned}$$

as desired.

A similar result can be proved for the exterior power functors using the same techniques.

Theorem 2.11. *Let $1 \le k \le n$. Then*

$$D_k\Lambda^n \simeq \bigoplus_{[\mathbf{t}]\in[V_k(n)]} (D\Lambda^{\mathbf{t}})^{\sigma}_{h\operatorname{st}(\mathbf{t})}$$

where for $\mathbf{t} = (t_1,\dots,t_k)$,

$$D\Lambda^{\mathbf{t}} = D_1\Lambda^{t_1}\otimes\cdots\otimes D_1\Lambda^{t_k},$$

and $(-)^\sigma$ indicates that $\operatorname{st}(\mathbf{t})$ acts with the signature.

3. Stable derived functors of symmetric and exterior powers. With Theorems 2.9 and 2.11 we saw that the kth layers in the Taylor towers of Sp^n and Λ^n are determined by tensor products of terms of the form $D_1\operatorname{Sp}^t$ or $D_1\Lambda^t$ for $t \le n$. The functors $D_1\operatorname{Sp}^t$ and $D_1\Lambda^t$ have been analyzed in detail by Dold and Puppe ([DP]), Simson and Tyc ([S], [ST]), Bousfield ([Bo]), and Betley ([Be]). We review some of their results in this section, and use these results to determine the first nontrivial homology groups of $D_k\operatorname{Sp}^n$ and $D_k\Lambda^n$ in Section 4.

The language used in the references listed in the previous paragraph is not that of Taylor towers. Instead, the functor D_1F is used to construct "stable derived functors." Dold and Puppe were the first to consider such functors. For a nonadditive functor $F : \mathcal{A} \to \mathcal{B}$ between abelian categories $\mathcal{A}$ and $\mathcal{B}$, they sought to construct derived functors for F. They did so by first stabilizing F to produce the functor $DF : \operatorname{Ch}_{\ge 0}\mathcal{A} \to \operatorname{Ch}_{\ge 0}\mathcal{B}$ given by

$$DF = \lim_n \operatorname{sh}_{-n} F \operatorname{sh}_n$$

where sh_n is the functor that shifts a chain complex by n degrees and F has been replaced by its prolongation. The functor DF preserves direct sums up to quasi-isomorphism, and when F is the prolongation of a functor from $\mathcal{A}$ to $\mathcal{B}$, $D_1 F$ is naturally equivalent to DF (see [JM1, 7.5], [JM2, 2.14]). When $\mathcal{A}$ has enough projectives, the nth stable derived functor of F evaluated at an object A in $\mathcal{A}$, denoted $L_n^{\mathrm{s}} F(A)$, is obtained from DF by evaluating it on a projective resolution P of A and taking the nth homology group. Thus,

$$L_n^{\mathrm{s}} F(A) := H_n(DF(P)) \cong H_n(D_1 F(P)).$$

As with derived functors, stable derived functors are independent of the choice of projective resolution. When the original functor F is additive (or degree one), the stable derived functors of F are isomorphic to the derived functors of F.

In this section we determine the degree in which the first nontrivial stable derived functors of Sp^n and Λ^n arise and identify these groups. These results date back to Dold and Puppe's original paper on stable derived functors ([DP]). We provide proofs in this section in terms of the Taylor tower constructions defined in [JM1] and [JM2]. To determine the location of the lowest nontrivial group we use the following lemma.

LEMMA 3.1 ([DP], [P]). *Let F and G be functors from the abelian category $\mathcal{A}$ to the abelian category $\mathcal{B}$. If $F(0) \cong G(0) \cong 0$, then $D_1(F \otimes G) \cong 0$ as a functor from $\mathrm{Ch}_{\geq 0}\,\mathcal{A}$ to $\mathrm{Ch}_{\geq 0}\,\mathcal{B}$.*

Proof. Let P be a chain complex in $\mathrm{Ch}_{\geq 0}\,\mathcal{A}$. From Example 2.14 of [JM2], we know that

$$D_1(F \otimes G)(P) \simeq D(F \otimes G)(P) \cong \lim_n \mathrm{sh}_{-n}(F \otimes G)(\mathrm{sh}_n P).$$

Consider $(F \otimes G)(\mathrm{sh}_n P)$. By the Eilenberg–Zilber theorem, this is equivalent to the total complex of the bicomplex $F(\mathrm{sh}_n P) \otimes G(\mathrm{sh}_n P)$. But $F(\mathrm{sh}_n P)$ and $G(\mathrm{sh}_n P)$ will be 0 in degrees less than n. As a result, the total complex $(F \otimes G)(\mathrm{sh}_n P)$ is 0 in degrees less than $2n$, and $\mathrm{sh}_{-n}(F \otimes G)(\mathrm{sh}_n P)$ is 0 in degrees less than n. Hence, in the limit, $D_1(F \otimes G)(P)$ is quasi-isomorphic to 0.

To determine the connectivity and first nontrivial homology of $D_1 \mathrm{Sp}^n$ and $D_1 \Lambda^n$, we introduce functors that are closely related to the symmetric and exterior power functors, the divided powers.

DEFINITION 3.2. Let R be a commutative ring and $\mathcal{M}_R$ be the category of modules over R. The *divided power functor* or *gamma functor* is the functor that takes an R-module A to the commutative ring $\Gamma(A)$ generated by elements of the form $x^{(r)}$ for each $x \in A$ and integer $r \geq 0$, subject to the following relations:

(1) $x^{(0)} = 1$,
(2) $x^{(r)}x^{(s)} = \binom{r+s}{r}x^{(r+s)}$,
(3) $(x+y)^{(t)} = \sum_{r+s=t} x^{(r)}y^{(s)}$,
(4) $(nx)^{(r)} = n^r x^{(r)}$ for $n \in R$.

An element of $\Gamma(A)$ of the form $x_1^{(r_1)} \ldots x_n^{(r_n)}$ has degree $r_1 + \cdots + r_n$. We use $\Gamma^n(A)$ to denote the R-module generated (additively) by the degree n elements of $\Gamma(A)$. This yields a functor, Γ^n, from $\mathcal{M}_R$ to itself. We refer to this functor as the *nth divided power functor*. Clearly,

$$\Gamma(A) \cong \bigoplus_{i=1}^{\infty} \Gamma^i(A).$$

(For more about divided powers see [E, Appendix 2], or [EM, §18].)

Remark 3.3. The cross effects of the divided power functors can be identified in a fashion similar to that of Example 1.4. For $n \geq 1$, Γ^n is a degree n functor. For $k \leq n$, and R-modules $A_1, \ldots, A_k$,

$$\operatorname{cr}_k \Gamma^n(A_1, \ldots, A_k) \cong \bigoplus_{\substack{t_1+\cdots+t_k=n \\ t_i \geq 1}} \Gamma^{t_1}(A_1) \otimes \cdots \otimes \Gamma^{t_k}(A_k).$$

In the case $k = 2$, the isomorphism comes from the isomorphism

$$\Gamma^n(A_1 \oplus A_2) \cong \bigoplus_{\substack{t_1+t_2=n \\ t_i \geq 0}} \Gamma^{t_1}(A_1) \otimes \Gamma^{t_2}(A_2)$$

that assigns $m_1^{(t_1)} \otimes m_2^{(t_2)} \in \Gamma^{t_1}(A_1) \otimes \Gamma^{t_2}(A_2)$ to $(m_1, 0)^{(t_1)}(0, m_2)^{(t_2)} \in \Gamma^n(A_1 \oplus A_2)$. We will make use of this isomorphism in the proof of Proposition 3.7.

The stable derived functors of the symmetric and exterior powers can be obtained from those of the divided powers via a dimension shift. We prove this next.

Proposition 3.4 ([DP]). *Let $n \geq 2$. Then*

$$H_i D_1 \Gamma^n \cong H_{i+n-1} D_1 \Lambda^n \cong H_{i+2n-2} D_1 \operatorname{Sp}^n.$$

Proof. To establish the second isomorphism, recall that the symmetric and exterior power functors are related by the following Koszul complex:

$$0 \to \Lambda^n \to \Lambda^{n-1} \otimes \operatorname{Sp}^1 \to \Lambda^{n-2} \otimes \operatorname{Sp}^2 \to \cdots \to \Lambda^1 \otimes \operatorname{Sp}^{n-1} \to \operatorname{Sp}^n \to 0.$$

By Proposition 1.6(1), applying D_1 to the Koszul complex produces the exact sequence

$$(3.5) \quad 0 \to D_1 \Lambda^n \to D_1(\Lambda^{n-1} \otimes \operatorname{Sp}^1) \to \cdots$$
$$\cdots \to D_1(\Lambda^1 \otimes \operatorname{Sp}^{n-1}) \to D_1 \operatorname{Sp}^n \to 0.$$

By Lemma 3.1, terms of the form $D_1(\Lambda^i \otimes \mathrm{Sp}^{n-i})$ for $0 < i < n$ are quasi-isomorphic to 0. So if we consider the spectral sequence associated to the bicomplex (3.5) we see that it has two nontrivial columns in the E_2 term:

$$\begin{array}{cccccc}
\vdots & \vdots & \dots & \vdots & \\
H_2 D_1 \mathrm{Sp}^n & 0 & \dots & 0 & H_2 D_1 \Lambda^n \\
H_1 D_1 \mathrm{Sp}^n & 0 & \dots & 0 & H_1 D_1 \Lambda^n \\
H_0 D_1 \mathrm{Sp}^n & 0 & \dots & 0 & H_0 D_1 \Lambda^n \\
0 & & & & n
\end{array}$$

Since (3.5) is exact, the spectral sequence converges to 0. The only nontrivial differentials occur in E_n and provide the isomorphisms

$$H_i D_1 \Lambda^n \cong H_{i+n-1} D_1 \mathrm{Sp}^n .$$

In a similar fashion, one can use the Koszul complex relating the exterior and divided powers

$$0 \to \Gamma^n \to \Gamma^{n-1} \otimes \Lambda^1 \to \cdots \to \Gamma^1 \otimes \Lambda^{n-1} \to \Lambda^n \to 0$$

to obtain the isomorphism $H_i D_1 \Gamma^n \cong H_{i+n-1} D_1 \Lambda^n$.

Since $D_1 \Gamma^n$ has no homology below degree 0, Proposition 3.4 also yields the following results.

COROLLARY 3.6. *For $n \geq 2$, $H_t(D_1 \mathrm{Sp}^n) \cong 0$ when $t < 2n-2$ and $H_s(D_1 \Lambda^n) \cong 0$ when $s < n-1$.*

By calculating $H_0 D_1 \Gamma^n(P)$ explicitly for a projective resolution P, we can identify the first nontrivial stable derived functors of Γ^n, Λ^n, and Sp^n.

PROPOSITION 3.7 ([DP]). *Let A be a finitely generated abelian group and P be a projective resolution of A. If $n = p^r$ for a prime p and $r \geq 1$, then*

$$H_{2n-2} D_1 \mathrm{Sp}^n(P) \cong H_{n-1} D_1 \Lambda^n(P) \cong H_0 D_1 \Gamma^n(P) \cong A \otimes_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z}.$$

If $n = 1$, we have

$$H_0 D_1 \Gamma^n(P) \cong H_0 D_1 \Lambda^n(P) \cong H_0 D_1 \mathrm{Sp}^n(P) = H_0(P) \cong A.$$

Otherwise,

$$H_{2n-2} D_1 \mathrm{Sp}^n(P) \cong H_{n-1} D_1 \Lambda^n(P) \cong H_0 D_1 \Gamma^n(P) \cong 0.$$

Proof. By Proposition 3.4, it suffices to determine $H_0 P_1 \Gamma^n(P)$, and since $P_1 \Gamma^n$ is right exact, it is enough to compute $H_0 P_1 \Gamma^n$ on a free module B. Since A is a finitely generated abelian group and $P_1 \Gamma^n$ preserves direct sums, we may restrict our attention to the case where $B = \mathbb{Z}$.

It follows from the construction of the Taylor tower in [JM2, §2] that $H_0 P_1 \mathit{\Gamma}^n(\mathbb{Z})$ is the cokernel of the map

$$\operatorname{cr}_2 \mathit{\Gamma}^n(\mathbb{Z},\mathbb{Z}) \xrightarrow{\mathit{\Gamma}^n(+)} \mathit{\Gamma}^n(\mathbb{Z}). \tag{3.8}$$

From [JM2, 2.8], the homomorphism $\mathit{\Gamma}^n(+)$ is the composite

$$\operatorname{cr}_2 \mathit{\Gamma}^n(\mathbb{Z},\mathbb{Z}) \hookrightarrow \mathit{\Gamma}^n(\mathbb{Z}\oplus\mathbb{Z}) \xrightarrow{+} \mathit{\Gamma}^n(\mathbb{Z})$$

where $+$ is induced by the fold map $\mathbb{Z}\oplus\mathbb{Z}\to\mathbb{Z}$ that takes (a,b) to $a+b$. Recall from Remark 3.3 that $\operatorname{cr}_2 \mathit{\Gamma}^n(\mathbb{Z},\mathbb{Z}) \cong \bigoplus_{i=1}^{n-1} \mathit{\Gamma}^i(\mathbb{Z})\otimes\mathit{\Gamma}^{n-i}(\mathbb{Z})$ and that this isomorphism takes $m_1^{(i)}\otimes m_2^{(n-i)} \in \mathit{\Gamma}^i(\mathbb{Z})\otimes\mathit{\Gamma}^{n-i}(\mathbb{Z})$ to $(m_1,0)^{(i)}(0,m_2)^{(n-i)} \in \operatorname{cr}_2 \mathit{\Gamma}^n(\mathbb{Z},\mathbb{Z}) \subseteq \mathit{\Gamma}^n(\mathbb{Z}\oplus\mathbb{Z})$. Composing with the fold map yields

$$m_1^{(i)}\otimes m_2^{(n-i)} \to (m_1,0)^{(i)}(0,m_2)^{(n-i)} \xrightarrow{+} (m_1)^{(i)}(m_2)^{(n-i)}.$$

Using the defining relations for $\mathit{\Gamma}$ (Definition 3.2), one finds that

$$(m_1)^{(i)}(m_2)^{(n-i)} = m_1^i m_2^{n-i} 1^{(i)} 1^{(n-i)} = m_1^i m_2^{n-i}\binom{n}{i} 1^{(n)}.$$

Since $1^{(n)}$ generates $\mathit{\Gamma}^n(\mathbb{Z})\cong\mathbb{Z}$ it follows that the image of $\mathit{\Gamma}^n(+)$ in $\mathit{\Gamma}^n(\mathbb{Z})$ is generated by the set $\{\binom{n}{i}1^{(n)} \mid 1\le i\le n-1\}$. Since the greatest common divisor of $\{\binom{n}{1},\binom{n}{2},\ldots,\binom{n}{n-1}\}$ is p when $n=p^r$ for some prime p, $r\ge 1$, and is 1 otherwise, we see that

$$\operatorname{im}\mathit{\Gamma}^n(+)(\mathbb{Z}) \cong \begin{cases} p\mathbb{Z} & \text{if } n=p^r, \\ \mathbb{Z} & \text{otherwise.}\end{cases} \tag{3.9}$$

Hence, we have

$$H_0 D_1 \mathit{\Gamma}^n(P) \cong \begin{cases} \mathbb{Z}/p\mathbb{Z} & \text{if } n=p^r \text{ for a prime } p, \text{ and } r\ge 1, \\ 0 & \text{otherwise.}\end{cases}$$

4. The first nontrivial homology of $D_k \operatorname{Sp}^n$ and $D_k \mathit{\Lambda}^n$. In Section 2 we showed that $D_k \operatorname{Sp}^n$ and $D_k\mathit{\Lambda}^n$ can be expressed in terms of linearizations of lower symmetric and exterior powers. Coupling this with the calculations of Section 3 enables us to completely determine the first nontrivial homology groups of $D_k \operatorname{Sp}^n$ and $D_k\mathit{\Lambda}^n$. We begin with the symmetric powers.

THEOREM 4.1. *Let A be a finitely generated abelian group, P be a projective resolution of A, and $n,k>1$. The lowest degree in which $D_k \operatorname{Sp}^n(P)$ can have nontrivial homology is $2n-2k$. For $k=n$,*

$$H_{2n-2k} D_k \operatorname{Sp}^n(P) \cong \operatorname{Sp}^n(A).$$

For $k < n$,

$$H_{2n-2k} D_k \operatorname{Sp}^n(P) \cong \bigoplus_{\substack{a_1+a_2p^{r_2}+\cdots+a_sp^{r_s}=n \\ p \text{ is prime}, 0<r_2<\cdots<r_s \\ a_1+\cdots+a_s=k \\ a_2,a_3,\ldots,a_s\geq 1; a_1\geq 0}} \operatorname{Sp}^{a_1}(A) \otimes \bigotimes_{j=2}^{s} \operatorname{Sp}^{a_j}(A/p).$$

By A/p we mean $A \otimes_{\mathbb{Z}} \mathbb{Z}/p$. For $a_1 = 0$, we set $\operatorname{Sp}^{a_1}(A) = \mathbb{Z}$.

Hence, to find $H_{2n-2k} D_k \operatorname{Sp}^n(P)$ one need only determine the ways in which n can be expressed as the sum of k (not necessarily distinct) powers of the same prime p. We illustrate this with the set of examples below.

Examples 4.2. Let P be a projective resolution of A.

(1) Consider $k = 2$ and $n = 12$. In this case, n can be written as a sum of two powers of the same prime in three ways, and the a_is and r_js have the indicated values:

Sum	a_1	a_i	r_j
$2^2 + 2^3$	0	$a_2 = 1, a_3 = 1$	$r_2 = 2$, $r_3 = 3$
$3^1 + 3^2$	0	$a_2 = 1$, $a_3 = 1$	$r_2 = 1$, $r_3 = 2$
$11^0 + 11^1$	1	$a_2 = 1$	$r_2 = 1$

It follows that

$$H_{20} D_2 \operatorname{Sp}^{12}(P) \cong (A/2 \otimes A/2) \oplus (A/3 \otimes A/3) \oplus (A \otimes A/11).$$

(2) Consider $k = 3$ and $n = 12$. In this case, n can be written as a sum of three powers of the same prime in two ways and the a_is and r_js have the indicated values:

Sum	a_1	a_i	r_j
$2^1 + 2^1 + 2^3$	0	$a_2 = 2$, $a_3 = 1$	$r_2 = 1$, $r_3 = 3$
$2^2 + 2^2 + 2^2$	0	$a_2 = 3$	$r_2 = 2$

It follows that

$$H_{18} D_3 \operatorname{Sp}^{12}(P) \cong (\operatorname{Sp}^2(A/2) \otimes A/2) \oplus \operatorname{Sp}^3(A/2).$$

(3) Consider $k = 4$ and $n = 12$. In this case, n can be written as a sum of four powers of the same prime in five ways:

$$\begin{aligned} 12 &= 2 \cdot 2^0 + 2^1 + 2^3 \\ &= 2 \cdot 2^1 + 2 \cdot 2^2 \\ &= 3 \cdot 3^0 + 3^2 \\ &= 4 \cdot 3^1 \\ &= 2 \cdot 5^0 + 2 \cdot 5^1. \end{aligned}$$

Hence,

$$\begin{aligned} H_{16}D_4\operatorname{Sp}^{12}(P) \cong\ & (\operatorname{Sp}^2(A)\otimes A/2\otimes A/2)\oplus(\operatorname{Sp}^2(A/2)\otimes\operatorname{Sp}^2(A/2))\\ &\oplus(\operatorname{Sp}^3(A)\otimes A/3)\oplus\operatorname{Sp}^4(A/3)\\ &\oplus(\operatorname{Sp}^2(A)\otimes\operatorname{Sp}^2(A/5)).\end{aligned}$$

(4) Suppose n is odd and $k=2$. Then n can be expressed as a sum of two prime powers if and only if n has the form $n=2^k+1$ for some $k\geq 1$. It follows that

$$H_{2n-4}D_2\operatorname{Sp}^n(P)=\begin{cases} A\otimes A/2 & \text{if } n=2^k+1 \text{ for some } k\geq 1,\\ 0 & \text{otherwise.}\end{cases}$$

Proof of Theorem 4.1. Much of the work for this proof has been done in Sections 2 and 3. We start with the formula of Theorem 2.9 and determine the needed homotopy orbits directly, before using the calculations of Proposition 3.7 to obtain the result.

Recall from Theorem 2.9 that

$$D_k\operatorname{Sp}^n\simeq\bigoplus_{[\mathbf{t}]\in[V_k(n)]}(D\operatorname{Sp}^{\mathbf{t}})_{h\mathrm{st}(\mathbf{t})}$$

where for $\mathbf{t}=(t_1,\dots,t_k)$, $D\operatorname{Sp}^{\mathbf{t}}=D_1\operatorname{Sp}^{t_1}\otimes\cdots\otimes D_1\operatorname{Sp}^{t_k}$. The Künneth theorem, Corollary 3.6, and the fact that homotopy orbits preserve connectivity guarantee that $H_tD_k\operatorname{Sp}^n(P)\cong 0$ for $t<(2t_1-2)+\cdots+(2t_k-2)=2n-2k$. Moreover, it follows directly from Corollary 3.6, the definition of homotopy orbits (see, for example, [JM2, 3.5]), and Künneth that

$$\begin{aligned} H_{2n-2k}D_k\operatorname{Sp}^n(P) &\cong\bigoplus_{[\mathbf{t}]\in[V_k(n)]}H_{2n-2k}(D\operatorname{Sp}^{\mathbf{t}}(P)_{h\mathrm{st}(\mathbf{t})})\\ &\cong\bigoplus_{[\mathbf{t}]\in[V_k(n)]}(H_{2n-2k}D\operatorname{Sp}^{\mathbf{t}}(P))_{\mathrm{st}(\mathbf{t})}\\ &\cong\bigoplus_{[(t_1,\dots,t_k)]\in[V_k(n)]}\Big(\bigotimes_{i=1}^{k}H_{2t_i-2}D_1\operatorname{Sp}^{t_i}(P)\Big)_{\mathrm{st}(\mathbf{t})}.\end{aligned}$$

Recall that $\mathbf{t}=(t_1,\dots,t_k)\in V_k(n)$ represents a partition of n, i.e., $t_1+\cdots+t_k=n$. By grouping together equal t_is, such a partition can be written in the form

$$c_1u_1+c_2u_2+\cdots+c_su_s=n \tag{4.3}$$

with $u_i\neq u_j$ for $i\neq j$, $c_1+\cdots+c_s=k$, and $c_1,\dots,c_s\geq 1$. Let $\mathbf{u}$ denote the element of $[\mathbf{t}]$ corresponding to this rearrangement. Then clearly $\mathrm{st}(\mathbf{u})=$

$\Sigma_{c_1} \times \cdots \times \Sigma_{c_s}$. Moreover,

$$(H_{2n-2k}(D\,\mathrm{Sp}^{\mathbf{u}}))_{\mathrm{st}(\mathbf{u})} \cong \bigotimes_{j=1}^{s} (\otimes^{c_j} H_{2u_j-2} D_1 \,\mathrm{Sp}^{u_j})_{\Sigma_{c_j}}.$$

To determine the orbits $(\otimes^{c_i} H_{2u_i-2} D_1 \,\mathrm{Sp}^{u_i})_{\Sigma_{c_i}}$, we consider the action of an arbitrary transposition $(jl) \in \Sigma_{c_i}$ on the complex $\otimes^{c_i} D_1 \,\mathrm{Sp}^{u_i}$. The transposition (jl) acts on $\otimes^{c_i} D_1 \,\mathrm{Sp}^{u_i}$ by interchanging the jth and lth tensorands. The careful reader will recognize that there must be a sign associated to the (jl)-action in order to produce a chain map from $\otimes^{c_i} D_1 \,\mathrm{Sp}^{u_i}$ to itself that is consistent with the sign convention for total complexes. In particular, (jl) acts by taking the element $a_1 \otimes \cdots \otimes a_j \otimes \cdots \otimes a_l \otimes \cdots \otimes a_{c_i}$ of multidegree $(b_1, \ldots, b_j, \ldots, b_l, \ldots, b_{c_i})$ to the element

$$(-1)^{b_j(b_{j+1}+\cdots+b_l)+b_l(b_{j+1}+\cdots+b_{l-1})} a_1 \otimes \cdots \otimes a_l \otimes \cdots \otimes a_j \otimes \cdots \otimes a_{c_i}.$$

However, for the present computation, we are interested only in elements of multidegree $(2u_i-2, 2u_i-2, \ldots, 2u_i-2)$ and so the sign is $+1$. As a result,

$$(H_{2n-2k}(D\,\mathrm{Sp}^{\mathbf{u}}(P)))_{\mathrm{st}(\mathbf{u})} \cong \bigotimes_{j=1}^{s} \mathrm{Sp}^{c_j}(H_{2u_j-2} D_1 \,\mathrm{Sp}^{u_j}(P)). \tag{4.4}$$

When $u_1 = 1$, it follows by Proposition 3.7 that (4.4) is isomorphic to

$$\mathrm{Sp}^{c_1}(A) \otimes \mathrm{Sp}^{c_2}(H_{2u_2-2} D_1 \,\mathrm{Sp}^{u_2}(P)) \otimes \cdots \otimes \mathrm{Sp}^{c_s}(H_{2u_s-2} D_1 \,\mathrm{Sp}^{u_s}(P)). \tag{4.5}$$

If $k = n$, the only partition of n into k positive integers is the trivial partition $n = 1 + \cdots + 1$. In this case, (4.5) gives us

$$H_{2n-2k} D_k \,\mathrm{Sp}^n(P) \cong \mathrm{Sp}^n(A).$$

For $k < n$, recall from Proposition 3.7 that $H_{2t-2} D_1 \,\mathrm{Sp}^t(P)$ is nontrivial if and only if t is 1 or t is a positive power of a prime. In particular, $H_{2t-2} D_1 \,\mathrm{Sp}^t(P) \cong A/p$ when t is a positive power of a prime p, and $H_{2t-2} D_1 \,\mathrm{Sp}^t(P) \cong A$ when $t = 1$. As a result, the summands (4.4) and (4.5) are nontrivial if and only if $u_1, \ldots, u_s$ are all nonnegative powers of the same prime, that is, if the partition (4.3) has the form $c_1 p^{r_1} + \cdots + c_s p^{r_s}$ where $r_i \neq r_j$ if $i \neq j$. Applying Proposition 3.7 to (4.4) and (4.5), and setting a_1 equal to the number of p^0s appearing in a partition, we obtain the desired result

$$H_{2n-2k} D_k \,\mathrm{Sp}^n(P) \cong \bigoplus_{\substack{a_1 + a_2 p^{r_2} + \cdots + a_s p^{r_s} = n \\ p \text{ is prime},\, 0 < r_2 < \cdots < r_s \\ a_1 + \cdots + a_s = k \\ a_2, \ldots, a_s \geq 1;\, a_1 \geq 0}} \mathrm{Sp}^{a_1}(A) \otimes \bigotimes_{j=2}^{s} \mathrm{Sp}^{a_j}(A/p).$$

Using the exact sequence $D_k \,\mathrm{Sp}^n \to P_k \,\mathrm{Sp}^n \to P_{k-1} \,\mathrm{Sp}^n$, we determine the first nontrivial homology of $P_k \,\mathrm{Sp}^n$ as well.

LEMMA 4.6. *Let P be a projective resolution of the abelian group A and $n, k \geq 2$. Then*

$$H_* P_k \operatorname{Sp}^n(P) \cong H_* D_k \operatorname{Sp}^n(P)$$

for $* \leq 2n - 2k$.

Proof. We begin by noting that $D_1 \operatorname{Sp}^n \cong P_1 \operatorname{Sp}^n$ since $P_0 \operatorname{Sp}^n = \operatorname{Sp}^n(0) \cong 0$. Hence, the first nontrivial homology group of $P_1 \operatorname{Sp}^n(P)$ occurs no lower than degree $2n-2$. From the sequence of functors $D_2 \operatorname{Sp}^n \to P_2 \operatorname{Sp}^n \to P_1 \operatorname{Sp}^n$, we obtain a long exact sequence of homology groups

$$\cdots \to H_* D_2 \operatorname{Sp}^n(P) \to H_* P_2 \operatorname{Sp}^n(P) \to H_* P_1 \operatorname{Sp}^n(P) \to \cdots .$$

Since $D_2 \operatorname{Sp}^n(P)$ is $(2n-5)$-connected and $P_1 \operatorname{Sp}^n(P)$ is $(2n-3)$-connected, it must be the case that $H_* P_2 \operatorname{Sp}^n(P) \cong H_* D_2 \operatorname{Sp}^n(P)$ for $* \leq 2n-4$. The result follows inductively in this fashion, by using the long exact homology sequence associated to $D_k \operatorname{Sp}^n \to P_k \operatorname{Sp}^n \to P_{k-1} \operatorname{Sp}^n$ and the fact that $H_* D_k \operatorname{Sp}^n$ vanishes for $* < 2n - 2k$.

COROLLARY 4.7. *Let A be a finitely generated abelian group, P be a projective resolution of A and $n, k \geq 2$. The lowest degree in which $P_k \operatorname{Sp}^n(P)$ can have nontrivial homology is $2n - 2k$. For $k = n$,*

$$H_{2n-2k} P_k \operatorname{Sp}^n(P) \cong \operatorname{Sp}^n(A).$$

For $k < n$,

$$H_{2n-2k} P_k \operatorname{Sp}^n(P) \cong \bigoplus_{\substack{a_1 + a_2 p^{r_2} + \cdots + a_s p^{r_s} = n \\ p \text{ is prime}, 0 < r_2 < \cdots < r_s \\ a_1 + \cdots + a_s = k \\ a_2, \ldots, a_s \geq 1;\, a_1 \geq 0}} \operatorname{Sp}^{a_1}(A) \otimes \bigotimes_{j=2}^{s} \operatorname{Sp}^{a_j}(A/p).$$

The lowest nontrivial homology of $D_k \Lambda^n(P)$ can be determined in a manner similar to that used for $D_k \operatorname{Sp}^n(P)$. We note that for odd primes p one works with even-dimensional homology classes and the Σ_k action is the usual action with signature. When $p = 2$, one works with odd-dimensional homology classes, except for $D_1 \Lambda^1 \cong \mathrm{Id}$, and so the permutation signature is cancelled by the sign arising from the action on the tensor product of chain complexes in homology above degree 0. This gives us the following.

THEOREM 4.8. *Let A be a finitely generated abelian group, P be a projective resolution of A, and $n, k > 1$. The lowest degree in which $D_k \Lambda^n(P)$ can have nontrivial homology is $n - k$. For $k = n$,*

$$H_{n-k} D_k \Lambda^n(P) \cong \Lambda^n(A).$$

For $k < n$,

$$H_{n-k} D_k \Lambda^n(P) \cong \bigoplus_{\substack{a_1 + a_2 p^{r_2} + \cdots + a_s p^{r_s} = n \\ p \text{ is prime}, 0 < r_2 < \cdots < r_s \\ a_1 + \cdots + a_s = k \\ a_2, \ldots, a_s \geq 1;\, a_1 \geq 0}} \Lambda^{a_1}(A) \otimes \Lambda_{a_2 p^{r_2}, \ldots, a_s p^{r_s}}(A)$$

where

$$\Lambda_{a_2 p^{r_2}, \ldots, a_s p^{r_s}}(A) \cong \begin{cases} \Lambda^{a_2}(A/p) \otimes \cdots \otimes \Lambda^{a_s}(A/p) & \text{if } p \text{ is odd}, \\ Sp^{a_2}(A/2) \otimes \cdots \otimes Sp^{a_s}(A/2) & \text{if } p = 2. \end{cases}$$

REMARK 4.9. The gap between the lowest nontrivial homology of $D_k \operatorname{Sp}^n$ and $D_{k-1} \operatorname{Sp}^n$ is just large enough for us to conclude that

$$H_{2n-2k} P_k \operatorname{Sp}^n(P) \cong H_{2n-2k} D_k \operatorname{Sp}^n(P).$$

An analogous result does not appear to hold for the exterior powers. To determine $H_{n-k} P_k \Lambda^n(P)$, one must consider possible extensions as well. We leave this to another paper.

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